GROUP REPRESENTATION THEORY AND RADAR AMBIGUITY FUNCTIONS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

EREN BERK KAMA

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
ELECTRICAL AND ELECTRONICS ENGINEERING

FEBRUARY 2022

Approval of the thesis:

GROUP REPRESENTATION THEORY AND RADAR AMBIGUITY FUNCTIONS

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ABSTRACT

GROUP REPRESENTATION THEORY AND RADAR AMBIGUITY FUNCTIONS

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February 2022, 80 pages

In this thesis, representations of Heisenberg group are applied to ambiguity functions and their properties are investigated in with an information theoretic perspective with applications in telecommunications and signal processing. Algebraic properties of ambiguity functions were investigated through application of representation theory. Novel approaches on phase space tiling are given and some existing methods for traditional ambiguity functions were extended to MIMO ambiguity functions. Irreducible representations are used to obtain an orthonormal basis of $L_2(\mathbb{R}^2)$. An identity on different functions having same ambiguity function is used in MIMO case. Uncertainty relations on ambiguity functions and certain time frequency distributions are studied. Relations between norms of MIMO ambiguity functions and norms of signals creating them are given. A local uncertainty relation on MIMO ambiguity functions and a bound on delay Doppler support is given. Lieb uncertainty is used in MIMO ambiguity functions to obtain a sharp uncertainty relation. These uncertainty relations are connected with applications in time frequency analysis, compressed sensing and integrated sensing and communication applications. Moreover, uncertainty relations on Wigner distributions and marginalizable time frequency distributions are given. Uncertainty relation of de Bruijn was used on time frequency distributions with signal processing examples. Effect of symplectic transformations on Wigner Distributions was investigated. Actions of generators of $SL(2,\mathbb{R})$ are tied with common signal processing operations and their effect on uncertainty was investigated. This effect and its applications in time frequency analysis and localization tasks are discussed. Furthermore, applications of time frequency analysis in quantum information theory and quantum harmonic analysis are discussed.

Keywords: harmonic analysis, ambiguity functions, time frequency analysis, representation theory, uncertainty relations

ÖZ

GRUP REPRESENTATİON TEORİSİ VE RADAR BELİRSİZLİK FONKSİYONLARI

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Şubat 2022 , 80 sayfa

Bu tezde, Heisenberg grubunun representasyonları belirsizlik fonksiyonlarına uygulanmış ve onların özellikleri bilgi teorisi çerçevesinde telekomunikasyon ve sinyal işleme uygulamalarıyla incelenmiştir. Belirsizlik fonksiyonlarının cebirsel özellikleri representasyon teorisi uygulamasıyla incelenmiştir. Faz uzayı bölüntülemesi üzerine yeni bir yaklaşım verilmiş ve bazı var olan yaklaşımlar MIMO belirsizlik fonksiyonlarına taşınmıştır. Parçalanamaz representasyonlar kullanılarak $L_2(\mathbb{R}^2)$ uzayına ortonormal bazlar türetilmiştir. Aynı belirsizlik fonksiyonuna sahip iki fonksiyonun üzerine bir ilişki MIMO belirsizlik fonksiyınlarına taşınmıştır. Belirsizlik fonksiyonları ve zaman frekans dağılımları üzerine belirsizlik ilişkileri çalışılmıştır. MIMO belirsizlik fonksiyonlarının normları ile onları üreten fonksiyonların normları arasında bir ilişki türetilmiştir. MIMO belirsizlik fonksiyonu üzerinde Lokal bir belirsizlik ilişkisi verilmiş ve delay Doppler genişliği üzerine bir sınır verilmiştir. MIMO belirsizlik fonksiyonları üzerinde Lieb belirsizlik ilişkisi kullanılmış ve keskin belirsizlik ilişkisi elde edilmiştir. Bu belirsizlik ilişkileri zaman frekans analizi, sıkıştırılmış algılama, entegre algılama ve telekomunikasyon uygulamalarıyla bağlanmıştır. Buna ek olarak, Wigner dağılımları ve marjinalize edilebilir zaman frekans dağılımları üzerinde belirsizlik ilişkileri verilmiştir. de Bruijn belirsizlik ilişkisi zaman frekans dağılımlarında sinyal işleme uygulamaları için kullanılmıştır. Simplektik transformların Wigner dağılımları üzerine etkisi incelenmiştir. $SL(2, \mathbb{R})$ grubunun generatorlerinin aksiyonları sinyal işlemede kullanılan operasyonlarla ilişkilendirilmiş ve bunların belirsizlik ilişkileri üzerine etkileri incelenmiştir. Bu etkinin zaman frekans analizindeki ve lokalizasyon konusundaki uygulamarı tartışılmıştır. Bununla birlikte, zaman frekans analizinin kuantum bilgi teorisi ve quantum harmonik analiz üzerine uygulamaları tartışılmıştır.

Anahtar Kelimeler: harmonik analiz, belirsizlik fonksiyonları, zaman frekans analizi, representasyon teorisi, belirsizlik bağıntıları

To my parents

ACKNOWLEDGMENTS

I want to thank my supervisor professor Mustafa Kuzuoğlu, for his endless support and guidance. He always gave me great insights in engineering and abstract mathematics. He encouraged me throughout the research research process. I have learned a great variety of topics in electrical and electronics engineering and abstract mathematics. I am very glad and grateful to have him as my supervisor.

I would like to thank professor Umut Orguner for his continuous support during my master's period. I am grateful for his valuable ideas on every aspect of research. I thank professor Gökhan Güvensen for his guidance. He gave me great insights in telecommunications field and helped me shape my ideas on the field.

I want to thank my family for their continuous help, love and support during my all life.

This work was supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK) through the 2210-A National Scholarship Programme for MSc Students. This work was supported by "Bilgi Teknolojileri ve İletişim Kurumu", BTK, and "Türk Telekom". I am thankful for the opportunities they provided.

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LIST OF ABBREVIATIONS

MIMO	Multi Input Multi Output
SIMO	Single Input Multi Output
WVD	Wigner Ville Distribution
TF	Time frequency
TFD	Time frequency distribution
STFT	Short time frequency transform
FT	Fourier transform
LFM	Linear frequency modulated
AF	Ambiguity function

NOMENCLATURE

χ	Symmetric Ambiguity function
A	Ambiguity function
W	Wigner Distribution
S_s^w	Spectrogram of signal s with window w
$\mathcal{A}$	Filtered ambiguity function
ρ_a	Time frequency distribution
σ	Symplectic Form
J	Symplectic Identity
S	Symplectic matrix
Γ	Symplectic matrix
T	Representation of a group
G	Group
e	Group identity
n	Dimension and group dimension
$H_{\mathbb{R}}$	Real Heisenberg Group
C_H	Center of group H
μ	Measure
E	Borel set
$\|E\|$	Measure of set E
$GL(V)$	Group of all linear transformations from V to V. (Group of endomorphisms of V)
G	Group
$\mathcal{H}$	Center of group H
$\mathcal{F}$	Fourier transform operator

$\mathcal{H}$	Hilbert transform operator
z	Element of phase space
u_m	mth waveform
$\mathbf{R_{ij}}$	ijth element of covariance matrix
δ_{mn}	Kronecker delta
p	Norm degree
$\hbar$	Planck constant

CHAPTER 1

INTRODUCTION

1.1 Motivation

Time frequency analysis is a field of signal processing where signals are studied in both time and frequency domains. Instead of studying a 1 dimensional signal and its dual such as in Fourier transform separately, signal is studied in both domain simultaneously. Common time frequency distributions are Wigner distribution, Short time Fourier transform, ambiguity functions. Wavelet analysis and its subclasses are used to analyze signals in time and frequency although they are functions of time and a scale rather than time and frequency. Compressed sensing is also a study of time frequency analysis.

Time frequency analysis methods are frequently used in many signal processing applications. Example applications are Intermediate frequency estimation, time frequency filtering, signal decomposition, various sampling methods, acoustics, biomedical applications and radar and sonar signal processing. LTV system analysis methods frequently use time frequency analysis in order to investigate time varying signals and filter them in time frequency domain. These methods are used for source separation and source localization.

In Radar signal processing, waveforms posing certain traits such as high resolution in delay domain or Doppler frequency domain are used. In order to design and analyze different waveforms, ambiguity functions are used. Ambiguity functions are also time frequency distributions on the time delay and Doppler frequency axes. They are related with Wigner distributions with a 2-dimensional type Fourier transform. With this relation, they are used in time frequency filter design and analysis. Ambiguity functions are generalized to MIMO case in [52]. It is well known that, MIMO systems enhance performance such as in detection and estimation, transmit beam design and they introduce degrees of freedom.

Wigner distributions and ambiguity functions have applications in quantum physics for a long time. Wigner distributions are used to identify pure and mixed states on the position momentum axes. Certain properties of these distributions such as non-negativity and uncertainties are studied frequently in quantum physics and harmonic analysis. However, crucial properties in quantum physics and signal processing applications differ. Here, we will address some of these properties and use them in signal processing context.

Harmonic analysis is the study of relations between functions and their duals such as time domain signal and its Fourier domain counterpart. Harmonic analysis can be separated to two parts as commutative and noncommutative harmonic analysis. In commutative harmonic analysis, norm inequalities, convergence relations and uncertainty relations are studied in relation with signal processing. In noncommutative harmonic analysis again in relation with signal processing, representation of non-commutative groups, their algebraic properties and uncertainty relations are studied. Ambiguity functions and wavelets are most famous uses of representations of non-commutative groups. Naturally methods of harmonic analysis are frequently used in time frequency analysis. Main topics of harmonic analysis that we will use here are representation of groups with their algebraic properties, uncertainty relations and norm inequalities.

Harmonic analysis of ambiguity functions was given in terms of representations of Heisenberg group in [6]. There, algebraic properties of ambiguity functions were given in relation to representation theory. Here, we use representations of Heisenberg group in MIMO ambiguity functions to investigate their properties.

As it is not possible to localize a signal in time and frequency simultaneously, methods for controlling the behavior of the signal on time frequency domain are sought. Wavelet analysis, windowed Fourier transform, Cohen's class functions are some examples of these studies. In this thesis, we apply uncertainty relations on MIMO ambiguity functions to see their localization properties. In addition, we use uncertainty relations and bounds from quantum physics in time frequency analysis.

1.2 Brief Problem Definition

Here, we consider connections of time frequency distributions with representations of Heisenberg group and uncertainty relations. In representations of Heisenberg groups, irreducible representations are found by method of Frobenius. After finding the representations, they are tied with ambiguity functions by using representations as operators. By using this relation, properties of representation theory are carried to ambiguity functions. These relations will be used in MIMO ambiguity functions to study their algebraic properties.

A variety of uncertainty relations will be given on both MIMO ambiguity functions and marginalizable time frequency distributions. A Donoho type uncertainty will be given for MIMO ambiguity functions to investigate their localization on measurable sets. A sharp uncertainty relation called by [40] will be related with MIMO ambiguity functions. Moreover, norm inequalities concerning MIMO ambiguity functions will be studied. An uncertainty of [15] will be used in marginalizable time frequency distributions. In addition, uncertainty relations from quantum physics will be used in time frequency distributions in signal processing context.

Throughout this study, we will treat ambiguity functions as both radar ambiguity functions and time frequency distributions. Moreover, we will give properties in relation with Wigner distributions with the dual relation between the two.

1.3 Literature Survey

1.3.1 Time Frequency Analysis

Time frequency methods are frequently used in many field such as audio signal processing, biomedical, radar, sonar, compressed sensing, telecommunications and wavelet analysis. Time frequency analysis methods are studied widely in [11]. Ambiguity functions were generalized to MIMO ambiguity functions in [52]. Some properties concerning delay and Doppler space are given in [13],[39],[56], [1], [57], [14]. In all these papers, MIMO ambiguity functions were investigated in terms of radar signal processing methods. We are approaching the problem directly from a mathematical viewpoint while using and mentioning certain properties given there.

1.3.2 Harmonic Analysis and Representation Theory

Some classical books for commutative harmonic analysis can be given as [35], [31], [32], [24], [60]. Harmonic analysis of noncommutative groups are given in [24], to which, we will follow a similar approach. Representation theory of noncommutative Lie groups used here can be found in [25], [58]. Ambiguity functions were connected with representations of Heisenberg group in [6]. In [46], a deeper analysis of the topic, starting from group theory is given. [47] also discusses the same problem with a general radar signal processing perspective. In [2], a method of designing ambiguity functions are given

Discrete ambiguity functions were studied in terms of harmonic analysis in [12]. Discrete Heisenberg group was used in a coding application in [33] via linking to quantum mechanics. [55] gives action of metaplectic group on ambiguity functions. Some other approaches to ambiguity functions using Heisenberg group can be found in [54], [53]. Some use cases of Heisenberg group in signal processing tasks can be found in

[23], [9], [8]. Representation theory applications in signal processing can be found in [7], [3], [4].

The use of Heisenberg group was not as popular as that of affine group. Affine group was used in wavelet analysis. It has been used frequently both in practical applications and abstract mathematics since its connection to wavelet analysis was discovered. The relationship between Heisenberg group and ambiguity functions are analogous to that of Affine group and wavelets. However, Heisenberg group representations didn't see much attention. The reason for that is the flexibility and effect affine group creating is not coming from Heisenberg group. Moreover, the application area didn't admit Heisenberg group.

Connection of representation theory to wavelet analysis with a similar approach to ambiguity functions can be found in [46]. General uses of representation theory in wavelet analysis can be found in [29], [18], [30] together with many other sources on the topic.

Uses of Heisenberg in group theory can be found in [25]. Heisenberg group has many uses in quantum mechanics. Similar ideas that we will use in the thesis can be found in many quantum mechanics papers. However, as the question studied is different in quantum mechanics, important points in quantum mechanics might be unuseful in signal processing applications and vice versa.

1.3.3 Uncertainty Relations

Uncertainty relations are one of the most frequently studied topics in harmonic analysis. A general look at uncertainty relations can be found in [26],[49],[29]. Some modern applications of uncertainty relations can be found in [36], [51]. Uncertainty relations were used in foundational papers in compressed sensing such as [22], [21].

In [40] a sharp uncertainty on ambiguity functions were given in terms of norms of signals used. [15] gives an uncertainty relation on Wigner distribution using the marginalization relation.

1.3.4 Methods of Quantum Mechanics

Uncertainty relations in quantum physics can be found in [19], [44]. Invariant uncertainty relations were given for mixed quantum states using symplectic forms and Williamson invariants in [41], [48]. Some uncertainty relations using symplectic form and Wigner distributions are given in [16], [17], [28], [27].

Heisenberg uncertainty relation and Robertson Schrödinger uncertainty relations are tied with Wigner Yanase Skew information and Fisher information in [44]. This analysis is furhter studied in many papers such as in [38], [42], [43], [45].

Studies on quantum harmonic analysis is taking attention in new years. There are many studies on connecting Heisenberg uncertainty relation, Wigner-Yanase Skew information and Quantum Fisher information. This connection may bring interesting approaches both to the studies of information theory and harmonic analysis. Through connections of classical physics and quantum physics, these relations can be connected. With these connections, further analysis results can be obtained. Current research is on understanding the connection, structure and effects of these information measures. Research is studying the noncommutative nature of quantum observables. Because of this noncommutative nature, the information content of one observable can be written in terms of the information content of the other. Therefore, an observable carries information about the other in relation with the amount of noncommutativity between the two. By using this relation and further analysis, more on information theory of quantum entities can be understood.

1.4 Contributions

This thesis studies harmonic analysis relations in time frequency distributions. Mainly, results on ambiguity functions and their dual, Wigner distributions are given. We look at representation theoretic aspects of them and uncertainty relation on them. We carry some existing results in MIMO case and give some new relations. Moreover, we use results from quantum physics in signal processing problems with proper alterations.

Throughout the thesis, we first give results from the analysis viewpoint, then give their use in engineering applications in time frequency analysis, quantum information theory, quantum harmonic analysis, compressed sensing and integrated sensing and communication. The thesis follows an information theoretic approach on ambiguity functions with their applications in telecommunication and signal processing problems. Main contributions are as follows,

- Representations of real Heisenberg group are used on MIMO ambiguity functions and their algebraic properties on $L_2(\mathbb{R})$ are given. This relation is tied with phase space tiling.
- Action of generators of $SL(2, \mathbb{R})$ on MIMO ambiguity functions are given in relation with common signal processing operations.
- Donoho's uncertainty relation is given in MIMO ambiguity functions and their localization on a measurable set is investigated.
- A sharp uncertainty, given in [40] is tied with MIMO ambiguity functions.
- Uncertainty relations for MIMO ambiguity functions are tied with compressed sensing and integrated sensing and communications applications.
- Norm inequalities on MIMO ambiguity functions are given.
- An uncertainty relation for time and frequency marginalizable TFDs is given. Possible uses of this relation in time frequency analysis is discussed.

- Robertson-Schrödinger uncertainty relation on time frequency distributions are studied.
- Invariance of certain time frequency distributions under action of generators of $SL(2,\mathbb{R})$ is given with their effects in terms of signal processing operations.
- Uncertainty relations given here are connected to quantum information theory and quantum harmonic analysis applications.

In the literature, given relations are for traditional ambiguity functions. Most results are for self ambiguity functions. Here, results for MIMO ambiguity functions contain cross ambiguity functions. By using MIMO systems, combined effect of signals are investigated algebraically and in terms of their uncertainty relations.

