DYNAMICS OF FLEXIBLE MEMBRANES WITH CIRCULAR HOLES DUE TO TRANSVERSE IMPACT

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
THE MIDDLE EAST TECHNICAL UNIVERSITY

BY

ÖMÜR UMUT

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF
DOCTOR OF PHILOSOPHY
IN
THE DEPARTMENT OF MATHEMATICS

JANUARY 1996

Approval of the Graduate School of Natural and Applied Sciences

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I certify that this thesis satisfies all the requirements as a thesis for the degree of Doctor of Philosophy.

Prof.Dr. Cemal KOÇ
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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosophy

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ABSTRACT

DYNAMICS OF FLEXIBLE MEMBRANES WITH HOLES DUE TO TRANSVERSE IMPACT

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January 1996, 96 pages

In the thesis, the problems of transverse impact to the flexible elastic membranes possessing a hole are investigated. In general the dynamics of the membrane is described by three sets of partial differential equations in three different regions with a priori unknown boundaries. The impact velocity is assumed to be high, so that one of the regions degenerates into a line of a strong discontinuity across which some nonlinear relations , following from the equations of motion, hold. Two different type problems, appropriate for two different type of boundary conditions on the hole are studied. The exact solutions to the problems on the radial wavefront are obtained, which, in particular, show that for a rigid hole the radial wavefront is a strong discontinuity wave, while for a free hole the derivatives of unknown functions are continuous. The transverse wavefront is the strong discontinuity wave for both cases.

The principle difference of the problems studied in the thesis from the already solved problems in dynamics of membranes is that the problems considered in the thesis do not possess a similarity solution like the problems studied previously and hence one has to deal with the nonlinear initial and boundary value problems for a set of partial differential equations in regions with a priori unknown boundaries.

The algorithms for the numerical solution to the problems are worked out. The numerical algorithms are based on the characteristic method of solution of the hyperbolic type of systems by making use of the jump conditions on the unknown boundary of regions.

The equations that appear in two different regions which are investigated in the thesis can also be viewed as a set of equations in one large region but in the latter case the coefficients become the impulsive type of functions. In the transition through the strong discontinuity wave the coefficients of the equations change impulsively. In this sense the problems considered in the thesis are related to the study of initial and boundary value problems for impulsive partial differential equations.

In the thesis the numerical solutions to the problems are obtained and the different graphs of the unknown quantities are brought. The convergence and stability of the numerical solutions have been established.

Keywords: Transverse impact, flexible elastic membrane, impact velocity, strong discontinuity, radial wavefront, transverse wavefront, similarity solution, characteristic method, hyperbolic type of system.

ÖZ

ÇAPRAZ ETKİDEN DOLAYI DELİKLİ ZARLARIN DİNAMİĞİ

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Bu tezde delikli esnek zara uygulanan dik çarpmanın neden olduğu problemlerle uğraşıldı. Genelde zarın dinamiği, sınırları önceden bilinmeyen üç değişik bölgedeki üç kısmi diferansiyel denklem sistemi ile tanımlanır. Çarpma hızının yüksek kabul edildiği durumda ise bölgelerden biri kuvvetli süreksizliğin karşılaşıldığı bir eğri haline gelir. Burada delik üzerindeki iki farklı sınır koşulu ile tanımlanan iki farklı problem üzerinde çalışıldı. Radyal dalga-önünde problemlerin kesin çözümleri elde edildi. Bu çözümler, çarpma ile çapı değişmeyen dalga probleminde radyal dalga-önünün kuvvetli süreksizlik olduğunu gösterdi. Diğer yandan delikteki mukavemetin sıfır olduğu durumda ise fonksiyonların kısmi türevlerinin bu dalga önünde sürekli olduğu sonucunu elde ettik. Her iki problemde de dik dalga-önü kuvvetli süreksizliktir.

Tezde çalışılan problemlerin zarların dinamiği ile ilgili çözülmüş problemlerden temel farkı, benzerlik çözümlerinin olmamasıdır. Bu problemler, sınırları önceden bilinmeyen bölgelerdeki kısmi diferansiyel denklem sistemi için başlangıç ve sınır koşullu problemler olarak ele alındı.

Problemlerin sayısal çözümlerini elde etmek için sayısal algoritmalar verildi. Bu algoritmalar hiperbolik denklemlerin karakteristikler yardımıyla çözümü yöntemine dayanmaktadır.

Bu tezde iki farklı bölgede incelenen denklemlere bir büyük bölgedeki denklem sistemi olarak bakılabılır. Bu amaçla yeni bir fonksiyon tanımı ile katsayılar, impulsif fonksiyonlar haline getirildi. İncelediğimiz iki problem, bu görüş ile başlangıç ve sınır koşullu impulsif kısmi diferansiyel denklemler olarak ele alındı.

Tezde calışılan problemlerin sayısal çözümleri elde edildi. Sayısal yöntemlerin yakısaması ve kararlılığı incelendi. İlgili grafikler verildi.

Anahtar Kelimeler: Dik çarpma, esnek zar, çarpma hızı, kuvvetli süreksizlik, radyal dalga-önü, enine dalga-önü, benzerlik çözümü, karakteristikler yöntemi, hiperbolik sistem.

ACKNOWLEDGEMENTS

It is a great pleasure for me to express my gratitude and deepest regards to my advisor Prof.Dr. Barat Nuriyev for his encouragement and guidance all through this thesis. It was possible for me to learn from him all about the contents of this thesis and "a lot more" as a result of his patience and continuous support, thus allowing the conclusion of this thesis.

I would like to thank to Asst.Prof.Dr.Tanıl Ergenç for his help and encouragement during the preparation of this study.

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CHAPTER 1

INTRODUCTION

Classical linear wave equations in one and two space dimensions describe wave propagation in flexible strings and membranes, respectively, and have the forms

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{1.1}$$

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{1.2}$$

where the unknown function u is the transverse displacement of particles of the string or membrane, a is some physical constant known as wave speed in the material of the string or membrane, t is time, x and y are Cartesian coordinates in some reference system.

Both equations (1.1) and (1.2) are derived under the following assumptions (see, for instance Tikhonov, A. and Samarski, A. [32])

1. All particles of the string or membrane move only in transverse direction and hence displacements of all particles are described by one scalar function u.
2. Deflections are small, in other words $\left|\frac{\partial u}{\partial x}\right| \ll 1$, $\left|\frac{\partial u}{\partial y}\right| \ll 1$ and $\left(1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right)^{-1/2} \simeq 1.$

3. The law of deformation is linear elastic and hence tractions are proportional to the strains.

From the first two assumptions it follows that the length of an initial rectilinear element of the string AB (or the area of the initial plane element of the membrane) does not change during the motion, in the process of which those elements become curvilinear (element $A'B'$); this fact is an obvious

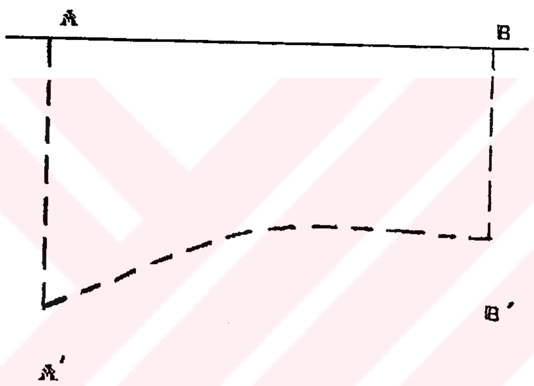

Figure 1.1: Change in the lenght of the string AB after impact

contradiction. All these assumptions of linear theory are not correct and hence cannot be used to describe the motion of strings and membranes subjected to the high transient loading.

To study the large displacements of strings under the transverse impact Rakhmatulin derived exact equations of plane motion of string [26, 27] in 1945, which have the form

$$\rho_0 \frac{\partial^2 x}{\partial t^2} = \frac{\partial}{\partial s_0}(\sigma \cos \varphi) \tag{1.3}$$

$$\rho_0 \frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial s_0}(\sigma \sin\varphi) \tag{1.4}$$

$$\frac{\partial x}{\partial s_0} = (1+\varepsilon)\cos\varphi \tag{1.5}$$

$$\frac{\partial y}{\partial s_0} = (1+\varepsilon)\sin\varphi \tag{1.6}$$

$$\sigma = \sigma(\varepsilon) \tag{1.7}$$

where ρ_0 is the initial density of the string, t is time, s_0 is the Lagrangian coordinate, x and y are Eulerian coordinates of the particle s_0 at time t, ε is the strain, σ is the stress and φ is the angle that tangent to the string at (s_0, t) makes with the initial rectilinear position of the string. The last equation is the state equation which for linear elastic deformation takes the form

$$\sigma = E\varepsilon \tag{1.8}$$

where E is some constant of the string's material, known as -Young's modulus. It should be noted that the equations (1.3)–(1.6) as well as the equations (1.1) and (1.2) are derived for a flexible string or membrane, the mathematical formulation of which is the statement that tension force is tangent to the current position of the string or membrane.

