3D CHERN-SIMONS-LIKE GRAVITY MODELS FOR $N = 2, 3, 4$

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

BÜŞRA DEDEOĞLU

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
PHYSICS

JANUARY 2022

Approval of the thesis:

3D CHERN-SIMONS-LIKE GRAVITY MODELS FOR $N = 2, 3, 4$

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ABSTRACT

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January 2022, 74 pages

Our objective in this thesis is to find and study three-dimensional gravity models which are broadly termed as “Chern-Simons-like models” for $N = 2, 3, 4$ cases. First, the term “Chern-Simons-like" is discussed and the difference between the Chern-Simons theory and Chern-Simons-like theory is given. Starting from the most generic Lagrangians which are invariant under local Lorentz transformations, the conditions that must be fulfilled so that the Bianchi identity (in the first order formalism) holds is found for each case. To determine whether the models found by using local Lorentz invariance and Bianchi identity are physical, Hamiltonian analysis is used.

Keywords: 3D gravity, Chern-Simons-like gravity, massive gravity

ÖZ

$N = 2, 3, 4$ İÇİN 3 BOYUTLU CHERN-SİMONS BENZERİ KÜTLEÇEKİM MODELLERİ

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Ocak 2022 , 74 sayfa

Bu tezin amacı, üç boyutta, $N = 2, 3, 4$ durumları için genel olarak “Chern-Simons benzeri modeller” olarak isimlendirilmiş modeller bulup çalışmaktır. Önce, “Chern-Simons-benzeri" terimi tartışılıp, Chern-Simons teorisiyle farklılıkları verilir. Lokal Lorentz dönüşümleri altında değişmez kalan en genel Lagrangianlardan başlayarak, birinci derece formalizminde Bianchi özdeşliğini sağlayan koşullar her bir durum için bulunur. Lokal Lorentz dönüşümleri ve Bianchi özdeşliği ile bulunan modellerin fiziksel olup olmadıklarını anlamak için Hamilton analizi yapılır.

Anahtar Kelimeler: 3D kütleçekim, Chern-Simons benzeri kütleçekim, kütleli kütleçekim

To Berrin & Turgut Dedeoğlu

ACKNOWLEDGMENTS

First and foremost, I would like to thank my supervisor Prof. Bahtiyar Özgür Sarıoğlu. Thanks to his guidelines, I have improved myself more than I could imagine. It is a great honor being his student.

I would like to thank Assoc. Prof. Dr. Mehmet Özkan for the discussion on the algebra of the group of Chern-Simons-like theory.

Also, I am thankful to Kemal Gümüş and Berk Özcan for help with Mathematica.

I am grateful to my family who supports and believes in me all the time. If I were reborn, I would have choosen them as my family. I thank Gonca Ünder for being the best friend in the world. I am also grateful to Burak Daylak, without his support this thesis could not be finished.

Finally, I am thankful to my cats Neuron as my study-buddy, Tensor who wrote the first words of this thesis and Carbon for his love.

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LIST OF ABBREVIATIONS

$\eta_{\mu\nu} = \text{diag}(-1, 1, 1)$	3D Minkowski metric
3D	3-Dimensional
4D	4-dimensional
GR	General relativity
MB	Mielke–Baekler
TMG	Topological massive gravity
MMG	Minimal massive gravity
CFT	Conformal field theory
NMG	New massive gravity
GMG	General massive gravity
EMG	Exotic massive gravity
EGMG	Exotic general massive gravity
ZDG	Zwei-dreibein gravity
LLT	Local Lorentz transformation

CHAPTER 1

INTRODUCTION

From the perspective of particle physics, general relativity (GR) can be seen as a massless spin-2 theory. The restriction on mass was due to the difficulty of finding a physical interactive theory whose linearization gives Fierz-Pauli theory of a massive spin-2 particle [1, 2]. Fierz and Pauli published the requirements for positivity of energy and Lorentz invariance of the action for a free massive spin-2 particle [3, 4]. However, the massless limit of Fierz-Pauli theory does not give GR. This is known as the van Dam-Veltman-Zakharov (vDVZ) discontinuity shown by van Dam and Veltman [5] and Zakharov [6] independently. When a massless spin-2 particle carries two degrees of freedom, the massive one carries five: a helicity 0 state which is a scalar, the helicity ± 1 modes which is a vector and helicity ± 2 modes of massless graviton. The massless limit of Fierz-Pauli theory gives extra decoupled vector modes with the helicity ± 2 states give linearized GR. The coupling of the scalar mode and the trace of the stress tensor describes the matter part. These couplings give two different physical models for massive and massless cases. To solve this problem, the non-linear effects are used. The nonlinear realization of Fierz-Pauli theory gives six degrees of freedom instead of five. The new degree of freedom is a scalar-ghost [1]. To get rid of this scalar-ghost, higher-order interactions should be added to the theory. Such a gravity theory is known as de Rham, Gabadaze, Tolley (dRGT) massive gravity [7, 8].

Even though we live in a four dimensional (4D) world and all of the above theories were studied for $D = 4$, there are many reasons to study the models for $D = 3$. Three dimensions can be used as a toy model to understand the quantum gravity models since the calculations are easier than those in the $D = 4$ case. Also, three dimensions (3D) is special since there are no degrees of freedom for the massless case. When the mass term is added to the model, a propagating dynamical model is obtained. To construct a ghost-free theory, apart from the addition of explicit mass term to the theory, another way is to add higher order derivatives of the metric [8]. When the mass term is added to the model, a propagating dynamical model is obtained. The mass term is related to the gravitational Chern-Simons term. Separately it was also shown that the Einstein-Hilbert action can be written as a pure gauge theory in 3D but not in 4D [9]. In noncoordinate basis language, this gauge theory (this is the so-called *Chern-Simons* action and is an $N = 2$ model) can be written by using the dreibein and the spin connection on the assumption of the invertibilty of the dreibein. Chern-Simons model is not the only way to construct a massive gravity model. Except for the dreibein and the spin connection, $N - 2$ auxiliary fields can be added to the action. Then the integral of a Lagrangian three form constructed with N Lorentz vector valued one-forms is defined without an explicit space-time metric. The model is called a *Chern-Simons-like* theory. In a Chern-Simons-like theory, the individual fields are henceforth Lie-algebra valued one forms [10]. Under the assumption of the invertibilty of the dreibein, the degrees of freedom can be found. For $N = 2$ the models are Witten's "exotic" gravity [9] and Mielke–Baekler (MB) action [11]. The equations of motion of both models have a source of Lorentzian curvature and torsion. It is Lorentzian not Riemannian due to the first order spin connection ω. If the curvature equation is written in terms of metric, $R_{\mu\nu} \propto 2\Lambda g_{\mu\nu}$ is obtained. This means that MB model gives a spacetime with constant curvature Λ. It is known that the condition which makes MB action torsionless gives Witten's "exotic" action.

When one auxiliary field is added to the model, TMG which is obtained by adding a parity violating Chern-Simons term [10, 12, 13] and minimal massive gravity (MMG) [13, 14] are obtained. TMG is the simplest Chern-Simons-like model. It gives a massive, propagating, spin-2 degrees of freedom. Although the addition of the topological term (Chern-Simons term) gives third order time derivative, the model is ghost-free and unitary [12]. On the boundary of AdS_3 vacuum of TMG is the holomorphically dual conformal field theory (CFT). The central charge of CFT of TMG is found negative. It is non-unitary. MMG is an alternative which solves this problem. It has the same minimal local structure as TMG, this is the reason why the model is so-called "Minimal Massive Gravity". The difference between field equations of MMG and TMG is that MMG field equation includes a curvature-squared symmetric term. MMG action is obtained by adding a term to TMG in dreibein formulation. This action cannot be captured adding higher order derivatives of metric to the TMG action written in terms of metric and its derivatives.

New massive gravity (NMG), general massive gravity (GMG), exotic massive gravity (EMG) and exotic general massive gravity (EGMG) are the $N = 4$ models written with two auxiliary fields. NMG includes curvature-square-terms, due to this term the field equations include parity-preserving fourth-order derivatives of the metric. In Minkovski vacuum of NMG, there are two massive spin-2 modes propagating unitarily. NMG is a special case of a more general, parity violating model called GMG [10]. GMG Lagrangian is obtained by including the Chern-Simons term to NMG Lagrangian. GMG has two propagating massive gravitons with different masses [10]. In Minkowski vacuum, linearized limit of NMG and EMG, GMG and EGMG are the same. EMG is an "exotic" model since it has parity-preserving field equations although its action is parity-odd. EGMG is obtained by the addition of a parity-violating Cotton tensor term to the field equation of EMG. EGMG is the parity-violating generalization of the EMG. The field equations of both models include fourth order derivative of the metric. Both models are not unitary since one of the spin-2 modes is a ghost. If the holographic dual CFT of these models are analysed, it is observed that the product of two central charges is negative. Thus, the CFT of these models are also non-unitary [2].

The algebra of the Chern-Simons-like models for arbitrary but finite number of fields are studied in [15]. The method is to write Lagrangians whose field equations are linear in auxiliary fields. The lowest degree auxiliary fields are proportional to Schouten tensor and Cotton tensor respectively. If the number of the auxiliary fields is increased, then their solutions are highest order derivatives of Schouten and Cotton tensors. Thus Schouten and Cotton tensors are used as building blocks to construct physical models that satisfy Bianchi identities as discussed in [16]. If the hierarchy of the field equations for N fields is extended to infinite number of flavours, the algebra of the model does not give any dynamics [15].

In this thesis, we aim to find if there exist other theories for $N = 2, 3, 4$. We start from the most generic Lagrangians for each case and then use the Bianchi identity in the first order formalism to find new models. Next, the degrees of freedom are counted to determine if the model is physical or not.

Chapter 2 of this thesis will serve as a review, in which we will discuss the motivation to study 2+1 dimensional gravity and its mathematical background. The non-coordinate bases will be introduced for the special case $D = 3$. The relation between the metric and the dreibein, the definitions of torsion, curvature and covariant derivative in a non-coordinate basis will be given. The Chern-Simons theory will be studied. An invertible dreibein e can be seen as an isometry between the tangent bundle of the manifold that we study and an abstract D-dimensional vector bundle. By using this isomorphism, Einstein-Hilbert action can be written as a gauge theory. However this is not possible in every dimension. In 3D, it is possible with the group $ISO(2, 1)$. The algebra of this group will be studied. Then we will try to generalise the Chern-Simons gravity as a massive gravity theory by using Lorentz-vector valued 1-forms. The theory is called "Chern-Simons-like" since the action includes the dreibein, the spin connection and higher derivatives of dreibein are defined without metric. For this first order formalism, the Cartan structure equations will be reviewed and the algorithm to check if the Bianchi identity is satisfied will be given for a generic N. To do this, the cases that the torsion vanishes or not will be examined separately. In order to determine if the theory is physical, i.e. if it allows the correct number of degrees of freedom, we will make a Hamiltonian analysis. We will calculate the primary and secondary constraints and discuss if they are first-class or second-class by following the assumptions given in [8]. These assumptions hold for all known models. First of all, we will assume that all Poisson brackets of secondary constraints with other secondary constraints vanish on the constraint surface. The second assumption is that when the secondary constraints are added to the total Poisson bracket matrix, they are linearly independent of the sub-matrix calculated by the Poisson brackets of the primary constraints. This means that, the addition of the secondary constraints does not lead to new first-class constraints. All secondary constraints are assumed to be second-class. After obtaining the Poisson bracket matrix, we will count the degrees of freedom.

In Chapter 3, we will start by applying what we have learnt from Chapter 2 for $N = 2$, i.e. the study of the models that include only the dreibein and the spin connection. At the beginning, the most generic Lagrangian will be written and by using the algorithm to check Bianchi identity the condition(s) to write a physical model will be found. Then the degrees of freedom will be counted by using Hamiltonian analysis.

In Chapter 4, we focus on the problem when we add only one auxiliary field to Lagrangian. For this case the known models are TMG and MMG. We will examine if there is another model by using Bianchi identity. After finding Lagrangians, the auxiliary fields and the metric equations will be found for each model. Then to determine the number of the spin-2 modes that propagates, the Hamiltonian analysis will be done.

In Chapter 5, we will add two auxiliary fields, i.e. work at the $N = 4$ case. The models will be found with and without torsion by using the Bianchi identity separately. The auxiliary fields and the metric equations will be found in terms of metric and its higher order derivatives for each case. Hamiltonian analysis will be done to find the number of the local degrees of freedom.

In Chapter 6, we will give a summary of our findings presented in this thesis, and indicate a few directions for future research.

CHAPTER 2

THREE DIMENSIONAL GRAVITY

In this chapter, we introduce the non-coordinate basis, try to generalise Bianchi identity and Hamiltonian analysis for an arbitrary number of Lorentz-valued one form fields to use in next chapters.

2.1 Non-coordinate basis

On a (pseudo)Riemannian manifold with a metric $(\mathcal{M}, g)$, tangent vectors defined at point p form tangent space $T_p\mathcal{M}$ which is locally spanned by $\{e_\mu\} = \{\partial/\partial x^\mu\}$ and the dual vectors form the cotangent space $T_p^*\mathcal{M}$ which is locally spanned by $\{dx^\mu\}$. There is an alternative choice which is obtained by a rotation of the coordinate basis [17]

$$e_a = e_a^\mu \frac{\partial}{\partial x^\mu} \qquad \{e_a^\mu\} \in GL(D, \mathbb{R}), \tag{2.1}$$

where $D = \dim(\mathcal{M})$. For the Lorentzian metric, the general relation between the metric and the basis is

$$g(e_a, e_b) = e_a^\mu e_b^\nu g_{\mu\nu} = \eta_{ab}. \tag{2.2}$$

If the metric is Euclidean, η is replaced by δ. The inverse of e_a^μ is $e_a^\mu e_\nu^a = \delta_\nu^\mu$ and $e_a^\mu e_\mu^b = \delta_a^b$. Then the dual basis $\{e^a\}$ is defined by inner product $\langle e^a, e_b\rangle = \delta_b^a$. The bases $\{e^a\}$ and $\{e_a\}$ are called non-coordinate bases. The coefficients e_a^μ are called vielbeins for a finite but many dimensional manifold and in $D = 3$ it is called dreibein [17]. Greek indices are used for denoting the coordinate basis of the curved space whereas the Latin indices denote non-coordinate basis of flat space. Vielbeins can be thought as the square root of the metric [8]

$$g_{\mu\nu}(x) = e_\mu^a(x)\eta_{ab}e_\nu^b(x). \tag{2.3}$$

Even if the definition of non-coordinate bases and the related definitions below are valid in any dimension D, in the rest of the thesis $D = 3$ will be used. The notation for an arbitrary D in [18] is adopted for $D = 3$. Thus, in (2.3), $\eta_{ab} = diag(-1, 1, 1)$ is the metric of 3-dimensional Minkowski spacetime. The dreibein is not unique. One can introduce new non-coordinate basis without changing the coordinate basis. This transformation should preserve the orthogonality condition and flat metric. This transformation should be done at any point p, thus it is a local Lorentz transformation [8]. The dreibein transforms under local Lorentz transformations (LLT) as

$$e'^a_\mu(x) = \Lambda^{-1a}{}_b(x) e^b_\mu(x) \qquad \Lambda \in SO(2,1). \tag{2.4}$$

The difference of the LLT in curved spacetime and the global Lorentz transformation is that there is no coordinate change, only the frame indices $a, b, \cdots$ transform in LLT. The dreibein transforms as a covector under diffeomorphisms

$$e'^a_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} e^a_\nu(x). \tag{2.5}$$

Introduce a 2-form by using the 1-form e^a_μ

$$de^a = (\partial_\mu e^a_\nu - \partial_\nu e^a_\mu) dx^\mu \wedge dx^\nu. \tag{2.6}$$

Under LLT, this 2-form transforms as

$$de'^a = d\big(\Lambda^{-1a}{}_b e^b\big) = \Lambda^{-1a}{}_b de^b + d\Lambda^{-1a}{}_b \wedge e^b. \tag{2.7}$$

Due to the second term, the 2-form de^a cannot transform as a local Lorentz vector. To avoid it, the spin connection is added and the torsion vector is obtained

$$de^a + \omega^a{}_b \wedge e^b \equiv T^a, \tag{2.8}$$

which is known as the first Cartan structure equation. The spin connection is transformed as

$$\omega'^a{}_b = \Lambda^{-1a}{}_c d\Lambda^c{}_b + \Lambda^{-1a}{}_c \omega^c{}_d \Lambda^d{}_b. \tag{2.9}$$

The component of the spin connection is $\omega_\mu{}^a{}_b$ and it transforms as a covector under diffeomorphisms. Spin connection is used for defining the local Lorentz covariant derivative as

$$\begin{aligned} D_\mu V^a &= \partial_\mu V^a + \omega_\mu{}^a{}_b V^b, \\ D_\mu U_a &= \partial_\mu U_a + \omega_\mu{}_a{}^b U_b. \end{aligned} \tag{2.10}$$

The covariant derivative of the Minkovski metric vanishes since it is an invariant tensor of the Lorentz group

$$D_\mu \eta_{ab} = \partial_\mu \eta_{ab} + \omega_{\mu\ a}^{\ \ \ c} \eta_{cb} + \omega_{\mu\ b}^{\ \ \ c} \eta_{ac} = \omega_{\mu\ a\ b} + \omega_{\mu\ b\ a} = 0, \tag{2.11}$$

thus the spin connection is antisymmetric in frame indices.

In equation (2.10) the local Lorentz covariant derivatives of local Lorentz covariant and contravariant vectors are given. There is a transformation of Lorentz covariant derivatives to covariant derivatives with respect to the general coordinate transformation [18]

$$\begin{aligned} \nabla_\mu V^\nu &:= e^\nu_a D_\mu V^a \\ &= e^\nu_a D_\mu (e^a_\lambda V^\lambda) \\ &= \partial_\mu V^\nu + e^\nu_a (\partial_\mu e^a_\lambda + \omega_\mu{}^a{}_b e^b_\lambda) V^\lambda \\ &= \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda . \end{aligned} \tag{2.12}$$

Then the affine connection is related to the spin connection by

$$\Gamma^\nu_{\mu\lambda} = e^\nu_a (\partial_\mu e^a_\lambda + \omega_\mu{}^a{}_b e^b_\lambda). \tag{2.13}$$

This allows one to write the vielbein postulate that is related to the metricity property

$$\nabla_\mu e^a_\nu = \partial_\mu e^a_\nu + \omega_\mu{}^a{}_b e^b_\nu - \Gamma^\lambda_{\mu\nu} e^a_\lambda = 0. \tag{2.14}$$

Both the vielbein postulate and the metric compatibility are independent of the existence of torsion.

In terms of spin connection, the Riemann curvature tensor is

$$R_{\mu\nu ab} := \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} + \omega_{\mu ac} \omega_\nu{}^c{}_b - \omega_{\nu ac} \omega_\mu{}^c{}_b . \tag{2.15}$$

Note that, it transforms as a $(0, 2)$ type tensor under LLT, and as a $(0, 2)$ type tensor under general coordinate transformations. The curvature 2-form can be defined by using the Riemann curvature tensor:

$$\begin{aligned} R^{ab} &= \frac{1}{2} R_{\mu\nu}{}^{ab}(x) dx^\mu \wedge dx^\nu \\ &= d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}, \end{aligned} \tag{2.16}$$

this is the second Cartan structure equation. In $D = 3$ the dualised curvature form can be written as [8]

$$R^a = \frac{1}{2} \epsilon^{abc} R_{bc}, \tag{2.17}$$

where

$$R^a = d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b \wedge \omega_c, \tag{2.18}$$

where ϵ^{abc} is the Levi-Civita symbol in local frame and its indices are lowered (raised) by Minkowski metric η_{ab} (η^{ab}). Note that we use and define $\epsilon^{012} = -1$. The relations of Levi-Civita tensor in local frame and the tensor density $\varepsilon^{\mu\nu\lambda}$ is given in [18]

$$\begin{aligned} \varepsilon_{\mu_1\mu_2\mu_3} &:= e^{-1}\epsilon_{a_1a_2a_3}e^{a_1}_{\mu_1}e^{a_2}_{\mu_2}e^{a_3}_{\mu_3}, \\ \varepsilon^{\mu_1\mu_2\mu_3} &:= e\epsilon^{a_1a_2a_3}e^{\mu_1}_{a_1}e^{\mu_2}_{a_2}e^{\mu_3}_{a_3}, \end{aligned} \tag{2.19}$$

where $e = \det e^a_\mu = \sqrt{-\det g}$. The canonical volume form is

$$\begin{aligned} dV &:= e^0 \wedge e^1 \wedge e^2 \\ &= \frac{1}{3!}\epsilon_{a_1a_2a_3}e^{a_1}e^{a_2}e^{a_3} \\ &= \frac{1}{3!}e\varepsilon_{\mu_1\mu_2\mu_3}dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \\ &= d^3x\sqrt{-\det g}. \end{aligned} \tag{2.20}$$

2.2 Chern-Simons theory

Suppose that the tangent bundle of a smooth manifold $\mathcal{M}$ with dimension m is denoted by $T\mathcal{M}$ An abstract D-dimensional vector bundle $\mathcal{V}$ with a structure group $SO(D-1,1)$ can be introduced with a metric $\eta_{ab} = \text{diag}\underbrace{(-1,1,\cdots,1)}_{D}$ and a volume element $\epsilon_{a_1\cdots a_d}$. It is assumed that $T\mathcal{M}$ and $\mathcal{V}$ are topologically the same, then one can show that they are isomorphic. However this isomorphism is not unique. One choice of the isomorphism is vielbein e, if it is invertible. Then, the spin connection becomes the connection of the vector bundle $\mathcal{V}$. When the metric η_{ab} on $\mathcal{V}$ and $g_{\mu\nu}$ on $T\mathcal{M}$ are related as in (2.3), the Einstein-Hilbert action can be written by using the vielbein and the spin connection. In four dimensions Einstein action is $\int e \wedge e \wedge (d\omega + \omega \wedge \omega)$. It can be rewritten as a gauge action $\int A \wedge A \wedge (dA + A \wedge A)$ but there is no such an action in gauge theory. Thus, in four dimensions there is no Chern-Simons action, in other words we cannot interpret gravity as a gauge theory. However in three dimensions, Einstein-Hilbert action without the cosmological constant can be written as a Chern-Simons action that is a gauge theory with the group

$ISO(2,1)$ [9]

$$I_{\text{CS}} = \frac{1}{2}\int_{\mathcal{M}} \text{Tr}\Big(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\Big) \tag{2.21}$$

For an arbitrary basis, the gauge field A can be written as $A = A^a T_a$. Then the quadratic part of the equation (2.21) gives the kinetic terms of the action

$$\text{Tr}(T_a T_b)\cdot\int_{\mathcal{M}}(A^a\wedge dA^b). \tag{2.22}$$

The trace part gives the non-degenerate bilinear form of the algebra. For three dimensions, the invariant, non-degenerate bilinear form is $W = \epsilon_{abc}P^a J^{bc}$ where P^a are translations and J^{ab} are Lorentz generators. Using the duals of the generators $J^a = \frac{1}{2}\epsilon^{abc}J_{bc}$ simplifies the calculations. The invariant forms are

$$\langle\, J_a, P_b\rangle = \delta_{ab}, \qquad \langle\, J_a, J_b\rangle = \langle\, P_a, P_b\rangle = 0. \tag{2.23}$$

And the commutation relations are

$$[J_a, J_b] = \epsilon_{abc}J^c, \qquad [J_a, P_b] = \epsilon_{abc}P^c, \qquad [P_a, P_b] = 0. \tag{2.24}$$

The gauge field A can be written as

$$A_\mu = e^a_\mu P_a + \omega^a_\mu J_a. \tag{2.25}$$

If the cosmological constant is added to the Lagrangian, then the space-time becomes dS or AdS instead of Minkowski spacetime with a generalised action

$$I = \int_{\mathcal{M}}\Big(e\wedge R + \frac{\Lambda}{6}e\wedge e\wedge e\Big). \tag{2.26}$$

The action is equivalent to the action (2.21) if the gauge field is defined as

$$A^{a\pm}_\mu = \omega^a_\mu \pm \sqrt{\Lambda}e^a_\mu, \tag{2.27}$$

with

$$J^\pm_a = \frac{1}{2}\Big(J_a \pm \frac{1}{\sqrt{\Lambda}}P_a\Big). \tag{2.28}$$

The Lie algebra is

$$[J^\pm_a, J^\pm_b] = \epsilon_{abc}J^{c\pm}, \qquad [J^+_a, J^-_b] = 0. \tag{2.29}$$

Three dimensional GR can be thought of as a gauge theory if the dreibein is invertible. The field equations show that the degrees of freedom of the theory is zero which means that there is no propagating gravitons. One way to add propagating gravitons to the 3D GR is by constructing "massive" gravity theories that includes higher derivatives of the metric. In the first order formalism, this is equivalent to adding an auxiliary field which can be taken as a Lorentz-vector valued 1-form. Then the action becomes "Chern-Simons-like" since the action is defined without a metric. Consider the most general Lagrangian written as a collection of N Lorentz vector-valued 1-form fields $\{\Omega_\mu^{aR} dx^\mu\}$ [8]

$$L = \frac{1}{2} g_{RS} \Omega^{aR} \wedge d\Omega_a^S + \frac{1}{6} f_{RST} \epsilon_{abc} \Omega^{aR} \wedge \Omega^{bS} \wedge \Omega^{cT}, \tag{2.30}$$

where the indices a, b, c are flat indices of 1-forms and run from 0 to 2 whereas the indices R, S, T label different 1-forms and run from 1 to N. The first two of them are dreibein e^a and the spin connection $\omega^a{}_b$, the others are auxiliary fields that include higher derivatives of the dreibein. The coefficients f_{RST} and g_{RS} are assumed to be totally symmetric tensors of an N-dimensional "flavor space" with g_{RS} an invertible "metric" for this space [2, 10, 8]. The transformations of the fields under LLT are given:

$$\Omega'^a_\mu(x) = \Lambda^{-1a}{}_b(x) \Omega^b_\mu(x); \quad \Omega \neq \omega, \quad \omega'^a{}_b = \Lambda^{-1a}{}_c d\Lambda^c{}_b + \Lambda^{-1a}{}_c \omega^c{}_d \Lambda^d{}_b. \tag{2.31}$$

The spin connection is not a local Lorentz tensor because of the transformation. Since the Lagrangian (2.30) is invariant under LLT, the spin connection can be seen in the Lagrangian only in covariant derivatives, in the curvature form and the Chern-Simons term $\omega \wedge d\omega + \frac{1}{3}\omega \wedge \omega \wedge \omega$.

2.3 Bianchi identities in the first order formalism

The algorithm for checking the Bianchi identity for the first ordered system is given in [16]:

★ Order the field equations from the lowest degree field (dreibein) to the highest degree (auxiliary field). In the known theories (TMG, MMG, NMG, EMG, EEMG) the auxiliary fields arise linearly as unknowns so they can be solved until the last equation which gives the metric equation of the model.

