ON THE BMY INEQUALITY ON SURFACES

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

SADIK TERZİ

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

FEBRUARY 2022

Approval of the thesis:

ON THE BMY INEQUALITY ON SURFACES

submitted by **SADIK TERZİ** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Mathematics Department, Middle East Technical University** by,

Prof. Dr. Halil Kalıpçılar
Dean, Graduate School of **Natural and Applied Sciences** ________________

Prof. Dr. Yıldıray Ozan
Head of Department, **Mathematics** ________________

Prof. Dr. M. Hurşit Önsiper
Supervisor, **Department of Mathematics, METU** ________________

Examining Committee Members:

Assoc. Prof. Dr. A. U. Özgür Kişisel
Department of Mathematics, METU ________________

Prof. Dr. M. Hurşit Önsiper
Department of Mathematics, METU ________________

Assoc. Prof. Dr. Mesut Şahin
Department of Mathematics, Hacettepe University ________________

Assoc. Prof. Dr. Tolga Karayayla
Department of Mathematics, METU ________________

Assoc. Prof. Dr. Alp Bassa
Department of Mathematics, Boğaziçi University ________________

Date:10.02.2022

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Surname: Sadık Terzi

Signature :

ABSTRACT

ON THE BMY INEQUALITY ON SURFACES

Terzi, Sadık
Ph.D., Department of Mathematics
Supervisor: Prof. Dr. M. Hurşit Önsiper

February 2022, 36 pages

In the thesis, we are concerned with the relation between the ordinarity of surfaces of general type and the failure of the BMY inequality in positive characteristic. We consider semistable fibrations $\pi : S \longrightarrow C$ where S is a smooth projective surface and C is a smooth projective curve. Using the exact sequence relating the locally exact differential forms on S, C, and S/C, we prove an inequality relating c_1^2 and c_2 for ordinary surfaces which admit generically ordinary semistable fibrations. This inequality differs from the BMY inequality by a correcting term which vanishes if the fibration is ordinary. In the last chapter we discuss an alternative approach which relates the failure of the Bogomolov inequality ($c_1^2 \leq 4c_2$) on a surface X to the existence of torsors for some special X-group schemes.

Keywords: BMY inequality, ordinarity, semistable fibrations, torsors

ÖZ

YÜZEYLERDEKİ BMY EŞİTSİZLİĞİ ÜZERİNE

Terzi, Sadık
Doktora, Matematik Bölümü
Tez Yöneticisi: Prof. Dr. M. Hurşit Önsiper

Şubat 2022 , 36 sayfa

Bu tezde, positif mertebede genel tipdeki sıradan yüzeyler ve BMY eşitsizliğinin geçersizliği arasındaki ilişki incelendi. Pürüzsüz projektif yüzey S ve pürüzsüz projektif eğri C için $\pi : S \longrightarrow C$ yarı kararlı liflemeleri üzerinde çalışılmıştır. S, C ve S/C üzerindeki yerel diferansiyel tam formlarla alakalı tam dizileri kullanarak, jenerik sıradan yarı kararlı lifleme gösteren sıradan yüzeylerin c_1^2 ve c_2 değişmezleri üzerine bir eşitsizlik ispatlandı. Bu eşitsizlik, sıradan lifleme söz konusu olduğunda yok olan bir düzeltme terimiyle BMY eşitsizliğinden farklılaşır. Son bölümde X yüzeyi üzerindeki Bogomolov eşitsizliğinin ($c_1^2 \leq 4c_2$) geçersizliği ile bazı özel X-grup şemaları için kıvrımların varlığını alakalandıran alternatif bir yaklaşım yorumlandı.

Anahtar Kelimeler: BMY eşitsizliği, sıradan, yarı kararlı liflemeler, kıvrımlar

To my father and my mother

ACKNOWLEDGMENTS

I would like to express my sincere gratitude to my supervisor Prof. Dr. M. Hurşit Önsiper for the continuous support of my Ph.D study and related research, for his patience, motivation, and immense knowledge. His guidance encouraged me in all the time of research and writing of this thesis. I could not have imagined having a better supervisor and mentor for my Ph.D study. I consider myself lucky for having the oppurtunity to study with him.

I would like to thank the professors of the Mathematics Department of METU from whom I learned almost all the mathematics I know. I especially would like to thank to Prof. Dr. Yıldıray Ozan, Prof. Dr. Mahmut Kuzucuoğlu and Assoc. Prof. Dr. Ali Doğanaksoy who had always generously shared their knowledge, experience and valuable time with me. I would like to thank Assoc. Prof. Dr. Mesut Şahin, Assoc. Prof. Dr. Tolga Karayayla and Assoc. Prof. Dr. A. U. Özgür Kişisel for their time to listen to my research works and for their suggestions. I would like to thank the examining committee members for their comments and suggestions. I would like to thank Dr. Muhiddin Uğuz for making the assistant jobs easier for me in the mathematics department and also thank him for funny jokes. I would like to thank my friends Özgür Karabayır and Emre Taştüner for generously sharing their precious time to me whenever I need a help for daily assistant work.

I would like to thank to my family for everything they have given me, especially to my father Sultan Terzi, whose endless support gave me most of the comfort and confidence I needed while I was preparing this thesis.

I gratefully acknowledge the financial support of The Scientific and Technological Research Council of Turkey (TÜBİTAK) with the program BİDEB 2211 (National Graduate Scholarship Programme).

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

In this thesis, we are concerned with the failure of Bogomolov-Miyaoka-Yau (BMY) inequality

$$c_1^2(X) \leq 3c_2(X)$$

on surfaces in positive characteristic.

We know that for any complex projective smooth minimal surface X of general type, the following holds:

- $c_1^2(X) > 0$ (self-intersection number of the canonical bundle),
- $c_2(X) > 0$ (topological Euler characteristic),
- $c_1^2(X) \leq 3c_2(X)$ (BMY inequality).

Over fields of characteristic $p > 0$, there exist surfaces of general type which violate BMY inequality. In fact, we have surfaces of general type with $c_2 < 0$; in what follows we will consider only surfaces with positive c_2. It was conjectured that a modified form of the BMY inequality which contains a correction term measuring non-smoothness of the Picard scheme holds for all surfaces in characteristic p ([14]). This conjecture was disproved by Jang. In fact, he proved that there is no positive constant M such that the relation $c_1^2 \leq Mc_2$ holds on all surfaces of general type with smooth Picard scheme.

Theorem 1. *[6, Section 3, Theorem] For any $M > 0$, there is a smooth proper surface of general type X over a finite field whose Picard scheme is smooth and*

$c_1^2(X) > Mc_2(X)$.

To get a better insight, we quot some recent results leading to further counter examples to BMY inequality in characteristic p.

In [2], Easton uses a concrete geometric approach to produce a family of counter-examples to BMY inequality for surfaces. These surfaces are abelian covers of $\mathbb{P}^2$ branched over particular arrangements of lines which exist only in positive characteristic.

Theorem 2. *[2, Section 3, Theorem] For each prime $q \geq 3$ and all sufficiently large primes $p \cong -1$ (mod q), there exists a surface S in characteristic p, nonsingular and of general type, with $\dfrac{K_S^2}{\chi(\mathcal{O}_S)} > 9$.*

In [20], Urzua constructs surfaces of general type (which are étale simply connected) with smooth Picard schemes for which c_1^2/c_2 is dense in $[2, \infty)$.

Theorem 3. *[20, Theorem 6.6] Let k be an algebraically closed field of characteristic $p > 0$. Then, for any real number $x \geq 2$, there are minimal surfaces of general type X over k such that*

1. *$c_1^2(X) > 0, c_2(X) > 0$,*
2. *$\pi_1^{ét}(X)$ is trivial,*
3. *$H^1(X, \mathcal{O}_X) = 0$,*
4. *and $c_1^2(X)/c_2(X)$ is arbitrarily close to x.*

These phenomena give rise to two intriguing problems in positive characteristic.

1) To determine the precise conditions on surfaces of general type in characteristic $p > 0$, under which the BMY inequality (or a version with an additive correcting term) holds.

2) To find relations of BMY type between the invariants c_1^2, c_2 of surfaces of general type under varying hypotheses.