The exact solution to the equations (1.3)–(1.7) for an arbitrary dependence of $\sigma = \sigma(\varepsilon)$ corresponding to transverse impact with a constant velocity is obtained by Rakhmatulin [26, 27, 32]. The space motion of strings was investigated by Agalarov, Nuriyev and Rakhmatulin [1], [17], [18], etc. Since then the analogous problem for a membrane has been investigated by many authors [28], [5], [8], [10]–[11], [19]–[24], [29]–[31].

The exact equations of the axi–symmetrical motion of a membrane derived by Rakhmatulin, have the form

$$\rho_0 \frac{\partial^2 x}{\partial t^2} = \frac{1}{r}\frac{\partial(\sigma_r r \cos\gamma)}{\partial r} - \frac{\sigma_\theta}{r} \tag{1.9}$$

$$\rho_0 \frac{\partial^2 y}{\partial t^2} = -\frac{1}{r}\frac{\partial(\sigma_r r \sin\gamma)}{\partial r} \tag{1.10}$$

$$\frac{\partial x}{\partial r} = (1+\varepsilon_r)\cos\gamma \tag{1.11}$$

$$\frac{\partial y}{\partial r} = -(1+\varepsilon_r)\sin\gamma \tag{1.12}$$

$$\varepsilon_\theta = \frac{x}{r} - 1 \tag{1.13}$$

where the independent variables t and r are time and Lagrangian coordinate of particles of a membrane, the distance of an arbitrary particle of the membrane from its some fixed point–point of impact, at the initial instant of time; x is the radial coordinate, while y is the vertical coordinate in the direction of impact velocity v_0; ε_r and ε_θ are meridional and circumferential components of the strain tensor, while σ_r and σ_θ are corresponding stress tensor components, γ is the angle made by the tangent plane to the membrane with x–axis (Fig. 1.2). In (1.9)– (1.13) the unknowns are the functions $x(r,t), y(r,t), \gamma(r,t), \varepsilon_r(r,t), \varepsilon_\theta(r,t), \sigma_r(r,t)$ and $\sigma_\theta(r,t)$.

The lacking equations are state equations of the membrane. For linear elastic deformations these equations are generalized Hook's law–linear relations between the stress and strain tensor components. These relations under the assumption of $\sigma_3 \simeq 0$, where σ_3 is the stress tensor component along the thickness of the membrane, yield

$$\sigma_r = \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta) \tag{1.14}$$

$$\sigma_\theta = \frac{E}{1-\nu^2}(\varepsilon_\theta + \nu\varepsilon_r) \tag{1.15}$$

$$\varepsilon_3 = -\nu(\varepsilon_r + \varepsilon_\theta) \tag{1.16}$$

where the material constants E and ν are Young's modulus and Poisson's ratio, while ε_3 is the normal strain along the thickness of the membrane

$$\varepsilon_3 = \frac{\delta - \delta_0}{\delta_0} \tag{1.17}$$

where δ_0 and δ are the initial and current thicknesses of the membrane.

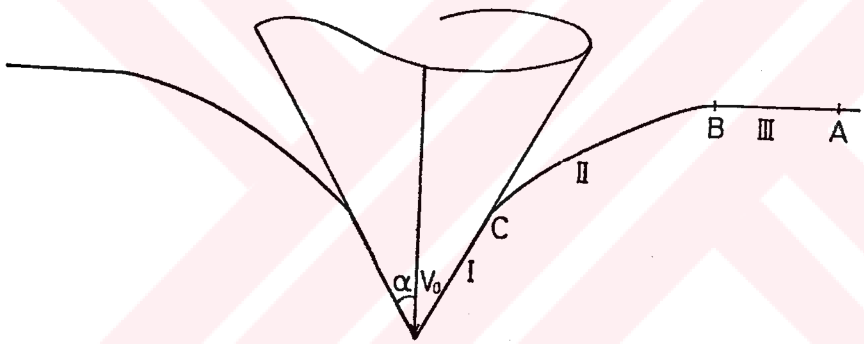

Figure 1.2: Wave scheme of motion of a membrane

In [10] a simplified solution was obtained by neglecting the circumferential component of stress tensor. But as shown in [8] in this problem the circumferential stress cannot be neglected by any means. In [8] an approximate solution to the problem is obtained. But the equations of motion of membrane under transverse impact given by (1.9)–(1.15) was derived in [28]. Further studies of the problem were devoted to the propagation of linear elastic waves in the membrane subjected to a transverse impact [8], [10]-[11],

[19]-[24], [29]-[31]. Many authors have considered various dynamic problems of membrane by making use of some simplifying assumptions [24, 29]. In [11] the numerical method of solution to the problem has been suggested. But all this authors and many others did not succeed in getting neither analytical nor numerical solution to the problem even for linearly elastic membrane. Numerical method suggested in [11] considers the transverse wavefront as a strong discontinuity wave like the analogous problem for strings. But as shown in [21] the transverse wavefront in the membrane subjected to the transverse impact for any constitutive equation is not a strong discontinuity wave, precisely it is shown that the conditions of dynamic and kinematic compatibility for the strong discontinuity wave on the transverse wavefront contradicts the equations of motion. Hence the numerical method suggested in [11] could not lead to the solution of the problem.

The solution to the problem for elastic membrane has obtained recently in [21]. In [22] the solution to the problem for perfectly plastic membrane is obtained. The analogous problem, but with the assumption that the radial displacement is small, has studied also in [31], where it is assumed that the radial stress is greater than the circumferential one. But in [22, 23] making use of the exact equations of motion it is shown that there is no region of plastic deformations where the radial stress exceeds the circumferential one. In [23] the solution to the problem for elastic-plastic membranes is obtained.

Equations (1.9)–(1.15) compose a closed set of nonlinear equations with respect to the unknown functions $x(r,t)$, $y(r,t)$, $\gamma(r,t)$, $\varepsilon_r(r,t)$, $\varepsilon_\theta(r,t)$, $\sigma_r(r,t)$ and $\sigma_\theta(r,t)$. After solving these equations under the appropriate initial and boundary conditions, ε_3 as well as current thickness of the membrane are easily evaluated by the formulae (1.16)–(1.17) which yield

$$\delta(r,t) = \delta_0(1 - \nu\varepsilon_r - \nu\varepsilon_\theta) \tag{1.18}$$

Besides the equation of conservation of mass yields the equation for defining the current density of the membrane [18]

$$\rho(r,t) = \frac{\rho_0}{(1+\varepsilon_r)(1+\varepsilon_\theta)(1+\varepsilon_3)} \tag{1.19}$$

In all of investigated problems the membrane is assumed to be initially at rest, that is initial conditions are homogeneous

$$x(r,0) = r, \quad y(r,0) = 0, \quad \gamma(r,0) = 0, \quad \varepsilon_r(r,0) = 0, \quad \varepsilon_\theta(r,0) = 0 \tag{1.20}$$

$$\frac{\partial x}{\partial t}(r,0) = 0, \quad \frac{\partial y}{\partial t}(r,0) = 0 \tag{1.21}$$

It was also assumed that the membrane is impacted by a costant velocity v_0 and hence the boundary conditions have the form

$$\frac{\partial x}{\partial t}(0,t) = 0, \quad \frac{\partial y}{\partial t}(0,t) = -v_0 \tag{1.22}$$

Thus the problem leads to the solution of equations (1.9)–(1.15) subject to the initial and boundary conditions (1.20)–(1.22) in the region $0 \leq r < \infty$, $t > 0$.

A brief discussion of the attempts of many authors to solve the above stated problem will be done in the following chapter. Here we only note that the above stated problem turns to have only similarity solution. The last circumstance allows one to reduce the set of nonlinear partial differential equations (1.9)–(1.15) to a set of nonlinear ordinary differential equations for some similarity variables and then investigate the corresponding boundary value problem for a set of ordinary differential equations in the regions with unknown boundaries.

