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Diversity Analysis of Hierarchical Modulation in Wireless Relay Networks

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***Abstract*—In communication systems, hierarchical modulation is used to increase system robustness and to send different information flows simultaneously. However, when used in cooperative communication systems, hierarchical modulation is prone to error propagation, which is the most important problem that prevents the achievement of full diversity gains. Depending on the signal-to-noise ratio between the source and the relay, threshold digital relaying can be used to mitigate error propagation. Mitigating error propagation and achieving full diversity depend on setting the thresholds at the relay properly. In this paper, the first and second threshold values are determined so that full diversity gains are attained for both base and enhancement layer bits. Analytical and simulation results are provided to show that the threshold values must depend on the hierarchy parameter.**

***Index Terms*—Cooperative communication, diversity, hierarchical modulation, relay networks.**

I. Introduction

In wireless channels, fading severely degrades system performance. To mitigate fading and to enhance robustness, spatial diversity techniques, multiple antennas, or relaying can be used [1]. Another method that increases system robustness in wireless channels is superposition coding [2]. In superposition coding, messages are layered on each other so that the destination receives more information as the channel state gets better and less information as it gets worse [2]. In this way, the transmission rate is adapted to the channel conditions even if there is no channel state information at the transmitter. Superposition coding is investigated for the relay channel in [3] and [4]. In [3], the source node sends its message in two layers. If the source–relay channel has good quality, the relay can decode and forward both layers. If the channel quality is intermediate, then the relay can decode only one layer. Finally, if neither is the case, the relay does not participate in the communication. Thus, superposition coding at the source allows the relay to have a more adaptive behavior and results in a better performance in cooperative systems.

Hierarchical modulation is a practical way of applying superposition coding [5], [6]. Allowing nonuniform spacing between signal points on the signal constellation, hierarchical modulation protects base layer bits more than enhancement layer bits. Under good channel conditions, both base and enhancement layer bits are demodulated, whereas under poor channel conditions, only base layer bits are demodulated. In the literature, [7]–[9] and [10] investigate the performance of hierarchical modulation in relay networks. In [7], the explicit closed-form expressions for the exact bit error rate (BER) are derived for cooperative systems with hierarchical modulation over additive white Gaussian noise (AWGN) and Rayleigh fading channels. The BER performance of hierarchical modulation is studied in cooperative broadcast channels in [8] and [9].

Error propagation is the most vital problem that prevents the achievement of full diversity gain in a relay network. As demodulation at the relay is not error free, the relay can transmit erroneous symbols to the destination. If the destination knows the probability of error at the relay, then it can implement a maximum-likelihood decoder [11], [12] and attain full diversity. However, if the probability of error at the relay is unknown, maximal ratio combining (MRC) the incorrectly demodulated symbols at the relay and the correct source symbols at the destination incurs losses in the achieved diversity gain [13]. To mitigate error propagation, the relay should forward the demodulated symbols only if the average reliability is high enough. To enhance reliability, error detecting/correcting codes can be used. However, channel coding increases the complexity and the processing power at the relay and introduces delays. Another approach that keeps relaying as simple as possible is threshold digital relaying (TDR) [14]. In TDR, if the instantaneous signal-to-noise ratio (SNR) between the source and the relay is lower than a threshold, the relay does not assist the source. On the other hand, if it is larger than the threshold, the relay forwards its decision to the destination. To achieve full diversity gain with TDR schemes, the threshold value must ensure that detection errors are quite unlikely when the relay transmits.

In [10], Nguyen *et al.* derived the BER equations for both base and enhancement layer bits in relay networks that employ hierarchical modulation and TDR. Assuming that the first and second threshold values are equal to $c_1\mathrm{SNR}$ and $c_2\mathrm{SNR}$, $(c_1 < c_2 \in \mathbb{R}^+)$, respectively, where SNR is a reference SNR that will be formally defined in the next section, they show that using hierarchical modulation with TDR improves the BER performance. Note that one could directly minimize BER expressions as in [10]; however, this requires all the average channel gains to be known both at the source and the relay, which is impractical for systems that change frequently. In this paper, we focus on maximizing the diversity gains for the same system investigated in [10]. To achieve full diversity gains, one does not need any kind of channel state information at the transmitter. We show that setting the thresholds as $c_i\mathrm{SNR}$, $i = 1, 2$, is not sufficient to achieve full diversity gains. The thresholds should be determined in the form of $c_i \log \mathrm{SNR}$,

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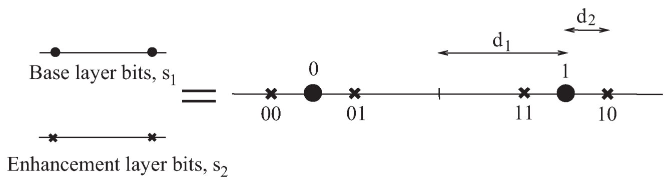

Fig. 1. Hierarchical 2/4 ASK constellation.

$i = 1, 2$, where c_i must depend on the hierarchy parameter α (which will be also defined in the next section). We give analytical proofs and simulation results to verify this claim. Moreover, we discuss that our analyses apply to hierarchical modulation schemes with an arbitrary number of transmission layers and an arbitrary number of threshold values as well.

II. System Model

This paper studies the relay network with one source, one relay, and one destination. We assume that there are two time slots of equal duration. In the first time slot, only the source broadcasts its information to the relay and the destination, and the relay remains silent. The received signal at the relay and the destination in the first time slot are written as $y_{r,1} = \sqrt{E_s}h_{sr}x + n_{r,1}$, $y_{d,1} = \sqrt{E_s}h_{sd}x + n_{d,1}$, respectively, where x denotes the modulated source symbol. In the second time slot, the relay demodulates its received signal and transmits $\hat{x}$ in the second time slot. The received signal at the destination in the second time slot is then $y_{d,2} = \sqrt{E_r}h_{rd}\hat{x} + n_{d,2}$, where $\sqrt{E_i}$, $i \in \{s, r\}$, denotes the average symbol energy at node i; h_{ij}, $i \in \{s, r\}$, $j \in \{r, d\}$, is the channel gain between nodes i and j; and $n_{j,k}$, $j \in \{r, d\}$, $k \in \{1, 2\}$, is the noise at node j in the kth time slot. There is Rayleigh fading, and the channel gain magnitude squares, i.e., $|h_{ij}|^2$, are independent exponential random variables with mean λ_{ij}. The noise components $n_{j,k}$ are independent and identically distributed complex Gaussian random variables with zero mean and variance N_0. Under these conditions, without loss of generality, we assume that $E_s = E_r = E$ and define the reference SNR as $\text{SNR} = E/N_0$. Then, the instantaneous received SNR per symbol between nodes i and j is $\gamma_{ij} = E|h_{ij}|^2/N_0 = |h_{ij}|^2\text{SNR}$. We will denote the expected value of γ_{ij} with σ_{ij}^2, where $\sigma_{ij}^2 = \lambda_{ij}\text{SNR}$.