1.5 The Outline of the Thesis

Principles of harmonic analysis and time frequency analysis are given in Chapter 2 to create a basis for the thesis. Ambiguity functions, time frequency distributions and main harmonic analysis tools are given. In Chapter 3 , harmonic analysis of MIMO ambiguity functions are studied in terms of their algebraic and measure theoretic aspects. Representations of Heisenberg groups are used on ambiguity functions to analyze their algebraic traits. Moreover, Various uncertainty relations are given on ambigutiy functions. Chapter 4 gives effects of symplectic transforms on time frequency distributions and invariant relations of them. In addition, several uncertainty relations are given on time frequency distributions. Chapter 5 concludes the theses and gives future research directions.

Throughout the thesis, scalars, column vectors, and matrices will be shown as x, $\mathbf{x}$, and $\mathbf{X}$, respectively. ith entry of vector $\mathbf{x}$ is $\mathbf{x}_i$. Also, $(\mathbf{X})_{ij}$ will be used to indicate the entry in the ith row and jth column. Transpose of a vector and matrix are given by $x^T, \mathbf{X}^T$ Inner product space is given as an indice if it is not clear from the context.

CHAPTER 2

TIME FREQUENCY DISTRIBUTIONS

In this chapter, we will give preliminary ideas for the thesis. We will define methods we will use from harmonic analysis. Briefly, representation theory, certain important groups, uncertainty relations and some norm inequalities will be defined. We will provide necessary details for the thesis and give references where needed.

2.1 Ambiguity Functions

Ambiguity functions are useful signal processing tools, used in radar waveform design and time frequency signal processing. They were first given in [61] In radar, they are used to analyze signal's behavior on the delay Doppler (τ, ν) domain. In time frequency analysis, as they are related to Wigner distributions via a 2D-type Fourier transform (one Fourier transform and one inverse Fourier transform), they are used as the filter domain. More on use of ambiguity functions in time frequency analysis can be found in [11],[29]. As they are used frequently in these areas, there is a need to investigate their properties. Harmonic analysis studies the relations between a function and its Fourier transform counterpart, a good reference giving both commutative and noncommutative harmonic analysis is [24]. Harmonic analysis methods are frequently used to analyze time frequency distributions. SIMO ambiguity functions were tied with harmonic analysis by using irreducible representations of Heisenberg group in [6], [46],[47]. Some applications of representations of Heisenberg group in signal processing are [2],[55],[3],[33],[59],[5]. More on Heisenberg group and harmonic analysis can be found in [25],[34].

Ambiguity functions were generalized to MIMO in [52]. As MIMO systems are capable of using separable, incoherent signals, performance in detection, estimation and beampattern design is better compared with SIMO systems. In [13], a variety of MIMO ambiguity functions were given. Here, we will use representations of Heisenberg group to investigate properties of MIMO ambiguity functions. We will use some results in [6], [46],[47] in MIMO setting. As MIMO systems use multiple signals together, resulting ambiguity function consists of multiple self and cross ambiguity functions. Therefore, the combined effect of input signals is seen in the results. This brings new properties in terms of harmonic analysis, together with some constraints on the operations we can perform.

An important point here is the localization properties of a signal in the delay Doppler domain. This issue is related with the uncertainty principles. Uncertainty principles are statements of how well can a signal and its Fourier transform counterpart can be localized on the respective time frequency domain. There are many varieties of uncertainty principles [26], [49] gives examples of them. Uncertainty relations, related with time frequency distributions can be found in [29], [26]. Uncertainty relations on the Wigner distribution which is closely related with ambiguity functions can be found in [15]. In [40], sharp uncertainty relations on ambiguity functions were given in terms of norms of the signals which create the ambiguity function. Some uncertainty relations from quantum mechanics can be found in [16],[27]. All these relations are given for traditional ambiguity functions (single input systems) or time frequency distributions. Harmonic analysis of SIMO ambiguity functions and some results regarding their algebraic structure was given in terms of representations of Heisenberg group in [6], [46],[47].

From a time frequency analysis perspective, ambiguity functions are related with Wigner distributions via a 2D-type Fourier transform (one Fourier transform and one inverse Fourier transform). Together with this relation, MIMO ambiguity functions are also useful as time frequency distributions in designing and analyzing time frequency filters. Here, we will use the MIMO ambiguity function of [52] in the time frequency distribution sense. More on use of ambiguity functions in time frequency analysis can be found in [11],[29]. We will use the uncertainty relations with the MIMO Ambiguity functions and give results for MIMO case. We note here that most uncertainty relations concerning ambiguity functions are for self ambiguity functions. MIMO systems introduce cross ambiguity functions. This situation doesn't allow direct generalization of uncertainty results. In addition to giving relations for MIMO ambiguity functions, we will give matrix norm relations on the MIMO correlation matrix.

2.1.1 Definition

We will give the definition of ambiguity functions here, more on radar signal processing can be found in [50],[37]. We may write the traditional self ambiguity functions as follows [37]

$$\chi(\tau,\nu) = \int_{-\infty}^{+\infty} u(t)u^*(t+\tau)e^{j2\pi\nu}dt \tag{2.1}$$

$u(t)$ being the signal used as input. This function shows the matched filter output where there is a delay mismatch τ and Doppler mismatch ν. In order to achieve high resolution in both range and Doppler, one desires to make the function $\chi(\tau,\nu)$ sharp around $(0,0)$ point. The cross ambiguity function is defined as follows,

$$\chi(u,v)(\tau,\nu) = \int_{-\infty}^{+\infty} u(t)v^*(t+\tau)e^{j2\pi\nu}dt \tag{2.2}$$

A very closely related time frequency distribution is the Wigner distribution which is defined as,

$$W(u,v)(t,f) = \int_{-\infty}^{+\infty} u(t+\tau/2)v(t-\tau/2)e^{-j2\pi\nu t}d\tau \tag{2.3}$$

We can write the ambiguity function in terms of representations of $H_{\mathbb{R}}$ as,

$$\langle \mathbf{u}, T(\tau,\nu)\mathbf{v}\rangle = \int_{-\infty}^{+\infty} u(t)v^*(t+\tau)e^{i2\pi\nu t}dt \tag{2.4}$$

Here, we used the operation on the second signal for notational convention. It gives conjugate of the Doppler shift when we use the first representation we presented, though we may assume here as the x_2 axis is inverted. This doesn't change any result, it merely is done for notational convention.

Two very commonly used ambiguity functions are ambiguity function of a rect signal and LFM signal, which are given in Figure 2.1, 2.2, 2.3 and 2.4. One can observe the delay and Doppler resolutions in both cases.

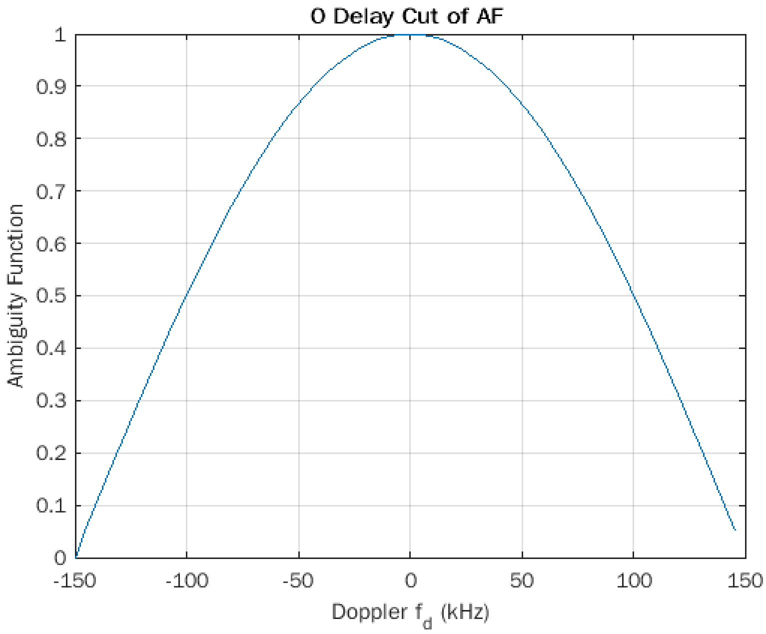

Figure 2.1: Ambiguity function of rect signal, 0 Doppler cut

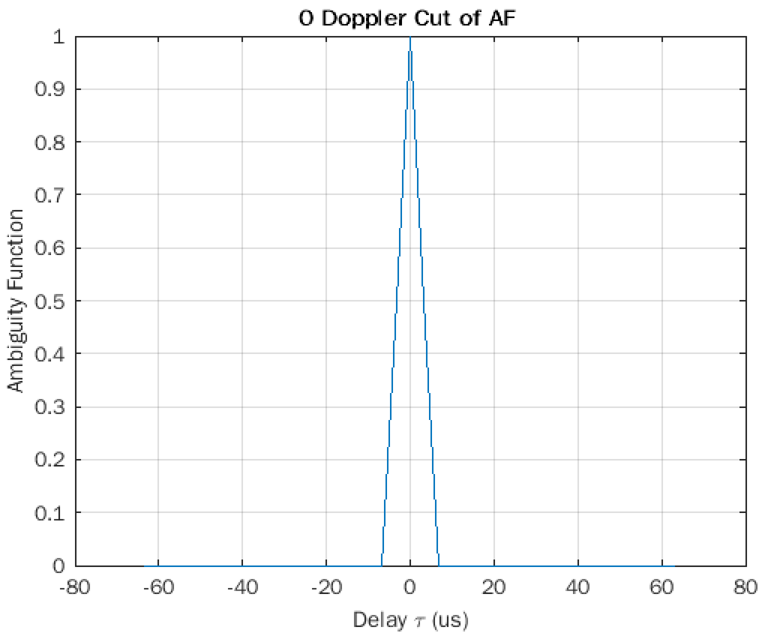

Figure 2.2: Ambiguity function of rect signal, 0 delay cut

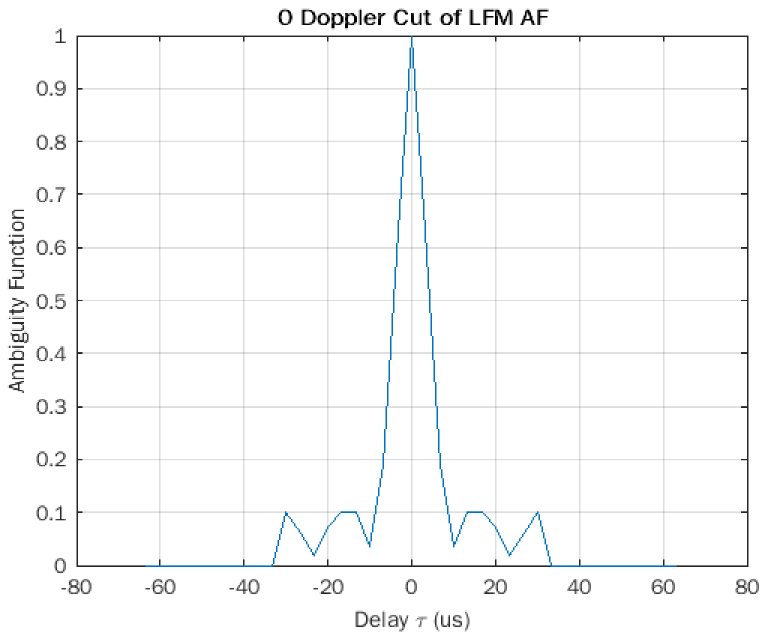

Figure 2.3: Ambiguity function of LFM signal, 0 Doppler cut

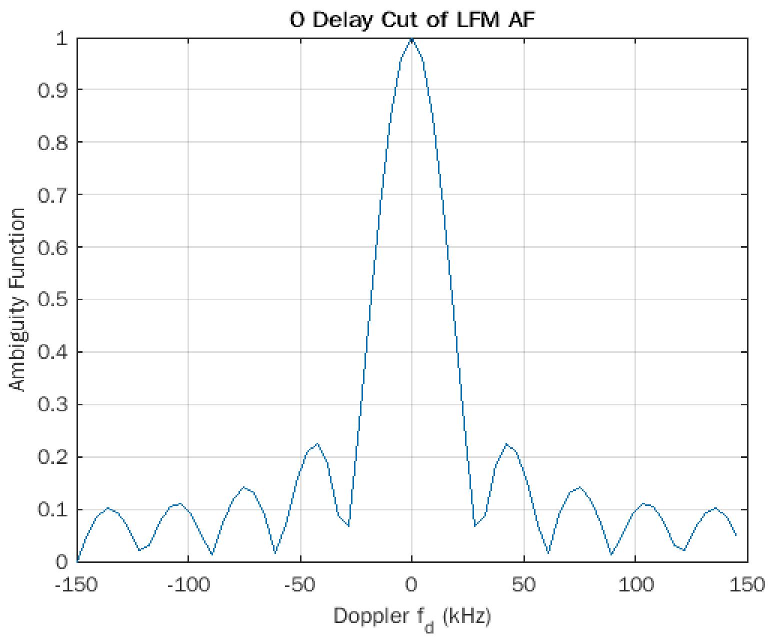

Figure 2.4: Ambiguity function of LFM signal, 0 delay cut

2.1.2 MIMO Ambiguity Functions

MIMO ambiguity functions were given by [52]. Here we will review the definiton. As we have mentioned in the introduction, [13] gave first properties in terms of waveform design. We will use the notation of [13], [37]. Assumptions on ambiguity functions change the form of the function such as in narrowband and wideband radars. We will assume that sensors are close, such that target's velocity vector has similar effect in each sensor. In addition, target is in the farfield and bandwith is narrow to allow narrowband case. Together with these assumptions, in [52], it was shown that elements of the MIMO covariance matrix are the Woodward ambiguity functions depending only on time delay and Doppler shift. We can write the MIMO correlation matrix as,

$$\mathbf{R}_{ij}(\tau,\nu) = A_{ij}(\tau,\nu) = \int_{-\infty}^{+\infty} u_i(t)u_j^*(t-\tau)e^{j2\pi\nu}dt \tag{2.5}$$

Elements $\mathbf{R}_{ij}(\tau,\nu)$ are the Woodward ambiguity functions. The MIMO ambiguity function is obtained by using the MIMO correlation matrix with the corresponding steering vectors.

$$\chi(\tau,\nu,f_s,f_s^{'}) = \sum_{m=0}^{M-1}\sum_{m'=0}^{M-1} \chi_{m,m'}(\tau,\nu)e^{i2\pi\gamma(f_s m - f_s^{'} m^{'})} \tag{2.6}$$

where $\chi_{m,m'}(\tau,\nu) = \langle u_m, T(\tau,\nu)u_{m'}\rangle$ is the cross ambiguity function. As we see, in addition to delay and Doppler, in MIMO there is the normalized spatial frequency of the target f_s. If there is no mismatch, the MIMO ambiguity function is $\chi(0,0,f_s,f_s)$. Therefore, for high resolution, we would like $\chi(\tau,\nu,f_s,f_s^{'})$ to be sharp around the point $(0,0,f_s,f_s)$. Here, we note instead of discarding x_3 term in T, the representation could be written with $x_3 = \gamma(f_s m - f_s^{'} m^{'})$ to give the relation $e^{i2\pi\gamma(f_s m - f_s^{'} m^{'})}$ with the use of representation. If we denote the new representation $T^{'}$ we have,

$$\chi(\tau,\nu,f_s,f_s^{'}) = \sum_{m=0}^{M-1}\sum_{m'=0}^{M-1} \langle u_m, T^{'}(\tau,\nu,x)u_{m'}\rangle \tag{2.7}$$

But using the x_3 term violates square integrability of ambiguity functions. Therefore, it is not very useful for our purposes, and we will not be using it.

2.2 Time Frequency Distributions

Time frequency distributions (TFDs) are used in many signal processing applications such as in telecommunications, radar, sonar, audio signal processing and biomedical areas. They are used for nonstationary signal analysis, blind source separation and time varying filter design. Examples of applications can be found in [11]. A very important issue in time frequency analysis is the uncertainty relations between time and frequency variables. It puts a constraint on how well a signal and its Fourier transform counterpart can be localized in the phase space ((t,f) domain). Studies on harmonic analysis bring useful tools to time frequency signal processing.

There are different types of uncertainty relations in the literature, a variety of examples are surveyed in [26], [49]. Uncertainty relations related with time frequency analysis can be found in [29],[10]. Implications of these relations on time frequency signal processing are given in [11]. We are interested in the relations regarding time frequency distributions. As other TFDs can be obtained by filtering the Wigner distribution, we will be mainly working on it with relations to some marginalizable TFDs. In [48], [41], [16],[27] covariance matrix of Wigner distributions was investigated from a quantum mechanical viewpoint. It was shown that, with the action of a symplectic matrix, a new Wigner distribution is created from an existing one, under certain assumptions. Moreover, [48] showed that the covariance matrix of a Wigner distribution transfroms in relation with the same symplectic matrix. This gives an invariance result on uncertainty relation. [55], [20] investigates some actions of symplectic group on ambiguity functions and Wigner distributions. In [6],[46],[47] harmonic analysis of ambiguity functions was studied in terms of the representations of the Heisenberg group. In [6], action of special linear group $SL(2,\mathbb{R})$ on the ambiguity functions was given. Ambiguity functions are related with the Wigner distributions via a symplectic Fourier transform. Here, we will use the action of $SL(2,\mathbb{R})$ on Wigner distributions to see which signals could be used to obtain Wigner distributions with unchanged uncertainty relations.