The topics of this thesis are the analogous problems for the membranes with a hole of finite radius R. In the latter case the problem ceases to possess a similarity solution and we have to deal with the initial and boundary value problem in the region $R \leq r < \infty$, $t > 0$ for a set of nonlinear partial differential equations (1.9)–(1.15) and subject to the initial conditions (1.20)-(1.21) and some boundary conditions at the finite boundary $r = R$, which of course, differ from the conditions (1.22).

The boundary $r = R$ may be rigidly fixed or free of stresses. In the first case on the boundary we have the boundary condition

$$\varepsilon_\theta(R, t) = 0 \tag{1.23}$$

In the latter case the boundary condition has the form

$$\sigma_r(R, t) = 0 \tag{1.24}$$

Solution of the problem with the boundary condition (1.23) is given in Chapter 3, while Chapter 4 is devoted to the analytico– numeric solution of the equations (1.9)– (1.15) subject to the initial and boundary conditions (1.20), (1.21) and (1.24).

In the both cases the characteristic form of the equations of motion are derived, the possible wave schemes are established, the exact solution of the problem on the radial wavefront are obtained and the algorithms of numerical solutions of the problems are worked out. The questions of convergence and stability of numerical schemes are investigated as well. At the end of each chapter the numerical results obtained by using computers are discussed and the necessary graphs are plotted.

CHAPTER 2

NONLINEAR THEORY OF ENTIRE MEMBRANES SUBJECT TO A TRANSVERSE IMPACT

The chapter is devoted to the description of motion of entire membranes subject to the transverse impact by a cone. The pecularities of the problem as well as the possible wave schemes of motion are analyzed in full detail with necessary references. At the end the open problems are brought and outline of the work done in the thesis is given.

2.1 On Wave Propagation in Linear Elastic Membranes due to Transverse Impact

As was noted in Chapter 1 exact equations of motion of particles of an elastic membrane subject to the transverse impact are described by the following set of equations

$$\rho_0 \frac{\partial^2 x}{\partial t^2} = \frac{1}{r}\frac{\partial(\sigma_r r \cos\gamma)}{\partial r} - \frac{\sigma_\theta}{r} \tag{2.1}$$

$$\rho_0 \frac{\partial^2 y}{\partial t^2} = -\frac{1}{r}\frac{\partial(\sigma_r r \sin\gamma)}{\partial r} \tag{2.2}$$

$$\frac{\partial x}{\partial r} = (1 + \varepsilon_r)\cos\gamma \tag{2.3}$$

$$\frac{\partial y}{\partial r} = -(1 + \varepsilon_r)\sin\gamma \tag{2.4}$$

$$\varepsilon_\theta = \frac{x}{r} - 1 \tag{2.5}$$

$$\sigma_r = \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta) \tag{2.6}$$

$$\sigma_\theta = \frac{E}{1-\nu^2}(\varepsilon_\theta + \nu\varepsilon_r) \tag{2.7}$$

where we have used the same notations as in Chapter 1. In the equations (2.1)–(2.7) the unknowns are the functions $x(r,t)$, $y(r,t)$, $\gamma(r,t)$, $\sigma_r(r,t)$, $\sigma_\theta(r,t)$, $\varepsilon_r(r,t)$ and $\varepsilon_\theta(r,t)$ and they compose a closed set of nonlinear partial differential equations. It is assumed that the membrane initially was at rest. Consequently initial conditions take the form at t=0

$$x(r,0) = r, \quad y(r,0) = 0, \quad \gamma(r,0) = 0, \quad \varepsilon_r(r,0) = 0, \quad \varepsilon_\theta(r,0) = 0$$

$$\frac{\partial x}{\partial t}(r,0) = 0, \quad \frac{\partial y}{\partial t}(r,0) = 0 \tag{2.8}$$

The membrane is assumed to be of infinite extend and subject to the transverse 'point' impact at the point $r = 0$ with the constant velocity v_0. Hence the boundary conditions take the form

$$x(0,t) = 0, \quad y(0,t) = v_0 t \tag{2.9}$$

Thus the problem leads to the solution of equations (2.1) – (2.7) under the initial and boundary conditions (2.8) and (2.9).

As was shown in [28] the solution to the problem is the similarity solution, i.e. all dimensionless functions $x_0 = \dfrac{x}{a_0 t}$, $y_0 = \dfrac{y}{a_0 t}$, γ, ε_r, ε_θ where $a_0^2 = \dfrac{E}{\rho_0(1-\nu^2)}$ depend only on $z = \dfrac{r}{a_0 t}$.

Consequently the equations (2.1) – (2.7) in new variables become the set of ordinary differential equations and one gets

$$z^2 x_0'' = \frac{1}{z}\frac{d(\sigma_{r0} z \cos\gamma_0)}{dz} - \frac{\sigma_{\theta 0}}{z} \tag{2.10}$$

$$z^2 y_0'' = -\frac{1}{z}\frac{d(\sigma_{r0} z \sin\gamma_0)}{dz} \tag{2.11}$$

$$x_0' = (1+\varepsilon_{r0})\cos\gamma_0 \tag{2.12}$$

$$y_0' = (1+\varepsilon_{r0})\sin\gamma_0 \tag{2.13}$$

$$\varepsilon_{\theta 0} = \frac{x_0}{z} - 1 \tag{2.14}$$

$$\sigma_{r0} = \varepsilon_{r0} + \nu\varepsilon_{\theta 0} \tag{2.15}$$

$$\sigma_{\theta 0} = \varepsilon_{\theta 0} + \nu\varepsilon_{r0} \tag{2.16}$$

where $\sigma_{r0} = \dfrac{\sigma_r(1-\nu^2)}{E}$, $\sigma_{\theta 0} = \dfrac{\sigma_\theta(1-\nu^2)}{E}$, $\gamma_0 = \gamma(\dfrac{r}{a_0 t})$, etc. Initial conditions (2.8) yield

$$\lim_{z\to\infty}\frac{x_0}{z} = 1, \quad y_0(\infty) = 0, \quad \gamma_0(\infty) = 0$$
$$\varepsilon_{r0}(\infty) = 0, \quad \varepsilon_{\theta 0}(\infty) = 0 \tag{2.17}$$

The boundary conditions (2.9) take the form

$$x_0(0) = 0, \quad y_0(0) = \frac{v_0}{a_0} \tag{2.18}$$

Thus, one has to solve the problem (2.10) – (2.18). Till the first half of the 1980's it was assumed that the boundary conditions (2.9) and hence (2.18) are correct (see, [28], [10], [8] etc.). But as shown by Nuriyev (see, [21]) the 'point' impact is impossible. In other words, if one considers an impact by a cone with the vertex angle 2α, for all values of v_0 and other physical parameters ρ_0, E, ν there will exist some region of contact around the point $r = 0$. Furthermore the measure of the contact region does not approach zero when $\alpha \to 0$. The proof is based on the asymptotic behaviour of solutions of the equations (2.10) – (2.16) under the boundary conditions (2.18). In [21] the following asymptotic expression for $z \to 0$ is obtained:

$$\tan\frac{\gamma_0}{2} = \text{const}\left(\frac{1-z^2}{z^2}\right)^{\nu/2} \tag{2.19}$$

From (2.19) it follows that

$$\lim_{z\to 0}\gamma_0(z) = \pi \tag{2.20}$$

which contradicts the real geometrical picture of the phenomenon, since $\sup\gamma_0(z) \le \frac{\pi}{2}$.

To resolve the contradiction Nuriyev suggested that near the impact point there appears some region of contact with unknown boundary. In this region of contact $\gamma(r,t) = \text{const.} = \frac{\pi}{2} - \alpha$ and hence is known. But in the contact region a new unknown quantity—the reaction force of the cone (pressure) comes into play. Hence the full set equations of motion in the region of contact takes the form (see [29])

$$\rho_0\frac{\partial^2 x}{\partial t^2} = \frac{1}{r}\frac{\partial(\sigma_r r\sin\alpha)}{\partial r} - \frac{\sigma_\theta}{r} + \frac{P\cos\alpha}{\delta_0} \tag{2.21}$$

$$\rho_0\frac{\partial^2 y}{\partial t^2} = -\frac{1}{r}\frac{\partial(\sigma_r r\cos\alpha)}{\partial r} + \frac{P\sin\alpha}{\delta_0} \tag{2.22}$$

$$\frac{\partial x}{\partial r} = (1 + \varepsilon_r) \sin \alpha \tag{2.23}$$

$$\frac{\partial y}{\partial r} = -(1 + \varepsilon_r) \cos \alpha \tag{2.24}$$

$$\varepsilon_\theta = \frac{x}{r} - 1 \tag{2.25}$$

$$\sigma_r = \frac{E}{1 - \nu^2}(\varepsilon_r + \nu \varepsilon_\theta) \tag{2.26}$$

$$\sigma_\theta = \frac{E}{1 - \nu^2}(\varepsilon_\theta + \nu \varepsilon_r) \tag{2.27}$$

where δ_0 is the initial thickness of the membrane, α is the semivertex angle of the cone, P is the pressure exerted upon the membrane by a cone. The unknowns in (2.21) – (2.27) are $x, y, \sigma_r, \sigma_\theta, P, \varepsilon_r$ and ε_θ.