In this paper, we assume that the relay and the destination have only receiver-side channel state information; i.e., the relay knows h_{sr} and the destination knows h_{sd} and h_{rd} perfectly. For simplicity, hierarchical 2/4 amplitude-shift keying (ASK) constellation shown in Fig. 1 is used at both the source and the relay. In the rest of this paper, we will denote the base and enhancement layer bits with s_1 and s_2, respectively. The parameters $2d_1$ and $2d_2$ shown in Fig. 1 are the minimum distance values for base and enhancement layer bits, respectively. We define the hierarchy parameter $\alpha = d_1/d_2$ as the ratio of these minimum distances. Note that all the calculations and the simulations in this paper can be generalized to any type of modulation scheme.

In our work, we investigate TDR [14] in cooperative networks that employ hierarchical modulation. In this scheme, we assume that there are two thresholds, i.e., γ_1^{th} and γ_2^{th} ($\gamma_1^{\text{th}} < \gamma_2^{\text{th}}$), and three operating regions. The relay knows and utilizes γ_{sr} to make a decision whether to remain silent or not. If $\gamma_{sr} < \gamma_1^{\text{th}}$, the system is in Region 1 ($\mathcal{R}_1$). In $\mathcal{R}_1$, as γ_{sr} is not large enough to demodulate the signal correctly, the relay remains silent. The destination demodulates both s_1 and s_2 by using only $y_{d,1}$. If γ_{sr} is between the first and second thresholds ($\gamma_1^{\text{th}} < \gamma_{sr} \ll \gamma_2^{\text{th}}$), the system is in Region 2 ($\mathcal{R}_2$). In $\mathcal{R}_2$, the relay demodulates only s_1 and remodulates them as $\hat{x}$ by using binary phase-shift keying. Note that despite the fact that $\gamma_1^{\text{th}} < \gamma_{sr} < \gamma_2^{\text{th}}$, there may still be detection errors at the relay. Thus, there are two subregions for $\mathcal{R}_2$: the relay demodulates s_1 correctly ($\mathcal{R}_{2,1}$) or incorrectly ($\mathcal{R}_{2,2}$). In $\mathcal{R}_2$, the destination demodulates s_1 via MRC $y_{d,1}$ and $y_{d,2}$ and demodulates s_2 by using only $y_{d,1}$. Finally, if $\gamma_2^{\text{th}} < \gamma_{sr}$, the system is in Region 3 ($\mathcal{R}_3$). In $\mathcal{R}_3$, the relay demodulates both s_1 and s_2 and remodulates them by using the same modulation scheme as the source, i.e., 2/4 ASK. In $\mathcal{R}_3$, there are four subregions, namely, $\mathcal{R}_{3,1}$, $\mathcal{R}_{3,2}$, $\mathcal{R}_{3,3}$, and $\mathcal{R}_{3,4}$, due to detection errors at the relay. In $\mathcal{R}_{3,1}$, the relay demodulates both s_1 and s_2 correctly; in $\mathcal{R}_{3,2}$, s_1 is correct, but s_2 is incorrect; in $\mathcal{R}_{3,3}$, s_1 is incorrect and s_2 is correct; and finally, in $\mathcal{R}_{3,4}$, neither s_1 nor s_2 is correct. In $\mathcal{R}_3$, the destination combines $y_{d,1}$ and $y_{d,2}$ by using MRC to demodulate both s_1 and s_2. These operating regions are summarized in Table I.

TABLE I
Operating Regions and Possible Subregions

Region	Relay Behavior	Sub-region
$\mathcal{R}_1$	$(\gamma_{sr} < \gamma_1^{th})$ Relay silent	-
$\mathcal{R}_2$	$(\gamma_1^{th} < \gamma_{sr} < \gamma_2^{th})$ Relay transmits only s_1	$\mathcal{R}_{2,1}$: s_1 correct $\mathcal{R}_{2,2}$: s_1 incorrect
$\mathcal{R}_3$	$(\gamma_2^{th} < \gamma_{sr})$ Relay transmits both s_1 and s_2	$\mathcal{R}_{3,1}$: s_1 correct, s_2 correct $\mathcal{R}_{3,2}$: s_1 correct, s_2 incorrect $\mathcal{R}_{3,3}$: s_1 incorrect, s_2 correct $\mathcal{R}_{3,4}$: s_1 incorrect, s_2 incorrect

In the next section, we assume that the first and second thresholds are chosen as $c_1 \log \text{SNR}$ and $c_2 \log \text{SNR}$, respectively ($c_1 \ll c_2, c_1, c_2 \in \mathbb{R}^+$), and find the conditions on c_1 and c_2 so that full diversity can be achieved for both s_1 and s_2. Note that the diversity gain is defined as $\delta = -\lim_{\text{SNR}\to\infty}((\log P_e(\text{SNR}))/(\log \text{SNR}))$[1] [1], where $P_e(\text{SNR})$ denotes the BER.[2]

III. Diversity Analysis of the Base and Enhancement Layer Bits

It is shown in [10] that the average probability of error for bit s_l, $l = 1, 2$, can be written as

$$\begin{aligned}
&\text{BER}\left(\gamma_1^{\text{th}}, \gamma_2^{\text{th}}, \alpha, s_l\right) \\
&= P(\epsilon_d, s_l|\mathcal{R}_1)P(\mathcal{R}_1) \\
&\quad + P(\epsilon_d, s_l|\mathcal{R}_{2,1})P(\mathcal{R}_{2,1}|\mathcal{R}_2)P(\mathcal{R}_2) \\
&\quad + P(\epsilon_d, s_l|\mathcal{R}_{2,2})P(\mathcal{R}_{2,2}|\mathcal{R}_2)P(\mathcal{R}_2) \\
&\quad + P(\epsilon_d, s_l|\mathcal{R}_{3,1})P(\mathcal{R}_{3,1}|\mathcal{R}_3)P(\mathcal{R}_3) \\
&\quad + P(\epsilon_d, s_l|\mathcal{R}_{3,2})P(\mathcal{R}_{3,2}|\mathcal{R}_3)P(\mathcal{R}_3) \\
&\quad + P(\epsilon_d, s_l|\mathcal{R}_{3,3})P(\mathcal{R}_{3,3}|\mathcal{R}_3)P(\mathcal{R}_3) \\
&\quad + P(\epsilon_d, s_l|\mathcal{R}_{3,4})P(\mathcal{R}_{3,4}|\mathcal{R}_3)P(\mathcal{R}_3) \qquad (1)
\end{aligned}$$

where $P(\epsilon_d, s_l|\mathcal{R}_{m,n})$ denotes the probability of error of bit s_l at the destination given the system is in region $\mathcal{R}_{m,n}$, $P(\mathcal{R}_{m,n}|\mathcal{R}_m)$ denotes the probability that the system is in subregion n given region m, and $P(\mathcal{R}_m)$ denotes the probability that the system is in region m. In the following, we will find the asymptotic behavior of all the terms in (1) to determine the diversity gains

[1] Throughout this paper, all the logarithms are in the natural base.