One may conjecture that the modified form of BMY inequality, if exists, should have a term which accounts for the pathological behaviour of the sheaves of locally exact differentials in characteristic p (non-ordinarity of the surface and of the Picard scheme of the surface) (*). This is the point of view we adapt in this thesis.

Recently, K. Joshi obtained inequalities relating c_1^2, c_2 for surfaces of general type under additional hypotheses. A notable relation is the inequality $c_1^2 \leq 5c_2$ for minimal surfaces of general type which are Hodge-Witt and satisfy certain extra hypotheses.

Theorem 4. *[7, Theorem 4.7.8] Let X be a smooth, projective, minimal surface of general type. Assume*

1. $c_2(X) > 0$,
2. $p_g > 0$,
3. *X is Hodge-Witt,*
4. $\mathrm{Pic}(X)$ *is reduced or* $H^2_{cris}(X/W)$ *is torsion free,*
5. *and* $H^2_{cris}(X/W)$ *has no slope* $< 1/2$.

Then $c_1^2(X) \leq 5c_2(X)$.

Now, we briefly explain the content of each chapter of the thesis.

The background material in Chapter 2 contains definitions and some basic facts about:

- semistable fibrations on surfaces and their invariants
- absolute, relative, and arithmetic Frobenius morphisms
- locally exact differential forms on varieties
- ordinarity of varieties, relative ordinarity for surfaces fibered over curves and the notion of being generically ordinary
- group schemes and group torsors

In Chapter 3, we first prove the following Proposition.

Proposition 5. *Let $\pi : S \longrightarrow C$ be a fibration. Then there exists a morphism of $\mathcal{O}_S$-modules $\phi : \pi^* F_{C*}\Omega^1_{C/k} \longrightarrow F_{S*}\pi^*\Omega^1_{C/k}$.*

Next we prove

Lemma 6. *Let $\pi : S \longrightarrow C$ be a fibration on a smooth projective surface S and let W be the arithmetic Frobenius morphism (Definition 17). Then one has a short exact sequence*

$$0 \longrightarrow \pi^* B^1_{C/k} \longrightarrow B^1_{S/k} \longrightarrow W_* B^1_{S/C} \longrightarrow 0. \tag{4}$$

This exact sequence is the main tool in proving our main result in Chapter 4, which concerns ordinary smooth projective surfaces of general type which admit generically ordinary semistable fibrations. For such a surface we prove an inequality relating c_1^2 and c_2, which differs from the BMY inequality by an additive term that vanishes if the fibration is ordinary.

Theorem 7. *Let S be an ordinary smooth projective surface which admits a generically ordinary semistable fibration $\pi : S \longrightarrow C$ of genus $g \geq 2$ over a smooth projective curve of genus $q \geq 1$. Then the invariants c_1^2 and c_2 satisfy the following equation*

$$c_1^2 = 2c_2 + \frac{12}{p-1}.h^1(B^1_{S/C}) - 3\delta$$

where δ is the total number of singular points in the fibers of π.

Corollary 8. *Under the hypotheses of Theorem 7, the following inequality holds*

$$c_1^2 \leq 3c_2 + \frac{12}{p-1}.h^1(B^1_{S/C}) - 4\delta.$$

We also have the following lemma which provides examples of surfaces for which the BMY inequality holds. In the proof of this lemma, we will use the basic properties of the Ekedahl-Oort strata in $\mathfrak{A}^*_{g,1,n}$ ([13], Sections 2-6).

Lemma 9. *Let $\pi : S \longrightarrow C$ be a semistable fibration of genus $g \geq 2$ such that the fibers are all (i) ordinary, or (ii) of p-rank $g-1$, or (iii) supersingular but not superspecial, or (iv) superspecial. Then π is isotrivial. Hence, the BMY inequality holds on the surface S.*

In Chapter 5; we address the existence of a special type of inseparable Galois covers, namely the principal homogeneous spaces on X for groups of type $\mathrm{G}_{a,b}^{\mathscr{L}}$ defined in [19]. These covers contain as a particular case the $\alpha_{\mathscr{L}}-$covers (Definition 37) of X.

This approach is motivated by the work of Mukai ([12]) on surfaces of general type (with negative c_2) which violate Kodaira vanishing theorem (KVT) and the paper by Shepherd-Barron ([15]) on the Bogomolov inequality on surfaces. In both of these papers one of the key ingredients is the construction of a purely inseparable finite cover of the given surface.

We prove an elementary triviality result for $\alpha_{\mathscr{L}}-$covers arising from suitable pairs (X, L) of a surface X and a line bundle L on X (Lemma 10).

Lemma 10. *Let X be a projective smooth surface and L be a line bundle on X such that one of the following conditions holds.*
a) X lifts to $W_2(k)$ and L^{-1} is numerically effective,
b) X is ordinary in dimension one, $H^0(X, \mathscr{L}) \neq 0$ where $\mathscr{L}$ is an invertible sheaf corresponding to L,
c) L^{-1} is ample and either

- *Pic_X^0 is smooth or*
- *(X, L^{-1}) lifts to characteristic zero with ramification index $e < p - 1$.*

Then there exists no non-trivial $\alpha_{\mathscr{L}}-$torsors on X. □

This result ties in with the fact that in characteristic $p = 2$, if X lifts to $W_2(k)$ then a vector bundle E of rank two which violates the Bogomolov inequality is unstable ([15], Corollary 11). We also remark that since the existence of infinitesimal unipotent covers of a variety is related to the non-ordinarity of its Picard scheme, the triviality results obtained may be considered as some weak evidence in support of the conjectural statement (*) given above.

Throughout this thesis, our notation is as follows:

Our base field is $k = \overline{F}_p$ for some prime $p > 0$.

S is a projective smooth surface of general type over k.

C is a projective smooth curve of genus $q \geq 1$.

$\pi : S \longrightarrow C$ is a semistable fibration of genus $g \geq 2$.

$\omega_{S/C}$ is the relative canonical bundle.

$\mathfrak{M}_g$ is the moduli space of smooth genus g curves.

$\overline{\mathfrak{M}}_g$ is the moduli space of stable genus g curves.

$\mathfrak{A}_{g,1,n}$ is the moduli space of principally polarized abelian varieties of dimension g with a symplectic level-n-structure.

$\mathfrak{A}^*_{g,1,n}$ is the Satake compactification of $\mathfrak{A}_{g,1,n}$.

F_X and F are absolute and relative Frobenius morphisms, respectively, for a variety X.

F also denotes Frobenius morphism on the cohomology groups of a variety X.

$B^1_{X/k}$ is the locally exact differential forms for a variety X.

$B^1_{X/Y}$ is the relative locally exact differential forms for a morphism $f : X \longrightarrow Y$.

Pic_X is the Picard scheme of X.

Pic^0_X is the connected component of Pic_X containing identity.

$\mathrm{NS}(X)$ is the Neron-Severi group of X.

$C_{++}(X)$ is the positive cone in $\mathrm{NS}(X)$.

$W(k)$ (resp. $W_2(k)$) is the ring of Witt vectors (resp. Witt vectors of length two).

μ_p, α_p are the standard infinitesimal group schemes.

For an abelian variety A, $A[p]$ is the subgroup scheme *kernel of multiplication by* p.

For an algebraic group scheme G, $\tau(G) = \dim_k Hom(\alpha_p, G)$.

CHAPTER 2

PRELIMINARIES

In this part, we state basic facts about semistable fibrations, ordinarity of varieties (especially of surfaces), group schemes and group torsors. We also include basic cohomological tools that we shall use in the rest of the paper.

2.1 Semistable fibrations

Definition 11. Let C be a projective curve over an algebraically closed field k. We say that C is $stable$ (resp., $semistable$) if:

- $\mathbf{a})$ C is connected and reduced,
- $\mathbf{b})$ all singular points are normal crossings, and
- $\mathbf{c})$ an irreducible component, isomorphic to $\mathbb{P}^1$, meets the other components in at least three (resp., two) points.

The relative version is given in the following definition.

Definition 12. A proper flat morphism of schemes $f : X \longrightarrow Y$ is said to be a $(semi)stable$ curve over Y if every geometric fiber of f is a (semi)stable curve.