Since the impacting cone has a constant velocity v_0, then the normal velocity of the particles in the contact region is also constant and equal to $v_0 \sin \alpha$. Hence

$$x \cos \alpha + y \sin \alpha = v_0 t \sin \alpha \tag{2.28}$$

Upon employing (2.28) from (2.21) – (2.22) we get

$$P = \frac{\sigma_\theta}{r} \delta_0 \cos \alpha \tag{2.29}$$

$$\frac{\rho_0}{\sin \alpha} \frac{\partial^2 x}{\partial t^2} = \frac{1}{r} \frac{\partial(\sigma_r r)}{\partial r} - \frac{\sigma_\theta}{r} \sin \alpha \tag{2.30}$$

Equations (2.23), (2.24) by virtue of (2.28) yield

$$\frac{1}{\sin \alpha} \frac{\partial x}{\partial r} = 1 + \varepsilon_r \tag{2.31}$$

$$y(r,t) = v_0 t - x(r,t)\cot\alpha \tag{2.32}$$

Thus in the contact region aside from (2.29) and (2.32) we have following closed set of equations with respect to the unknown functions $x(r,t), \varepsilon_r(r,t), \varepsilon_\theta(r,t), \sigma_r(r,t)$ and $\sigma_\theta(r,t)$.

$$\frac{\rho_0}{\sin\alpha}\frac{\partial^2 x}{\partial t^2} = \frac{1}{r}\frac{\partial(\sigma_r r)}{\partial r} - \frac{\sigma_\theta}{r}\sin\alpha \tag{2.33}$$

$$\frac{1}{\sin\alpha}\frac{\partial x}{\partial r} = 1 + \varepsilon_r \tag{2.34}$$

$$\varepsilon_\theta = \frac{x}{r} - 1 \tag{2.35}$$

$$\sigma_r = \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta) \tag{2.36}$$

$$\sigma_\theta = \frac{E}{1-\nu^2}(\varepsilon_\theta + \nu\varepsilon_r) \tag{2.37}$$

Initial and boundary conditions (2.8) and (2.9) retain their form.

The region of contact, where the equations (2.33) – (2.37) are valid we denote as region I. By region II, we denote the region of free transverse motion, where we have the nonlinear set of equations (2.1) – (2.7)

On the common unknown boundary of regions I and II, $r = r^\star(t)$ we have the continuity of the displacements and smoothness of the membrane, which is equivalent to the continuity on $\gamma(r,t)$. Hence on the unknown boundary $r = r^\star(t)$

$$\gamma(r^\star(t), t) = \frac{\pi}{2} - \alpha \tag{2.38}$$

$$x(r^\star(t)-0,t)=x(r^\star(t)+0,t) \tag{2.39}$$

$$y(r^\star(t)+0,t)=v_0t-x(r^\star(t)-0,t)\cot\alpha \tag{2.40}$$

Thus in real statement of the problem of transverse impact by a cone to an elastic membrane leads to the integration of pair of system equations (2.1) – (2.7) and (2.33)–(2.37) under the initial and boundary conditions (2.8), (2.9) and (2.38) – (2.40).

2.2 Characteristics and Characteristic Relations

First we investigate the nonlinear set of equations (2.1) – (2.7). In [20] upon introducing two new functions $u(r,t)$ and $w(r,t)$, which denote the normal and tangential velocities of particles of the membrane, by the formulae

$$u(r,t)=\frac{\partial x}{\partial t}\cos\gamma-\frac{\partial y}{\partial t}\sin\gamma \tag{2.41}$$

$$w(r,t)=-\frac{\partial x}{\partial t}\sin\gamma-\frac{\partial y}{\partial t}\cos\gamma \tag{2.42}$$

$$\rho_0\left(\frac{\partial u}{\partial t}-w\frac{\partial\gamma}{\partial t}\right)=\frac{\partial\sigma_r}{\partial r}+\frac{\sigma_r-\sigma_\theta\cos\gamma}{r} \tag{2.43}$$

$$\rho_0\left(\frac{\partial w}{\partial t}+u\frac{\partial\gamma}{\partial t}\right)=\sigma_r\frac{\partial\gamma}{\partial r}+\frac{\sigma_\theta}{r}\sin\theta \tag{2.44}$$

$$\frac{\partial u}{\partial r}-w\frac{\partial\gamma}{\partial r}=\frac{\partial\varepsilon_r}{\partial t} \tag{2.45}$$

$$\frac{\partial w}{\partial r}+u\frac{\partial\gamma}{\partial r}=(1+\varepsilon_r)\frac{\partial\gamma}{\partial t} \tag{2.46}$$

$$\frac{\partial \varepsilon_\theta}{\partial t} = \frac{1}{r}(u\cos\gamma - w\sin\gamma) \tag{2.47}$$

By making use of (2.6) – (2.7) equations (2.43) – (2.47) constitute a closed set of five nonlinear first order partial differential equations with respect to the unknown functions $u(r,t), w(r,t), \gamma(r,t), \varepsilon_r(r,t), \varepsilon_\theta(r,t)$

Equations (2.43) – (2.47) can be written as

$$A\frac{\partial V}{\partial t} + B\frac{\partial V}{\partial r} = C \tag{2.48}$$

where

$$A = \begin{bmatrix} 1 & 0 & -w & 0 & 0 \\ 0 & 1 & u & 0 & 0 \\ 0 & 0 & 1+\varepsilon_r & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.49}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & -a_0^2 & -a_0^2\nu \\ 0 & 0 & \frac{-\sigma_r}{\rho_0} & 0 & 0 \\ 0 & -1 & -u & 0 & 0 \\ -1 & 0 & w & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.50}$$

$$C = \begin{bmatrix} \dfrac{\sigma_r - \sigma_\theta\cos\gamma}{r\rho_0} \\ \dfrac{\sigma_\theta}{r\rho_0} \\ 0 \\ 0 \\ \dfrac{1}{r}(u\cos\gamma - w\sin\gamma) \end{bmatrix}, \quad V = \begin{bmatrix} u \\ w \\ \gamma \\ \varepsilon_r \\ \varepsilon_\theta \end{bmatrix} \tag{2.51}$$

Wave speed $\lambda = \frac{dr}{dt}$ obtained from $|\lambda A - B| = 0$ are

$$\lambda_{1,2} = \pm a_0 \tag{2.52}$$

$$\lambda_{3,4} = \pm\sqrt{\frac{\sigma_r}{\rho_0(1+\varepsilon_r)}} \tag{2.53}$$

$$\lambda_5 = 0 \tag{2.54}$$

Characteristic relations along the each characteristic curves are

$$du - \frac{1}{\rho_0\lambda}d\sigma_r - dw = \frac{\sigma_r - \sigma_\theta\cos\gamma}{r\rho_0\lambda}dt \quad \text{along} \quad \lambda = \pm a_0 \tag{2.55}$$

$$dw + [u - \lambda(1+\varepsilon)]d\gamma = \frac{\sigma_\theta}{r\rho_0}\sin\gamma dt \quad \text{along} \quad \lambda = \pm\sqrt{\frac{\sigma_r}{\rho_0(1+\varepsilon_r)}} \tag{2.56}$$

$$d\varepsilon_\theta = \frac{1}{r}(u\cos\gamma - w\sin\gamma)dt \quad \text{along} \quad \lambda = 0 \tag{2.57}$$

Thus equations (2.43) – (2.47) are totally hyperbolic.

It is clear that $\frac{\partial x}{\partial t} < 0$ and $\varepsilon_\theta < 0$. This follows from the fact that the wave is stretching one, i.e. particles of the membrane move towards to the impact point $r = 0$. Hence we have

$$b_0 = \sqrt{\frac{\sigma_r}{\rho_0(1+\varepsilon_r)}} = a_0\sqrt{\frac{\varepsilon_r + \nu\varepsilon_\theta}{1+\varepsilon_r}} < a_0\sqrt{\frac{\varepsilon_r}{1+\varepsilon_r}} < a_0 \tag{2.58}$$

Consequently $\sup b_0 < a_0$

But $\sup b_0 = b_\star$ is the velocity of transverse wavefront. Then in the region $b_\star < \frac{dr}{dt} < a_0$ transverse displacement $y(r,t) \equiv 0$. Therefore in this region $w(r,t) \equiv 0, \gamma(r,t) \equiv 0$. This region we denote by region III.