[2] In the rest of this paper, the notation $g(\mathbf{SNR}) \doteq \text{SNR}^c$ will be used if $\lim_{\text{SNR}\to\infty}((\log g(\text{SNR}))/(\log \text{SNR})) = c$.

for s_l, $l=1,2$. For compactness, we first define $f_1(a_1,x) \triangleq Q(\sqrt{2a_1x})$, $f_2(a_1,a_2,x,y) \triangleq Q((a_1x+a_2y)/(\sqrt{(x+y)/2}))$, and $f_3(a_1,a_2,x,y) \triangleq Q((a_1x-a_2y)/(\sqrt{(x+y)/2}))$, where $Q(.)$ is the well-known Q-function [1], x and y are independent exponential random variables, and $a_1,a_2 \in \mathbb{R}^+$ are given constants. In addition, we define $k_1=(\alpha+1)^2/(\alpha^2+1)$, $k_2=(\alpha-1)^2/(\alpha^2+1)$, $k_3=1/(\alpha^2+1)$, $k_4=(2\alpha+1)^2/(\alpha^2+1)$, $k_5=(2\alpha-1)^2/(\alpha^2+1)$, and $k_6=1$. Next, we will restate the asymptotic behavior of $\mathbb{E}\{f_1(.)\}$, $\mathbb{E}\{f_2(.)\}$, and $\mathbb{E}\{f_3(.)\}$ in terms of average link SNRs, where $\mathbb{E}\{.\}$ denotes the expected value.

Theorem 1 [1, Sec. 3.2]: The average error probability $\mathbb{E}\{f_1(a_1,\gamma_{ij})\}$ satisfies

$$\mathbb{E}\{f_1(a_1,\gamma_{ij})\} = \frac{1}{2}\left(1-\sqrt{\frac{a_1\sigma_{ij}^2}{1+a_1\sigma_{ij}^2}}\right) \doteq \text{SNR}^{-1} \tag{2}$$

where $i \in \{s,r\}$, $j \in \{r,d\}$.

Theorem 2 [10], [14]: The average error probability $\mathbb{E}\{f_3(a_1,a_2,\gamma_{sd},\gamma_{rd})\}$ satisfies

$$\mathbb{E}\{f_3(a_1,a_2,\gamma_{sd},\gamma_{rd})\} \approx \frac{a_2\sigma_{rd}^2}{a_1\sigma_{sd}^2+a_2\sigma_{rd}^2} \doteq 1. \tag{3}$$

Theorem 3: The average error probability $\mathbb{E}\{f_2(a_1,a_2,\gamma_{sd},\gamma_{rd})\}$ satisfies $\mathbb{E}\{f_2(a_1,a_2,\gamma_{sd},\gamma_{rd})\} \doteq \text{SNR}^{-2}$.

Proof: Without loss of generality, assume that $a_1 \geq a_2$. Then $\mathbb{E}\{f_2(a_1,a_2,\gamma_{sd},\gamma_{rd})\} \geq \mathbb{E}\{f_2(a_1,a_1,\gamma_{sd},\gamma_{rd})\}$ and $\mathbb{E}\{f_2(a_1,a_1,\gamma_{sd},\gamma_{rd})\} \doteq \text{SNR}^{-2}$ [1]. Similarly, $\mathbb{E}\{f_2(a_1,a_2,\gamma_{sd},\gamma_{rd})\} \leq \mathbb{E}\{f_2(a_2,a_2,\gamma_{sd},\gamma_{rd})\} \doteq \text{SNR}^{-2}$ [1]. Thus, $\mathbb{E}\{f_2(a_1,a_2,\gamma_{sd},\gamma_{rd})\} \doteq \text{SNR}^{-2}$. ∎

Note that, in Theorem 2, in f_3, $a_2\gamma_{rd}$ is subtracted from $a_1\gamma_{sd}$ in the numerator. This occurs when the relay's decision is wrong: its transmission confuses the destination; thus, the diversity gain is 0. However, in Theorem 3, in f_2, $a_1\gamma_{sd}$ and $a_2\gamma_{rd}$ are added in the numerator. This occurs when the relay's decision is correct: its transmission is helpful to the destination; thus, the diversity gain is 2.

A. Region Probabilities

Given γ_{sr} is an exponential random variable with mean σ_{sr}^2, we can write the region probabilities as

$$P(\mathcal{R}_1) = 1-e^{-\gamma_1^{\text{th}}/\sigma_{sr}^2} \tag{4}$$

$$P(\mathcal{R}_2) = e^{-\gamma_1^{\text{th}}/\sigma_{sr}^2} - e^{-\gamma_2^{\text{th}}/\sigma_{sr}^2} \tag{5}$$

$$P(\mathcal{R}_3) = e^{-\gamma_2^{\text{th}}/\sigma_{sr}^2}. \tag{6}$$

After applying Taylor series expansion and substituting $\gamma_1^{\text{th}} = c_1 \log \text{SNR}$ and $\gamma_2^{\text{th}} = c_2 \log \text{SNR}$, we can find the asymptotic behavior of the region probabilities as

$$P(\mathcal{R}_1) \doteq \text{SNR}^{-1}\log \text{SNR} \tag{7}$$

$$P(\mathcal{R}_2) \doteq \text{SNR}^{-1}\log \text{SNR} \tag{8}$$

$$P(\mathcal{R}_3) \doteq 1. \tag{9}$$

Note that if the thresholds were chosen as $c_i \log \text{SNR}$, $i=1,2$, then $P(\mathcal{R}_1) \doteq P(\mathcal{R}_2) \doteq P(\mathcal{R}_3) \doteq 1$.