2.3 TF Preliminaries

Ambiguity functions and TFDs

In this section, we will give definitions of TFDs and ambiguity functions in relation to each other. More on the topic can be found in [11]. The Wigner Distribution of $u, v \in L_2(\mathbb{R})$ is,

$$W(u,v)(t,f) = \int_{-\infty}^{+\infty} u(t+\tau/2)v(t-\tau/2)e^{-j2\pi\nu t}d\tau \tag{2.8}$$

If $u = v$, this relation is called Wigner Distribution of u. We recall the symmetric ambiguity function for $u, v \in L_2(\mathbb{R})$ as,

$$A(u,v)(\tau,\nu) = \int_{-\infty}^{+\infty} u(t+\tau/2)v(t-\tau/2)e^{j2\pi\nu t}dt \tag{2.9}$$

for the $u = v$ case, this relation is called the self ambiguity function.

One can use the analytic signal $a(t) = u(t) + jH\{u(t)\}$ in Wigner Distributions to avoid negative frequency interference. When $a(t)$ is used the distribution is called Wigner Ville Distribution (WVD) $W_a(t,f)$. A general TFD $\rho_a(t,f)$ can be defined as

$$\rho_a(t,f) = \gamma(t,f) \underset{t,f}{**} W_a(t,f) \tag{2.10}$$

where $\gamma(t,f)$ is a 2D filter and $**$ denotes convolution in time and frequency. A TFD has time marginal property if,

$$\int_{-\infty}^{+\infty} \rho_a(t,f)df = |a(t)|^2 \tag{2.11}$$

and it has frequency marginal property if it satisfies,

$$\int_{-\infty}^{+\infty} \rho_a(t,f)dt = |A(f)|^2 \tag{2.12}$$

Another important property certain TFDs satisfy is the nonnegativity property. It simply holds if $\rho_a(t,f) \geq 0$ for all t, f. Another important type of TFD is the spectrogram. The spectrogram is defined as,

$$S_s^w(t,f) = |\int_{-\infty}^{+\infty} s(\tau)w(\tau - t)e^{-j2\pi\nu t}d\tau|^2 \tag{2.13}$$

The spectrogram satisfies the nonnegativity property, but it does not satisfy the marginal properties. As the TFDs can be written in terms of Wigner distributions by filtering, we use the following relation to realise ambiguity functions as the filter domain for TFD design. Between ambiguity functions and WVD, the following relation exists,

$$W_a(t,f) = \mathcal{F}\{\mathcal{F}^{-1}\{A_a(\tau,\nu)\}\} \tag{2.14}$$

We can define a filtered ambiguity function $\mathcal{A}(\tau,\nu)$, similar to filtering the WVD,

$$\mathcal{A}(\tau,\nu) = g(\tau,\nu)A_a(\tau,\nu) \tag{2.15}$$

Then, we can write,

$$\rho_a(t,f) = \mathcal{F}\{\mathcal{F}^{-1}\{\mathcal{A}(\tau,\nu)\}\} \tag{2.16}$$

Here, we can see that $**$ convolution in time and frequency became multiplication in delay and Doppler. Because of this, ambiguity domain is useful in designing filters. If the kernel $g(\tau,\nu)$ is in the form,

$$g(\tau,\nu) = G_1(\nu)g_2(\tau) \tag{2.17}$$

it is called a seperable kernel.

WVD can be seen as a main TFD to modify and obtain other TFDs. WVD satisfies certain plausible properties. It has time marginal and frequency marginal properties, highest resolution in (t,f) among all TFDs. However, it can take negative values and as it is not a linear transformation (bilinear), cross terms deteriorate its performance. On the other hand, spectrogram is nonnegative and has good cross term suppression. Although, it is sensitive to the window size as it increases resolution in time domain by shorter windows, while sacfrificing from the frequency domain, and vice versa. Cross Wigner distributions have applications in quantum mechanics, such as in [28],[17]. In figures 2.5 and 2.6, we see examples of WVD and STFT.

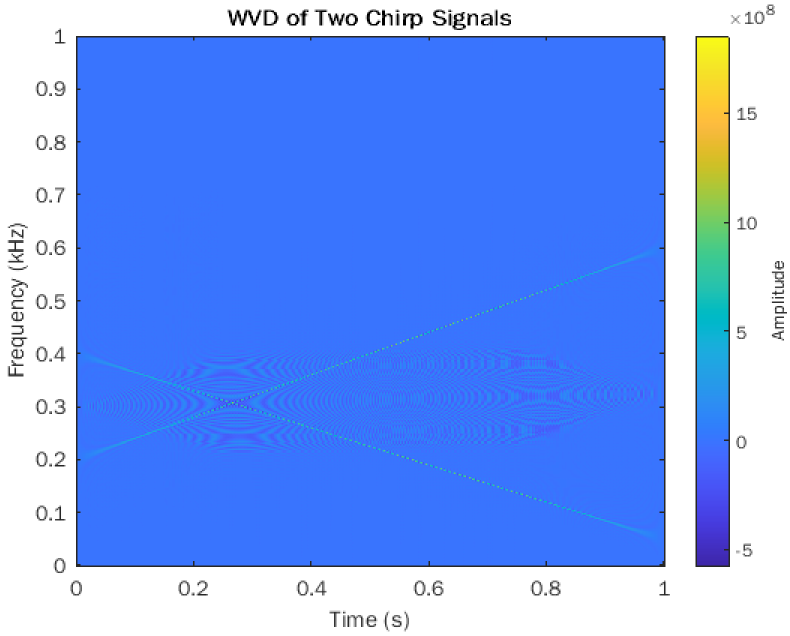

Figure 2.5: WVD of sum of two chirp signals

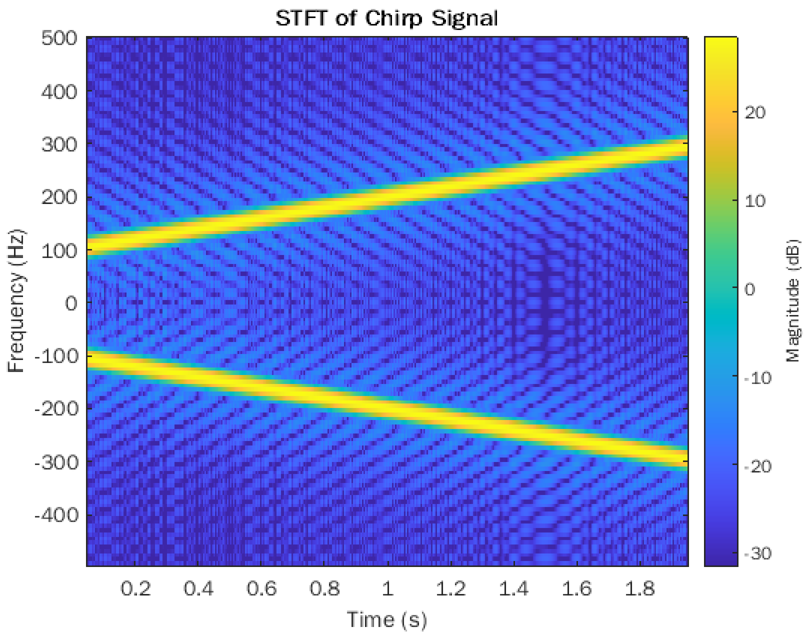

Figure 2.6: STFT of a chirp signal

A comparison of important properties of WVD and STFT is given below. While using one, the tradeoffs between these are considered. These two TFDs are used as main TFDs for design. The other TFDs are between the two, which are effected by

Table 2.1: Comparison of properties of WVD and STFT

Comparison of WVD and STFT		
Properties	WVD	STFT
Resolution	Depends on the window	Highest resolution among TFDs
Cross term suppression	Best suppression	Worst suppression
Window use	Yes	No

the choose of window and other design parameters. Table 2.1 gives a comparison of important properties of WVD and STFT.

CHAPTER 3

HARMONIC ANALYSIS PRINCIPLES

In this chapter, we will give preliminary ideas for the thesis. We will define methods we will use from harmonic analysis. Briefly, representation theory, certain important groups, uncertainty relations and some norm inequalities will be defined. We will provide necessary details for the thesis and give references where needed.

3.1 Group Theory

In this section we will define the main algebraic structure we will use throughout the thesis.

3.1.1 Definition

A group G is a set of objects with an operation which maps elements of G into nother element of G. This operation is called the group multiplication. The group with its multiplication should hold the following,

1. There exists $e \in G$ such that $\forall g \in G,\ ge = eg = g$
2. For every $g \in G$, $\exists g^{-1} \in G$, such that $gg^{-1} = g^{-1}g = e$
3. Associative law, where the identity $(gh)k = g(hk)$ is satisfied $\forall g, h, k \in G$

If, in addition, for any $g, h \in G$, $gh = hg$ holds, we say that the group is commutative, otherwise noncommutative. A subgroup H of G is a subset of G which again is a group with the same group multiplication and the identity. Next, some examples of groups which are used frequently in signal processing are given.

3.1.2 Examples of Groups

1. The real numbers $\mathbb{R}$ with addition is a commutative group. The identity of it is 0 and the inverse of $x \in \mathbb{R}$ is $-x$.

2. The integers $\mathbb{Z}$, with addition is a proper subgroup of $\mathbb{R}$. Its identity is 0 and the inverse of $x \in \mathbb{Z}$ is $-x$.

3. The general linear group $GL(n, \mathbb{R})$ is a noncommutative group where,

$$GL(n, \mathbb{R}) = \{A = (A_{ij}),\ 1 < i, j < n : A_{ij} \in \mathbb{R},\ det(A) \neq 0\} \tag{3.1}$$

4. Affine group is a subgroup of $GL(n, \mathbb{R})$ with the elements,

$$(a, b) = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}, \quad a, b \in \mathbb{R},\ a > 0 \tag{3.2}$$

   and group product

$$(a, b)(c, d) = \begin{bmatrix} ac & ad + b \\ 0 & 1 \end{bmatrix} \tag{3.3}$$

   The identity $e = (1, 0)$ and the inverse $(a, b)^{-1}$=$(1/a, -b/a)$

5. Heisenberg group is also a subgroup of $GL(n, \mathbb{R})$ defined by,

$$H_{\mathbb{R}} = \{(x_1, x_2, x_3) = \begin{bmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{bmatrix} : x_i \in \mathbb{R}\} \tag{3.4}$$

   with group product

$$(x_1, x_2, x_3)(y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1 y_2) \tag{3.5}$$

   The identity $e = (0, 0, 0)$ and the inverse $(x_1, x_2, x_3)^{-1}$=$(-x_1, -x_2, x_1 x_2 - x_3)$

$H_{\mathbb{R}}$ is non Abelian. The center of $H_{\mathbb{R}}$, C_H, whose all elements commute with each element of $H_{\mathbb{R}}$, is,

$$C_H = \{h \in H_{\mathbb{R}} : hg = gh, \forall g \in H_{\mathbb{R}}\} \tag{3.6}$$

consists of elements $(0, 0, x_3)$, $x_3 \in \mathbb{R}$.

$SL(n, \mathbb{R})$, $O(n)$, $GL(n, \mathbb{C})$, $SL(n, \mathbb{C})$,$U(n)$,$SO(n)$,$SE(n)$ can also be given as some non Abelian group examples. These groups similarly are matrix groups which have their specified definitons and have the group operation as matrix multiplication. The operations are realized in real life as rotation for $O(n)$, $SO(n)$ and as a complex set rotation for $U(n)$, $SU(n)$.These groups are used in harmonic analysis studies. We will relate ambiguity functions with Heisenberg groups in the next sections and use its properties. Throughout this text we will use real valued three dimensional Heisenberg Group.

Group $GL(V)$

Let V be a finite dimensional vector space. $GL(V)$ is the set of all invertible linear transformations of V onto V, T: $V \to V$

Group product: $T_1, T_2 \in GL(V)$ (product of linear transformations)

$$(T_1T_2)\mathbf{v} = T_1(T_2\mathbf{v}) \tag{3.7}$$

Identity: There exist an identity of the group $GL(V)$, e, such that

$$e\mathbf{v} = \mathbf{v} \tag{3.8}$$

Inverse: Inverse of $T \in GL(V)$ is

$$T\mathbf{v} = \mathbf{w} \Rightarrow T^{-1}\mathbf{w} = \mathbf{v} \tag{3.9}$$

$GL(V)$ will be used frequently in constructing representations.

3.1.3 Topological Groups

A type of groups which we will use frequently is topological groups. A topological group is a group, which has a topology, with respect to which the group multiplication is continuous.

Topological groups are used commonly in engineering applications. $\mathbb{R}$ with the usual topology, $\mathbb{R}^n$ under addition and the group $\mathbb{T}$ of complext numbers with modulus 1 , which is the unit circle, can be given as commutative topological group examples. $GL(n, \mathbb{R})$ as the subspace of $\mathbb{R}^{n\times n}$ is a noncommutative topological group example. The groups given here are Lie groups. A Lie group is a group which is also a finite dimensional manifold. Which means that the group multiplication and inversion are smooth maps in addition to being continuous.

3.1.4 Haar Measure

Let G be a locally compact group. A left invariant Borel measure μ on G satisfying,

1. $\mu(xE) = \mu(E)$ for every $x \in G$ and every Borel set $E \subset G$
2. $\mu(K) < \infty$ for all compact $K \subseteq G$
3. $\mu(L) > 0$ for every non-empty open subset $L \subseteq G$

Haar measures are commonly used in engineering problems where an invariance in certain shift exists. Another commonly used measure is the Lebesgue measure. Lebesgue measure is Haar measure on $\mathbb{R}^n$, and it is left and right invariant.

3.1.5 Representations of $H_{\mathbb{R}}$

In this section, we will give a review of representation theory without going into too much detail. Ideas we present here, their proofs and more on representations of Heisenberg group can be found in harmonic analysis books which cover noncommutative harmonic analysis topics, a good reference is [24]. Especially representations of Heisenberg group in relation with ambiguity functions can be found in [6], [46],[47].

In [46], ideas are constructed starting from group theory and the relationship with wavelets is given (via representation of affine group).

A representation of a group G, with its representation space V is a homomorphism $T : g \mapsto T(g)$ of G into $GL(V)$. $T : G \to GL(V)$. $GL(V)$ is the set of all linear transformations from V to V. We will use representations as operators here to keep familiarity with operations in signal processing applications. Matrix representation could be obtained by use of an orthonormal basis. We will be mainly working on $L_2(\mathbb{R})$, which is the space of signals with finite energy. A unitary operator on $L_2(\mathbb{R})$ is a linear mapping satisfying

$$\langle T\mathbf{v}, T\mathbf{w}\rangle = \langle \mathbf{v}, \mathbf{w}\rangle \qquad \forall \mathbf{v}, \mathbf{w} \in V \tag{3.10}$$

A very important class of representations is irreducible representations. A representation is reducible if there exists a subspace W of V such that $T(g)w \in W$ for every $g \in G$, $w \in W$. Otherwise it is irreducible. There are different constructions of Heisenberg groups, here we will use the following,
Heisenberg group is a group of matrices

$$H_{\mathbb{R}} = \{(x_1, x_2, x_3) = \begin{bmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{bmatrix} : x_i \in \mathbb{R}\} \tag{3.11}$$

with group product

$$(x_1, x_2, x_3)(y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1 y_2) \tag{3.12}$$

Representations of Heisenberg group can be found by induction. Construction of representations can be found in [24]. Here, we will not go into details of constructing representations, rather, we will use them. We will use the following relation in writing ambiguity functions. The Schroedinger representation of $H_{\mathbb{R}}$ is,

$$T(x_1, x_2, x_3)f(t) = e^{i2\pi(x_3 + x_2 t)} f(t + x_1) \tag{3.13}$$

T is an irreducible representation of $H_{\mathbb{R}}$ on $L_2(\mathbb{R})$. In ambiguity functions, we won't be using the element x_3, we will use $T(x_1, x_2, 0)$. We will construct ambiguity functions from irreducible representations on the next subsection. Now, we will mention some properties of representations. Representations are closely related with positive definite functions. A positive definite function is a function which satisfies,

$$\sum_{ij=1}^{N} \rho(x_j^{-1}x_i)\bar{c}_j c_i \geq 0 \tag{3.14}$$

for any set of complex numbers $c_1, c_2, ..., c_N$, and elements x_i of the group G. Let T be a unitary representation of G on $L_2(\mathbb{R})$ (or any Hilbert space $\mathcal{H}$). Positive definite functions can be written via representations,

$$p = \langle T\mathbf{v}, \mathbf{v}\rangle \tag{3.15}$$

These relations will be used on ambiguity functions in the next section.

If T is an irreducible representation, then $\{T(x)\mathbf{u}\}$ spans a dense subspace of $L_2(\mathbb{R})$. An important property of irreducible representations is that, positive definite functions created by an irreducible representation are indecomposable. A positive definite function p is indecomposable if any other positive definite function p', where $p - p'$ is also positive definite, is a scalar multiple of p.

3.1.6 The Special Linear Group $SL(2,\mathbb{R})$

In [6], action of the special linear group on SIMO ambiguity functions was given. In section 4, we will use this relations in MIMO setting and relate to signals of interest in time frequency analysis. $SL(2,\mathbb{R})$ is the group of 2×2 real matrices, with determinant 1. It is called the special linear group.

$$SL(2,\mathbb{R}) = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad a,b,c,d \in \mathbb{R}\} \tag{3.16}$$

We will look at the following generators of $SL(2,\mathbb{R})$,

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{3.17}$$

$$t(a) = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}, \quad a \in \mathbb{R} \tag{3.18}$$

$$m(b) = \begin{bmatrix} b & 0 \\ 0 & 1/b \end{bmatrix}, \quad b > 0 \tag{3.19}$$

3.2 Uncertainty Relations Preliminaries

3.2.1 Uncertainty relations

Heisenberg's uncertainty is an inequality about how localized can a signal and its Fourier transform can be on the time frequency plane. Here, we will give the Heisenberg inequality. More on the topic can be found in [26]. We will keep the notation and terminology parallel with signal processing applications.

