SOUND VELOCITY IN DENSE MATTER SUCH AS NEUTRON STARS

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ABSTRACT

SOUND VELOCITY IN DENSE MATTER SUCH AS NEUTRON STARS

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Properties of matter at ultra-high density, called dense matter, is an important subject that has been studied theoretically and experimentally in recent years. In a very dense system, a composite matter consisting of nucleons, pions, hadrons would overlap, so the new form of matter constituting quarks and gluons would occur at a baryon density of around ten times the ordinary nuclear density. Such a transition could have appeared in the early universe during the first microsecond of the Big Bang, in the core of a Neutron Star, or during high energy collisions of massive nuclei in terrestrial particle accelerators. In this thesis, we will analyze some approaches addressing the equation of state (EoS) of dense matter. Specifically, the speed of sound, which is a complicated function of state functions such as density and pressure in dense matter, will be studied within various approaches by considering the fact that the speed of sound should not exceed the speed of light, which puts a constraint on possible equations of state.

Keywords: Speed of Sound, Dense Matter, Compact Stars, Equation of State, Relativistic Fluid Systems, Quantum Chromodynamics, Quark-Gluon Plasma

ÖZ

NÖTRON YILDIZLARI GİBİ YOĞUN MADDELERDE SES HIZI

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Yoğun madde olarak adlandırılan yüksek yoğunluktaki maddelerin özellikleri teorik ve deneysel olarak son zamanlarda ele alınan önemli bir konudur. Normal nükleer yoğunluğunun yaklaşık on katı kadar baryon yoğunluğunun olduğu çok yoğun bir sistemde, nukleonlardan, pionlardan, hadronlardan oluşan kompozit madde birbiri üstüne çökmesi sonucu quark ve gluonlardan oluşan yeni bir madde formu oluşabilir. Bu geçiş, evrenin oluşumu sırasında büyük patlamanın ilk mikrosaniyesinde, Nötron yıldızının çekirdek kısmında veya parçacık hızlandırıcılarında kütleli çekirdeklerin çok yüksek enerjide çarpıştırılması sırasında görülebilir. Bu tezde, yoğun maddenin hal denklemini ele almak için bazı yaklaşımları analiz edeceğiz. Bu yaklaşımlar doğrultusunda ses hızının ışık hızını aşmaması gerekliliği göz önünde bulundurularak yoğun maddede yoğunluk ve basınçla ilişkili olan ses hızı hesaplanacak. Diğer taraftan, bahsedilen bu gereklilik yoğun maddenin olası hal denklemleri için bir sınırlandırma getirir.

Anahtar Kelimeler: Ses Hızı, Yoğun Madde, Kompakt Yıldızlar, Hal Denklemi, Gö-

receli Akışkan Sistemler, Kuantum Kromodinamiği, Kuark-Gluon Plazma

To art, philosophy, and science

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LIST OF ABBREVIATIONS

EoS	Equation of State
TOV	Tolman-Oppenheimer-Volkoff
MCFR	Momentarily Comoving Reference Frame
RMF	Relativistic Mean Field
QCD	Quantum Chromodynamics
QGP	Quark-Gluon Plasma
GWs	Gravitational Waves

CHAPTER 1

INTRODUCTION

The speed of sound problem was initially posed by Isaac Newton in his book *Principia*, and he calculated it with a difference of almost twenty percent between theoretical and experimental values [1]. The reason for this discrepancy is that Newton did not realize heat in compression and that the sound vibrations took place so fast that this caused increasing local temperature and pressure. Laplace made the correlation by considering the adiabatic process rather than isothermal which was Newton's assumption. The known equation of state of a system (EoS) that is the relationship between pressure and energy density, allows one to find the speed of sound (v_s), which means matter itself plays an essential role in determining v_s whose value in dense media is one of the debated questions recently.

There are a variety of ways to describe dense and cold nuclear matter. In this thesis, we will examine some models in a particular order to understand the dense matter by considering Neutron Stars, the most valuable arena for this purpose. Before going on to these approaches, the background will be given in the first three chapters. The definition of the speed of sound in the aspect of fluid dynamics is given in this chapter. In Chapter 2, we will study the story of the compact stars as degenerate fermion systems.

In Chapter 3, the Relativistic Mean Field Theory (RMF) will be used for the description of compact star's matter up to the critical density of the phase transition to quark matter. The RMF theory describes an interaction between nucleons in the matter along the two mesons called the scalar σ and vector w, so the model is also named as $(\sigma - w)$ model or nuclear field theory whose extensions are suitable for the description of neutron star matter. After constructing the Lagrangian density which should be a

Lorentz scalar, one can calculate the partition function using mathematical tools such as Grassmann variables. The EoS and v_s for this model can be obtained by following the relationship between thermodynamic properties and partition functions. Then, to check the result of the theory, the bulk nuclear properties which are binding energy, saturation density, incompressible modulus and effective nucleon will be used. If the results do not match with the bulk properties, one can add cubic and quadratic terms to the Lagrangian density to explain the known qualities of nuclear matter.

In Chapter 4, the fundamental theory of quarks and gluons or quantum chromodynamics (QCD) will be studied. The aim of this chapter is to find the EoS of QCD plasma for the One-Loop level. To do this, the functional integral representation of the partition function (Z) containing ghosts will be constructed first. Then, the perturbated thermodynamical potential (Ω) up to the $\alpha^2 \ln \alpha$ in which α represents gauge parameter will be calculated by using the relation between Z and Ω, where the renormalization group is used in order to improve the expansion by letting α to be a function of the temperature and chemical potential. Finally, the EoS and v_s can be found by using the thermodynamical quantities associated with Ω. In Chapter 5, an other approach examined for dense matter is Quark-Gluon plasma (QGP) by considering the MIT Bag Model, meaning that the effect of confining vacuum structure is going to be included.

In chapter 6, the neutron core is assumed as a perfect fluid, and the speed of sound in a static homogeneous relativistic fluid will be calculated by using the fundamental equations of the relativistic hydrodynamics. In the fluid approach, there are more assumptions such that the neutron star's core is a mixture including both dark matter and ordinary nuclear matter. This dark matter admixed with neutron star model can be considered in some different perspectives such that dark matter is assumed to be free Fermi gas or mirror dark matter [2, 3]. Its effect on neutron star's properties like mass-radius relation by solving two-fluid Tolman-Oppenheimer-Volkoff equations can be observed, which enables one to present the mass-radius diagram. The mass, amount of the dark matter, and the interaction between them can also be taken into account [4]. In some cases, the dark matter can be considered as mirror baryons and present modified mass-radius relation in the existence of a percent mirror baryons can be obtained by using the minimal parity symmetry extension of the standard model [5].

The effect of the asymmetric behavior of the dark matter on the mixed neutron stars can give us the opportunity to compare the mass or radius of it with the ordinary neutron stars [6, 7]. The properties of the mixed neutron stars made up of quark matter and fermionic dark matter are also analyzed by getting mass-radius relation from TOV equations for two fluids [8].

Lastly, one of the exciting approaches being searched recently is related to the Gravitational waves (GWs), which should be mentioned here, although it is not included as a separate title in the thesis. The above-mentioned quark-gluon plasma situation may occur in the hyper-massive neutron stars, which take shape in the aftermath of the merger of two neutron stars and exist for a short while before collapsing into a black hole [9]. The key signatures of the features of neutron stars' core and the hot dense matter equation of state are carried by GWs emitted during the last few cycles of neutron star inspiral and the ringdown of the remnant. In a binary system, a neutron star can tidally deform its companion, causing the system to inspiral and combine more quickly. For neutron stars with larger radii, or cores with stiffer equations of state, such as hadronic matter, the tidal deformation is stronger. Unbounded quark matter cores, on the other hand, will have smaller radii and be more difficult to deform, so tidal effects will have less of an impact on orbital evolution. Tidal effects are directly stored in the radiated GWs; however, they appear at the fifth-post-Newtonian order. Furthermore, the (initially) extremely deformed hyper-massive neutron star could produce GW emission during the post-merger period; the spectral characteristics of the produced signal can be directly mapped to the equation of state of 'hot' dense matter, including probable phase transitions [10, 11].

1.1 Speed of Sound

1.1.1 Conservation Laws and Constitutive Equations of Fluid Dynamics

Gas and liquids are considered as a continuum in the fluid dynamics. The fluid motion can be delineated by the conservation laws of mass, momentum and energy applied to an elementary fluid particles [12, 13]. The mass conservation law in differential form is

$$\frac{\partial \rho}{\partial t} + \frac{\partial (v_i \rho)}{\partial x_i} = m, \tag{1.1}$$

where v_i is the flow velocity at position x_i and time t, ρ is the fluid density. The momentum conservation law in differential form or Euler equation is,

$$\frac{\partial (v_i \rho)}{\partial t} + \frac{\partial (P_{ij} + v_i v_j \rho)}{\partial x_j} = f_i + m v_i, \tag{1.2}$$

where is f_i is an external force density such as gravitational force and P_{ij} is the minus the fluid stress tensor relating to the viscous stress tensor τ_{ij} and the pressure p, that is, $P_{ij} = p\delta_{ij} - \tau_{ij}$.

After substituting the mass conservation into momentum conservation, one gets

$$\rho \frac{\partial v_i}{\partial t} + \frac{\partial P_{ij}}{\partial x_j} + \rho v_j \frac{\partial v_i}{\partial x_j} = f_i. \tag{1.3}$$

The viscous stresses related to the deformation rate of the fluid element can be ignored in most of the applications. As long as this relation is linear, It is possible to mention about the Newtonian fluid, so Navier-Stokes equation as the conservation of the momentum. By assuming the Stoke's hypothesis, the thermodynamic pressure and the pressure are equal to the each other for fluid in LTE (Local Thermodynamic Equilibrium). In this situation, one gets constitutive the equation.

$$\tau_{ij} = \xi \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) - \frac{2}{3} \xi \left(\frac{\partial v_k}{\partial x_k} \right) \delta_{ij}, \tag{1.4}$$

where ξ is the dynamic viscosity relying on the pressure and temperature. The energy conservation law in differential form or internal energy equation is

$$\frac{\partial}{\partial t} \rho (e + \frac{1}{2} v^2) + \frac{\partial}{\partial x_i} \rho v_i (e + \frac{1}{2} v^2) = -\frac{\partial}{\partial x_i} p v_i - \frac{\partial q_i}{\partial x_i} + \frac{\partial (\tau_{ij} v_j)}{\partial x_i} + f_i v_i, \tag{1.5}$$

where q is the heat flux, e is the internal energy per unit of mass and $v = |\vec{v}|$. The relationship between heat flux and heat conductivity $K(p, T)$ can be set up with Fourier's law

$$q = -K\nabla T. \tag{1.6}$$

The equation for the entropy can be obtained by using the fundamental law of thermodynamics for a reversible process and the equation for mechanical energy, so

$$\left(\frac{\partial s}{\partial t} + \frac{\partial s}{\partial x_i} v_i\right) T\rho = \tau_{ij}\frac{\partial v_j}{\partial x_i} - \frac{\partial q_i}{\partial x_i}, \tag{1.7}$$

where s is the entropy per unit of mass. The flow is isentropic if heat conduction and viscous dissipation are ignored, so entropy of a fluid particle stays constant;

$$\frac{\partial s}{\partial t} + v \cdot \nabla s = 0. \tag{1.8}$$

The equation of state can be designated as $p = p(\rho, s)$, and the differential form of it is

$$dp = (\frac{\partial p}{\partial \rho})_s d\rho + (\frac{\partial p}{\partial s})_\rho ds, \tag{1.9}$$

One can get the $c^2 = (\frac{\partial p}{\partial \rho})_s$ which relates the wave speed with the rate of change of pressure with the density.

1.1.2 Formalization of the Speed of Sound

Newton who was the first to formalize the rate of change of pressure with density supposed that the temperature stay the same. He presumed that the conduction of heat from one region to the another so fast that the temperature remained unchanged, that is, the process is the isothermal. The correction comes from Laplace who suppose that the pressure and temperature alter adiabatically in a sound wave. Thus, one can get the relation in the adiabatic variation

$$PV^{\gamma} = constant,$$

where V is volume. The density is inversely proportional to the volume, so the relation becomes

$$P = const.\rho^{\gamma},$$

Thus, the speed of sound takes the final form as

$$c_s^2 = \gamma \frac{P}{\rho}. \tag{1.10}$$

where γ is the adiabatic index.

CHAPTER 2

DEGENERATE FERMION SYSTEMS

In accordance with Pauli's exclusion principle, two or more fermions cannot occupy the same quantum state, which determines the distribution probability of identical fermion particles. Identical particles can reach the same energy at the different states combinations called degeneracy, whose pressure determines the stellar evolution by comparing with gravitational pressure. Here, the pressing important question is what the critical density or degeneracy pressure preventing compact stars like neutron stars from collapsing is or what the critical radius, or mass of the stars is to determine whether gravitational collapse occurs or not. The EoS of cold dense matter determines the radii and masses of neutron stars by using Tolman-Oppenheimer-Volkov (TOV) [14, 15] relativistic stellar structure equations. Moreover, the EoS can be found from the observed mass and radius of the neutron stars by using the TOV equations which is the bridge between the microphysics described by EoS and macroscopic properties of the neutron stars [16, 17].

2.1 TOV equations

For a static and spherically symmetric spacetime, the metric is

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\gamma(r)}dr^2 + r^2 d\theta^2 + r^2 \sin\theta^2 d\phi^2. \tag{2.1}$$

By using the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.2}$$

$\Phi(r)$ and $\gamma(r)$ will be solved. Christoffel symbols

$$\Gamma^{\eta}_{\mu\nu} = \frac{1}{2} g^{\eta\alpha}(\partial_{\mu} g_{\nu\alpha} + \partial_{\nu} g_{\alpha\nu} - \partial_{\alpha} g_{\mu\nu}) \tag{2.3}$$

can be computed to be

$$\begin{aligned}
&\Gamma^{t}_{tr} = \Phi^{'} && \Gamma^{r}_{tt} = \Phi^{'} e^{2(\Phi-\gamma)} && \Gamma^{r}_{rr} = \gamma^{'} \\
&\Gamma^{\theta}_{r\theta} = r^{-1} && \Gamma^{r}_{\theta\theta} = -re^{-2\Phi} && \Gamma^{\phi}_{r\phi} = r^{-1} \\
&\Gamma^{r}_{\phi\phi} = -re^{-2\gamma}\sin^2\theta && \Gamma^{\theta}_{\phi\phi} = -\frac{\sin 2\theta}{2} && \Gamma^{\phi}_{\theta\phi} = \cot\theta
\end{aligned}$$

where prime represents derivation with respect to the r. Substituting these into Riemann tensor formula

$$R^{\rho}_{\alpha\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\nu\alpha} - \partial_{\nu}\Gamma^{\rho}_{\mu\alpha} + \Gamma^{\rho}_{\mu\eta}\Gamma^{\eta}_{\nu\alpha} - \Gamma^{\rho}_{\nu\eta}\Gamma^{\eta}_{\mu\alpha}, \tag{2.4}$$

the nonvanishing terms can be calculated to be

$$\begin{aligned}
&R^{t}_{rtr} = -(\Phi^{''} + (\Phi^{'})^2) + \Phi^{'}\gamma^{'}, && R^{t}_{\theta t\theta} = -r\Phi^{'} e^{-2\gamma}, \\
&R^{t}_{\phi t\phi} = -r\Phi^{'}\sin^2\theta e^{-2\gamma}, && R^{r}_{\theta r\theta} = r\gamma^{'} e^{-2\gamma}, \\
&R^{r}_{\phi r\phi} = r\gamma^{'}\sin^2\theta e^{-2\gamma}, && R^{\theta}_{\phi\theta\phi} = -\sin^2\theta(e^{-2\gamma} - 1).
\end{aligned}$$

Then, using the Ricci tensor defined as

$$R_{\nu\mu} = R_{\mu\nu} = R^{\eta}_{\mu\eta\nu}, \tag{2.5}$$

one finds

$$
\begin{aligned}
R_{tt} &= e^{2(\Phi-\gamma)}(\Phi'' + (\Phi')^2 - \Phi'\gamma' + \frac{2}{r}\Phi'),\\
R_{rr} &= -\Phi'' - (\Phi')^2 + \Phi'\gamma' + \frac{2}{r}\gamma',\\
R_{\theta\theta} &= 1 - e^{-2\gamma}(r(\Phi' - \gamma')),\\
R_{\phi\phi} &= \sin^2\theta R_{\theta\theta}.
\end{aligned}
$$

The Ricci scalar defined by

$$
R = R^{\mu}_{\mu} = g^{\mu\nu}R_{\mu\nu}, \tag{2.6}
$$

can be obtained by inserting Ricci tensor into it

$$
R = -2e^{-2\gamma}\left(\Phi'' + (\Phi')^2 - \Phi'\gamma' - \frac{2}{r}(\gamma' - \Phi') - (e^{2\gamma} - 1)\right). \tag{2.7}
$$

The Einstein tensor (2.2) has the following components

$$
\begin{aligned}
G_{tt} &= r^{-2}e^{2(\Phi-\gamma)}(2r\gamma' + e^{2\gamma} - 1) = 8\pi G T_{tt},\\
G_{rr} &= r^{-2}(2r\Phi' - e^{2\gamma} + 1) = 8\pi G T_{rr},\\
G_{\theta\theta} &= r^2 e^{-2\gamma}\left(\Phi'' + (\Phi')^2 - \Phi'\gamma' - r^{-1}(\gamma' - \Phi')\right) = 8\pi G T_{\theta\theta},\\
G_{\phi\phi} &= \sin^2\theta G_{\theta\theta} = 8\pi G T_{\phi\phi}.
\end{aligned} \tag{2.8}
$$

Now, the right side of the Einstein equation needs to be calculated. The star is considered as the perfect fluid with the energy momentum tensor

$$
T_{\mu\nu} = (p + \rho)u_\mu u_\nu + p g_{\mu\nu}, \tag{2.9}
$$

where the four velocity is taken as

$$
u_\mu = (e^{\Phi}, 0, 0, 0). \tag{2.10}
$$

for a static perfect fluid one. Hence, the components of the energy momentum tensor are

$$T_{\mu\nu} = \begin{pmatrix} \rho e^{2\Phi(r)} & 0 & 0 & 0 \\ 0 & p e^{2\gamma(r)} & 0 & 0 \\ 0 & 0 & r^2 p & 0 \\ 0 & 0 & 0 & r^2 p \sin^2\theta \end{pmatrix}. \tag{2.11}$$