From now on, we are concerned with semistable fibrations $\pi : S \longrightarrow C$ of genus $g \geq 2$ on smooth projective surfaces S where the base curve C is smooth of genus $q \geq 1$. Let $T \subset C$ be the set of points over which the fiber is not smooth and t be the cardinality of T.

Definition 13. A semistable fibration $\pi : S \longrightarrow C$ is called *isotrivial* if there exists a finite morphism $\phi : C' \longrightarrow C$ such that the fiber product $S \times_C C'$ is, birationally on C', isomorphic to the trivial fibration. In this case, one can assume that ϕ is étale on $C - T$.

Remark 14. For a given semistable fibration $\pi : S \longrightarrow C$, the following hold:

1) We have a morphism $\alpha_\pi : C \longrightarrow \overline{\mathfrak{M}}_g$ and α_π is constant iff π is isotrivial.

2) The uniqueness of the semistable model implies that if the fibration π is isotrivial, then it is smooth.

Next we discuss some basic invariants of semistable fibrations. For a semistable fibration $\pi : S \to C$, we define the following invariants:

1) $d = \text{degree}(\pi_* \omega_{S/C})$,

2) $\delta = \sum_{P \in T} \delta_P$ where δ_P is the number of singular points in a fiber,

3) c_1 and c_2 are the first and the second Chern classes of S,

4) $N = \bigoplus_{P\text{:sing in a fiber}} i_{P*}(k(P))$.

These invariants satisfy the following relations [17, Sections 0-1, pp.46-49].

i) $\chi(\mathcal{O}_S) = \dfrac{c_1^2 + c_2}{12} = \chi(\pi_* \mathcal{O}_S) - \chi(R^1 \pi_* \mathcal{O}_S)$,

ii) $c_1^2 = 12d - \delta + 8(g-1)(q-1)$,

iii) $c_2 = \delta + 4(g-1)(q-1)$.

From the equations in **ii**) and **iii**), we obtain the equality $c_1^2 = 2c_2 + 12d - 3\delta$.

We also have the following exact sequences (loc. cit.).

5) $0 \longrightarrow \pi^* \Omega^1_C \longrightarrow \Omega^1_{S/k} \longrightarrow \Omega^1_{S/C} \longrightarrow 0$

6) $0 \longrightarrow \Omega^1_{S/C} \longrightarrow \omega^1_{S/C} \longrightarrow N \longrightarrow 0$

2.2 Ordinarity

In this section, we first consider the absolute, the arithmetic and the relative Frobenius morphisms for a morphism $f : X \longrightarrow Y$ of varieties in characteristic $p > 0$.

Definition 15. The *absolute Frobenius morphism* $F_Y : Y \longrightarrow Y$ is given by the identity map on the underlying topological space and the p-th power map on the structure sheaf.

Remark 16. The following diagram is commutative:

$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d, "f"] & X \arrow[d, "f"] \\ Y \arrow[r, "F_Y"] & Y \end{tikzcd}$$

Definition 17. The *arithmetic Frobenius morphism* is the morphism

$$W : X^{(p)} = X \times_{(Y, F_Y)} Y \longrightarrow X$$

obtained from F_Y by the base extension.

It follows from Remark 16 that there exists a unique morphism $F : X \longrightarrow X^{(p)}$ over Y fitting into the following commutative diagram.

$$\begin{tikzcd} X \arrow[dr, dotted, "F"] \arrow[drr, bend left, "F_X"] \arrow[ddr, bend right, "f"'] & & \\ & X^{(p)} \arrow[r, "W"] \arrow[d, "f^{(p)}"] & X \arrow[d, "f"] \\ & Y \arrow[r, "F_Y"] & Y \end{tikzcd}$$

Diagram 1

Definition 18. $F : X \longrightarrow X^{(p)}$ is called the *relative Frobenius morphism*.

We next discuss the concepts of ordinarity and generic ordinarity. For a variety X of dimension n, we have a complex

$$0 \longrightarrow F_{X*}\mathcal{O}_X \xrightarrow{d^1} F_{X*}\Omega^1_{X/k} \xrightarrow{d^2} F_{X*}\Omega^2_{X/k} \xrightarrow{d^3} \cdots \xrightarrow{d^n} F_{X*}\Omega^n_{X/k} \longrightarrow 0.$$

The sheaf $\mathrm{Im}(d^i)$ is called the sheaf of locally exact i-th differential forms and is denoted by $B^i_{X/k}$. Notice that $B^1_{X/k} = \mathrm{Coker}(F_X)$ and thus sits in the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{F_X} F_{X*}\mathcal{O}_X \longrightarrow B^1_{X/k} \longrightarrow 0. \tag{1}$$

Definition 19. ([5], Definition 1.1) We say that X is *ordinary* if $H^i(X, B^j_{X/k}) = 0$ for all i and j.

We recall the characterization of ordinary curves and surfaces in terms of the action of Frobenius on the cohomology.

1) Let X be a smooth projective curve of genus g. Set $V = H^1(X, \mathcal{O}_X)$.
   Let $V_n = \{\xi \in V \mid F^m(\xi) = 0 \text{ for some positive integer m}\}$ be the subspace on which F is nilpotent and V_s the complement of V_n in V. In fact,

   $$V_s = \text{Span}(\{\xi \in V \mid F(\xi) = \xi\}).$$

   The natural number $\sigma_X = \dim_k(V_s)$ is called the p-rank of X. The following facts are well-known:

   a) X is an ordinary curve if and only if $\sigma_X = g$.
   b) The p-rank of a curve coincides with the p-rank σ_J of its Jacobian.
   c) For a semistable curve X, the Jacobian sits in an extension of group schemes

      $$0 \longrightarrow G_m^s \longrightarrow J_X \longrightarrow A \longrightarrow 0$$

      where A is an abelian variety. We define the p-rank of X by setting

      $$\sigma_X = s + \sigma_A.$$

2) Let S be a surface. By using the short exact sequence (1) and Serre duality, we see that S is ordinary if and only if $H^i(S, B^1_{S/k}) = 0$ for all i. This condition is equivalent to requiring

   $$F : H^i(S, \mathcal{O}_S) \to H^i(S, \mathcal{O}_S)$$

   be bijective for $i = 1, 2$.

Definition 20. ([6], Definition 2.8) Let $\pi : S \longrightarrow C$ be a proper semistable fibration. We say that π is *generically ordinary* if at least one closed fiber of π is ordinary. (Hence almost all closed fibers of π are ordinary.)

Recall that the BMY inequality is the relation

$$c_1^2(X) \leq 3c_2(X)$$

between the Chern classes of the smooth projective surface X.

Remark 21. We can verify the BMY inequality on certain minimal surfaces of general type by computing directly c_1^2 and c_2. For example:

1) Let $\pi : X \longrightarrow C$ be any smooth isotrivial fibration with $g(C) \geq 2$ and $g(F) \geq 2$. Then we have

$$c_1^2(X) = 8(g(C) - 1)(g(F) - 1)$$
$$c_2(X) = 4(g(C) - 1)(g(F) - 1)$$

and so $c_1^2(X) = 2c_2(X)$.

2) Let X be any complete intersection smooth surface of general type of degree $d = (d_1, d_2, \cdots, d_{n-2})$ in $\mathbf{P}^n$. We assume that $\sum_i d_i > n + 1$ and we set $\xi = \mathcal{O}_X(1)$. Then we have

$$c_1^2(X) = \left[-n - 1 + \sum_i d_i\right]^2 \xi^2 \tag{2.2.1}$$

$$c_2(X) = \left[\frac{(n+1)n}{2} - (n+1)\sum_i d_i + \sum_{i \leq j} d_i d_j\right] \xi^2 \tag{2.2.2}$$

and so $c_1^2(X) \leq c_2(X)$.

In particular, the BMY inequality holds on any smooth surface of degree $d \geq 5$ in $\mathbf{P}^3$.