★To check the Bianchi identity, the covariant derivative of the last equation is calculated. When doing that, using all previous equations and their covariant derivative is allowed. The final result should be zero.

★If the Bianchi identity is satisfied without using the last equation of motion itself (off-shell), then the theory is diffeomorphism invariant. If it is satisfied only on-shell (after using the last equation of motion itself), then the theory is *third way* consistent. The consistency condition implies $G_{\mu\nu} \sim T_{\mu\nu}$. Then the conservation is satisfied by (i) stress-energy tensor as an effect of matter equations of motion (ii) identically by other tensors including metric tensor and its derivatives. There is also a third way, it involves a "non-geometrical" action which is not constructed from metric and its derivatives alone [19].

Consider the model with N Lorentz valued 1-forms $\{\Omega_\mu^{aR} dx^\mu\}$. Two of these 1-forms are the dreibein and the spin connection and there are $(N-2)$ auxiliary fields which are denoted by capital latin letters. To apply the procedure, the field equations will be written in terms of the covariant derivatives

$$
\begin{aligned}
De &= \varphi_0 e \wedge e + \sum_{I=1}^{N-2} \varphi_I e \wedge f_I + \sum_{I \le J}^{N-2} \varphi_{IJ} f_I \wedge f_J, \\
R &= \chi_0 e \wedge e + \sum_{I=1}^{N-2} \chi_I e \wedge f_I + \sum_{I \le J}^{N-2} \chi_{IJ} f_I \wedge f_J, \\
Df_K &= \xi_{K,0} e \wedge e + \sum_{I=1}^{N-2} \xi_{K,I} e \wedge f_I + \sum_{I \le J}^{N-2} \xi_{K,IJ} f_I \wedge f_J, \quad K = 1, \cdots, N-2.
\end{aligned}
\tag{2.32}
$$

Since the torsion and its covariant derivative is used in the derivative of the last equation, the models with and without torsion are analysed separately.

2.3.1 Vanishing torsion

When torsion vanishes, the Bianchi identities become

$$
\begin{aligned}
0 &= DDe = R \wedge e, \\
0 &= DR = \sum_{I=1}^{N-2} \chi_I (-e \wedge Df_I) + \sum_{I \le J}^{N-2} \chi_{IJ} (Df_I \wedge f_J - f_I \wedge Df_J), \\
0 &= DDf_{N-2} = \sum_{I=1}^{N-2} \xi_{(N-2),I} (-e \wedge Df_I) + \sum_{I \le J}^{N-2} \xi_{(N-2),IJ} (Df_I \wedge f_J - f_I \wedge Df_J),
\end{aligned}
\tag{2.33}
$$

where $De = 0$ is used whenever it appears. The first equations gives the constraint equations of the model:

$$0 = DDe_a = R_{ab}\wedge e^b = 2\chi_0 e_a(e\cdot e)+\sum_{I=1}^{N-2}\chi_I e_a(f_I\cdot e)+\sum_{I\leq J}^{N-2}\chi_{IJ}\big(f_{Ia}(f_J\cdot e)-(e\cdot f_I)f_{Ja}\big), \tag{2.34}$$

where $\Omega^R \cdot \Omega_S := \Omega^{aR} \wedge \Omega^S_a$, $R, S = 1, \cdots, N$. Constraint equations are used to calculate the auxiliary fields f_K with $K = 1, \cdots, N-2$. These $N-2$ equations are

$$e \cdot f_K := e_a \wedge f^a_K = e_{a\mu} f^a_{K\nu} dx^\mu \wedge dx^\nu = 0, \qquad \forall K = 1, \cdots, N-2. \tag{2.35}$$

Therefore totally symmetric tensors can be defined by $f_{K\mu\nu} := f^a_{K\mu} e_{a\nu}$. The covariant derivative of the curvature is

$$\begin{aligned} DR_a &= \epsilon_{abc}\Bigg(2\chi_0 De^b \wedge e^c + \sum_{I=1}^{N-2}\chi_I\big(De^b \wedge f^c_I - e^b \wedge Df^c_I\big) \\ &\qquad\qquad + \sum_{I\leq J}^{N-2}\chi_{IJ}\big(Df^b_I \wedge f^c_J - f^b_I \wedge Df^c_J\big)\Bigg) \\ &= \sum_{I\leq J}^{N-2}\chi_{IJ}\Bigg(\xi_{I,K} e_a(f_K \cdot f_J) + \xi_{J,K} e_a(f_K \cdot f_I) + \sum_{K\leq M}^{N-2}\Big(\xi_{I,KM}\Big(f_{Ka}(f_M \cdot f_J) \\ &\qquad - \big(f_{Ma}(f_J \cdot f_K)\big) - \xi_{J,KM}\big(f_{Ma}(f_I \cdot f_K) - f_{Ka}(f_M \cdot f_I)\big)\Big)\Bigg) \\ &= 0. \end{aligned} \tag{2.36}$$

For generic auxiliary fields, the dot product $f_K \cdot f_J$ does not vanish. In known models (TMG, MMG, NMG, GMG, E(G)MG), the coefficient χ_{IJ} is zero. Setting $\chi_{IJ} = 0$ satisfies the Bianchi identity and linearizes the equation in terms of auxiliary fields so it can be solved. The covariant derivative of the last equation determines if the model is on-shell or off-shell. To find the Bianchi identity for an on-shell model, the covariant derivative of the last equation is calculated:

$$
\begin{aligned}
DDf^a_{N-2} = \epsilon^{abc}\Bigg(& 2\xi_{(N-2),0}De_b\wedge e_c + \sum_{I=1}^{N-2}\xi_{(N-2),I}\big(De_b\wedge f_{Ic} - e_b\wedge Df_{Ic}\big) \\
& + \sum_{I\leq J}^{N-2}\xi_{(N-2),IJ}\big(Df_{Ib}\wedge Df_{Jc} - f_{Ib}\wedge Df_{Jc}\big)\Bigg) \\
= & \sum_{I\leq J}^{N-2}\xi_{(N-2),IJ}\Bigg(\sum_{K=1}^{N-2}\big[\xi_{I,K}e^a(f_K\cdot f_J) + \xi_{J,K}e^a(f_K\cdot f_I)\big] \\
& + \sum_{K\leq M}^{N-2}\Big(\xi_{I,KM}\big[f^a_K(f_M\cdot f_J) - f^a_M(f_J\cdot f_K)\big] \\
& + \xi_{J,KM}\big[f^a_K(f_M\cdot f_I) - f^a_M(f_I\cdot f_K)\big]\Big)\Bigg) \\
= & \; 0.
\end{aligned}
\tag{2.37}
$$

Since the term $f_K \cdot f_J$ does not vanish if $K \neq J$, Bianchi identity is satisfied if the summation is zero.

To find the Bianchi identity for an off-shell model, the last term in the summation in field equation should be written separately such that

$$
\begin{aligned}
Df_{N-2} = & \xi_{(N-2),0}e\wedge e + \sum_{I=1}^{N-3}\xi_{(N-2),I}e\wedge f_I + \xi_{(N-2),(N-2)}e\wedge f_{N-2} \\
& + \sum_{I\leq J}^{N-3}\xi_{(N-2),IJ}f_I\wedge f_J + \sum_{I=1}^{N-2}\xi_{(N-2),I(N-2)}f_I\wedge f_{N-2}.
\end{aligned}
\tag{2.38}
$$

Note that this equation can be written by using the curvature form:

$$
\begin{aligned}
Df_{N-2} = & \Big(\xi_{(N-2),0} - \frac{\chi_0}{\chi_{N-2}}\xi_{(N-2),(N-2)}\Big)e\wedge e + \frac{\xi_{(N-2),(N-2)}}{\chi_{N-2}}R \\
& + \sum_{I=1}^{N-3}\Big(\xi_{(N-2),I} - \frac{\chi_I}{\chi_{N-2}}\xi_{(N-2),(N-2)}\Big)e\wedge f_I \\
& + \sum_{I\leq J}^{N-3}\Big(\xi_{(N-2),IJ} - \frac{\chi_{IJ}}{\chi_{N-2}}\xi_{(N-2),(N-2)}\Big)f_I\wedge f_J \\
& + \sum_{I=1}^{N-2}\Big(\xi_{(N-2),I(N-2)} - \frac{\chi_{I(N-2)}}{\chi_{N-2}}\xi_{(N-2),(N-2)}\Big)f_I\wedge f_{N-2},
\end{aligned}
\tag{2.39}
$$

thus there are two independent cases $(i)\chi_{N-2} = 0$ and $(ii)\chi_{N-2} \neq 0$. For $N = 3$, TMG and MMG are in the second case and for $N = 4$, NMG, GMG are in first case but E(G)MG is in the second case. The covariant derivative for the first case (2.38) is calculated by using $De = 0$

$$\begin{aligned} DDf_{N-2} = &- \sum_{I=1}^{N-3} \xi_{(N-2),I}(e \wedge Df_I) - \xi_{(N-2),(N-2)}(e \wedge Df_{N-2}) \\ &+ \sum_{I \leq J}^{N-3} \xi_{(N-2),IJ}(Df_I \wedge f_J - f_I \wedge Df_J) \\ &+ \sum_{I=1}^{N-2} \xi_{(N-2),I(N-2)}(Df_I \wedge f_{N-2} - f_I \wedge Df_{N-2}) \\ = &\ 0. \end{aligned} \tag{2.40}$$

To write an off-shell model, the field equation of the highest order term Df_{N-2} should vanish in its covariant derivative. Thus the coefficients $\xi_{(N-2),(N-2)}$ and $\xi_{(N-2),I(N-2)}$ are zero $\forall I$. The covariant derivative for the second case (2.39) is calculated by using $De = 0$:

$$\begin{aligned} DDf_{(N-2)} = &\sum_{I=1}^{N-3} \left(\xi_{(N-2),I} - \frac{\chi_I}{\chi_{N-2}}\xi_{(N-2),(N-2)}\right)(-e \wedge Df_I) + \frac{\xi_{(N-2),(N-2)}}{\chi_{N-2}} DR \\ &+ \sum_{I=1}^{N-2} \left(\xi_{(N-2),I(N-2)} - \frac{\chi_{I(N-2)}}{\chi_{N-2}}\xi_{(N-2),(N-2)}\right)\left(Df_I \wedge f_{N-2} - f_I \wedge Df_{N-2}\right) \\ &+ \sum_{I \leq J}^{N-3} \left(\xi_{(N-2),IJ} - \frac{\chi_{IJ}}{\chi_{N-2}}\xi_{(N-2),(N-2)}\right)\left(Df_I \wedge f_J - f_I \wedge Df_J\right). \end{aligned} \tag{2.41}$$

To write an off-shell model, the field equation of the highest order term Df_{N-2} should vanish in its covariant derivative. Thus we have the relation $\xi_{(N-2),I(N-2)} = \frac{\chi_{I(N-2)}}{\chi_{N-2}}\xi_{(N-2),(N-2)}$. Thus, the third way consistent, LLT invariant model has the following field equations

$$\begin{aligned} &De = 0, \\ &R = \chi_0 e \wedge e + \sum_{I=1}^{N-2} \chi_I e \wedge f_I, \\ &Df_k = \xi_{K,0} e \wedge e + \sum_{I=1}^{N-2} \xi_{K,I} e \wedge f_I + \sum_{I \leq J}^{N-2} \xi_{K,IJ} f_I \wedge f_J, \quad K = 1, \cdots, N-3, \\ &Df_{N-2} = \xi_{(N-2),0} e \wedge e + \sum_{I=1}^{N-2} \xi_{(N-2),I} e \wedge f_I + \sum_{I \leq J}^{N-2} \xi_{(N-2),IJ} f_I \wedge f_J. \end{aligned} \tag{2.42}$$

If $\xi_{(N-2),I(N-2)} = \frac{\chi_{I(N-2)}}{\chi_{N-2}}\xi_{(N-2),(N-2)}$ holds, then the model is off-shell, otherwise there is an on-shell model.

2.3.2 Non-vanishing torsion

The covariant derivative of torsion is

$$\begin{aligned} DDe_a = \epsilon_{abc}\Bigg(& 2\varphi_0 De^b \wedge e^c + \sum_{I=1}^{N-2} \varphi_I \big(De^b \wedge f_I^c - e^b \wedge Df_I^c\big) \\ & + \sum_{I\leq J}^{N-2} \varphi_{IJ}\big(Df_I^b \wedge f_J^c - f_I^b \wedge Df_J^c\big)\Bigg) \\ = {} & R_{ab} \wedge e^b \\ = {} & \sum_{I=1}^{N-2} \chi_I \big(e_a(f_I \cdot e)\big) + \sum_{I\leq J}^{N-2} \big(f_{Ia}(f_J \cdot e) - (e \cdot f_I) f_{Ja}\big). \end{aligned} \tag{2.43}$$

The covariant derivative of curvature is the same as in the torsionless case. So the constraint equations $e \cdot f_I = 0\ \forall I = 1, \cdots, N-2$ and the discussion on vanishing of the coefficient χ_{IJ} still holds. After using these results, the covariant derivative of torsion gives

$$\begin{aligned} DDe_a = & \left(\sum_{K=1}^{N-2}\sum_{I\leq J}^{N-2}\Big(\varphi_I\varphi_K + \varphi_{IJ}\xi_{J,K}\Big) + \sum_{J\leq I}^{N-2}\varphi_{JI}\xi_{J,K}\right) e_a(f_K \cdot f_I) \\ & + \sum_{K\leq M}\Big(\sum_{I=1}^{N-2}\varphi_I\varphi_{KM} + \sum_{J\leq I}^{N-2}\varphi_{JI}\xi_{J,KM} + \sum_{I\leq J}^{N-2}\varphi_{IJ}\xi_{I,KM}\Big) f_{Ka}(f_M \cdot f_I) \\ & + \sum_{M\leq K}\Big(\sum_{I=1}^{N-2}\varphi_I\varphi_{MK} + \sum_{J\leq I}^{N-2}\varphi_{JI}\xi_{J,MK} + \sum_{I\leq J}^{N-2}\varphi_{IJ}\xi_{I,MK}\Big) f_{Ka}(f_M \cdot f_I) = 0. \end{aligned} \tag{2.44}$$

Due to the product of symmetric and antisymmetric indices, the term $\varphi_I\varphi_K e_a(f_K \cdot f_I)$ gives zero. The simplest solution of (2.44) is found by setting $\varphi_{IJ} = 0\ \forall I, J = 1, \cdots, N-2$. Then the torsion and curvature become

$$\begin{aligned} De &= \varphi_0 e \wedge e + \sum_{I=1}^{N-2} \varphi_I e \wedge f_I, \\ R &= \chi_0 e \wedge e + \sum_{I=1}^{N-2} \chi_I e \wedge f_I. \end{aligned} \tag{2.45}$$

To find the Bianchi identity for an off-shell model, we need to calculate the covariant derivative of the field equation for the highest order field (2.38). As in the torsionless models, there are two different cases $(i)\chi_{N-2} = 0$ and $(ii)\chi_{N-2} \neq 0$. For the first case, to write an off-shell model, we need both $\xi_{(N-2),(N-2)}$ and $\xi_{(N-2),I(N-2)}$ to be zero $\forall I$. Then the last Bianchi identity is satisfied if

$$
\begin{aligned}
DDf^a_{N-2} =& \sum_{I\leq J}^{N-3} \sum_{K\leq M}^{N-2} \xi_{(N-2),IJ}\Big(\xi_{I,K}e^a(f_K\cdot f_J) + \xi_{I,KM}\big(f^a_K(f_M\cdot f_J) - f^a_M(f_J\cdot f_K)\big) \\
&+ \xi_{J,K}e^a(f_K\cdot f_I) + \xi_{J,KM}\big(f^a_K(f_M\cdot f_I) - f^a_M(f_I\cdot f_K)\big)\Big) \\
&+ \sum_{K=1}^{N-2}\sum_{I=1}^{N-3} \varphi_K \xi_{(N-2),I} e^a (f_K\cdot f_I) = 0.
\end{aligned}
\tag{2.46}
$$

For the case $\chi_{N-2} \neq 0$, the curvature form is used in the last field equation given in (2.39). To write an off-shell model, the condition is $\xi_{(N-2),I(N-2)} = 0 \ \forall I = 1, \cdots, N-2$. Then the Bianchi identity is satisfied if

$$
\begin{aligned}
DDf^a_{N-2} =& \sum_{K\leq M}^{N-2} \sum_{I\leq J}^{N-3} \xi_{(N-2),IJ}\Bigg(\xi_{I,KM}\bigg(f^a_K(f_M\cdot f_J) - f^a_M(f_J\cdot f_K)\bigg) \\
&+ \xi_{J,KM}\bigg(f^a_K(f_M\cdot f_I) - f^a_M(f_I\cdot f_K)\bigg)\Bigg) \\
&+ \sum_{K=1}^{N-2}\sum_{I\leq J}^{N-3} \xi_{(N-2),IJ}\Big(\xi_{I,K}e^a(f_K\cdot f_J) + \xi_{J,K}e^a(f_K\cdot f_I)\Big) \\
&+ \sum_{I=1}^{N-3}\sum_{J=1}^{N-2} \zeta_i \varphi_J e^a(f_J\cdot f_I) = 0,
\end{aligned}
\tag{2.47}
$$

where $\zeta_I := \xi_{(N-2),I} - \frac{\chi_I}{\chi_{N-2}}\xi_{(N-2),(N-2)}$, and the structure equation $DR = 0$ is used. For the on-shell case, vanishing of the coefficient χ_{N-2} does not effect the Bianchi identity. Thus the Bianchi identity is satisfied if

$$\begin{aligned} DDf^a_{N-2} =& \sum_{I\leq J}^{N-2}\sum_{K\leq M}^{N-2} \xi_{(N-2),IJ}\Bigg(\xi_{I,KM}\Big(f^a_K(f_M\cdot f_J) - f^a_M(f_J\cdot f_K)\Big) \\ &+ \xi_{J,KM}\Big(f^a_K(f_M\cdot f_I) - f^a_M(f_I\cdot f_K)\Big)\Bigg) \\ &+ \sum_{I\leq J}^{N-2}\sum_{K=1}^{N-2} \xi_{(N-2),IJ}\Big(\xi_{I,K}e^a(f_K\cdot f_J) + \xi_{J,K}e^a(f_K\cdot f_I)\Big) \\ &+ \sum_{I,J=1}^{N-2} \xi_{(N-2),I}\varphi_J e^a(f_J\cdot f_I) = 0. \end{aligned} \tag{2.48}$$

For an LLT invariant model with N fields, the field equation should be

$$\begin{aligned} De &= \varphi_0 e\wedge e + \sum_{I=1}^{N-2}\varphi_I e\wedge f_I, \\ R &= \chi_0 e\wedge e + \sum_{I=1}^{N-2}\chi_I e\wedge f_I, \\ Df_K &= \xi_{K,0} e\wedge e + \sum_{I=1}^{N-2}\xi_{K,I} e\wedge f_I + \sum_{I\leq J}^{N-2}\xi_{K,IJ} f_I\wedge f_J, \quad K = 1,\cdots,N-3, \\ Df_{N-2} &= \xi_{(N-2),0} e\wedge e + \sum_{I=1}^{N-2}\xi_{(N-2),I} e\wedge f_I + \sum_{I\leq J}^{N-2}\xi_{(N-2),IJ} f_I\wedge f_J, \end{aligned} \tag{2.49}$$

the conditions to satisfy Bianchi identity off-shell are (2.46), $\xi_{(N-2),(N-2)} = 0$ and $\xi_{(N-2),I(N-2)} = 0\ \forall I$ for $\chi_{N-2} = 0$, (2.47) and $\xi_{(N-2),I(N-2)} = 0\ \forall I$ for $\chi_{N-2} \neq 0$. The Bianchi identity is satisfied on-shell if the coefficients satisfy the equation (2.48). In the next chapters, these identities will be used for the specific cases $N = 2, 3, 4$.

2.4 Hamiltonian analysis

The separation of space and time indices of a generic 1-form is

$$\Omega^{Ra} = \Omega^{Ra}{}_{\mu} dx^{\mu} = \Omega^{Ra}_0 dt + \Omega^{Ra}_i dx^i. \tag{2.50}$$

Greek letters refer to the curved spacetime indices and range over $0, 1, 2$, whereas i, j, k are spatial curved indices and range over $1, 2$. When the 1-form (2.50) is put into the Lagrangian (2.30)

$$
\begin{aligned}
L =& \frac{1}{2} g_{RS} \Big((\Omega_0^{Ra} dt + \Omega_i^{Ra} dx^i) \wedge d(\Omega_{0a}^S dt + \Omega_{ja}^S dx^j) \Big) \\
& + \frac{1}{6} f_{RST} \epsilon_{abc} (\Omega_0^{Ra} dt + \Omega_i^{Ra} dx^i) \wedge (\Omega_0^{Sb} dt + \Omega_j^{Sb} dx^j) \wedge (\Omega_0^{Tc} dt + \Omega_k^{Tc} dx^k) \\
=& \, dt \wedge dx^j \wedge dx^i \Big(g_{RS} (\Omega_{0a}^R \partial_j \Omega_i^{Sa} + \Omega_{ia}^R \dot{\Omega}_j^{Sa} - \Omega_{ia}^S \partial_j \Omega_0^{sa}) + \frac{1}{2} f_{rst} \epsilon_{abc} \Omega_0^{rR} \Omega_j^{Sb} \Omega_i^{Tc} \Big) \\
=& \, dt \, d^2x \, \varepsilon^{ij} \Big(\frac{1}{2} g_{RS} (\Omega_{0a}^R \partial_i \Omega_j^{Sa} + \Omega_{0a}^S \partial_i \Omega_j^{Ra} - \Omega_i^{Ra} \dot{\Omega}_{ja}^S) + \frac{1}{2} f_{RST} \epsilon_{abc} \Omega_0^{Ra} \Omega_j^{Sb} \Omega_i^{Tc} \Big) \\
& - dt \, d^2x \, \partial_i \Big(\varepsilon^{ij} \frac{1}{2} g_{RS} \Omega_j^{Ra} \Omega_{0a}^S \Big).
\end{aligned}
\tag{2.51}
$$

Where $\dot{\Omega} \equiv \partial_0 \Omega$ and $\varepsilon^{ij} \equiv \varepsilon^{0ij}$. The exterior product is antisymmetric, thus the term $dt \wedge dt$ gives zero. Since the indices i, j can take 1 or 2, the triple product of $dx^i \wedge dx^j \wedge dx^k$ is also zero. Then the Lagrangian density becomes (up to the boundary term)

$$
\mathcal{L} = -\frac{1}{2} \varepsilon^{ij} g_{RS} \Omega_i^{Ra} \dot{\Omega}_{ja}^S + \Omega_0^{Ra} \phi_{Ra}. \tag{2.52}
$$

The term Ω_0^{Ra} behaves as a Lagrange multiplier. Since a can be $a = 0, 1, 2$ and R runs from 1 to N, there are $3N$ constraints. The primary constraints of the theory are given by

$$
\phi_{Ra} = \varepsilon^{ij} \Big(g_{RS} \partial_i \Omega_{ja}^S + \frac{1}{2} f_{RST} \epsilon_{abc} \Omega_i^{Sb} \Omega_j^{Tc} \Big). \tag{2.53}
$$

The conjugate momenta p_{ja}^R of the dynamical variables Ω_{ja}^R are given by

$$
p_a^{Rj} := \frac{\partial \mathcal{L}}{\partial \dot{\Omega}_{ja}^R} = -\frac{1}{2} \varepsilon^{ij} g_{SR} \Omega_{ia}^S. \tag{2.54}
$$

Therefore, the Hamiltonian density becomes

$$
\mathcal{H} = -\Omega_0^{Ra} \phi_{Ra}. \tag{2.55}
$$

The Lagrangian density can be written in terms of the Hamiltonian

$$
\mathcal{L} = \theta_S^{ja}(\Omega) \dot{\Omega}_{ja}^S - \mathcal{H}(\Omega), \tag{2.56}
$$

where $\theta^{ja}_S(\Omega) := -\frac{1}{2}\varepsilon^{ij} g_{RS}\Omega^{Ra}_i$. The Euler-Lagrange equations become as in [20]

$$f^{SR,ij}_{ab}\dot{\Omega}^S_{ja} = \frac{\partial \mathcal{H}}{\partial \Omega^R_{ib}}, \qquad\qquad f^{SR,ij}_{ab} := \frac{\partial \theta^{ja}_S}{\partial \Omega^b_{Ri}} - \frac{\partial \theta^{ib}_R}{\partial \Omega^a_{Sj}}. \tag{2.57}$$

The tensor $f^{SR,ij}_{ab}$ is called the *symplectic tensor* [20]. The Lagrangian density in equation (2.56) is in the form $p_i q^i - H$, where the pair (q^i, p_i) are canonical coordinates. Therefore, the Euler-Lagrange equations in (2.57) give also Hamiltonian equations if the symplectic tensor is invertible. By using the Hamiltonian equations of motion and the invertibility of $f^{SR,ij}_{ab}$, the Poisson bracket can be defined. For an arbitrary canonical variable Ω^{ia}_R the equation of motion is given by

$$\dot{\Omega}^{ia}_R = \{\mathcal{H}, \Omega^{ia}_R\} = \frac{\partial \mathcal{H}}{\partial \Omega^{jb}_S}\{\Omega^{jb}_S, \Omega^{ia}_R\}. \tag{2.58}$$

The comparison of equations (2.57) and (2.58) gives the Poisson bracket

$$\{\Omega^b_{Sj}(\mathbf{x}), \Omega^a_{Ri}(\mathbf{x}')\} = (f^{SR,ij}_{ab})^{-1} = \varepsilon_{ij} g^{RS}\eta^{ab}\delta^2(\mathbf{x}-\mathbf{x}'). \tag{2.59}$$