In order to define Heisenberg's uncertainty, we may first define the variance. Suppose p is a probability measure on $\mathbb{R}$.

$$\sigma_x^2 = Var(x) = \int (x - m_x)^2 p_x(x) dx \tag{3.20}$$

where m_x is the mean,

$$m_x = \int x p_x(x) dx \tag{3.21}$$

The variance gives how much a signal spreads around its mean. By using this, we can construct Heisenberg's inequality. Proof of Heisenberg's inequality can be found in many books related with harmonic analysis and time frequency analysis. We will give a simple proof, similar to [26].

3.2.2 Heisenberg's Inequality

Let $x \in L_2(\mathbb{R})$ with energy $\|x\|_2^2$, then assuming zero mean in both cases,

$$\left(\int t^2 |x(t)|^2 dt \right)^{1/2} \left(\int f^2 |X(f)|^2 df \right)^{1/2} \geq \frac{\|x\|_2^2}{4\pi} \tag{3.22}$$

Proof. First, we note the relation bewtween the Fourier transfrom of derivative of a signal and signal's Fourier transform counterpart,

$$\mathcal{F}\{x'(t)\} = i2\pi f X(f) \tag{3.23}$$

We will use this relation in the proof. If we take the derivative of magnitude squared of the signal $x(t)$ we obtain,

$$(|x(t)|^2)' = (xx^*)' = 2Re\{xx'^*\} \tag{3.24}$$

Using integration by parts we have,

$$2Re\int_a^b tx(t)x'^*(t)dt = t|x(t)|^2|_a^b - \int_a^b |x(t)|^2 dt \tag{3.25}$$

Here, we see $tx(t), x'^*(t)$ terms together. As we assume the integrals in the Heisenberg inequality is finite, $tx(t), x'^*(t)$ are elements of $L_2(\mathbb{R})$. The terms, $a|x(a)|$ and $b|x(b)|$ are finite as $x(t), tx(t), x'^*(t)$ are elements of $L_2(\mathbb{R})$. But if they are finite they have to be zero, as otherwise $|x(t)|^2$ and t^{-1} should be comparable in limits $t \to a$ and $t \to b$. This contradicts with $x(t)$ being in $L_2(\mathbb{R})$. Therefore, we have,

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = -2Re\int_{-\infty}^{\infty} tx(t)x'^*(t)dt \tag{3.26}$$

If we apply Schwarz inequality to the right hand side,

$$(2Re\int_{-\infty}^{\infty} tx(t)x'(t)dt)^2 \leq 4\int_{-\infty}^{\infty} t^2|x(t)|^2 dt \int_{-\infty}^{\infty} |x^{'}(t)|^2 dt \tag{3.27}$$

We can apply Plancherel theorem to the last integral,

$$\int_{-\infty}^{\infty} |x^{'}(t)|^2 dt = 4\pi^2 \int_{-\infty}^{\infty} f^2|X(f)|^2 df \tag{3.28}$$

We also have,

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \|x\|_2^2 \tag{3.29}$$

Combining the results we have,

$$\|x\|_2^4 \leq 16\pi^2 \int t^2|x(t)|^2 dt \int f^2|X(f)|^2 df \tag{3.30}$$

Taking the square root, the uncertainty relation follows,

$$\left(\int t^2|x(t)|^2 dt\right)^{1/2} \left(\int f^2|X(f)|^2 df\right)^{1/2} \geq \frac{\|x\|_2^2}{4\pi} \tag{3.31}$$

□

This inequality says that, a function and its Fourier dual cannot be sharply localized in (t, f). The inequality holds with equation if and only if $x(t)$ is a Gaussian. That is because Schwarz inequality hold with equality iff $x'(t) = \alpha(tx(t))$ for $\alpha \in \mathbb{R}^+$, giving $x(t) = \beta e^{-\frac{\alpha}{w}t^2}$. A different proof of this relation, using infinitesimal generators of the Heisenberg algebra can be found in [29].

3.2.3 Uncertainty Relations on Ambiguity Functions

Here, we will give some uncertainy relaitons on ambiguity functions which can be found in literature or simply derived. If we look at the MIMO ambiguity function given in [13] with a shifted form.

$$A(\tau, \nu, f_s, f_s^{'}) = \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} A(u_m, u_{m'})(\tau, \nu) e^{i2\pi\gamma(f_s m - f_s^{'} m^{'})} \tag{3.32}$$

One can note that, unlike SIMO ambiguity functions, in MIMO, there are cross terms and summation of them. Cross ambiguity functions and cross Wigner distributions can be related with STFT in time frequency analysis. They also have applications in quantum mechanics [28],[17]. Uncertainty relations for ambiguity functions can be found by estimating the L^p norms of $A(u, v)(\tau, \nu)$.
Firstly, one can obtain the relation

$$\|A(u, v)\|_2 = \|u\|_2 \|v\|_2 \tag{3.33}$$

proof of this relation can be found in radar signal processing books. In [40] there are a variety of relations between the norms of ambiguity functions and the norms of the signals creating the ambiguity functions. Here, we will repeat the ones useful for our purpose. From the definition of ambiguity functions,

$$|A(u, v)| \leq \|u\|_2 \|v\|_2 \tag{3.34}$$

and more generally, for $\frac{1}{a} + \frac{1}{b} = 1$, Hölder's inequality gives, for $u \in L^a$, $v \in L^b$,

$$|A(u, v)| \leq \|u\|_a \|v\|_b \tag{3.35}$$

Moreover, by the Schwarz inequality,

$$\|A(u,v)\|_\infty \leq \|u\|_2\|v\|_2 \tag{3.36}$$

Combining the relations above, we can obtain the following uncertainty relation similar to Donoho's uncertainty relation [22], [29], [26],

$$\int\int_E |A(u,v)|^2 \leq \|A(u,v)\|_\infty^2|E| \leq \|A(u,v)\|_2^2|E| \tag{3.37}$$

This relation is sometimes called the local uncertainty relation. It states that the measure of the set $|E|$ is lower bounded by $\int\int_E |A(u,v)|^2$. This means that we cannot localize $A(u,v)$ in a small set on (τ,ν) plane. In [40], the relation $\int\int |A(u,v)|^p$ is considered and upper and lower bounds were found for $p \geq 2$ and $1 \leq p \leq 2$ cases.

$$\int\int_{\mathbb{R}^2} |A(u,v)|^p d\tau d\nu \leq \frac{2}{p}\|u\|_2^p\|v\|_2^p \quad for\ p \geq 2 \tag{3.38}$$

$$\int\int_{\mathbb{R}^2} |A(u,v)|^p d\tau d\nu \geq \frac{2}{p}\|u\|_2^p\|v\|_2^p \quad for\ 1 \leq p \leq 2 \tag{3.39}$$

This relation is a strong inequality. It has a technical proof, we will omit it here, for the proof one can refer to [40]. Lieb [40] also shows that these bounds hold with equality if u and v are a matched Gaussian pair(u and v has the form $e^{-\alpha t^2+\beta t+\theta}$ with same $\alpha > 0$). Inequalities with norms $\|u\|_a^p\|v\|_b^p$, with $\frac{1}{a}+\frac{1}{b}=1$, were also given in the same paper.

All the bounds in this chapter were given for single input ambiguity functions. In the next chapter, we will use these in the MIMO case.

CHAPTER 4

HARMONIC ANALYSIS OF MIMO AMBIGUITY FUNCTIONS

4.1 Properties of MIMO Ambiguity Functions

We will relate MIMO ambiguity functions with irreducible unitary representations of Heisenberg group and give a varierty of properties in various situations. We will use some results from [6],[46],[47] in MIMO ambiguity functions. We note that relations in these references were given for SIMO case. Therefore, they were about self ambiguity of a single signal or cross ambiguity of two signals. We are using M self ambiguity functions and $M(M-1)$ cross ambiguity functions, and we are investigating in their combined effect. Throughout the paper, properties without prime are for traditional ambiguity functions and properties with prime on them are for MIMO ambiguity functions.

Property 4.1

For $\chi(u_m, u_{m'}) \in L_2(\mathbb{R}^2)$ and $u_m, u_{m'} \in L_2(\mathbb{R})$

$$\|\chi_{m,m'}(\tau,\nu)\|_2^2 = \|u_m\|^2\|u_{m'}\|^2 \tag{4.1}$$

Proof. Fix τ,

$$\|\chi_{m,m'}(\tau,\nu)\|_2^2 = \int\int_{-\infty}^{+\infty} |\mathcal{F}\{u_m(t)u_{m'}^*(t+\tau)\}|d\nu d\tau \tag{4.2}$$

By Plancherel theorem (sometimes called generalized Parseval theorem in signal processing books),

$$= \int\int_{-\infty}^{+\infty} |u_m(t)u_{m'}(t+\tau)|^2 dt d\tau = \|u_m\|^2\|u_{m'}\|^2 \tag{4.3}$$

$\square$

This property means that, norm of $\chi_{m,m'}(\tau,\nu)$ is equal to waveforms' energies and it is a constant, only depends on waveforms' energy. The following property takes this property to the MIMO case. A very similar property for unit energy signals was given in [13],

Property 4.1'

We will assume γ is an integer as in [13],

$$\int_0^1\int_0^1\int\int_{-\infty}^{+\infty}|\chi(\tau,\nu,f_s,f_s^{'})|^2 d\tau d\nu df_s df_s^{'} = \sum_{m=0}^{M-1}\sum_{m'=0}^{M-1}\|u_m\|^2\|u_{m'}\|^2 \tag{4.4}$$

Proof. Again we will use Parseval's relation

$$\int_0^1\int_0^1\int\int_{-\infty}^{+\infty}|\chi(\tau,\nu,f_s,f_s^{'})|^2 d\tau d\nu df_s df_s^{'} = \tag{4.5}$$

$$\int\int_{-\infty}^{+\infty}\int_0^1\int_0^1 |\sum_{m=0}^{M-1}\sum_{m'=0}^{M-1}\chi_{m,m'}(\tau,\nu)e^{i2\pi\gamma(f_s m - f_s^{'}m^{'})}|^2 df_s df_s^{'} d\tau d\nu \tag{4.6}$$

By a change of variables,

$$1/\gamma^2\int\int_{-\infty}^{+\infty}\int_0^{\gamma}\int_0^{\gamma} |\sum_{m=0}^{M-1}\sum_{m'=0}^{M-1}\chi_{m,m'}(\tau,\nu)e^{i2\pi(f_s m - f_s^{'}m^{'})}|^2 df_s df_s^{'} d\tau d\nu \tag{4.7}$$

$$= \int\int_{-\infty}^{+\infty}\sum_{m=0}^{M-1}\sum_{m'=0}^{M-1}|\chi_{m,m'}(\tau,\nu)|^2 dt d\tau = \tag{4.8}$$

$$= \sum_{m=0}^{M-1}\sum_{m'=0}^{M-1}\|\chi_{m,m'}(\tau,\nu)\|^2 = \sum_{m=0}^{M-1}\sum_{m'=0}^{M-1}\|u_m\|^2\|u_{m'}\|^2 \tag{4.9}$$

$\square$

Here, we see that the energy of MIMO ambiguity is directly determined by energies of input signals. The following property gives the inner product relation between two ambiguity functions.

Property 4.2

Let $u_1, u_2, u_3, u_4 \in L_2(\mathbb{R})$, then

$$\langle \chi(u_1, u_3), \chi(u_2, u_4) \rangle_{L_2(\mathbb{R}^2)} = \langle u_1, u_2 \rangle \langle u_3, u_4 \rangle \tag{4.10}$$

Proof. If $\chi(u_i, u_j) \in L_2(\mathbb{R})$ for $i = 1, 2, 3, 4$

$$\int\int_{-\infty}^{+\infty} \chi(u_1, u_3)\chi(u_2, u_4) d\tau d\nu \tag{4.11}$$

$$= \int\int_{-\infty}^{+\infty} \mathcal{F}\{u_1(t)u_3^*(t+\tau)\}\mathcal{F}\{u_2(t)u_4^*(t+\tau)\} d\nu d\tau \tag{4.12}$$

Again, by similar arguments to Property 4.1,

$$= \langle u_1, u_2 \rangle \langle u_3, u_4 \rangle \tag{4.13}$$

□

This is a frequently used identity. Using this in definition of ambiguity functions,

$$\int\int_{-\infty}^{+\infty} \langle u_1, Tu_3 \rangle \langle u_2, Tu_4 \rangle d\tau d\nu = \langle u_1, u_2 \rangle \langle u_3, u_4 \rangle \tag{4.14}$$

Miller [46] stated that, if we use an orthonormal basis $\{u_i\}$ of $L_2(\mathbb{R})$ in constructing ambiguity functions, they form an orthonormal basis for $L_2(\mathbb{R}^2)$. In order for ambiguity functions to be an orthonormal basis of $L_2(\mathbb{R}^2)$, they need to be dense in $L_2(\mathbb{R}^2)$. To show that ambiguity functions are dense in $L_2(\mathbb{R}^2)$, it is enough to show that $\langle \chi(u_i, u_j)(\tau, \nu), h(\tau, \nu) \rangle_{L_2(\mathbb{R}^2)} = 0$ for all i, j implies $h(\tau, \nu) = 0$ almost everywhere. A simple proof of this can be found in [6]. The normality part here is obtained by using signals with unit norm. We may drop the normality here. We note that by similar arguments Wigner distributions can be shown to hold the same relation. This relation allows us to write signals in $L_2(\mathbb{R}^2)$ with basis expansions. This can be used in time frequency detection, estimation of LTV or nonstationary systems and filtering applications.

As we use a combination of different signals in MIMO ambiguity functions, we are actually using M^2 elements of the orthogonal basis of $L_2(\mathbb{R}^2)$. This relation, together with MIMO Wigner distributions can be used in time frequency signal processing problems.

Property 4.2'

Let $u_i, v_j \in L_2(\mathbb{R})$ for all i, j and let $\chi_u(\tau, \nu, f_s, f_s^{'})$ and $\chi_v(\tau, \nu, f_s, f_s^{'})$ denote the ambiguity function from waveforms u_i and v_j respectively, then

$$\langle \chi_u(\tau, \nu, f_s, f_s^{'}), \chi_v(\tau, \nu, f_s, f_s^{'}) \rangle = \tag{4.15}$$

$$= \sum_{m,m^{'}} \sum_{n,n^{'}} \langle u_m, v_n \rangle \langle v_{n^{'}}, u_{m^{'}} \rangle e^{i2\pi\gamma[(f_s m - f_s^{'} m^{'}) - (f_s n - f_s^{'} n^{'})]} \tag{4.16}$$

Proof. Proof can be done by inserting Property 4.2 in this relation

$$\langle \chi_u(\tau, \nu, f_s, f_s^{'}), \chi_v(\tau, \nu, f_s, f_s^{'}) \rangle = \tag{4.17}$$

$$= \left(\sum_{m,m^{'}} \chi_{m,m^{'}}(\tau, \nu) e^{i2\pi\gamma(f_s m - f_s^{'} m^{'})} \right) \left(\sum_{n,n^{'}} \chi_{n,n^{'}}(\tau, \nu) e^{i2\pi\gamma(f_s n - f_s^{'} n^{'})} \right)^* \tag{4.18}$$

$$= \sum_{m,m^{'}} \sum_{n,n^{'}} \chi_{m,m^{'}}(\tau, \nu) \chi_{n,n^{'}}(\tau, \nu) e^{i2\pi\gamma(f_s m - f_s^{'} m^{'})} e^{i2\pi\gamma(f_s n - f_s^{'} n^{'})} \tag{4.19}$$

$$= \sum_{m,m^{'}} \sum_{n,n^{'}} \langle u_m, v_n \rangle \langle v_{n^{'}}, u_{m^{'}} \rangle e^{i2\pi\gamma[(f_s m - f_s^{'} m^{'}) - (f_s n - f_s^{'} n^{'})]} \tag{4.20}$$

□

Here, we again see that if $\{u_i\}\{v_j\}$ together form an orthonormal basis for $L_2(\mathbb{R})$, and if we fix $f_s, f_s^{'}$, then $\chi_u(\tau, \nu, f_s, f_s^{'})$ is an orthonormal basis in $L_2(\mathbb{R}^2)$. That they are dense in $L_2(\mathbb{R}^2)$ follows from the same argument for Property 4.2. This relations can be used for practical waveform design in integrated sensing and communications and localization applications. Their orthogonality relations can be used for creating a time frequency basis using all signals together. Here, the tradeoff is using many signals from many sensors and achieving an orthonormal system with more control on the time frequency axis. In the next section, effect of this signals on the time frequency plane clearly. They can be used for comparison with practical waveforms in the radar and integrated sensing and communication literature. We may look at the following case of two MIMO ambiguity functions created by using all orthonormal signals.