Finally, the Einstein equations are

$$\begin{aligned} r^{-2} e^{-2\gamma}(2r\gamma' + e^{2\gamma} - 1) &= 8\pi G\rho, \\ r^{-2} e^{-2\gamma}(2r\Phi' - e^{2\gamma} + 1) &= 8\pi G p, \\ e^{-2\gamma}\Big(\Phi'' + (\Phi')^2 - \Phi'\gamma' - r^{-1}(\gamma' - \Phi')\Big) &= 8\pi G p. \end{aligned} \tag{2.12}$$

It's not necessary to write the $\phi\phi$ component being proportional to the $\theta\theta$ component. Since R_{tt} and R_{rr} vanish independently, it is possible to write

$$\begin{aligned} 0 &= R_{tt} e^{-2(\Phi-\gamma)} + R_{rr} \\ &= 2r^{-1}(\Phi' + \gamma') \rightarrow \Phi = -\gamma, \end{aligned} \tag{2.13}$$

where constant coming from integral is taken as zero. $R_{\theta\theta} = 0$ gives

$$\begin{aligned} e^{-2\gamma}(2r\gamma' + 1) &= 1, \\ \partial_r(re^{2\gamma}) &= 1, \end{aligned} \tag{2.14}$$

$$e^{2\gamma} = \left(1 - \frac{R_S}{r}\right)^{-1}, \tag{2.15}$$

where $R_S = 2GM$ is called the Schwarzschild radius. After rewriting,

$$e^{2\gamma} = \left(1 - \frac{2Gm(r)}{r}\right)^{-1}, \tag{2.16}$$

substituting the last one into rr component of the Einstein equation, one gets

$$\Phi' = \frac{m(r)G + 4\pi r^3 Gp}{r^2 - 2Gm(r)r}. \tag{2.17}$$

From the conservation of the energy momentum tensor,

$$\begin{aligned}
\nabla_\mu T^{\mu\nu} &= 0, \\
\nabla_\mu \bigg((p+\rho) u^\mu u^\nu + p g^{\mu\nu} \bigg) &= 0, \\
u_\nu u^\mu \nabla_\mu (\rho + p) + (\rho + p) u^\mu \nabla_\mu u_\nu + (\rho + p) u_\nu \nabla_\mu u^\mu + \nabla_\nu p &= 0, \\
u_\nu u^\mu \nabla_\mu (\rho + p) + (\rho + p) u^\mu \partial_\mu u_\nu - (\rho + p) u^\mu \Gamma^\lambda_{\mu\nu} u_\lambda + & \\
(\rho + p) u_\nu g^{-1/2} \partial_\mu (g^{1/2} u^\mu) + \partial_\nu p &= 0.
\end{aligned} \tag{2.18}$$

Nonvanishing terms are

$$\begin{aligned}
-(p+\rho) u^\mu \Gamma^\lambda_{\mu\nu} u_\lambda + \partial_\nu p &= 0, \\
-(p+\rho) u^t \Gamma^t_{tr} u_t + \partial_r p &= 0, \\
(p+\rho) \partial_r \Phi + \partial_r p &= 0.
\end{aligned} \tag{2.19}$$

Thus, one has

$$\frac{dp}{dr} = -(p+\rho) \frac{d\Phi}{dr}. \tag{2.20}$$

Here, pressure is the relativistic correction to the Newtonian expression of Gravitational Potential that is the reason why Φ is used at the beginning of the story. Inserting the equation (2.17) into last one gets

$$\frac{dp}{dr} = -\frac{(p+\rho)(4\pi r^3 pG + m(r)G)}{r^2 - 2GRm(r)}, \tag{2.21}$$

which is known as Tolman-Oppenheimer-Volkoff (TOV) equation. The relationships between the mass and radius in rough calculation,

$$\rho(r) = \begin{cases} \rho_0, & r < R, \\ 0, & r > R, \end{cases}$$

and

$$m(r) = \begin{cases} \frac{4}{3}\pi r^3 \rho_0, & r < R, \\ \frac{4}{3}\pi R^3 \rho_0 = M, & r > R. \end{cases}$$

TOV equation for $r < R$ becomes

$$p(r) = \rho_0 \left(\frac{(1 - 2GMr^2/R^3)^{1/2} - (1 - 2GM/R)^{1/2}}{3(1 - 2GM/R)^{1/2} - (1 - 2GMr^2/R^3)^{1/2}} \right). \tag{2.22}$$

At $r = 0$, the dominator is

$$3(1 - \frac{2GM}{R})^{1/2} = 1 \rightarrow R = \frac{9}{4}GM, \tag{2.23}$$

where $M = \frac{4}{3}\pi R^3 \rho_0$.

2.2 Polytropes and White Dwarfs

Newtonian physics describe the most of the stars, which helps one to both get the limiting cases for more exotic objects like neutron star, black hole and to understand the properties of these objects. By considering the properties of the Newtonian stellar structure physics such as the internal energy and pressure are much less than the rest mass density ($nm_N \gg e, p$), the gravitational potential is small everywhere ($\frac{2MG}{r} \ll 1$), TOV equation can be reduced to

$$\frac{d}{dr}\frac{r^2}{\rho(r)}p'(r) = -4\pi G r^2 \rho(r). \tag{2.24}$$

In the polytropic picture, the equation of state is

$$p = K\rho(r)^{1+\frac{1}{n}}. \tag{2.25}$$

By introducing the dimensionless radius ζ and function $\vartheta(\zeta)$,

$$\begin{aligned} r &= b\zeta, \\ \rho(r) &= \rho_c \vartheta(\zeta)^n, \end{aligned} \tag{2.26}$$

where

$$\begin{aligned} b &= (\frac{(n+1)K\rho_c^{\frac{1}{n}-1}}{4\pi G})^{1/2}, \\ \rho_c &= \rho_c(r=0). \end{aligned} \tag{2.27}$$

(2.24) can be transformed into dimensionless form named Lane-Emden equation

$$\frac{1}{\zeta}\frac{d}{d\zeta}(\zeta^2\frac{d\vartheta(\zeta)}{d\zeta}) = -\vartheta^n, \tag{2.28}$$

by using the boundary conditions at center ($\vartheta(0) = 1$, $\vartheta^{'}(0) = 0$), $M(r)$ and R are computed to be respectively

$$\begin{aligned} M(r) &= 4\pi b^3 \rho_c \int_0^{\zeta_1} \zeta^2 \vartheta^n d\zeta \\ &= 4\pi \left(\frac{(n+1)K}{4\pi G}\right)^{3/2} \rho_c^{\frac{3-n}{2n}} \zeta_1^2 |\vartheta^{'}(\zeta_1)| \\ R &= \left(\frac{(n+1)K}{4\pi G}\right)^{1/2} \rho_c^{\frac{1-n}{2n}} \zeta_1 \end{aligned} \tag{2.29}$$

leaving alone ρ_c from R, then substituting into $M(r)$ gives relation between M and R

$$M = 4\pi R^{\frac{3-n}{1-n}} \left(\frac{(n+1)K}{4\pi G}\right)^{\frac{n}{n-1}} \zeta_1^{\frac{n-3}{1-n}} \zeta_1^2 |\vartheta^{'}(\zeta)|. \tag{2.30}$$

The energy density, pressure and number density can be written respectively by summing over the occupied states [18]

$$
\begin{aligned}
n &= \frac{g}{2\pi^2\hbar^3}\int_0^{k_F} k^2 dk = \frac{k_F^3}{3\pi^2\hbar^3}, \\
p &= \frac{1}{3}\frac{g}{2\pi^2\hbar^3}\int_0^{k_F} \frac{k^2}{\sqrt{k^2+m^2}} k^2 dk \\
&= \frac{1}{12\pi^2\hbar^3}(k_F\sqrt{m^2+k_F^2}\frac{(2k_F^2-3m^2)}{2}) + \frac{3}{2}m^4\ln(\frac{k_F^2+\sqrt{k^2+m^2}}{m}), \qquad (2.31)\\
\epsilon &= \frac{g}{2\pi^2\hbar^3}\int_0^{k_F} k^2\sqrt{k^2+m^2}dk \\
&= \frac{1}{4\pi^2\hbar^3}(k_F^2(\frac{m^2+2k_F^2}{2})\sqrt{m^2+k_F^2} - \frac{m^4}{2}\ln(\frac{k_F^2+\sqrt{k^2+m^2}}{m}),
\end{aligned}
$$

$$
\rho = n m_N \eta. \qquad (2.32)
$$

where g is the degeneracy factor formulated as $g = 2s + 1$ and its value is 2 for fermions because of having spin $\frac{1}{2}$. The pressure of an electron is calculated for an isotropic distribution of momenta, so the factor $\frac{1}{3}$ in front of the pressure formula arises from isotropy. Also, η is the number of nucleons per electron; η is approximately equal to the 2 for stars having exhausted their hydrogen. The maximum momenta can be found by using the number density and the mass density formulas;

$$
k_F = \hbar\left(\frac{3\pi^2\rho}{m_N\eta}\right)^{1/3}. \qquad (2.33)
$$

The EoS can be reduced to the polytrope in two cases which are $\rho_c \gg \rho$ and $\rho \gg \rho_c$, where ρ_c is the critical density coming from $k_F = m_e$, so

$$
\rho_c = \frac{m_N\eta {m_e}^3}{3\pi^2\hbar^3} \approx 1\times 10^6\eta \quad g/cm^3. \qquad (2.34)
$$

For the relativistic limit $k \gg m$ and $\rho \gg \rho_c$, the pressure and energy density becomes respectively

$$
p \approx \frac{1}{12\pi^2\hbar^3}(k_F^4 + \frac{3m^4}{2}\ln\frac{2k_F}{m}), \qquad (2.35)
$$

$$\epsilon \approx \frac{1}{4\pi^2\hbar^3}(k_F^4 - \frac{m^4}{2}\ln\frac{2k_F}{m}). \tag{2.36}$$

The logarithmic terms can be ignored since they are small compared to k^4 in the ultrarelativistic limit, so the equation of the state is

$$\epsilon \to 3p \approx \frac{(3\pi^2 n)^{\frac{4}{3}}}{4\pi^2}. \tag{2.37}$$

The speed of sound for the degenerate ideal fermion gas can be calculated by using the EoS

$$v_s^2 = \frac{dp}{d\epsilon} = \frac{1}{3}. \tag{2.38}$$

By using k_F, the relation between pressure and mass density gives the polytrope case

$$p = \frac{\hbar^3}{12\pi^2}\left(\frac{3\pi^2}{m_N\eta}\right)^{4/3}\rho^{4/3}, \tag{2.39}$$

where $\gamma = \frac{4}{3}$ and $K = \frac{\hbar^3}{12\pi^2}\left(\frac{3\pi^2}{m_N\eta}\right)^{4/3}$. The radius and the mass of the white dwarf can be calculated by considering the critical density and pluging the above equations into (2.29) and (2.30) equations, respectively

$$\begin{aligned} R &= \frac{\hbar^{3/2}}{2}\left(\frac{3\pi}{cGm_e^2m_n^2\eta}\right)^{1/2}\left(\frac{\rho_c}{\rho(0)}\right)^{1/3}(6.89685) \\ &= 5.3\times 10^4\left(\frac{\rho_c}{\rho(0)}\right)^{1/3}\eta^{-1}km, \\ M &= \frac{(3\pi)^{1/2}}{2}\left(\frac{\hbar c}{Gm_N^{4/3}\eta^{4/3}}\right)^{3/2}(2.01824) = 5.87\eta^{-2}M_\odot, \end{aligned} \tag{2.40}$$

where ζ and $-\zeta_1^2\theta'(\zeta_1)$ are 6.89685 and 2.01824, respectively.

Also, the last part of the (2.40) is the maximum mass for white dwarfs, which is called as the Chandrasekhar limit. A white dwarf against gravitational collapse by the pressure of cold degenerate electrons cannot be stable once its mass exceeds the Chandrasekhar limit.

2.3 Neutron Stars

Neutron stars having probably with large fraction of neutrons are compact stars containing matter of the supranuclear density in their interiors [19, 20]. Neutron stars's mass M ~ 1.4 M$_\odot$ is close to the solar mass M$_\odot$ = 1.989×10^{33} g while their radius $R \sim$ 10 km are $\sim$ 10^5 times smaller than the solar radius R$_\odot$ =6.96$\times10^5$ km. Their average mass density is

$$\bar{\rho} \simeq \frac{M}{\frac{4}{3}\pi R^3} \simeq 7 \times 10^{14} gcm^{-3} \sim (2-3)\rho_0 \tag{2.41}$$

where $\rho_0 = 2.8 \times 10^{14} gcm^{-3}$ is the normal nuclear density. Indeed, the central density of neutron stars can reach $(10-20)\rho_0$, which means neutron stars are the most compact stars known in the universe. While a white dwarf is supported by the degeneracy electron pressure, neutron star is supported by degeneracy neutron pressure against gravitational collapse. Thus, total energy density and pressure of an ideal Fermi gas consisting of neutrons can respectively be written as

$$\epsilon_n = g\frac{4\pi}{(2\pi\hbar^3)}\int_0^{k_F}\sqrt{(k^2+m_n^2)}k^2 dk = 3\rho_c\int_0^{k_F/m_n}(z^2+1)^1/2z^2 dz, \tag{2.42}$$

$$p_n = g\frac{4\pi}{(2\pi\hbar^3)}\int_0^{k_F}\frac{k^2}{\sqrt{(k^2+m_n^2)}}k^2 dk = \rho\int_0^{k_F/m_n}(z^2+1)^{-1/2}z^4 dz, \tag{2.43}$$

where the critical density for neutrons is

$$\rho_c = \frac{8\pi c^3 m_n^4}{3(2\pi\hbar^3)} = 6.1 \times 10^{15} g/cm^3. \tag{2.44}$$

For relativistic approach $k_F \gg m_n$ and $\rho(0) \gg \rho_c$, the total energy and pressure becomes

$$\epsilon_n = \frac{3\rho_c}{4}(\frac{k_F}{m_n})^5, \qquad\qquad p_n = \frac{\rho_c}{4}(\frac{k_F}{m_n})^5,$$

$$p = \frac{\epsilon}{3}. \tag{2.45}$$

CHAPTER 3

RELATIVISTIC MEAN FIELD THEORY

So far, a simple Fermi gas model of dense matter has been discussed. Relativistic Mean Field Theory (RMF) is an extension that contains interaction of the baryons along the mean fields of certain mesons. The ground state in both circumstances is a degenerate state composing of all particle momentum levels filled to Fermi momentum. The model called Walecka whose background given in the Appendix A is based on four particles fields which are nucleons, a scalar meson σ and omega vector meson ω. The reason of using these two mesons instead of the others such as pion and kaon is that the mean values of them in the normal ground state do not vanish in the approximation in which we work.

3.1 Walecka Model

In accordance with the model, the partition function can be calculated after constructing the Lagrangian density. The partition function (Appendix A) is

$$Z = \int [D\bar{\varphi}][D\varphi][D\sigma][D\omega] \exp \int_0^\beta d\tau \int d^3x \mathcal{L} \tag{3.1}$$

where the Lagrangian density is

$$\begin{aligned}\mathcal{L} &= \mu_n \varphi_n^\dagger \varphi_n + \mu_p \varphi_p^\dagger \varphi_p + \mathcal{L}_{\mathcal{W}} \\ &= \bar{\varphi}(i\gamma^\mu \partial_\mu - m_N)\varphi + \frac{1}{2}(\partial_\mu \sigma \partial^\mu \sigma - m_\sigma^2 \sigma^2) - \frac{1}{4}\omega_{\mu\nu}\omega^{\mu\nu} \\ &+ \frac{1}{2} m_\omega^2 \omega_\mu \omega^\mu + (g_\sigma \bar{\varphi}\sigma\varphi - g_\omega \bar{\varphi}\gamma^\mu \omega_\mu \varphi) + \mu_n \varphi_n^\dagger \varphi_n + \mu_p \varphi_p^\dagger \varphi_p,\end{aligned} \tag{3.2}$$

where $\varphi = \begin{pmatrix} \varphi_n \\ \varphi_p \end{pmatrix}$ with the φ_n neutron spinor and the φ_p proton spinor, $\bar{\varphi} = \varphi \dagger \gamma^0$, the first term in $\mathcal{L}_W$ is the free nucleon Lagrangian density and 2^{nd}, 3^{th}, 4^{th} comes from free mesonic Lagrangian density. The last two terms of the $\mathcal{L}_W$ is the interaction Lagrangian with Yukawa interaction between mesons and nucleons. Also, the chemical potential of the neutron (μ_n) and proton (μ_p) are taken as equal each one because of considering symmetry matter, which means $\mu_n = \mu_p = \mu$. In the meson equations, nucleons behaves as sources, which suppose that a net baryon density create vector and meson condensates, so σ and ω can have nonzero expectation values

$$\begin{aligned} \sigma &\rightarrow \bar{\sigma} + \sigma', \\ \omega_\mu &\rightarrow \delta_{\mu 0}\bar{\omega_0} + \omega'_\mu, \end{aligned} \tag{3.3}$$

where the bar indicates average value of the fields whereas prime represents the fluctuations. If the fluctuations are ignored, the mean field approximation is obtained. When the equation (3.3) is substituted into the equation (3.2) and then removed all derivative terms of the mesons, the Lagrangian density becomes

$$\mathcal{L} = \left(\bar{\varphi}(i\gamma^\mu \partial_\mu - m_N^* + \mu * \gamma_0)\varphi - \frac{1}{2}m_\sigma^2 \bar{\sigma}^2 + \frac{1}{2}m_\omega^2 \bar{\omega_0}^2 \right), \tag{3.4}$$

where

$$\begin{aligned} m_N^* &= m_N - \bar{\sigma} g_\sigma, \\ \mu* &= \mu - \bar{\omega_0} g_\omega, \end{aligned} \tag{3.5}$$

whereas μ is the actual chemical potential related to the nucleon number, the new effective chemical potential μ has the physical meaning because of determining the Fermi energy. After inserting Lagrange density into partition function (3.1), one gets

$$\begin{aligned} Z = (\exp \frac{V}{T}(\frac{1}{2}m_\sigma^2 \bar{\sigma}^2 + \frac{1}{2}m_\omega^2 \bar{\omega_0}^2)) \int [D\bar{\varphi}][D\varphi] \\ \times \exp \int_0^\beta d\tau \int d^3x \bar{\varphi}(i\gamma^\mu \partial_\mu - m_N^* + \mu^* \gamma_0)\varphi. \end{aligned} \tag{3.6}$$