B. Probability of Error at the Relay

The subregion probabilities at the relay when the system is in $\mathcal{R}_2$ is calculated in [10] as

$$P(\mathcal{R}_{2,2}|\mathcal{R}_2) = \mathbb{E}\{f_1(k_1,\gamma_{sr})|\mathcal{R}_2\} + \mathbb{E}\{f_1(k_2,\gamma_{sr})|\mathcal{R}_2\} \tag{10}$$

$$P(\mathcal{R}_{2,1}|\mathcal{R}_2) = 1-P(\mathcal{R}_{2,2}|\mathcal{R}_2) \leq 1. \tag{11}$$

Theorem 4: When $\gamma_1^{\text{th}} = c_1 \log \text{SNR}$ and $\gamma_2^{\text{th}} = c_2 \log \text{SNR}$, the average error probability $\mathbb{E}\{f_1(a_1,\gamma_{sr})|\mathcal{R}_2\}$ satisfies $\mathbb{E}\{f_1(a_1,\gamma_{sr})|\mathcal{R}_2\} \dot{\leq} \text{SNR}^{-a_1c_1}(\log \text{SNR})^{-1}$.

Proof: See the Appendix. ∎

As a result of Theorem 4, we find an upper bound to (10) as

$$\begin{aligned} &P(\mathcal{R}_{2,2}|\mathcal{R}_2) \\ &\quad \dot{\leq} \text{SNR}^{-k_1c_1}(\log \text{SNR})^{-1} + \text{SNR}^{-k_2c_1}(\log \text{SNR})^{-1} \\ &\quad \doteq \text{SNR}^{-k_2c_1}(\log \text{SNR})^{-1}. \end{aligned} \tag{12}$$

The last step in (12) follows as $k_1 > k_2$. As a result of (12), $P(\mathcal{R}_{2,1}|\mathcal{R}_2) \doteq 1$ in (11). In (53), one can substitute $c_i \log \text{SNR}$, $i=1,2$, into γ_i^{th} to find that $P(\mathcal{R}_{2,2}|\mathcal{R}_2) \dot{\leq} \text{SNR}^{-1}e^{-k_2c_1 \log \text{SNR}}$. Observe that this is a reliability comparable to AWGN channels. Let $P(\epsilon_r,s_l|\mathcal{R}_3)$, $l=1,2$, denote the conditional probability of error of s_l at the relay when the system is in $\mathcal{R}_3$. Then, as the events $\{\epsilon_r,s_1|\mathcal{R}_3\}$ and $\{\epsilon_r,s_2|\mathcal{R}_3\}$ are independent, [10] finds the conditional subregion probabilities $P(\mathcal{R}_{3,n}|\mathcal{R}_3)$, $n=1,2,3,4$, as

$$\begin{aligned} P(\mathcal{R}_{3,1}|\mathcal{R}_3) &= [1-P(\epsilon_r,s_1|\mathcal{R}_3)]\,[1-P(\epsilon_r,s_2|\mathcal{R}_3)] \\ &\leq 1 \end{aligned} \tag{13}$$

$$\begin{aligned} P(\mathcal{R}_{3,2}|\mathcal{R}_3) &= [1-P(\epsilon_r,s_1|\mathcal{R}_3)]\,P(\epsilon_r,s_2|\mathcal{R}_3) \\ &\leq P(\epsilon_r,s_2|\mathcal{R}_3) \end{aligned} \tag{14}$$

$$\begin{aligned} P(\mathcal{R}_{3,3}|\mathcal{R}_3) &= P(\epsilon_r,s_1|\mathcal{R}_3)\,[1-P(\epsilon_r,s_2|\mathcal{R}_3)] \\ &\leq P(\epsilon_r,s_1|\mathcal{R}_3) \end{aligned} \tag{15}$$

$$P(\mathcal{R}_{3,4}|\mathcal{R}_3) = P(\epsilon_r,s_1|\mathcal{R}_3)P(\epsilon_r,s_2|\mathcal{R}_3) \tag{16}$$

$$P(\epsilon_r,s_1|\mathcal{R}_3) = \mathbb{E}\{f_1(k_1,\gamma_{sr})|\mathcal{R}_3\} + \mathbb{E}_{\gamma_{sr}}\{f_1(k_2,\gamma_{sr})|\mathcal{R}_3\} \tag{17}$$

$$\begin{aligned} P(\epsilon_r,s_2|\mathcal{R}_3) &= \mathbb{E}\{2f_1(k_3,\gamma_{sr})|\mathcal{R}_3\} - \mathbb{E}\{f_1(k_4,\gamma_{sr})|\mathcal{R}_3\} \\ &\quad + \mathbb{E}\{f_1(k_5,\gamma_{sr})|\mathcal{R}_3\} \\ &\leq \mathbb{E}\{2f_1(k_3,\gamma_{sr})|\mathcal{R}_3\} + \mathbb{E}\{f_1(k_5,\gamma_{sr})|\mathcal{R}_3\}. \end{aligned} \tag{18}$$

Theorem 5 [14]: The average error probability satisfies $\mathbb{E}\{f_1(a_1,\gamma_{sr})|\mathcal{R}_3\} \dot{\leq} \text{SNR}^{-(1+a_1c_2)}$.

Proof: In (53), substitute $\gamma_1^{\text{th}} = c_2 \log \text{SNR}$ and $\gamma_2^{\text{th}} = \infty$. Then the result follows. □

Using Theorem 5, we conclude that (17) and (18), respectively, are upper bounded as

$$\begin{aligned} P(\epsilon_r,s_1|\mathcal{R}_3) &\dot{\leq} \text{SNR}^{-(1+k_1c_2)} + \text{SNR}^{-(1+k_2c_2)} \\ &\doteq \text{SNR}^{-(1+k_2c_2)} \end{aligned} \tag{19}$$

$$\begin{aligned} P(\epsilon_r,s_2|\mathcal{R}_3) &\dot{\leq} \text{SNR}^{-(1+k_3c_2)} + \text{SNR}^{-(1+k_5c_2)} \\ &\doteq \text{SNR}^{-(1+k_3c_2)} \end{aligned} \tag{20}$$

since $\alpha = d_1/d_2 > 1$, and thus, $k_3 < k_5$. Finally, substituting (19) and (20) into (13)–(16), we have

$$P(\mathcal{R}_{3,1}|\mathcal{R}_3) \dot{\leq} 1 \tag{21}$$

$$P(\mathcal{R}_{3,2}|\mathcal{R}_3) \dot{\leq} \text{SNR}^{-(1+k_3c_2)} \tag{22}$$

$$P(\mathcal{R}_{3,3}|\mathcal{R}_3) \dot{\leq} \text{SNR}^{-(1+k_2c_2)} \tag{23}$$

$$P(\mathcal{R}_{3,4}|\mathcal{R}_3) \dot{\leq} \text{SNR}^{-(2+k_2c_2+k_3c_2)}. \tag{24}$$