Property 4.2"

If $\{u_i\}\{v_j\}$ together is an orthonormal basis of $L_2(\mathbb{R})$, then the inner product of two MIMO ambiguity functions created with these signals is,

$$\sum_{m,m'}\sum_{n,n'}\langle u_m, v_n\rangle\langle v_{n'}, u_{m'}\rangle e^{i2\pi\gamma[(f_s m - f_s' m') - (f_s n - f_s' n')]} = M^2 \tag{4.21}$$

Proof. If we use the orthogonality of u_i, v_j,

$$\langle u_m, v_n\rangle\langle v_{n'}, u_{m'}\rangle = \delta_{mn}\delta_{m'n'} \tag{4.22}$$

$$\sum_{m,m'}\sum_{n,n'}\delta_{mn}\delta_{m'n'} e^{i2\pi\gamma[(f_s m - f_s' m') - (f_s n - f_s' n')]} = \sum_{m,m'}\sum_{n,n'} e^{i0} = M^2 \tag{4.23}$$

□

Properties 4.3, 4.4 and 4.5 are given in [6],[46],[47], we will repeat them here and use them in MIMO case. The following property uses positive definiteness relation of representations mentioned in section 2 in ambiguity functions.

Property 4.3

$\chi(\tau,\nu)$ is a positive definite function.

Proof. T is a unitary representation of the Heisenberg group and $\chi(u,v)(\tau,\nu) = \langle T(x)u, u\rangle$. Therefore, from the definition in Section 2, $\chi(\tau,\nu)$ is a positive definite function. □

From this property, we see that self ambiguity functions are positive definite as a result of representations. An intereseting point we can investigate about MIMO ambiguity functions is the no spatial frequency mismatch part of the function's support, which is the part corresponding to (τ,ν,f_s,f_s). As there are cross terms in MIMO ($u_m \neq u_{m'}$), rather than self terms, we are interested in finding cases where we can find self terms or cases where $u_m = u_{m'}$. In the next property, we will look at self terms by taking the integral on the no spatial frequency mismatch part similar to [13]. But we will still have the delay Doppler variables, we won't look at the identity point.

Property 4.3'

If γ is an integer $\int_0^1 \chi(\tau,\nu,f_s,f_s)df_s$ is a positive definite function

Proof.

$$\int_0^1 \sum_{m=0}^{M-1}\sum_{m'=0}^{M-1} \chi_{m,m'}(\tau,\nu)e^{i2\pi\gamma(f_s m - f_s m')}df_s = \tag{4.24}$$

$$= \sum_{m=0}^{M-1}\sum_{m'=0}^{M-1} \chi_{m,m'}(\tau,\nu)\delta_{m,m'} = \sum_{m=0}^{M-1} \chi_{m,m}(\tau,\nu) \tag{4.25}$$

Last equation follows as delta will only leave the self terms. By this integration on no spatial frequency mismatch part, self ambiguity terms are obtained. In Property 4.3, we have shown that self ambiguity functions $\chi_{m,m}$ are positive definite. Therefore, $\sum_{m=0}^{M-1} \chi_{m,m}(\tau,\nu)$ is also a positive definite function. Here we see that $\int_0^1 \chi(\tau,\nu,f_s,f_s)df_s$ is positive definite. □

Here, we note that the relation $\sum_{m=0}^{M-1} \chi_{m,m}(\tau,\nu)$ is equal to the trace of the MIMO correlation matrix $\mathrm{Tr}(\mathbf{R}(\tau,\nu))$. Therefore, the result holds for the trace of the correlation matrix too. Next two properties use positive definiteness to give a constraint on signals creating the same ambiguity function. We will give a proof similar to that in [6] for Property 4.4 as it is simpler. A different proof, with more comment on the algebraic structure can be found in [46], we will not be using the proof, we give it here for completeness.

Property 4.4

If $u, v \in L_2(\mathbb{R})$ and $\chi(u,u) = \chi(v,v)$, then

$$u = \lambda v \tag{4.26}$$

for a constant $\lambda \in \mathbb{C}$., with $|\lambda| = 1$

Proof. As given in section 2, the span of $\{T_x u\}$ is dense in $L_2(\mathbb{R}^2)$ for an irreducible T. Let $x, y, z \in H_{\mathbb{R}}$, define S such that, $S(T_x u) = T_x v$. We have $\langle T_x u, u\rangle = \langle T_x v, v\rangle$ then, $\langle T_y u, T_x u\rangle = \langle T_y v, T_x v\rangle$ holds for all $x, y \in H_{\mathbb{R}}$. Suppose, $T_y u = T_z u$, $\langle T_y v, T_x v\rangle = \langle T_y v, T_x v\rangle$ for all $x \in H_{\mathbb{R}}$

As $\{T_x v\}$ spans a dense subspace of $L_2(\mathbb{R})$, $T_y v = T_x v$ also holds. Therefore, $S : \{T_x u\} \to \{T_x v\}$ is a unitary operator of $L_2(\mathbb{R})$ satisfying $ST(x) = T(x)S$ for all $x \in H_{\mathbb{R}}$. As a result, we have $ST(x)S^{-1} = T(x)$ and $S = \lambda I$, $|\lambda| = 1$, which implies the result we stated. □

Property 4.5

If $\chi(u_1,u_1) = \chi(u_2,u_2) + \chi(u_3,u_3)$ for $u_1, u_2, u_3 \in L_2(\mathbb{R})$ then $u_2 = \alpha u_3$ for $\alpha \in \mathbb{C}$.

Proof. Let $p_i = \chi(u_i,u_i) = \langle u_i, Tu_i\rangle$, then we know that $p_i(x)$ is a positive definite function by the arguments in Property 4.3. As $\chi(u_1,u_1) = \chi(u_2,u_2) + \chi(u_3,u_3)$, we have $p_1 = p_2 + p_3$. Therefore, p_1 dominates both p_2 and p_3. As we mentioned in section 2, positive definite functions constructed by irreducible representations are indecomposable, which implies that $p(x) = \langle u, Tu\rangle$ is indecomposable. As a result, the positive definite functions are related as, $p_1 = \beta_2^2 p_2 = \beta_3^2 p_3$, and $\langle u_1, Tu_1\rangle = \langle \beta_2 u_2, T\beta_2 u_2\rangle = \langle \beta_3 u_3, T\beta_3 u_3\rangle$. Similar to Property 4.4, we have that $\beta_2 u_2 = \lambda\beta_3 u_3$, combining terms, we obtain $\alpha = \beta_2/\beta_3\lambda$. Therefore, sum of two ambiguity functions is again an ambiguity function iff $u_2 = \alpha u_3$ □

In the next property, we will use Property 4.5 on MIMO ambiguity functions.

Property 4.5'

If γ is an integer,

$$\int_0^1 \chi(\tau, \nu, f_s, f_s) df_s \tag{4.27}$$

Can be interpreted as an ambiguity function iff $u_i = \lambda_{ij} u_j$, $|\lambda_{ij}| = 1$ for all $i, j = 0, 1, ..., M-1$

Proof. Proof of this property uses the same arguments with that of Property 4.5'

$$\int_0^1 \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \chi_{m,m'}(\tau, \nu) e^{i2\pi\gamma(f_s m - f_s m')} df_s = \tag{4.28}$$

$$= \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \chi_{m,m'}(\tau, \nu) \delta_{m,m'} = \sum_{m=0}^{M-1} \chi_{m,m}(\tau, \nu) \tag{4.29}$$

We can denote $\sum_{m=0}^{M-1} \chi_{m,m}(\tau, \nu)$ as p_A. We know that $\chi_{m,m}$ are positive definite and indecomposable and p_A dominates all $\chi_{m,m}$. Therefore, p_A is an ambiguity function iff $u_i = \lambda_{ij} u_j$, $|\lambda_{ij}| = 1$ for all $i, j = 0, 1, ..., M-1$

□

Again, we can see that the result holds for the trace of correlation matrix $\mathrm{Tr}(\mathbf{R}(\tau, \nu))$. This means that, the sum of self ambiguity functions can be treated as a single ambiguity function iff input signals satisfy $u_i = \lambda_{ij} u_j$, $|\lambda_{ij}| = 1$ for all $i, j = 0, 1, ..., M-1$. This may be seen as a reduction in the number of terms by $M-1$ on the trace of correlation matrix.

4.2 Symmetry Properties of MIMO Ambiguity Functions

Here, we will give the actions of the group $SL(2, (\mathbb{R}))$ on ambiguity functions. The actions can be realized as $\chi(u, v)(M(\tau, \nu)) = \chi(u, v)(\tau, \nu) \circ M$, where M is an

2×2 invertible matrix with determinant 1. We will observe which signal creates the signal after transform. This will help to realise the set of signals and corresponding ambiguity functions that can be obtained via the combined actions of invertible 2×2 matrices.

Property 4.6

Let $u, v \in L_2(\mathbb{R})$, then

$$\chi(u,v)(\tau,\nu) \circ J = \chi(\mathcal{F}\{u\}, \mathcal{F}\{v\})(\tau,\nu)e^{j2\pi\nu\tau} \tag{4.30}$$

Proof. We will use definition of ambiguity functions

$$\chi(u,v)(\tau,\nu) = e^{j2\pi\nu\tau}\langle u(t), v(t+\tau)e^{j2\pi(t+\tau)}\rangle \tag{4.31}$$

$$= e^{j2\pi\nu\tau}\langle \mathcal{F}\{u\}(\xi), \mathcal{F}\{v\}(\xi-\nu)e^{j2\pi\tau(\xi-\nu)}\rangle \tag{4.32}$$

$$= e^{j2\pi\nu\tau}\chi(\mathcal{F}\{u\}, \mathcal{F}\{v\})(-\nu,\tau) \tag{4.33}$$

Therefore,

$$\chi(\mathcal{F}\{u\}, \mathcal{F}\{v\})(\tau,\nu) = e^{-j2\pi\nu\tau}\chi(u,v)(\nu,-\tau) \tag{4.34}$$

$$= e^{j2\pi\nu\tau}\chi(u,v)(\tau,\nu) \circ J \tag{4.35}$$

The property follows □

Next, we will use this in MIMO ambiguity functions,

Property 4.6'

If we fix f_s, f_s' and denote $\chi_u(\tau,\nu,f_s,f_s')$ the MIMO ambiguity function with the set of waveforms $\{u_m\}$,

$$\chi_u(\tau,\nu,f_s,f_s') \circ J = \chi_{\mathcal{F}\{u\}}(\tau,\nu,f_s,f_s')e^{j2\pi\nu\tau} = \chi_u(\nu,-\tau,f_s,f_s') \tag{4.36}$$

Proof. We will use definition of ambiguity functions

$$\chi_{\mathcal{F}\{u\}}(\tau,\nu,f_s,f_s') = \sum_{m=0}^{M-1}\sum_{m'=0}^{M-1} \chi(\mathcal{F}\{u_m\},\mathcal{F}\{u_m'\})(\tau,\nu)e^{i2\pi\gamma(f_s m - f_s' m')} \tag{4.37}$$

$$= \sum_{m=0}^{M-1}\sum_{m'=0}^{M-1} \chi(u_m,u_m')(\nu,-\tau)e^{i2\pi\gamma(f_s m - f_s' m')} \tag{4.38}$$

$$= \sum_{m=0}^{M-1}\sum_{m'=0}^{M-1} \chi(u_m,u_m')(\tau,\nu)\circ Je^{-j2\pi\nu\tau}e^{i2\pi\gamma(f_s m - f_s' m')} \tag{4.39}$$

$$= \chi_u(\tau,\nu,f_s,f_s')\circ Je^{-j2\pi\nu\tau} = \chi_u(\nu,-\tau,f_s,f_s') \tag{4.40}$$

By moving the exponential to the other side of equations, the property follows. □

This property shows that J acts on MIMO ambiguity functions to give ambiguity function of the Fourier transforms of $\{u_m\}$. Next property will be twice application of J on ambiguity functions.

Property 4.7

$$(\chi(u,v)(\tau,\nu)\circ J)\circ J = \chi(u,v)(-\tau,-\nu) = \chi^*(v,u)(\tau,\nu) \tag{4.41}$$

Proof. Similar to property 4.1's proof, we will use the definition of ambiguity functions

$$\chi(u,v)(\tau,\nu)\circ J = \chi(u,v)(\nu,-\tau)e^{j2\pi\nu\tau} \tag{4.42}$$

$$(\chi(u,v)(\nu,-\tau)e^{j2\pi\nu\tau})\circ J = \chi(u,v)(-\tau,-\nu) \tag{4.43}$$

For the second equality, we can use a change of variables,

$$\chi(u,v)(-\tau,-\nu) = \int_{-\infty}^{+\infty} u(t)v^*(t+\tau)e^{j2\pi\nu t}dt \tag{4.44}$$

$$= \int_{-\infty}^{+\infty} u(t'+\tau)v^*(t')e^{j2\pi\nu(t'+\tau)}dt' \tag{4.45}$$

$$= \chi^*(v,u)(\tau,\nu)e^{-j2\pi\nu\tau} \tag{4.46}$$

□

Next property is application of this result to the MIMO case.

Property 4.7'

If we fix f_s, f_s'

$$(\chi(\tau,\nu,f_s,f_s')\circ J)\circ J = \chi(-\tau,-\nu,f_s,f_s') = \chi^*(\tau,\nu,f_s',f_s)e^{-j2\pi\nu\tau} \tag{4.47}$$

Proof. For both equations, we can use property 4.2 in definition of MIMO ambiguity functions.

$$\chi^*(\tau,\nu,f_s',f_s)e^{-j2\pi\nu\tau} = (\sum_{m=0}^{M-1}\sum_{m'=0}^{M-1}\chi(u_m^{'},u_m)(\tau,\nu)e^{i2\pi\gamma(f_s^{'}m^{'}-f_sm)})^*e^{-j2\pi\nu\tau} \tag{4.48}$$

$$= \sum_{m=0}^{M-1}\sum_{m'=0}^{M-1}\chi^*(u_m^{'},u_m)(\tau,\nu)e^{-j2\pi\nu\tau}e^{i2\pi\gamma(f_sm-f_s^{'}m^{'})} \tag{4.49}$$

$$= \sum_{m=0}^{M-1}\sum_{m'=0}^{M-1}\chi(u_m,u_m^{'})(-\tau,-\nu)e^{i2\pi\gamma(f_sm-f_s^{'}m^{'})} \tag{4.50}$$

$$= \chi(-\tau,-\nu,f_s,f_s') = (\chi(\tau,\nu f_s,f_s')\circ J)\circ J \tag{4.51}$$

□

This property shows that, application of J on MIMO ambiguity functions gives a delay and Doppler mirror image. Next property shows the LFM effect.

Property 4.8

$$\chi(u,v)(\tau,\nu)\circ t(-k) = \chi(u,v)(\tau,\nu-k\tau)e^{-j\pi k\tau^2} = \chi^{LFM}(u,v)(\tau,\nu) \tag{4.52}$$

Proof. The first equation can be proved by diract substitution. The LFM effect is,

$$\chi^{LFM}(u,v)(\tau,\nu) = \int_{-\infty}^{+\infty}(u(t)e^{j2\pi k\nu t^2})(v(t+\tau)e^{j2\pi k\nu(t+\tau)^2})^*e^{j2\pi\nu t}dt \tag{4.53}$$

$$\chi(u,v)(\tau,\nu-k\tau)e^{-j2\pi k\tau^2} \tag{4.54}$$

□

The next property shows the effect of $t(-k)$ on MIMO ambiguity funtions.

Property 4.8'

If we fix f_s, f_s'

$$\chi(\tau,\nu,f_s,f_s') \circ t(-k) = \chi(\tau,\nu-k\tau,f_s,f_s')e^{-j\pi k\tau^2} = \chi^{LFM}(\tau,\nu,f_s,f_s') \tag{4.55}$$

Proof. We will use property 4.3 in definition of MIMO ambiguity functions,

$$\chi^{LFM}(\tau,\nu,f_s,f_s') = (\sum_{m=0}^{M-1}\sum_{m'=0}^{M-1}\chi^{LFM}(u_m,u_m^{'})(\tau,\nu)e^{i2\pi\gamma(f_s m - f_s^{'} m^{'})})^* e^{-j2\pi\nu\tau} \tag{4.56}$$

$$= (\sum_{m=0}^{M-1}\sum_{m'=0}^{M-1}\chi^{LFM}(u_m,u_m^{'})(\tau,\nu-k\tau)e^{-j\pi k\tau^2}e^{i2\pi\gamma(f_s m - f_s^{'} m^{'})})^* e^{-j2\pi\nu\tau} \tag{4.57}$$

$$= \chi(\tau,\nu-k\tau,f_s,f_s')e^{-j\pi k\tau^2} = \chi(\tau,\nu,f_s,f_s') \circ t(-k) \tag{4.58}$$

□

LFM signals improve range resolution with the ridge effect. Here, we see that $t(-k)$ acts on the ambiguity function to create an LFM effect. Next property shows the effect of $m(b)$, which scales the waveforms.

Property 4.9

Let $\tilde{u(t)} = u(bt)$, $\tilde{v(t)} = v(bt)$ then,

$$\chi(u,v)(\tau,\nu) \circ m(b) = \tfrac{1}{b}\chi(u,v)(b\tau,\tfrac{\nu}{b}) = \chi(\tilde{u},\tilde{v})(\tau,\nu) \tag{4.59}$$

Proof. We will use a change of variables,

$$\chi(\tilde{u},\tilde{v})(\tau,\nu) = \int_{-\infty}^{+\infty} u(bt)v^*(b(t+\tau))e^{j2\pi\nu t}dt \tag{4.60}$$

$$= 1/b\int_{-\infty}^{+\infty} u(t')v^*(t'+b\tau)e^{j2\pi\frac{\nu}{b}t'}dt' \tag{4.61}$$

Where t' dummy variable is not related with that in proof of property 4.2

$$= \tfrac{1}{b}\chi(u,v)(b\tau,\tfrac{\nu}{b}) \tag{4.62}$$

Left hand side is by direct substitution. □

Next, we will use this relation in MIMO ambiguity functions.