Introducing the Fourier transforms

$$\varphi(x) = \frac{1}{V^{\frac{1}{2}}} \sum_p e^{-ip.x} \varphi(p), \qquad \bar{\varphi}(x) = \frac{1}{V^{\frac{1}{2}}} \sum_p e^{-ip.x} \bar{\varphi}(p),$$

where $p = (-i\omega_n, \boldsymbol{p})$, $x = (-i\tau, \boldsymbol{x})$ and $p = p_0 x_0 - \boldsymbol{p} \cdot \boldsymbol{x} = -(\omega_n \tau + \boldsymbol{p} \cdot \boldsymbol{x})$ with frequencies $\omega_n = (2n+1)\pi T$. After taking integral in the exponential,

$$Z = (\exp \frac{V}{T}(\frac{1}{2}m_\sigma^2 \bar{\sigma}^2 + \frac{1}{2}m_\omega^2 \bar{\omega_0}^2)) \int [D\varphi^\dagger][D\varphi] \exp\left(- \sum_p \varphi^\dagger(p) \frac{H^{-1}(p)}{T} \varphi(p) \right), \tag{3.7}$$

where the inverse nucleon propagator is

$$H^{-1}(p) = -\gamma^\mu p_\mu - \gamma_0 \mu^* + m_N^*,$$

by using the functional integral over the Grassmann variables, Z becomes

$$Z = (\exp \frac{V}{T}(\frac{1}{2}m_\sigma^2 \bar{\sigma}^2 + \frac{1}{2}m_\omega^2 \bar{\omega_0}^2)) \det \frac{H^{-1}(p)}{T}, \tag{3.8}$$

where the momentum space, the Dirac space and the neutron proton space are taken into account in determinant. To obtain pressure, log of the the partition function must be taken, so

$$\begin{aligned} \ln Z = &\frac{V}{T}\left(\frac{1}{2}m_\sigma^2 \bar{\sigma}^2 + \frac{1}{2}m_\omega^2 \bar{\omega_0}^2\right) \\ &+ 4V \int \frac{d^3p}{(2\pi)^3}\left(\frac{E_p}{T} + \ln(1 + e^{-\frac{E_p - \mu^*}{T}}) + \ln(1 + e^{-\frac{E_p + \mu^*}{T}})\right) \end{aligned} \tag{3.9}$$

where

$$E_p = \sqrt{p^2 + (m_N^*)^2}.$$

and P_{FG} is the Fermi nucleon pressure and the factor 4 is the sum of the two spin degrees of freedom and the two baryon (proton and neutron) degrees of freedom. Then, the pressure is

$$P(\mu, T) = \frac{T}{V} \ln Z = P_{FG}(\mu^*, T) + \frac{1}{2} m_\sigma^2 \bar{\sigma}^2 + \frac{1}{2} m_\omega^2 \bar{\omega_0}^2, \tag{3.10}$$

By maximizing the pressure, the meson condensate can be found and substituted into the above equation (3.10)

$$\begin{aligned} 0 &= \frac{\partial P(\mu, T)}{\partial \bar{\sigma}} = -m_\sigma^2 \bar{\sigma} - g_\sigma \frac{\partial P_{FG}}{\partial m_N^*}, \\ 0 &= \frac{\partial P(\mu, T)}{\partial \bar{\omega}} = m_\omega^2 \bar{\omega} - g_\omega \frac{\partial P_{FG}}{\partial \mu^*}. \end{aligned} \tag{3.11}$$

By using $\mathcal{L}_W$, the Euler-Lagrange equations in the presence of the interactions are

$$\begin{aligned} (\partial_\mu \partial^\mu + m_\sigma^2) \sigma(x) &= g_\sigma \bar{\varphi}(x) \varphi(x), \\ (\partial_\mu \partial^\mu + m_\omega^2) \omega_\mu(x) - \partial_\mu \partial^\nu \omega_\nu(x) &= g_\omega \bar{\varphi}(x) \gamma_\mu \varphi(x), \\ \Big(\gamma_\mu (i\partial^\mu - g_\omega \omega^\mu(x)) - (m - g_\sigma \sigma(x)) \Big) \varphi(x) &= 0. \end{aligned} \tag{3.12}$$

These can be reduced to the simple equations by considering the mean field approximations;

$$\begin{aligned} m_\sigma^2 \langle \sigma \rangle &= g_\sigma \langle \bar{\varphi} \varphi \rangle, \\ m_\omega^2 \langle \omega_0 \rangle &= g_\omega \langle \varphi^\dagger \varphi \rangle, \\ m_\omega^2 \langle \omega_\nu \rangle &= g_\omega \langle \bar{\varphi} \gamma_\nu \varphi \rangle, \end{aligned} \tag{3.13}$$

where "$\langle\rangle$" represents average. When the equations (3.11) are substituted into the equations (3.13), one gets the baryon and scalar densities

$$n_B = \langle \varphi^\dagger \varphi \rangle = \frac{\partial P_{FG}}{\partial \mu^*}$$
$$= 4 \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{e^{\beta(E^*-\mu^*)}+1} + \frac{1}{e^{\beta(E^*+\mu^*)}+1} \right), \tag{3.14}$$

$$n_S = \langle \bar{\varphi} \varphi \rangle = -\frac{\partial P_{FG}}{\partial m_N^*}$$
$$= 4 \int \frac{d^3p}{(2\pi)^3} \frac{m_N^*}{E^*} \left(\frac{1}{e^{\beta(E^*-\mu^*)}+1} + \frac{1}{e^{\beta(E^*+\mu^*)}+1} \right), \tag{3.15}$$

where $E^* = \sqrt{p^2 + m_N^{*2}}$. The relationship between average mesons and baryon-scalar densities is

$$\bar{\sigma} = \frac{g_\sigma}{m_\sigma^2} n_S, \tag{3.16}$$

$$\bar{\omega} = \frac{g_\omega}{m_\omega^2} n_B. \tag{3.17}$$

Equation (3.5) can be rewritten in terms of the effective mass

$$m_N^* = m_N - \frac{g_\sigma^2}{m_\sigma^2} n_S. \tag{3.18}$$

When the zero temperature limit is taken as $T \ll \mu, m_N$ and by using basic thermodynamics identity, equation (3.10) becomes

$$P = \frac{1}{4\pi^2} \left((\frac{2}{3} p_F^2 - m_N^{*2}) E_F^* p_F + m_N^{*4} \ln \frac{p_F + E_F^*}{m_N^*} \right)$$
$$+ \frac{g_\omega^2}{2m_\omega^2} n_B^2 - \frac{g_\sigma^2}{2m_\sigma^2} n_S^2, \tag{3.19}$$

where $E_F = \mu^* = \sqrt{p_F^2 + m_N^{*2}}$. To find the equation of the state, the energy density need also to be found. As mentioned earlier, the actual potential related to the baryon number n_B is μ, which means the pressure can be written as $P = -\epsilon + \mu n_B$ at zero temperature. The first term of the pressure equation (3.10) arises from $\epsilon_0+\mu^* n_B$. That is,

$$
\begin{aligned}
P_N = -\epsilon_0 + \mu^* n_B &= -\epsilon_0 + \mu n_B - g_\omega n_B \bar{\omega}_0 \\
&= -\epsilon_0 + \mu n_B - \frac{g_\omega^2}{m_\omega^2} n_B^2,
\end{aligned}
$$

where the effective chemical potential (3.5) and the vector meson condensate (3.18) are substituted into the above P_N equation. After this equation is replaced with the first term of the equation (3.10), one gets

$$
P = -\left(\epsilon_0 + \frac{g_\omega^2}{2m_\omega^2} n_B^2 + \frac{g_\sigma^2}{2m_\sigma^2} n_S^2\right) + \mu n_B. \tag{3.20}
$$

Thus, by using $P = -\epsilon + \mu n_B$, energy density becomes

$$
\epsilon = \epsilon_0 + \frac{g_\omega^2}{2m_\omega^2} n_B^2 + \frac{g_\sigma^2}{2m_\sigma^2} n_S^2, \tag{3.21}
$$

which yields

$$
\begin{aligned}
\epsilon = &\frac{1}{4\pi^2}\left((2p_F^2 + m_N^{*2})E_F^* p_F - m_N^{*4} \ln \frac{p_F + E_F^*}{m_N^*}\right) \\
&+ \frac{g_\omega^2}{2m_\omega^2} n_B^2 + \frac{g_\sigma^2}{2m_\sigma^2} n_S^2,
\end{aligned} \tag{3.22}
$$

where

$$
n_B = \frac{2p_F^3}{3\pi^2},
$$

$$
n_S = \frac{m_N^* p_F E_F}{\pi^2} - \frac{m_N^{*2}}{\pi^2} \ln\left(\frac{E_F^* + p_F}{m_N^*}\right). \tag{3.23}
$$

Before finding the relationship between the pressure P and the energy density ϵ, the pressure and energy density will be analyzed for both at low and high densities. At low density $p_F \to 0$, the quantities become

$$P = \frac{2}{15\pi^2}\frac{p_F^5}{m_N}, \quad (3.24)$$

$$\epsilon = (m_N + \frac{3}{10}\frac{p_F^2}{m_N})n_B. \quad (3.25)$$

At high density, $p_F \to \infty$

$$P = \epsilon = \frac{1}{2}(\frac{g_\omega^2}{m_\omega^2})n_B^2, \quad (3.26)$$

Finally, the speed of sound $c_s^2 = \frac{\partial P}{\partial \epsilon}$ approaches the speed of light at very high density.

3.1.1 Some Properties of Nuclear Matter

Nuclear matter is a saturated systems whose density at which pressure is zero and at which matter would stay static unless disturbed, because of the short range of the nuclear force, strong repulsion and Pauli principle. Even if more nucleons are inserted into the saturated nuclear matter, the density of the central region stays the same, as $n_0 = \frac{2p_F^3}{3\pi^2} = 0.153\ fm^{-3}$. The nuclear matter of the saturation density per nucleon is stable at zero pressure, so the density-dependent binding energy per nucleon is minimal and its value is $E_0 = (\frac{\epsilon}{n_B} - m_N)_{n_0} = -16.3$ MeV. The saturation density and the binding energy per nucleon enable one to determine the coupling constants as $g_\omega^2/4\pi = 14.717$ and $g_\sigma^2/4\pi = 9.537$. The other associated parameter is the incompressible modulus K measuring the stiffness of nuclear matter at saturation. Its value in the given model is $K = p_F^2 \frac{\partial^2(\epsilon/n_B)}{\partial p_F^2} \approx 550$ MeV about two times larger than the experimental value which is $K \approx 250$ MeV. The effective nucleon mass in matter is the other relevant property playing role like incompressible modulus K because of the high density behavior of EOS. Its value in the Walecka model is $m_N^* = 0.57 m_N$ which is smaller than the empirical range $m_N^* \approx 0.7 - 0.8 m_N$. On the other hand, the Landau mass m_L is more useful than the mass parameter m_N* because it is an effective mass for fermions at the Fermi surface where the location of all low energy excitations is, which makes the Landau mass more accessible than the effective nucleon mass. The Landau mass is described as $m_L = p_F/v_F$ where $v_F = (\frac{\partial E_F}{\partial p})_{p=p_F}$

and its value is $m_L = 0.83m_N$. Consequently, the given model is not expected to extrapolate well enough to high density in symmetric or asymmetric matter of neutron stars because incompressible modulus and the effective nucleon mass do not match with empirical values whereas saturation density and binding energy remains the same with experimental results.

3.1.2 The Walecka Model with the Scalar Interaction

In the relativistic mean field theory, the scalar interaction will be inserted into the model by adding the cubic and quadratic terms to explain the known qualities of nuclear matter. Besides this necessity, this model will be renormalizable by inserting these terms into the Lagrange density. The pressure and the effective nucleon mass can be rewritten respectively as

$$P = -\frac{1}{2}m_\sigma^2\bar{\sigma}^2 - \frac{1}{2}m_\omega^2\bar{\omega}^2 - \frac{1}{3}bm_N(g_\sigma\bar{\sigma})^3 - \frac{1}{4}c(g_\sigma\bar{\sigma})^4 + P_N, \tag{3.27}$$

$$m_N^* = m_N - n_S\frac{g_\sigma^2}{m_\sigma^2} + \frac{g_\sigma^2}{m_\sigma^2}\left(bm_N(m_N - m_N^*)^2 + c(m_N - m_N^*)^3\right), \tag{3.28}$$

where b and c are the new dimensionless constants and σ^3, σ^4 represent a three-body and four-body interactions respectively. While σ^3 is necessary to define the bound nuclear matter in non-relativistic potential approach, σ^4 does not have more information on it. The experimental values can be fitted with the Walecka model by choosing $g_\sigma^2/(4\pi)$ =6.003, $g_\omega^2/(4\pi)$=5.948, $b = 7.950 \times 10^{-3}$ and $c = 6.952 \times 10^{-4}$. Adding the scalar interaction to the model has an important impact on the behaviour at large density.

3.1.3 Hyperons

In the core of the compact stars, densities can be higher several times than nuclear saturation density, which leads to presenting baryons with hyperons whose lowest states are the baryon octet consisting of $n, p, \Xi^-, \Xi^0, \Sigma^-, \Sigma^0, \Sigma^+, \Lambda$. To calculate EOS, the hyperons will be incorporated into the given model. In the RMF approximation, the new model includes protons and neutrons in beta equilibrium which interact via the exchange of ω, σ and ρ. The reason of adding ρ meson is to recreate the nuclear matter's measured charge-symmetry energy. Also, the spin 1/2 six hyperons and the vector meson ϕ will be included in the model. The ϕ couples to the the hyperons and symbolizes vector repulsion among them. In accordance with these situations, the Lagrange density can be rewritten as

$$\begin{aligned}\mathcal{L} = &\sum_j \bar{\varphi}_j(i\sigma^\mu\partial_\mu - m_j + g_{\sigma j}\omega - g_{\omega j}\sigma^\mu\omega_\mu - g_{\phi j}\gamma^\mu\phi_\mu - g_{\rho j}\gamma^\mu\rho^a_\mu T_a)\varphi_j \\ &+ \frac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma - m_\sigma^2\sigma^2) - \frac{1}{3}bm_N(g_\sigma\sigma)^3 - \frac{1}{4}c(g_\sigma\sigma)^4 \\ &- \frac{1}{4}\omega^{\mu\nu}\omega_{\mu\nu} + \frac{1}{2}m_\omega^2\omega_\mu\omega^\mu - \frac{1}{4}\phi^{\mu\nu}\phi_{\mu\nu} + \frac{1}{2}m_\phi^2\phi_\mu\phi^\mu \\ &- \frac{1}{4}\rho_a^{\mu\nu}\rho^a_{\mu\nu} + \frac{1}{2}m_\rho^2\rho^a_\mu\rho^\mu_a,\end{aligned} \tag{3.29}$$

where j represents baryon species and T^a is the isospin generator. From this Lagrange density, the effective baryon masses and the effective baryon chemical potential are respectively

$$\begin{aligned} m_j^* &= m_j - g_{\sigma j}\bar{\sigma} \\ \mu_j^* &= \mu_j - g_{wj}\bar{\omega} - g_{\phi j}\bar{\phi} - I_{3j}g_{\rho j}\bar{\rho}_0^3, \end{aligned} \tag{3.30}$$

where I_{3j} is the 3^{th} component of the isospin of the j^{th} baryon. There are relationships between particle density and fermi momenta, also the fermi momenta are associated with the effective chemical potential by respectively

$$\begin{aligned} n_j &= \frac{p_{Fj}^3}{3\pi^2}, \\ \mu_j^* &= \sqrt{m_j^* + p_{Fj}^2}. \end{aligned}$$

In compact stars, the matter is electrically neutral and has chemical equilibrium in accordance the weak interactions. The conditions in the case of the hyperons are

$$\begin{aligned} \mu_p &= \mu_n - \mu_e, & \mu_\Lambda &= \mu_n, \\ \mu_{\Sigma^+} &= \mu_n - \mu_e, & \mu_{\Sigma^0} &= \mu_n, \\ \mu_{\Sigma^-} &= \mu_n + \mu_e, & \mu_{\Xi^0} &= \mu_n, \\ \mu_{\Xi^-} &= \mu_n + \mu_e. \end{aligned}$$

Then, the constraint of the electric neutrality can be given as

$$n_p = n_{\Sigma^+} = n_e + n_\mu + n_{\Sigma^-} + n_\Xi, \tag{3.31}$$

It is possible to say that hyperons can be interpolated to the Walecka model and they seem for large densities since baryon chemical potential gives sufficiently a non-vanishing Fermi momentum. The total pressure and energy density with muon can be calculated the same way which used in the Walecka model, so

$$\begin{aligned} P &= P_{FG}(\mu_e, m_e) + P_{FG}(\mu_\mu) + \sum_j P_{FG}(\mu_j^*, m_j^*) \\ &- \frac{1}{2}m_\sigma^2\bar{\sigma}^2 + \frac{1}{2}m_\omega^2\bar{\omega}_0^2 + \frac{1}{2}m_\phi^2\bar{\phi}_0^2 + \frac{1}{2}m_\rho^2(\bar{\rho}_0{}^3)^2 \\ &- \frac{1}{3}bm_N(g_\sigma\bar{\sigma})^3 - \frac{1}{4}c(g_\sigma\bar{\sigma})^4 \\ \epsilon &= \epsilon_{FG}(\mu_e, m_e) + \epsilon_{FG}(\mu_\mu, m_\mu) + \sum_j, \epsilon_{FG}(\mu_j^*, m_j^*) \\ &+ \frac{1}{2}m_\sigma^2\bar{\sigma}^2 + \frac{1}{2}m_\omega^2\bar{\omega}_0^2 + \frac{1}{2}m_\phi^2\bar{\phi}_0^2 + \frac{1}{2}m_\rho^2(\bar{\rho}_0{}^3)^2 \\ &+ \frac{1}{3}bm_N(g_\sigma\bar{\sigma})^3 + \frac{1}{4}c(g_\sigma\bar{\sigma})^4. \end{aligned} \tag{3.32}$$

The mean value of the vector fields can be expressed as

$$\begin{aligned} m_\omega^2\bar{\omega}_0 &= \sum_j g_{\omega j}n_j, \\ m_\phi^2\bar{\phi}_0 &= \sum_j g_{\phi j}n_j, \\ m_\rho^2\bar{\rho}_0^3 &= \sum_j I_{3j}g_{\rho j}n_j. \end{aligned} \tag{3.33}$$

The mean value of the scalar field need to be obtained numerically from the condition

$$\sum_j g_{\sigma j} n_j^S = m_\sigma^2 \bar{\sigma} + b m_N g_{\sigma N}^3 \bar{\sigma}^2 + c g_{\sigma N}^4 \bar{\sigma}^3, \tag{3.34}$$

where n_j^S represents the scalar density of the j^{th} baryon. After determining suitable constants, the equation of the state (P versus ϵ) can be plotted for electrically nuclear dense matter. The pressure of non-interacting nucleons at low density is greater than interacting one since attractive interactions decrease the pressure. Indeed, the pressure becomes zero at the saturation density of isospin-symmetric nuclear matter. This situation changes vice versa for the high density, which means repulsive interactions including vector mesons lead to rise in the pressure. When hyperons are counted in nuclear matter, the pressure decreases, and so obtained EoS is soft.