2.3 Group schemes and group torsors

Definition 22. ([18], Definition 1.5) Let X be a scheme and G be a scheme over X. Let $m : G \times G \longrightarrow G$ be a morphism. We say that (G, m) is a *group scheme* if

$m(Y) : G(Y) \times G(Y) \longrightarrow G(Y)$ gives $G(Y)$ a group structure in the category $(\mathbf{Gr})$ of groups for any scheme Y over X. A *homomorphism* between group schemes G and H is a morphism $\phi : G \longrightarrow H$ such that $\phi(Y) : G(Y) \longrightarrow H(Y)$ is a group homomorphism.

Remark 23. ([18], Section 2.2) Let A be an R-algebra. let $G = \mathrm{Spec}(A)$ be an affine scheme over $X = \mathrm{Spec}(R)$. To give a group structure on G is equivalent to give a Hopf R-algebra structure on A. The last one is equivalent to give three R-algebra homomorphisms

- $s : A \longrightarrow A \otimes_R A$
- $e : A \longrightarrow R$
- $i : A \longrightarrow A$

which satisfy the following rules for composition of R-homomorphisms

- $(\mathrm{id}_A \otimes s) \circ s = (s \otimes \mathrm{id}_A) \circ s$
- $(\mathrm{id}_A \otimes e) \circ s = \alpha \circ \mathrm{id}_A$
- $(s, \mathrm{id}_A) \circ s = \epsilon \circ e$

where $\alpha : A \longrightarrow A \otimes_R R$ is an isomorphism given by $\alpha(ra) = a \otimes r$ and ϵ is the map which gives the R-algebra structure on A.

We have the following well-known examples:

Example 24. Affine additive group $\mathbb{G}_a$

1. $A = R[x]$ is polynomial ring over R.
2. $s(x) = 1 \otimes x + x \otimes 1$ (comultiplication)
3. $e(x) = 0$ (counit)
4. $i(x) = -x$ (antipodal map)

Example 25. Affine multiplicative group $\mathbb{G}_m$

1. $A = R[x, x^{-1}]$ is polynomial ring over R.

2. $s(x) = x \otimes x$ (comultiplication)

3. $e(x) = 1$ (counit)

4. $i(x) = x^{-1}$ (antipodal map)

Let $R = k$ be an algebraically closed field of positive characteristic p.

Definition 26. The infinitesimal group schemes α_p and μ_p are defined by the following short exact sequences

$$0 \longrightarrow \alpha_p \longrightarrow \mathbb{G}_a \xrightarrow{F} \mathbb{G}_a \longrightarrow 0, \tag{2.3.1}$$

$$1 \longrightarrow \mu_p \longrightarrow \mathbb{G}_m \xrightarrow{[p]} \mathbb{G}_m \longrightarrow 1, \tag{2.3.2}$$

respectively where F (Frobenius morphism) and $[p]$ (the p-th power map) are obviously group scheme homomorphisms.

Let G be a group scheme which is flat and locally of finite type over X and S be a scheme over X.

Definition 27. We say that G *acts* on S if there is a morphism $S \times_X G \longrightarrow S$ so that $G(Y)$ acts on $S(Y)$ in the usual sense for all scheme Y over X.

Definition 28. ([10], Definition 4.1.) Let S be an S-scheme on which G acts. S is said to be a G *principal homogenous space* or a G *torsor* if there is a covering $(U_i \longrightarrow X)$ for the flat topology on X such that $S \times_X U_i$ is isomorphic with its $G \times_X U_i$-action to $G \times_X U_i$.

The following proposition gives a characterization of the α_p and μ_p torsors in terms of differential forms on X.

Proposition 29. *([10], Proposition 4.14) Let X be a smooth variety over a perfect field k of positive characteristic p. Then*

$$H^1(X_{fl}, \alpha_p) = \{\omega \in H^0(X, \Omega^1_X) \mid d\omega = 0, \mathscr{C}(\omega) = 0\}$$

and

$$H^1(X_{fl}, \mu_p) = \{\omega \in H^0(X, \Omega^1_X) \mid d\omega = 0, \mathscr{C}(\omega) = \omega\}$$

where $\mathscr{C}$ is the $(p^{-1})-$linear Cartier operator.

The $Cartier\ operator$ $\mathscr{C}$ defined in [8, 7.2] is a $1/p-$linear operator fitting on the short exact sequence

$$\Omega_X^{i-1} \xrightarrow{d} Z\Omega_X^i \xrightarrow{\mathscr{C}} \Omega_X^i$$

where $Z\Omega_X^i$ is the closed differential i-forms.
The operator $\mathscr{C}$ satisfies the following properties

1) $\mathscr{C}(1) = 1$

2) $\mathscr{C}(df) = 0$

3) $\mathscr{C}(f^p\omega) = f\mathscr{C}(\omega)$

4) $\mathscr{C}(f^{p-1}df) = df$

5) $\mathscr{C}(df/f) = df/f$

6) $\mathscr{C}(\alpha \wedge \beta) = \mathscr{C}(\alpha) \wedge \mathscr{C}(\beta)$

for all closed forms α, β, ω and for all functions f.

CHAPTER 3

SOME LOCAL ALGEBRA

Let $\pi : S \longrightarrow C$ be a fibration as in Chapter 1. Then one has the following short exact sequence of $\mathcal{O}_S$-modules

$$0 \longrightarrow \pi^*\Omega^1_{C/k} \longrightarrow \Omega^1_{S/k} \longrightarrow \Omega^1_{S/C} \longrightarrow 0. \tag{2}$$

Since F_S is a finite morphism, we obtain the following exact sequence

$$0 \longrightarrow F_{S*}\pi^*\Omega^1_{C/k} \longrightarrow F_{S*}\Omega^1_{S/k} \longrightarrow F_{S*}\Omega^1_{S/C} \longrightarrow 0 \tag{3}$$

applying F_{S*} in (2).

In Lemma 6, we prove the existence of a short exact sequence of sheaves of locally exact differential forms (analogous to (3)). To this end, we first recall the definition of the relative locally exact differential forms $B^1_{S/C}$. Using the relative deRham complex

$$0 \longrightarrow \mathcal{O}_S \xrightarrow{\ d\ } \Omega^1_{S/C} \longrightarrow 0$$

and the relative Frobenius map $F : S \longrightarrow S^{(p)}$, we obtain a complex

$$0 \longrightarrow F_*\mathcal{O}_S \xrightarrow{F_*d} F_*\Omega^1_{S/C} \longrightarrow 0.$$

Definition 30. The sheaf $B^1_{S/C} := \mathrm{Im}(F_*d)$ is called the *relative locally exact differential forms* for the fibered surface $\pi : S \longrightarrow C$.

Remark 31. $B^1_{S/C}$ sits in the following short exact sequence

$$0 \longrightarrow \mathcal{O}_{S^{(p)}} \longrightarrow F_*\mathcal{O}_S \longrightarrow B^1_{S/C} \longrightarrow 0.$$

Remark 32. Let X be a scheme over a perfect field of positive characteristic p and W be the arithmetic Frobenius morphism. Then $X^{(p)} \simeq X$ via the map W, because $F_k : \mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(k)$ is an isomorphism. Therefore, $B^1_{X/k}$ can be viewed as a sheaf of $\mathcal{O}_X$-modules on X.