Equations, that describe the motion of a membrane in region III, can be obtained from (2.1) – (2.7) upon putting $y(r,t) \equiv 0, \gamma(r,t) \equiv 0$ thereon, which yield

$$\rho_0 \frac{\partial^2 x}{\partial t^2} = \frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} \tag{2.59}$$

$$\varepsilon_r = \frac{\partial x}{\partial r} - 1 \tag{2.60}$$

$$\varepsilon_\theta = \frac{x}{r} - 1 \tag{2.61}$$

$$\sigma_r = \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta) \tag{2.62}$$

$$\sigma_\theta = \frac{E}{1-\nu^2}(\varepsilon_\theta + \varepsilon_r) \tag{2.63}$$

The common boundary of regions II and III where $\frac{dr}{dt} = b_*$ is also unknown in advance. The question arises and this question is what kind of wave is the boundary of regions II–III. The matter is that for the strings this is a strong discontinuity wave, since on that wave at least γ has a jump for any value of impact velocity v_0 (see,[28]). With great due to this fact until the first half of 1980's it was accepted that the same is true also for the membranes, i.e., transverse wavefront in membranes is a strong discontinuity wave. Using this fact Grigoryan,S. and Grigoryan,D. [11] suggested some numerical algorithm of solution to the problem, but never published the results of computations. In 1982 Nuriyev [20] upon employing the equations (2.43) – (2.46) exactly proved that on the transverse wavefront

$$\sigma_\theta \sin\gamma = 0 \tag{2.64}$$

From (2.64) it follows that if $\sigma_\theta \neq 0$ then $\gamma = 0$ on the transverse wavefront. But ahead of transverse wavefront $\gamma \equiv 0$ and hence γ is continuous thereon. Then it can easily be shown that the continuity of γ yields the continuity of u, w, ε_r. Hence the transverse wavefront is not the strong discontinuity wave. The last circumstance turned to be the main obstacle, that didn't allow the authors [11] and many others to get any numerical results. The solutions to the problem for linear elastic, perfectly plastic and elastic—plastic membranes are given in [21], [22] and [23].

2.3 Wave Schemes of Motion

Thus in general, study of the problem of transverse impact by a cone to an elastic membrane leads to the solution of three set of equations: (2.1) – (2.7), (2.33) – (2.37) and (2.59) – (2.63) in three different regions I, II and III with a prior unknown boundaries. The initial conditions are homogeneous, while the boundary conditions have the form (2.9), (2.38) – (2.40) plus the conditions of continuity of $\gamma, w, u, \varepsilon_\theta$ (and hence ε_r, σ_r) on the transverse wavefront. As the formula (2.58) shows $\sup b_0 = b_\star < a_0$ and hence is bounded for all values of α and v_0. The velocity of the points of the cone corresponding to $y = 0$ has the velocity component along x axis equal to $v_0 \tan\alpha$. Upon increasing v_0 and/or α one easily may get

$$v_0 \tan\alpha > b_\star \tag{2.65}$$

In the latter case region II disappears altogether and we get a problem, where the whole region of transverse motion turns to be in contact with an impacting cone. In this region we have a closed set of equations (2.33) –

(2.37). Ahead of the transverse wavefront the motion of particles of the membrane is pure radial motion which is described by the equations (2.59) – (2.63). But in this case the common boundary of the regions-transverse wavefront becomes strong discontinuity wave, since at least γ has a jump thereon

$$\gamma^- = \frac{\pi}{2} - \alpha, \quad \gamma^+ = 0 \tag{2.66}$$

The boundary conditions on the transverse wavefront have the form ([28], [20])

$$\frac{\partial x^+}{\partial t} + b_0 \frac{\partial x^+}{\partial r} = v_0 \tan \alpha \tag{2.67}$$

$$\frac{\partial x^-}{\partial t} + b_0 \frac{\partial x^-}{\partial r} = v_0 \tan \alpha \tag{2.68}$$

$$\rho_0 b_0^2 \left[(1 + \varepsilon_r^-) - (1 + \varepsilon_r^+) \sin \alpha\right] = \sigma_r^- - \sigma_r^+ \sin \alpha \tag{2.69}$$

The first two conditions (2.67) and (2.68) are the results of continuity of displacements (equivalently, the law of conservation of mass) on the transverse wavefront, while the relation (2.69) is the law of conservation of the momentum on the transverse wavefront.

On the longitudinal wavefront where $r = a_0 t$ we have the condition of continuity of displacements

$$x(a_0 t, t) = a_0 t \tag{2.70}$$

The problem for linear elastic membrane has been solved analytically by Pavlenko, A. [24], while for pure plastic and elastoplastic membranes by Nuriyev, B. [18].

In the case of plastic deformations it has been proven in [18] that $\sigma_\theta > \sigma_r$ while for linear elastic deformations we have the inverse relation: $\sigma_r > \sigma_\theta$.

In the following two chapters we will give the solution of two new problems for membranes subjected to transverse impact by a cone: namely we will assume that the membrane initially possesses a hole of radius R. Chapter 3 will be devoted to the solution of problem of transverse impact to membranes with a rigid hole, while in Chapter 4 we study the dynamics of membrane with a free hole. The latter both problems differ from the previous solved problems in that, that dynamics of membranes with holes do not possess a similarity solution and hence one has to deal with the nonlinear initial-boundary value problems for two sets of partial differential equations, in different regions with unknown boundaries.

CHAPTER 3

DYNAMICS OF A MEMBRANE WITH A FIXED HOLE DUE TO TRANSVERSE IMPACT

In this chapter, we deal with the solution of a problem of transverse impact to membranes with a rigid hole. The possible wave schemes of motion of the membranes are analyzed. Since problem does not possess similarity solution, an algorithm of numerical solution to the problem is suggested. Convercency and stability of the numerical scheme are investigated. At the end the numerical results obtained by using computers are discussed and the necessary grahs are plotted.

3.1 Problem Statement

We consider an infinite and initially planar membrane with a rigid hole which is at rest. We assume that at the moment $t = 0$ the membrane is subjected to the transverse impact by cone with constant velocity v_0. The motion of the membrane is covered by two different regions. In the region of transverse motion, labelled by I, membrane is in contact with the cone. In region II, the motion of the particles of the membrane is pure radial (see Fig 3.1). It is of interest to note that transverse wavefront RB is a strong discontinuity wave, since the angle $\gamma(r,t)$ is discontinuous thereon. Besides this common boundary of regions I and II is a priori unknown.

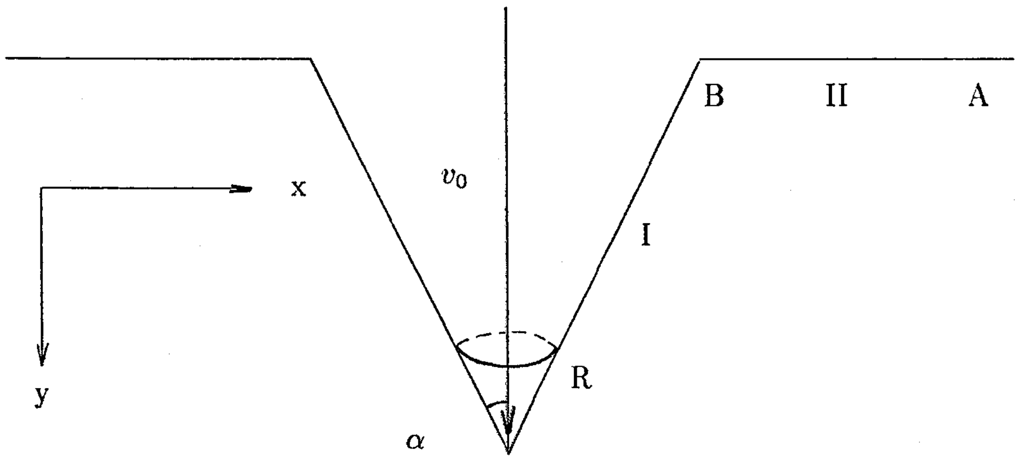

Figure 3.1: Wave scheme of motion of a membrane with hole

The equations of transverse motion in region I, can be written as

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta \sin\alpha}{r} \tag{3.1}$$

$$\varepsilon_r = \frac{\partial u}{\partial r} - 1, \quad \varepsilon_\theta = \frac{u}{r}\sin\alpha - 1 \tag{3.2}$$

$$\sigma_r = \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta), \quad \sigma_\theta = \frac{E}{1-\nu^2}(\varepsilon_\theta + \nu\varepsilon_r) \tag{3.3}$$

where $u = \dfrac{x}{\sin\alpha}$.