The consistency equation is

$$\{\phi^a_R, H\} \approx 0, \tag{2.60}$$

where H is the total Hamiltonian, which is the integral of the Hamiltonian density (2.55). The weak equality $\approx$ means that it is satisfied on the constraint surface.
The Poisson bracket of the primary constraints gives three different terms $\{\partial\Omega, \partial\Omega\}$, $\{\partial\Omega, \Omega\Omega\}$ and $\{\Omega\Omega, \Omega\Omega\}$. For the generic case, they are calculated as

$$\begin{aligned}
&\{\varepsilon^{ij} g_{RS}\partial_i\Omega^{Sa}_j(\mathbf{x}), \varepsilon^{km} g_{PT}\partial'_k\Omega^{Tb}_m(\mathbf{x}')\} \\
&\qquad = \varepsilon^{ij}\varepsilon^{km} g_{RS} g_{PT}\partial_i\partial'_k\{\Omega^{Sa}_j(\mathbf{x}), \Omega^{Tb}_m(\mathbf{x}')\} \\
&\qquad = \varepsilon^{ij}\varepsilon^{km} g_{RS} g_{PT}\partial_i\partial'_k\Big(\varepsilon_{jm}\eta^{ab} g^{ST}\delta^2(\mathbf{x}-\mathbf{x}')\Big) \\
&\qquad = \varepsilon^{ij}\delta^k_j\eta^{ab} g_{RS} g_{PT}\partial_i\partial'_k\Big(g^{ST}\delta^2(\mathbf{x}-\mathbf{x}')\Big) \\
&\qquad = -\varepsilon^{ik}\eta^{ab} g_{RS} g_{PT}\partial_i\partial_k\Big(g^{ST}\delta^2(\mathbf{x}-\mathbf{x}')\Big) \\
&\qquad = 0,
\end{aligned} \tag{2.61}$$

where ∂'_k means derivative with respect to x'^k. Here $\mathbf{x}, \mathbf{x}'$ are spatial vectors. In the second step, the equation (2.59) is used. In the fourth step the variable x'^k is changed with x^k. The Poisson bracket of the term $\{\partial\Omega, \Omega\Omega\}$ is

$$\begin{aligned}
&\{\varepsilon^{ij} g_{RS} \partial_i \Omega^S_{ja}(\mathbf{x}), \varepsilon^{km} f_{PVU} \epsilon_{bdg} \Omega^{Vd}_k(\mathbf{x}') \Omega^{Ug}_m(\mathbf{x}')\} \\
&\quad = \varepsilon^{ij} \varepsilon^{km} g_{RS} f_{PVU} \epsilon_{bdg} \{\partial_i \Omega^S_{ja}(\mathbf{x}), \Omega^{Vd}_k(\mathbf{x}') \Omega^{Ug}_m(\mathbf{x}')\} \\
&\quad = \varepsilon^{ij} \varepsilon^{km} g_{RS} f_{PVU} \epsilon_{bdg} \Big(\Omega^{Vd}_k(\mathbf{x}') \{\partial_i \Omega^S_{ja}(\mathbf{x}), \Omega^{Ug}_m(\mathbf{x}')\} \\
&\qquad\qquad\qquad\qquad + \Omega^{Ug}_m(\mathbf{x}') \{\partial_i \Omega^S_{ja}(\mathbf{x}), \Omega^{Vd}_k(\mathbf{x}')\} \Big) \\
&\quad = \varepsilon^{ij} \varepsilon^{km} g_{RS} f_{PVU} \epsilon_{bdg} \Big(\Omega^{Vd}_k(\mathbf{x}')\, \eta^g_a \varepsilon_{jm} g^{SU} \partial_i \delta^2(\mathbf{x} - \mathbf{x}') \\
&\qquad\qquad\qquad\qquad + \Omega^{Ug}_m(\mathbf{x}')\, \eta^d_a \varepsilon_{jk} g^{SV} \partial_i \delta^2(\mathbf{x} - \mathbf{x}') \Big) \\
&\quad = \varepsilon^{ik} f_{PVU} \epsilon_{abc} \Omega^{Vc}_k(\mathbf{x}') g_{RS} g^{SU} \partial_i \delta^2(\mathbf{x} - \mathbf{x}').
\end{aligned} \tag{2.62}$$

In the third line symmetry of the coefficient f_{PUV} is used. Lastly, the Poisson bracket of the term $\{\Omega\Omega, \Omega\Omega\}$ is given by

$$\begin{aligned}
&\{\varepsilon^{ij} f_{RST} \epsilon_{ach} \Omega^{Sc}_i(\mathbf{x}) \Omega^{Th}_j(\mathbf{x}), \varepsilon^{km} f_{PVU} \epsilon_{bdg} \Omega^{Vd}_k(\mathbf{x}') \Omega^{Ug}_m(\mathbf{x}')\} \\
&\quad = \varepsilon^{ij} f_{RST} \epsilon_{ach} \varepsilon^{km} f_{PVU} \epsilon_{bdg} \{\Omega^{Sc}_i(\mathbf{x}) \Omega^{Th}_j(\mathbf{x}), \Omega^{Vd}_k(\mathbf{x}') \Omega^{Ug}_m(\mathbf{x}')\} \\
&\quad = 4\varepsilon^{ij} f_{RST} \epsilon_{ach} \varepsilon^{km} f_{PVU} \epsilon_{bdg} \Omega^{Th}_j(\mathbf{x}) \Omega^{Ug}_m(\mathbf{x}') \{\Omega^{Sc}_i(\mathbf{x}), \Omega^{Vd}_k(\mathbf{x}')\} \\
&\quad = 4\varepsilon^{ij} f_{RST} \epsilon_{ach} \varepsilon^{km} f_{PVU} \epsilon_{bdg} \Omega^{Th}_j(\mathbf{x}) \Omega^{Ug}_m(\mathbf{x}') \varepsilon_{ik} g^{SV} \eta^{cd} \delta^2(\mathbf{x} - \mathbf{x}') \\
&\quad = 8\varepsilon^{ik} f_{RST} f_{PVU} g^{SV} \big(-\eta_{ab} \Omega_{iTc}(\mathbf{x}) \Omega^c_{kU}(\mathbf{x}') + \Omega_{iTb}(\mathbf{x}) \Omega_{kUa}(\mathbf{x}) \big) \delta^2(\mathbf{x} - \mathbf{x}') \\
&\quad = 4\varepsilon^{ik} g^{SV} \Big(f_{SU[R} f_{P]TV} \eta_{ab} \Omega_{iTc}(\mathbf{x}) \Omega^c_{kU}(\mathbf{x}') \\
&\qquad\qquad\qquad + f_{SR[P} f_{T]UV} \Omega_{iTb}(\mathbf{x}) \Omega_{kUa}(\mathbf{x}) \Big) \delta^2(\mathbf{x} - \mathbf{x}').
\end{aligned} \tag{2.63}$$

When we calculate the consistency equation (2.60), the term (2.62) gives a boundary term in integral. The term (2.63) is rewritten in [2] as

$$\mathcal{P}^{ab}_{RS} = g^{UT} (f_{UQ[R} f_{S]PT} \eta^{ab} \Delta^{PQ} + 2 f_{UR[S} f_{Q]PT} (V^{ab})^{PQ}), \tag{2.64}$$

with

$$\Delta^{PQ} = \varepsilon^{ij} \Omega^P_i \cdot \Omega^Q_j, \qquad V^{abPQ} = \varepsilon^{ij} \Omega^{aP}_i \Omega^{bQ}_j. \tag{2.65}$$

The integrability condition (2.64) can be found by using the equation of motion itself [8]. For the generic Lagrangian (2.30), the equation of motion is

$$g_{RS} d\Omega^{Sa} + \frac{1}{2} f_{RST} \epsilon^{abc} \Omega^S_b \wedge \Omega^T_c = 0. \tag{2.66}$$

If the exterior derivative of the equation of motion is taken by using the nilpotency, the conditions become

$$g^{TU} f_{UQ[R} f_{S]PT} \Omega^{Ra} \Omega^{P} \cdot \Omega_{Q} = 0. \tag{2.67}$$

By using the space-time separation of the 1-forms given in (2.50) the equation (2.67) can be rewritten as

$$0 = g^{TU} f_{UQ[R} f_{S]PT} \Omega^{Rb} \Omega^{P} \cdot \Omega_{Q} = \Omega^{R}_{0a} \mathcal{P}^{ab}_{RS}. \tag{2.68}$$

If the matrix $\mathcal{P}^{ab}_{RS}$ does not vanish identically, this means that the linear combination of the Lagrange multipliers Ω^{R}_{0a} should be zero. On the other hand, for identically vanishing $\mathcal{P}^{ab}_{RS}$ all primary contraints are first-class and the rank of the matrix is nonzero. To solve the field equations, it is assumed that some of the fields are invertible. This invertibility condition makes the 3-form equation (2.68) a 2-form contraint equation. Thus the secondary constraints must be added to the theory. Since the linear combination of the invertible fields may not be invertible in general, sums over the values R in (2.68) with some invertible fields does not imply new constraints in general. As in [8], it is assumed that sum over R is nonzero only for $R = e$ for an invertible dreibein e^{a}. Then the condition (2.68) becomes

$$g^{TU} f_{UQ[e} f_{S]PT} \Omega^{P} \cdot \Omega^{Q} = 0. \tag{2.69}$$

Using the space-time separation of the 1-forms, this can be written

$$g^{TU} f_{UQ[e} f_{S]PT} \Omega^{P} \cdot \Omega^{Q}$$
$$= g^{TU} f_{UQ[e} f_{S]PT} \Big(\big(\Omega^{P}_{0a} \Omega^{Qa}_{i} - \Omega^{Q}_{0a} \Omega^{Pa}_{i}\big) dt \wedge dx^{i} + \Omega^{P}_{ia} \Omega^{a}_{j} dx^{i} \wedge dx^{j} \Big) = 0. \tag{2.70}$$

The space-space part of the above equation depends only on the canonical variables (not Lagrange multipliers). Thus, for different values of S, the above equation holds. Then the secondary constraints are

$$\Psi_{I} := f_{I,RQ} \Delta^{PQ}, \qquad I = 1, \cdots, M. \tag{2.71}$$

where the definition of Δ^{PQ} is given in (2.65). Then there are $3N + M$ contraints on the system. The Poisson bracket of the primary and secondary contraints are

$$\{\phi[\eta], \Psi_{I}\} = \varepsilon^{ij} \Big(f_{I,RP} \partial_{i}(\eta^{R}) \cdot \Omega^{P}_{j} + g^{tu} f_{URS} f_{I,PT} \epsilon_{abc} \eta^{Ra} \Omega^{Sb}_{i} \Omega^{Pc}_{j} \Big), \tag{2.72}$$

where η is the test function. Finally the Poisson bracket of the secondary constraints are

$$\{\Psi_I, \Psi_J\} = 4 f_{I,PQ} f_{J,RS} \Delta^{PR} g^{QS}. \tag{2.73}$$

To count the degrees of freedom, we will follow the same path as in [8]. On the constraint surface, the Poisson bracket of two secondary constraints is zero. Then the total Poisson bracket matrix of all $3N + M$ constraints (where N is the number of the fields thus $3N$ is the number of primary constraints evaluated on the contraint surface and M is the number of the secondary constraints) becomes

$$\boldsymbol{P} = \begin{bmatrix} \mathcal{P} & -\{\phi, \Psi\}^T \\ \{\phi, \Psi\} & 0 \end{bmatrix}. \tag{2.74}$$

It is also assumed that secondary constraints are second-class which means that addition of the secondary constraints does not give rise to new first-class contraints. Then the rank of the matrix $\boldsymbol{P}$ is M+rank$(\mathcal{P}) + M$, where the second term is the number of the linearly independent columns of the matrix

$$\begin{bmatrix} \mathcal{P} \\ \{\phi, \Psi\} \end{bmatrix}. \tag{2.75}$$

The number of the undetermined Lagrange multipliers are $3N -$ rank$(\mathcal{P}) - M$ which corresponds to the first-class constraints. There are also $2M +$ rank$(\mathcal{P})$ second-class constraints. Then the degrees of freedom is counted as

$$\begin{aligned} D =& (\text{number of canonical variables}) - 2(\text{number of the first-class constraints}) \\ & - (\text{number of the second-class constraints}). \end{aligned} \tag{2.76}$$

The number of the canonical variables Ω_i^{Ra} is $6N$ since R can take N values, $a = 0, 1, 2$ and $i = 1, 2$. Then the number of degrees of freedom becomes

$$D = 6N - 2(3N - \text{rank}(\mathcal{P}) - M) - (2M + \text{rank}(\mathcal{P})) = \text{rank}(\mathcal{P}). \tag{2.77}$$

CHAPTER 3

$N = 2$ MODEL

Let us start with the most general $N = 2$ Lagrangian 3-form with the dreibein e and connection ω [10]

$$L = \alpha_2 e \wedge De + \alpha_1 e \wedge R + \alpha_0 \Big(\omega \wedge d\omega + \frac{1}{3} \omega \wedge \omega \wedge \omega \Big) + \frac{1}{6} \beta_3 e \wedge e \wedge e. \quad (3.1)$$

Taking e as a dimensionless object, ω has dimensions of mass (or 1/length). Keeping in mind that the three-dimensional Newton's constant G_3 has dimensions of 1/mass (or length), the Lagrangian L must have dimensions of mass^2 (or 1/length^2). The coupling constants α_n and β_n have been chosen such that each has dimensions of mass$^{(n-1)}$ (or length$^{(1-n)}$), with $n \in \{0, 1, 2, 3\}$.
The cosmological Einstein theory is captured when one sets $\alpha_n = 0$ for $n = 0, 2$ and $\beta_3 = \Lambda$.
The Mielke–Baekler (MB) Lagrangian [11] is obtained by $\alpha_2 = \sigma_3$, $\alpha_1 = 2\sigma_1$, $\alpha_0 = \sigma_2$ and $\beta_3 = 2\sigma_0$.

3.1 The field equations

The field equations that follow are

$$\underbrace{\begin{bmatrix} 2\alpha_2 & \alpha_1 \\ \alpha_1 & 2\alpha_0 \end{bmatrix}}_{A} \begin{bmatrix} De \\ R \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{\beta_3}{2} \\ \alpha_2 \end{bmatrix}}_{B} e \wedge e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (3.2)$$

Assume that $\det A = 4\alpha_0\alpha_2 - \alpha_1^2 \neq 0$ which is the condition given in [11]. Then the field equations are rewritten as

$$De = \frac{\alpha_1\alpha_2 - \alpha_0\beta_3}{4\alpha_0\alpha_2 - \alpha_1^2} e \wedge e, \qquad R = \frac{\alpha_1\beta_3 - 4\alpha_2^2}{2(4\alpha_0\alpha_2 - \alpha_1^2)} e \wedge e. \tag{3.3}$$

The covariant derivative of the torsion and the curvature are

$$DDe = 2\frac{\alpha_1\alpha_2 - \alpha_0\beta_3}{4\alpha_0\alpha_2 - \alpha_1^2} De \wedge e, \qquad DR = \frac{\alpha_1\beta_3 - 4\alpha_2^2}{4\alpha_0\alpha_2 - \alpha_1^2} De \wedge e. \tag{3.4}$$

When we calculate DDe and DR, we will not use the curvature itself or its covariant derivative. Thus, there are only off-shell models.

3.1.1 Vanishing torsion

If we set torsion to zero, then there is one more algebraic condition on the coefficients

$$De = 0 \Rightarrow \alpha_1\alpha_2 = \alpha_0\beta_3. \tag{3.5}$$

Then the covariant derivative of the curvature becomes

$$DR = -\frac{\alpha_2}{\alpha_0} De \wedge e = 0. \tag{3.6}$$

The structure equation is

$$0 = DDe = R \wedge e = -\frac{\alpha_2}{2\alpha_0} e \wedge e \wedge e. \tag{3.7}$$

The Lagrangian of the model is (3.1) with the condition $\alpha_1\alpha_2 = \alpha_0\beta_3$. This Lagrangian is Witten's "exotic" Lagrangian [9, 11].

3.1.2 Non-vanishing torsion

The second structure equation implies that

$$DR_a = \frac{2(\alpha_1\beta_3 - 4\alpha_2^2)(\alpha_1\alpha_2 - \alpha_0\beta_3)}{(4\alpha_0\alpha_2 - \alpha_1)^2} e_a(e \cdot e) = 0, \tag{3.8}$$

where $e \cdot e = e^b \wedge e_b$. The covariant derivative of torsion is

$$DDe_a = \frac{4(\alpha_1\alpha_2 - \alpha_0\beta_3)}{(4\alpha_0\alpha_2 - \alpha_1)^2} e_a(e \cdot e) = R_{ab} \wedge e^b = \frac{(\alpha_1\beta_3 - 4\alpha_2^2)}{(4\alpha_0\alpha_2 - \alpha_1)^2} e_a(e \cdot e) = 0. \quad (3.9)$$

The Lagrangian of the model is (3.1) which is MB Lagrangian with the condition $\alpha_1\alpha_2 \neq \alpha_0\beta_3$.

3.2 Hamiltonian analysis

For the $N = 2$ case, the coefficients are

$$\begin{aligned} &g_{ee} = 2\alpha_2, \quad g_{e\omega} = \frac{1}{2}\alpha_1, \quad g_{\omega\omega} = 2\alpha_0, \\ &f_{eee} = \beta_3, \quad f_{ee\omega} = 2\alpha_2, \quad f_{e\omega\omega} = \alpha_1, \quad f_{\omega\omega\omega} = 2\alpha_0. \end{aligned} \quad (3.10)$$

The term $g^{UT} f_{UR[S} f_{Q]PT} (V^{ab})^{PQ} = 0$ for these coefficients. Thus the model (3.1) has no local degrees of freedom independent of whether torsion exists.

The most general Lagrangian written by using only the dreibein and the spin connection (3.1) gives only off-shell models, thus there is no third-way consistent model for the $N = 2$ case. If torsion is set to zero, the Lagrangian is called Witten's "exotic" Lagrangian. This model is a special case of the MB Lagrangian whose equations of motion give a torsionless model. When the Hamiltonian analysis is applied to count the local degrees of freedom, it is observed that the $N = 2$ models have no local degrees of freedom. This is an expected result. Since the way to construct massive gravity is adding auxiliary field(s) to the Lagrangian.

CHAPTER 4

$N = 3$ MODELS

Let us start with the most general $N = 3$ Lagrangian 3-form with the dreibein e, the connection ω and an auxiliary field f which is used for introducing higher derivative terms [10]

$$\begin{aligned} L =& \alpha_2 e \wedge De + \alpha_1 e \wedge R + \alpha_0(\omega \wedge d\omega + \frac{1}{3}\omega \wedge \omega \wedge \omega) + \alpha_{2+k} f \wedge De + \alpha_{1+k} f \wedge R \\ &+ \alpha_{2k+2} f \wedge Df + \frac{1}{6}(\beta_3 e \wedge e \wedge e + \beta_{3+3k} f \wedge f \wedge f) \\ &+ \frac{1}{2}(\beta_{3+k} e \wedge e \wedge f + \beta_{3+2k} e \wedge f \wedge f), \end{aligned} \tag{4.1}$$

where

$$De := de + \omega \wedge e, \qquad R := d\omega + \frac{1}{2}\omega \wedge \omega, \qquad Df := df + \omega \wedge f. \tag{4.2}$$

Taking f has dimensions of mass^{-k} (or length^k). The coupling constants α_n and β_n have been chosen such that each has dimensions of mass$^{(n-1)}$ (or length$^{(1-n)}$).
The known models with $N = 3$ fields are TMG that is sum of the conformal gravity and cosmological Einstein theory and MMG which is an addition of eff term to TMG. For TMG, the auxiliary field behaves as a Lagrange multiplier for the torsion constraint [8]. The Lagrangians of these theories can be obtained from the generic Lagrangian given in (4.1).
TMG Lagrangian L_{TMG} can be obtained by taking $\alpha_0 = \frac{1}{2\mu}$, $\alpha_1 = -\sigma$, $\alpha_{2+k} = \frac{1}{\mu}$, $\beta_3 = \Lambda$ and $\beta_{2+k} = \frac{1}{\mu}$, where $\sigma = \pm 1$ is sign for convention.
The terms in MMG Lagrangian L_{MMG} are the same with L_{TMG} up to an addition of the term $\beta_{3+2k} = \frac{\varrho}{\mu^2}$, where ϱ is a constant [8].

4.1 Field equations

The field equations that follow are

$$\underbrace{\begin{bmatrix} 2\alpha_2 & \alpha_1 & \alpha_{2+k} \\ \alpha_1 & 2\alpha_0 & \alpha_{1+k} \\ \alpha_{2+k} & \alpha_{1+k} & 2\alpha_{2k+2} \end{bmatrix}}_{A} \begin{bmatrix} De \\ R \\ Df \end{bmatrix} + \frac{1}{2} \underbrace{\begin{bmatrix} \beta_3 & 2\beta_{3+k} & \beta_{3+2k} \\ 2\alpha_2 & 2\alpha_{2+k} & 2\alpha_{2+2k} \\ \beta_{3+k} & 2\beta_{3+2k} & \beta_{3+3k} \end{bmatrix}}_{2B} \begin{bmatrix} e \wedge e \\ e \wedge f \\ f \wedge f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{4.3}$$

Let the matrix A be invertible, then its determinant should be nonzero

$$\det A = 8\alpha_0\alpha_2\alpha_{2+2k} - 2\alpha_2\alpha_{1+k}^2 + 2\alpha_1\alpha_{1+k}\alpha_{2+k} - 2\alpha_1^2\alpha_{2+2k} - 2\alpha_0\alpha_{2+k}^2 \neq 0. \tag{4.4}$$

Then the field equations become

$$\begin{bmatrix} De \\ R \\ Df \end{bmatrix} + A^{-1}B \begin{bmatrix} e \wedge e \\ e \wedge f \\ f \wedge f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{4.5}$$

They can be rewritten as

$$\begin{aligned} De &= \varphi_0 e \wedge e + \varphi_1 e \wedge f + \varphi_{11} f \wedge f, \\ R &= \chi_0 e \wedge e + \chi_1 e \wedge f + \chi_{11} f \wedge f, \\ Df &= \psi_0 e \wedge e + \psi_1 e \wedge f + \psi_{11} f \wedge f, \end{aligned} \tag{4.6}$$

where the coefficients are given in appendix A.

4.2 Bianchi identities in the first order formalism

Here we apply the strategy discussed in Chapter 2 to study Bianchi identities. The covariant derivative of the last term in equation (4.6) is

$$\begin{aligned} DDf &= 2\psi_0 De \wedge e + \psi_1 D(e \wedge f) + 2\psi_{11} Df \wedge f \\ &= 2\psi_0 De \wedge e + \frac{\psi_1}{\chi_1}\Big(DR - 2\chi_0 De \wedge e - 2\chi_{11} Df \wedge f\Big) + 2\psi_{11} Df \wedge f \\ &= 0, \end{aligned} \tag{4.7}$$

where in the third term the curvature 2-form is used. There are also Cartan structure equations

$$DDe = R \wedge e, \qquad\qquad DR = 0. \tag{4.8}$$

As discussed in Chapter 2, the classification of the models is the following: The model can be torsionless or not; and the model can be diffeomorphism invariant or third-way consistent. These cases will be examined separately.

4.2.1 Vanishing torsion

We want to keep Chern-Simons term so α_0 cannot vanish. If one takes $f = 0$, then the Einstein-Hilbert Lagrangian ($N = 2$ case) should be obtained. This means that $\alpha_1 \neq 0$. It is also assumed that $\alpha_{2+k} \neq 0$. To write a torsionless model, the algebraic conditions on the coefficients are

$$\begin{aligned} &\alpha_{1+k}\alpha_{2+k} - 2\alpha_1\alpha_{2+2k} = 0, \qquad 4\alpha_0\alpha_{2+2k} - \alpha_{1+k}^2 = 0, \\ &\beta_{3+k} = 0, \qquad \beta_{3+2k} = 0, \qquad \beta_{3+3k} = 0. \end{aligned} \tag{4.9}$$

which give $\alpha_{1+k} = 0$ and $\alpha_{2+2k} = 0$ or $\alpha_{1+k} = \frac{2\alpha_0\alpha_{2+k}}{\alpha_1}$ and $\alpha_{2+2k} = \frac{\alpha_0\alpha_{2+k}^2}{\alpha_1^2}$. However, the second condition makes the coefficient matrix A singular, so this cannot be the case. The Lagrangian for this model is

$$L_1 = \alpha_2 e \wedge De + \mathrm{L}_{\mathrm{TMG}}, \tag{4.10}$$

where

$$\mathrm{L}_{\mathrm{TMG}} = \alpha_1 e \wedge R + \alpha_{2+k}\, f \wedge De + \alpha_0(\omega \wedge d\omega + \frac{1}{3}\omega \wedge \omega \wedge \omega) + \frac{1}{6}\beta_3 e \wedge e \wedge e. \tag{4.11}$$

The field equations are

$$\begin{aligned} De &= 0, \\ R &= \frac{1}{2\alpha_0}\big(\alpha_2 e \wedge e + \alpha_{2+k} e \wedge f\big), \\ Df &= \frac{\big(\alpha_0\beta_3 - \alpha_1\alpha_2\big)}{2\alpha_0\alpha_{2+k}} e \wedge e - \frac{\alpha_1}{2\alpha_0} e \wedge f. \end{aligned} \tag{4.12}$$

The covariant derivative of Df is

$$DDf_a = \frac{(\alpha_0\beta_3 - \alpha_1\alpha_2)}{2\alpha_0\alpha_{2+k}} 2\epsilon_{abc} De^b \wedge e^c + D\bigg(-\frac{\alpha_1}{\alpha_{2+k}} R_a + \frac{\alpha_1\alpha_2}{2\alpha_0\alpha_{2+k}} \bigg) \epsilon_{abc} e^b \wedge e^c$$
$$= \frac{\beta_3}{\alpha_{2+k}} \epsilon_{abc} De^b \wedge e^c - \frac{\alpha_1}{\alpha_{2+k}} DR_a = 0. \tag{4.13}$$

Thus, the Bianchi identity is satisfied off-shell. Next step is checking if Cartan structure equations are satisfied

$$0 = DDe_a = R_{ab} \wedge e^b = \frac{\alpha_2}{\alpha_0} e_a(e\cdot e) + \frac{\alpha_{2+k}}{\alpha_0} e_a(f\cdot e) = \frac{\alpha_{2+k}}{\alpha_0} e_a(f\cdot e) = 0, \tag{4.14}$$

which gives the constraint equation

$$e_a(f\cdot e) = 0, \tag{4.15}$$

where $f \cdot e := f^b \wedge e_b$.

For the torsionless case, there is no third-way consistent model. The most generic Lagrangian of such a model is (4.10) with the constraint (4.15).