Proposition 5. *Let $\pi : S \longrightarrow C$ be a fibration. Then there exists a morphism of $\mathcal{O}_S$-modules $\phi : \pi^* F_{C*}\Omega^1_{C/k} \longrightarrow F_{S*}\pi^*\Omega^1_{C/k}$.*

Proof. Since the statement is local on the both source and target, we may assume that $S = \mathrm{Spec}(R)$ and $C = \mathrm{Spec}(A)$. Therefore, $\pi^* F_{C*}\Omega^1_{C/k} = F_{A*}\widetilde{\Omega_{A/k} \bigotimes_A} R$ and $F_{S*}\pi^*\Omega^1_{C/k} = F_{R*}(\widetilde{\Omega^1_{A/k} \bigotimes_A} R)$ where F_A and F_R denote the relevant Frobenius morphisms. Define an R-module homomorphism

$$\varphi : F_{A*}\Omega_{A/k} \otimes_A R \longrightarrow F_{R*}(\Omega^1_{A/k} \otimes_A R)$$

$$\sum_{i=1}^{n} a_1^i.a_2^i da_3^i \otimes r_i \longmapsto \sum_{i=1}^{n} (a_1^i)^p a_2^i da_3^i \otimes r_i^p.$$

φ is a well-defined R-module homomorphism due to the equality

$$\sum_{i=1}^{n} a_1^i.a_2^i da_3^i \otimes r_i^p = \sum_{i=1}^{n} r_i.((a_1^i)^p a_2^i da_3^i \otimes 1).$$

Hence, we have a morphism of $\mathcal{O}_S$-modules

$$\widetilde{\varphi} : F_{A*}\widetilde{\Omega_{A/k} \otimes_A} R \longrightarrow F_{R*}(\widetilde{\Omega^1_{A/k} \otimes_A} R).$$

Thus, we obtain the required morphism of $\mathcal{O}_S$-modules

$$\phi : \pi^* F_{C*}\Omega^1_{C/k} \longrightarrow F_{S*}\pi^*\Omega^1_{C/k}.$$

$\square$

Remark 33. Since $B^1_{A/k}$ is a submodule of $F_{A*}\Omega^1_{A/k}$, $B^1_{A/k} \otimes_A R$ is a submodule of $F_{A*}\Omega^1_{A/k} \otimes_A R$. Therefore, we get a morphism of R-modules

$$\overline{\varphi} : B^1_{A/k} \otimes_A R \longrightarrow F_{R*}(\Omega^1_{C/k} \otimes_A R)$$

by restricting φ to $B^1_{A/k} \otimes_A R$. For a given differential form $a_1.a_2da_3 \otimes 1 \in B^1_{A/k} \otimes_A R$, there exists $a_0 \in A$ such that $a_2da_3 = da_0$ as $a_2da_3 \in B^1_{A/k}$. Hence, $a_1.a_2da_3 =$

$a_1.da_0 = a_1^p da_0 = da_1^p a_0$. As a result, any element $\omega \in B^1_{A/k} \otimes_A R$ can be written as $\omega = da \otimes r$ for some $a \in A$ and $r \in R$. Thus, we see that $\overline{\varphi}$ is an injective homomorphism of R-modules. Therefore, we get an injective morphism of $\mathcal{O}_S$-modules

$$\overline{\phi} : \pi^* B^1_{C/k} \hookrightarrow F_{S*}\pi^*\Omega^1_{C/k}.$$

By the exact sequence (3) and Remark 33, we obtain a complex of $\mathcal{O}_S$-modules

$$0 \longrightarrow \pi^* B^1_{C/k} \longrightarrow F_{S*}\Omega^1_{S/k} \longrightarrow F_{S*}\Omega^1_{S/C} \longrightarrow 0.$$

Restricting to the the open subscheme $\mathrm{Spec}(A) \subset C$, we obtain the following complex of R-modules

$$0 \longrightarrow B^1_{A/k} \otimes_A R \xrightarrow{\overline{u}} F_{R*}\Omega^1_{R/k} \xrightarrow{\overline{v}} F_{R*}\Omega^1_{R/A} \longrightarrow 0$$

where $\overline{u} = F_{R*}u \circ \overline{\varphi}$ and $\overline{v} = F_{R*}v$.

We will work out the details of the R-module structure on $F_{R*}\Omega^1_{R/A}$.

Remark 34. One has the following commutative diagram (which corresponds to Diagram 1 applied to the morphism $f : S \longrightarrow C$ over $\mathrm{Spec}(A) \subset C$).

$$\begin{tikzcd}
A \arrow[r, "F_A"] \arrow[d, "\psi"] & A \arrow[d, "\psi^{(p)}"] \arrow[rdd, bend left, "\psi"] & \\
R \arrow[r, "W"] \arrow[rrd, bend right, "F_R"'] & R^{(p)} \arrow[rd, dotted, "F"] & \\
 & & R
\end{tikzcd}$$

Diagram 2

In Diagram 2,

a) $R^{(p)} = R \otimes_{A,F_A} A$,

b) $\psi : A \longrightarrow R$ is a ring homomorphism which corresponds to $\pi : S \longrightarrow C$,

c) $F : R^{(p)} \longrightarrow R$ given by $r \otimes a \longmapsto ar^p$ corresponding to $F : S \longrightarrow S^{(p)}$,

d) $W : R \longrightarrow R^{(p)}$ given by $ar \longmapsto r \otimes a^p$ corresponding to $W : S^{(p)} \longrightarrow S$,

e) $W \circ F = F_R$.

Therefore, $F_{R*}\Omega^1_{R/A} = W_*(F_*\Omega^1_{R/A})$. Then for any $\omega = (a_1r_1).r_2dr_3 \in F_{R*}\Omega^1_{R/A}$, we have

$$\omega = (a_1r_1).r_2dr_3 = (r_1 \otimes a_1^p).r_2dr_3 \text{ via } W$$

and

$$(r_1 \otimes a_1^p).r_2dr_3 = a_1^p r_1^p r_2 dr_3 \text{ via } F$$

i.e., first we make $\Omega^1_{R/A}$ an $R^{(p)}$-module via F and then via the map $W : R \longrightarrow R^{(p)}$, $\Omega^1_{R/A}$ becomes an R-module. Moreover, we may view $W_*B^1_{R/A}$ as a subsheaf of $F_{R*}\Omega^1_{R/A} = W_*(F_*\Omega^1_{R/A})$ because $B^1_{R/A}$ is the subsheaf of $F_*\Omega^1_{R/A}$.

Lemma 6. *Let $\pi : S \longrightarrow C$ be a fibration on a smooth projective surface S. Then one has a short exact sequence*

$$0 \longrightarrow \pi^*B^1_{C/k} \longrightarrow B^1_{S/k} \longrightarrow W_*B^1_{S/C} \longrightarrow 0. \tag{4}$$

Proof. Let $\psi : A \longrightarrow R$ be the ring homomorphism corrosponding to π. Let $da \otimes r$ be in $B^1_{A/k} \otimes_A R$. We may restrict $\overline{v}$ to the subsheaf $B^1_{R/k}$ of $F_{R*}\Omega^1_{R/k}$ as $\overline{u}(da \otimes r) = r^p d\psi(a) = dr^p\psi(a) \in B^1_{R/k}$. Also for a given $dr \in B^1_{R/k}$ since $\overline{v}(dr) = dr \in W_*B^1_{R/A}$, we have the following sequence of R-modules:

$$0 \longrightarrow B^1_{A/k} \otimes_A R \xrightarrow{\overline{u}} B^1_{R/k} \xrightarrow{\overline{v}} W_*B^1_{R/A} \longrightarrow 0.$$

To complete the proof, we need to prove the following claims:

Claim (1) : $\overline{u}$ is injective,

Claim (2) : $\mathrm{Im}(\overline{u}) = \mathrm{Ker}(\overline{v})$,

Claim (3) : $\overline{v}$ is surjective.

The first claim follows from Remark 33.

Clearly, $\mathrm{Im}(\overline{u}) \subseteq \mathrm{Ker}(\overline{v})$ by the short exact sequence (2). Let dr be in $\mathrm{Ker}(\overline{v})$. We have $dr = 0$ in $B^1_{R/A}$ which implies that $r \in A$ i.e., there exists $a \in A$ such that $r = \psi(a)$. Therefore, $dr = d\psi(a) = \overline{u}(da \otimes 1)$. As a result, we have

$$\mathrm{Ker}(\overline{v}) = \mathrm{Im}(\overline{u})$$

which completes the proof of the second claim.