If we substitute equations (3.2) and (3.3) into (3.1) we obtain

$$\frac{1}{a_0^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\sin^2\alpha - \frac{1+\nu}{r}(1-\sin\alpha) \tag{3.4}$$

In region II, the radial motion is described by the equations

$$\rho_0 \frac{\partial^2 x}{\partial t^2} = \frac{\partial \sigma_{r1}}{\partial r} + \frac{\sigma_{r1} - \sigma_{\theta 1}}{r} \tag{3.5}$$

$$\varepsilon_{r1} = \frac{\partial x}{\partial r} - 1, \quad \varepsilon_{\theta 1} = \frac{x}{r} - 1 \tag{3.6}$$

$$\sigma_{r1} = \frac{E}{1-\nu^2}(\varepsilon_{r1} + \nu\varepsilon_{\theta 1}), \quad \sigma_{\theta 1} = \frac{E}{1-\nu^2}(\varepsilon_{\theta 1} + \nu\varepsilon_{r1}) \tag{3.7}$$

or by a single equation if one employs the relations (3.6) – (3.7) in (3.5)

$$\frac{1}{a_0^2}\frac{\partial^2 x}{\partial t^2} = \frac{\partial^2 x}{\partial r^2} + \frac{1}{r}\frac{\partial x}{\partial r} - \frac{x}{r^2} \tag{3.8}$$

Initial conditions at $t = 0$ are

$$x(r,0) = r, \quad \frac{\partial x}{\partial t}(r,0) = 0, \quad u(r,0) = \frac{r}{\sin\alpha}, \quad \frac{\partial u}{\partial t}(r,0) = 0 \tag{3.9}$$

Boundary condition at $r = R$

$$\varepsilon_\theta(R,t) = 0 \tag{3.10}$$

If we introduce a new variable

$$w(r,t) = u(r,t) - \frac{1+\nu}{1+\sin\alpha}r \tag{3.11}$$

then the governing equations as well as initial and boundary conditions of motion become:

In region I

$$\frac{1}{a_0^2}\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{w}{r^2}\sin^2\alpha \tag{3.12}$$

In region II

$$\frac{1}{a_0^2}\frac{\partial^2 x}{\partial t^2} = \frac{\partial^2 x}{\partial r^2} + \frac{1}{r}\frac{\partial x}{\partial r} - \frac{x}{r^2} \tag{3.13}$$

Initial conditions at $t = 0$ are

$$\begin{aligned} x(r,0) = r, &\quad \frac{\partial x}{\partial t}(r,0) = 0 \\ w(r,0) = \frac{r}{\sin\alpha} - \frac{1+\nu}{1+\sin\alpha}r, &\quad \frac{\partial w}{\partial t}(r,0) = 0 \end{aligned} \tag{3.14}$$

Boundary condition at $r = R$ is

$$\varepsilon_\theta(R, t) = 0 \tag{3.15}$$

The last condition implies that the boundary $r = R$ of the membrane is fixed rigidly, i.e. the circle $r = R$ all time of motion has the same lenght.

3.2 Characteristics and Characteristic Relations

Upon introducing u and v such that

$$u = w_t, \quad v = w_r$$

equation (3.12) can be written in the form

$$A_1 \frac{\partial V_1}{\partial t} + B_1 \frac{\partial V_1}{\partial r} = C_1 \tag{3.16}$$

where

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & -a_0^2 \\ -1 & 0 \end{bmatrix} \tag{3.17}$$

$$C_1 = \begin{bmatrix} \frac{1}{r} w_r - \frac{w}{r^2} \sin^2 \alpha \\ 0 \end{bmatrix}, \quad V_1 = \begin{bmatrix} u \\ v \end{bmatrix} \tag{3.18}$$

Characteristic speeds $\lambda = \frac{dr}{dt}$ obtained by $|\lambda A_1 - B_1| = 0$ are

$$\lambda = \pm a_0 \tag{3.19}$$

Characteristic relations which replace the equations (3.16) and hence (3.12) take form

$$dw_t = \pm a_0 dw_r + a dt \quad \text{along} \quad \lambda = \pm a_0 \tag{3.20}$$

where $a = a_0^2(\frac{1}{r}w_r - \frac{w}{r^2}\sin^2\alpha)$

Equation (3.13) can also be written as a system of first order equations by introducing two new functions by $u' = x_t, \quad v' = x_r$

$$A_2\frac{\partial V_2}{\partial t} + B_2\frac{\partial V_2}{\partial r} = C_2 \tag{3.21}$$

where A_2 and B_2 are same as A_1 and B_1, respectively, given in (3.17) and

$$C_2 = \begin{bmatrix} \frac{1}{r}x_r - \frac{x}{r^2} \\ 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} u' \\ v' \end{bmatrix} \tag{3.22}$$

Characteristic velocities are

$$\lambda = \pm a_0 \tag{3.23}$$

and characterictic relations become

$$dx_t = \pm a_0 dx_r + a_1 dt \quad \text{along} \quad \lambda = \pm a_0 \tag{3.24}$$

where $a_1 = a_0^2(\frac{1}{r}x_r - \frac{x}{r^2})$

3.3 Solution on the Radial Wavefront

Since the displacement $x(r,t)$ is continuous on the radial wavefront RA as well as everywhere else, along the wavefront RA where $r = R + a_0 t$ we have

$$x(R + a_0 t, t) = R + a_0 t \tag{3.25}$$

Let's consider the characteristic relation (3.24) on $r = R + a_0 t$

$$dx_t = a_0 dx_r + a_1 dt \tag{3.26}$$

Total derivative of the equation (3.25) is

$$\frac{dx}{dt}|_{r=R+a_0t} = \frac{\partial x}{\partial t} + a_0\frac{\partial x}{\partial r} = a_0 \tag{3.27}$$

Equation (3.26) can be written as

$$\frac{d}{dt}\left(\frac{\partial x}{\partial t}\right) = a_0\frac{d}{dt}\left(\frac{\partial x}{\partial r}\right) + a_1 \tag{3.28}$$

and

$$a_1|_{r=R+a_0t} = a_0^2\left(\frac{1}{r}x_r - \frac{x}{r^2}\right)|_{r=R+a_0t} = \frac{a_0^2}{R+a_0t}\left(\frac{\partial x}{\partial r} - 1\right) \tag{3.29}$$

From the equation (3.27) we get

$$\frac{\partial x}{\partial t} = -a_0\left(\frac{\partial x}{\partial r} - 1\right) \tag{3.30}$$

Let

$$\frac{\partial x}{\partial r} - 1 = \psi \quad \text{along} \quad \lambda = a_0 \tag{3.31}$$

then,

$$\frac{\partial x}{\partial t} = -a_0\psi \tag{3.32}$$

Substitution of the equations (3.29), (3.32) and (3.31) into (3.28) yields the first order linear ordinary differential equation with respect to $\psi(t)$

$$\frac{d\psi}{dt} + \frac{a_0}{2(R+a_0t)}\psi = 0 \tag{3.33}$$

General solution to the equation (3.33) is

$$\psi = C\frac{1}{\sqrt{R+a_0t}} \tag{3.34}$$

If we choose $C = C_0\sqrt{R}$, then

$$\frac{\partial x}{\partial r} = 1 + C_0\sqrt{\frac{R}{R + a_0 t}} \tag{3.35}$$

$$\frac{\partial x}{\partial t} = -a_0 C_0\sqrt{\frac{R}{R + a_0 t}} \tag{3.36}$$

Thus, by using the equations (3.25) and (3.35) we obtained the value of ε_{r1} and $\varepsilon_{\theta 1}$ depending on C_0 and t on the radial wavefront RA.