The torsion constraint allows to write the spin connection ω as a function of the dreibein e for (4.10). Thus, by using the curvature two-form, the auxiliary field f can be found in terms of dreibein e. After calculating $f(e)$, the covariant derivative of the auxiliary field can be written in terms of metric and its derivatives. The constraint equation for both Lagrangians gives

$$0 = e\cdot f = e^a \wedge f_a = e^a_\mu dx^\mu \wedge f_{a\nu} dx^\nu = e^a_\mu \eta_{ab} f^b_\nu dx^\mu \wedge dx^\nu \Rightarrow f_{\nu\mu} := e^a_\mu \eta_{ab} f^b_\nu. \tag{4.16}$$

Since the term $dx^\mu \wedge dx^\nu$ is antisymmetric in μ, ν, the tensor $f_{\nu\mu}$ has only symmetric part. To calculate this symmetric tensor, the curvature 2-form will be used. The components of the curvature 2-form $R(\Omega)$ are

$$R_{\mu\nu ab} = \frac{\alpha_2}{\alpha_0} e_{a\mu} e_{b\nu} + \frac{\alpha_{2+k}}{2\alpha_0} (e_{a\mu} f_{b\nu} + e_{b\nu} f_{a\mu}). \tag{4.17}$$

The solution for $f_{\mu\nu}$ is given by

$$
\begin{aligned}
&e_{a\mu}f_{b\nu}+e_{b\nu}f_{a\mu}=\frac{2\alpha_0}{\alpha_{2+k}}\big(R_{\mu\nu ab}+\frac{\alpha_2}{\alpha_0}e_{a\nu}e_{b\mu}\big) \qquad \setminus e_{c\lambda}\eta^{ac}\\
&g_{\mu\lambda}f_{b\nu}+e_{b\nu}f_{\mu\lambda}=\frac{2\alpha_0}{\alpha_{2+k}}\big(R_{\mu\nu ab}e^a_\lambda+\frac{\alpha_2}{\alpha_0}g_{\nu\lambda}e_{b\mu}\big) \qquad \setminus e^{b\nu}\\
&g_{\mu\lambda}f+f_{\mu\lambda}=\frac{2\alpha_0}{\alpha_{2+k}}\big(R_{\mu\lambda}+\frac{\alpha_2}{\alpha_0}g_{\mu\lambda}\big) \qquad \setminus g^{\mu\lambda}\\
&4f=4f^\mu_\mu=\frac{2\alpha_0}{\alpha_{2+k}}(R+3\frac{\alpha_2}{\alpha_0}), \qquad \text{then the third equation gives}\\
&f_{\mu\lambda}=\frac{2\alpha_0}{\alpha_{2+k}}\left(R_{\mu\lambda}-\frac{R}{4}g_{\mu\lambda}+\frac{\alpha_2}{2\alpha_0}g_{\mu\lambda}\right)=f_{\mu\lambda}=\frac{2\alpha_0}{\alpha_{2+k}}\left(S_{\mu\lambda}+\frac{\alpha_2}{2\alpha_0}g_{\mu\lambda}\right).
\end{aligned}
\tag{4.18}
$$

Note that the relation between the local Lorentz covariant derivative of f given in (5.2) and the covariant derivative defined by affine connection is the following

$$
\begin{aligned}
\nabla_\lambda f_{\mu\nu}&=\nabla_\lambda(f^a_\mu\eta_{ab}e^b_\nu)=\eta_{ab}e^b_\nu\nabla_\lambda f^a_\mu=\eta_{ab}e^b_\nu(\partial_\lambda f^a_\mu+\omega_\lambda{}^a{}_b f^b_\mu-\Gamma^\alpha_{\lambda\mu}f^a_\alpha)\\
&=\eta_{ab}e^b_\nu\big(D_\lambda f^a_\mu-e^c_\alpha(\partial_\lambda e_{c\mu}+\omega_\lambda{}^c{}_k e^k_\mu)\big)\\
&=\eta_{ab}e^b_\nu D_\lambda f^a_\mu\\
&=\frac{2\alpha_0}{\alpha_{2+k}}\nabla_\lambda S_{\mu\nu}.
\end{aligned}
\tag{4.19}
$$

Then the covariant derivative of the auxiliary field is

$$
\begin{aligned}
Df^a&=\psi_0\epsilon^{abc}e_b\wedge e_c+\psi_1\epsilon^{abc}e_b\wedge f_c\\
&=\psi_0\epsilon^{abc}e_b\wedge e_c-\psi_1\epsilon^{abc}e_c\wedge f_b \qquad \setminus\epsilon_{klm}e_{a\nu}e^\lambda_i\eta^{ik}e^{l\mu}e^m_\tau\\
\frac{1}{e}\varepsilon^{\lambda\mu}{}_\tau D_\lambda f_{\mu\nu}&=-4\psi_0 g_{\nu\tau}-\psi_1(f_{\nu\tau}-fg_{\nu\tau})\\
C_{\mu\nu}&=\frac{1}{\alpha_0^2}\Big(\frac{5}{4}\alpha_1\alpha_2-\frac{1}{2}\alpha_0\beta_3\Big)g_{\mu\nu}+\frac{\alpha_1}{2\alpha_0}G_{\mu\nu},
\end{aligned}
\tag{4.20}
$$

where the coefficients in (4.12) are used and $C^\mu{}_\nu$ is symmetric traceless Cotton tensor defined by

$$
C^\mu{}_\nu:=\frac{1}{e}\varepsilon^{\mu\tau\rho}\nabla_\tau S_{\rho\nu}. \tag{4.21}
$$

If the coefficient α_2 in (4.20) vanishes, the metric equation for (cosmological) TMG is obtained.

4.2.2 Non-vanishing torsion

For simplicity, the coefficients $\varphi_k(\alpha_n,\beta_n),\ \chi_k(\alpha_n,\beta_n),\ \psi_k(\alpha_n,\beta_n)$ given in appendix A will be used in this subsection. Start with the covariant derivative of Df:

$$DDf = 2\psi_0 De\wedge e+\frac{\psi_1}{\chi_1}\Big(DR-2\chi_0 De\wedge e-2\chi_{11}Df\wedge f\Big)+2\psi_{11}Df\wedge f = 0. \quad (4.22)$$

For a model with torsion, there are both on-shell and off-shell models. I will analyse these cases seperately.

4.2.2.1 Off-shell model

Let $\chi_1 \neq 0$. The off-shell model can be found if $\chi_{11} = 0,\ \psi_{11} = 0$. To obtain these conditions, the terms $\alpha_{2+2k},\ \beta_{3+2k},\ \beta_{3+3k}$ should vanish. This makes φ_{11} zero. The Lagrangian of such a model is the following

$$L_2 = \mathrm{L}_{\text{TMG}} + \alpha_2 e\wedge De + \alpha_{1+k} f\wedge R + \frac{1}{2}\beta_{3+k}\, e\wedge e\wedge f. \quad (4.23)$$

The Lagrangian L_{TMG} is given in (4.11). To obtain MMG, we need eff term. The corresponding field equations are

$$\begin{aligned} De &= \varphi_0 e\wedge e + \varphi_1 e\wedge f,\\ R &= \chi_0 e\wedge e + \chi_1 e\wedge f,\\ Df &= \psi_0 e\wedge e + \psi_1 e\wedge f. \end{aligned} \quad (4.24)$$

To check if the Bianchi identity is satisfied, one can start with the structure equations

$$\begin{aligned} DDe_a &= \epsilon_{abc}\Big(2\varphi_0 De^b\wedge e^c + \frac{\varphi_1}{\chi_1}\big(DR_a - 2\chi_0 De^b\wedge e^c\big)\Big)\\ &= 2\Big(\varphi_0 - \frac{\chi_0\varphi_1}{\chi_1}\Big)(2\varphi e_a(e\cdot e) + \varphi_1 e_a(f\cdot e) + \varphi_1 f_a(e\cdot e))\\ &= 2\Big(\varphi_0 - \frac{\chi_0\varphi_1^2}{\chi_1}e_a(f\cdot e)\Big)\\ &= R_{ab}\wedge e^b = 2\chi_0 e_a(e\cdot e) + 2\chi_1 e_a(f\cdot e) = 2\chi_1 e_a(f\cdot e). \end{aligned} \quad (4.25)$$

The curvature 2-form is used in the first line. This gives the constraint equation

$$e_a(f\cdot e) = 0. \quad (4.26)$$

The second structure equation is

$$DR = 2\chi_0 De \wedge e + \frac{\chi_1}{\varphi_1}(DDe - 2\varphi_0 De \wedge e) = 0. \tag{4.27}$$

In the first step, torsion in (4.24) is used and in the last step the constraint equation (4.26) is used. The covariant derivative of the last equation is

$$\begin{aligned} DDf_a &= 2\psi_0 \epsilon_{abc} De^b \wedge e^c + \frac{\psi_1}{\chi_1}\Big(DR_a - 2\chi_0 \epsilon_{abc} De^b \wedge e^c\Big) \\ &= 2\Big(\psi_0 - \frac{\psi_1 \chi_0}{\chi_1}\Big)\big(2\varphi_0 e_a (e\cdot e) + \varphi_1 e_a (f \cdot e) + \varphi_1 f_a (e \cdot e)\big) \\ &= 2\varphi_1 \Big(\psi_0 - \frac{\psi_1\chi_0}{\chi_1}\Big) e_a (f\cdot e) = 0. \end{aligned} \tag{4.28}$$

In the last step, the constraint equation (4.26) is used. For the models with torsion, the transformation $\omega \to \omega - \varphi_0 e - \varphi_1 f =: \Omega$ gives a torsionless model. By using this new connection, the covariant derivative for an arbitrary one form a and the curvature are redefined such that $d_\Omega a = da + \Omega \wedge a,\ R(\Omega) = d\Omega + \frac{1}{2}\Omega \wedge \Omega$. Then the procedure for torsionless model is applied to find $f(e)$ and the metric equation. After this transformation, the field equations become

$$\begin{aligned} &d_\Omega e = de + \Omega \wedge e = 0, \\ &R(\Omega) = (\chi_0 - \varphi_0^2 - \psi_0\varphi_1) e \wedge e + (\chi_1 - \varphi_1\psi_1) e \wedge f - \frac{\varphi_1^2}{2} f \wedge f, \\ &d_\Omega f = \psi_0 e \wedge e + (\psi_1 - \varphi_0) e \wedge f - \varphi_1 f \wedge f. \end{aligned} \tag{4.29}$$

Note that the coefficient of the term ff is $-\frac{1}{2}\varphi_1^2$. Therefore there are two different cases. If φ_1 vanishes, then the auxiliary field is found as in the previous case which is equal to

$$f_{\mu\lambda} = \frac{1}{\chi_1 - \psi_1\varphi_1}\left(S_{\mu\lambda} + \frac{1}{2}(\chi_0 - \varphi_0^2 - \psi_0\varphi_1) g_{\mu\lambda}\right). \tag{4.30}$$

And the metric equation is

$$C_{\mu\nu} = -4\chi_1\psi_0 g_{\mu\nu} - (\psi_1 - \varphi_0)\Big(G_{\mu\nu} - (\chi_0 - \varphi_0^2) g_{\mu\nu}\Big). \tag{4.31}$$

The coefficient φ_1 vanishes if α_{1+k} vanishes or $\alpha_{1+k}\beta_{3+k} = \alpha_{2+k}^2$. However, the second one makes $\chi_1 = 0$ in (4.24) which is assumed nonzero. Thus, the Lagrangian (4.23) that gives the field equations (4.24) with $\varphi_0 = 0$ does not have fR term. Then the metric equation becomes

$$C_{\mu\nu} = \frac{1}{\alpha_0^2}\left(\frac{5}{4}\alpha_1\alpha_2 - \alpha_0\beta_3 + \frac{\beta_{3+k}}{\alpha_{2+k}+\alpha_{2+k}}\Big(\frac{9}{4}\alpha_0\alpha_2 - \frac{3}{8}\alpha_1^2\Big) - \frac{\alpha_0^2\beta_{3+k}^3}{8(\alpha_{2+k}+\alpha_{2+k})^3}\right) g_{\mu\nu}$$
$$+ \left(\frac{\alpha_1}{2\alpha_0} - \frac{\beta_{3+k}}{2(\alpha_{2+k}+\alpha_{2+k})}\right) G_{\mu\nu}. \tag{4.32}$$

It is obvious that if $\beta_{3+k} = 0$, the Lagrangian (4.23) becomes (4.10) and the metric equation becomes (4.20).

The second case in which the coefficient φ_1 is nonzero gives the components of the curvature 2-form $R(\Omega)$ such that

$$R_{\mu\nu ab} = 2\big(\chi_0 - \varphi_0^2 - \psi_0\varphi_1\big)e_{a\mu}e_{b\nu} + (\chi_1 - \psi_1\varphi_1)(e_{a\mu}f_{b\nu} + e_{b\nu}f_{a\mu}) + \varphi_1^2 f_{a\mu}f_{b\nu}. \tag{4.33}$$

The solution for $f_{\mu\nu}$ is given by

$$e_{a\mu}f_{b\nu} + e_{b\nu}f_{a\mu} = \frac{1}{(\chi_1 - \psi_1\varphi_1)}\big(R_{\mu\nu ab} - 2\big(\chi_0 - \varphi_0^2 - \psi_0\varphi_1\big)e_{a\mu}e_{b\nu} - \varphi_1^2 f_{a\mu}f_{b\nu}\big) \quad \setminus e_{c\lambda}\eta^{ac}$$
$$g_{\mu\lambda}f_{b\nu} + e_{b\nu}f_{\mu\lambda} = \frac{1}{(\chi_1 - \psi_1\varphi_1)}\big(R_{\mu\nu ab}e^a_\lambda - 2\big(\chi_0 - \varphi_0^2 - \psi_0\varphi_1\big)g_{\mu\nu}e_{b\nu} - \varphi_1^2 f_{\mu\lambda}f_{b\nu}\big) \quad \setminus e^{b\nu}$$
$$g_{\mu\lambda}f + f_{\mu\lambda} = \frac{1}{(\chi_1 - \psi_1\varphi_1)}\big(R_{\mu\lambda} - 2\big(\chi_0 - \varphi_0^2 - \psi_0\varphi_1\big)g_{\mu\lambda} - \varphi_1^2 f_{\mu\lambda}f\big) \qquad \setminus g^{\mu\lambda}$$
$$4f = 4f^\mu_\mu = \frac{1}{(\chi_1 - \psi_1\varphi_1)}(R - 6\big(\chi_0 - \varphi_0^2 - \psi_0\varphi_1\big) - \varphi_1^2 f^2)$$
$$\Rightarrow f^2 + \frac{4}{\varphi_1^2}(\chi_1 - \psi_1\varphi_1)f - \frac{1}{\varphi_1^2}(R - 6\big(\chi_0 - \varphi_0^2 - \psi_0\varphi_1\big)) = 0$$
$$\Rightarrow f = \frac{2(\chi_1 - \psi_1\varphi_1)}{\varphi_1^2}\Big(-1 \pm \sqrt{1 + \frac{\varphi_1^2}{4(\chi_1 - \psi_1\varphi_1)^2}(R - 6\big(\chi_0 - \varphi_0^2 - \psi_0\varphi_1\big)}\Big)$$
$$f_{\mu\lambda} = \frac{1}{(\chi_1 - \psi_1\varphi_1) + f\varphi_1^2}\big(R_{\mu\lambda} - \big(2(\chi_0 - \varphi_0^2 - \psi_0\varphi_1) + f(\chi_1 - \psi_1\varphi_1)\big)g_{\mu\lambda}\big). \tag{4.34}$$

Then the metric equation is found by using the last equation and the relation between the covariant derivative and the affine connection which is given by

$$\frac{1}{e}\varepsilon^{\lambda\mu\tau}\nabla_\lambda R_{\mu\nu} = -4\left((\zeta_3 + 2f\zeta_6)\psi_0 + \Big(2f\zeta_3 - \frac{R}{2} + 4\zeta_1\Big)(\psi_1 - \varphi_0)\right) g_\nu{}^\tau$$
$$+ (\psi_1 - \varphi_0)G_\nu{}^\tau - \frac{\varphi_1}{2e^2}\varepsilon^{\lambda\mu\tau}\varepsilon_{\nu\rho\kappa}f_\lambda{}^\rho f_\mu{}^\kappa. \tag{4.35}$$

4.2.2.2 On-shell model

When I take $\varphi_{11} \neq 0$, i.e. the torsion depends on the ff term, both the covariant derivative of the curvature and torsion include Df. To avoid this situation, we take $\varphi_{11} = 0$. This means that $\alpha_{2+2k} = 0$, $\alpha_{1+k} = 0$, $\beta_{3+3k} = 0$ which make χ_{11} zero. The Lagrangian of such a theory is the following

$$L_3 = \mathrm{L}_{\mathrm{MMG}} + \alpha_2 e \wedge De + \frac{1}{2}\beta_{3+k}\, e \wedge e \wedge f, \tag{4.36}$$

where

$$\mathrm{L}_{\mathrm{MMG}} = \mathrm{L}_{\mathrm{TMG}} + \frac{1}{2}\beta_{3+2k}\, e \wedge f \wedge f. \tag{4.37}$$

The field equations are

$$\begin{aligned} De &= \varphi_0 e \wedge e + \varphi_1 e \wedge f, \\ R &= \chi_0 e \wedge e + \chi_1 e \wedge f. \\ Df &= \psi_0 e \wedge e + \psi_1 e \wedge f + \psi_{11} f \wedge f. \end{aligned} \tag{4.38}$$

To find the constraint equation, the structure equation is used:

$$\begin{aligned} DDe_e &= 2\varphi_0 e_a(e\cdot e) + \varphi_1 e_a(f\cdot e) + \varphi_1 f_a(e\cdot e) = \varphi_1 e_a(f\cdot e) \\ &= R_{ab}\wedge e^b = 2\chi_0 e_a(e\cdot e) + \chi_1 e_a(f\cdot e) + \chi_1 f_a(e\cdot e) = \chi_1 e_a(f\cdot e). \end{aligned} \tag{4.39}$$

This gives

$$e_a(f\cdot e) = 0. \tag{4.40}$$

The second structure equation is

$$DR = 2\chi_0 De \wedge e + \frac{\chi_1}{\varphi_1}(DDe - 2\varphi_0 De \wedge e) = \cdots e_a(f\cdot e) = 0. \tag{4.41}$$

In the second term, torsion is used. The last equation gives

$$\begin{aligned} DDf_a &= 2\psi_0\epsilon_{abc}De^b\wedge e^c + \frac{\psi_1}{\chi_1}(DR_a - 2\chi_0\epsilon_{abc}De^b\wedge e^c) + 2\psi_{11}\epsilon_{abc}Df^b\wedge f^c \\ &= 2\Big(\psi_0 - \frac{\psi_1\chi_0}{\chi_1}\Big)\big(2\varphi_0 e_a(e\cdot e) + \varphi_1 e_a(f\cdot e) + \varphi_1 f_a(e\cdot e)\big) \\ &\quad + 2\psi_{11}\big(\psi_0 e_a(e\cdot f) + \psi_1 e_a(f\cdot f) + \psi_1 f_a(e\cdot e) + \psi_{11} f_a(f\cdot f)\big) \\ &= \left(2\Big(\psi_0 - \frac{\psi_1\chi_0}{\chi_1}\Big)\varphi_1 - 2\psi_0\psi_{11}\right)e_a(f\cdot e) = 0, \end{aligned} \tag{4.42}$$

where the constraint (4.40) is used in the final step. Since this model includes torsion, the transformation $\omega \to \omega - \varphi_0 e - \varphi_1 f =: \Omega$ will be applied as in the previous case to obtain a torsionless model. By using this new connection, the covariant derivative for an arbitrary one form a and the curvature are redefined such that $d_\Omega a = da + \Omega \wedge a,\ R(\Omega) = d\Omega + \frac{1}{2}\Omega \wedge \Omega$. Then the procedure for torsionless model is applied to find $f(e)$ and the metric equation. After this transformation, the field equations become

$$\begin{aligned}
&d_\Omega e = de + \Omega \wedge e = 0,\\
&R(\Omega) = (\chi_0 - \varphi_0^2 - \psi_0\varphi_1) e \wedge e + (\chi_1 - \varphi_1\psi_1) e \wedge f + \Big(\frac{\varphi_1^2}{2} - \varphi_1\psi_{11}\Big) f \wedge f,\\
&d_\Omega f = \psi_0 e \wedge e + (\psi_1 - \varphi_0) e \wedge f - (\psi_{11} - \varphi_1) f \wedge f.
\end{aligned} \tag{4.43}$$

Note that the coefficient of the ff term vanishes. For $\varphi_1 = 2\psi_{11}$, then the auxiliary field is given by

$$f_{\mu\nu} = \frac{1}{\chi_0 - \varphi_1\psi_1}\left(S_{\mu\nu} + \frac{1}{2}\left(\chi_0 - \varphi_0^2 - \psi_0\varphi_1\right) g_{\mu\nu}\right), \tag{4.44}$$

and the metric equation becomes

$$C_{\mu\nu} = -4(\chi_1 - \varphi_1\psi_1)\psi_0 g_{\mu\nu} - (\psi_1 - \varphi_0)\Big(G_{\mu\nu} - (\chi_0 - \varphi_0^2 - \psi_0\varphi_1) g_{\mu\nu}\Big). \tag{4.45}$$

It is obvious that if $\beta_{3+2k} = 0$ the Lagrangian (4.36) becomes (4.23) and the metric equation becomes (4.32).

4.3 Hamiltonian analysis

There are three families when $N = 3$. For the first Lagrangian (4.10) the coefficients are

$$\begin{aligned}
&g_{ee} = 2\alpha_2, \quad g_{e\omega} = \alpha_1, \quad g_{ef} = \alpha_{2+k}, \quad g_{\omega\omega} = 2\alpha_0,\\
&f_{eee} = \beta_3, \quad f_{ee\omega} = 2\alpha_2, \quad f_{e\omega\omega} = \alpha_1, \quad f_{e\omega f} = \alpha_{2+k}, \quad f_{\omega\omega\omega} = 2\alpha_0.
\end{aligned} \tag{4.46}$$

The coefficients for the Lagrangian (4.23) are the terms in (4.46) with

$$g_{f\omega} = \alpha_{1+k} = f_{f\omega\omega}, \quad f_{eef} = \beta_{3+k}. \tag{4.47}$$

Finaly, the coefficients for the Lagrangian (4.36) are the terms in (4.46) with

$$f_{eef} = \beta_{3+k}, \quad f_{eff} = \beta_{3+2k}. \tag{4.48}$$

Then the submatrix of the Poisson bracket becomes

$$\mathcal{P}^{ab}_{RS} = p_i \begin{pmatrix} -V^{ff} & 0 & V^{fe} \\ 0 & 0 & 0 \\ V^{ef} & 0 & -V^{ee} \end{pmatrix}, \qquad i = 1, 2, 3, \tag{4.49}$$

where $V^{abPQ} = \varepsilon^{ij}\Omega_i^{aP}\Omega_j^{bQ}$. The form of the Posisson bracket submatrix is same for both Lagrangians. The only difference is the coefficients p_i. The coefficients for Lagrangians are

$$\begin{aligned} p_1 &= \frac{\alpha_{2+k}^2}{2\alpha_0}, \\ p_2 &= \frac{\left(\alpha_{2+k}^2 - \alpha_{1+k}\beta_{3+k}\right)^2}{2\left(\alpha_0\alpha_{2+k}^2 + \alpha_{1+k}(\alpha_2\alpha_{1+k} - \alpha_1\alpha_{2+k})\right)}, \\ p_3 &= \frac{\alpha_{2+k}^4 - 2\alpha_1\alpha_{2+k}^2\beta_{3+2k} + 2\alpha_0\alpha_{2+k}\beta_{3+k}\beta_{3+2k} + (\alpha_1^2 - 4\alpha_0\alpha_2)\beta_{3+2k}^2}{2\alpha_0\alpha_{2+k}^2}. \end{aligned} \tag{4.50}$$

The Poisson bracket of primary and secondary constraints are

$$\{\phi[\xi], \Psi\} = \epsilon^{ij}\left(f_{fRP}\partial_i\xi^R \cdot \Omega_j^P + g^{TU} f_{RSU} f_{fPT}\xi^R \cdot \Omega_i^S \times \Omega_j^P\right), \tag{4.51}$$

where ξ^R is the test function. The vector calculated from the Poisson bracket of primary and secondary constraints are linearly independent of the colums of the submatrix of Poisson bracket. The matrices (4.49) have rank 2. Then the rank of the full 10×10 Poisson bracket matrix (2.74) becomes 4. Four of the constraints are second-class and the rest of the six constraints are first-class. So the number of the degrees of freedom is calculated as $6 \times 3 - 2 \times 6 - 4 = 2$, which is the single bulk degrees of freedom [8].

When only one auxiliary field is added to the Lagrangian, three families are found (4.10), (4.23) and (4.36). These models are invariant under LLT and satisfy the Bianchi identities. The curvature form is used to find the auxiliary field in terms of the metric. The auxiliary field is a linear combination of the Schouten tensor and the metric. It is given in (4.18) for (4.10), in (4.30) for (4.23) and in (4.44) for (4.36). If the metric equation is calculated by using the field equation of the highest order term i.e. the auxiliary field, it is observed that the highest order term is Cotton tensor which involves the third order derivative of the metric. From the Hamiltonian analysis, the dimension of the physical phase space per space point is found as 2 for all Lagrangians. This implies a single bulk degree of freedom as TMG and MMG.

CHAPTER 5

$N = 4$ MODEL

Consider the most general $N = 4$ Lagrangian 3-form with the dreibein e, the connection ω and auxiliary fields f, h

$$
\begin{aligned}
L = & \alpha_1 e \wedge De + \alpha_2 e \wedge R + \alpha_3 h \wedge De + \alpha_4 f \wedge De + \alpha_5 f \wedge R + \alpha_6 f \wedge Df \\
& + \alpha_7 h \wedge R + \alpha_8 h \wedge Df + \alpha_9 h \wedge Dh + \alpha_0 (\omega \wedge d\omega + \frac{1}{3}\omega \wedge \omega \wedge \omega) \\
& + \frac{1}{2}(\beta_4 e \wedge e \wedge f + \beta_5 e \wedge f \wedge f + \beta_6 e \wedge e \wedge h + \beta_7 e \wedge h \wedge h + \beta_8 e \wedge f \wedge h) \\
& + \frac{1}{2}(\beta_9 f \wedge f \wedge h + \beta_{10} f \wedge h \wedge h) + \frac{1}{6}(\beta_3 e \wedge e \wedge e + \beta_2 f \wedge f \wedge f) \\
& + \frac{1}{6}\beta_1 h \wedge h \wedge h,
\end{aligned}
\tag{5.1}
$$

where

$$
D\Omega := d\Omega + \omega \wedge \Omega, \qquad \Omega := e, f, h \qquad R := d\omega + \frac{1}{2}\omega \wedge \omega \tag{5.2}
$$

There are four known models with $N = 4$ fields. The first two models are NMG which is a parity preserving model [8], GMG which is obtained by the addition of Chern-Simons term to NMG [8]. There is also one more family, the models are EMG whose field equations preserves parity even though the action does not and EGMG the addition of eee term to EMG [2]. The Lagrangians of these theories can be obtained from the generic Lagrangian given in (5.1):

NMG Lagrangian L_{NMG} can be obtained by taking $\alpha_2 = -\sigma$, $\alpha_3 = 1$, $\beta_3 = \Lambda_0$ and $\alpha_5 = \beta_5 = -\frac{1}{m^2}$.

The terms in GMG Lagrangian L_{GMG} are the same with L_{NMG} with $\alpha_0 = \frac{1}{\mu}$.

EMG Lagrangian L_{EMG} is obtained by taking $\alpha_5 = 1$, $\alpha_6 = -\frac{1}{2m^2}$, $\alpha_3 = -m^2$,

$\beta_2 = \dfrac{1}{6m^4}$, $\beta_4 = -\Lambda$, $\alpha_0 = -(m + 2 + \Lambda)$.