Similarly, substituting $\gamma_1^{\rm th} = c_2{\rm SNR}$ and $\gamma_2^{\rm th} = \infty$ into (53), we can find that $P(\mathcal{R}_{3,1}|\mathcal{R}_3)\dot{\leq}1$, $P(\mathcal{R}_{3,2}|\mathcal{R}_3)\dot{\leq}{\rm SNR}^{-1}e^{-k_3c_2{\rm SNR}}$, $P(\mathcal{R}_{3,3}|\mathcal{R}_3)\dot{\leq}{\rm SNR}^{-1}e^{-k_2c_2{\rm SNR}}$, and $P(\mathcal{R}_{3,4}|\mathcal{R}_3)\dot{\leq}{\rm SNR}^{-2}e^{-(k_2+k_3)c_2{\rm SNR}}$ if the thresholds are assumed to be c_i SNR, $i = 1, 2$.

C. Probability of Error at the Destination

The probability of error equations of s_1 and s_2 at the destination are found in [10]. Here, we restate the results and find the asymptotic behavior. In $\mathcal{R}_1$, due to Theorem 1, we have

$$\begin{aligned} P(\epsilon_d, s_1|\mathcal{R}_1) &= \mathbb{E}\left\{\frac{f_1(k_1,\gamma_{sd})}{2} + \frac{f_1(k_2,\gamma_{sd})}{2}\right\} \\ &\doteq {\rm SNR}^{-1} \end{aligned} \tag{25}$$

$$\begin{aligned} P(\epsilon_d, s_2|\mathcal{R}_1) &= \mathbb{E}\left\{f_1(k_3,\gamma_{sd}) - \frac{f_1(k_4,\gamma_{sd})}{2} + \frac{f_1(k_5,\gamma_{sd})}{2}\right\} \\ &\leq \mathbb{E}\left\{f_1(k_3,\gamma_{sd}) + \frac{f_1(k_5,\gamma_{sd})}{2}\right\} \\ &\doteq {\rm SNR}^{-1}. \end{aligned} \tag{26}$$

In addition, $k_5 < k_4$, $f_1(k_5,\gamma_{sd}) > f_1(k_4,\gamma_{sd})$, and $P(\epsilon_d, s_2|\mathcal{R}_1) \geq \mathbb{E}\{f_1(k_3,\gamma_{sd})\} \doteq {\rm SNR}^{-1}$. Thus, $P(\epsilon_d, s_2|\mathcal{R}_1) \doteq {\rm SNR}^{-1}$. In $\mathcal{R}_2$, the average BER of s_1 given subregion $\mathcal{R}_{2,n}$, $n = 1, 2$, at the destination can be written as

$$\begin{aligned} P(\epsilon_d, s_1|\mathcal{R}_{2,1}) =& \mathbb{E}\left\{f_2(k_1,k_6,\gamma_{sd},\gamma_{rd})\right\} \\ &+ \mathbb{E}\left\{f_2(k_2,k_6,\gamma_{sd},\gamma_{rd})\right\} \end{aligned} \tag{27}$$

$$\begin{aligned} P(\epsilon_d, s_1|\mathcal{R}_{2,2}) =& \mathbb{E}\left\{f_3(k_1,k_6,\gamma_{sd},\gamma_{rd})\right\} \\ &+ \mathbb{E}\left\{f_3(k_2,k_6,\gamma_{sd},\gamma_{rd})\right\}. \end{aligned} \tag{28}$$

As a result of Theorems 2 and 3, we have

$$P(\epsilon_d, s_1|\mathcal{R}_{2,1}) \doteq {\rm SNR}^{-2} \tag{29}$$

$$P(\epsilon_d, s_1|\mathcal{R}_{2,2}) \doteq 1. \tag{30}$$

In this region, the probability of error of s_2 $P(\epsilon_d, s_2|\mathcal{R}_{2,1})$ and $P(\epsilon_d, s_2|\mathcal{R}_{2,2})$ are the same as $P(\epsilon_d, s_2|\mathcal{R}_1)$ because the relay does not assist the transmission of s_2. Hence

$$P(\epsilon_d, s_2|\mathcal{R}_1) \doteq {\rm SNR}^{-1} \tag{31}$$

$$P(\epsilon_d, s_2|\mathcal{R}_{2,1}) = P(\epsilon_d, s_2|\mathcal{R}_{2,2}) \doteq {\rm SNR}^{-1}. \tag{32}$$

In $\mathcal{R}_3$, the relay demodulates both s_1 and s_2, and the destination combines the source and relay signals. The average probability of errors of s_1 and s_2 in $\mathcal{R}_{3,n}$, $n = 1, 2, 3, 4$, can be computed as

$$\begin{aligned} P(\epsilon_d, s_1|\mathcal{R}_{3,1}) =& \frac{1}{2}\mathbb{E}\left\{f_2(k_1,k_1,\gamma_{sd},\gamma_{rd})\right\} \\ &+ \frac{1}{2}\mathbb{E}\left\{f_2(k_2,k_2,\gamma_{sd},\gamma_{rd})\right\} \end{aligned} \tag{33}$$

$$\begin{aligned} P(\epsilon_d, s_2|\mathcal{R}_{3,1}) =& \mathbb{E}\left\{f_2(k_3,k_3,\gamma_{sd},\gamma_{rd})\right\} \\ &+ \frac{1}{2}\mathbb{E}\left\{f_2(k_4,k_4,\gamma_{sd},\gamma_{rd})\right\} \\ &+ \frac{1}{2}\mathbb{E}\left\{f_2(k_5,k_5,\gamma_{sd},\gamma_{rd})\right\} \end{aligned} \tag{34}$$

$$\begin{aligned} P(\epsilon_d, s_1|\mathcal{R}_{3,2}) =& \frac{1}{2}\mathbb{E}\left\{f_2(k_1,k_2,\gamma_{sd},\gamma_{rd})\right\} \\ &+ \frac{1}{2}\mathbb{E}\left\{f_2(k_2,k_1,\gamma_{sd},\gamma_{rd})\right\} \end{aligned} \tag{35}$$

$$\begin{aligned} P(\epsilon_d, s_2|\mathcal{R}_{3,2}) =& \mathbb{E}\left\{f_3(k_3,k_3,\gamma_{sd},\gamma_{rd})\right\} \\ &+ \frac{1}{2}\mathbb{E}\left\{f_2(k_4,k_5,\gamma_{sd},\gamma_{rd})\right\} \\ &+ \frac{1}{2}\mathbb{E}\left\{f_2(k_5,k_4,\gamma_{sd},\gamma_{rd})\right\} \end{aligned} \tag{36}$$