CHAPTER 4

QCD PLASMA

When the energy density exceeds a certain hadronic value $(\sim 1GeV/fm^3)$[21], matter no longer made up of separate hadrons, but of their fundamantal constitutents, quarks and gluons. We can call this phase of matter the QCD (or quark-gluon) plasma because of the apparent connection with analogous events in atomic physics. Thus, the EoS of QCD plasma phase with the help of the thermodynamical potentials up to the order $O(\alpha^2 \ln \alpha^2)$ will be calculated.

4.1 Partition Function

In non-gauge many body theories, all thermodynamic quantities can be found by using the partition function

$$Tr \exp[-\beta(H - \mu \cdot N)]. \tag{4.1}$$

In gauge theories, the definition of this function relies on which gauge is used [22]. The reason for this is the existence of nonphysical particles in some gauges. The way to avoid this problem is either to evaluate the partition function only in physical gauge which does not include non-physical states or take into account only the physical states for trace. The calculations can be easy if one can find a Lorentz and gauge covariant computational scheme which depend on Feynman's path integral formulation of statistical mechanics [22].Thus, the partition function [23] can be written as a functional integral over whole fields φ

$$Z = N(\beta)\int[\varphi]\exp i\int_0^{-i\beta}dx_0\int d^3xL_{eff}(\varphi(x),\partial_\mu\varphi(x);\mu) \tag{4.2}$$

where $N(\beta)$ is the normalization factor. In QCD, the ghost field named Faddeev-Popov appear in order to remove the effect of studying on unphysical gauges. These are nonphysical spin-zero particles that have a fermionlike minus sign for loops. Approximately [24],

$$Z \approx \int[A_\mu][\phi][\bar{\phi}][W][\bar{W}]\exp S_{eff} \tag{4.3}$$

where A_μ is a periodic boson and ϕ is an antiperiodic fermion

$$\begin{aligned} S_{eff} = \int_0^\beta d\tau\int d^3x\Big(& -\frac{1}{4}F_{\mu\nu}^{a\ 2} - \frac{1}{2\alpha}(\partial_\mu A_a^\mu) + \partial^\mu\bar{W}_a\partial_\mu W_a \\ & + gf^{abc}\partial^\mu\bar{W}_aA_\mu^bW^c + \sum_f\bar{\phi}_f(i\not{D} + \mu_f\gamma^0 - m_f)\phi_f\Big), \end{aligned} \tag{4.4}$$

where the field tensor F and the covariant derivative D can be described respectively

$$\begin{aligned} F_a^{\mu\nu} &= \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - gf_{abc}A_b^\mu A_c^\nu, \\ D_\mu &= \partial_\mu + igA_\mu, \end{aligned} \tag{4.5}$$

and α is the gauge fixing parameter. ϕ, A and W are the fermion, gluon and ghost fields, respectively. One can derive the finite temperature Feynman rules by substituting external forces into the partition function and dividing the Lagrangian into an integration part and a kinetic part shown as $L = L_0 + L_I$.Thus,

$$\begin{aligned} Z[j_\mu,...] \approx \int[A_\mu][\phi][\bar{\phi}][\bar{W}][W]\exp\Big(d\tau\int d^3xL_I(A_\mu,...) \\ -\frac{1}{2}A^\mu[g_{\mu\nu}\partial^2 - (1-\frac{1}{\alpha})\partial_\mu\partial_\nu]A^\nu - j_\mu A^\mu + ...\Big), \end{aligned} \tag{4.6}$$

where the dots represent fermions and ghosts terms.When one converts the above equation into the momentum space, the finite size of the space in the τ directions together with the periodic/antiperiodic boundry conditions causes the energy Fourier series. One gets

$$
\begin{aligned}
Z[j_\mu,..] \approx \int [A_\mu][\phi][\bar{\phi}][\bar{W}][W] \exp S_I(\frac{-i\delta}{\delta j_\mu},..) \\
\times \exp\Big(-T\sum_n \int \frac{d^3p}{(2\pi)^3}\Big(\frac{1}{2}A^\mu(n,\bar{k})(k_\mu k_\nu - k^2 g_{\mu\nu})A^\nu(n,\bar{k}) \\
+ j_\mu(n,\bar{k})A^\mu(n,\bar{k}) + ..\Big),
\end{aligned} \tag{4.7}
$$

$$
Z \approx \exp S_I(-i\frac{\delta}{\delta J_\mu},..) \exp\Big(T\sum_n \int \frac{d^3k}{(2\pi)^3} \times (\frac{1}{2}j_\mu(n,\bar{k})\Delta^{\mu\nu}j_\nu(n,\bar{k}))\Big), \tag{4.8}
$$

where $\Delta^{\mu\nu}$ is the Feynman propagator and $k_0 = 2\pi niT$. Hence, the finite temperature Feynman rules are the same as the $T = \mu = 0$ rules with the substitution

$$
\int \frac{d^4k}{(2\pi)^4} \to iT\sum_{k_0} \int \frac{d^3k}{(2\pi)^3} \tag{4.9}
$$

where

$$
\begin{aligned}
k_0 &= 2n\pi iT, \qquad (bosons, ghosts) \\
&= (2n+1)\pi iT + \mu, \qquad (fermions) \\
(2\pi)^4\delta^{(4)}(k_1 + ...k_N) &\to \frac{-i}{T}(2\pi)^3\delta_{k_1^0+...k_N^0}\delta^3(\bar{k}_1 + ...\bar{k}_N).
\end{aligned}
$$

In order to perform the frequency shown in the above equation, they can be converted to the contour integrals (Appendix B)

4.2 Thermodynamic Potential in One-Loop Level

4.2.1 Ideal Gas of Quarks and Gluons

The calculation of the thermodynamic potential in the perturbation theory up to the few lowest orders can be found using the partition function. By applying the Gaussian integration to the equation (4.14) and considering $g = 0$ for the ideal gas limit, the integration becomes

$$\Omega^{ideal} = -\frac{T}{V}\ln\left(det^{-1/2}(\partial^2 g^{\mu\nu}\delta^{ab})\det(\partial^2\delta^{ab})\Pi_f \det(i[\gamma^0(\partial_\tau-\mu_f)+\bar{\gamma}.\nabla)]-m_f)\right) \tag{4.10}$$

where the determinants come from the gauge, the ghost and fermion fields, respectively. Π represents the polarization tensor (Appendix C). The polarization tensor Whereas the fermion determinant will be evaluated on the space of antiperiodic functions, the gauge and ghost determinants will be evaluated on the space of periodic functions. By using the method shown in the Appendix B and Appendix C, one can get in the momentum space

$$\Omega^{ideal} = -\frac{\pi^2(N^2-1)}{45\beta} - \frac{N}{3\pi^2}\sum_f \int_0^\infty dp\frac{p^4}{E_p}\left(\frac{1}{e^{\beta(E_p-\mu_f)}+1} + \frac{1}{e^{\beta(E_p+\mu_f)}+1}\right). \tag{4.11}$$

For the ultrarelativistic limit($m = 0$),

$$\Omega^{ideal}_{(m=0)} = -\frac{\pi^2}{45\beta^4}\left[-1+N^2+\frac{7NN_f}{4}+15N\sum_f\left(\frac{\mu_f^2\beta^2}{2\pi^2}(1+\frac{\mu_f^2\beta^2}{2\pi^2})\right)\right]. \tag{4.12}$$

4.2.2 The Exchange Energy

In order to consider the perturbative correction to the above ideal gas formula, one can differentiate Ω with respect to the coupling constant.

$$\Omega(g) = \Omega(0) + \int_0^g \frac{\partial\Omega}{\partial g'}dg'. \tag{4.13}$$

The derivation of it can be written in the form [25]

$$
\begin{aligned}
\frac{\partial\Omega}{\partial g} &= \frac{T}{2g}\sum_{k_0}\int\frac{d^3k}{(2\pi)^3}\Bigg(\frac{T}{6}\sum_{q_0}\int\frac{d^3q}{(2\pi)^3}\Gamma^{abc}_{0\mu\nu\alpha}(k,q,-k-q)D^{\mu\mu'}_{aa'}(k) \\
& D^{\nu\nu'}_{bb'}(q)D^{\alpha\alpha'}_{cc'}(k+q)\Gamma^{a'b'c'}_{\mu'\nu'\alpha'}(-k,-q,k+q)+\Bigg(\Pi^{\mu\nu}_{ab}(k)D^{ab}_{\mu\nu}(k) \\
& -\sum_f S_f(k)\sum_f(k)-G(k)\sum_G(k)\Bigg)\Bigg),
\end{aligned}
\tag{4.14}
$$

where Π is the polarization tensor calculated in the Appendix B; (k), $S_f(k)$ and $G(k)$ are respectively exact gauge, fermion and ghost propagators. $\sum_f$ and $\sum_G$ are respectively the self-energies of the fermion and ghost. Γ and Γ_0 are the exact-three point and bare functions. To get the first perturbative correction $O(g^2)$, the exact propagators and vertex function are replaced with the bare ones. Thermodynamic potential with the correction is

$$
\begin{aligned}
\Omega^{exch} &= \frac{g^2N(N^2-1)T^4}{144}+\frac{(N^2-1)g^2T^2}{24\pi^2}\sum_f\int_0^\infty\frac{p^2n_p}{E_p}dp \\
&+(N^2-1)\frac{g^2}{32\pi^5}\sum_f\int_0^\infty\frac{p^2q^2}{E_pE_q}dqdp\Bigg((n_p^+n_q^++n_p^-n_q^-) \\
&\times(\frac{m_f^2}{qp}\ln\frac{E_pE_q-m_f^2-qp}{E_pE_q-m_f^2+qp}+2) \\
&+(n_p^+n_q^++n_p^-n_q^-)(\frac{m_f^2}{qp}\ln\frac{E_pE_q+m_f^2+qp}{E_pE_q+m_f^2-qp}+2)\Bigg).
\end{aligned}
\tag{4.15}
$$

For the massless limit,

$$
\Omega^{exch}_{m=0}=\frac{q^2T^4(N^2-1)}{144}\left(\frac{5N_f}{4}+N+g\sum_f\left(\frac{\mu_f^2}{2\pi^2T^2}(1+\frac{\mu_f^2}{2\pi^2T^2})\right)\right). \tag{4.16}
$$

4.2.3 Correlation Correction

In the case of perturbative calculation up to the three loop level intead of two loop level which is evaluated at the previous section, one encounters infrared singularities.

The exact gluon propagator in equation (5.14) will be extended in a relative to the polarization tensor and the bare propagator in accordance with

$$D = D^0 \sum_{n=0}^{\infty} (-\prod D^0)^n. \tag{4.17}$$

Directly, the thermodynamic potential can be written in the simpler form for a zero-mass limit

$$\begin{aligned}\Omega_{m=0}^{plasma} &= \frac{T^4(1-N^2)}{12\pi}\left(\frac{N+N_f}{6} + \frac{1}{2\pi^2 T^2}\sum_f \mu_f^2\right)^{3/2} g^3 \\ &+ \frac{T^4(N-N^3)}{32\pi^2}\left(\frac{N+N_f}{6} + \frac{1}{2\pi^2 T^2}\sum_f \mu_f^2\right) g^4 \ln g + O(g^4).\end{aligned} \tag{4.18}$$

4.3 Thermodynamic Quantities and EoS of QCD in One-Loop Level

Finally, the pressure is obtained by using the total thermodynamic potential getting up to $O(g^4 \ln g)$ by summing the above results for arbitrary m_f and μ_f.

$$\begin{aligned} p &= -\Omega^{Total} \\ &= -(\Omega_{m=0}^{ideal} + \Omega_{m=0}^{exchange} + \Omega_{m=0}^{plasma}) \\ &= \frac{\pi^2 T^4}{45}\left((N^2 - 1 + \frac{7}{4} N N_f) + \frac{15}{T^4}\sum_f (T^2 \vartheta_f^2 + \vartheta_f^4)\right) \\ &+ \frac{(1-N^2)}{144} g^2 T^4 \left(N + \frac{5}{4} N_f + \frac{9}{T^4}\sum_f (T^2 \vartheta_f^2 + \vartheta_f^4)\right) \\ &+ \frac{(N^2-1)}{12\pi} g^3 T^3 \left(\frac{1}{6}(2N + N_f) + \frac{1}{T^2}\sum_f \vartheta_f^2\right)^{3/2} \\ &+ \frac{N(N^2-1)}{32\pi^2} T^4 g^4 \ln g \left(\frac{1}{6}(2N + N_f) + \frac{1}{T^2}\sum_f \vartheta_f^2\right) + O(g^4), \end{aligned} \tag{4.19}$$

where $\vartheta_f^2 = \mu_f^2 / 2\pi^2$. To improve the calculation, one needs take into account the renormalization group that is improved coupling constant $\alpha(\frac{T}{\Lambda_{QCD}}, \frac{\mu}{\Lambda_{QCD}})$.The chemical potential can be ignored for simplicity, and so have for massless QCD [26],

$$\frac{N\alpha(T)}{\pi} = \frac{Ng^2(T)}{4\pi^2}. \tag{4.20}$$

The thermodynamic potential perturbatively calculated in the previous chapter is not analytic in the α coupling constant. However, it has contributions of both nonanalytic terms like $\alpha^{(n+1)/2}$ and logarithmic type $\alpha^{n/2}\ln\alpha$. Thus, the pressure can be expressed as

$$p = T^4\left(a + \sum_{n=0}^{\infty} b_n(\alpha/\pi)^{1+n/2} + \sum_{n=0}^{\infty} c_n(\alpha/\pi)^{2+n/2}(\ln\alpha)/\pi\right). \tag{4.21}$$

The coefficients a, b_o, b_1 and c_0 are known for QCD. When one put $N = 3$, take u,d and s quarks as massless and disregards the contribution of the heavier flavors in equation (4.19) and comparing with the equation (4.21), one obtains

$$p = \pi^2 T^4\left(\frac{19}{36} - \frac{3\alpha}{2\pi} - \sqrt{96}(\frac{\alpha}{\pi})^{3/2} + 6(\frac{\alpha}{\pi})^2\ln\alpha/\pi + O(\alpha^2)\right). \tag{4.22}$$

By using the equation

$$\varepsilon = T^2\frac{\partial}{\partial T}(\frac{p}{T}). \tag{4.23}$$

The equation of the state can be found

$$\varepsilon = 3p + \pi^2 T^4\left(\frac{27}{4}(\frac{\alpha}{\pi})^2 - 18\sqrt{6}(\frac{\alpha}{\pi})^{5/2} - 54(\frac{\alpha}{\pi})^3\ln\frac{\alpha}{\pi}\right) + O(\alpha^3), \tag{4.24}$$

and so the speed of sound is

$$v_s^2 = \frac{dp}{d\varepsilon} = \frac{1}{3} - \frac{81}{19}(\frac{\alpha}{\pi})^2 + \frac{216\sqrt{6}}{19}(\frac{\alpha}{\pi})^{5/2} + \frac{648}{19}(\frac{\alpha}{\pi})^3\ln\frac{\alpha}{\pi} + O(\alpha^3). \tag{4.25}$$

CHAPTER 5

QUARK-GLUON PLASMA

Here, hadron is pictured as a small domain in the new phase with quark and gluon, which is called the bag. The field equations of QCD carry out the dynamics of the quark-gluon fields inside the bag. Thus, the following consideration is going to be limited to the lowest order perturbation term connect with the vacuum term B.