To obtain the EGMG Lagrangian L_{EGMG} eee term is added to L_{EMG} with a slight modification such that $\alpha_5 = 1$, $\alpha_6 = -\dfrac{1}{2m^2}$, $\alpha_3 = -m^2$, $\beta_2 = \dfrac{1}{6m^4}$, $\beta_4 = -\nu$, $\alpha_0 = \nu - m^2$ and $\beta_3 = \dfrac{2\nu m^4}{\mu}$.

5.1 Field equations

The field equations that follow are

$$\underbrace{\begin{bmatrix} 2\alpha_1 & \alpha_2 & \alpha_4 & \alpha_3 \\ \alpha_2 & 2\alpha_0 & \alpha_5 & \alpha_7 \\ \alpha_4 & \alpha_5 & 2\alpha_6 & \alpha_8 \\ \alpha_3 & \alpha_7 & \alpha_8 & 2\alpha_9 \end{bmatrix}}_{A} \begin{bmatrix} De \\ R \\ Df \\ Dh \end{bmatrix} + \frac{1}{2} \underbrace{\begin{bmatrix} \beta_3 & 2\beta_4 & 2\beta_6 & \beta_5 & \beta_8 & \beta_7 \\ 2\alpha_1 & 2\alpha_4 & 2\alpha_3 & 2\alpha_6 & 2\alpha_8 & 2\alpha_9 \\ \beta_4 & 2\beta_5 & \beta_8 & \beta_2 & 2\beta_9 & \beta_{10} \\ \beta_6 & \beta_8 & 2\beta_7 & \beta_9 & 2\beta_{10} & \beta_1 \end{bmatrix}}_{2B} \begin{bmatrix} e \wedge e \\ e \wedge f \\ e \wedge h \\ f \wedge f \\ f \wedge h \\ h \wedge h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{5.3}$$

The matrix A is assumed invertible. Then the field equations become

$$\begin{aligned} De &= \varphi_0 e \wedge e + \varphi_1 e \wedge f + +\varphi_2 e \wedge h + \varphi_{11} f \wedge f + \varphi_{12} f \wedge h + \varphi_{22} h \wedge h, \\ R &= \chi_0 e \wedge e + \chi_1 e \wedge f + \chi_2 e \wedge h + \chi_{11} f \wedge f + \chi_{12} f \wedge h + \chi_{22} h \wedge h, \\ Df &= \psi_0 e \wedge e + \psi_1 e \wedge f + \psi_2 e \wedge h + \psi_{11} f \wedge f + \psi_{12} f \wedge h + \psi_{22} h \wedge h, \\ Dh &= \gamma_0 e \wedge e + \gamma_1 e \wedge f + \gamma_2 e \wedge h + \gamma_{11} f \wedge f + \gamma_{12} f \wedge h + \gamma_{22} h \wedge h. \end{aligned} \tag{5.4}$$

The coefficients are given in appendix B. Note that for GMG the coefficients are $\varphi_i = 0\ \forall i$, $\chi_1 = -1$, $\psi_1 = \dfrac{m^2}{\mu}$, $\psi_2 = m^2$, $\gamma_0 = -\dfrac{\lambda_0}{2}$, $\gamma_1 = \sigma$, $\gamma_{11} = \dfrac{1}{2m^2}$ and for EGMG they are $\varphi_i = 0\ \forall i$, $\chi_0 = -\dfrac{m^2}{2}$, $\chi_2 = \dfrac{m^2}{\nu}$, $\psi_0 = \dfrac{m^2}{2}(\nu - m^2)$, $\psi_2 = -\dfrac{m^4}{\nu}$, $\psi_{11} = \dfrac{1}{2m^2}$, $\gamma_0 = \dfrac{2m^2\nu}{\mu}$ and $\gamma_1 = \dfrac{\nu}{m^2}$.

5.2 Bianchi identities in the first order formalism

Here we apply the strategy discussed in chapter 2 to study Bianchi identities. The covariant derivative of the last term in equation (5.4) is

$$\begin{aligned} DDh =& 2\gamma_0 De \wedge e + \gamma_1(De \wedge f - e \wedge Df) + \gamma_2(De \wedge h - e \wedge Dh) + 2\gamma_{11} Df \wedge f \\ & + \gamma_{12}(Df \wedge h - f \wedge Dh) + 2\gamma_{22} Dh \wedge h = 0. \end{aligned} \tag{5.5}$$

There are also Cartan structure equations

$$DDe = R \wedge e, \qquad DR = 0. \tag{5.6}$$

The classification of the models is the following: the model can be torsionless or not and the model can be diffeomorphism invariant or third way consistent. These cases will be examined separately.

5.2.1 Vanishing torsion

In NMG and GMG the coefficient χ_1 does not vanish whereas in EMG and EGMG the coefficient χ_2 does not vanish. Thus we can start our analysis by assuming that there are at least two families in torsionless case, one of them includes NMG/GMG and the other one does E(G)MG. From the Cartan structure equation, the constraint equation is obtained

$$\begin{aligned} R_a^b \wedge e_b = & 2\chi_0 e_a(e \cdot e) + \chi_1\Big(e_a(f \cdot e) - f_a(e \cdot e)\Big) + \chi_2\Big(e_a(h \cdot e) - h_a(e \cdot e)\Big) \\ & + \chi_{11}\Big(f_a(f \cdot e) - f_a(e \cdot f)\Big) + \chi_{12}\Big(f_a(h \cdot e) - h_a(e \cdot f)\Big) + 2\chi_{22} h_a(h \cdot e) \\ = & DDe_a = 0, \end{aligned} \tag{5.7}$$

where $\Omega \cdot \Omega = \Omega^a \wedge \Omega_a$ is used. Due to the antisymmetry of the wedge product, the dot product terms give zero when the same fields are used. Two constraint equations are obtained from the equation (5.7)

$$e \cdot f = 0, \qquad e \cdot h = 0. \tag{5.8}$$

The constraint equation gives

$$0 = e \cdot \Omega = e^a \wedge \Omega_a = e^a_\mu dx^\mu \wedge \Omega_{a\nu} dx^\nu = e^a_\mu \eta_{ab} \Omega^b_\nu dx^\mu \wedge dx^\nu \Rightarrow \Omega_{\nu\mu} := e^a_\mu \eta_{ab} \Omega^b_\nu, \tag{5.9}$$

where $\Omega = f, h$. Since the term $dx^\mu \wedge dx^\nu$ is antisymmetric in μ, ν, the tensor $\Omega_{\nu\mu}$ has only symmetric part. To calculate this symmetric tensor, the curvature 2-form will be used.

If the covariant derivative of the curvature 2-form is calculated, the second Cartan structure equation gives

$$\begin{aligned} DR_a = e_a(f \cdot h)\Big(&- 2\chi_{22}\psi_2 + \chi_{12}(\psi_1 - \gamma_2) + 2\chi_{22}\gamma_1\Big) \\ &+ f_a(f \cdot h)\Big(- 2\chi_{11}\psi_{12} + \chi_{12}(2\psi_{11} - \gamma_{12}) + 4\chi_{22}\gamma_{11}\Big) \\ &+ h_a(f \cdot h)\Big(- 4\chi_{11}\psi_{22} + \chi_{12}(\psi_{12} - 2\gamma_{22}) + 2\chi_{22}\gamma_{12}\Big) = 0. \end{aligned} \tag{5.10}$$

Note that the product $f \cdot h$ does not vanish. The second structure equation is satisfied if the coefficients of the three forms vanishes respectively. The simplest solution is to take $\chi_{11} = \chi_{12} = \chi_{22} = 0$. Therefore the curvature 2-form becomes

$$R = \chi_0 e \wedge e + \chi_1 e \wedge f + \chi_2 e \wedge h, \tag{5.11}$$

which should give one of the auxiliary fields as a function of dreibeins. Thus, we will assume that one of the coefficients χ_1, χ_2 vanishes. Up to now, we have not used the covariant derivative of the auxiliary field h used to determine if the model is on-shell or off-shell. We will analyse these two cases separately.

5.2.1.1 Off-shell

In this section, the cases $(i)\chi_2 = 0$ and $(ii)\chi_1 = 0$ will be analysed separately. In order to find one of the auxiliary fields, it is assumed that the coefficient χ_2 vanishes and ψ_2 is nonzero. Therefore the covariant derivative in (5.5) becomes

$$
\begin{aligned}
DDh =& \left(\frac{\psi_1\gamma_2}{\psi_2} - \gamma_1\right) e\wedge Df + \gamma_2 DDf + 2\left(\gamma_{11} - \frac{\psi_{11}\gamma_2}{\psi_2}\right) Df\wedge f \\
&+ \left(\gamma_{12} - \frac{\psi_{12}\gamma_2}{\psi_2}\right)\Big(Df\wedge h - f\wedge Dh\Big) + 2\left(\gamma_{22} - \frac{\psi_{22}\gamma_2}{\psi_2}\right) Dh\wedge h \\
=& \left(\frac{\psi_1\gamma_2}{\psi_2} - \psi_1\gamma_2 - \gamma_1\right) e\wedge Df - \psi_2\gamma_2 e\wedge Dh \\
&+ \left(\frac{\psi_{11}\gamma_2}{\psi_2} - \psi_{11}\gamma_2 - \gamma_{11}\right) f\wedge Df + 2\left(\frac{\psi_{22}\gamma_2}{\psi_2} - \psi_{22}\gamma_2 - \gamma_{22}\right) h\wedge Dh \\
&+ \left(\frac{\psi_{12}\gamma_2}{\psi_2} - \psi_{12}\gamma_2 - \gamma_{12}\right)(f\wedge Dh - Df\wedge h),
\end{aligned}
\tag{5.12}
$$

the torsion constraint and the explicit form of DDf term is used throughout. For the off-shell model the last field equation should not be used. This means that the coefficients of the eDh, fDh and hDh terms must vanish. Since we assumed that the coefficient ψ_2 is nonzero, we have $\gamma_2 = 0$. Torsionless, off-shell model is obtained by choosing the coefficients in Lagrangian (5.1) such that $\chi_i = 0$ unless $i = 0, 1$, $\varphi_i = 0$ $\forall i$, $\gamma_2 = \gamma_{12} = \gamma_{22} = 0$. The simplest Lagrangian whose field equations satisfy these conditions is

$$
L_1 = \mathrm{L}_{\mathrm{GMG}} + \alpha_1 e\wedge De + \alpha_4 f\wedge De + +\frac{1}{2}\beta_4 e\wedge e\wedge f, \tag{5.13}
$$

where

$$
\begin{aligned}
\mathrm{L}_{\mathrm{GMG}} =& \alpha_2 e\wedge R + \alpha_5 f\wedge R + \alpha_3 h\wedge De + \frac{1}{6}\beta_3 e\wedge e\wedge e + \alpha_0(\omega\wedge d\omega + \frac{1}{3}\omega\wedge\omega\wedge\omega) \\
&+ \beta_5 e\wedge f\wedge f.
\end{aligned}
\tag{5.14}
$$

The field equations are

$$
\begin{aligned}
&De = 0,\\
&R = -\frac{\beta_4}{2\alpha_5} e\wedge e - \frac{\beta_5}{\alpha_5} e\wedge f,\\
&Df = \Big(\frac{\beta_4\alpha_0}{\alpha_5^2} - \frac{\alpha_1}{\alpha_5}\Big) e\wedge e + \Big(2\frac{\beta_5\alpha_0}{\alpha_5^2} - \frac{\alpha_4}{\alpha_5}\Big) e\wedge f - \frac{\alpha_3}{\alpha_5} e\wedge h,\\
&Dh = \frac{1}{2\alpha_3}\Bigg(\Big(\frac{\beta_4\alpha_2}{\alpha_5} - \frac{2\alpha_4}{\alpha_5^2}(\alpha_0\beta_4 - \alpha_1\alpha_5) - \beta_3\Big) e\wedge e + + \frac{\alpha_4}{\alpha_5} e\wedge h - \beta_5 f\wedge f\\
&\qquad \Big(\frac{2\beta_5\alpha_2}{\alpha_5} - \frac{2\alpha_4}{\alpha_5^2}(2\alpha_0\beta_5 - \alpha_4) - \beta_4\Big) e\wedge f\Bigg).
\end{aligned}
\tag{5.15}
$$

The solution of the first equation gives the spin connection as a function of dreibein $\omega(e)$. One can find the auxiliary field f by using the curvature form. From the third equation, other auxiliary field h can be obtained. The last equation gives the metric equation. The components of the curvature 2-form obtained from the Lagrangian (5.13) is

$$
R_{\mu\nu ab} = 2\zeta_1 e_{a\mu} e_{b\nu} + \zeta_2 (e_{a\mu} f_{b\nu} + e_{b\nu} f_{a\mu}). \tag{5.16}
$$

The coefficients ζ_1, ζ_2 can be found by comparison with the equation (5.15). The solution for $f_{\mu\nu}$ is given by

$$
\begin{aligned}
&e_{a\mu} f_{b\nu} + e_{b\nu} f_{a\mu} = \frac{1}{\zeta_3}\big(R_{\mu\nu ab} + 2\zeta_1 e_{a\nu} e_{b\mu}\big) && \setminus e_{c\lambda}\eta^{ac}\\
&g_{\mu\lambda} f_{b\nu} + e_{b\nu} f_{\mu\lambda} = \frac{1}{\zeta_3}\big(R_{\mu\nu ab} e^a_\lambda + 2\zeta_1 g_{\nu\lambda} e_{b\mu}\big) && \setminus e^{b\nu}\\
&g_{\mu\lambda} f + f_{\mu\lambda} = \frac{1}{\zeta_3}\big(R_{\mu\lambda} + 2\zeta_1 g_{\mu\lambda}\big) && \setminus g^{\mu\lambda}\\
&4f = 4f^\mu_\mu = \frac{1}{\zeta_3}(R + 6\zeta_1) \qquad \text{then the third equation gives}\\
&f_{\mu\lambda} = \frac{1}{\zeta_3}\left(R_{\mu\lambda} - \frac{R}{4} g_{\mu\lambda} + \frac{1}{2}\zeta_1 g_{\mu\lambda}\right) = \frac{1}{\zeta_3}\left(S_{\mu\lambda} + \frac{1}{2}\zeta_1 g_{\mu\lambda}\right)\\
&\quad = -\frac{\alpha_5}{\beta_5} S_{\mu\nu} - \frac{\beta_4}{4\beta_5} g_{\mu\nu},
\end{aligned}
\tag{5.17}
$$

where $S_{\mu\lambda}$ is the Schouten tensor. Note that in GMG β_4 vanishes. The relation between the local Lorentz covariant derivative of f given in (5.15) and the covariant derivative defined by affine connection is the following

$$\begin{aligned}\nabla_\lambda f_{\mu\nu} &= \nabla_\lambda (f^a_\mu \eta_{ab} e^b_\nu) \\ &= \eta_{ab} e^b_\nu \nabla_\lambda f^a_\mu = \eta_{ab} e^b_\nu (\partial_\lambda f^a_\mu + \omega^a_{\lambda\ b} f^b_\mu - \Gamma^\alpha_{\lambda\mu} f^a_\alpha) \\ &= \eta_{ab} e^b_\nu \big(D_\lambda f^a_\mu - e^c_\alpha (\partial_\lambda e_{c\mu} + \omega^c_{\lambda\ k} e^k_\mu)\big) \\ &= \eta_{ab} e^b_\nu D_\lambda f^a_\mu \\ &= \frac{\alpha_5}{\beta_5} \nabla_\lambda S_{\mu\nu}.\end{aligned} \tag{5.18}$$

In the last step, (5.17) is used.

$$\begin{aligned}&D_\lambda f^a_\mu = \epsilon^{abc}(\psi_0 e_{b\lambda} \wedge e_{c\mu} + \psi_1 e_{b\lambda} \wedge f_{c\mu} + \psi_2 e_{b\lambda} \wedge h_{c\mu}) \qquad \diagdown \epsilon_{klm} e_{a\nu} e^\mu_i \eta^{ik} e^{l\nu} e^m_\tau \\ &\frac{\alpha_5}{\beta_5} \frac{1}{e} \varepsilon^{\mu\nu}{}_\tau \nabla_\mu S_{\nu\lambda} = 2\psi_0 g_{\lambda\tau} + \psi_1 (f_{\lambda\tau} - f g_{\lambda\tau}) + \psi_2(-h g_{\lambda\tau} + h_{\lambda\tau}) \\ &\frac{\alpha_5}{\beta_5} C_{\tau\lambda} = 2\psi_0 g_{\lambda\tau} + \psi_1 (f_{\lambda\tau} - f g_{\lambda\tau}) + \psi_2(-h g_{\lambda\tau} + h_{\lambda\tau}) \qquad \diagdown g^{\lambda\tau} \\ &0 = 6\psi_0 + \psi_1 (f - 3f) g + \psi_2(-3h + h) \qquad \Rightarrow \qquad h = \frac{1}{\psi_2}(3\psi_0 - \psi_1 f),\end{aligned} \tag{5.19}$$

the coefficients given in equation (5.15) are used in the last line. The explicit form of the auxiliary field is

$$h_{\mu\nu} = -\frac{\alpha_5}{\alpha_3}\left(\frac{\alpha_5}{\beta_5} C_{\mu\nu} + \Big(\frac{3\beta_4\alpha_0}{2\alpha_5^2} - \frac{2\alpha_1}{\alpha_5} - \frac{\beta_4\alpha_4}{4\beta_5\alpha_5}\Big) g_{\mu\nu} + \Big(\frac{2\alpha_0}{\alpha_5} - \frac{\alpha_4}{\beta_5} S_{\mu\nu}\Big)\right). \tag{5.20}$$

The metric equation is obtained by using the fields and the last equation of (5.15), which is

$$\begin{aligned}H_{\mu\nu} = &-\frac{\beta_5(\alpha_3)}{\alpha_6^2}\left(\Big(\frac{\alpha_4}{\beta_5} - \frac{2\alpha_0}{\alpha_5} + \frac{\alpha_5\alpha_4}{\beta_5\alpha_3}\Big) C_{\mu\nu} + \frac{\alpha_5^2}{\beta_5^2} J_{\mu\nu}\right. \\ &+ \left(\frac{\alpha_5\beta_4}{2\beta_5^2} + \frac{1}{2\alpha_3}\Big(\frac{2\beta_5\alpha_2}{\alpha_5} - \frac{2\alpha_4}{\alpha_5^2}(2\alpha_0\beta_5 - \alpha_4) - \beta_4\Big)\right. \\ &\left.\left. - \big(2\beta_5\alpha_0 - \alpha_4\alpha_5\big)\frac{\alpha_4}{\alpha_5^2\alpha_3}\right) G_{\mu\nu} - \left(\frac{\beta_4^2}{4\beta_5^2} + \gamma_0 - \frac{\psi_0\gamma_2}{\psi_2} + \frac{\gamma_1\beta_4}{2\beta_5} - \frac{\psi_1\gamma_2\beta_4}{2\beta_5\psi_2}\Big) g_{\tau\lambda}\right),\end{aligned} \tag{5.21}$$

where

$$H^{\mu\nu} := \frac{1}{e}\varepsilon^{\mu\rho\sigma}\nabla_\rho C^\nu{}_\sigma. \tag{5.22}$$

As a second case, assume that $\chi_i = 0$ unless $i = 0, 2$. The covariant derivative of the last equation in (5.15) is

$$DDh = 2\Big(\gamma_0 - \chi_0\frac{\gamma_2}{\chi_2}\Big)De\wedge e + \gamma_1\big(De\wedge f - e\wedge Df\big)\frac{\gamma_2}{\chi_2}DR + 2\gamma_{11}Df\wedge f + \gamma_{12}\big(Df\wedge h - f\wedge Dh\big) + 2\gamma_{22}Dh\wedge h, \tag{5.23}$$

for the off-shell case we have $\gamma_{12} = \gamma_{22} = 0$. If we use the structure equation $DR = 0$ and the torsion constraint $De = 0$, we obtain

$$DDh_a = 2\gamma_{11}(\psi_2 e^a(h\cdot f) + \psi_{12}f^a(h\cdot f) - 2\psi_{22}h^a(h\cdot f)) = 0. \tag{5.24}$$

If γ_{11} is taken zero, then the simplest Lagrangian whose field equations that satisfy the conditions $\chi_i = 0$ unless $i = 0, 2$, $\varphi_i = 0\ \forall i$ and $\gamma_{11} = 0$ is

$$L_2 = \mathrm{L}_{\mathrm{EGMG}} + \alpha_1 e\wedge De + \alpha_2 e\wedge R + \alpha_4 f\wedge De + \frac{1}{2}\Big(\frac{2\alpha_6\alpha_4}{\alpha_5}e\wedge f\wedge f\Big) + \frac{1}{6}\Big(\frac{4\alpha_6^2}{\alpha_5}f\wedge f\wedge f\Big), \tag{5.25}$$

where

$$\mathrm{L}_{\mathrm{EGMG}} = \alpha_5 f\wedge R + \alpha_6 f\wedge Df + \alpha_3 h\wedge De + \frac{1}{2}\beta_4 e\wedge e\wedge f + \alpha_0\Big(\omega\wedge d\omega + \frac{1}{3}\omega\wedge\omega\wedge\omega\Big) + \frac{1}{6}\beta_3 e\wedge e\wedge e. \tag{5.26}$$

The field equations are

$$\begin{aligned}
De &= 0,\\
R &= -\frac{1}{2(4\alpha_6\alpha_0 - \alpha_5^2)}\Big((4\alpha_1\alpha_6 - \alpha_5\beta_4)e\wedge e + 4\alpha_6\alpha_3 e\wedge h\Big),\\
Df &= -\frac{1}{4\alpha_6\alpha_0 - \alpha_5^2}\Big((\alpha_0\beta_4 - \alpha_1\alpha_5)e\wedge e - \alpha_5\alpha_3 e\wedge h\Big) - \frac{\alpha_4}{\alpha_5}e\wedge f - \frac{\alpha_6}{\alpha_5}f\wedge f,\\
Dh &= \left(-\frac{\beta_4\left(2\alpha_0\alpha_4 - \alpha_2\alpha_6\right)}{2\left(\alpha_6^2 - 4\alpha_0\alpha_7\right)\alpha_3} - \frac{\alpha_1\left((2\alpha_2\alpha_7 - \alpha_6\alpha_4\right)}{\left(\alpha_6^2 - 4\alpha_0\alpha_7\right)\alpha_3} - \frac{\beta_3}{2\alpha_3}\right)e\wedge e\\
&\quad + \frac{\alpha_4^2 - \alpha_6\beta_4}{\alpha_6\alpha_3}e\wedge f + \frac{\alpha_5\alpha_4 - 2\alpha_2\alpha_7}{\alpha_5^2 - 4\alpha_0\alpha_7}e\wedge h.
\end{aligned} \tag{5.27}$$

The solution of the first equation gives the spin connection as a function of dreibein $\omega(e)$. One can find the auxiliary field h by using the curvature form. From the last equation, the other auxiliary field f can be obtained. The third equation gives the metric equation. The components of the curvature 2-form obtained from the Lagrangian (5.13) is

$$R_{\mu\nu ab} = 2\zeta_1 e_{a\mu}e_{b\nu} + \zeta_3(e_{a\mu}h_{b\nu} + e_{b\nu}h_{a\mu}). \tag{5.28}$$

The coefficients ζ_1, ζ_3 can be found by comparison with the equation (5.27). The solution for $h_{\mu\nu}$ is given by

$$h_{\mu\nu} = -\frac{2(4\alpha_6\alpha_0 - \alpha_5^2)}{4\alpha_6\alpha_3}\Big(S_{\mu\nu} + \frac{4\alpha_1\alpha_6 - \alpha_5\beta_4}{4(4\alpha_6\alpha_0 - \alpha_5^2)}g_{\mu\nu}\Big) = kS_{\mu\nu} + lg_{\mu\nu}. \tag{5.29}$$

By using the last equation in (5.27), the auxiliary field f is found as in (5.19). The explicit form of the field f is given

$$\begin{aligned} f_{\mu\nu} =& \frac{\alpha_5^2 - 4\alpha_0\alpha_6}{\alpha_5\beta_4}\Bigg[- C_{\mu\nu} - \frac{2\alpha_2\alpha_6}{4\alpha_0\alpha_6 - \alpha_5^2}S_{\mu\nu} \\ &+ \Bigg(\Big(\frac{1}{\alpha_3} - \frac{\alpha_6}{4\alpha_0\alpha_6 - \alpha_5^2}\Big)\frac{\alpha_2(4\alpha_1\alpha_6 - \alpha_5\beta_4)}{2(4\alpha_0\alpha_6 - \alpha_5^2)} - \frac{\beta_3}{2\alpha_3}\Bigg)g_{\mu\nu}\Bigg] \\ =& \, aC_{\mu\nu} + bS_{\mu\nu} + cg_{\mu\nu}. \end{aligned} \tag{5.30}$$

The metric equation is obtained by using the auxiliary fields and the third equation in (5.27) as

$$\begin{aligned} aH_{\mu\nu} =& - a^2\frac{\alpha_6}{\alpha_5}L_{\mu\nu} - 4b^2\frac{\alpha_6}{\alpha_5}J_{\mu\nu} - 2ab\frac{\alpha_6}{\alpha_5}K_{\mu\nu} + C_{\mu\nu}\Big(- b - \frac{2}{e^2}\frac{\alpha_6}{\alpha_5}ac\Big) \\ &+ S_{\mu\nu}\Big(- k\frac{\alpha_5\alpha_3}{4\alpha_0\alpha_6 - \alpha_5^2} - \frac{2}{e^2}\frac{\alpha_6}{\alpha_5}bc\Big) \\ &+ g_{\mu\nu}\Bigg(- 2l\frac{\alpha_5\alpha_3}{4\alpha_0\alpha_6 - \alpha_5^2} + 2c^2\frac{\alpha_6}{\alpha_5} + \frac{R}{4}\Big(k\frac{\alpha_5\alpha_3}{4\alpha_0\alpha_6 - \alpha_5^2} - \frac{1}{e^2}2bc\Big)\Bigg), \end{aligned} \tag{5.31}$$

where

$$\begin{aligned} H_{\mu\nu} &:= \frac{1}{e}\varepsilon_\mu{}^{\rho\sigma}\nabla_\rho C_{\nu\sigma}, \\ L_{\mu\nu} &:= \frac{1}{2e^2}\varepsilon_\mu{}^{\rho\sigma}\varepsilon_\nu{}^{\lambda\tau}C_{\rho\lambda}C_{\sigma\tau}, \\ J_{\mu\nu} &:= \frac{1}{2e^2}\varepsilon_\mu{}^{\rho\sigma}\varepsilon_\nu{}^{\lambda\tau}S_{\rho\lambda}S_{\sigma\tau}, \\ K_{\mu\nu} &:= \frac{1}{e^2}\varepsilon_\mu{}^{\rho\sigma}\varepsilon_\nu{}^{\lambda\tau}C_{\rho\lambda}S_{\sigma\tau}. \end{aligned} \tag{5.32}$$

5.2.1.2 On-shell

If the covariant derivative of the last field equation in (5.15) is calculated by using the field equation itself, an on-shell model is found. As in the off-shell case, one can assume two separate cases which are $\chi_1 = 0$ and $\chi_2 = 0$. For the case $\chi_2 = 0$, Bianchi identity is calculated by using the equation (5.12) and the field equations and the conditions are

$$e^a(h\cdot f)(2\gamma_{11}\psi_2+\gamma_{12}(\gamma_2-\psi_1)-2\gamma_1\gamma_{22})=0$$
$$f^a(h\cdot f)(\gamma_{11}(2\psi_{12}-4\gamma_{22})+\gamma_{12}(\gamma_{12}-2\psi_{11}))+h^a(h\cdot f)(4\gamma_{11}\psi_{22}-\gamma_{12}\psi_{12})=0. \tag{5.33}$$

The simplest solution is to take $\gamma_1=\gamma_{11}=\gamma_{12}=0$. We have also $\chi_i=0$ unless $i=0,1$ and $\varphi_i=0\ \forall i$. However these conditions give inconsistent equalities. Thus, we can conclude that there is no on-shell model that the coefficients of field equations are $\chi_i=0$ unless $i=0,1$, $\varphi_i=0\ \forall i$, $\gamma_1=\gamma_{11}=\gamma_{12}=0$.
The second case is to take $\chi_1=0$ and $\chi_2\neq 0$. If we put the field equations (5.4) into (5.12), then

$$\begin{aligned}DDh^a&=e^a(f\cdot h)\big(2\psi_1\gamma_{11}-\gamma_2\gamma_{12}+2\gamma_1\gamma_{22}\big)+f^a(f\cdot h)\big(4(\psi_{11}+\gamma_{22})\gamma_{11}-\gamma_{12}^2\big)\\&\qquad+h^a(f\cdot h)2\psi_{12}\gamma_{11}\\&=0.\end{aligned} \tag{5.34}$$

Since the term $f\cdot h$ does not vanish, the Bianchi identity is satisfied if $2\psi_1\gamma_{11}-\gamma_2\gamma_{12}+2\gamma_1\gamma_{22}=0$, $4(\psi_{11}+\gamma_{22})\gamma_{11}-\gamma_{12}^2=0$ and $\psi_{12}\gamma_{11}=0$. The coefficients in field equations (5.4) are highly complicated, thus we looked for simple cases which are $(i)\gamma_1=\gamma_{11}=\gamma_{12}=0$ with $\psi_{12}\neq 0$, $(ii)\gamma_1=\gamma_2=\gamma_{11}=\gamma_{12}=0$ with $\psi_{12}\neq 0$ and $(iii)\gamma_1=\gamma_{11}=\gamma_{12}=0$ with $\psi_{12}=0$. Then the torsion solves the spin connection in terms of dreibein $\omega=\omega(e)$, the curvature equation gives the auxiliary field h and the last equation which is the covariant derivative of the auxiliary field h gives the metric equation. The equation that includes the covariant derivative of the other auxiliary field f is a nonlinear equation, and according to the procedure given in [16] to find the auxiliary fields, the field equations should be linear. If the coefficient of the nonlinear term is set to vanish, there is no new model different from (5.13) and (5.25).