Property 4.9'

If we fix f_s, f_s' and let $\chi_{\tilde{u}}(\tau,\nu,f_s,f_s')$ denote the MIMO ambiguity function with $\tilde{u}_m = u_m(bt)$,

$$\chi_u(\tau,\nu,f_s,f_s') \circ m(b) = \tfrac{1}{b}\chi(b\tau,\tfrac{\nu}{b},f_s,f_s') = \chi_{\tilde{u}}(\tau,\nu,f_s,f_s') \tag{4.63}$$

Proof. We will use property 4.4 in definition of MIMO ambiguity functions

$$\chi_{\tilde{u}}(\tau,\nu,f_s,f_s') = \sum_{m=0}^{M-1}\sum_{m'=0}^{M-1} \chi_{\tilde{u}}(u_m,u_m^{'})(\tau,\nu)e^{i2\pi\gamma(f_s m - f_s^{'} m^{'})} \tag{4.64}$$

$$= \sum_{m=0}^{M-1}\sum_{m'=0}^{M-1} \tfrac{1}{b}\chi(u_m,u_m^{'})(b\tau,\tfrac{\nu}{b})e^{i2\pi\gamma(f_s m - f_s^{'} m^{'})} \tag{4.65}$$

$$= \tfrac{1}{b}\chi(b\tau,\tfrac{\nu}{b},f_s,f_s') \tag{4.66}$$

Left hand side is proved by substituting $\frac{1}{b}\chi(u_m,u_m^{'})(b\tau,\frac{\nu}{b})$with $\chi(u_m,u_m^{'})(\tau,\nu) \circ m(b)$ □

This property shows the effect of $m(b)$ on the MIMO ambiguity functions which is turning waveforms $u_m(t)$ to the scaled ones $u_m(bt)$.

4.3 Uncertainty Relations for MIMO Ambiguity Functions

We will give the inequalities in the previous chapter in MIMO setting. We will be using the symmetric ambiguity function. This first relation is showing the relation between the 2-norm of MIMO ambiguity function and 2-norms of signals. A similar proof was given for equal norm signals in [13].

Relation 4.1

Let $u_m(t) \in L_2(\mathbb{R})$,

$$\|A(\tau, \nu, f_s, f_s^{'})\|_2^2 = \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \|u_m\|^2 \|u_{m'}\|^2 \tag{4.67}$$

Proof. We can use the relation $\|A(u,v)\|_2 = \|u\|_2 \|v\|_2$, and Parseval's relation

$$\int_0^1 \int_0^1 \int \int_{-\infty}^{+\infty} |A(\tau, \nu, f_s, f_s^{'})|^2 d\tau d\nu df_s df_s^{'} = \tag{4.68}$$

$$1/\gamma^2 \int \int_{-\infty}^{+\infty} \int_0^{\gamma} \int_0^{\gamma} |\sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} A_{m,m'}(\tau, \nu) e^{i2\pi(f_s m - f_s^{'} m^{'})}|^2 df_s df_s^{'} d\tau d\nu \tag{4.69}$$

$$= \int \int_{-\infty}^{+\infty} \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} |A_{m,m'}(\tau, \nu)|^2 dt d\tau = \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \|A_{m,m'}(\tau, \nu)\|^2 \tag{4.70}$$

$$= \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \|u_m\|^2 \|u_{m'}\|^2 \tag{4.71}$$

□

Next, we will combine the first relation with a bound given in the previous section.

Relation 4.2

Let $u_m(t) \in L_2(\mathbb{R})$,

$$\|A(\tau, \nu, f_s, f_s^{'})\|_2^2 \geq \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} |A_{m,m'}(\tau, \nu)|^2 \tag{4.72}$$

Proof. We can use the first relation and $|A(u, v)| \leq \|u\|_2 \|v\|_2$

$$\|A(\tau, \nu, f_s, f_s^{'})\|_2^2 = \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \|u_m\|^2 \|u_{m'}\|^2 \geq \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} |A_{m,m'}(\tau, \nu)|^2 \tag{4.73}$$

□

The following relation uses the uncertainty relation on the set E and give its relation to signals used in MIMO ambiguity,

Relation 4.3

$$\int_0^1 \int_0^1 \int_E |A(\tau, \nu, f_s, f_s^{'})|^2 d\tau d\nu df_s df_s^{'} \leq |E| \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \|A_{m,m'}(\tau, \nu)\|_\infty^2 \tag{4.74}$$

$$\leq |E| \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \|A_{m,m'}(\tau, \nu)\|_2^2 = |E| \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \|u_m\|^2 \|u_{m'}\|^2 \tag{4.75}$$

Proof. Using relation $\int_E |A(u, v)|^2 \leq \|A(u, v)\|_\infty^2 |E| \leq \|A(u, v)\|_2^2 |E|$ and relation 1

$$\int_0^1 \int_0^1 \int_E |A(\tau, \nu, f_s, f_s^{'})|^2 d\tau d\nu df_s df_s^{'} = \tag{4.76}$$

$$= \int \int_E \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} |A_{m,m'}(\tau, \nu)|^2 d\tau d\nu \tag{4.77}$$

$$= \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \int \int_E |A_{m,m'}(\tau, \nu)|^2 d\tau d\nu \tag{4.78}$$

$$\leq |E| \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \|A_{m,m'}(\tau,\nu)\|_{\infty}^2 \leq |E| \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \|A_{m,m'}(\tau,\nu)\|_2^2 \tag{4.79}$$

$$= |E| \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \|u_m\|^2 \|u_{m'}\|^2 \tag{4.80}$$

□

By using this relation, we obtain the following lower bound on the (τ, ν) support of MIMO ambiguity function,

$$\text{supp}(A(\tau,\nu,f_s,f_s^{'})) \geq \frac{\int_0^1 \int_0^1 \int_E |A(\tau,\nu,f_s,f_s^{'})|^2 d\tau d\nu df_s df_s^{'}}{\sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \|u_m\|^2 \|u_{m'}\|^2} \tag{4.81}$$

As we can see, this bound is related with the norms of signals used in MIMO. By varying the norms we may alter the result. This relation shows how much can the MIMO ambiguity functions be localized in a set of (τ, ν) plane. The next relation uses Relation 4 and waveforms with same norms, which do not have to be identical otherwise. This is usually the case in many time frequency analysis problems.

Relation 4.4

If we choose all $u_m(t)$ to have equal norm, such that $\|u_m(t)\|_2^2 = \|u_{m'}(t)\|_2^2 = \|u(t)\|_2^2$ for all $m, m' = 0, 1, ..., M-1$,

$$\int_0^1 \int_0^1 \int_E |A(\tau,\nu,f_s,f_s^{'})|^2 d\tau d\nu df_s df_s^{'} \tag{4.82}$$

$$\leq |E| \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \|A_{m,m'}(\tau,\nu)\|_2^2 = M|E| \|u(t)\|_2^4 \tag{4.83}$$

Proof. Using Relation 4, with $\|u_m(t)\|_2^2 = \|u_{m'}(t)\|_2^2 = \|u(t)\|_2^2$

$$\int_0^1 \int_0^1 \int_E |A(\tau,\nu,f_s,f_s^{'})|^2 d\tau d\nu df_s df_s^{'} = \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \int \int_E |A_{m,m'}(\tau,\nu)|^2 d\tau d\nu \tag{4.84}$$

$$\leq |E| \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \|A_{m,m'}(\tau,\nu)\|_2^2 = M|E|\|u(t)\|_2^4 \tag{4.85}$$

Last equality follows from $\|A_{u,v}(\tau,\nu)\|_2^2 = \|u(t)\|_2^2 \|v(t)\|_2^2$ □

We may give a lower bound on the (τ,ν) support similar to the one above as follows,

$$\text{supp}(A(\tau,\nu,f_s,f_s^{'})) \geq \frac{\int_0^1 \int_0^1 \int_E |A(\tau,\nu,f_s,f_s^{'})|^2 d\tau d\nu df_s df_s^{'}}{M\|u(t)\|_2^4} \tag{4.86}$$

Next, we will use the main results in [40] together with MIMO ambiguity functions. We will look at the case where $|A_{m,m'}| \leq 1$. If we choose unit norms on signals $\|u_m(t)\|_2^2 = \|u_{m'}(t)\|_2^2 = \|u(t)\|_2^2 = 1$, we have $|A_{m,m'}| \leq \|u_m\|_2 \|u_{m'}\|_2 = 1$. But, this leads to a trivial case we can already observe from relation 1. Therefore, we won't put restrictions on the norms. We have the following two cases,

For $p \geq 2$ we have $|A_{m,m'}|^2 \geq |A_{m,m'}|^p$,

$$\int_0^1 \int_0^1 \int \int_{-\infty}^{+\infty} |A(\tau,\nu,f_s,f_s^{'})|^2 d\tau d\nu df_s df_s^{'} = \tag{4.87}$$

$$\sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \int \int_{-\infty}^{+\infty} |A_{m,m'}(\tau,\nu)|^2 dt d\tau \geq \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \int \int_{-\infty}^{+\infty} |A_{m,m'}(\tau,\nu)|^p \tag{4.88}$$

Also, we have by uncertainty in [40],

$$\int \int_{\mathbb{R}^2} |A_{m,m'}(\tau,\nu)|^p d\tau d\nu \leq \frac{2}{p} \|u_m\|_2^p \|u_{m'}\|_2^p \quad for\ p \geq 2 \tag{4.89}$$

For $1 \leq p \leq 2$ we have $|A_{m,m'}|^2 \leq |A_{m,m'}|^p$,

$$\int_0^1 \int_0^1 \int \int_{-\infty}^{+\infty} |A(\tau,\nu,f_s,f_s^{'})|^2 d\tau d\nu df_s df_s^{'} = \tag{4.90}$$

$$\sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \int \int_{-\infty}^{+\infty} |A_{m,m'}(\tau,\nu)|^2 dt d\tau \leq \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \int \int_{-\infty}^{+\infty} |A_{m,m'}(\tau,\nu)|^p \tag{4.91}$$

Again by the uncertainty relation,

$$\int \int_{\mathbb{R}^2} |A_{m,m'}(\tau,\nu)|^p d\tau d\nu \geq \frac{2}{p} \|u_m\|_2^p \|u_{m'}\|_2^p \quad for\ 1 \leq p \leq 2 \tag{4.92}$$

From these two relations, we see that we cannot continue directly, but we may use the equality in the Lieb's uncertainty. We have given that the uncertainty relation holds with equality iff the two signals are a matched Gaussian pair. Therefore, by choosing $u_m(t)$ and $u_{m'}(t)$ a matched Gaussian pair, we reach the following relation.

Relation 4.5

Let $u_m(t)$ and $u_{m'}(t)$ be a matched Gaussian pair as given in section 2. Also, let $|A_{m,m'}| \leq 1$. Then, we have the following two relations,
For $p \geq 2$,

$$\int_0^1 \int_0^1 \int \int_{-\infty}^{+\infty} |A(\tau, \nu, f_s, f_s^{'})|^2 d\tau d\nu df_s df_s^{'} = \tag{4.93}$$

$$\sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \int \int_{-\infty}^{+\infty} |A_{m,m'}(\tau, \nu)|^2 dt d\tau \geq \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \int \int_{-\infty}^{+\infty} |A_{m,m'}(\tau, \nu)|^p \tag{4.94}$$

$$\frac{2}{p} \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \|u_m\|_2^p \|u_{m'}\|_2^p \tag{4.95}$$

For $1 \leq p \leq 2$,

$$\int_0^1 \int_0^1 \int \int_{-\infty}^{+\infty} |A(\tau, \nu, f_s, f_s^{'})|^2 d\tau d\nu df_s df_s^{'} = \tag{4.96}$$

$$\sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \int \int_{-\infty}^{+\infty} |A_{m,m'}(\tau, \nu)|^2 dt d\tau \leq \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \int \int_{-\infty}^{+\infty} |A_{m,m'}(\tau, \nu)|^p \tag{4.97}$$

$$\frac{2}{p} \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \|u_m\|_2^p \|u_{m'}\|_2^p \tag{4.98}$$

Proof. Follows by using the relations above together. □

This relation bounds the norm of MIMO ambiguity function from below and above for $p \geq 2$ and $1 \leq p \leq 2$ cases respectively when we use matched Gaussian pairs in signals.
From these relations, we see the effect of MIMO signals together in in the time frequency plane. These relations can be used to design new waveforms in integrated

sensing and communications. They can be compared with performance of commonly used methods such as OFDM signals, or with constraints on PAPR, out of band radiation. These relaitons can be generalized to the spatial domain via use of three variable uncertainty relations.

4.4 Uncertainty Relations for MIMO Covariance Function

In this section, we will give norm relations concerning MIMO covariance functions. The first relation is a relation on the Frobenius norm on MIMO covariance matrix.

Relation 4.4.1

$$\int\int_{-\infty}^{+\infty} \|\mathbf{R}(\tau,\nu)\|_F^2 d\tau d\nu = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \|u_i\|^2\|u_j\|^2 \tag{4.99}$$

$\|\mathbf{R}(\tau,\nu)\|_F$ denoting the Frobenius norm.

Proof. As we have,

$$\|A_{ij}(\tau,\nu)\|_2^2 = \|u_i\|^2\|u_j\|^2 \tag{4.100}$$

$$\int\int_{-\infty}^{+\infty} \|\mathbf{R}(\tau,\nu)\|_F^2 d\tau d\nu = \int\int_{-\infty}^{+\infty} \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} |A_{ij}(\tau,\nu)|^2 d\tau d\nu \tag{4.101}$$

$$\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \|A_{ij}(\tau,\nu)\|_2^2 = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \|u_i\|^2\|u_j\|^2 \tag{4.102}$$

□

The next relation is an upper bound on the 1-norm of MIMO covariance matrix in terms of 2-norms of signals used.

Relation 4.4.2

$$\|\mathbf{R}(\tau,\nu)\|_1 \leq \left(\sum_{i=1}^{N} \|u_i\|_2\right) \max_{j=1,\dots,N} \|u_j\|_2 \tag{4.103}$$

$\|\mathbf{R}(\tau,\nu)\|_1$ denoting the 1-norm.

Proof. As we have $|A_{ij}(\tau,\nu)| \leq \|u_i\|_2\|u_j\|_2$,

$$\|\mathbf{R}(\tau,\nu)\|_1 = \max_{j=1,\dots,N} \sum_{i=1}^{N} |A_{ij}(\tau,\nu)| \tag{4.104}$$

$$\leq \max_{j=1,\dots,N} \sum_{i=1}^{N} \|u_i\|_2\|u_j\|_2 = \left(\sum_{i=1}^{N} \|u_i\|_2\right) \max_{j=1,\dots,N} \|u_j\|_2 \tag{4.105}$$

□

By using the above result, we can observe the following. Let $1/p + 1/q = 1$. Then, by $|A_{ij}(\tau,\nu)|_2^2 \leq \|u_i\|_p\|u_j\|_q$, which comes from Hölder's inequality,

$$\|\mathbf{R}(\tau,\nu)\|_1 \leq \left(\sum_{i=1}^{N} \|u_i\|_p\right) \max_{j=1,\dots,N} \|u_j\|_q \tag{4.106}$$

Similar to relation 4.2, the next relation is an upper bound on the ∞-norm of MIMO covariance matrix by using 2-norms of signals.

Relation 4.4.3

$$\|\mathbf{R}(\tau,\nu)\|_\infty \leq \left(\sum_{j=1}^{N} \|u_j\|_2\right) \max_{i=1,\dots,N} \|u_i\|_2 \tag{4.107}$$

$\|\mathbf{R}(\tau,\nu)\|_\infty$ denoting the ∞-norm.

Proof. As we have $|A_{ij}(\tau,\nu)| \leq \|u_i\|_2\|u_j\|_2$,

$$\|\mathbf{R}(\tau,\nu)\|_\infty = \max_{i=1,\dots,N} \sum_{j=1}^{N} |A_{ij}(\tau,\nu)| \tag{4.108}$$

$$\leq \max_{i=1,\dots,N} \sum_{j=1}^{N} \|u_i\|_2\|u_j\|_2 = \left(\sum_{j=1}^{N} \|u_j\|_2\right) \max_{i=1,\dots,N} \|u_i\|_2 \tag{4.109}$$

□

Similar to Relation 4.4.2, here also by the above result, we can observe the following. Let $1/p + 1/q = 1$. Then, by $|A_{ij}(\tau, \nu)|_2^2 \leq \|u_i\|_p \|u_j\|_q$, which comes from Hölder's inequality,

$$\|\mathbf{R}(\tau, \nu)\|_\infty \leq \left(\sum_{j=1}^{N} \|u_j\|_p \right) \max_{i=1,\ldots,N} \|u_i\|_q \tag{4.110}$$

Next relation is using the local uncertainty with MIMO correlation matrix. By doing so we give a lower bound on the delay Doppler support of the Frobenius norm of MIMO correlation matrix.