5.1 The Fermi and Bose Quantum States

The grand partition function of the quantum static approach for a particle of mass m and degeneracy g by including the existence of antiparticles can be shown[27]

$$\ln Z_{F,B} = \pm gV \int \frac{d^3p}{(2\pi)^3} \Bigg(\ln\left(1 \pm \alpha\Lambda e^{-\beta\varepsilon)} + \ln\left(1 \pm \alpha\Lambda^{-1} e^{-\beta\varepsilon)}\right)\right) \Bigg), \tag{5.1}$$

where $\beta = 1/T$, $\varepsilon = \sqrt{p^2 + m^2}$, α is related with the number of members of the ensemble and Λ is called the fugacity factor associated with the number of particles, written as

$$\Lambda = e^{\mu/T}. \tag{5.2}$$

By adding the different gas fractions f

$$\ln Z = \sum_f \ln Z_f, \tag{5.3}$$

the equation (5.1) becomes

$$\ln Z_{F,B} = \pm g_{F,B} V \int \frac{d^3p}{(2\pi)^3} \Bigg(\ln\left(1 \pm \alpha \Lambda e^{-\beta\varepsilon}\right)] + \ln\left(1 \pm \alpha \Lambda^{-1} e^{-\beta\varepsilon}\right) \Bigg), \tag{5.4}$$

where $g_F = g_s g_c$, $g_s = 2s + 1 = 2$ for spin 1/2 degeneracy and $g_c = 3$ for color. By using

$$f_{F,B} = -\frac{1}{\beta} \frac{\partial}{\partial \varepsilon} \ln Z_{F,B}, \tag{5.5}$$

the single particle distribution functions for fermions and antifermions are

$$f_{F,\bar{F}} = \frac{1}{\alpha^{-1} e^{\beta(\varepsilon \mp \mu)} + 1}, \tag{5.6}$$

for bosons and antibosons

$$f_{B,\bar{B}} = \frac{1}{\alpha^{-1} e^{\beta(\varepsilon \mp \mu)} - 1}. \tag{5.7}$$

In short-hand notation,

$$f^{\pm}_{F,B} = f_{F,B} \pm \bar{f}_{F,B}. \tag{5.8}$$

The particle densities and energy density can be calculated respectively

$$\rho_{F,B} = \frac{N_{F,B}}{V} = \frac{1}{V} \Lambda \frac{d}{d\Lambda} \ln Z_{F,B} = g_{F,B} \int \frac{d^3p}{(2\pi)^3} f^{-}_{F,B}, \tag{5.9}$$

$$\rho_g = \frac{N}{V} = \frac{1}{V} \lim_{\Lambda \to 1} \Lambda \frac{d}{d\Lambda} \ln Z_g = g_g \int \frac{d^3p}{(2\pi)^3} f_g, \tag{5.10}$$

$$\epsilon = -\frac{1}{V} \frac{\partial}{\partial \beta} \ln Z_{F,B} = g_{F,B} \int \frac{d^3p}{(2\pi)^3} \varepsilon f^{+}_{F,B}, \tag{5.11}$$

$$\epsilon_g = g_g \int \frac{d^3p}{(2\pi)^3} \varepsilon f_g. \tag{5.12}$$

Total energy density is

$$\epsilon = \sum_j = \epsilon_j, \tag{5.13}$$

where ρ_g and ϵ_g represents particle density and energy density for gluons respectively.

5.2 The Relativistic Phase Space Integral

To analyze the features of ideal relativistic gases, the relativistic momentum integral encountered with all phase space integrals in a similar form need to be evaluated by considering the Bessel function,

$$K_\upsilon(z) = \frac{(z/2)^\upsilon \sqrt{\pi}}{\Gamma(\upsilon + \frac{1}{2})} \int_1^\infty (t^2 - 1)^{\upsilon - \frac{1}{2}} e^{-zt} dt, \quad Re\upsilon > -1/2 \tag{5.14}$$

valid for $|argz| < \pi/2$. After substituting

$$t \to \sqrt{p^2 + m^2}/2, \qquad\qquad z \to \beta m,$$

into the K_υ, one gets

$$K_\upsilon(\beta m) = (\frac{\beta}{2m})^\upsilon \frac{\sqrt{\pi}}{\Gamma(\upsilon + \frac{1}{2})} \int_0^\infty \frac{p^{2\upsilon}}{\varepsilon} e^{-\beta\varepsilon} dp. \tag{5.15}$$

5.3 EoS of the QGP

The quantum nature of the massless, relativistic quark-gluon gases need to be considered in the deconfined QGP. The integral part of the $\ln Z_{F,B}$ becomes

$$\pm \int \frac{d^3p}{(2\pi)^3} \ln(1 \pm \alpha\Lambda e^{-\beta\epsilon}) = \frac{\beta}{3} \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{\varepsilon} f_{F,B}. \tag{5.16}$$

Pressure comes from

$$
\begin{aligned}
P &= \frac{1}{\beta}\frac{\partial}{\partial V}\ln Z_F + \frac{1}{\beta}\frac{\partial}{\partial V}\ln Z_B \\
&= \frac{g_F}{3}\int \frac{d^3p}{(2\pi)^3}\frac{p^2}{\varepsilon}(f_F+\bar{f}_F) - \frac{g_B}{3}\int \frac{d^3p}{(2\pi)^3}\frac{p^2}{\varepsilon}(f_B+\bar{f}_B) \\
&= g_F\int \frac{d^3p}{(2\pi)^3}(\varepsilon - \frac{m^2}{\varepsilon})f_F^+ + g_B\int \frac{d^3p}{(2\pi)^3}(\varepsilon - \frac{m^2}{\varepsilon})f_B^+.
\end{aligned}
\tag{5.17}
$$

By considering energy density, the total pressure can be rewritten

$$
\epsilon - 3P = g_F\int \frac{d^3p}{(2\pi)^3}\frac{m^2}{\varepsilon}f_F^+ + g_B\int \frac{d^3p}{(2\pi)^3}\frac{m^2}{\varepsilon}f_B^+. \tag{5.18}
$$

The Boltzmann term of the equation with antiparticle by using Bessel function

$$
\epsilon - 3P = \frac{gT^4}{\pi^2}x^3K_1(x). \tag{5.19}
$$

where $x = m/T$. The relativistic EOS for high temperature relative to the mass

$$
3P \to \epsilon, \qquad\qquad \beta m \to 0. \tag{5.20}
$$

More precisely,

$$
\ln Z_{Total} = \ln Z_q + \ln Z_g + \ln Z_{vac}, \tag{5.21}
$$

which can be demonstrated as Figure 5.1.

Thus, the partition functions can be written

$$
\ln Z_F|_{m=0} = \frac{g_F V\beta^{-3}}{6\pi^2}\left(\frac{\pi^2}{2}\ln^2\Lambda + \frac{1}{4}\ln^4\Lambda + \frac{7\pi^4}{60}\right), \tag{5.22}
$$

by including interaction between them

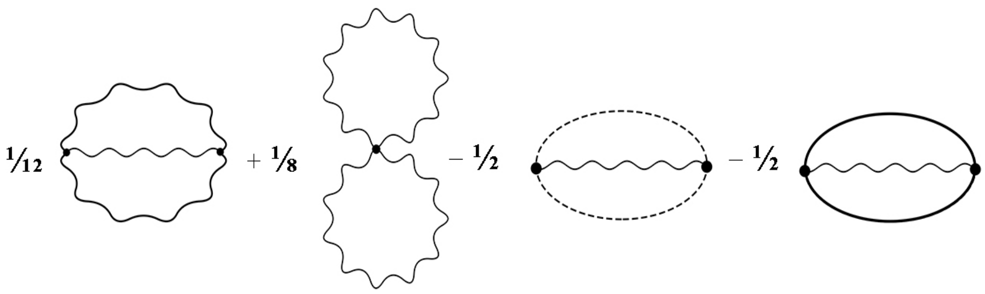

Figure 5.1: Feynmann diagrams that contribute to the EoS of the Quark-Gluon Plasma in order α_s. Gluons, Quarks and Ghost substruction of non physical dof are represented by wavy lines, solid lines and dashed lines, respectively.

$$\ln Z_q(\beta,\Lambda) = \frac{g_q V}{12\pi^2\beta^3}\left((1-\frac{2\alpha_s}{\pi})(\pi^2\ln^2\Lambda + \frac{1}{2}\ln^4\Lambda) + (1-\frac{50\alpha_s}{21\pi})\frac{7\pi^4}{30}\right), \quad (5.23)$$

where $g_q = n_s n_c n_f = (2s+1)\times 3\times 2(u,d) = 12$ and $\Lambda_q^3 = \Lambda = e^{\mu_q/T}$ since each quark has 1/3 baryon numbers and $\mu = 3\mu_q$ due to the conservation of the baryon numbers.The gluons contribution

$$\ln Z_g(\beta,\Lambda) = \frac{8\pi^2 V}{45\beta^3}(1-\frac{15\alpha_s}{4\pi}), \quad (5.24)$$

and the vacuum term

$$\ln Z_{vac} = -B\beta V. \quad (5.25)$$

So, the total partition function with respect to μ and T

$$\ln Z_{Total}(\mu,T) = \frac{2VT^3}{\pi^2}\left((1-\frac{2\alpha_s}{\pi})(\frac{1}{4}(\frac{\mu}{3T})^4 + \frac{\pi^2}{2}(\frac{\mu}{3T})^2) + (1-\frac{50\alpha_s}{21\pi})\frac{7\pi^4}{60}\right) + \frac{8\pi^2 V}{45}T^3(1-\frac{15\alpha_s}{4\pi}) - BVT^{-1}. \quad (5.26)$$

The pressure and energy density can be obtained respectively

$$P = \frac{T}{V}\ln Z_{Total} = \frac{2T^4}{\pi^2}\left((1-\frac{2\alpha_s}{\pi})(\frac{1}{4}(\frac{\mu}{3T})^4 + \frac{\pi^2}{2}(\frac{\mu}{3T})^2) + (1-\frac{50\alpha_s}{21\pi})\frac{7\pi^4}{60}\right) + \frac{8\pi^2}{45}T^4(1-\frac{15\alpha_s}{4\pi}) - B, \tag{5.27}$$

$$\epsilon = -\frac{1}{V}\frac{d}{d\beta}\ln Z_{Total} = \frac{6T^4}{\pi^2}\left((1-\frac{2\alpha_s}{\pi})(\frac{1}{4}(\frac{\mu}{3T})^4 + \frac{\pi^2}{2}(\frac{\mu}{3T})^2) + (1-\frac{50\alpha_s}{21\pi})\frac{7\pi^4}{60}\right) + \frac{8\pi^2}{15}T^4(1-\frac{15\alpha_s}{4\pi}) + B. \tag{5.28}$$

Thus, the relationship between them is

$$P = \frac{1}{3}(\epsilon - 4B). \tag{5.29}$$

The baryon number density and entropy density are respectively

$$n = \frac{2}{3\pi^2}\left((1-\frac{2\alpha_s}{\pi})((\pi T)^2\frac{\mu}{3} + (\frac{\mu}{3})^3\right), \tag{5.30}$$

$$s = \frac{2}{\pi^2}(1-\frac{2\alpha_s}{\pi})\frac{\mu^3}{3}(\pi T) + \frac{4}{15\pi}(1-\frac{50\alpha_s}{21\pi})(\pi T)^3 + \frac{32}{45\pi}(1-\frac{15\alpha_s}{4\pi})(\pi T)^3. \tag{5.31}$$

When finite massless quarks are included, the EOS of the QGP shown above is slightly modified.

CHAPTER 6

FLUID APPROACH

6.1 Relativistic Hydrodynamics

Many macroscopic systems like neutron stars can be regarded as a perfect fluid. Each point has a velocity $\boldsymbol{v}$ and a moving observer with this velocity sees the fluid around itself as isotropic. This case occurs when $l_{mean} \ll d_{observer}$, which means the scale of the length used by observer is bigger than the mean free path between collisions of the fluid elements. Suppose that the fluid is in a momentarily co-moving reference frame (MCRF), so the energy momentum tensor becomes [20, 28]

$$\begin{aligned} \tilde{T^{ij}} &= p\delta_{ij}, \\ \tilde{T^{i0}} &= T^{0i} = 0, \\ \tilde{T^{00}} &= \epsilon, \end{aligned} \tag{6.1}$$

where p is the pressure and ϵ is the proper energy density. In the lab frame, $T^{\alpha\beta}$ can directly be written as

$$T^{\alpha\beta} = \Lambda^{\alpha}_{\sigma}\Lambda^{\beta}_{\delta}\tilde{T^{\sigma\delta}}, \tag{6.2}$$

where $\Lambda^{\alpha}_{\sigma}$ is constant and described as $\Lambda^0_0 = (1-v^2)^{\frac{-1}{2}} = \gamma$, $\Lambda^i_0 = v_i(1-v^2)^{\frac{-1}{2}} = v_i\gamma$. The above equation can be written in components as;

$$
\begin{aligned}
T^{ij} &= p\delta_{ij} - (p+\epsilon)\frac{v_i v_j}{v^2-1}, \\
T^{i0} &= -(p+\epsilon)\frac{v_i}{v^2-1}, \\
T^{00} &= -\frac{(\epsilon + pv^2)}{v^2-1}.
\end{aligned}
\tag{6.3}
$$

These equations can be reduced to a single equation;

$$
T^{\sigma\alpha} = p\eta^{\sigma\alpha} + wu^\sigma u^\alpha, \tag{6.4}
$$

where $w = \epsilon + p$ is the heat function per volume, u^σ is the velocity four-vector

$$
\mathbf{u} = \frac{d\mathbf{x}}{d\tau} = \gamma\mathbf{v}, \qquad u^0 = \frac{d\mathbf{x}^0}{d\tau} = \gamma, \tag{6.5}
$$

so the equations of conservation of energy-momentum tensor is;

$$
\frac{\partial T^{\sigma\alpha}}{\partial x^\alpha} = \frac{\partial p}{\partial x_\sigma} + \frac{\partial (wu^\sigma u^\alpha)}{\partial x^\alpha} = 0. \tag{6.6}
$$

To get the Euler equation;

$$
\begin{aligned}
0 &= \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x^0}[wu^i u^0] + \frac{\partial}{\partial x_j}[wu^i u^j], \\
0 &= \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x^0}[wv^i\gamma^2] + \frac{\partial}{\partial x_j}[w\gamma^2 v^i v^j], \\
0 &= \frac{\partial p}{\partial x_i} + \gamma^2 v^i\partial_t w + wv^i\partial_t\gamma^2 + w\gamma^2\partial_t v^i + \gamma^2 v^i v^j\partial_j w + w\gamma^2\partial_j(v^i v^j) + wv^i v^j\partial_j\gamma^2.
\end{aligned}
\tag{6.7}
$$

Now, let's multiply equation (6.7) by v^i after rewritting it for $\sigma = 0$, then

$$
\begin{aligned}
0 &= \frac{\partial p}{\partial x_0} + \frac{\partial}{\partial x^0}[wu^0 u^0] + \frac{\partial}{\partial x^j}[wu^0 u^j] \\
0 &= -v^i\frac{\partial p}{\partial t} + v^i\frac{\partial}{\partial t}[w\gamma^2] + v^i\frac{\partial}{\partial x^j}[w\gamma^2 v^j] \\
0 &= -v^i\frac{\partial p}{\partial t} + v^i\gamma^2\partial_t w + v^i w\partial_t\gamma^2 + v^i v^j w\partial_j\gamma^2 + v^i v^j w\partial_j\gamma^2 + v^i\gamma^2 v^j\partial_j w
\end{aligned}
\tag{6.8}
$$

When the equation (6.8) is substracted from equation (6.7),

$$\begin{aligned}
0 &= \frac{\partial p}{\partial x_i} + \cancelto{1}{\gamma^2 v^i \partial_t w} + \cancelto{2}{w v^i \partial_t \gamma^2} + w\gamma^2 \partial_t v^i + \cancelto{3}{\gamma^2 v^i v^j \partial_j w} + w\gamma^2 \partial_j (v^i v^j) + \cancelto{4}{w v^i v^j \partial_j \gamma^2} \\
&+ v^i \frac{\partial p}{\partial t} - \cancelto{1}{v^i \gamma^2 \partial_t w} - \cancelto{2}{v^i w \partial_t \gamma^2} - \cancelto{4}{v^i v^j w \partial_j \gamma^2} - v^i v^j w \partial_j \gamma^2 - \cancelto{3}{v^i \gamma^2 v^j \partial_j w} \\
0 &= \frac{\partial p}{\partial x_i} + w\gamma^2 \partial_t v^i + w\gamma^2 \partial_j (v^i v^j) + v^i \frac{\partial p}{\partial t} - v^i v^j w \partial_j \gamma^2
\end{aligned} \tag{6.9}$$

$$\gamma^2 w (\partial_t v^i + v^j \partial_j v^i) = -\frac{\partial p}{\partial x^i} - v^i \frac{\partial p}{\partial t}, \tag{6.10}$$

$$\frac{w}{v^2 - 1} \left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \boldsymbol{v} \right) = \boldsymbol{\nabla} p + \boldsymbol{v} \frac{\partial p}{\partial t}. \tag{6.11}$$

which is the relativistic version of Euler equation. In order to obtain fluid equations, the conservation of particle numbers is also needed. In MCRF, the particle current four vector;

$$\tilde{n}^i = 0 \quad \tilde{n}^0 = n, \tag{6.12}$$

where the time component is the number density of the particles and its three spatial components are the three dimensional particle current vector. In the lab frame, the particle current four vector is related to the four-velocity;

$$n^i = \Lambda^i_\alpha(\boldsymbol{v}) \tilde{n}^\alpha = v^i n (1 - v^2)^{\frac{-1}{2}}, \tag{6.13}$$

$$n^o = \Lambda^0_\alpha(\boldsymbol{v}) \tilde{n}^\alpha = n (1 - v^2)^{\frac{-1}{2}}, \tag{6.14}$$

explicitly,

$$n^\beta = n u^\beta. \tag{6.15}$$

The equation of conservation of particle number is;

$$\frac{\partial N^{\beta}}{\partial x^{\beta}} = \frac{\partial(nu^{\beta})}{\partial x^{\beta}} = \frac{\partial(nu^{0})}{\partial x^{0}} + \frac{\partial(nu^{i})}{\partial x^{i}}, \tag{6.16}$$

so, the continuity equation is obtained

$$\frac{\partial(\gamma n)}{\partial t} + \boldsymbol{\nabla} \cdot (n\gamma \boldsymbol{v}) = 0. \tag{6.17}$$

To find the scalar equation, multiply equation of conservation energy-momentum with u_σ

$$\begin{aligned}
0 &= u_\sigma \frac{\partial p}{\partial x_\sigma} + u_\sigma \frac{\partial}{\partial x^\alpha}(wu^\sigma \alpha) \\
0 &= u_\sigma \frac{\partial p}{\partial x_\sigma} + u_\sigma u^\sigma \frac{\partial}{\partial x^\alpha} wu^\alpha + u_\sigma u^\alpha w \frac{\partial u^\sigma}{\partial x^\alpha} \\
0 &= u^\alpha \frac{\partial p}{\partial x^\alpha} - w\frac{\partial u^\alpha}{\partial x^\alpha} - u^\alpha \frac{\partial w}{\partial x^\alpha},
\end{aligned} \tag{6.18}$$

when (6.16) is inserted into the second term of (6.18)

$$\begin{aligned}
0 &= u^\alpha\left(\frac{\partial p}{\partial x^\alpha} + \frac{w}{n}\frac{\partial n}{\partial x^\alpha} - \frac{\partial w}{\partial x^\alpha}\right) \\
0 &= u^\alpha\left(\frac{\partial p}{\partial x^\alpha} - n\frac{\partial}{\partial x^\alpha}\left(\frac{w}{n}\right)\right) \\
0 &= -nu^\alpha\left(-\frac{1}{n}\frac{\partial p}{\partial x^\alpha} + \frac{\partial}{\partial x^\alpha}\left(\frac{\rho}{n}\right) + \frac{\partial}{\partial x^\alpha}\left(\frac{p}{n}\right)\right) \\
0 &= -nu^\alpha\left(\frac{\partial}{\partial x^\alpha}\left(\frac{\rho}{n}\right) + p\frac{\partial}{\partial x^\alpha}\left(\frac{1}{n}\right)\right).
\end{aligned} \tag{6.19}$$

Substituting the 2^{nd} law thermodynamics

$$kTds = p\frac{\partial}{\partial x^\alpha}\left(\frac{1}{n}\right) + \frac{\partial}{\partial x^\alpha}\left(\frac{\rho}{n}\right), \tag{6.20}$$

into the (6.19)

$$\begin{aligned}
0 &= -nu^\alpha\left(kT\frac{\partial s}{\partial x^\alpha}\right) \\
0 &= u^\alpha\left(kT\frac{\partial s}{\partial x^\alpha}\right) = u^0\frac{\partial s}{\partial x^0} + u^i\frac{\partial s}{\partial x^i},
\end{aligned} \tag{6.21}$$

which gives the energy equation

$$\frac{\partial s}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) s = 0. \tag{6.22}$$

The Euler equations (6.11), the continuity equation (6.17), the energy equation (6.22) and together with the equation of state that related p and ϵ in terms of n and s are the fundamental equations of the relativistic hydrodynamics.