For the first case, the Lagrangian is

$$
\begin{aligned}
L_3 =& \alpha_1 e\wedge De + \frac{2\alpha_1\alpha_7}{\alpha_3} e\wedge R + \alpha_5 f\wedge R + \alpha_6 f\wedge Df + \alpha_3 h\wedge De \\
&+ \alpha_7 h\wedge R + \alpha_9 h\wedge Dh + \alpha_0\big(\omega\wedge d\omega + \frac{1}{3}\omega\wedge\omega\wedge\omega\big) \\
&+ \frac{1}{6}\Bigg(\beta_3 e\wedge e\wedge e + \frac{4\alpha_6^2}{\alpha_5^2}\Big(\alpha_5 - \frac{\alpha_7^2\beta_4}{m_+}\Big) f\wedge f\wedge f\Bigg) \\
&+ \frac{x\alpha_3}{6\alpha_1\alpha_7}\Big(\frac{\alpha_5(\alpha_3+\alpha_3)^2\beta_4}{16\alpha_1^2\alpha_6} - \alpha_9\Big) h\wedge h\wedge h \\
&+ \frac{1}{2}\Bigg(\beta_4 e\wedge e\wedge f - \frac{2\beta_4\alpha_6 x}{\alpha_5 m_+} e\wedge f\wedge f + \frac{2\beta_4\alpha_7}{\alpha_2} e\wedge f\wedge h\Bigg) \\
&+ \frac{1}{2}\Bigg(\frac{\beta_4\alpha_5 x}{2\alpha_2\alpha_6\alpha_7} e\wedge e\wedge h + \Big(\frac{\beta_4\alpha_5 x}{2\alpha_2^2\alpha_6} - \frac{x\alpha_3}{\alpha_2\alpha_7}\Big) e\wedge h\wedge h\Bigg) \\
&+ \frac{1}{2}\Bigg(-\frac{2\beta_4\alpha_6\alpha_7 x}{\alpha_2\alpha_5 m_+} f\wedge f\wedge h + \frac{\beta_4\alpha_{10}^2}{\alpha_2^2} f\wedge h\wedge h\Bigg),
\end{aligned}
\tag{5.35}
$$

where $x := \alpha_7\alpha_3 - 2\alpha_2\alpha_9$ and $m_\pm := 4\alpha_1\alpha_9 \pm \alpha_3^2$. The field equations are

$$
\begin{aligned}
&De = 0, \\
&R = \chi_0 e\wedge e + \chi_2 e\wedge h, \\
&Df = \psi_0 e\wedge e + \psi_1 e\wedge f + \psi_2 e\wedge h + \psi_{11} f\wedge f + \psi_{12} f\wedge h + \psi_{22} h\wedge h, \\
&Dh = \gamma_0 e\wedge e + \gamma_2 e\wedge h + \gamma_{22} h\wedge h.
\end{aligned}
\tag{5.36}
$$

The solution of the first equation is $\omega = \omega(e)$. By using the curvature form, the auxiliary field h can be calculated as in (5.29). The last equation gives the metric equation. By using the third equation, the auxiliary field f can be found but the equation is nonlinear in f. The second case again give a nonlinear field equation. When the coefficient that causes the non-linearity is set to zero, then the Lagrangians (5.13) and (5.25) are obtained.

5.2.2 Non-vanishing torsion

In the literature, there is no model with torsion for $N = 4$. To find models, the results of Section 2.3 will be used by taking $f_1 = f$, $f_2 = h$ and $\xi_{1,a} = \psi_a$, $\xi_{2,a} = \gamma_a$ ($a = 0, I, IJ$). The Bianchi identities are satisfied if the torsion and curvature is written as in (2.45)

$$De = \varphi_0 e \wedge e + \varphi_1 e \wedge f + \varphi_3 e \wedge h, \\ R = \chi_0 e \wedge e + \chi_1 e \wedge f + \chi_3 e \wedge h. \tag{5.37}$$

As in the torsionless model, there are two cases $(i)\chi_2 = 0$ and $(ii)\chi_2 \neq 0$. For the first case we need $\gamma_2 = 0$ and $\gamma_{I2} = 0$ for $I = 1, 2$. Then the last Bianchi identity is calculated by using (2.46) such that

$$e^a (h \cdot f)\big(\gamma_1 \varphi_2 + 2\gamma_{11} \left(\psi_2 + \psi_{12} + 2\psi_{22}\right)\big) = 0. \tag{5.38}$$

In order to solve the field equations, the transformation $\omega \to \omega - \varphi_0 e - \varphi_1 f - \varphi_2 h =:$ Ω is used as in MMG. Then the field equations become

$$\begin{aligned}
&d_\Omega e := de + \Omega \wedge e = 0, \\
&R(\Omega) = \big(\chi_0 - \varphi_1\psi_0 - \varphi_2\gamma_0 - \frac{1}{2}\varphi_0^2\big) e \wedge e + \big(\chi_1 - \varphi_1\psi_1 - \varphi_2\gamma_1\big) e \wedge f \\
&\qquad - \varphi_1\psi_2 e \wedge h + \big(\frac{1}{2}\varphi_1^2 - \varphi_1\psi_{11} - \varphi_2\gamma_{11}\big) f \wedge f + \varphi_1(\varphi_2 - \psi_{12}) f \wedge h \\
&\qquad + \Big(\frac{1}{2}\varphi_2^2 - \varphi_1\psi_{22}\Big) h \wedge h, \\
&d_\Omega f = \psi_0 e \wedge e + (\psi_1 - \varphi_0) e \wedge f + \psi_2 e \wedge h + (\psi_{11} - \varphi_1) f \wedge f + (\psi_{12} - \varphi_2) f \wedge h \\
&\qquad + \psi_{22} h \wedge h, \\
&d_\Omega h = \gamma_0 e \wedge e + \gamma_1 e \wedge f - \varphi_0 e \wedge h + \gamma_{11} f \wedge f - \varphi_1 f \wedge h - \varphi_2 h \wedge h.
\end{aligned} \tag{5.39}$$

The solution of the first equation gives $\Omega = \Omega(e)$. In order to find one of the auxiliary fields as functions of dreibein, the curvature form is used. However, the curvature form includes quadratic terms ff, fh and hh. Consider the conditions which make the curvature form linear in f and h,

$$\frac{1}{2}\varphi_1^2 - \varphi_1\psi_{11} - \varphi_2\gamma_{11} = 0; \qquad \varphi_1(\varphi_2 - \psi_{12}) = 0; \qquad \frac{1}{2}\varphi_2^2 - \varphi_1\psi_{22} = 0. \tag{5.40}$$

The simplest solution is choosing $\varphi_1 = \varphi_2 = 0$. Then the auxiliary field f can be found

$$f_{\mu\nu} = \frac{1}{\chi_1}\left(S_{\mu\nu} + \frac{1}{2}\left(\chi_0 - \frac{1}{2}\varphi_0^2\right) g_{\mu\nu}\right). \tag{5.41}$$

By using $f_{\mu\nu}$ and the third equation in (5.39), other auxiliary field can be found. To solve this equation we need $\psi_2 = \psi_{22} = 0$ or $\psi_{12} = \psi_{22} = 0$. The Bianchi identity is satisfied if $\gamma_{11} = 0$ or $\psi_2 + \psi_{12} + 2\psi_{22} = 0$. The second condition implies that

$\psi_2 = \psi_{12} = \psi_{22} = 0$. Then the third equation in (5.39) does not give a solution for $h = h(e)$ since all coefficients of h is zero. Thus, to satisfy the Bianchi identity γ_{11} is set to zero. To solve the equation in terms of $h = h(e)$, two cases will be analysed separately $(i)\psi_{12} = \psi_{22} = 0$ and $(ii)\psi_2 = \psi_{22} = 0$. For the second case there are no consistent relations between coefficients. Thus only the first case will be considered. There are two different Lagrangians

$$\begin{aligned} L_4 =& \alpha_1 e \wedge De + \alpha_2 e \wedge R + \alpha_3 h \wedge De + \alpha_4 f \wedge De + \alpha_5 f \wedge R + \alpha_7 h \wedge R \\ &+ \alpha_0 \left(d\omega + \frac{1}{3}\omega \wedge \omega \wedge \omega \right) \\ &+ \frac{1}{6}\beta_3 e \wedge e \wedge e + \frac{1}{2}\left(\beta_4 e \wedge e \wedge f + \beta_6 e \wedge e \wedge h\right). \end{aligned} \tag{5.42}$$

The coefficients $\alpha_3 = 0$, $\beta_6 = \frac{\alpha_3^2}{\alpha_7}$ and $\alpha_7 = 0$ gives 3 more Lagrangians. The other Lagrangian which give the same field equations is

$$\begin{aligned} L_5 =& \alpha_1 e \wedge De + \alpha_2 e \wedge R + \alpha_4 f \wedge De + \alpha_5 f \wedge R + \alpha_6 f \wedge Df + \alpha_7 h \wedge R \\ &+ \alpha_0 \left(d\omega + \frac{1}{3}\omega \wedge \omega \wedge \omega \right) + \frac{1}{6}\left(\beta_3 e \wedge e \wedge e + \frac{2\beta_5\alpha_6}{\alpha_4} f \wedge f \wedge f \right) \\ &+ \frac{\beta_5}{2}\left(\frac{\alpha_4}{2\alpha_6} e \wedge e \wedge f + e \wedge f \wedge f \right). \end{aligned} \tag{5.43}$$

By setting $\alpha_6 = \frac{\alpha_4^2}{4\alpha_1}$ or $\alpha_6 = \frac{\beta_5\alpha_5}{2\alpha_4}$, two different Lagrangians can be obtained. The field equation in terms of the transformed spin connection are

$$\begin{aligned} & d_\Omega e = 0, \\ & R(\Omega) = \left(\chi_0 - \frac{1}{2}\varphi_0^2\right) e \wedge e + \chi_1 e \wedge f, \\ & d_\Omega f = \psi_0 e \wedge e + (\psi_1 - \varphi_0) e \wedge f + \psi_2 e \wedge h + \psi_{11} f \wedge f, \\ & d_\Omega h = \gamma_0 e \wedge e + \gamma_1 e \wedge f - \varphi_0 e \wedge h. \end{aligned} \tag{5.44}$$

The first equation implies that $\Omega = \Omega(e)$. The solution of the first one is

$$f_{\mu\nu} = \frac{1}{\chi_1}\left(S_{\mu\nu} + \frac{1}{2}\left(\chi_0 - \frac{1}{2}\varphi_0^2 \right) g_{\mu\nu} \right). \tag{5.45}$$

By using the covariant derivative of auxiliary field f with respect to the transformed spin connection, second auxiliary field can be found

$$
\begin{aligned}
\psi_2 h_{\mu\nu} =& \frac{1}{\chi_1} C_{\mu\nu} + \frac{1}{4} g_{\mu\nu} \left(\psi_0 - \frac{1}{2\chi_1^2} \left[\frac{R^2}{4} + 3R \left(\chi_0 - \frac{\varphi_0^2}{2} \right) + 9 \left(\chi_0 - \frac{\varphi_0^2}{2} \right)^2 \right] \right) \\
&+ \frac{1}{\chi_1} \left(S_{\mu\nu} + \frac{1}{2} \left(\chi_0 - \frac{1}{2} \varphi_0^2 \right) g_{\mu\nu} \right) \left(\psi_1 - \varphi_0 - \frac{\psi_{11}}{4\chi_1} \left(R + 6 \big(\chi_0 - \frac{\varphi_0^2}{2} \big) \right) \right) \\
&+ \frac{\psi_{11}}{2\chi_1^2} \left(S_\mu^\tau S_{\tau\nu} + \left(1 + \frac{1}{4} \big(\chi_0 - \frac{\varphi_0^2}{2} \big) \right) \left(\chi_0 - \frac{\varphi_0^2}{2} \right) S_{\mu\nu} \right).
\end{aligned}
\tag{5.46}
$$

5.3 Hamiltonian analysis

For the Lagrangian (5.13), the coefficients are

$$
\begin{aligned}
&g_{ee} = 2\alpha_1, \quad g_{e\omega} = \alpha_2, \quad g_{ef} = \alpha_4, \quad g_{eh} = \alpha_3, \quad g_{\omega f} = \alpha_5, \quad g_{\omega\omega} = 2\alpha_0, \\
&f_{eee} = \beta_3, \quad f_{ee\omega} = 2\alpha_1, \quad f_{e\omega\omega} = \alpha_2, \quad f_{e\omega f} = \alpha_4, \quad f_{f\omega\omega} = \alpha_5, \\
&f_{e\omega h} = \alpha_3, \quad f_{\omega\omega\omega} = 2\alpha_0, \quad f_{eef} = \beta_4, \quad f_{eff} = \beta_5.
\end{aligned}
\tag{5.47}
$$

Then the Poisson bracket submatrix $\mathcal{P}_{RS}^{ab}$ becomes

$$
\begin{pmatrix}
\alpha_1 V^{\omega\omega} + \frac{\beta_5 \left(2\alpha_0 \beta_5 V^{ff} - 2\alpha_5 \alpha_6 V^{ff} - \alpha_9 \alpha_6 (V^{fh} + V^{hf}) \right)}{\alpha_6^2} & -\alpha_1 V^{\omega e} & \frac{\beta_5 \left(-2\alpha_0 \beta_5 V^{fe} + 2\alpha_5 \alpha_6 V^{fe} + \alpha_9 \alpha_6 V^{he} \right)}{\alpha_6^2} & \frac{\alpha_9 \beta_5 V^{fe}}{\alpha_6} \\
-\alpha_1 V^{e\omega} & \alpha_1 V^{ee} & 0 & 0 \\
\frac{\beta_5 \left(-2\alpha_0 \beta_5 V^{ef} + 2\alpha_5 \alpha_6 V^{ef} + \alpha_9 \alpha_6 V^{eh} \right)}{\alpha_6^2} & 0 & \frac{2\beta_5 V^{ee} (\alpha_0 \beta_5 - \alpha_5 \alpha_6)}{\alpha_6^2} & -\frac{\alpha_9 \beta_5 V^{ee}}{\alpha_6} \\
\frac{\alpha_9 \beta_5 V^{ef}}{\alpha_6} & 0 & -\frac{\alpha_9 \beta_5 V^{ee}}{\alpha_6} & 0
\end{pmatrix}.
\tag{5.48}
$$

If the coefficient α_1 vanishes, then the rank of the matrix becomes 4 as in GMG. Thus there are two propagated massive spin-2 modes. For the second Lagrangian (5.25) the coefficients are given in (5.47) with addition

$$
g_{ff} = \alpha_6, \quad f_{ff\omega} = \alpha_6, \quad f_{fff} = \frac{4\alpha_6^2}{\alpha_5}.
\tag{5.49}
$$

Then the Poisson bracket submatrix $\mathcal{P}_{RS}^{ab}$ becomes

$$
\mathcal{P}_{RS}^{ab} = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{pmatrix},
\tag{5.50}
$$

where the matrix elements a_{mn} are given in appendix C. This is a 12×12 matrix. The rank of the Poisson bracket submatrix is greater than 4. This means that apart from massive spin-2 states, there are also ghost states. Thus, the model given with the Lagrangian (5.25) is not physical.

The coefficients for the Lagrangian (5.42) are the terms in (5.47) with

$$g_{\omega h} = \alpha_7 \qquad f_{eeh} = \beta_6, \qquad f_{h\omega\omega} = \alpha_7, \qquad f_{eff} = 0. \tag{5.51}$$

Then the Poisson bracket submatrix becomes

$$\mathcal{P}^{ab}_{RS} = \alpha_1 \begin{pmatrix} V^{\omega\omega} & -V^{\omega e} & 0 & 0 \\ -V^{e\omega} & V^{ee} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.52}$$

Since the rank of the Poisson bracket submatrix is 2, there is only 1 propagating massive spin-2 modes for the model (5.42). However, we expected two propagating masive spin-2 modes since we added 2 auxiliary fields to the model.

Finally the coefficients of Lagrangian (5.43) are given by (5.47) with

$$\begin{aligned} &g_{ff} = \alpha, \quad g_{\omega h} = \alpha_7, \quad g_{eh} = 0, \\ &f_{f\omega f} = \alpha_6, \quad f_{h\omega\omega} = \alpha_7, \quad f_{fff} = \frac{\alpha_6 \beta_5}{3\alpha_4} \qquad f_{e\omega h} = 0. \end{aligned} \tag{5.53}$$

The Poisson bracket submatrix becomes

$$\mathcal{P}^{ab}_{RS} = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}, \tag{5.54}$$

where the matrix elements b_{mn} are given in appendix C. The rank of Poisson bracket submatrix is greater than 4. This means that apart from massive spin-2 states, there are also ghost states. Thus, the model given with the Lagrangian (5.43) is not physical.

When we add auxiliary fields to model, we have found five Lagrangians invariant under LLT which satisfy Bianchi identity given in (5.13), (5.25), (5.35), (5.42), and (5.43). One of the auxiliary fields is a linear combination of Schouten tensor and the metric. The other one is proportional with the Cotton tensor for both Lagrangians. The metric equations are also found. To determine if the models are physical, the

Hamiltonian analysis was done. The degrees of freedom for the model (5.13) was found 4 which means that there are 2 massive spin-2 modes. This model includes NMG and GMG, thus this is an expected result. For model (5.25), the rank of the submatrix of Poisson bracket is 2. Thus, this model has massive spin-2 modes more than two. It includes E(G)MG models, and Hamiltonian analysis showed that there is no new model from this family which does not give ghost-mode. The field equations for the Lagrangian (5.35) are nonlinear in auxiliary fields thus the auxiliary fields cannot be determined. When the degrees of freedom were counted for the models with torsion whose Lagrangians are in (5.42) and (5.43), we have found that there is no massive spin-2 modes for (5.42) but there are two spin-2 modes with ghost-modes for (5.43).

CHAPTER 6

CONCLUSION

The purpose of this thesis was to find Chern-Simons-like models for $N = 2, 3, 4$ cases. To do this, Bianchi identity in the first-order formalism and Hamiltonian analysis are used. In Chapter 2 we gave a review on noncoordinate basis, introduce dreibein and associated spin connection, torsion and curvature one-forms. 3D is one of the special dimensions, since Einstein-Hilbert action can be seen as a gauge theory in this dimension if the dreibein is invertible. For the model with $N = 2$ i.e. the action is written in terms of the dreibein e and the spin connection ω, the algebra of the gauge theory was studied. Since the degrees of freedom is zero in 3D one way to obtain a non-trivial theory is to consider massive gravity. A massive theory can be obtained by adding an explicit mass term or the higher order derivative terms of the metric. To do this, the auxiliary fields that are Lorentz-vector valued one-forms are added to the Lagrangian and Chern-Simons-like model is obtained. This allows us to write a generic Lagrangian for N Lorentz-vector valued one-forms, two of them are the dreibein and the spin connection. These Lorentz vectors are local Lorentz vectors. This means that they transform under LLT as vector. To find the auxiliary fields as functions of metric and its derivatives, the field equations are ordered from the lowest degree field i.e. dreibein to the auxiliary field. If the auxiliary fields arise linearly as unknowns then they can be solved until the last equation. The last equation gives the metric equation. Thus, the auxiliary fields and the metric equation can be obtained if the field equations are linear in auxiliary fields. The Lagrangian should satisfy the Bianchi identity. The Bianchi identity for an arbitrary number of auxiliary field was studied. If the Bianchi identity is satisfied without using the last field equation or its covariant derivative, the model is off-shell, otherwise it is on-shell. Off-shell and on-shell cases were studied separately for generic N. Then the constraint equations were found to use in the specific $N = 2, 3, 4$ cases. The constraint equations imply that the auxiliary fields are symmetric tensors. The solutions of the field equations gave Schouten tensor, Cotton tensor and their derivatives depends on the number of the auxiliary fields. After writing the Lagrangian, we needed to show if it is physical by using Hamiltonian analysis. Hamiltonian analysis allows us to determine the number of degrees of freedom without using linearization of the field equations. The linearized approximation may give misleading results when degrees of freedom is counted. Even though the model has no ghost mode in linearized approximation, the Hamiltonian analysis may show that there is ghost-mode. The primary and secondary constraints are calculated for a generic Lagrangian and they were classified as first-class and second-class. Their Poisson brackets were calculated. There were two assumptions. The first one is that the Poisson brackets of secondary constraints with other secondary constraints vanish on the full constraint surface. Therefore, the non-vanishing components of the total Poisson bracket matrix are the Poisson brackets of primary constraints with primary and secondary constraints. For the model with N fields, the rank of this total matrix is $3N + M$, where M is the number of the secondary constraints. It was also assumed that the submatrix calculated by using the Poisson brackets of primary constraints with primary constraints are linearly independent of the column matrix which is the Poisson bracket of primary and secondary constraints. This assumption implies that all of the secondary constraints and their linear combinations are second-class. These assumptions hold for the known models.

In Chapter 3 we started to do the analysis given in Chapter 2 for $N = 2$. The model includes only the dreibein e and the spin-connection ω. By using these fields, LLT invariant most generic Lagrangian was written. The Bianchi identities were satisfied without giving any constraint equations. The Hamiltonian analysis showed that the Poisson bracket submatrix is trivial and as expected there are no local degrees of freedom.

In Chapter 4 we wrote the LLT invariant generic Lagrangian by adding only one auxiliary field to the model. After finding field equations, two separate cases were examined: Torsion vanishes or not. The model may be on-shell or off-shell independent of the existence of torsion. The torsionless case gave only an off-shell model, it gave the TMG Lagrangian with addition of the term eDe. The coefficient of this term appeared in the solution of the auxiliary field and in the metric equation. There is no on-shell torsionless model. For the models with torsion, two families were found. The first one was off-shell and it was obtained by adding eDe, fR and eef terms to TMG. Lastly the third Lagrangian was obtained by adding eDe and eef terms to MMG and this model was on-shell. The Hamiltonian analysis for $N = 3$ case showed that for all these three Lagrangians the local degrees of freedom is 2, this means that there are 2 massive spin-2 modes for all these models.

In Chapter 5 one more auxiliary field was added to the Lagrangian. The Bianchi identity was used to find the Lagrangians. The first Lagrangian is GMG with the addition of the terms eDe, fDe and eef. It gave torsionless off-shell model. The Hamiltonian analysis showed that if the term eDe vanishes in the Lagrangian (5.13), then two propagating massive spin-2 modes are obtained as in NMG and GMG. The second Lagrangian was obtained by adding eDe, eR, fDe, eff and fff terms to the EGMG Lagrangian. The rank of the sub-matrix of Poisson bracket is two. This means that there is no model with massive spin-2 modes obtained from the Lagrangian (5.25) except E(G)MG. For the models with torsion, the Lagrangians (5.42) and (5.43) are found by using Bianchi identity. The Hamiltonian analysis for (5.42) showed that there is no massive spin-2 modes whereas for (5.43), there are ghost-modes. When we try to find the conditions which makes (5.43) physical, the model reduces to the known models. Thus, the only physical models are given in (5.13) as a family of NMG and GMG and the models known as EMG and EGMG.

For the future work, the holographic dual CFT of the Lagrangians can be studied. It can be asked if CFT for each Lagrangian is unitary. To answer this question, the central charge of the holographic dual CFT on the boundary of AdS_3 vacuum of the models studied in the thesis should be calculated. Also supersymmetric extensions of the models can be studied. Even if the metric equations of each models were found in this thesis, their solutions were not studied. This can be another research topic. We have studied Chern-Simons-like models for $D = 3$ case. It can be wondered that if the algebra can be generalised for $D = 4$.