$$\begin{aligned} P(\epsilon_d, s_1|\mathcal{R}_{3,3}) =& \frac{1}{2}\mathbb{E}\left\{f_3(k_1,k_1,\gamma_{sd},\gamma_{rd})\right\} \\ &+ \frac{1}{2}\mathbb{E}\left\{f_3(k_2,k_2,\gamma_{sd},\gamma_{rd})\right\} \end{aligned} \tag{37}$$

$$\begin{aligned} P(\epsilon_d, s_1|\mathcal{R}_{3,4}) =& \frac{1}{2}\mathbb{E}\left\{f_3(k_1,k_2,\gamma_{sd},\gamma_{rd})\right\} \\ &+ \frac{1}{2}\mathbb{E}\left\{f_3(k_2,k_1,\gamma_{sd},\gamma_{rd})\right\} \end{aligned} \tag{38}$$

$$\begin{aligned} P(\epsilon_d, s_2|\mathcal{R}_{3,3}) =& \frac{1}{2}\mathbb{E}\left\{f_3(k_3,k_4,\gamma_{sd},\gamma_{rd})\right\} \\ &+ \frac{1}{2}\mathbb{E}\left\{f_3(k_4,k_3,\gamma_{sd},\gamma_{rd})\right\} \\ &+ \frac{1}{2}\mathbb{E}\left\{f_2(k_3,k_5,\gamma_{sd},\gamma_{rd})\right\} \\ &+ \frac{1}{2}\mathbb{E}\left\{f_2(k_5,k_3,\gamma_{sd},\gamma_{rd})\right\} \end{aligned} \tag{39}$$

$$\begin{aligned} P(\epsilon_d, s_2|\mathcal{R}_{3,4}) =& \frac{1}{2}\mathbb{E}\left\{f_3(k_3,k_5,\gamma_{sd},\gamma_{rd})\right\} \\ &+ \frac{1}{2}\mathbb{E}\left\{f_2(k_4,k_3,\gamma_{sd},\gamma_{rd})\right\} \\ &+ \frac{1}{2}\mathbb{E}\left\{f_2(k_3,k_4,\gamma_{sd},\gamma_{rd})\right\} \\ &+ \frac{1}{2}\mathbb{E}\left\{f_3(k_5,k_3,\gamma_{sd},\gamma_{rd})\right\}. \end{aligned} \tag{40}$$

Then, due to Theorems 2 and 3, the asymptotic orders of (33)–(40) are found as

$$P(\epsilon_d, s_1|\mathcal{R}_{3,1}) \doteq P(\epsilon_d, s_1|\mathcal{R}_{3,2}) \doteq {\rm SNR}^{-2} \tag{41}$$

$$P(\epsilon_d, s_2|\mathcal{R}_{3,1}) \doteq {\rm SNR}^{-2} \tag{42}$$

$$P(\epsilon_d, s_1|\mathcal{R}_{3,3}) \doteq P(\epsilon_d, s_1|\mathcal{R}_{3,4}) \doteq 1 \tag{43}$$

$$P(\epsilon_d, s_2|\mathcal{R}_{3,2}) \doteq P(\epsilon_d, s_2|\mathcal{R}_{3,3}) \doteq P(\epsilon_d, s_2|\mathcal{R}_{3,4}) \doteq 1. \tag{44}$$

Note that the conditional probabilities $P(\epsilon_d, s_l|\mathcal{R}_{m,n})$, $l = 1, 2$, $m = 1, 2, 3$, $n = 1, 2, 3, 4$, do not depend on the threshold selection. They are the same for both threshold types c_iSNR or $c_i \log$ SNR, $i = 1, 2$.

D. Overall Diversity Calculation

We can make overall diversity analysis by substituting (7)–(9), (11), (12), (21)–(24) (25)–(26), (29)–(30) (31)–(32), and (41)–(44) into (1). Then

$$\begin{aligned} {\rm BER}\left(\gamma_1^{\rm th}, \gamma_2^{\rm th}, \alpha, s_1\right) \dot{\leq}& {\rm SNR}^{-2}\log {\rm SNR} \\ &+ {\rm SNR}^{-(1+k_2c_1)} \end{aligned} \tag{45}$$

$$\begin{aligned} {\rm BER}\left(\gamma_1^{\rm th}, \gamma_2^{\rm th}, \alpha, s_2\right) \dot{\leq}& {\rm SNR}^{-(1+k_2c_2)} + {\rm SNR}^{-2}\log {\rm SNR} \\ &+ {\rm SNR}^{-(1+k_3c_2)} \end{aligned} \tag{46}$$

leading into a lower bound on the diversity gains for s_1 and s_2 as $\min\{1 + k_2c_1, 2\} \leq \delta_1$ and $\min\{1 + k_2c_2, 1 + k_3c_2, 2\} \leq \delta_2$, where δ_l is the diversity gain for s_l, $l = 1, 2$. Note that the ${\rm SNR}^{-2}\log$ SNR term also has two levels of diversity [14]. Second, in a Rayleigh fading relay channel, δ_1 and δ_2 are both upper bounded by 2 [15]. When $\alpha > 2$, $k_2 > k_3$ is satisfied, and if c_1 and c_2 are chosen such that $1 + k_2c_1 \geq 2$ and $1 + k_3c_2 \geq 2$, or $c_1 \geq ((\alpha^2 + 1)/(\alpha - 1)^2)$ and $c_2 \geq \alpha^2 + 1$, (and $c_2 \geq c_1$ by definition), the lower and upper bounds on diversity gains become equal, and the system presents two levels of diversity for both s_1 and s_2. When $1 < \alpha \leq 2$, we have $k_2 \leq k_3$, and the condition $1 + k_2c_1 \geq 2$ guarantees that $\min\{1 + k_2c_2, 1 + k_3c_2, 2\} = 2$ as $c_2 \geq c_1$. Then both s_1 and s_2 have two levels of diversity.