6.2 EoS for Simple a Gas of Point Particles

To obtain EoS for fluid composing of structureless point particles, energy-momentum tensor for a system of particles with energy momentum four vectors $p_n^\alpha(t)$ is

$$T^{\alpha\beta} = \sum_N \frac{p_N^\alpha p_N^\beta}{E_N} \delta^3(x - x_N), \tag{6.23}$$

and its MCFR has the isotropic form (6.1). Thus, while the particle number density is

$$n = \sum_N \delta^3(x - x_N), \tag{6.24}$$

pressure and the energy density are

$$p = \frac{1}{3}\sum_{i=1}^{3} T^{ii} = \frac{1}{3}\sum_N \frac{p_N^2}{E_N}\delta^3(x - x_N), \tag{6.25}$$

$$\epsilon = T^{00} = \sum_N \frac{p_N^0 p_N^0}{E_N}\delta^3(x - x_N) = \sum_N E_N \delta^3(x - x_N), \tag{6.26}$$

in this frame. Generally,

$$0 \leq p \leq \frac{\epsilon}{3}. \tag{6.27}$$

For a non-relativistic gas, the energy is approximately

$$E_N \simeq m + \frac{p_N^2}{2m}. \tag{6.28}$$

If the above equation is substituted into the equation (6.26), the energy density becomes

$$\epsilon \simeq \sum_N (m + \frac{p_N^2}{2m})\delta^3(x - x_N) \simeq nm + \frac{3p}{2}. \tag{6.29}$$

For extremely relativistic gas, energy is

$$E_N \simeq |p_N| \gg m. \tag{6.30}$$

After substituting this equation into (6.26), energy density is

$$\epsilon \simeq \sum_N p_N \delta^3(x - x_N) \simeq 3p \gg nm, \tag{6.31}$$

so, the above two energy density equations can be combined into a single one;

$$\epsilon - nm \simeq \frac{p}{\gamma - 1}, \tag{6.32}$$

where

$$\gamma = \begin{cases} 5/3 & \text{nonrelativistic ,} \\ 4/3 & \text{extreme relativistic .} \end{cases} \tag{6.33}$$

If the energy density (6.32) is substituted into the second law of thermodynamics which is

$$kTds = pd(\frac{1}{n}) + d(\frac{\epsilon}{n}), \tag{6.34}$$

the law takes the form

$$kTds = \frac{n^{\gamma-1}}{\gamma - 1} d(\frac{p}{n^\gamma}). \tag{6.35}$$

In accordance with above equation, the energy equation (6.22) is

$$\frac{\partial}{\partial t}(\frac{p}{n^\gamma}) + (\boldsymbol{v} \cdot \nabla)(\frac{p}{n^\gamma}) = 0. \tag{6.36}$$

6.3 The Speed of Sound in a Static Homogeneous Relativistic Fluid

In the unperturbed state, terms n_0, ϵ_0, p_0 and s_0 showing fluid properties are constant in space-time and $\boldsymbol{v}_0 = 0$. Whereas the sound wave creates slight changes n_1,ϵ_1,p_1,v_1 in unperturbated state, s_0 stays the same. When these are substituted into the Euler equations (6.11) and the Continuity equation (6.17), they becomes respectively;

$$\frac{\partial \boldsymbol{v_1}}{\partial t} = -\frac{\nabla p_1}{\epsilon + p}, \tag{6.37}$$

$$\frac{\partial n_1}{\partial t} + n\nabla \cdot \boldsymbol{v_1} = 0, \tag{6.38}$$

and the second law of thermodynamics (6.34) with $ds = 0$ reads

$$\epsilon_1 = \frac{(p+\epsilon)n_1}{n}. \tag{6.39}$$

By using $p = p(\epsilon)$, it is possible to write $p_1 = \epsilon_1(\frac{\partial p}{\partial \epsilon})_s$. Then, if substituting this relationship between pressure and energy density into Euler equation (6.37), one gets;

$$\frac{\partial \boldsymbol{v_1}}{\partial t} = -\frac{1}{(\epsilon + p)}(\frac{\partial p}{\partial \epsilon})_s \nabla \epsilon_1, \tag{6.40}$$

where $v_s^2 = (\frac{\partial p}{\partial \epsilon})_s$. Collecting the equations (6.39) and (6.40) into one equation;

$$\frac{\partial \boldsymbol{v_1}}{\partial t} = -\frac{v_s^2}{n}\nabla n_1. \tag{6.41}$$

Eventually, when the above equation is substituted into the continuity equation (6.38), a wave equation is obtained

$$[\frac{\partial^2}{\partial t^2} - v_s^2 \nabla^2]n_1 = 0, \tag{6.42}$$

showing that sound waves travel with the speed v_s. By using the relationship between pressure and energy density for extremely relativistic gas (6.31), the speed of sound can be found as

$$v_s = \frac{1}{\sqrt{3}}. \tag{6.43}$$

CHAPTER 7

CONCLUSIONS AND FINAL REMARKS

In this thesis, the sound velocity in dense matter is examined within various models where each of model brings different approaches to matter by considering some restrictions arising from known properties such as Pauli Exclusion Principle, saturation density, strong interactions, etc. We first focus on describing the dense media itself since the speed of sound is directly related to it. As the density increases, description of matter takes a new form according to the structure of the atom and the properties of particles that make it.

The Neutron Stars' cold and ultra-dense cores are natural laboratories to understand the dense matter. One of the most significant aims of mass-radius measurements is to constrain the EoS of the dense matter. The TOV equations, which reveal the direct connection between astronomical observations and nuclear physics, relate the microphysics described by the EoS to the macroscopic qualities of the neutron stars. The EoS of a White Dwarf as a Newtonian Star in the polytropic picture can be found as $p = \frac{\epsilon}{3}$ first by taking the TOV equations into account and considering occupied states for degenerate electrons. If the occupied states are rearranged for the Neutron Stars consisting of degenerate neutrons, one can reach the same EoS as $p = \frac{\epsilon}{3}$. Thus, the speed of sound for these degenerate systems is $\frac{1}{\sqrt{3}}$, i.e. smaller than the speed of light.

The density in the core of a Neutron Star can reach $\sim 10\rho_{sat}$. The neutrons overlap at $\sim 4\rho_{sat}$ where the matter may include a wide variety of hadronic degrees of freedom together with the nucleons. As the overlap between nucleons increases, transition to the non-nucleonic states of matter is expected. At this point, Relativistic Mean Field Theory which states that the interaction between nucleons mediated by the scalar σ and the vector mesons w can be taken into account since meson condensations can occur at higher densities. In accordance with this theory or $(\sigma - w)$ model, the pressure and energy density is equal to each other which means that the speed of sound approaches the speed of light at a very high density. However, the known bulk nuclear matter properties such as incompressible modulus and the effective nucleon mass associated stiffness of nuclear matter at saturation do not match with the experimental results. Thus, the new EoS can be obtained when the discrepancy is removed after adding the cubic and quadratic terms to the defined Lagrangian density, which causes the speed of sound to be smaller than the speed of light.

At high enough density, the ordinary matter is subjected to a transition into a phase composed of quarks and gluons instead of separated hadrons. This phase of matter can be called the QCD plasma, whose EoS in the form $p = p(\epsilon)$ is obtained by calculating the perturbative thermodynamical potentials up to the O($\alpha^2 \ln \alpha$), keeping the fermions's masses arbitrary. Thus, the sound velocity is calculated as a dependent gauge fixing parameter. The Quark-Gluon plasma is also evaluated by considering the MIT Bag Model, where quark is regarded as confined in the bag, so the vacuum structure of strong interactions is considered. In the process of summing all contributions of gas, $\ln Z_{vac}(\beta) = -\beta BV$ should be added to the $\ln Z_q$ and $\ln Z_g$. This vacuum term, including the bag constant, reflects on calculating pressure and energy density such that the energy density inside the bag is positive $(+B)$ while the pressure exerted on the surface of the bag is negative $(-B)$. Thus, the EoS of this model becomes $p = \frac{1}{3}(\epsilon - 4B)$ and sound velocity are still smaller than the speed of light.

The assumption that the Neutron Star is composed of a perfect fluid is another approach to work on dense matter. In accordance with the fundamental equations of Relativistic Hydrodynamics, which are Relativistic Euler, Continuity and Energy equations, the speed of sound in a static homogeneous relativistic fluid can be computed to be $\frac{1}{\sqrt{3}}$ arising from $v_s^2 = (\frac{\partial p}{\partial \epsilon})_\sigma$.

To sum up, the speed of sound calculated by following the EoS of the dense matter in all analyzed models is always smaller than the speed of light as expected.

REFERENCES

[1] S. F. Bernard, "Laplace and the speed of sound," *ISIS*, vol. 55, pp. 7–19, 1964.

[2] B. Kain, "Dark matter admixed neutron stars," *Physical Review D*, vol. 103, p. 043009, 2021.

[3] P. Ciarcelluti and F. Sandin, "Have neutron stars a dark matter core?," *Phys. Lett. B*, vol. 695, pp. 19–21, 2011.

[4] Q.-F. Xiang, W.-Z. Jiang, D.-R. Zhang, and R.-Y. Yang, "Effects of fermionic dark matter on properties of neutron stars," *Phys. Rev. C*, vol. 89, p. 025803, 2014.

[5] F. Sandin and P. Ciarcelluti, "Effects of mirror dark matter on neutron stars," *Astroparticle Physics*, vol. 32(5), pp. 278–284, 2009.

[6] I. Goldman, R. N. Mohapatra, S. Nussinov, D. Rosenbaum, and V. Teplitz, "Possible implications of asymmetric fermionic dark matter for neutron stars," *Phys. Lett. B*, vol. 725, pp. 200–207, 2013.

[7] W. Husain and A. W. Thomas, "Possible nature of dark matter," *JCAP*, vol. 10, p. 086, 2021.

[8] P. Mukhopadhyay and J. Schaffner-Bielich, "Quark stars admixed with dark matter," *Phys. Rev. D*, vol. 93, p. 083009, 2016.

[9] Arimoto, Makoto and others, "Gravitational Wave Physics and Astronomy in the nascent era," 2021.

[10] A. Bauswein and H. T. Janka, "Measuring neutron-star properties via gravitational waves from binary mergers," *Phys. Rev. Lett.*, vol. 108, p. 011101, 2012.

[11] K. Takami, L. Rezzolla, and L. Baiotti, "Constraining the Equation of State of Neutron Stars from Binary Mergers," *Phys. Rev. Lett.*, vol. 113, no. 9, p. 091104, 2014.

[12] L. Landau and E. Lifshitz, *Fluid Mechanics*. Pergamon, 1987.

[13] S. W. Rienstra and A. Hirschberg, "An introduction to acoustics," *Eindhoven University of Technology*, vol. 18, p. 19, 2004.

[14] R. C. Tolman, "Static solutions of Einstein's field equations for spheres of fluid," *Phys. Rev.*, vol. 55, pp. 364–373, Feb 1939.

[15] J. Oppenheimer and G. Volkoff, "On massive neutron stars," *Physical Review*, vol. 55, pp. 374–381, 1939.

[16] F. Özel and P. Freire, "Masses, Radii, and the Equation of State of Neutron Stars," *Ann. Rev. Astron. Astrophys.*, vol. 54, pp. 401–440, 2016.

[17] A. W. Steiner, J. M. Lattimer, and E. F. Brown, "The Equation of State from Observed Masses and Radii of Neutron Stars," *Astrophys. J.*, vol. 722, pp. 33–54, 2010.

[18] I. Gradshteyn and R. I.M, *Table of Integrals, Series and Products*. Elsevier, 2007.

[19] P. Hansel, A. Potekhin, and D. Yakovlev, *Neutron Stars 1: Equation State and Structure*. Springer, 2007.

[20] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*. John Wiley & Sons, Inc., 1972.

[21] E. V. Shuryak, "Quantum chromodynamics and the theory of superdense matter," *Physics Reports*, vol. 61, pp. 71–158, 1980.

[22] C. Bernard, "Feynman rules for gauge theories at finite temperature," *Physical Review D*, vol. 9, p. 3312, 1974.

[23] J. Kapusta, "QCD at high temperature," *Nuclear Physics B*, vol. 148, pp. 461–498, 1979.

[24] T. Toimela, "Perturbative QED and QCD at finite temperatures and densities," *International Journal of Theoretical Physics*, vol. 24, pp. 901–949, 1985.

[25] O. Kalashnikov and V. Klimov, "Infrared behavior of green's functions in yang-mills theory at finite temperatures," *Soviet Journal of Nuclear Physics*, vol. 33, pp. 443–447, 1981.

[26] W. E. Caswell, "Asymptotic behavior of non-abelian gauge theories to two-loop order," *Phys. Rev. Lett.*, vol. 33, pp. 244–246, 1974.

[27] J. Letessier and J. Rafelski, *Hadrons and Quark–Gluon Plasma*. Cambridge University Press, 2002.

[28] L. Rezzolla and O. Zanotti, *Relativistic Hydrodynamics*. Oxford University Press, 2013.

[29] N. K. Glendenning, *Compact Stars:Nuclear Physics, Particle Physics, and General Relativity*. Springer New York, 2000.

[30] J. Kapusta and C. Gale, *Finite-Temperature Field Theory: Principles and Applications*. Cambridge University Press, 2006.

[31] A. Schmitt, *Dense Matter in Compact Stars*. Springer, 2010.

[32] D. Gross, R. Pisarski, and L. Yaffe, "QCD and instantons at finite temperature," *Reviews of Modern Physics*, vol. 53, p. 43, 1981.

[33] K. Kajantie and J. Kapusta, "Behaviour of gluons at high temperature," *Annals of physics*, vol. 160, pp. 477–513, 1985.

[34] W. Celmaster and D. Sivers, "Studies in the renormalization-prescription dependence of perturbative calculations," *Physical Review D*, vol. 23, p. 227, 1981.

[35] O. Kalashnikov and V. Klimov, "Polarization operator in QCD at finite temperatures and densities," *Soviet Journal of Nuclear Physics*, vol. 31, pp. 699–706, 1980.

Appendix A

BACKGROUND FOR THE RELATIVISTIC MEAN FIELD THEORY

A.1 Lagrangian Formalism

The dynamics of the classical relativistic theory depending on a number of fields φ_i is determined through a Lagrangian density $\mathcal{L}[\varphi(x), \partial_\mu \varphi(x)]$, where $x \equiv x^\mu = (t, x, y, z)$. In accordance with the variation principle,

$$\begin{aligned} \delta S &= \delta \int_{t_1}^{t_2} d^4x \mathcal{L}[\varphi(x), \partial_\mu \varphi(x)] \\ &= \int_{t_1}^{t_2} d^4x (\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}) \delta\varphi(x) \cdot \\ &= 0 \end{aligned} \tag{A.1}$$

After taking the variations of the fields as arbitrary, one gets the Euler-Lagrange equation for the field $\varphi(x)$;

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = 0. \tag{A.2}$$

The Lagrangian density is a Lorentz scalar since it is only constucted by scalar functions of the fields and their derivatives. Thus, the equation of the motion of it is Lorentz covariant, which means the Lagrangian be scalar $\mathcal{L}'(x') = \mathcal{L}(x)$. To assemble the sensible Lagrangians, the certain symmetries can be used [29, 30, 31].