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Appendix A

COEFFICIENTS IN FIELD EQUATIONS FOR $N = 3$ CASE

The coefficients in the generic Lagrangian (4.6) are

$$
\begin{aligned}
\varphi_0 &= \frac{1}{4m}\left(\beta_3\alpha_{k+1}^2 - \alpha_1\alpha_{k+1}\beta_{k+3} + 2\alpha_0\left(\alpha_{k+2}\beta_{k+3} - 2\beta_3\alpha_{2k+2}\right)\right. \\
&\qquad \left. + \alpha_2\left(4\alpha_1\alpha_{2k+2} - 2\alpha_{k+1}\alpha_{k+2}\right)\right), \\
\varphi_1 &= \frac{1}{2m}\Big(\alpha_{k+1}^2\beta_{k+3} - \alpha_{k+1}\left(\alpha_1\beta_{2k+3} + \alpha_{k+2}^2\right) + 2\alpha_0\left(\alpha_{k+2}\beta_{2k+3} - 2\alpha_{2k+2}\beta_{k+3}\right) \\
&\qquad + 2\alpha_1\alpha_{k+2}\alpha_{2k+2}\Big), \\
\varphi_{11} &= \frac{1}{4m}\Big(\alpha_{k+1}^2\beta_{2k+3} - \alpha_{k+1}\left(\alpha_1\beta_{3k+3} + 2\alpha_{k+2}\alpha_{2k+2}\right) + 4\alpha_1\alpha_{2k+2}^2 \\
&\qquad + 2\alpha_0\left(\alpha_{k+2}\beta_{3k+3} - 2\alpha_{2k+2}\beta_{2k+3}\right)\Big), \\
\chi_0 &= \frac{1}{4m}\Big(2\alpha_2\left(\alpha_{k+1}\beta_{k+3} + \alpha_{k+2}^2\right) - \beta_3\alpha_{k+1}\alpha_{k+2} - \alpha_1\left(\alpha_{k+2}\beta_{k+3} - 2\beta_3\alpha_{2k+2}\right) \\
&\qquad - 8\alpha_2^2\alpha_{2k+2}\Big), \\
\chi_1 &= \frac{1}{2m}\Big(- \alpha_{k+2}\left(\alpha_{k+1}\beta_{k+3} + \alpha_1\beta_{2k+3} + 4\alpha_2\alpha_{2k+2}\right) + \alpha_{k+2}^3 \\
&\qquad + 2\left(\alpha_1\alpha_{2k+2}\beta_{k+3} + \alpha_2\alpha_{k+1}\beta_{2k+3}\right)\Big), \\
\chi_{11} &= \frac{1}{4m}\Big(- \alpha_{k+2}\left(\alpha_{k+1}\beta_{2k+3} + \alpha_1\beta_{3k+3}\right) + 2\alpha_1\alpha_{2k+2}\beta_{2k+3} + 2\alpha_{2k+2}\alpha_{k+2}^2 \\
&\qquad + 2\alpha_2\left(\alpha_{k+1}\beta_{3k+3} - 4\alpha_{2k+2}^2\right)\Big), \\
\psi_0 &= \frac{1}{4m}\Big(\alpha_1^2\beta_{k+3} - \alpha_1\beta_3\alpha_{k+1} + 2\alpha_0\beta_3\alpha_{k+2} - 2\alpha_2\left(2\alpha_0\beta_{k+3} + \alpha_1\alpha_{k+2}\right) + 4\alpha_2^2\alpha_{k+1}\Big) \\
\psi_1 &= \frac{1}{2m}\Big(- \alpha_1\left(\alpha_{k+1}\beta_{k+3} + \alpha_{k+2}^2\right) + 2\alpha_0\alpha_{k+2}\beta_{k+3} + 2\alpha_2\left(\alpha_{k+1}\alpha_{k+2} - 2\alpha_0\beta_{2k+3}\right), \\
&\qquad \alpha_1^2\beta_{2k+3}\Big), \\
\psi_{11} &= \frac{1}{4m}\Big(\alpha_1^2\beta_{3k+3} - \alpha_1\left(\alpha_{k+1}\beta_{2k+3} + 2\alpha_{k+2}\alpha_{2k+2}\right) + 2\alpha_0\alpha_{k+2}\beta_{2k+3} \\
&\qquad + 4\alpha_2\left(\alpha_{k+1}\alpha_{2k+2} - \alpha_0\beta_{3k+3}\right)\Big),
\end{aligned}
\tag{A.1}
$$

with $m := \alpha_1^2\alpha_{2k+2} - \alpha_1\alpha_{k+1}\alpha_{k+2} + \alpha_0\alpha_{k+2}^2 + \alpha_2\left(\alpha_{k+1}^2 - 4\alpha_0\alpha_{2k+2}\right)$.

Appendix B

COEFFICIENTS IN FIELD EQUATIONS FOR $N = 4$ CASE

The coefficients in the generic Lagrangian (5.4) are

$$\begin{aligned}
\varphi_0 = \frac{1}{\det A}\Big(&- 2\alpha_9\alpha_5^2\beta_3 + \alpha_3\alpha_5^2\beta_6 + 2\alpha_7\alpha_8\alpha_5\beta_3 - \alpha_3\alpha_7\alpha_5\beta_4 + 2\alpha_2\alpha_9\alpha_5\beta_4 \\
&- \alpha_4\alpha_7\alpha_5\beta_6 - \alpha_2\alpha_8\alpha_5\beta_6 - 2\alpha_0\alpha_8^2\beta_3 + \alpha_4\alpha_7^2\beta_4 + 2\alpha_0\alpha_3\alpha_8\beta_4 - \alpha_2\alpha_7\alpha_8\beta_4 \\
&- 4\alpha_0\alpha_4\alpha_9\beta_4 + 2\alpha_0\alpha_4\alpha_8\beta_6 + \alpha_6\left(-2\alpha_7^2\beta_3 + 2\alpha_2\alpha_7\beta_6 + 4\alpha_0\left(2\alpha_9\beta_3 - \alpha_3\beta_6\right)\right) \\
&+ 2\alpha_1\left(\alpha_3\left(2\alpha_6\alpha_7 - \alpha_5\alpha_8\right) + \alpha_4\left(2\alpha_5\alpha_9 - \alpha_7\alpha_8\right) + \alpha_2\left(\alpha_8^2 - 4\alpha_6\alpha_9\right)\right)\Big), \\
\varphi_1 = \frac{1}{\det A}\Big(&\alpha_4\left(2\alpha_7^2\beta_5 - \alpha_5\alpha_7\beta_8 + 2\alpha_0\left(\alpha_8\beta_8 - 4\alpha_9\beta_5\right) + 2\alpha_2\left(\alpha_8^2 - 4\alpha_6\alpha_9\right)\right) \\
&- 4\alpha_6\alpha_7^2\beta_4 - 4\alpha_0\alpha_8^2\beta_4 + 4\alpha_5\alpha_7\alpha_8\beta_4 - 4\alpha_5^2\alpha_9\beta_4 + 16\alpha_0\alpha_6\alpha_9\beta_4 - 2\alpha_2\alpha_7\alpha_8\beta_5 \\
&+ 4\alpha_2\alpha_5\alpha_9\beta_5 + 2\alpha_2\alpha_6\alpha_7\beta_8 - \alpha_2\alpha_5\alpha_8\beta_8 + \alpha_3(\alpha_5^2\beta_8 - 2\alpha_7\alpha_5\beta_5 \\
&+ 4\alpha_0\left(\alpha_8\beta_5 - \alpha_6\beta_8\right) + \alpha_4\left(4\alpha_6\alpha_7 - 2\alpha_5\alpha_8\right)) + \left(4\alpha_5\alpha_9 - 2\alpha_7\alpha_8\right)\alpha_4^2\Big),
\end{aligned} \tag{B.1}$$

$$\begin{aligned}
\varphi_2 = \frac{1}{\det A}\Big(&\alpha_3(2\alpha_5^2\beta_7 - \alpha_7\alpha_5\beta_8 - 8\alpha_0\alpha_6\beta_7 + 2\alpha_0\alpha_8\beta_8 + \alpha_4\left(4\alpha_5\alpha_9 - 2\alpha_7\alpha_8\right) \\
&+ 2\alpha_2\left(\alpha_8^2 - 4\alpha_6\alpha_9\right)) - 4\alpha_0\alpha_8^2\beta_6 + 4\alpha_5\alpha_7\alpha_8\beta_6 - 4\alpha_5^2\alpha_9\beta_6 - 2\alpha_4\alpha_5\alpha_7\beta_7 \\
&+ 4\alpha_0\alpha_4\alpha_8\beta_7 - 2\alpha_2\alpha_5\alpha_8\beta_7 + 4\alpha_6\left(\alpha_7^2\left(-\beta_6\right) + \alpha_2\alpha_7\beta_7 + 4\alpha_0\alpha_9\beta_6\right) \\
&+ \alpha_4\alpha_7^2\beta_8 - \alpha_2\alpha_7\alpha_8\beta_8 - 4\alpha_0\alpha_4\alpha_9\beta_8 + 2\alpha_2\alpha_5\alpha_9\beta_8 + \left(4\alpha_6\alpha_7 - 2\alpha_5\alpha_8\right)\alpha_3^2\Big), \\
\varphi_{11} = \frac{1}{\det A}\Big(&- 2\alpha_9\alpha_5^2\beta_5 + 2\alpha_2\alpha_9\alpha_5\beta_2 + 2\alpha_7\alpha_8\alpha_5\beta_5 - \alpha_2\alpha_8\alpha_5\beta_9 - \alpha_2\alpha_7\alpha_8\beta_2 \\
&- 2\alpha_6\alpha_7^2\beta_5 - 2\alpha_0\alpha_8^2\beta_5 + 8\alpha_0\alpha_6\alpha_9\beta_5 + 2\alpha_2\alpha_6\alpha_7\beta_9 - 8\alpha_2\alpha_6^2\alpha_9 + 2\alpha_2\alpha_6\alpha_8^2 \\
&+ \alpha_3\left(\alpha_5^2\beta_9 - \alpha_7\alpha_5\beta_2 + 2\alpha_0\alpha_8\beta_2 - 2\alpha_6\left(2\alpha_0\beta_9 + \alpha_5\alpha_8\right) + 4\alpha_6^2\alpha_7\right) \\
&+ \alpha_4\left(\alpha_7^2\beta_2 - \alpha_5\alpha_7\beta_9 + 2\alpha_0\left(\alpha_8\beta_9 - 2\alpha_9\beta_2\right) + \alpha_6\left(4\alpha_5\alpha_9 - 2\alpha_7\alpha_8\right)\right)\Big),
\end{aligned} \tag{B.2}$$

$$\varphi_{12} = -\frac{2}{\det A}\Big(\alpha_0\alpha_8^2\beta_8 - \alpha_5\alpha_7\alpha_8\beta_8 + \alpha_2\alpha_7\alpha_8\beta_9 + \alpha_2\alpha_5\alpha_8\beta_{10} + \alpha_6\alpha_7^2\beta_8$$
$$+ \alpha_5^2\alpha_9\beta_8 - 4\alpha_0\alpha_6\alpha_9\beta_8 - 2\alpha_2\alpha_5\alpha_9\beta_9 - 2\alpha_2\alpha_6\alpha_7\beta_{10} + \alpha_3(\alpha_5^2\,(-\beta_{10})$$
$$+ \alpha_5(\alpha_7\beta_9 + \alpha_8^2) - 2\alpha_0\alpha_8\beta_9 + \alpha_6(4\alpha_0\beta_{10} - 2\alpha_7\alpha_8)) + \alpha_4(\alpha_7^2\,(-\beta_9)$$
$$+ \alpha_7\left(\alpha_5\beta_{10} + \alpha_8^2\right) - 2(\alpha_0\left(\alpha_8\beta_{10} - 2\alpha_9\beta_9\right) + \alpha_5\alpha_8\alpha_9)) - \alpha_2\alpha_8^3 + 4\alpha_2\alpha_6\alpha_9\alpha_8\Big),$$
$$\varphi_{22} = \frac{1}{\det A}\Big(- 2\alpha_9\alpha_5^2\beta_7 - \alpha_2\alpha_8\alpha_5\beta_1 + 2\alpha_7\alpha_8\alpha_5\beta_7 + 2\alpha_2\alpha_9\alpha_5\beta_{10} + 2\alpha_2\alpha_6\alpha_7\beta_1$$
$$- 2\alpha_6\alpha_7^2\beta_7 - 2\alpha_0\alpha_8^2\beta_7 + 8\alpha_0\alpha_6\alpha_9\beta_7 - \alpha_2\alpha_7\alpha_8\beta_{10} + \alpha_3(\alpha_5^2\beta_1 - \alpha_5\left(\alpha_7\beta_{10} + 2\alpha_8\alpha_9\right)$$
$$+ 4\alpha_6\left(\alpha_7\alpha_9 - \alpha_0\beta_1\right) + 2\alpha_0\alpha_8\beta_{10}) + \alpha_4(\alpha_7^2\beta_{10} - \alpha_7\left(\alpha_5\beta_1 + 2\alpha_8\alpha_9\right)$$
$$+ 2\alpha_0(\alpha_8\beta_1 - 2\alpha_9\beta_{10}) + 4\alpha_5\alpha_9^2) - 8\alpha_2\alpha_6\alpha_9^2 + 2\alpha_2\alpha_8^2\alpha_9\Big),$$
(B.3)

$$\chi_0 = \frac{1}{\det A}\Big(\alpha_1\left(2\alpha_7\alpha_8\beta_4 - 4\alpha_5\alpha_9\beta_4 - 4\alpha_6\alpha_7\beta_6 + 2\alpha_5\alpha_8\beta_6 - 4\alpha_6\alpha_3^2 + 4\alpha_4\alpha_8\alpha_3 - 4\alpha_4^2\alpha_9\right)$$
$$+ \alpha_2\alpha_8^2\beta_3 - \alpha_4\alpha_7\alpha_8\beta_3 + 2\alpha_4\alpha_5\alpha_9\beta_3 - 4\alpha_2\alpha_6\alpha_9\beta_3 + \alpha_3^2\alpha_5\beta_4 + 2\alpha_2\alpha_4\alpha_9\beta_4$$
$$+ \alpha_4^2\alpha_7\beta_6 - \alpha_2\alpha_4\alpha_8\beta_6 - \alpha_3\left(\left(\alpha_4\alpha_7 + \alpha_2\alpha_8\right)\beta_4 - 2\alpha_6(\alpha_7\beta_3 + \alpha_2\beta_6)\right.$$
$$\left.+ \alpha_5\left(\alpha_8\beta_3 + \alpha_4\beta_6\right)\right) - 4\left(\alpha_8^2 - 4\alpha_6\alpha_9\right)\alpha_1^2\Big),$$
$$\chi_1 = \frac{1}{\det A}\Big(\alpha_7\alpha_4^2\beta_8 - 2\alpha_7\alpha_8\alpha_4\beta_4 + 4\alpha_5\alpha_9\alpha_4\beta_4 + 4\alpha_2\alpha_9\alpha_4\beta_5 - \alpha_2\alpha_8\alpha_4\beta_8$$
$$+ 2\alpha_2\alpha_8^2\beta_4 - 8\alpha_2\alpha_6\alpha_9\beta_4 + \alpha_3^2\left(2\alpha_5\beta_5 - 4\alpha_4\alpha_6\right) + \alpha_3(-\alpha_4(2\alpha_7\beta_5 + \alpha_5\beta_8)$$
$$- 2\alpha_8(\alpha_5\beta_4 + \alpha_2\beta_5) + 2\alpha_6(2\alpha_7\beta_4 + \alpha_2\beta_8) + 4\alpha_8\alpha_4^2) + 2\alpha_1(2\alpha_7(\alpha_8\beta_5 - \alpha_6\beta_8)$$
$$+ \alpha_5\left(\alpha_8\beta_8 - 4\alpha_9\beta_5\right) - 2\alpha_4\left(\alpha_8^2 - 4\alpha_6\alpha_9\right)) - 4\alpha_9\alpha_4^3\Big),$$
(B.4)

$$
\begin{aligned}
\chi_2 = \frac{1}{\det A}\Big(& \alpha_3^2(\alpha_5\beta_8 + 4\alpha_4\alpha_8) - \alpha_3(\alpha_4\left(2\alpha_5\beta_7 + \alpha_7\beta_8\right) - 4\alpha_6\alpha_7\beta_6 + 2\alpha_5\alpha_8\beta_6 \\
& - 4\alpha_2\alpha_6\beta_7 + \alpha_2\alpha_8\beta_8 + 4\alpha_9\alpha_4^2 + 4\alpha_1\left(\alpha_8^2 - 4\alpha_6\alpha_9\right)) + \alpha_2\left(\alpha_8^2 - 4\alpha_6\alpha_9\right)\beta_6 \\
& + 2\left(\alpha_7\alpha_4^2\beta_7 + \alpha_4\left(-\alpha_7\alpha_8\beta_6 + 2\alpha_5\alpha_9\beta_6 + \alpha_2(\alpha_9\beta_8 - \alpha_8\beta_7)\right)\right. \\
& \left. + \alpha_1\left(-4\alpha_6\alpha_7\beta_7 + \alpha_7\alpha_8\beta_8 + 2\alpha_5\left(\alpha_8\beta_7 - \alpha_9\beta_8\right)\right)\right) - 4\alpha_6\alpha_3^3\Big), \\
\chi_{11} = \frac{1}{\det A}\Big(& \alpha_3^2(\alpha_5\beta_2 - 4\alpha_6^2) + \alpha_3((2\alpha_6\alpha_7 - \alpha_5\alpha_8)\beta_5 + \alpha_4\left(-\alpha_7\beta_2 - \alpha_5\beta_9 + 4\alpha_6\alpha_8\right) \\
& + \alpha_2\left(2\alpha_6\beta_9 - \alpha_8\beta_2\right)) + 2\alpha_2\alpha_4\alpha_9\beta_2 + \alpha_2\alpha_8^2\beta_5 - \alpha_4\alpha_7\alpha_8\beta_5 + 2\alpha_4\alpha_5\alpha_9\beta_5 \\
& - 4\alpha_2\alpha_6\alpha_9\beta_5 + \alpha_4^2\alpha_7\beta_9 - \alpha_2\alpha_4\alpha_8\beta_9 + 2\alpha_1\left(-2\alpha_6(\alpha_7\beta_9 + \alpha_8^2\right) + \alpha_7\alpha_8\beta_2 \\
& + \alpha_5\left(\alpha_8\beta_9 - 2\alpha_9\beta_2\right) + 8\alpha_9\alpha_6^2) - 4\alpha_4^2\alpha_6\alpha_9\Big),
\end{aligned}
\tag{B.5}
$$

$$
\begin{aligned}
\chi_{12} = \frac{1}{\det A}\Big(& \alpha_3^2\left(2\alpha_5\beta_9 - 4\alpha_6\alpha_8\right) + \alpha_3(-\alpha_8\left(\alpha_5\beta_8 + 2\alpha_2\beta_9\right) + 2\alpha_6(\alpha_7\beta_8 + 2\alpha_2\beta_{10}) \\
& - 2\alpha_4\left(\alpha_7\beta_9 + \alpha_5\beta_{10} - 2\alpha_8^2\right)) + \alpha_2\alpha_8^2\beta_8 - \alpha_4\alpha_7\alpha_8\beta_8 + 2\alpha_4\alpha_5\alpha_9\beta_8 \\
& - 4\alpha_2\alpha_6\alpha_9\beta_8 + 4\alpha_2\alpha_4\alpha_9\beta_9 + 2\alpha_4^2\alpha_7\beta_{10} - 2\alpha_2\alpha_4\alpha_8\beta_{10} \\
& - 4\alpha_1\left(-\alpha_8\left(\alpha_7\beta_9 + \alpha_5\beta_{10} + 4\alpha_6\alpha_9\right) + 2\left(\alpha_5\alpha_9\beta_9 + \alpha_6\alpha_7\beta_{10}\right) + \alpha_8^3\right) - 4\alpha_4^2\alpha_8\alpha_9\Big), \\
\chi_{22} = \frac{1}{\det A}\Big(& \alpha_3^2\left(\alpha_5\beta_{10} - 4\alpha_6\alpha_9\right) + \alpha_3((2\alpha_6\alpha_7 - \alpha_5\alpha_8)\beta_7 + \alpha_4^2\alpha_7\beta_1 - \alpha_2\alpha_4\alpha_8\beta_1 \\
& + \alpha_4\left(-\alpha_5\beta_1 - \alpha_7\beta_{10} + 4\alpha_8\alpha_9\right) + \alpha_2\left(2\alpha_6\beta_1 - \alpha_8\beta_{10}\right)) + \alpha_2\alpha_8^2\beta_7 - \alpha_4\alpha_7\alpha_8\beta_7 \\
& + 2\alpha_4\alpha_5\alpha_9\beta_7 - 4\alpha_2\alpha_6\alpha_9\beta_7 + 2\alpha_2\alpha_4\alpha_9\beta_{10} + 2\alpha_1(\alpha_8\left(\alpha_5\beta_1 + \alpha_7\beta_{10}\right) \\
& - 2\left(\alpha_6\left(\alpha_7\beta_1 - 4\alpha_9^2\right) + \alpha_5\alpha_9\beta_{10}\right) - 2\alpha_9\alpha_8^2) - 4\alpha_4^2\alpha_9^2\Big),
\end{aligned}
\tag{B.6}
$$

$$
\begin{aligned}
\psi_0 = \frac{1}{\det A}\Big(&2\alpha_1(-\alpha_7^2\beta_4 + 4\alpha_0\alpha_9\beta_4 + \alpha_5\alpha_7\beta_6 - 2\alpha_0\alpha_8\beta_6 + \alpha_5\alpha_3^2 - (\alpha_4\alpha_7 + \alpha_2\alpha_8)\,\alpha_3 \\
&+ 2\alpha_2\alpha_4\alpha_9) + \alpha_4\alpha_7^2\beta_3 - \alpha_2\alpha_7\alpha_8\beta_3 - 4\alpha_0\alpha_4\alpha_9\beta_3 + 2\alpha_2\alpha_5\alpha_9\beta_3 - 2\alpha_0\alpha_3^2\beta_4 \\
&- 2\alpha_2^2\alpha_9\beta_4 - \alpha_2\alpha_4\alpha_7\beta_6 + \alpha_2^2\alpha_8\beta_6 + \alpha_3\,(2\alpha_2\alpha_7\beta_4 - \alpha_5\,(\alpha_7\beta_3 + \alpha_2\beta_6) \\
&+ 2\alpha_0\,(\alpha_8\beta_3 + \alpha_4\beta_6) \qquad + \qquad 4\,(\alpha_7\alpha_8 - 2\alpha_5\alpha_9)\,\alpha_1^2\Big), \\
\psi_1 = \frac{1}{\det A}\Big(&- 4\alpha_9\alpha_2^2\beta_5 + \alpha_8\alpha_2^2\beta_8 - 2\alpha_7\alpha_8\alpha_2\beta_4 + 4\alpha_5\alpha_9\alpha_2\beta_4 - \alpha_4\alpha_7\alpha_2\beta_8 + 2\alpha_4\alpha_7^2\beta_4 \\
&- 8\alpha_0\alpha_4\alpha_9\beta_4 + 2\alpha_3^2\,(\alpha_4\alpha_5 - 2\alpha_0\beta_5) - \alpha_3(2\alpha_4\,(\alpha_2\alpha_8 - \alpha_0\beta_8) - 4(\alpha_0\alpha_8\beta_4 + \alpha_2\alpha_7\beta_5) \\
&+ \alpha_5(2\alpha_7\beta_4 + \alpha_2\beta_8) + 2\alpha_7\alpha_4^2) + 2\alpha_1(-2\alpha_7^2\beta_5 + \alpha_5\alpha_7\beta_8 + 2\alpha_0(4\alpha_9\beta_5 - \alpha_8\beta_8) \\
&+ 2\alpha_4(\alpha_7\alpha_8 - 2\alpha_5\alpha_9)) + 4\alpha_4^2\alpha_9\alpha_2\Big),
\end{aligned}
\tag{B.7}
$$

$$
\begin{aligned}
\psi_2 = -\frac{2}{\det A}\Big(&\alpha_3^2\,(\alpha_0\beta_8 + \alpha_4\alpha_7 + \alpha_2\alpha_8) + \alpha_3(\alpha_5\alpha_7\beta_6 - 2\alpha_0\alpha_8\beta_6 - 2\alpha_0\alpha_4\beta_7 \\
&+ \alpha_2\,(\alpha_5\beta_7 - \alpha_7\beta_8 - 2\alpha_4\alpha_9) + \alpha_1\,(4\alpha_5\alpha_9 - 2\alpha_7\alpha_8)) + \alpha_2\alpha_7\alpha_8\beta_6 - 2\alpha_2\alpha_5\alpha_9\beta_6 \\
&- 2\alpha_1\alpha_5\alpha_7\beta_7 - \alpha_2^2\alpha_8\beta_7 + 4\alpha_0\alpha_1\alpha_8\beta_7 + \alpha_4(\alpha_7^2\,(-\beta_6) + \alpha_2\alpha_7\beta_7 + 4\alpha_0\alpha_9\beta_6) \\
&+ \alpha_1\alpha_7^2\beta_8 + \alpha_2^2\alpha_9\beta_8 - 4\alpha_0\alpha_1\alpha_9\beta_8 - \alpha_5\alpha_3^3\Big), \\
\psi_{11} = \frac{1}{\det A}\Big(&- 2\alpha_9\alpha_2^2\beta_2 + \alpha_8\alpha_2^2\beta_9 - \alpha_7\alpha_8\alpha_2\beta_5 + 2\alpha_5\alpha_9\alpha_2\beta_5 - \alpha_4\alpha_7\alpha_2\beta_9 \\
&+ 2\alpha_3^2\,(\alpha_5\alpha_6 - \alpha_0\beta_2) + \alpha_4\alpha_7^2\beta_5 - 4\alpha_0\alpha_4\alpha_9\beta_5 - \alpha_3((\alpha_5\alpha_7 - 2\alpha_0\alpha_8)\,\beta_5 \\
&+ 2\alpha_4\,(\alpha_6\alpha_7 - \alpha_0\beta_9) + \alpha_2(-2\alpha_7\beta_2 + \alpha_5\beta_9 + 2\alpha_6\alpha_8)) + 2\alpha_1(-\alpha_7^2\beta_2 \\
&+ \alpha_5\alpha_7\beta_9 + 2\alpha_0\,(2\alpha_9\beta_2 - \alpha_8\beta_9) + 2\alpha_6\,(\alpha_7\alpha_8 - 2\alpha_5\alpha_9)) + 4\alpha_4\alpha_6\alpha_9\alpha_2\Big),
\end{aligned}
\tag{B.8}
$$