Instead of choosing $\gamma_i^{\rm th}$ as $c_i \log$ SNR and determining c_i as described above, if the thresholds are set to c_iSNR, $i = 1, 2$ ($c_1 <$

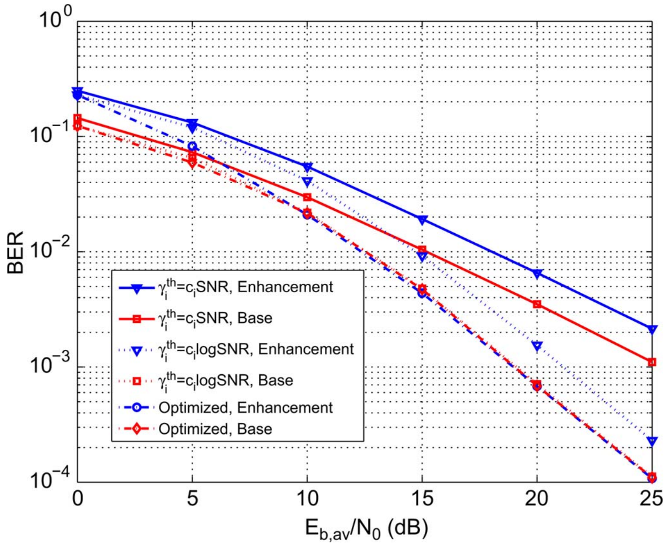

Fig. 2. BER of s_1 and s_2 versus average SNR per bit for $\gamma_i^{\text{th}} = c_i \log \text{SNR}$ and $\gamma_i^{\text{th}} = c_i\text{SNR}$, $i = 1, 2$, using hierarchical 2/4 ASK.

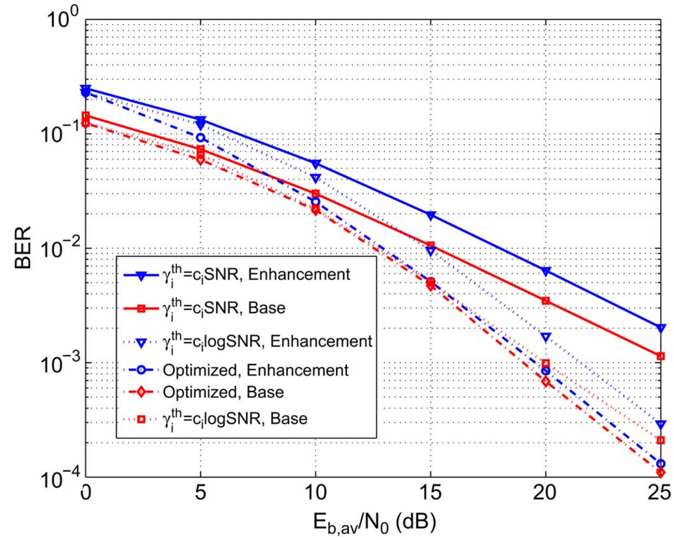

Fig. 3. BER of s_1 and s_2 versus average SNR per bit for $\gamma_i^{\text{th}} = c_i \log \text{SNR}$ and $\gamma_i^{\text{th}} = c_i\text{SNR}$, $i = 1, 2$, using hierarchical 4/16 quadrature amplitude modulation.

c_2, $c_1, c_2 \in \mathbb{R}^+$), full diversity gain could not be achieved since the first term $P(\epsilon_d, s_l|\mathcal{R}_1)P(\mathcal{R}_1)$ in (1) will become on the order of SNR^{-1} for both s_1 and s_2. Second, for s_2, the term $P(\epsilon_d, s_2|\mathcal{R}_{2,1})P(\mathcal{R}_{2,1}|\mathcal{R}_2)P(\mathcal{R}_2)$ will become on the order of SNR^{-1}.

E. Simulation Results and Discussion

Here, we confirm the analytical results for both s_1 and s_2 with simulation results. For $\alpha = 2$, $c_1 = 5$, $c_2 = 5.1$, we compare two systems: the thresholds are determined as 1) $c_i \log \text{SNR}$, and 2) $c_i\text{SNR}$ $(i = 1, 2)$. Figs. 2 and 3 show that if one determines the thresholds in the form of $c_i\text{SNR}$, it is not possible to achieve full diversity gain. To achieve full diversity, the thresholds should be chosen as $c_i \log \text{SNR}$. When thresholds are set to $c_i\text{SNR}$, the relay's transmission is inherently more reliable than the case when thresholds are $c_i \log \text{SNR}$, but the relay does not assist the source transmission as much as needed.

Note that it is possible to further improve the BER results by optimizing over γ_1^{th}, γ_2^{th}, and α. To provide the optimum BER performance as a benchmark, we solve the constrained optimization problem defined in (49) in [10]. In other words, given the constraint BER_1 on the base layer BER, we minimize the enhancement layer BER via the parameters $\gamma_1^{\text{th}}, \gamma_2^{\text{th}}$, and α. In Figs. 2 and 3, we assumed that BER_1 is equal to the base layer BER achieved when $\alpha = 2$, $\gamma_1^{\text{th}} = 5 \log \text{SNR}$, and $\gamma_2^{\text{th}} = 5.1 \log \text{SNR}$. We observe that there is an extra 2-dB gain with respect to the case obtained when the parameters are fixed to $\alpha = 2$, $\gamma_1^{\text{th}} = 5 \log \text{SNR}$, and $\gamma_2^{\text{th}} = 5.1 \log \text{SNR}$.

In this paper, our focus was on two-layered transmission with two threshold values. However, we can also study two-layered transmission with a single threshold $\gamma_1^{\text{th}} = \gamma_2^{\text{th}}$, resulting in only two regions $\mathcal{R}_1$ and $\mathcal{R}_3$ and dispensing with $\mathcal{R}_2$. In this case, the diversity results of this paper will not change, i.e., choosing $\gamma_1^{\text{th}} = \gamma_2^{\text{th}} = c \log \text{SNR}$, one can still achieve the full diversity gain of 2. Similarly, if $\gamma_1^{\text{th}} = \gamma_2^{\text{th}} = c\text{SNR}$, the diversity gain is still limited to 1.

It is also possible that there are N layers in the hierarchical modulation scheme in use and, thus, $N - 1$ hierarchy parameters. For such a system, we conjecture that it is sufficient to choose a single threshold value as $c \log \text{SNR}$ and to determine c as a function of hierarchy parameters to achieve full diversity. If the number of thresholds is increased, there is obviously more freedom in the system, and the probability of error can be improved for the same diversity gain.

IV. Conclusion

In this paper, we have studied the asymptotic probability of error of TDR over a three-node relay network with hierarchical modulation. Assuming hierarchical 2/4 ASK and Rayleigh fading, we showed that to mitigate error propagation and to achieve full diversity gains for both the base and enhancement layer bits, thresholds used at the relay should be chosen in the form of $c_i \log \text{SNR}$. Furthermore, c_i must depend on the hierarchy parameter α.