Let $[r_1.(r_2 \otimes a)].dr_3$ be in $W_*B^1_{R/A}$. Then the last claim follows by the equality

$$\begin{aligned} [r_1.(r_2 \otimes a)].dr_3 &= [(r_1 \otimes 1)(r_2 \otimes a)].dr_3 \\ &= (r_1r_2 \otimes a).dr_3 \\ &= ar_1^p r_2^p dr_3 \\ &= d\psi(a)r_1^p r_2^p r_3 \\ &= \overline{v}(ar_1^p r_2^p r_3). \end{aligned} \tag{3.0.1}$$

Thus, the sequence

$$0 \longrightarrow B^1_{A/k} \otimes_A R \xrightarrow{\overline{u}} B^1_{R/k} \xrightarrow{\overline{v}} W_*B^1_{R/A} \longrightarrow 0$$

is a short exact sequence of R-modules. This implies the following is a short exact sequence of $\mathcal{O}_S$-modules

$$0 \longrightarrow \widetilde{B^1_{A/k} \otimes_A R} \longrightarrow \widetilde{B^1_{R/k}} \longrightarrow \widetilde{W_*B^1_{R/A}} \longrightarrow 0.$$

Therefore, we have the following short exact sequence of $\mathcal{O}_S$-modules

$$0 \longrightarrow \pi^* B^1_{C/k} \longrightarrow B^1_{S/k} \longrightarrow W_*B^1_{S/C} \longrightarrow 0.$$

□

CHAPTER 4

THE MAIN RESULT

In this chapter, we prove our main result, Theorem 7. The main ingredient is the short exact sequence of locally exact differential forms constructed in the preceding chapter. We will use the following well-known result [4, Chapter 3, Exercises 8.3] in the proof of Theorem 7.

Proposition 35. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces, let $\mathcal{F}$ be an $\mathcal{O}_X$-module and let $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module of finite rank. Then*

$$R^i f_*(\mathcal{F} \otimes f^*\mathcal{E}) = R^i f_*(\mathcal{F}) \otimes \mathcal{E}$$

for all $i \geq 0$.

Remark 36. Let $\pi : S \longrightarrow C$ be a semistable fibration as in the statement of 7. Then $B^1_{C/k}$ is a locally free $\mathcal{O}_C$-module of rank $p-1$ and of degree $(p-1)(q-1)$. By Proposition 35, we have

$$R^i \pi_* \pi^* B^1_{C/k} = R^i \pi_*(\mathcal{O}_S \otimes \pi^* B^1_{C/k}) = R^i \pi_* \mathcal{O}_S \otimes B^1_{C/k}.$$

We calculate the rank and the degree of the sheaf $\mathcal{M} = R^1\pi_*\mathcal{O}_S \otimes B^1_{C/k}$ and we get

$$\text{rank}(\mathcal{M}) = r(p-1) \text{ and } \deg(\mathcal{M}) = (p-1)e + r(p-1)(q-1)$$

where $r = \text{rank}(R^1\pi_*\mathcal{O}_S)$ and $e = \deg(R^1\pi_*\mathcal{O}_S)$.

Consider the Leray spectral sequence attached to the sheaf $\mathcal{F} = \pi^*(B^1_{C/k})$ on the fibration $\pi : S \longrightarrow C$, namely

$$E_2^{pq} = H^p(C, R^q\pi_*\pi^* B^1_{C/k}) = H^p(C, R^q\pi_*\mathcal{O}_S \otimes B^1_{C/k}) \implies H^{p+q}(S, \pi^* B^1_{C/k}). \qquad (*)$$

We have the following properties:

a) Since C is a curve, $H^p(C, -) = 0$ for $p > 1$.

b) By Corollary 11.2 in [4, Chapter 3], $R^q\pi_*\mathcal{O}_S = 0$ for $q > 1$.

c) By proposition 35 and since $\pi_*\mathcal{O}_S = \mathcal{O}_C$, we have $\pi_*\pi^*B^1_{C/k} = B^1_{C/k}$.

Therefore, we get

$$H^0(S, \pi^*B^1_{C/k}) = H^0(C, B^1_{C/k}) \text{ and } H^2(S, \pi^*B^1_{C/k}) = H^1(C, R^1\pi_*\mathcal{O}_S \otimes B^1_{C/k}).$$

If we assume that C is an ordinary curve, then we also have

$$H^1(S, \pi^*B^1_{C/k}) = H^0(C, R^1\pi_*\mathcal{O}_S \otimes B^1_{C/k}).$$

Now we prove Theorem 7. We remark that the hypothesis in Theorem 7 differs from the hypothesis in Lemma 9; we remove the p-rank condition on the non-smooth fibers, but now we assume that S is ordinary.

Theorem 7. *Let S be an ordinary smooth projective surface which admits a generically ordinary semistable fibration $\pi : S \longrightarrow C$ of genus $g \geq 2$ over a smooth projective curve of genus $q \geq 1$. Then the invariants c_1^2 and c_2 satisfy the following equation*

$$c_1^2 = 2c_2 + \frac{12}{p-1}.h^1(B^1_{S/C}) - 3\delta$$

where δ is the total number of singular points in the fibers of π.

Proof. Recall that for semistable fibrations, we have the equality:

$$c_1^2 = 2c_2 + 12d - 3\delta.$$

We will prove that $(p-1)d = h^1(B^1_{S/C})$. For this purpose, we will use the short exact sequence (4) proved in Lemma 6 :

$$0 \longrightarrow \pi^*B^1_{C/k} \longrightarrow B^1_{S/k} \longrightarrow W_*B^1_{S/C} \longrightarrow 0.$$

We have the long exact sequence

$$0 \longrightarrow H^0(S, \pi^* B^1_{C/k}) \longrightarrow H^0(S, B^1_{S/k}) \longrightarrow H^0(S, W_* B^1_{S/C}) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^1(S, \pi^* B^1_{C/k}) \longrightarrow H^1(S, B^1_{S/k}) \longrightarrow H^1(S, W_* B^1_{S/C}) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^2(S, \pi^* B^1_{C/k}) \longrightarrow H^2(S, B^1_{S/k}) \longrightarrow H^2(S, W_* B^1_{S/C}) \longrightarrow 0.$$

We note the following:

1) Since S is an ordinary surface,

$$H^i(S, B^1_{S/k}) = 0$$

for all $i \geq 0$. It is easily concluded that $H^2(S, W_* B^1_{S/C}) = 0$.

2) Since π is a generically ordinary semistable fibration, $\pi^{(p)}_* B^1_{S/C} \mid_U = 0$ where U is the ordinary locus of π. However, $B^1_{S/C}$ is flat over $\mathcal{O}_C$ so $\pi^{(p)}_* B^1_{S/C} = 0$. It follows that $H^0(S, W_* B^1_{S/C}) = H^0(S^{(p)}, B^1_{S/C}) = H^0(C, \pi^{(p)}_* B^1_{S/C}) = 0$ as W is a finite morphism.

Therefore, by the long exact sequence

$$H^0(S, \pi^* B^1_{C/k}) = H^1(S, \pi^* B^1_{C/k}) = 0$$

and

$$H^1(S, W_* B^1_{S/C}) = H^2(S, \pi^* B^1_{C/k}).$$

Now, recall that S is assumed to be an ordinary surface. Then C is an ordinary curve and hence by using the Leray spectral sequence $(*)$ we have an equality:

$$\chi(\mathcal{M}) = h^0(C, \mathcal{M}) - h^1(C, \mathcal{M}) = h^1(S, \pi^* B^1_{C/k}) - h^2(S, \pi^* B^1_{C/k}).$$

On the other hand,

$$\chi(\mathcal{M}) = \deg(\mathcal{M}) - (q-1)\,\mathrm{rank}(\mathcal{M}).$$

Therefore, we have

$$-h^2(S, \pi^* B^1_{C/k}) = \chi(\mathcal{M})$$

and so

$$-h^2(S, \pi^* B^1_{C/k}) = (p-1)e + r(p-1)(q-1) - (q-1)r(p-1).$$

This implies that $(p-1)d = h^2(S, \pi^* B^1_{C/k}) = h^1(S, W_* B^1_{S/C}) = h^1(B^1_{S/C})$ where $d = -e$. Recall that since $\pi : S \longrightarrow C$ is a semistable fibration, we have $c_1^2 = 2c_2 + 12d - 3\delta$. Substituting $d = \dfrac{1}{p-1} \cdot h^1(B^1_{S/C})$, we obtain the equality:

$$c_1^2 = 2c_2 + \frac{12}{p-1}.h^1(B^1_{S/C}) - 3\delta.$$

□

Since for the fibration in Theorem 7

$$c_2 - \delta = 4(g-1)(q-1) \geq 0$$

we obtain the inequality given in Corollary 8, namely

$$c_1^2 \leq 3c_2 + \frac{12}{p-1}.h^1(B^1_{S/C}) - 4\delta.$$

Next we shall prove Lemma 9 which relates the ordinarity of the fibers in a semistable fibration on a surface S and the BMY inequality on S. The lemma provides other examples for which the BMY inequality holds simply because under each of the hypotheses (i)-(iv) on the Ekedahl-Oort type of the fibers, the given fibration is isotrivial (cf. Remark 21). In the proof of this lemma, we will use the basic properties of the Ekedahl-Oort strata in $\mathfrak{A}^*_{g,1,n}$ ([13], Sections 2-6).