In this problem transverse wavefront RB with velocity $\dfrac{dr_\star}{dt} = b_0$ is a strong discontinuity wave and shock condition on RB is

$$\rho_0(-b_0)\left(\frac{\partial w}{\partial t} - \frac{\partial x}{\partial t}\sin\alpha - v_0\cos\alpha\right) = \sigma_r - \sigma_{r1}\sin\alpha \tag{3.37}$$

If we solve (3.37) we obtain

$$\rho_0 b_0^2 = \frac{\sigma_r - \sigma_{r1}\sin\alpha}{(1+\varepsilon_r) - (1+\varepsilon_{r1})\sin\alpha} \tag{3.38}$$

On the transverse wavefront, x and w can be defined as

$$x(r_\star(t), t) = R + v_0 t\tan\alpha \tag{3.39}$$

$$w(r_\star(t), t) = \frac{R + v_0 t\tan\alpha}{\sin\alpha} - \frac{1+\nu}{1+\sin\alpha}r_\star \tag{3.40}$$

By taking total derivatives of the equations (3.39) and (3.40) we get partial time derivatives of x and w, respectively, as

$$\frac{\partial x}{\partial t} = v_0\tan\alpha - b_0\frac{\partial x}{\partial r} \tag{3.41}$$

$$\frac{\partial w}{\partial t} = \frac{v_0}{\cos\alpha} - b_0\left(\frac{\partial w}{\partial r} + \frac{1+\nu}{1+\sin\alpha}\right) \tag{3.42}$$

On the boundary $r = R$ from the equations (3.2) (3.10) and (3.11) we get

$$w(R,t) = \frac{R}{\sin\alpha} - \frac{1+\nu}{1+\sin\alpha}R \tag{3.43}$$

This means that w is time independent on Rt

$$\frac{\partial w}{\partial t}(R,t) = 0 \tag{3.44}$$

Thus we have the following boundary conditions

On Rt where r=R

$$\frac{\partial w}{\partial t}(R,t) = 0 \tag{3.45}$$

On RB where $\dfrac{dr_\star}{dt} = b_0$

$$\frac{\partial x}{\partial t} = v_0 \tan\alpha - b_0(1+\varepsilon_{r1}) \tag{3.46}$$

$$\frac{\partial w}{\partial t} = \frac{v_0}{\cos\alpha} - b_0(1+\varepsilon_r) \tag{3.47}$$

$$\rho_0 b_0^2 = \frac{\sigma_r - \sigma_{r1}\sin\alpha}{(1+\varepsilon_r) - (1+\varepsilon_{r1})\sin\alpha} \tag{3.48}$$

On RA where $r = R + a_0 t$

$$\frac{\partial x}{\partial r} = 1 + C_0\sqrt{\frac{R}{R+a_0 t}} \tag{3.49}$$

$$\frac{\partial x}{\partial t} = -a_0 C_0\sqrt{\frac{R}{R+a_0 t}} \tag{3.50}$$

Now we define the values of unknown functions at $r = R$ and $t = 0^+$ that is immediately after the impact. (It should be noted that impact loading causes the discontinuity of some quantities like velocity, radial strain at $t = 0$).

Since the displacement is assumed to be continuous (the main assumption of continuum mechanics), due to the (3.2) we have

$$\lim_{t \to 0^+} \varepsilon_\theta(r,t) = 0 \tag{3.51}$$

and hence, in particular

$$\varepsilon_\theta(R, 0^+) = 0 \tag{3.52}$$

The same is true in region II

$$\varepsilon_{\theta 1}(R, 0^+) = 0 \tag{3.53}$$

In relations (3.46), (3.47) and (3.48) passing to the as limit $r \to R$, $t \to 0^+$ we get

$$\frac{\partial x}{\partial t}(R, 0^+) = v_0 \tan\alpha - b_0 \frac{\partial x}{\partial r}(R, 0^+) \tag{3.54}$$

$$\frac{\partial x}{\partial r}(R, 0^+) = 1 + C_0 \tag{3.55}$$

$$\frac{\partial x}{\partial t}(R, 0^+) = -a_0 C_0 \tag{3.56}$$

The same operation in equations (3.6) and (3.55) yield

$$\varepsilon_{r1}(R, 0^+) = C_0 \tag{3.57}$$

Upon substituting the equations (3.55) and (3.56) into (3.54) we will get

$$-a_0 C_0 = v_0 \tan\alpha - b_0(1 + C_0) \tag{3.58}$$

Now approaching $r \to R$ and $t \to 0^+$ in (3.47) and taking into account (3.45) we obtain

$$b_0(1 + \varepsilon_r) = \frac{v_0}{\cos\alpha} \tag{3.59}$$

Besides from (3.48) by making use of (3.57) we get

$$b_0^2 = a_0^2 \frac{\varepsilon_r - C_0 \sin\alpha}{(1+\varepsilon_r) - (1+C_0)\sin\alpha} \tag{3.60}$$

Equations (3.58), (3.59) and (3.60) constitute three equations with respect to C_0, b_0 and ε_r (for given v_0, a_0 and α). By solving these three equations we define $b_0(R, 0^+), C_0$ and $\varepsilon_r(R, 0^+)$.

Thus the equations (3.58), (3.59), (3.60), (3.52), (3.54), (3.57) define the values of all unknown quantities at the boundary $r = R$ immediately after impact. Besides according to the formulae (3.49), (3.50) the functions $\frac{\partial x}{\partial t}$ and $\frac{\partial x}{\partial r}$ on the whole line RA become known functions of t for all time of motion.

3.4 Algorithm of Numerical Solution to the Problem

The above stated problem has no similarity solution. Due to this reason, we want to obtain numerical solution to the problem by using the characteristic method.

The equations of characteristic curves and radial wavefront RA are known in advance but the transverse wavefront RB is unknown. Our aim is to construct unknown curve RB and to find values of all unknown functions at nodal points in regions I and II as well as on the transverse wavefront RB.

Initially, we construct characteristic curves with the slopes a_0 and $-a_0$ and radial wavefront with a slope a_0. Since these are straight lines, coordinates of the nodal points, r_{ij} and t_{ij}, can be easily calculated as the intersection of two lines.

We need some values at the point R, namely b_0, v_0, C_0 and ε_r. These are evaluated by using the equations (3.58), (3.59) and (3.60).

On the radial wavefront RA, the nodal values of x_t and x_r are obtained by using the equations (3.49) and (3.50).

At the first step, we draw the characteristic line with a slope $-a_0$. Intersection of this line with $t-$axis and with radial wavefront are labelled by C_1 and A_1, respectively. Its intersection with $r-$axis denoted by $R+h$, where h is a chosen step size along the $r-$axis which is constant. From C_1, we draw the characteristic line with the slope a_0 which is parallel to the radial wavefront. Now we can construct RB with initial slope $b_0(R, 0^+)$. Intersection of RB with C_1A_1 is labelled by B_1. Coordinates of B_1, i.e.,r_{B_1} and t_{B_1} can be calculated by Euler's method (see Fig.3.2).

From the characteristic relation (3.24), we write the equation along B_1A_1

$$x_t(B_1) - x_t(A_1) = -a_0(x_r(B_1) - x_r(A_1)) + a_1(A_1)(t_{B_1} - t_{A_1}) \qquad (3.61)$$

where

$$a_1(A_1) = a_0^2\left(\frac{x_r(A_1)}{r_{A_1}} - \frac{x(A_1)}{r_{A_1}^2}\right)$$
$$x(A_1) = x_t(A_1)t_{A_1} + x_r(A_1)(r_{A_1} - R) + x(R)$$

Second equation at point B_1 can be written by using (3.41)

$$x_t(B_1) + b_0(R)x_r(B_1) = v_0 \tan\alpha \qquad (3.62)$$

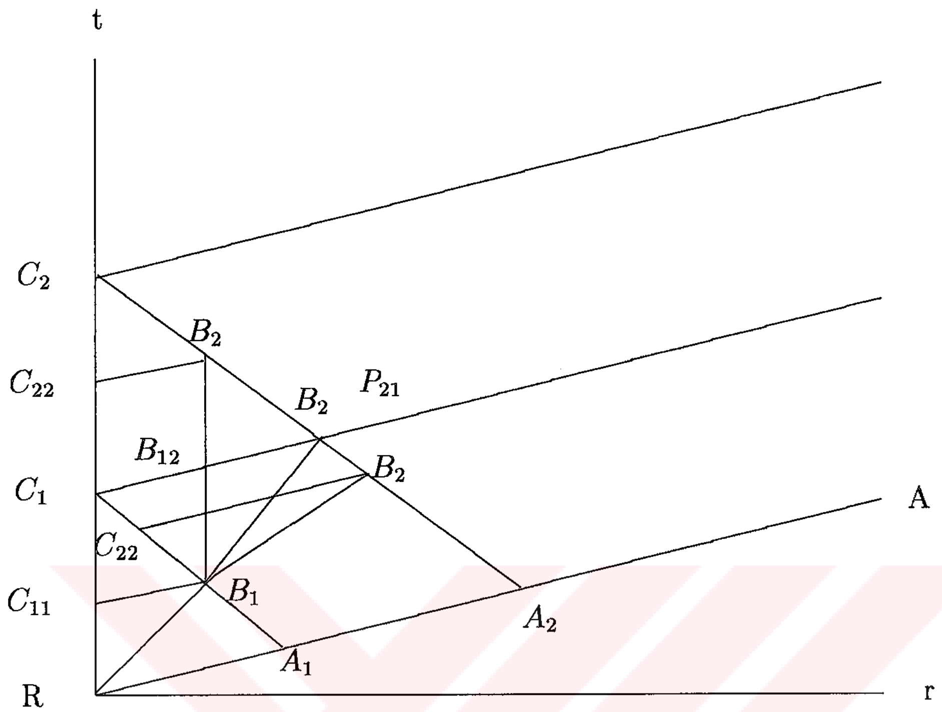

Figure 3.2: Construction of scheme at first two steps

Thus at the first step we get two equations in two unknowns, $x_t(B_1)$ and $x_r(B_1)$. The determinant of the coefficients is not zero. Indeed