Appendix

Here, we prove the asymptotic behavior of the term $\mathbb{E}\{f_1(a_1, \gamma_{sr})|\mathcal{R}_2\}$, where

$$\mathbb{E}\left\{f_1(a_1, \gamma_{sr})|\mathcal{R}_2\right\} = \frac{1}{e^{-\gamma_1^{\text{th}}/\sigma_{sr}^2} - e^{-\gamma_2^{\text{th}}/\sigma_{sr}^2}} \frac{1}{2\sigma_{sr}^2} \times \int_{\gamma_1^{\text{th}}}^{\gamma_2^{\text{th}}} Q(\sqrt{2a_1\gamma_{sr}})e^{-\gamma_{sr}/\sigma_{sr}^2}\, d\gamma_{sr}. \quad (47)$$

By substituting Craig's formula $Q(z) = (1/\pi) \int_0^{(\pi/2)} e^{-z^2/(2 \sin^2 \theta)} d\theta$ [16] into (47), we have

$$\begin{aligned}
&\mathbb{E}\left\{f_1(a_1, \gamma_{sr})|\mathcal{R}_2\right\} \\
&= \frac{1}{e^{-\gamma_1^{\text{th}}/\sigma_{sr}^2} - e^{-\gamma_2^{\text{th}}/\sigma_{sr}^2}} \frac{1}{2\pi\sigma_{sr}^2} \int_{\gamma_1^{\text{th}}}^{\gamma_2^{\text{th}}} \exp\left(-\gamma_{sr}/\sigma_{sr}^2\right) \\
&\quad \times \int_0^{\frac{\pi}{2}} \exp\left(-\frac{a_1\gamma_{sr}}{\sin^2\theta}\right) d\theta\, d\gamma_{sr} \qquad (48) \\
&= \frac{1}{e^{-\gamma_1^{\text{th}}/\sigma_{sr}^2} - e^{-\gamma_2^{\text{th}}/\sigma_{sr}^2}} \frac{1}{2\pi} \int_0^{\frac{\pi}{2}} \sin^2\theta \\
&\quad \times \frac{\exp\left(\frac{-\gamma_1^{\text{th}}}{\sigma_{sr}^2}\right)\exp\left(\frac{-a_1\gamma_1^{\text{th}}}{\sin^2\theta}\right) - \exp\left(\frac{-\gamma_2^{\text{th}}}{\sigma_{sr}^2}\right)\exp\left(\frac{-a_1\gamma_2^{\text{th}}}{\sin^2\theta}\right)}{\sin^2\theta + \sigma_{sr}^2 a_1} d\theta \qquad (49)
\end{aligned}$$

$$\leq \frac{e^{-\gamma_1^{\text{th}}/\sigma_{sr}^2}}{e^{-\gamma_1^{\text{th}}/\sigma_{sr}^2} - e^{-\gamma_2^{\text{th}}/\sigma_{sr}^2}} \frac{1}{2\pi} \times \int_0^{\frac{\pi}{2}} \exp\left(\frac{-a_1\gamma_1^{\text{th}}}{\sin^2\theta}\right) \frac{\sin^2\theta}{\sin^2\theta + \sigma_{sr}^2 a_1} \, d\theta. \quad (50)$$

Since $((\sin^2\theta)/(\sin^2\theta + \sigma_{sr}^2 a_1)) < (1/(\sigma_{sr}^2 a_1))$, we can arrange (50) to get the new upper bound as

$$\mathbb{E}\left\{f_1(a_1, \gamma_{sr})|\mathcal{R}_2\right\} \leq \frac{e^{-\gamma_1^{\text{th}}/\sigma_{sr}^2}}{e^{-\gamma_1^{\text{th}}/\sigma_{sr}^2} - e^{-\gamma_2^{\text{th}}/\sigma_{sr}^2}} \frac{1}{2\pi\sigma_{sr}^2 a_1} \int_0^{\frac{\pi}{2}} \exp\left(\frac{-a_1\gamma_1^{\text{th}}}{\sin^2\theta}\right) d\theta. \quad (51)$$

Applying Craig's formula once again, to arrange (51), we get

$$\mathbb{E}\left\{f_1(a_1, \gamma_{sr})|\mathcal{R}_2\right\} \leq \frac{e^{-\gamma_1^{\text{th}}/\sigma_{sr}^2}}{e^{-\gamma_1^{\text{th}}/\sigma_{sr}^2} - e^{-\gamma_2^{\text{th}}/\sigma_{sr}^2}} \frac{1}{2\sigma_{sr}^2 a_1} Q\left(\sqrt{2a_1\gamma_1^{\text{th}}}\right). \quad (52)$$

When the upper bound given in [17] for the Q-function $Q(x) < (1/2)e^{-x^2/2}$ is applied to (52), (52) becomes

$$\mathbb{E}\left\{f_1(a_1, \gamma_{sr})|\mathcal{R}_2\right\} \leq \frac{e^{-\gamma_1^{\text{th}}/\sigma_{sr}^2}}{e^{-\gamma_1^{\text{th}}/\sigma_{sr}^2} - e^{-\gamma_2^{\text{th}}/\sigma_{sr}^2}} \frac{1}{4\sigma_{sr}^2 a_1} \exp\left(-a_1\gamma_1^{\text{th}}\right). \quad (53)$$

As the first and second thresholds are $\gamma_1^{\text{th}} = c_1 \log \text{SNR}$ and $\gamma_2^{\text{th}} = c_2 \log \text{SNR}$, we get

$$\mathbb{E}\left\{f_1(a_1, \gamma_{sr})|\mathcal{R}_2\right\} \leq \frac{e^{-\gamma_1^{\text{th}}/\sigma_{sr}^2}}{e^{-\gamma_1^{\text{th}}/\sigma_{sr}^2} - e^{-\gamma_2^{\text{th}}/\sigma_{sr}^2}} \frac{1}{4\lambda_{sr} a_1 \text{SNR}} \frac{1}{\text{SNR}^{a_1 c_1}}. \quad (54)$$

The high SNR limit of the first term in (54) can be computed via the Taylor series expansion as

$$\frac{e^{-\gamma_1^{\text{th}}/\sigma_{sr}^2}}{e^{-\gamma_1^{\text{th}}/\sigma_{sr}^2} - e^{-\gamma_2^{\text{th}}/\sigma_{sr}^2}} \doteq \frac{\lambda_{sr}\text{SNR}}{(c_2 - c_1)\log \text{SNR}}. \quad (55)$$

Then, (54) becomes

$$\mathbb{E}\left\{f_1(a_1, \gamma_{sr})|\mathcal{R}_2\right\} \dot{\leq} \text{SNR}^{-a_1 c_1}(\log \text{SNR})^{-1}. \quad (56)$$

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