Lemma 9. *Let $\pi : S \longrightarrow C$ be a semistable fibration of genus g such that the fibers are all (i) ordinary, or (ii) of p-rank $g-1$, or (iii) supersingular but not superspecial, or (iv) superspecial. Then π is isotrivial. Hence, the BMY inequality holds on the surface S.*

Proof. Let $U \subseteq S$ be the union of the smooth fibers of π and set $C' = \pi(U)$. The canonical principal polarization on the relative Jacobian $J_{U/C'}$ extends to a principal cubic structure on $J_{S/C}$ ([11], Chapter II, Theorem 3.5) which we denote by Θ. Thus, we obtain a morphism

$$h : C \longrightarrow \mathfrak{A}^*_{g,1,n}$$

given by

$$p \mapsto (J_{S_p}, \Theta_p).$$

In each of the cases $(\mathbf{i})$, $(\mathbf{ii})$, $(\mathbf{iii})$ and $(\mathbf{iv})$ we verify that the image $h(C)$ lies in an Ekedahl-Oort stratum determined by a unique elementary sequence φ ([13], Definition 2.1.).

$\mathbf{Case}\ (\mathbf{i})$: $|\varphi| = g \cdot (g+1)/2$ correspondes only to the elementary sequence

$$\varphi = \{1, 2, \cdots, g\}.$$

$\mathbf{Case}\ (\mathbf{ii})$: $|\varphi| = g \cdot (g+1)/2 - 1$ is obtained only from the elementary sequence

$$\varphi = \{1, 2, \cdots, g-1, g-1\}.$$

$\mathbf{Case}\ (\mathbf{iii})$: If the fibers are supersingular, but not superspecial, then we have $|\varphi| = 1$ for which the elementary sequence is

$$\varphi = \{0, 0, \cdots, 0, 1\}.$$

$\mathbf{Case}\ (\mathbf{iv})$: If the fibers are superspecial, then we have $|\varphi| = 0$ which arises only from the elementary sequence

$$\varphi = \{0, 0, \cdots, 0\}.$$

It follows from ([13], Theorem 6.4) that the image $h(C)$ is quasi-affine, and so the morphism h is constant on C. Then we see that the smooth fibers of π are all isomorphic by applying the Torelli theorem. That is, $\pi : S \longrightarrow C$ is an isotrivial semistable fibration. Therefore, π is smooth and $d = 0,\ \delta = 0$. It follows that $c_1^2 = 2c_2$ by Remark 21. $\square$

CHAPTER 5

DISCUSSION

Let X be a smooth projective surface over a field of arbitrary characteristic and E be a rank two (semi)stable vector bundle on X. The Bogomolov inequality for E is the inequality $c_1^2(E) \leq 4c_2(E)$. In characteristic 0, since the cotangent bundle Ω_X of a surface of general type is stable, we obtain the inequality $c_1^2(X) \leq 4c_2(X)$ between the Chern classes $c_1^2(X)$ and $c_2(X)$. This relation was refined by Miyaoka (and independently by Yau) to what is known as the BMY inequality

$$c_1^2(X) \leq 3c_2(X).$$

In characteristic $p > 0$, we know by the results and examples quoted in the Introduction that, for any $M > 0$ there are minimal surfaces of general type (with smooth Picard scheme) X such that the inequality

$$c_1^2(X) \leq Mc_2(X)$$

fails to hold on X.

In last chapter of the thesis, we will discuss a geometric approach applied in investigating the failure of Bogomolov inequality in positive characteristic. Using the main results in [15] we will relate the method first to $\alpha_{\mathscr{L}}$-torsors and then to ordinarity of the surface under consideration.

Definition 37. Let X be a scheme and $\mathscr{L}$ be an invertible sheaf on X and $s \in H^0(X, \mathscr{L}^{p-1})$ be a section. The affine group scheme α_s over X is defined by the following short exact sequence in flat topology :

$$0 \longrightarrow \alpha_s \longrightarrow L \xrightarrow{F-s} L^p \longrightarrow 0.$$

where L is the line bundle corresponding to $\mathscr{L}$.
In particular, when $s = 0$, we obtain the exact sequence defining the affine group scheme $\alpha_{\mathscr{L}}$:

$$0 \longrightarrow \alpha_{\mathscr{L}} \longrightarrow L \xrightarrow{F} L^p \longrightarrow 0.$$

Lemma 10. *Let X be a projective smooth surface and L be a line bundle on X such that one of the following conditions holds.*
a) X lifts to $W_2(k)$ and L^{-1} is numerically effective,
b) X is ordinary in dimension one, $H^0(X, \mathscr{L}) \neq 0$ where $\mathscr{L}$ is an invertible sheaf corresponding to L,
c) L^{-1} is ample and either

- *Pic^0_X is smooth or*
- *(X, L^{-1}) lifts to characteristic zero with ramification index $e < p - 1$.*

Then there exists no non-trivial $\alpha_{\mathscr{L}}-$torsors on X.

Proof. We check that in each case $H^1(X_{fl}, \alpha_{\mathscr{L}}) = 0$.

From the short exact sequence in (X_{fl})

$$0 \to \alpha_{\mathscr{L}} \to L \to L^p \to 0$$

we obtain

$$H^1(X_{fl}, \alpha_{\mathscr{L}}) = \text{Ker}(H^1(X_{fl}, L) \to H^1(X_{fl}, L^p)).$$

In fact, by ([10], Chapter III, Proposition 3.7) we have

$$H^1(X_{fl}, \alpha_{\mathscr{L}}) = \text{Ker}(H^1(X, L) \to H^1(X, L^p)).$$

a) If X lifts to $W_2(k)$ and L^{-1} is numerically effective, then $H^1(X, L) = 0$ ([1], Corollary 2.8(ii)) and the conclusion follows.

b) Our aim is to show that $F : H^1(X, L) \to H^1(X, L^p)$ is injective. Let D be the divisor on X corresponding to $\mathscr{L}$. Since $H^0(X, \mathscr{L}) \neq 0$ we may assume that D is an effective divisor. We fix $n \in \mathbb{N}$ and tensor the exact sequence

$$0 \to \mathcal{O}_X \to F_*\mathcal{O}_X \to B^1_X \to 0$$

by the invertible sheaf $\mathscr{L}^{-n} = \mathcal{O}_X(-nD)$ and get the following short exact sequence

$$0 \to \mathscr{L}^{-n} \to F_*\mathcal{O}_X \otimes \mathscr{L}^{-n} \to B^1_X(-nD) \to 0.$$

In the following relevant part of the corresponding long exact sequence of cohomology

$$\cdots \to H^0(X, B^1_X(-nD)) \to H^1(X, \mathcal{O}_X(-nD)) \to H^1(X, \mathcal{O}_X(-npD)) \to \cdots$$

we have $H^0(X, B^1_X(-nD)) = 0$. To see this we observe that

$$H^0(X, B^1_X(-nD)) = \{df \in \Omega_{\mathbf{Q}(X)} | (df) \geq npD\} \subseteq H^0(X, B^1_X)$$

and that $H^0(X, B^1_X) = 0$ since X is ordinary in dimension one. Hence for all $n \geq 1$,

$$\text{Ker}(H^1(X, \mathscr{L}^{-n}) \to H^1(X, \mathscr{L}^{-np})) = 0.$$

It follows that $F : H^1(X, L) \to H^1(X, L^p)$ is injective and so $H^1(X_{fl}, \alpha_{\mathscr{L}}) = 0$.

c) $H^1(X, L) = 0$ by KVT (in the second case, we refer to [16], Theorem 4.5.1).