A.2 Symmetries

There are symmetries called as external related to the spacetime and internal symmetries like phase transformations. For first one, Lagrangian has fixed spacetime symmetries or invariances like translation of the spacetime coordinates because physical theories in flat spacetime should be Lorentz covariant and so Lagrange density have to be a Lorentz scalar. The change in $\mathcal{L}(x)$ for an infinitesimal translation $x'^{\mu} = x^{\mu} + \xi^{\mu}$

$$\begin{aligned} \mathcal{L} &= \mathcal{L}[\varphi(x), \partial_\mu \varphi(x)] \\ \delta\mathcal{L} &= \mathcal{L}[\varphi(x^\nu + \xi^\nu), \partial_\mu \varphi(x^\nu + \xi^\nu)] - \mathcal{L}[\varphi(x^\nu), \partial_\mu \varphi(x^\nu)] \\ &= \xi_\nu \partial^\nu \mathcal{L}. \end{aligned} \tag{A.3}$$

where Taylor expansion is applied and then kept first order of ξ. $\mathcal{L}$ variation depending on scalar φ and $\partial_\mu \varphi$ is

$$\begin{aligned} \delta\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \varphi}\delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\delta(\partial_\mu \varphi) \\ &= \frac{\partial \mathcal{L}}{\partial \varphi}\xi_\nu \partial^\nu \varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\xi_\nu \partial_\mu \partial^\nu \varphi \\ &= (\frac{\partial \mathcal{L}}{\partial \varphi}\partial^\nu \varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\partial_\mu \partial^\nu \varphi)\xi_\nu. \end{aligned} \tag{A.4}$$

By using Euler-Lagrange equation for the first term,

$$\delta\mathcal{L} = \partial_\mu (\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\partial^\nu \varphi)\xi_\nu. \tag{A.5}$$

Equating two obtained Lagrangian variations (A.3) and (A.5), one gets

$$\partial_\mu (\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\partial^\nu \varphi)\xi_\nu = \xi_\nu \partial^\nu \mathcal{L} = \xi_\nu \eta^{\mu\nu} \partial_\mu \mathcal{L}, \tag{A.6}$$

$$\partial_\mu (\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\partial^\nu \varphi - \eta^{\mu\nu}\mathcal{L})\xi_\nu = 0. \tag{A.7}$$

ξ is the arbitrary translation, so

$$\partial_\mu T^{\mu\nu} = 0. \tag{A.8}$$

where

$$T^{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \partial^\nu \varphi_i - \eta^{\mu\nu} \mathcal{L}. \tag{A.9}$$

is the energy-momentum tensor. The sum on φ represent all boson and fermion fields. There are four conserved four currents and they are related to the charges. The canonical momentum and Hamiltonian density for the field are,

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi_i)}, \tag{A.10}$$

$$\begin{aligned} \mathcal{H} &= \pi \partial^0 \varphi - \mathcal{L} \\ &= \partial(\partial_0 \varphi_i) \partial^0 \varphi - \mathcal{L} \\ &= T^{00}. \end{aligned} \tag{A.11}$$

Taking its integral over three-space gives Hamiltonian and its value is the energy E as one of the four constants of (A.9);

$$H = \int_V dr T^{00}, \tag{A.12}$$

the spacelike components are

$$P^j = \int_V dr T^{0j}, \tag{A.13}$$

Thus, one has the Lorentz scalar

$$P_\mu P^\mu = E^2 - P^2 = M^2 = invariant. \tag{A.14}$$

For the second one, symmetries called as internal global symmetries, one consider an internal, infinitesimal and continuous transformation;

$$\varphi_i^{'} = \varphi_i(x) + \zeta_j F_i^j[\varphi_1(x), ..., \varphi_n(x)],$$

whereas ζ_j are infinitesimal parameters which are spacetime independent, F represents the transformation. Assume a $\mathcal{L}$ is invariant to a continuous symmetry transformation described by a certain set of F_i^j, so its variation have to vanish.In the formal expression,

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi_i}\delta\varphi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\delta(\partial_\mu\varphi_i), \tag{A.15}$$

by using $\delta\varphi_i = \zeta_j F_i^j$, $\delta(\partial_\mu\varphi_i) = \zeta_j\partial_\mu F_i^j$ and Euler-Lagrange equation step by step

$$\begin{aligned}\delta\mathcal{L} &= (\frac{\partial\mathcal{L}}{\partial\varphi_i}F_i^j + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\partial_\mu F_i^j)\zeta_j \\ &= (F_i^j\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\partial_\mu F_i^j)\zeta_j \\ &= \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}F_i^j)\zeta_j \\ &= 0.\end{aligned} \tag{A.16}$$

ζ_j is arbitrary, so the set of n four vectors is

$$J_j^\mu \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}F_i^j, \tag{A.17}$$

which satisfies

$$\partial_\mu J_j^\mu = 0(j = 1, 2, ...n), \tag{A.18}$$

which are referred to as Noether currents.

$$0 = \partial_\mu J^\mu = \frac{\partial J^0}{\partial t} + \nabla.J, \tag{A.19}$$

which is the continuity equation. By integrating the above equation over three-space and using Gauss's theorem, one gets

$$\begin{aligned} \frac{\partial}{\partial t}\int_V J^0 dr &= -\int_V \nabla \cdot J dr \\ &= -\int_S J dS. \end{aligned} \tag{A.20}$$

The surface integral goes to zero, so left hand side of the above equation is zero and one gets the conserved charges

$$Q_j = \int_V J_j^0 dr, \tag{A.21}$$

which means that invariance of the $\mathcal{L}$ to continuous internal symmetries refers to the existence of the conserved charges.

A.2.1 Transition amplitudes for bosons

In the field theory, the eigenstates of the Schrödinger picture field operator $\hat{\varphi}(x, 0)$ and its conjugate momentum operator $\hat{\pi}(x, 0)$ satisfy respectively

$$\begin{aligned} \hat{\varphi}(x, 0)|\varphi\rangle &= \varphi(x)|\varphi\rangle \\ \hat{\pi}(x, 0)|\pi\rangle &= \pi(x)|\pi\rangle \end{aligned}$$

where $\varphi(x)$ and $\pi(x)$ are the eigenvalues of the operators. The completeness and orthogonality conditions for each one are

$$\int d\varphi(x)|\varphi\rangle\langle\varphi| = 1, \qquad \langle\varphi_a|\varphi_b\rangle = \prod_x \delta(\varphi_a(x) - \varphi_b(x)),$$

$$\int \frac{d\pi(x)}{2\pi}|\pi\rangle\langle\pi| = 1 \qquad \langle\pi_a|\pi_b\rangle = \prod_x \delta(\pi_a(x) - \pi_b(x)).$$

To pass from the field space to the conjugate space,

$$\langle\varphi|\pi\rangle = \exp(i\int d^3x\varphi(x)\pi(x))$$

The dynamics of the field theory needs a Hamiltonian function

$$H = \int d^3x\mathcal{H}(\hat{\pi},\hat{\varphi})$$

The state of the system which is φ_a at $t=0$ becomes at the $e^{-iHt_f}|\varphi_a\rangle$ after a time t_f and when passing from φ_a to the φ_b after t_f the transition amplitude is $\langle\varphi_a|e^{-iHt_f}|\varphi_b\rangle$. The aim of this title for followings is to find where the system goes back to the its original state after t_f. To get the useful transition amplitude, one can divide time interval into N equal steps of duration, meaning $\Delta t = \frac{t_f}{N}$. by inserting a complete set of state into each time, one gets

$$\begin{aligned}\langle\varphi_a|e^{-iHt_f}|\varphi_a\rangle = \lim_{N\to\infty}\int(\sum_{i=1}^{N}\frac{d\pi_i d\varphi_i}{2\pi})\\ \times\langle\varphi_a|\pi_N\rangle\langle\pi_N|e^{-iH\Delta t}|\varphi_N\rangle\langle\varphi_N|\pi_{N-1}\rangle\\ \times\langle\pi_{N-1}|e^{-iH\Delta t}|\varphi_{N-1}\rangle..\\ \times\langle\varphi_2|\pi_1\rangle\langle\pi_1|e^{-iH\Delta t}|\varphi_1\rangle\langle\varphi_1|\varphi_a\rangle.\end{aligned} \quad \text{(A.22)}$$

Because $\Delta t \to 0$, the transition amplitude becomes

$$\begin{aligned}\langle\pi_i|e^{-iH_i\Delta t}|\varphi_i\rangle &\approx \langle\pi_i|(1-iH_i\Delta t)|\varphi_i\rangle\\ &= (1-iH_i\Delta t)\exp(-i\int d^3x\varphi(x)\pi(x)).\end{aligned}$$

Thus, by using the completeness and the orthogonality properties, equation (A.22) becomes

$$\langle\varphi_a|e^{-iHt_f}|\varphi_a\rangle = \lim_{N\to\infty}\int(\sum_{i=1}^{N}\frac{d\pi_i d\varphi}{2\pi})\delta(\varphi_1-\varphi_a)$$
$$\times\exp\bigg\{-i\Delta t\sum_{j=1}^{N}\int d^3x\frac{H(\pi_j,\varphi_j)-\pi_j(\varphi_{j+1}-\varphi_j)}{\Delta t}\bigg\} \tag{A.23}$$

where $\varphi_{N+1}=\varphi_a=\varphi_1$. One can get the final result by taking the continuum limit of the above equation;

$$\langle\varphi_a|e^{-iHt_f}|\varphi_a\rangle = \int[D\pi]\int_{\varphi_a(x)}^{\varphi_a(x)}[D\varphi]$$
$$\times\exp\bigg\{i\int_0^{t_f}\int d^3x\bigg(\pi(x,t)\frac{\partial\varphi(x,t)}{\partial t}-\mathcal{H}(\pi(x,t),\varphi(x,t))\bigg)\bigg\}, \tag{A.24}$$

where $D[\pi]$ and $D[\varphi]$ are the functional integral measures.

A.2.2 Partition Functon for Bosons

Basically, partition function is

$$Z = Tr\exp-\beta(H-\mu_i\hat{N}_i) = \sum_a\int d\varphi_a\langle\varphi_a|\exp-\beta(H-\mu_i\hat{N}_i)|\varphi_a\rangle \tag{A.25}$$

and this can be rewritten as an integral over fields and their conjugate momentum in accordance with equation (A.24) by introducing the imaginary time $\tau = it$. In addition to this, if the system accepts a conserved charge, the replacement have to be taken

$$\mathcal{H}(\pi,\varphi)=\mathcal{H}(\pi,\varphi)-\mu\mathcal{N}(\pi,\varphi)$$

where $\mathcal{N}$ is the conserved charge density and μ is a Lagrange multiplier called the chemical potential. Thus, the partition function becomes

$$Z = \int [D\pi] \int_{periodic} [D\varphi] \times \exp\left[\int_0^\beta d\tau \int d^3x \left(i\pi \frac{\partial \varphi}{\partial \tau} - \mathcal{H}(\pi, \varphi) + \mu \mathcal{N}(\pi, \varphi) \right)\right]. \tag{A.26}$$

.

A.3 Bosonic Field

Lagrangian density can be written for the complex scalar field φ with mass m and coupling constant η without including the chemical potential

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - m^2 |\varphi|^2 - \eta |\varphi|^4 \tag{A.27}$$

To introduce the chemical potential associated with the conserved charge, one must describe the conserved current related to the Lagrangian density's symmetry. $\mathcal{L}$ is invariant under rotations of the field

$$\varphi \to e^{-i\beta} \varphi$$

so the Noether current becomes

$$\begin{aligned} J^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \frac{\delta \varphi}{\delta \beta} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^*)} \frac{\delta \varphi^*}{\delta \beta} \\ &= i(\varphi^* \partial^\mu \varphi - \varphi \partial^\mu \varphi^*), \end{aligned}$$

by using $\partial_\mu J^\mu = 0$, the conserved charge density is

$$J^0 = i(\varphi^* \partial^0 \varphi - \varphi \partial^0 \varphi^*),$$

which is related to the chemical potential μ and in order to write the Lagrangian density with μ, the partition function (A.26) will be used. In that equation, the charge density is $\mathcal{N} = J^0$.

A.4 Fermionic Field

To define a system of fermions in the absence of the interactions, we take the Lagrangian density

$$\mathcal{L} = \bar{\varphi}(i\gamma^{\mu}\partial_{\mu} - m)\varphi, \tag{A.28}$$

where $\bar{\varphi} = \varphi^{\dagger}\gamma^{0}$ and the Dirac matrices are defined as

$$\gamma^{0} = \begin{pmatrix} 1 & 0, \\ 0 & -1 \end{pmatrix} \qquad \gamma = \begin{pmatrix} 0 & \sigma, \\ -\sigma & 0 \end{pmatrix}$$

here 1 represents the unit 2×2 matrix and σ implies the triplet of Pauli matrices. Rewritten the Lagrangian density is

$$\mathcal{L} = \varphi^{\dagger}\gamma^{0}\left(i\gamma^{0}\frac{\partial}{\partial t} + i\gamma \cdot \nabla - m\right)\varphi, \tag{A.29}$$

which has the global symmetry, that is Lagrangian is invariant for $\varphi \rightarrow e^{-i\alpha}\varphi$. Thus, in accordance with the Noether's theorem, there is a conserved current related to this symmetry;

$$J^{\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\varphi)}\frac{\delta\varphi}{\delta\alpha} = \bar{\varphi}\gamma^{\mu}\varphi, \tag{A.30}$$

$$J^{0} = \varphi^{\dagger}\varphi. \tag{A.31}$$

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_{0}\varphi)} = i\varphi^{\dagger} \tag{A.32}$$

$$\mathcal{H} = \pi\partial_{0}\varphi - \mathcal{L} = \bar{\varphi}(-i\gamma.\nabla + m)\varphi. \tag{A.33}$$

The last three equations are the conserved charge, the conjugate momentum and Hamiltonian respectively. Thus, the partition function is

$$
\begin{aligned}
Z &= Tr^{\dagger} e^{-\beta(\hat{H}-\mu\hat{N})} \\
&= \int_{anti-periodic} [D\varphi^{\dagger}][D\varphi] \exp\Big(- \int d\tau \int d^3x (\mathcal{H} - \mu\mathcal{N} - i\pi\partial_\tau \varphi)\Big) \\
&= \int_{anti-periodic} [D\varphi^{\dagger}][D\varphi] \exp\Big(- \int d\tau \int d^3x \bar{\varphi}(-\gamma^0\partial_\tau - i\gamma.\nabla + \gamma^0\mu - m)\varphi\Big)
\end{aligned}
\tag{A.34}
$$

In this case, it is not convenient to evaluate the integral by separating conjugate momentum from the field and one should work in the (p, ω_n) space instead of the (x, τ) space. Also, φ and $\varphi^{\dagger}$ are independent fields that have to be integrated independently. By introducing the Fourier-transformed fields

$$
\begin{aligned}
\varphi(x) &= \frac{1}{\sqrt{V}} \sum_p e^{-ip.x} \varphi(p), \\
\bar{\varphi}(x) &= \frac{1}{\sqrt{V}} \sum_p e^{-ip.x} \bar{\varphi}(p),
\end{aligned}
$$

where $p.x = -(\omega_n\tau + \boldsymbol{p}.\boldsymbol{x})$. For anti-periodicity, $\varphi(0, x) = -\varphi(\beta, x)$ suggests $e^{i\omega_n\beta} = -1$, so $\omega_n = (2n + 1)\pi T$. By using the Fourier decomposition, the term in the exponential becomes

$$
-\int d\tau \int d^3x \bar{\varphi}(-\gamma^0\partial_\tau - i\gamma.\nabla + \gamma^0\mu - m)\varphi = -\sum_p \varphi^{\dagger}(p)\frac{G^{-1}(p)}{T}\varphi(p). \tag{A.35}
$$

where

$$
G^{-1}(p) = -\gamma^{\mu}p_{\mu} - \gamma^0\mu + m
$$

By introducing the integrals over the Grassmann variables rule

$$\int d\eta_1^\dagger d\eta_1....d\eta_N^\dagger d\eta_N e^{\eta^\dagger D\eta} = \det D \tag{A.36}$$

Thus, the partition function becomes

$$Z = \det \frac{G^{-1}(p)}{T} = \det \frac{1}{T} \begin{pmatrix} -(p_0+\mu)+m & -\sigma\cdot p \\ \sigma\cdot p & (p_0+\mu)+m \end{pmatrix} \tag{A.37}$$

$$\ln Z = \sum_k \ln(\frac{E_p^2-(p_0+\mu)^2}{T^2})^2, \tag{A.38}$$

with $p_0 = -i\omega_n$

$$\begin{aligned} \ln Z &= \sum_p \ln\left(\frac{E_p^2+(\omega_n+i\mu)^2}{T^2}\right)^2 \\ &= \sum_p \left(\ln\frac{E_p^2+(\omega_n+i\mu)^2}{T^2} + \ln\frac{E_p^2+(-\omega_n+i\mu)^2}{T^2}\right) \\ &= \sum_p \left(\ln\frac{\omega_n^2+(E_p-\mu)^2)}{T^2} + \ln\frac{\omega_n^2+(E_p+\mu)^2)}{T^2}\right), \end{aligned} \tag{A.39}$$

where

$$\sum_n \ln\frac{\omega_n^2+(E_p-\mu)^2}{T^2} = \frac{(E_p-\mu)}{T} + 2\ln(1+e^{-\frac{(E_p-\mu)}{T}}) + constant$$

which is going to be used in Walecka model for calculations.

Appendix B

EVALUATION OF THE FREQUENCY SUMS

If the function $f(z)$ which vanish fast at infinity does not have singularities on the imaginary axis, one can have

$$T\sum_n f(2n\pi iT) = \int_{-i\infty}^{+i\infty}\frac{dp_0}{2\pi i}f(p_0) + \int_{-i\infty+\varepsilon}^{-i\infty+\varepsilon}\frac{f(p_0)+f(-p_0)}{\exp{(\beta p_0)}-1}\frac{dp_0}{2\pi i} \tag{B.1}$$

If $f(z)$ does not have other singularities than simple poles at the $z = w_{b'}$, the equation takes the form

$$\begin{aligned} T = \sum_n f(2n\pi iT) = \int_{-i\infty}^{+i\infty}\frac{dp_0}{2\pi i}f(p_0) - \sum_{Re(w_b)>0}\frac{1}{\exp(w_b/T)-1}Res_(p_0 = w_b)f(p_0) \\ + \sum_{Re(w_b)}\frac{1}{\exp(-w_b/T)-1}Res_(p_0 = w_b)f(p_0) \end{aligned} \tag{B.2}$$

For the fermions

$$\begin{aligned} T\sum_n f[(2n+1)\pi iT+\mu] = \int_{-i\infty}^{+i\infty}\frac{dk_0}{2\pi i}f(k_0) + \oint_C\frac{dk_0}{2\pi i}f(k_0) \\ \int_{-i\infty+\mu+\varepsilon}^{+i\infty+\mu+\varepsilon}\frac{dk_0}{2\pi i}\frac{f(k_0)}{\exp{(k_0-\mu)}/T+1} \\ \int_{-i\infty+\mu-\varepsilon}^{-i\infty+\mu-\varepsilon}\frac{dk_0}{2\pi i}\frac{f(k_0)}{\exp{(\mu-p_0)}/T+1} \end{aligned} \tag{B.3}$$

The following way of the the contour C on the k_0 plane is

$$\mu - i\infty \to \mu + i\infty \to 0 + i\infty \to 0 - i\infty \to \mu - i\infty$$

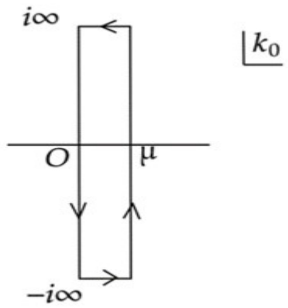

Figure B.1: Contour C appearing in the equation (B.3)

and represented by Figure B.1.