Relation 4.4.4

$$\int\int_E \|\mathbf{R}(\tau,\nu)\|_F^2 d\tau d\nu \leq |E| \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \|u_i\|^2\|u_j\|^2 \tag{4.111}$$

Proof. Similar to proof of Relation 4.4.1,

$$\int\int_E \|\mathbf{R}(\tau,\nu)\|_F^2 d\tau d\nu = \tag{4.112}$$

$$= \int\int_E \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} |A_{ij}(\tau,\nu)|^2 d\tau d\nu \leq \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} |E|\|A_{ij}(\tau,\nu)\|_\infty^2 \tag{4.113}$$

$$\leq \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} |E|\|A_{ij}(\tau,\nu)\|_2^2 = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} |E|\|u_i\|^2\|u_j\|^2 \tag{4.114}$$

□

We can write the bound, in a similar way we used in section 2 as follows,

$$\text{supp}(A(\tau,\nu,f_s,f_s^{'})) \geq \frac{\int\int_E \|\mathbf{R}(\tau,\nu)\|_F^2 d\tau d\nu}{\sum_{m=0}^{M-1}\sum_{m^{'}=0}^{M-1}\|u_m\|^2\|u_{m'}\|^2} \tag{4.115}$$

This relation bounds the support of the Frobenius norm of the correlation matrix, the bound is connected to the norms of signals. The relation again can be modified by using equal norm signals. The following relation connects Lieb's uncertainty [40] with the p-norm of correlation matrix.

Relation 4.4.5

$$\int\int_{-\infty}^{+\infty} \|\mathbf{R}(\tau,\nu)\|_p^p d\tau d\nu \leq \frac{2}{p}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \|u_i\|_2^2\|u_j\|_2^2 \quad for \quad p \geq 2 \tag{4.116}$$

$$\int\int_{-\infty}^{+\infty} \|\mathbf{R}(\tau,\nu)\|_p^p d\tau d\nu \geq \frac{2}{p}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \|u_i\|_2^2\|u_j\|_2^2 \;\; for \;\; 1 \leq p \leq 2 \tag{4.117}$$

$\|\mathbf{R}(\tau,\nu)\|_p$ denoting the p-norm.

Proof. We will use the property from [40] given in preliminaries. For $p \geq 2$,

$$\int\int_{-\infty}^{+\infty} \|\mathbf{R}(\tau,\nu)\|_p^p d\tau d\nu = \int\int_{-\infty}^{+\infty} \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} |A_{ij}(\tau,\nu)|^p d\tau d\nu \tag{4.118}$$

$$\leq \frac{2}{p}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \|u_i\|^2\|u_j\|^2 \tag{4.119}$$

Proof of the case $1 \leq p \leq 2$ follows same steps. □

Generalization to $\|u_i\|_a^p\|u_j\|_b^p$, for $1/a + 1/b = 1$, and more on the topic can be found in [40]. There, the result for 2-norm is generalized as,

$$\int\int_{-\infty}^{+\infty} |A_{ij}(\tau,\nu)|^p \leq C_1(p,a,b)\|u_i\|_a^p\|u_j\|_b^p \quad for\ p \geq 2 \tag{4.120}$$

$$\int\int_{-\infty}^{+\infty} |A_{ij}(\tau,\nu)|^p \geq C_2(p,a,b)\|u_i\|_a^p\|u_j\|_b^p \quad for\ 1 \leq p \leq 2 \tag{4.121}$$

Where, $C_1(p,a,b)$ and $C_2(p,a,b)$ are constants related with p, a, b given in. We will mention two interesting results from [40] combined with MIMO case are,

$$\int\int_{-\infty}^{+\infty} \|\mathbf{R}(\tau,\nu)\|_1 d\tau d\nu \geq 2\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \|u_i\|_2\|u_j\|_2 \tag{4.122}$$

Which asserts that 1-norm of $\mathbf{R}(\tau,\nu)$'s integral is larger than summation of 2-norms of all waveforms

$$\int\int_{-\infty}^{+\infty} \|\mathbf{R}(\tau,\nu)\|_1 d\tau d\nu \geq \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \|u_i\|_1\|u_j\|_\infty \tag{4.123}$$

Asserts that 1-norm of $\mathbf{R}(\tau,\nu)$'s integral is greater than summation of 1-norms of all signals times magnitude of maximum norm waveform. Here, 1-norm and ∞-norm can be used interchangably

CHAPTER 5

SYMPLECTIC TRANSFORMATIONS ON TIME FREQUENCY DISTRIBUTIONS

In this chapter, we will investigate effects of symplectic transforms on time frequency distributions. We will see the invariance effect of them. This effect can be used as a guideline for designing waveforms in telecommunications, integrated sensing and communications, radar, sonar and time frequency analysis tasks. We will use some relations used in quantum mechanics. These effects were used to understand the evolution of state of a quantum entity. Here, however, we will use it in a signal processing problem.

In signal processing tasks, important points vary from that of quantum mechanics. Examples of this situation can be nonnegativity of time frequency distributions and cross Wigner distributions. In quantum mechanics, nonnegativity has a significant impact as time frequency distributions resemble a real quantity. However, in signal processing tasks mild negativity in time frequency distributions do not have a significant effect. Cross Wigner distributions do not resemble a state in quantum mechanics, and not used frequently. Although, in signal processing, cross Wigner distributions are used frequently when there are more than one input signal present. Similar to these examples, in this chapter, we will give the effect of symplectic tranforms on marginalizable time frequency distributions for signal processing tasks. In this chapter, we will not go too much in the details of quantum mechanics. Rather, we will keep the discussion on signal processing and time frequency analysis. We will give references to quantum mechanics and mention the meaning of these relations for quantum mechanics shortly.

Here, first, we will define the symplectic form. Symplectic form can be seen as a type of product on the phase space from an engineering viewpoint. Similar to the usual inner product, here, we use it for the quantities on the phase space. However, one should not mix it with the inner product, symplectic form has a different structure.

5.1 Symplectic Form

In time frequency analysis and quantum systems, the phase space Γ is defined as $\mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}$. For n-dimensional case, we have $z = (q_1, ..., q_n : p_1, ...p_n)$. The symplectic form on phase space is defined as,

$$\sigma(z, z') = \sum_{i=1}^{n} q'_i p_i - q_i p'_i \tag{5.1}$$

For signal processing problems on 1-dimension, we can use $z = (t, f)$ and the rest of the definitions change accordingly. We will be using the 1-dimensional case here. A 2×2 matrix S is symplectic iff,

$$\sigma(Sz, Sz') = \sigma(z, z') \tag{5.2}$$

or, equivalently,

$$S^{\intercal} J S = J,\ for\ J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{5.3}$$

5.2 Covariance Matrix of a Wigner Distribution

The covariance of a Wigner distribution is directly related with the uncertainty relations, as they are statements on variances of time and frequency variables. That is the main reason we are interested in using them for time frequency signal processing applications. In [48], [41], it was shown that, the covariance matrix of a Wigner distribution has the invariance property under certain linear transformations. We will state some important results from [48], which are useful for our purposes here. We will not go into too much detail here, as they are in quantum mechanics context, we refer the interested reader for proofs to the paper. For a treatment in terms of quantum harmonic analysis with relations to Wigner distributions and ambiguity functions one can refer to [27].

As we have mentioned, the Wigner distribution holds the marginal properties. Integrating it over the time variable will yield the frequency marginal and vice versa. We will consider normalized Wigner distributions,

$$\int\int_{-\infty}^{+\infty} W(t,f)dtdf = 1 \tag{5.4}$$

Moreover, we will assume that the integral,

$$\int_{\Gamma}(1+|z|^2)W(z)dz < \infty \tag{5.5}$$

is finite. This assumption assures that Fourier transform of the Wigner distribution is twice continuously differentiable on the dual of the phase space $\hat{\Gamma}$. We may find the elements of the covariance matrix as,

$$C_{ij} = \int (z_i - z_{i0})(z_j - z_{j0})W(z)dz \tag{5.6}$$

Where, z_{i0}, z_{j0} are the means of z_i, z_j respectively. For one dimensional case, the covariance matrix is,

$$C = \begin{bmatrix} \Delta t^2 & \Delta(t,f) \\ \Delta(f,t) & \Delta f^2 \end{bmatrix} \tag{5.7}$$

Where,

$$\Delta t^2 = \int\int t^2 W(t,f)dtdf - \left[\int\int tW(t,f)dtdf\right]^2 =< t^2 > - < t >^2 \tag{5.8}$$

$$\Delta f^2 = \int\int f^2 W(t,f)dtdf - \left[\int\int fW(t,f)dtdf\right]^2 =< f^2 > - < f >^2 \quad (5.9)$$

$$\Delta(t,f) = \int\int tfW(t,f)dtdf - \int\int tW(t,f)dtdf \int\int fW(t,f)dtdf \quad (5.10)$$

$$=< tf > - < t >< f > \quad (5.11)$$

Here, we can note that time frequency distributions which satisfy the marginality conditions can also be used to obtain the C matrix. The dual of $\rho(t,f)$ is $A_u(\tau,\nu)g(\tau,\nu)$. Here, the kernel $g(\tau,\nu)$ must satisfy the conditions for marginality given in section 2. In order to satisfy the twice differentiability condition, we require $g(\tau,\nu)$ to be twice continuously differentiable in addition to $A_u(\tau,\nu)$.

A very important result of [48] is, if $W(z)$ holds the above given assumptions then, C is its covariance matrix iff it is real symmetric and the matrix,

$$C + \frac{i\hbar}{2}J \quad (5.12)$$

is Hermitian, and non-negative. This result is also called the strong uncertainty. From here, by looking at the determinant, we obtain

$$\Delta t^2 \Delta f^2 \geq \Delta(t,f) + \frac{\hbar^2}{4} \quad (5.13)$$

Importance of the covariance matrix for our purpose here is that it has an invariance under affine transformations. This relation was deeply investigated in [41] and [48]. We consider the linear transformation $A : W(z) \rightarrow W(S^{-1}z)$, where S is a symplectic matrix. Here, we omitted the constant part of the affine transformation as we are not going to use it in our analysis. We are interested in the behavior of the covariance matrix under the effect of the symplectic matrix. Simply, by using $S^{-1}z$ in the definition of covariance matrix above, we obtain the following matrix,

$$C' = \int [S(z_i - z_{i0})][S(z_j - z_{j0})]^{\intercal} W(z) dz \tag{5.14}$$

$$= S \left[\int (z_i - z_{i0})(z_j - z_{j0}) W(z) dz \right] S^{\intercal} = SCS^{\intercal} \tag{5.15}$$

We can see here that $C' + \frac{i\hbar}{2}J$ is also nonnegative, because,

$$C' + \frac{i\hbar}{2}J = SCS^{\intercal} + S(\frac{i\hbar}{2}J)S^{\intercal} = S\left[C + \frac{i\hbar}{2}J\right]S^{\intercal} \tag{5.16}$$

and we know that $C + \frac{i\hbar}{2}J$ is non negative.

5.3 Uncertainty Relations

Here, we will give relations about ambiguity functions and TFDs. The following relation is the usual Heisenberg uncertainty relation.

Heisenberg Uncertainty Relation

Heisenberg uncertainty relation is a well known statement from quantum mechanics. In signal processing context, it means that a signal and its Fourier domain counterpart, cannot be localized in (t, f) domain simultaneously. The uuncertainty relation can be given as follows,

$$\int (t-a)^2 |x(t)|^2 dt \int (f-b)^2 |X(f)|^2 df \geq \frac{\|x\|_2^4}{16\pi^2} \tag{5.17}$$

for $x \in L_2(\mathbb{R})$. Proof of this relation can be found in [26]. In section 2, time marginal and frequency marginal properties were given. Combining these with the uncertainty relations we can use them to prove theorems given for Wigner distributions. Here, by using a marginalizable TFD $\rho_a(t, f)$ we will obtain the following uncertainty relation,

Relation 1

For a $\rho_a(t, f)$ satisfying the time marginal and frequency marginal properties and the constants t_0 and f_0,

$$\int\int |t - t_0|^2 + |f - f_0|^2 \rho_a(t, f) dt df \geq \frac{\|a\|_2^2}{2\pi} \tag{5.18}$$

Proof. As $\rho_a(t, f)$ satisfies marginal conditions,

$$\int_{-\infty}^{+\infty} \rho_a(t, f) df = |a(t)|^2 \tag{5.19}$$

and,

$$\int_{-\infty}^{+\infty} \rho_a(t, f) dt = |A(f)|^2 \tag{5.20}$$

We have uncertainty relation given in section 2,

$$\int (t - a)^2 |x(t)|^2 dt \int (f - b)^2 |X(f)|^2 df \geq \frac{\|x\|_2^4}{16\pi^2} \tag{5.21}$$

Also, as $u^2 + v^2 \geq 2uv$, we obtain,

$$\int\int |t - t_0|^2 + |f - f_0|^2 \rho_a(t, f) dt df = \tag{5.22}$$

$$= \int |t - t_0|^2 |a(t)|^2 dt + \int |f - f_0|^2 |A(f)|^2 df \geq \frac{\|a\|_2^2}{2\pi} \tag{5.23}$$

□

This relation gives an uncertainty for all the TFDs wchich satisfy time marginal and frequency marginal properties. In order to satisfy the time marginal, the TFD's kernel should satisfy $g(0, \nu) = 1, \forall \nu$ and for the frequency marginal $g(\tau, 0) = 1, \forall \tau$. Some TFD's satisfying both marginal conditions are, WVD, Levin Distribution, Rihaczek Distribution, Born Jordan Distribution, Gaussian TFD and Page Distributions.
Gaussian transform of WVD , $W_{\alpha\beta}(t, f)$ can be defined as [15],

$$W_{\alpha\beta}(t, f) = \int\int_{-\infty}^{\infty} W_a(t, f) e^{-\frac{\pi}{\alpha}(t-t')^2} e^{-\frac{\pi}{\beta}(f-f')^2} dt' df' \tag{5.24}$$

for $z \in L_2(\mathbb{R})$. $W_{\alpha\beta}(t,f)$ is a filtered version of $W_a(t,f)$. If we take $\gamma_1(t) = e^{-\frac{\pi}{\alpha}t}$ and $\gamma_2(f) = e^{-\frac{\pi}{\beta}f}$, then,

$$W_{\alpha\beta}(t,f) = \gamma_1(t) \underset{t}{*} W_a(t,f) \underset{f}{*} \gamma_2(f) \tag{5.25}$$

Actually, $\gamma(t,f) = \gamma_1(t)\gamma_2(f)$ can be realized as a seperable kernel. A property of Gaussian transformed WVD is that, $W_{\alpha,\beta}(t,f)$ filtered with parameters α', β' gives $W_{\alpha+\alpha',\beta+\beta'}(t,f)$.

The following relation is important as it help to realize $W_{\alpha,\beta}(t,f)$ as a probability distribution. We will give it without the proof. The proof can be found in [15].

Relation 2

Let $\alpha > 0$, $\beta > 0$. Then, $W_{\alpha,\beta}(a,a)(t,f)$ is positive definite iff $\alpha\beta > 1/4$.

As WVD can have negative values, it cannot be considered as a probability distribution. This relation gives $W_{\alpha,\beta}(z,z)(t,f)$ the nonnegativity property, hence, allowing us to consider it as a probability distribution. The relation intuitively means that, if the distribution $W_{\alpha,\beta}(a,a)(t,f)$ is a measure of energy, being nonnegative, then the kernel $\gamma_1(t)\gamma_2(f)$ has to satisfy $\alpha\beta > 1/4$. As WVD is filtered with Gaussians, this distribution can be interpreted as joint distribution of time to an error α and frequency to an error β [26].
If we look at the ambiguity function counterpart of $W_{\alpha,\beta}(a,a)(t,f)$, and denote it with $A_{\alpha,\beta}(a,a)(t,f)$ we can see from,

$$W_{\alpha\beta}(t,f) = \gamma_1(t) \underset{t}{*} W_a(t,f) \underset{f}{*} \gamma_2(f) \tag{5.26}$$

and,

$$\mathcal{A}(\tau,\nu) = \mathcal{F}\{\mathcal{F}^{-1}\{\rho_a(t,f)\}\} \tag{5.27}$$

that,

$$\mathcal{A}_{\alpha,\beta}(\tau,\nu) = A_a(\tau,\nu)\mathcal{F}\{\gamma_1(t)\}\mathcal{F}^{-1}\{\gamma_2(f)\} \tag{5.28}$$

$$= A_a(\tau,\nu)G_1(\nu)g_2(\tau) \tag{5.29}$$

As Fourier transform of a Gaussian signal gives a Gaussian. $G_1(\nu)$ and $g_2(\tau)$ are also Gaussians. Here, we see that the effect of filtering the Wigner distribution with Gaussians in the dual domain is multiplication with Gaussians. This effect modifies the variance of the ambiguity distribution in the opposite way. This happens as the Fourier transform of the Gaussian will have the variance inverted.

5.4 $SL(2,\mathbb{R})$ Actions on Wigner Distributions

In [6], $SL(2,\mathbb{R})$ action on ambiguity functions were given. Here, we will use them in Wigner distributions and we will also give the ambiguity function counterparts. As we have mentioned earlier, the covariance matrix can be obtained by marginalizable TFDs. Therefore, after applying the group actions on a marginalizable TFD, one can still obtain the covariance matrix. Here, we will only solve for the Wigner distributions. Our purpose here is to observe the effect of generators of the group $SL(2,\mathbb{R})$. Using relations for generators, we can see the effect of other actions of $SL(2,\mathbb{R})$, as they can be written in terms of generators.