$$
\begin{aligned}
\psi_{12} = \frac{1}{\det A}\Big(& -4\alpha_9\alpha_2^2\beta_9 + 2\alpha_8\alpha_2^2\beta_{10} - \alpha_7\alpha_8\alpha_2\beta_8 + 2\alpha_5\alpha_9\alpha_2\beta_8 - 2\alpha_4\alpha_7\alpha_2\beta_{10} \\
& + \alpha_4\alpha_7^2\beta_8 - 4\alpha_0\alpha_4\alpha_9\beta_8 + 2\alpha_3^2\left(\alpha_5\alpha_8 - 2\alpha_0\beta_9\right) - \alpha_3\left(\left(\alpha_5\alpha_7 - 2\alpha_0\alpha_8\right)\beta_8\right. \\
& + 2\alpha_4\left(\alpha_7\alpha_8 - 2\alpha_0\beta_{10}\right) + 2\alpha_2\left(-2\alpha_7\beta_9 + \alpha_5\beta_{10} + \alpha_8^2\right) \\
& \left. - 4\alpha_1\left(\alpha_7^2\beta_9 - \alpha_7\left(\alpha_5\beta_{10} + \alpha_8^2\right) + 2\alpha_0\left(\alpha_8\beta_{10} - 2\alpha_9\beta_9\right) + 2\alpha_5\alpha_8\alpha_9\right) + 4\alpha_4\alpha_8\alpha_9\alpha_2\Big), \\
\psi_{22} = \frac{1}{\det A}\Big(& \alpha_8\alpha_2^2\beta_1 - 2\alpha_9\alpha_2^2\beta_{10} - \alpha_4\alpha_7\alpha_2\beta_1 - \alpha_7\alpha_8\alpha_2\beta_7 + 2\alpha_5\alpha_9\alpha_2\beta_7 \\
& + \alpha_4\alpha_7^2\beta_7 - 4\alpha_0\alpha_4\alpha_9\beta_7 + 2\alpha_3^2\left(\alpha_5\alpha_9 - \alpha_0\beta_{10}\right) - \alpha_3\left(2\alpha_4\left(\alpha_7\alpha_9 - \alpha_0\beta_1\right)\right. \\
& + \left(\alpha_5\alpha_7 - 2\alpha_0\alpha_8\right)\beta_7 + \alpha_2\left(\alpha_5\beta_1 - 2\alpha_7\beta_{10} + 2\alpha_8\alpha_9\right) \\
& \left. + \alpha_1\left(-2\alpha_7^2\beta_{10} + 2\alpha_7\left(\alpha_5\beta_1 + 2\alpha_8\alpha_9\right) - 4\alpha_0\left(\alpha_8\beta_1 - 2\alpha_9\beta_{10}\right) - 8\alpha_5\alpha_9^2\right) + 4\alpha_4\alpha_9^2\alpha_2\Big),
\end{aligned}
\tag{B.9}
$$

$$
\begin{aligned}
\gamma_0 = \frac{1}{\det A}\Big(& -2\alpha_1(-\alpha_5\alpha_7\beta_4 + 2\alpha_0\alpha_8\beta_4 + \alpha_5^2\beta_6 - 4\alpha_0\alpha_6\beta_6 - \alpha_7\alpha_4^2 \\
& + \alpha_2\alpha_8\alpha_4 + \alpha_3\left(\alpha_4\alpha_5 - 2\alpha_2\alpha_6\right)) - \alpha_4\alpha_5\alpha_7\beta_3 + 2\alpha_2\alpha_6\alpha_7\beta_3 \\
& + 2\alpha_0\alpha_4\alpha_8\beta_3 - \alpha_2\alpha_5\alpha_8\beta_3 - \alpha_2\alpha_4\alpha_7\beta_4 + \alpha_2^2\alpha_8\beta_4 \\
& + \alpha_3\left(\alpha_5^2\beta_3 - \alpha_2\alpha_5\beta_4 + 2\alpha_0\left(\alpha_4\beta_4 - 2\alpha_6\beta_3\right)\right) - 2\alpha_0\alpha_4^2\beta_6 + 2\alpha_2\alpha_4\alpha_5\beta_6 \\
& - 2\alpha_2^2\alpha_6\beta_6 + \left(4\alpha_5\alpha_8 - 8\alpha_6\alpha_7\right)\alpha_1^2\Big), \\
\gamma_1 = \frac{2}{\det A}\Big(& -\alpha_4^2\left(\alpha_0\beta_8 + \alpha_2\alpha_8\right) + \alpha_4(2\alpha_0\alpha_8\beta_4 - \alpha_2\alpha_7\beta_5 + \alpha_5\left(\alpha_2\beta_8 - \alpha_7\beta_4\right) \\
& + \alpha_1\left(2\alpha_5\alpha_8 - 4\alpha_6\alpha_7\right)) + 2\alpha_2\alpha_6\alpha_7\beta_4 - \alpha_2\alpha_5\alpha_8\beta_4 + 2\alpha_1\alpha_5\alpha_7\beta_5 + \alpha_2^2\alpha_8\beta_5 \\
& - 4\alpha_0\alpha_1\alpha_8\beta_5 + \alpha_3\left(2\alpha_4\left(\alpha_0\beta_5 + \alpha_2\alpha_6\right) + \alpha_5^2\beta_4 - 4\alpha_0\alpha_6\beta_4 - \alpha_2\alpha_5\beta_5 - \alpha_5\alpha_4^2\right) \\
& - \alpha_1\alpha_5^2\beta_8 - \alpha_2^2\alpha_6\beta_8 + 4\alpha_0\alpha_1\alpha_6\beta_8 + \alpha_7\alpha_4^3\Big),
\end{aligned}
\tag{B.10}
$$

$$\gamma_2 = \frac{1}{\det A}\Big(- 4\alpha_6\alpha_2^2\beta_7 + \alpha_8\alpha_2^2\beta_8 + 4\alpha_6\alpha_7\alpha_2\beta_6 - 2\alpha_5\alpha_8\alpha_2\beta_6 - 4\alpha_0\alpha_4^2\beta_7$$
$$- 4\alpha_1\alpha_5^2\beta_7 + 16\alpha_0\alpha_1\alpha_6\beta_7 + 2\alpha_1\alpha_5\alpha_7\beta_8 - 4\alpha_0\alpha_1\alpha_8\beta_8$$
$$+ \alpha_4 \left(4\alpha_0\alpha_8\beta_6 + \alpha_5 \left(4\alpha_2\beta_7 - 2\alpha_7\beta_6\right) - \alpha_2\alpha_7\beta_8\right) + \alpha_3 \left(\alpha_4 \left(2\alpha_0\beta_8 - 2\alpha_2\alpha_8\right)\right.$$
$$+ 2\alpha_5^2\beta_6 - 8\alpha_0\alpha_6\beta_6 - \alpha_2\alpha_5\beta_8 + 2\alpha_7\alpha_4^2 + \alpha_1 \left(4\alpha_5\alpha_8 - 8\alpha_6\alpha_7\right)$$
$$\left. + \alpha_3^2 \left(4\alpha_2\alpha_6 - 2\alpha_4\alpha_5\right) \Big),\right.$$
$$\gamma_{11} = \frac{1}{\det A}\Big(\alpha_8\alpha_2^2\beta_2 - 2\alpha_6\alpha_2^2\beta_9 + 2\alpha_6\alpha_7\alpha_2\beta_5 - \alpha_5\alpha_8\alpha_2\beta_5 + 2\alpha_1\alpha_5\alpha_7\beta_2$$
$$- 4\alpha_0\alpha_1\alpha_8\beta_2 + \alpha_3(\alpha_4 \left(2\alpha_0\beta_2 - 2\alpha_5\alpha_6\right) + \alpha_2 \left(4\alpha_6^2 - \alpha_5\beta_2\right)$$
$$+ \left(\alpha_5^2 - 4\alpha_0\alpha_6\right) \beta_5) - 2\alpha_1\alpha_5^2\beta_9 + 8\alpha_0\alpha_1\alpha_6\beta_9 + 2\alpha_4^2 \left(\alpha_6\alpha_7 - \alpha_0\beta_9\right)$$
$$- \alpha_4 \left(\left(\alpha_5\alpha_7 - 2\alpha_0\alpha_8\right) \beta_5 + \alpha_2 \left(\alpha_7\beta_2 - 2\alpha_5\beta_9 + 2\alpha_6\alpha_8\right)\right) - 8\alpha_1\alpha_6^2\alpha_7 + 4\alpha_1\alpha_5\alpha_6\alpha_8\Big), \tag{B.11}$$

$$\gamma_{12} = \frac{1}{\det A}\Big(2\alpha_8\alpha_2^2\beta_9 - 4\alpha_6\alpha_2^2\beta_{10} + 2\alpha_6\alpha_7\alpha_2\beta_8 - \alpha_5\alpha_8\alpha_2\beta_8 + 4\alpha_1\alpha_5\alpha_7\beta_9$$
$$- 8\alpha_0\alpha_1\alpha_8\beta_9 + \alpha_3(\left(\alpha_5^2 - 4\alpha_0\alpha_6\right) \beta_8 + \alpha_4 \left(4\alpha_0\beta_9 - 2\alpha_5\alpha_8\right)$$
$$+ \alpha_2 \left(4\alpha_6\alpha_8 - 2\alpha_5\beta_9\right)) - 4\alpha_1\alpha_5^2\beta_{10} + 16\alpha_0\alpha_1\alpha_6\beta_{10} + 2\alpha_4^2 \left(\alpha_7\alpha_8 - 2\alpha_0\beta_{10}\right)$$
$$- \alpha_4 \left(\left(\alpha_5\alpha_7 - 2\alpha_0\alpha_8\right) \beta_8 + 2\alpha_2 \left(\alpha_7\beta_9 - 2\alpha_5\beta_{10} + \alpha_8^2\right)\right) + 4\alpha_1\alpha_5\alpha_8^2 - 8\alpha_1\alpha_6\alpha_7\alpha_8\Big),$$
$$\gamma_{22} = \frac{1}{\det A}\Big(- 2\alpha_6\alpha_2^2\beta_1 + \alpha_8\alpha_2^2\beta_{10} + 2\alpha_6\alpha_7\alpha_2\beta_7 - \alpha_5\alpha_8\alpha_2\beta_7 - 2\alpha_1\alpha_5^2\beta_1$$
$$+ 8\alpha_0\alpha_1\alpha_6\beta_1 + 2\alpha_4^2 \left(\alpha_7\alpha_9 - \alpha_0\beta_1\right) + 2\alpha_1\alpha_5\alpha_7\beta_{10} - 4\alpha_0\alpha_1\alpha_8\beta_{10}$$
$$+ \alpha_3 \left(\left(\alpha_5^2 - 4\alpha_0\alpha_6\right) \beta_7 + \alpha_4 \left(2\alpha_0\beta_{10} - 2\alpha_5\alpha_9\right) + \alpha_2 \left(4\alpha_6\alpha_9 - \alpha_5\beta_{10}\right)\right)$$
$$- \alpha_4 \left(\left(\alpha_5\alpha_7 - 2\alpha_0\alpha_8\right) \beta_7 + \alpha_2 \left(-2\alpha_5\beta_1 + \alpha_7\beta_{10} + 2\alpha_8\alpha_9\right)\right) - 8\alpha_1\alpha_6\alpha_7\alpha_9 + 4\alpha_1\alpha_5\alpha_8\alpha_9\Big). \tag{B.12}$$

Appendix C

MATRIX ELEMENTS OF $\mathcal{P}^{ab}_{RS}$ FOR L_2 IN $N = 4$ CASE

The matrix elements of the submatrix of Poisson brackets (5.50) for the Lagrangian (5.25) are

$$
\begin{aligned}
a_{11} &= \frac{1}{\alpha_5^4 - 4\alpha_0\alpha_5^2\alpha_6}\big(\alpha_1\alpha_5^2\left(\alpha_5^2 V^{\omega\omega} + 2\alpha_6\left(\alpha_6 V^{ff} - 2\alpha_0 V^{\omega\omega}\right) - \alpha_6\alpha_5(V^{\omega f} + V^{f\omega})\right) \\
&\quad + \alpha_6(\alpha_5\beta_4\left(2\alpha_0\left(\alpha_5(V^{\omega f} + V^{f\omega}) - 4\alpha_6 V^{ff}\right) + \alpha_5^2 V^{ff}\right) - 2\left(\alpha_5^2 - 4\alpha_0\alpha_6\right)\alpha_4^2 V^{ff} \\
&\quad + 2\alpha_3^2\alpha_5^2 V^{hh})\big), \\
a_{12} &= -\alpha_1 V^{\omega e} + \frac{\alpha_6 V^{fe}\left(\alpha_1\alpha_5 - 2\alpha_0\beta_4\right)}{\alpha_5^2 - 4\alpha_0\alpha_6} + \frac{\alpha_4\alpha_6 V^{ff}}{\alpha_5} + \frac{\alpha_3\alpha_5\alpha_6 V^{fh}}{\alpha_5^2 - 4\alpha_0\alpha_6}, \\
a_{13} &= \frac{1}{\alpha_5^4 - 4\alpha_0\alpha_5^2\alpha_6}\big(\alpha_6(\alpha_5(-\beta_4\left(2\alpha_0\left(\alpha_5 V^{\omega e} - 4\alpha_6 V^{fe}\right) + \alpha_5^2 V^{fe}\right) \\
&\quad + \alpha_1\alpha_5\left(\alpha_5 V^{\omega e} - 2\alpha_6 V^{fe}\right) + \alpha_3\alpha_5\left(\alpha_5 V^{\omega h} - 2\alpha_6 V^{fh}\right)) + 2\left(\alpha_5^2 - 4\alpha_0\alpha_6\right)\alpha_4^2 V^{fe} \\
&\quad - \alpha_5\left(\alpha_5^2 - 4\alpha_0\alpha_6\right)\alpha_4 V^{f\omega})\big), \\
a_{14} &= \frac{\alpha_3\alpha_6\left(\alpha_5(V^{\omega f} + V^{f\omega}) - 2\alpha_6 V^{ff} + 2\alpha_3 V^{he}\right)}{4\alpha_0\alpha_6 - \alpha_5^2},
\end{aligned}
\tag{C.1}
$$

$$
\begin{aligned}
a_{21} &= \alpha_1(-V^{e\omega}) + \frac{\alpha_6 V^{ef}\left(\alpha_1\alpha_5 - 2\alpha_0\beta_4\right)}{\alpha_5^2 - 4\alpha_0\alpha_6} + \frac{\alpha_4\alpha_6 V^{ff}}{\alpha_5} + \frac{\alpha_3\alpha_5\alpha_6 V^{hf}}{\alpha_5^2 - 4\alpha_0\alpha_6}, \\
a_{22} &= \alpha_1 V^{ee} + \alpha_6\left(\frac{2\alpha_0\alpha_6}{\alpha_5^2 - 4\alpha_0\alpha_6} + 1\right) V^{ff}, \\
a_{23} &= \frac{}{\alpha_5^3 - 4\alpha_0\alpha_5\alpha_6}\big(\alpha_6(2\alpha_0\alpha_5\beta_4 V^{ee} - \alpha_1\alpha_5^2 V^{ee} - \alpha_3\alpha_5^2(V^{eh} + V^{he}) \\
&\quad - \alpha_4\alpha_5^2 V^{fe} + 4\alpha_0\alpha_4\alpha_6 V^{fe} + \alpha_5^3(-V^{f\omega}) + 2\alpha_0\alpha_6\alpha_5 V^{f\omega})\big), \\
a_{24} &= \frac{\alpha_3\alpha_5\alpha_6 V^{ef}}{\alpha_5^2 - 4\alpha_0\alpha_6},
\end{aligned}
\tag{C.2}
$$

$$
\begin{aligned}
a_{31} &= \frac{1}{\alpha_5^4 - 4\alpha_0\alpha_5^2\alpha_6}\Big(\alpha_6(\alpha_5(-\beta_4\left(2\alpha_0\left(\alpha_5 V^{e\omega} - 4\alpha_6 V^{ef}\right) + \alpha_5^2 V^{ef}\right) \\
&\quad + \alpha_1\alpha_5\left(\alpha_5 V^{e\omega} - 2\alpha_6 V^{ef}\right) + \alpha_3\alpha_5\left(\alpha_5 V^{h\omega} - 2\alpha_6 V^{hf}\right)) + 2\left(\alpha_5^2 - 4\alpha_0\alpha_6\right)\alpha_4^2 V^{ef} \\
&\quad - \alpha_5\left(\alpha_5^2 - 4\alpha_0\alpha_6\right)\alpha_4 V^{\omega f})\Big), \\
a_{32} &= \frac{1}{\alpha_5^3 - 4\alpha_0\alpha_5\alpha_6}\Big(\alpha_6(2\alpha_0\alpha_5\beta_4 V^{ee} - \alpha_1\alpha_5^2 V^{ee} - \alpha_4\alpha_5^2 V^{ef} + 4\alpha_0\alpha_4\alpha_6 V^{ef} \\
&\quad - \alpha_3\alpha_5^2(V^{eh} + V^{he}) + \alpha_5^3(-V^{\omega f}) + 2\alpha_0\alpha_6\alpha_5 V^{\omega f})\Big), \\
a_{33} &= \frac{1}{\alpha_5^4 - 4\alpha_0\alpha_5^2\alpha_6}\Big(\alpha_6(-2\left(\alpha_5^2 - 4\alpha_0\alpha_6\right)\alpha_4^2 V^{ee} \\
&\quad + \alpha_5(\alpha_5^2\beta_4 V^{ee} - 8\alpha_0\alpha_6\beta_4 V^{ee} \\
&\quad + 2\alpha_6\alpha_5\left(\alpha_1 V^{ee} + \alpha_3(V^{eh} + V^{he}) + \alpha_0(-V^{\omega\omega})\right) + \alpha_5^3 V^{\omega\omega}) + \alpha_5\left(\alpha_5^2 - 4\alpha_0\alpha_6\right)\alpha_4(V^{e\omega} + V^{\omega e}))\Big), \\
a_{34} &= \frac{\alpha_3\alpha_6\left(2\alpha_6 V^{ef} - \alpha_5 V^{e\omega}\right)}{4\alpha_0\alpha_6 - \alpha_5^2},
\end{aligned}
\tag{C.3}
$$

$$
\begin{aligned}
a_{41} &= \frac{\alpha_3\alpha_6\left(2\alpha_3 V^{eh} + \alpha_5(V^{\omega f} + V^{f\omega}) - 2\alpha_6 V^{ff}\right)}{4\alpha_0\alpha_6 - \alpha_5^2}, \\
a_{42} &= \frac{\alpha_3\alpha_5\alpha_6 V^{fe}}{\alpha_5^2 - 4\alpha_0\alpha_6}, \\
a_{43} &= \frac{\alpha_3\alpha_6\left(2\alpha_6 V^{fe} - \alpha_5 V^{\omega e}\right)}{4\alpha_0\alpha_6 - \alpha_5^2}, \\
a_{44} &= \frac{2\alpha_3^2\alpha_6 V^{ee}}{\alpha_5^2 - 4\alpha_0\alpha_6}.
\end{aligned}
\tag{C.4}
$$

The matrix elements of the submatrix of Poisson brackets (5.54) for the Lagrangian (5.43) are

$$
\begin{aligned}
b_{11} &= \frac{1}{\alpha_6^4 - 4\alpha_0\alpha_6^2\alpha_7}\Big(\alpha_1\alpha_6^2\left(\alpha_6^2 V^{\omega\omega} + 2\alpha_7\left(\alpha_7 V^{ff} - 2\alpha_0 V^{\omega\omega}\right) - \alpha_7\alpha_6(V^{\omega f} + V^{f\omega})\right) \\
&\quad + \alpha_7(\alpha_6\left(2\alpha_6\left(\alpha_0\beta_4(V^{\omega f} + V^{f\omega}) + \alpha_9^2 V^{hh}\right) + \alpha_6^2\beta_4 V^{ff} - 8\alpha_0\alpha_7\beta_4 V^{ff}\right)\Big) \\
&\quad - 2\alpha_5^2\left(\alpha_6^2 - 4\alpha_0\alpha_7\right)V^{ff})\Big), \\
b_{12} &= \alpha_1(-V^{\omega e}) + \frac{\alpha_7 V^{fe}\left(\alpha_1\alpha_6 - 2\alpha_0\beta_4\right)}{\alpha_6^2 - 4\alpha_0\alpha_7} + \frac{\alpha_5\alpha_7 V^{ff}}{\alpha_6} + \frac{\alpha_6\alpha_7\alpha_9 V^{fh}}{\alpha_6^2 - 4\alpha_0\alpha_7}, \\
b_{13} &= \frac{1}{\alpha_6^4 - 4\alpha_0\alpha_6^2\alpha_7}\Big(\alpha_7(\alpha_6(\alpha_1\alpha_6\left(\alpha_6 V^{\omega e} - 2\alpha_7 V^{fe}\right) - 2\alpha_6\left(\alpha_0\beta_4 V^{\omega e} + \alpha_7\alpha_9 V^{fh}\right) \\
&\quad + \alpha_6^2\left(\alpha_9 V^{\omega h} - \beta_4 V^{fe}\right) + 8\alpha_0\alpha_7\beta_4 V^{fe}) + 2\left(\alpha_6^2 - 4\alpha_0\alpha_7\right)\alpha_5^2 V^{fe} \\
&\quad - \alpha_6\left(\alpha_6^2 - 4\alpha_0\alpha_7\right)\alpha_5 V^{f\omega})\Big), \\
b_{14} &= \frac{\alpha_7\alpha_9\left(\alpha_6(V^{\omega f} + V^{f\omega}) - 2\alpha_7 V^{ff} + 2\alpha_9 V^{he}\right)}{4\alpha_0\alpha_7 - \alpha_6^2},
\end{aligned}
\tag{C.5}
$$

$$
\begin{aligned}
b_{21} &= \alpha_1(-V^{e\omega}) + \frac{\alpha_7 V^{ef}\left(\alpha_1\alpha_6 - 2\alpha_0\beta_4\right)}{\alpha_6^2 - 4\alpha_0\alpha_7} + \frac{\alpha_5\alpha_7 V^{ff}}{\alpha_6} + \frac{\alpha_6\alpha_7\alpha_9 V^{hf}}{\alpha_6^2 - 4\alpha_0\alpha_7}, \\
b_{22} &= \alpha_1 V^{ee} + \alpha_7\left(\frac{2\alpha_0\alpha_7}{\alpha_6^2 - 4\alpha_0\alpha_7} + 1\right)V^{ff}, \\
b_{23} &= \frac{1}{\alpha_6^3 - 4\alpha_0\alpha_6\alpha_7}\Big(\alpha_7(\alpha_1\alpha_6^2(-V^{ee}) - \alpha_6(-2\alpha_0\left(\beta_4 V^{ee} + \alpha_7 V^{f\omega}\right) \\
&\quad + \alpha_9\alpha_6(V^{eh} + V^{he}) + \alpha_6^2 V^{f\omega}) - \alpha_5\left(\alpha_6^2 - 4\alpha_0\alpha_7\right)V^{fe})\Big), \\
b_{24} &= \frac{1}{\alpha_6^4 - 4\alpha_0\alpha_6^2\alpha_7}\Big(\alpha_7(\alpha_6(\alpha_1\alpha_6\left(\alpha_6 V^{e\omega} - 2\alpha_7 V^{ef}\right) - 2\alpha_6\left(\alpha_0\beta_4 V^{e\omega} + \alpha_7\alpha_9 V^{hf}\right) \\
&\quad + 8\alpha_0\alpha_7\beta_4 V^{ef} + \alpha_6^2\left(\alpha_9 V^{h\omega} - \beta_4 V^{ef}\right)) + 2\left(\alpha_6^2 - 4\alpha_0\alpha_7\right)\alpha_5^2 V^{ef} \\
&\quad - \alpha_6\left(\alpha_6^2 - 4\alpha_0\alpha_7\right)\alpha_5 V^{\omega f})\Big),
\end{aligned}
\tag{C.6}
$$

$$
\begin{aligned}
b_{31} &= \frac{1}{\alpha_6^3 - 4\alpha_0\alpha_6\alpha_7}\Big(\alpha_7(\alpha_1\alpha_6^2(-V^{ee}) - \alpha_6(-2\alpha_0\left(\beta_4 V^{ee} + \alpha_7 V^{\omega f}\right) \\
&\quad + \alpha_9\alpha_6(V^{eh} + V^{he}) + \alpha_6^2 V^{\omega f}) - \alpha_5\left(\alpha_6^2 - 4\alpha_0\alpha_7\right)V^{ef})\Big), \\
b_{32} &= \frac{1}{\alpha_6^4 - 4\alpha_0\alpha_6^2\alpha_7}\Big(\alpha_7(-2\left(\alpha_6^2 - 4\alpha_0\alpha_7\right)\alpha_5^2 V^{ee} \\
&\quad + \alpha_6(\alpha_6^2\beta_4 V^{ee} - 8\alpha_0\alpha_7\beta_4 V^{ee} + 2\alpha_7\alpha_6(\alpha_1 V^{ee} + \alpha_9(V^{eh} + V^{he}) \\
&\quad + \alpha_0(-V^{\omega\omega})) + \alpha_6^3 V^{\omega\omega}) + \alpha_6(\alpha_6^2 - 4\alpha_0\alpha_7)\alpha_5(V^{e\omega} + V^{\omega e}))\Big), \\
b_{33} &= \frac{1}{\alpha_6^4 - 4\alpha_0\alpha_6^2\alpha_7}\Big(\alpha_7\left(-2\left(\alpha_6^2 - 4\alpha_0\alpha_7\right)\alpha_5^2 V^{ee} + \alpha_6\left(\alpha_6^2\beta_4 V^{ee} - 8\alpha_0\alpha_7\beta_4 V^{ee}\right.\right. \\
&\quad + 2\alpha_7\alpha_6\left(\alpha_1 V^{ee} + \alpha_9(V^{eh} + V^{he}) + \alpha_0(-V^{\omega\omega})\right) + \alpha_6^3 V^{\omega\omega} \\
&\quad \left.\left. + \alpha_6\alpha_5\left(\alpha_6^2 - 4\alpha_0\alpha_7\right)(V^{e\omega} + V^{\omega e})\right)\right)\Big), \\
b_{34} &= \frac{\alpha_7\alpha_9\left(2\alpha_7 V^{ef} - \alpha_6 V^{e\omega}\right)}{4\alpha_0\alpha_7 - \alpha_6^2},
\end{aligned}
\tag{C.7}
$$

$$
\begin{aligned}
b_{41} &= \frac{\alpha_7\alpha_9\left(2\alpha_9 V^{eh} + \alpha_6(V^{\omega f} + V^{f\omega}) - 2\alpha_7 V^{ff}\right)}{4\alpha_0\alpha_7 - \alpha_6^2}, \\
,b_{42} &= \frac{\alpha_6\alpha_7\alpha_9 V^{fe}}{\alpha_6^2 - 4\alpha_0\alpha_7}, \\
b_{43} &= \frac{\alpha_7\alpha_9\left(2\alpha_7 V^{fe} - \alpha_6 V^{\omega e}\right)}{4\alpha_0\alpha_7 - \alpha_6^2}, \\
b_{44} &= \frac{2\alpha_7\alpha_9^2 V^{ee}}{\alpha_6^2 - 4\alpha_0\alpha_7}.
\end{aligned}
\tag{C.8}
$$