While integral over the contour C disappears when $\mu \to 0$, the last two terms of the integral vanish at $T = 0$. In the simple poles case, the form can be simplified as

$$T \sum_n f[(2n+1)\pi i T + \mu] = \int_{-i\infty}^{+i\infty} \frac{dk_0}{2\pi i} f(k_0) + \sum_{Re(w_b>0)} \frac{1}{\exp{(w_b - \mu)/T + 1}} Res_{k_0 = w_b} f(k_0) - \sum_{Re(w_b<0)} \frac{1}{\exp(-w_b + \mu)/T + 1} Res_{k_0 = w_b} f(k_0) \tag{B.4}$$

Appendix C

POLARIZATION TENSOR

C.1 Tensor Structure

The polarization tensor is described by the Dyson equation

$$D_{\mu\nu} = D^0_{\mu\nu} - D^0_{\mu\alpha}\Pi^{\alpha\beta}D_{\beta\nu} \tag{C.1}$$

where $D_{\mu\nu}, D^0_{\mu\nu}, \Pi^{\alpha\beta}$ are the exact, bare gluon propagators and the polarization tensor, respectively. The polarization tensor can be defined by four independent symmetric O(3) which can be chosen as tensors [24, 32, 33]

$$\begin{aligned}
A_{00} &= A_{0i} = A_{i0} = O, \\
A_{ij} &= \delta_{ij} - \frac{k_i k_j}{k^2}, \\
B_{\mu\nu} &= \frac{k_\mu k_\nu}{k^2} - g_{\mu\nu} - A_{\mu\nu}, \\
C_{\mu\nu} &= \frac{1}{2^{1/2}|\bar{k}|}\left((g_{\mu 0} - \frac{k_\mu k_0}{k^2})k_\nu + k_\mu(g_{\nu 0} - \frac{k_0 k_\nu}{k^2})\right), \\
D_{\mu\nu} &= \frac{k_\mu k_\nu}{k^2}.
\end{aligned} \tag{C.2}$$

By using the Dyson equation and the Ward identity that is

$$k_\mu k_\nu D^{\mu\nu} = -\alpha. \tag{C.3}$$

$\Pi_{\mu\nu}$ takes form

$$\Pi_{\mu\nu} = (m - k^2)A_{\mu\nu} + (n - k^2)B_{\mu\nu} + sC_{\mu\nu} + \frac{s^2}{2n}D_{\mu\nu}, \tag{C.4}$$

where m, n and s are the functions of the variables k_0 and $\bar{k}$.

C.2 Polarization Tensor in One-Loop Level at T$\neq 0$

The polarization tensor for the one-loop level is

$$\Pi_{\mu\nu} = \Pi^{Vacuum}_{\mu\nu} + \Delta\Pi^{Quark}_{\mu\nu} + \Delta\Pi^{Gluon}_{\mu\nu}, \tag{C.5}$$

where the vacuum part is

$$\Pi^{vac}_{\mu\nu}(k) = (k^2 g_{\mu\nu} - k_\mu k_\nu)\Pi^{vac}(k^2), \tag{C.6}$$

arising from the constraint

$$k_\mu \Pi^{\mu\nu}(k) = 0. \tag{C.7}$$

The vacuum part can be written in the massless limit [34]

$$\Pi^{MOM}_{\mu\nu} = \frac{g^2}{96\pi^2}(k^2 g_{\mu\nu} - k_\mu k_\nu)\bigg(4N_f + (-13 + 3\alpha)\bigg)\ln(\frac{k^2}{-M_E^2}), \tag{C.8}$$

where $-M^2$ is the Euclidean subtraction point. The contribution of the quark loop is represented in Figure C.1 (i).

It reads

$$\Delta\Pi^Q_{\mu\nu} = -2g^2 \sum_f T \sum_{p_0} \int \frac{d^3p}{(2\pi)^3} \bigg(\frac{2p_\mu p_\nu + (k_\mu p_\nu + k_\nu p_\mu) - (p^2 + kp - m^2)g_{\mu\nu}}{(m^2 - p^2)(m^2 - (k+p)^2)}\bigg) - \Pi^{Q(vac)}_{\mu\nu} \tag{C.9}$$

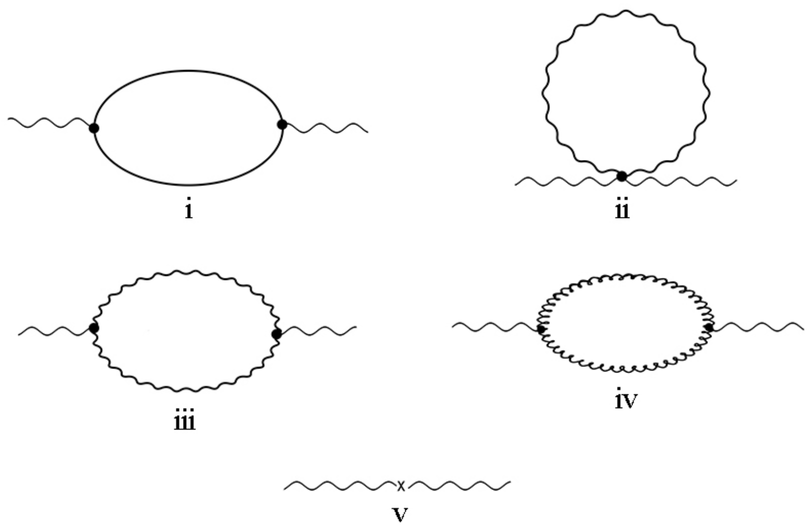

Figure C.1: The polarization tensor for one-loop level. (i) quark loop, (ii,iii) gauge loops, (iv) ghost loop, (v) counterterm

The gluon part of the polarization tensor is the non-abelian contribution coming from the (Figure C.1 (ii),(iii),(iv)). The form of it is [24, 35, 25]

$$
\begin{aligned}
\Delta\Pi^{G}_{\mu\nu} = \frac{1}{2}Ng^2T\sum_{q_0}\int\frac{d^3q}{(2\pi)^3}\Bigg(&\frac{4(q_\mu k_\nu + k_\mu q_\nu) - 2(k_\nu k_\mu - 4q_\nu q_\mu)}{(q+k)^2q^2} \\
&\frac{-4g_{\mu\nu}(q^2+2k.q)}{(q+k)^2q^2} - (\alpha-1)\frac{4g_{\mu\nu}[2(k.q)^2 + q^2k.q - q^2k^2/2]}{(q+k)^2q^4} \\
&-(\alpha-1)\frac{2k_\mu k_\nu q^2 + 4q_\mu q_\nu k^2 - 2(k_\mu q_\nu + k_\nu q_\mu)(q^2+3k.q)}{(q+k)^2q^4} \\
&+(\alpha-1)^2\frac{q_\nu q_\mu k^4 - k^2k.q(k_\mu q_\nu + q_\mu k_\nu) + (q.k)^2k_\mu k_\nu}{(q+k)^4q^4}\Bigg) - \Pi^{G(vac)}_{\mu\nu}.
\end{aligned}
\tag{C.10}
$$

For simplicity, one can work in the Feynman gauge ($\alpha = 1$) and the sum and integral can be evaluated over the angle variables. The polarization can be written the form as

$$
\Delta\Pi^{\mu\nu}(k) = (k^\mu k^\nu - g^{\mu\nu}k^2)\frac{\Delta\Pi^{00}}{\bar{k}^2} + \frac{A^{\mu\nu}}{2}(3\Delta\Pi^{00} + \Delta\Pi^{\mu}_{\mu}\frac{\bar{k}^2}{k^2}) \tag{C.11}
$$

where $\Delta\Pi^{00}(k_0,\bar{k})$ and $\Delta\Pi^{\mu}_{\mu}(k_0,\bar{k})$ can be expressed at the set of imaginary values $k_0 = 2\pi inT$, respectively [23]

$$
\begin{aligned}
\Delta\Pi^{00} = & \frac{g^2}{2\pi^2}\sum_f \int_0^\infty p^2 dp \frac{n_p}{E_p}\bigg(1 + \frac{4E_p^2 - a^2}{8pw} \times \ln\frac{(a^2+2\pi w)^2 + b^2E_p^2}{(a^2-2pw)^2+b^2E_p^2} \\
& - \frac{2\pi nTE_p}{wp}\tan^{-1}\frac{4pwb}{a^4+b^2E_p^2-4p^2w^2}\bigg) + \frac{g^2N}{\pi^2}\int_0^\infty qdqn_q \\
& \bigg(1 + \frac{4g^2 - w^2 - a^2}{8wq} \times \ln\frac{(a^2+2wq)^2+q^2b^2}{(a^2-2wq)^2+q^2b^2} \\
& - \frac{b}{2w}\tan^{-1}\frac{b^2wq^2}{a^2+q^2b^2-4q^2w^2}\bigg),
\end{aligned} \tag{C.12}
$$

$$
\begin{aligned}
\Delta\Pi^\mu_\mu = & \frac{g^2}{\pi^2}\sum_f \int_0^\infty p^2 dp \frac{n_p}{E_p}\bigg(1 + \frac{2m_f^2 - a^2}{4pw} \times \ln\frac{(a^2+2pw)^2+E_p^2b^2}{(a^2-2pw)^2+E_p^2b^2}\bigg) \\
& + \frac{Ng^2}{\pi^2}\int_0^\infty qdqn_q\bigg(2 - \frac{5a^2}{8qw} \times \ln\frac{(a^2+2wq)^2+q^2b^2}{(a^2-2qw)^2+q^2b^2}\bigg),
\end{aligned} \tag{C.13}
$$

where

$$
n_p = n_p^+ + n_p^- = \frac{1}{\exp{(E_p-\mu)/T+1}} + \frac{1}{\exp{(E_p+\mu)/T+1}},
$$

$$
n_q = \frac{1}{\exp{(q/T)}-1}, \qquad E_p = (p^2+m_f^2)^{1/2}, \qquad n = \frac{k_0}{2\pi iT},
$$

$$
w = |\bar{k}|, \qquad a^2 = w^2 + (2\pi nT)^2, \qquad b = 4n\pi T.
$$

C.3 Polarization Tensor in One-Loop Level at T=0

At $T = 0$, one needs to rewrite the polarization tensors for $q_0 = i\tilde{q}$, where $\tilde{q_0}$ is continous and real. The use of the Euclidean metric here is more convenient than the Minkowski metric. By using the Euclidian polar coordinates, one can get

$$
\begin{aligned}
q &= \sqrt{\tilde{q_0}^2 + \bar{q}^2}, \\
\varphi &= \tan^{-1}\left(\frac{|\bar{q}|}{\tilde{q_0}}\right).
\end{aligned} \tag{C.14}
$$

Thus, the equations (C.12) and (C.13) take the form associated with q, φ, μ, respectively

$$
\begin{aligned}
\Delta\Pi^{00} = \frac{g^2}{2\pi^2}\sum_f \int_0^{\sqrt{\mu_f^2-m_f^2}} \frac{p^2}{E_p}dp\Bigg(1-\frac{q^2/8-E_p^2/2}{qp\sin\varphi} \\
\times \ln\frac{(q+2p\sin\varphi)^2+4E_p^2\cos^2\varphi}{(q-2p\sin\varphi)^2+4E_p^2\cos^2\varphi} \\
-\frac{E_p\cot\varphi}{p}\tan^{-1}\frac{4pE_p\sin 2\varphi}{q^2+4(E_p^2\cos^2\varphi-p^2\sin^2\varphi)}\Bigg).
\end{aligned} \tag{C.15}
$$

$$
\begin{aligned}
\Delta\Pi^{\mu}_{\mu} = \frac{g^2}{\pi^2}\sum_f \int_0^{\sqrt{\mu_f^2-m_f^2}} \frac{p^2}{E_p}dp\Bigg(1-\frac{q^2/8-m_f^2/4}{pq\sin\varphi} \\
\times \ln\frac{(q+2p\sin\varphi)^2+4E_p^2\cos^2\varphi}{(q-2p\sin\varphi)^2+4E_p^2\cos^2\varphi}\Bigg).
\end{aligned} \tag{C.16}
$$

After evaluating the integral in a closed form, the polarization tensors can be rewritten in the case of the ultrarelativistic limit ($m = 0$), respectively

$$
\begin{aligned}
\Delta\Pi_{00} = \frac{g^2}{\pi^2}\sum_f\Bigg(\mu\frac{4\mu^2-3q^2}{24q\sin\varphi}\ln\frac{4\mu^2\cos^2\varphi+(2\sqrt{\mu^2-m^2}\sin\varphi+q)^2}{4\mu^2\cos^2\varphi+(2\sqrt{\mu^2-m^2}\sin\varphi-q)^2} \\
+\frac{2}{3}\mu\sqrt{(\mu^2-m^2)}-\frac{1}{6}q^2\sin^2\varphi\ln\frac{1}{m}(\mu+\sqrt{\mu^2-m^2}) \\
-\frac{\cot\varphi}{2}(\mu^2-q^2\frac{2\sin^2\varphi+1}{12})\times\tan^{-1}(\frac{\mu\sqrt{\mu^2-m^2}\sin 2\varphi}{q^2/4+m^2\sin^2\varphi+\mu^2\cos 2\varphi} \\
\ln\frac{\mu^2(q^2+2m^2)-m^2/2(q^2+4m^2\sin^2\varphi)-\mu q\sqrt{(q^2+4m^2)(\mu^2-m^2)}}{\mu^2(q^2+2m^2)-m^2/2(q^2+4m^2\sin^2\varphi)+\mu q\sqrt{(q^2+4m^2)(\mu^2-m^2)}} \\
(2m^2-q^2)\sqrt{4m^2+q^2}\frac{\sin^2\varphi}{24q}+)\Bigg).
\end{aligned} \tag{C.17}
$$

$$\Delta\Pi_{00}(m=0) = \frac{g^2}{2\pi^2}\sum_f \Bigg(\cot\varphi(q^2\frac{3-2\cos^2\varphi}{24} - \frac{\mu^2}{2})\tan^{-1}(\frac{\sin 2\varphi}{\cos 2\varphi + q^2/4\mu^2})$$
$$\times \frac{(4\mu^3 - 3q^2\mu)}{24q\sin\varphi}\ln\frac{1+(\tan\varphi + q\sec\varphi/2\mu)^2}{1+(\tan\varphi + q\sec\varphi/2\mu)^2} + \frac{2\mu^2}{3}$$
$$-\frac{q^2}{24}\sin^2\varphi\ln(1+\frac{16\mu^4}{q^4} + \frac{8\mu^2}{q^2}\cos 2\varphi)\Bigg) \tag{C.18}$$

$$\Delta\Pi^{\mu}_{\mu} = \frac{q^2}{2\pi^2}\sum_f \Bigg(\frac{2m^2 - q^2}{4q}\Bigg(-\tan^{-1}(\frac{\mu\sqrt{\mu^2-m^2}\sin 2\varphi}{(m^2+\mu^2)\sin^2\varphi + q^2})q\cot\varphi$$
$$+\mu\csc\varphi\ln\frac{1+(\mu^{-1}\tan\varphi\sqrt{\mu^2-m^2} + (2\mu)^{-1}q\sec\varphi)^2}{1+(\mu^{-1}\tan\varphi\sqrt{\mu^2-m^2} - (2\mu)^{-1}q\sec\varphi)^2} + \frac{\sqrt{q^2+4m^2}}{2}$$
$$\times\ln\frac{2\mu^2(q^2+2m^2) - m^2(q^2+4m^2\sin^2\varphi) - 2q\mu\sqrt{(\mu^2-m^2)(q^2+4m^2)}}{2\mu^2(q^2+2m^2) - m^2(q^2+4m^2\sin^2\varphi) + 2q\mu\sqrt{(\mu^2-m^2)(q^2+4m^2)}}\Bigg)$$
$$+\mu\sqrt{\mu^2-m^2} - \frac{q^2}{2}\ln(\frac{\mu}{m} + \sqrt{\frac{\mu^2}{m^2} - 1})\Bigg). \tag{C.19}$$

$$\Delta\Pi^{\mu}_{\mu}(m=0) = \frac{g^2}{2\pi^2}\sum_f \Bigg(\mu^2 - q_\mu\csc\varphi\ln\frac{1+(\tan\varphi + \mu q/2\sec\varphi)^2}{1+(\tan\varphi - \mu q/2\sec\varphi)^2}$$
$$+\frac{q^2\cot\varphi}{4}\tan^{-1}(\frac{1}{\cot 2\varphi + \frac{q^2}{4}\csc 2\varphi}) - \frac{q^2}{8}\ln(\frac{q^4 + 8\mu^2q^2\cos 2\varphi + 16\mu^4}{q^4})\Bigg). \tag{C.20}$$

For the zero momentum limit of the polarization,

$$\Delta\Pi_{00} = \frac{g^2}{2\pi^2}\sum_f \Bigg(\mu\sqrt{\mu^2-m^2} - \tan^{-1}(\frac{\sqrt{\mu^2-m^2}}{\mu\cot\varphi})\mu^2\cot\varphi\Bigg), \tag{C.21}$$

$$\Delta\Pi^{\mu}_{\mu} = \frac{g^2}{2\pi^2}\sum_f \Bigg(\mu\sqrt{\mu^2-m^2} - \tan^{-1}(\frac{\sqrt{\mu^2-m^2}}{\mu\cot\varphi})m^2\cot\varphi\Bigg). \tag{C.22}$$