Property 5.1

Let $a \in L_2(\mathbb{R})$, then

$$W_a(t,f) \circ J^{-1} = W_{\mathcal{F}\{a\}}(t,f) = W_a(-f,t) \tag{5.30}$$

Proof. The relation can be shown to hold by using the definiton of Wigner distributions and Parseval theorem. □

If we look at the corresponding ambiguity function,

$$\tilde{A}_a(\tau,\nu) = \mathcal{F}\{\mathcal{F}^{-1}\{W_a(-f,t)\}\} = \mathcal{F}\{\mathcal{F}^{-1}\{W_{\mathcal{F}\{a\}}(t,f)\}\} \tag{5.31}$$

$$= A_{\mathcal{F}\{a\}}(\tau,\nu) = A_{\mathcal{F}\{a\}}(\nu,-\tau) \tag{5.32}$$

Here, if we look at the covariance matrix,

$$C' = JCJ^{\intercal} \tag{5.33}$$

$$= J \begin{bmatrix} \Delta t^2 & \Delta(t,f) \\ \Delta(f,t) & \Delta f^2 \end{bmatrix} J^{\intercal} \tag{5.34}$$

$$= \begin{bmatrix} \Delta f^2 & -\Delta(t,f) \\ -\Delta(f,t) & \Delta t^2 \end{bmatrix} \tag{5.35}$$

We see that, $\Delta t' = \Delta f$, $\Delta f' = \Delta t$ and the uncertainty relation $\Delta t' \Delta f' \geq \frac{\hbar}{2}$ holds.

Property 5.2

$$(W_a(t,f) \circ J^{-1}) \circ J^{-1} = W_a(-t,-f) = W_a^*(t,f) \tag{5.36}$$

Proof. The relation can be proved by applying the operation directly and using the definition of Wigner distributions. □

The ambiguity function counterpart here is,

$$\tilde{A}_a(\tau,\nu) = \mathcal{F}\{\mathcal{F}^{-1}\{W_a(-t,-f)\}\} = \mathcal{F}\{\mathcal{F}^{-1}\{W_a^*(t,f)\}\} \tag{5.37}$$

$$= A_a(-\tau,-\nu) \tag{5.38}$$

The covariance matrix is,

$$C' = (JJ)C(JJ)^{\intercal} \tag{5.39}$$

As $JJ = (-1)\mathbf{I}$,

$$C' = (-1)\mathbf{I}C(-1)\mathbf{I}^{\intercal} = C \tag{5.40}$$

The covariance matrix doesnt change. Therefore, $\Delta t' = \Delta t$, $\Delta f' = \Delta f$ and the uncertainty relation $\Delta t' \Delta f' \geq \frac{\hbar}{2}$ is unchanged.

Property 5.3

$$W_a(t,f) \circ t^{-1}(k) = W_a(t, f-kt) = W_a^{LFM}(t,f) \tag{5.41}$$

Proof. This property can be shown to hold by direct substitution. □

Here, the effect of $t^{-1}(k)$ changes $a(t)$ to $a_3(t) = a(t)e^{-j\pi kt^2}$, which is the LFM effect. Ambiguity function here becomes,

$$A_{a_3}(t,f) = \mathcal{F}\{\mathcal{F}^{-1}\{W_a(t,f-kt)\}\} = \mathcal{F}\{\mathcal{F}^{-1}\{W_a^{LFM}(t,f)\}\} \tag{5.42}$$

$$= A_a(\tau, \nu + k\tau) = A_a^{LFM}(\tau,\nu) \tag{5.43}$$

Which is obtained by using LFM effect in ambiguity functions. We note that, the LFM effect here is given as $(\tau, \nu + k\tau)$, which is usually given as a negative shift, this can be solved by choosing a negative k. Here, the corresponding symplectic matrix S is,

$$= \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \tag{5.44}$$

If we look at the covariance matrix,

$$C' = SCS^{\intercal} \tag{5.45}$$

$$= S \begin{bmatrix} \Delta t^2 & \Delta(t,f) \\ \Delta(f,t) & \Delta f^2 \end{bmatrix} S^{\intercal} \tag{5.46}$$

$$= \begin{bmatrix} \Delta t^2 & \Delta(t,f) + k\Delta f^2 \\ \Delta(f,t) + k\Delta f^2 & \Delta f^2 + 2k\Delta(f,t) + k^2\Delta t^2 \end{bmatrix} \tag{5.47}$$

Here, we have $\Delta t' = \Delta t$, $\Delta f' = \sqrt{\Delta f^2 + 2k\Delta(f,t) + k^2\Delta t^2}$ and we can easily see that, $\Delta t'\Delta f' \geq \Delta t\Delta f \geq \frac{\hbar}{2}$. Therefore, the uncertainty is preserved.

Property 5.4

Let $\tilde{a(t)} = a(\frac{t}{c})$, then,

$$W_a(t,f) \circ m^{-1}(c) = W_a(\tfrac{t}{c}, cf) = W_{a_4}(\tau,\nu) \tag{5.48}$$

Proof. This property can be proved by directly applying the operation and using the definition of ambiguity functions. □

$m^{-1}(c)$ operation turns $a(t)$ to $a_4(t) = a(\frac{t}{c})$ here. The resulting ambiguity function counterpart is therefore,

$$A_{a_4}(t, f) = \frac{1}{c}\mathcal{F}\{\mathcal{F}^{-1}\{W_a(\tfrac{t}{c}, cf)\}\} = \mathcal{F}\{\mathcal{F}^{-1}\{W_{a_4}(t, f)\}\} \tag{5.49}$$

$$= A_{a_4}(t, f) = \frac{1}{c}A_a(c\tau, \tfrac{\nu}{c}) \tag{5.50}$$

The symplectic matrix here is,

$$S = \begin{bmatrix} c & 0 \\ 0 & \frac{1}{c} \end{bmatrix} \tag{5.51}$$

Then, the covariance matrix is

$$C' = SCS^{\intercal} \tag{5.52}$$

$$= S \begin{bmatrix} \Delta t^2 & \Delta(t, f) \\ \Delta(f, t) & \Delta f^2 \end{bmatrix} S^{\intercal} \tag{5.53}$$

$$= \begin{bmatrix} c^2\Delta t^2 & \Delta(t, f) \\ \Delta(f, t) & \frac{1}{c^2}\Delta f^2 \end{bmatrix} \tag{5.54}$$

Therefore, we have $\Delta t' = c\Delta t$, $\Delta f' = \frac{1}{c}\Delta f$ and we have, $\Delta t'\Delta f' = \Delta t\Delta f \geq \frac{\hbar}{2}$. Therefore, the uncertainty holds. In this section we gave the effect of three generators of $SL(2, \mathbb{R})$ on the Wigner distributions and therefore, on ambiguity functions and marginalizable TFDs. As matrices in $SL(2, \mathbb{R})$ can be written in terms of these three generators, the relations of the others can be deduced from the ones we gave here. We have also investigated which signal corresponds to the resulting Wigner distribution.

These relations can be used for waveform design in audio and biomedical applications where there are multi input and output sensors. Furhtermore, they can be used in MIMO radar and communications. In addition, they can be used in integrated sensing and communication problems for waveform design and coding. As these relations give invariant forms, they can be seen as guidelines for waveform design.

Besides their use in signal processing tasks, these relations are mainly used in quantum mechanics. These relations are used in quantum harmonic analysis to see the effect on Wigner distributions. However, the motivation in quantum mechanics is different from that of signal processing. Naturally, points of interest and use cases are different too. Therefore, important topics in quantum mechanics might be not useful for signal processing and vice versa. Here, we are adopting a signal processing approach to the problem. It lifts directly as the spaces used and the distributions are the same. Although, the meanings they carry are different. In quantum mechanics, we are interested in an object's physical state and we try to observe the probability of its' being in a position and momentum (or time and energy). However, in signal processing tasks, we are interested in a real world signal's properties. The properties we observe are the time and frequency localization of the signal.

In this Chapter, we have discussed effects of symplectic transformations on time frequency distributions. This topic can be further studied. In quantum harmonic analysis, in the recent years, there is an interest in studying uncertainty relations of time frequency distributions and relating them with quantum entities.

An interesting topic here is the connection between Heisenberg uncertainty, Wigner-Yanase skew information and Quantum Fisher information. The relation was shown in [44]. This relation can be further investigated in terms of algebraic relations, topology and geometry. This analysis will bring insights on information theoretic measures, their connection with uncertainty relations, and how we can make changes on information theoretic measures. These analyses can be used in quantum information theory and quantum harmonic analysis. It was shown in [44] that information between two quantum entities is related with the amount of noncommutativity between the two.

As there is a relation between quantum and classical phenomenon, relations for classical and quantum information measures can are also given. By using these relations, one can further proceed and relate classical information measures with Heisenberg uncertainty relation.

CHAPTER 6

CONCLUSIONS

In this thesis, harmonic analysis methods are applied to certain time frequency distributions to understand their nature and behavior. They are investigated in terms of representation theory of Heisenberg group. With this investigation, their algebraic aspects are observed. Further, uncertainty relations and their implications are discussed. Firstly, MIMO ambiguity functions are related with irreducible unitary representations of $H_{\mathbb{R}}$. Some harmonic analysis relations found for traditional radar in [6], [46], [47] are generalized to MIMO case. In Chapter 4, the norm of MIMO ambiguity functions is shown to be directly determined by norm of individual, seperate signals. It was shown that, the MIMO ambiguity functions create an orthonormal basis of $L_2(\mathbb{R}^2)$, which can be used for time frequency signal processing applications. Moreover, orthogonality of MIMO ambiguity functions arising from two orthonormal sets was given. In addition to these, results from [6] and [46], concerning positive definiteness and an identity between different functions with same ambiguity functions are used in MIMO ambiguity functions. Furthermore, some symmetry relations of ambiguity functions are investigated by use of the action of generators of the group $SL(2,\mathbb{R})$. Signals creating such relations and their corresponding ambiguity functions are studied.

We have also given uncertainty relations for MIMO ambiguity functions in chapter 4. We investigated relations between norms of MIMO ambiguity functions and signals creating the ambiguity function. We gave local uncertainty relation on MIMO ambiguity functions and gave a bound on the delay Doppler support of it. We have related the uncertainty relation of [40] with MIMO ambiguity functions. In addi-

tion to these, we gave a variety of norm relations on MIMO correlation matrix. We have tied Lieb's uncertainty with p-norm of MIMO correlation matrix. We have used the MIMO ambiguity relations in the time frequency sense. These relations can be applied to MIMO Wigner distribution by Fourier transforms and algebraic manipulations. Moreover, these relations can be used in compressed sensing or integrated sensing and communications research.
In the literature, these relations are given for traditional ambiguity functions and time frequency distributions. Most relations only consider self ambiguity and Wigner distributions. Here, in using MIMO systems, we generalized results to cross terms. Cross terms are used together with self terms. Effect of using MIMO systems is investigated. Most results for traditional ambiguity functions don't hold for MIMO case. Therefore, generalizations and algebraic manupilations are done on them.

Finally some uncertainty relations on Wigner distributions and marginalizable TFDs are given. We used some results in [15] with a signal processing approach in relation with TFD design. We have used the important relation on Wigner distributions in [48] in the TFD context. We have shown which signals are corresponding to the actions of the special linear group. Moreover, we have investigated the effect of the generators of $SL(2,\mathbb{R})$ on Wigner distributions as symplectic matrices. We have mentioned the relationship between these relations and quantum mechanics. Moreover, the uses of these relations in quantum harmonic analysis were clarified. Furthermore, we have tied these relations to applications of quantum information theory and quantum harmonic analysis. These fields are getting great interest in recent years, and we believe applications given here can find applications in quantum information theory research.

The relations we gave here are frequently used in quantum mechanics. However, as we mentioned in previous chapters, motivations of quantum mechanics and signal processing are different. Therefore, points of interest in these branches are naturally different. We have used all these relations mainly for signal processing tasks. The mathematics carry directly as the spaces and operations used in quantum mechanics and signal processing are similar. They can actually seen to be the same mathematically in many points. Although, the meaning these relations carry are different in these branches. We have used this difference and applied certain methods and operations to signal processing. This interaction can be done the reverse way too, which might yield interesting results in quantum harmonic analysis and information theory.

6.1 Future Work

These analyses can be used in many applications. Examples are time frequency analysis, compressed sensing, integrated sensing and communication, quantum harmonic analysis and quantum information theory. In time frequency analysis, these can be used in the multi input multi output sensor cases. This can be the case for many audio and biomedical problems. In compressed sensing, uncertainty relations for MIMO case can be used for foundational aspects of the topic. In integrated sensing and communication, these can be used for better localization and waveform design for such purposes. Moreover, it can be used to analyse information theoretic problems in integrated sensing and communication.

In quantum harmonic analysis, similar problems are frequently studied. An interesting further direction is to use the connection of Heisenberg uncertainty relation, Wigner-Yanase Skew information and Quantum Fisher information. By use of this connection and using MIMO structure given in the thesis, algebraic structure of MIMO can be further used with more important results. By studying these relations, more on the connection of the three and how to use the connection can be made clear. Results on their algebraic, topological and geometric structure can be reached.

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Appendix A

MATHEMATICAL PRELIMINARIES

Here, we will briefly give certain points of representation theory to have better clarity. More details can be found in most harmonic analysis books such as [46], [24]

A.1 Group Representations

Here, we will define group representations and some important points we have used in the analysis.

A.1.1 Definition

Definition A.1.1. *Let G be a group. A representation of G is a homorphism $\mathbf{T}$ from G into $GL(V)$, where V is the representation space. The dimension of $\mathbf{T}$ is the dimension of V. We have, for all $g, h \in G$,*

$$\mathbf{T}(g)\mathbf{T}(h) = \mathbf{T}(gh) \tag{A.1}$$

$$\mathbf{T}(g)^{-1} = \mathbf{T}(g^{-1}) \tag{A.2}$$

Usually, ρ or another greek letter for representations are used as they are homomorphisms. Here we will use $\mathbf{T}$ to denote representations as they will be used as operators on time frequency distributions. Denoting it as such is merely because it is more similar to signal processing notation. Representations can be realized with matrices and linear transformations by using a basis of V to pass from matrix to linear transformation. Different choices of bases give different representations of G.

Definition A.1.2. *Two representations* $\mathbf{T}, \mathbf{T}'$ *of* G *on* V, V' *respectively are equivalent if there exists an invertible matrix* $\mathbf{S}$ *from* V *to* V'*, such that,*

$$\mathbf{T}'(g) = \mathbf{S}\mathbf{T}(g)\mathbf{S}^{-1} \tag{A.3}$$

for all $g \in G$ *holds.*

We denote equivalance by $\mathbf{T} \cong \mathbf{T}'$

Definition A.1.3. *Let* U *be a subspace of* V*. If* $\mathbf{T}(g)\mathbf{u} \in U$ *for all* $u \in U$ *and* $g \in G$*,* U *is called invariant under* $\mathbf{T}$*.*

Definition A.1.4. *If there is a proper subspace* U *of* V*, which is invariant under* $\mathbf{T}$*, then* $\mathbf{T}$ *is called reducible. If no such subspace exists, then* $\mathbf{T}$ *is irreducible.*

If U is a proper subspace of V and $U^{\perp}$ the subspace orthogonal to U, we can show that $V = U \oplus U^{\perp}$. If $\mathbf{T}$ is a reducible representation of G on V and $V = U \oplus U^{\perp}$, we can decompose $\mathbf{T}$ into $\mathbf{T} = \mathbf{T}_1 \oplus \mathbf{T}_2$. Here, if $\mathbf{T}_1$ or $\mathbf{T}_2$ is still reducible, we can continue to decompose until we reach irreducible representations. The decomposition can be written as,

$$V = U_1 \oplus U_2 \oplus \ldots \oplus U_n \tag{A.4}$$

and

$$\mathbf{T} = \mathbf{T}_1 \oplus \mathbf{T}_2 \oplus \ldots \oplus \mathbf{T}_n \tag{A.5}$$

where U_i transform irreducibly under the action of $\mathbf{T}|U_i$.

A.2 Representations of the Heisenberg group $H_{\mathbb{R}}$

A.2.1 Induced representations of $H_{\mathbb{R}}$

Here we recall $H_{\mathbb{R}}$ group,

$$H_{\mathbb{R}} = \{(x_1, x_2, x_3) = \begin{bmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{bmatrix} : x_i \in \mathbb{R}\} \tag{A.6}$$

Here, we will use a subgroup H of G and we define a representation T_H on H and then induce a representation for G. This method is a well known method in harmonic analysis called the method of induction. We will give the outline, details can be found in [46].

A.2.2 The Schroedinger Representation

Let us consider the case $G = H_R$ and H is the subgroup of H_R. $H = X(0, x_2, x_3)$. Let us look at $X(0, x_2, x_3)X(0, x_2', x_3') = X(0, x_2 + x_2', x_3 + x_3')$. It is clear that $H = X(0, x_2, x_3)$ is Abelian. Abelian groups have one dimensional unitary irreducible representations. We can find $TX(0, x_2, x_3) = e^{i2\pi x_3}$ (which is a one dimensional, irreducible, unitary representation).
Since,

$$X(x_1, x_2, x_3) = X(0, x_2, x_3)X(x_1 0, 0) \tag{A.7}$$

we may parametrise $H\backslash G$ (which is the subgroup created with elements $X(x_1 0, 0)$) by the coordinate $x_1 = t$. We may consider the scalar functions $f(t)$. If $X(x_1, x_2, x_3)$ acts on $f(t)$, it transforms the function $f(t)$ to

$$f(t + x_1, x_2, x_3 + tx_2) = T(Y)f(t + x_1) \tag{A.8}$$

where $Y = X(0, x_2, x_3 + tx_2)$. (This is like the matrix multiplication and we induce a representation from T, we induce a representation from subgroup H of G)(T is a representation of H). Thus the action of $H_{\mathbb{R}}$ on this space is

$$TX(x_1, x_2, x_3)f(t) = e^{i2\pi(x_3 + x_2 t)}f(t + x_1) \tag{A.9}$$

T is a representation, and it is unitary with respect to the inner product on V.