□

Remark 38. Let X be a smooth projective surface and Pic^0_X be the connected component containing $0 \in \mathrm{Pic}_X$. Then one can replace the condition in Lemma 5 (b) by the condition that Pic^0_X is ordinary [9, Lemma 3].

An $\alpha_{\mathscr{L}}-$torsor on X is clearly a purely inseparable cover of degree p. In characteristic two, we have the following result of T.Ekedahl [3, Proposition 1.11] which characterizes all covers of degree two of smooth projective varieties as α_s-torsors.

Proposition 39. *Let X be a smooth projective variety over a field of characteristic two. Then any degree two covering $\pi : Y \longrightarrow X$, where Y is Cohen-Macaulay, is an α_s-torsor for a suitable line bundle $\mathscr{L}$ and a section $s \in H^0(X, \mathscr{L})$.*

Remark 40. It seems an interesting problem to investigate the generalization of the preceding proposition to covers in characteristic $p > 2$. Characterizing purely inseparable covers of degree p which arise as torsors for group schemes $\mathrm{G}_{a,b}^{\mathscr{L}}$ defined in [19] is particularly interesting.

Let us fix a minimal surface X of general type and let D be a divisor in $C_{++}(X)$. We set $\mathscr{L} = \mathcal{O}_X(D)$. An element $\rho \in H^1(X, \mathscr{L}^{-1})$ gives an extension

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow \mathscr{L} \longrightarrow 0.$$

From this extension we obtain the following geometric structure:

a. a section $X_2 \subset \mathbb{P}(E)$ with $\mathcal{O}_{\mathbb{P}}(1) = \mathcal{O}_{\mathbb{P}}(X_2)$

b. the normal bundle $N_{X_2/\mathbb{P}} = \mathscr{L}$

c. an $\mathscr{L}-$torsor $\mathbb{P}(E) - X_2$

This extension splits if and only if there is a section disjoint from X_2.
Note that if $c_1^2(E) > 4c_2(E)$, then there is an integral surface $Y \subset \mathbb{P}$, with composition $\pi : Y \longrightarrow X$ purely inseparable of degree p^n [15, Theorem 1] and Y is disjoint from X_2 [15, Lemma 16]. We may assume that $n = 1$.

Remark 41. If $c_1^2(E) > 4c_2(E)$ and E is semi-stable, then the cotangent bundle of X has an invertible subsheaf in $C_{++}(X)$ by [15, Proposition 6]. Therefore, the tangent bundle of X has an invertible subsheaf which provides a degree p purely inseparable covering $\pi : Y \longrightarrow X$. If one can show that this cover is an $\alpha_{\mathscr{L}}-$torsor for a suitable invertible sheaf $\mathscr{L}$, then we conclude that X is not an ordinary surface by Lemma 10.

Based on this analysis, one is lead to the following conjecture:

Conjecture 42. Let X be a smooth projective minimal surface of general type which is ordinary in dimension one. Then Bogomolov inequality $c_1^2(E) \leq 4c_2(E)$ holds for any semistable rank two vector bundle E on X. In particular, we have the inequality $c_1^2(X) \leq 4c_2(X)$ for the chern numbers c_1^2, c_2.

In characteristic two, without imposing ordinarity conditions but assuming that X is not ruled and that X lifts to $W_2(k)$, one proves that the conclusion of the preceding conjecture holds on X [15, Corollary 11]:

(i) any rank two vector bundle E on X which satisfies $c_1^2(E) > 4c_2(E)$ is unstable,

(ii) we have $c_1^2 \leq 4c_2$ for the Chern numbers of X if X is not ruled.

Remark 43. In any characteristic $p \geq 2$, S.Mukai constructs examples X of surfaces of general type which admit purely inseparable covers of degree p [12, Theorem 2]. Mukai proves that

(a) KVT does not hold on X

(b) $c_2(X) < 0$, hence BMY inequality fails on X.

We also observe that X is not ordinary, simply because we have nowhere vanishing locally exact 1-forms on X.

We end this part by proposing another conjecture:

Conjecture 44. Let X be an ordinary smooth projective minimal surface of general type in characteristic $p > 0$. Then KVT holds on X.

REFERENCES

[1] P.Deligne, L.Illusie, *Relevements modulo p^2 et decomposition du complexe de deRham*, Inv.Math. 89 (1987), 247-270.

[2] R. Easton, *Surfaces violating Bogomolov-Miyaoka-Yau ın positive characteristic*, Proc. of AMS, Volume 136 (2008), 2271-2278.

[3] T.Ekedahl, *Canonical models of surfaces of general type in positive characteristic*, Pub. Math. IHES 67 (1988), 97-144.

[4] R. Hartshorne, Algebraic Geometry, Springer-Verlag, New York Inc. 1977.

[5] L. Illusie, *Ordinarité des intersections complètes générales*, Progress in Math. 87 (1990), Birkhauser-Verlag, 375-405.

[6] J. Jang, *Generically ordinary fibrations and a counterexample to Parshin's conjecture*, Michigan Math. J. 59 (2010), 169-178.

[7] K. Joshi, *Crystalline aspects of geography of low dimensional varieties I: numerology*, European J. of Math., https://doi.org/10.1007/s40879-020-00416-x.

[8] N. Katz, *Nilpotent connections and the monodromy theorem; application of a result of Turrittin*, Publ. Math. I.H.E.S. 39 (1970) 232-355.

[9] C. Kaya, *On ordinarity and lifting fibrations on surfaces*, Comm. in Algebra v.43 (2015), 1915-1919.

[10] J.S.Milne, Etale Cohomology, Princeton University Press, (1980).

[11] L. Moret-Bailly, Pinceaux de Variétés Abéliennes, Astérisque 129, SMF, 1985.

[12] S.Mukai, *Counterexamples of Kodaira's vanishing and Yau's inequality in positive characteristic*, Kyoto Journal of Mathematics, Vol. 53, No. 2 (2013), 515-532.

[13] F. Oort, *A stratification of a moduli space of polarized abelian varieties in positive characteristic*, In: Moduli of Curves and Abelian Varieties, Aspects of Math. E 33, Vieweg 1999; p. 47-64.

[14] A. Parshin, *A letter to Don Zagier*, Progress in Math., 89 (1991), Birkhauser, 285-292.

[15] N.I.Shepherd-Barron, *Unstable vctor bundles and linear systems on surfaces in characteristic p*, Inv.Math. 106 (1991), 243-262.

[16] M.Raynaud, *p-torsion du schema de Picard*, in Asterisque 64 (1979), Soc. Math. France, 87-148.

[17] L. Szpiro, *Propriétés numériques du faisceau dualisant relatif*, Astérisque 86 (1981), p. 44-78.

[18] J. Tate, *Finite flat group schemes.* In: Gary Cornell, Joseph H. Silverman, Glenn Stevens, eds. Modular forms and Fermat last theorem. Springer-Verlag, New York 1997. pp. 121-154

[19] J. Tate, F. Oort, *Group schemes of prime order*, Annales scientifiques de l'E.N.S., 1970.

[20] G. Urzúa, *Chern slopes of surfaces of general type in positive characteristic*, Duke Math. J. v.166, no.5 (2017), 975-1004.

CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name: Terzi, Sadık

EDUCATION

Degree	Institution	Year of Graduation
Ph.D.	Department of Mathematics, Middle East Technical University	2022
B.S.	Department of Mathematics, Middle East Technical University	2012

PROFESSIONAL EXPERIENCE

Year	Place	Enrollment
2013-2022	Department of Mathematics, Middle East Technical University	Research Assistant

FOREIGN LANGUAGES

English

PUBLICATIONS

International Journal Publications

1. S. Terzi, *On the BMY inequality on surfaces.* Comm. in Algebra, DOI:10.1080/00927872.2021.1967367, 2021.

2. S. Terzi, *On the p-rank of curves* (Submitted for Publication)

AWARDS

- Middle East Technical University, Honor student degree, 2012

- The Scientific and Technological Research Council of Turkey (TÜBİTAK), National Ph.D. Fellowship Programme (2211-A), 2012-2017