$$\Delta_x = \begin{vmatrix} 1 & a_0 \\ 1 & b_0(R) \end{vmatrix}$$

$$\Delta_x = b_0(R) - a_0 < 0, \quad (b_0 < a_0) \tag{3.63}$$

So, the unknowns $x_t(B_1)$ and $x_r(B_1)$ are uniquely determined from (3.61) and (3.62).

The equation along B_1C_1 can be written by using the characteristic relation (3.20) as

$$w_t(B_1) - w_t(C_1) = -a_0(w_r(B_1) - w_r(C_1)) + a(C_1)(t_{B1} - t_{C_1}) \tag{3.64}$$

where

$$a(C_1) = a_0^2 \left(\frac{w_r(C_1)}{r_{C_1}} - \frac{w(C_1)}{r_{C_1}^2} \sin^2 \alpha \right)$$

$$w_t(C_1) = 0, \quad w(C_1) = w(R)$$

By using (3.42) we write the second equation along RB_1

$$w_t(B_1) + b_0(R) w_r(B_1) = \frac{v_0}{\cos \alpha} - \frac{1+\nu}{1+\sin \alpha} b_0(R) \tag{3.65}$$

Now we have two equations (3.64) – (3.65) in three unknowns, namely in $w_t(B_1)$, $w_r(B_1)$ and $w_r(C_1)$. Since system is totally hyperbolic we may construct an extra equation to obtain the unique solution. We draw another characteristic line with the slope a_0 which passes through B_1. Its intersection with $t-$axis is labelled by C_{11}. Then the characteristic relation along $B_1 C_{11}$ takes the form

$$w_t(B_1) - w_t(C_{11}) = a_0(w_r(B_1) - w_r(C_{11})) + a(C_{11})(t_{B_1} - t_{C_{11}}) \tag{3.66}$$

where

$$w_t(C_{11}) = 0$$

$$w_r(C_{11}) = w_r(R) \frac{t_{C_1} - t_{C_{11}}}{t_{C_1}} + w_r(C_1) \frac{t_{C_{11}}}{t_{C_1}}$$

Now we have three equations and three unknowns. The principal determinant is not zero.

$$\Delta_w = \begin{vmatrix} 1 & b_0(R) & 0 \\ 1 & -a_0 & A_{11} \frac{t_{C_{11}}}{h} \\ 1 & a_0 & -A_{12} \end{vmatrix}$$

$$\Delta_w = (a_0 + b_0(R))\, A_{12} - (a_0 - b_0(R))\, A_{11} \frac{t_{C_{11}}}{h} \tag{3.67}$$

where

$$A_{11} = a_0 - \frac{a_0}{r_{C_{11}}}(t_{B_1} - t_{C_1})$$

$$A_{12} = a_0 + \frac{a_0}{r_{11}}(t_{B_1} - t_{C_1})$$

Since $0 < t_{B_1} - t_{C_1} < 1$, $-1 < t_{B_1} - t_{C_1} < 0$, $r_{C_{11}} = R$, $r_{C_1} = R$ then A_{11} and A_{12} are bounded by 1 ($0 < A_{11} < a_0$, $0 < A_{12} < a_0$). Also $a_0 + b_0(R)$ and $a_0 - b_0(R)$ are bounded ($a_0 < a_0 + b_0(R) < 2a_0$, $0 < a_0 - b_0(R) < a_0$). All these imply that $\Delta_w > 0$. So, we determine the unknowns $w_t(B_1)$, $w_r(B_1)$ and $w_r(C_1)$ uniquely.

The values of x and w at the point B_1 can be easily evaluated by using values of their partial derivatives at that point, namely we put

$$x(B_1) = x_t(B_1)t_{B_1} + x_r(B_1)(r_{B_1} - R) + x(R) \tag{3.68}$$

$$w(B_1) = w_t(B_1)t_{B_1} + w_r(B_1)(r_{B_1} - R) + w(R) \tag{3.69}$$

By using these values, the unknown functions ε_r, ε_θ, σ_r, σ_θ, ε_{r1}, $\varepsilon_{\theta 1}$, σ_{r1}, $\sigma_{\theta 1}$ as well as b_0 are easily evaluated from equations (3.2), (3.3), (3.6), (3.7) and (3.48) at the point B_1.

At the second step we draw another characteristic line with the slope $-a_0$ which is parallel to C_1A_1 and intersects $r-$axis at the point $R + 2h$. Its $t-$intersection is labelled by C_2 and its intersection with radial wavefront is labelled by A_2. From the point C_2 , we draw a characteristic line with slope a_0. The unknown curve RB is constructed with the slope $b_0(B_1)$. Intersection of RB with C_2A_2 is labelled by B_2 (see Fig. 3.2). There are three possibilities for the position of B_2:

The first possibility of the location of B_2 is $r_{21} < r_{B_2} < r_{A_2}$;

The equation along B_2A_2 can be written from equation (3.24)

$$x_t(B_2) - x_t(A_2) = -a_0(x_r(B_2) - x_r(A_2)) + a_1(A_2)(t_{B_2} - t_{A_2}) \tag{3.70}$$

and the second equation at B_2 can be written by using (3.30) as

$$x_t(B_2) + b_0(B_1)x_r(B_2) = v_0 \tan\alpha \tag{3.71}$$

Since we have two equations in two unknowns , $x_t(B_2)$, and $x_r(B_2)$, we easily get the unknown values of $x_t(B_2)$ and $x_r(B_2)$.

$$\Delta_x = \begin{vmatrix} 1 & a_0 \\ 1 & b_0(B_1) \end{vmatrix}$$

$$\Delta_x = b_0(B_1) - a_0 < 0 \tag{3.72}$$

By using equations (3.20) and (3.42), we write the equations along B_2P_{21} , $P_{21}C_1$, $P_{21}C_2$ and at B_2 as

$$w_t(B_2) - w_t(2,1) = -a_0(w_r(B_2) - w_r(2,1)) + a(2,1)(t_{B_2} - t_{21}) \tag{3.73}$$

where

$$a(2,1) = a_0^2\left(\frac{w_r(2,1)}{r_{21}} - \frac{w(2,1)}{r_{21}^2}\sin^2\alpha\right)$$
$$w(2,1) = w_t(2,1)(t_{21} - t_{C_1}) + w_r(2,1)(r_{21} - r_{C_1}) + w(C_1)$$

.

$$w_t(2,1) - w_t(C_1) = a_0(w_r(2,1) - w_r(C_1)) + a(C_1)(t_{21} - t_{C_1}) \tag{3.74}$$

$$w_t(2,1) - w_t(C_2) = -a_0(w_r(2,1) - w_r(C_2)) + a(C_2)(t_{21} - t_{C_2}) \tag{3.75}$$

where

$$a(C_2) = a_0^2 \left(\frac{w_r(C_2)}{r_{C_2}} - \frac{w(C_2)}{r_{C_2}^2} \sin^2 \alpha \right)$$

$$w_t(C_2) = 0, \quad w(C_2) = w(R)$$

$$w_t(B_2) + b_0(B_1) w_r(B_2) = \frac{v_0}{\cos \alpha} - \frac{1+\nu}{1+\sin \alpha} b_0(B_1) \tag{3.76}$$

$$w_t(B_2) - w_t(C_{22}) = a_0(w_r(B_2) - w_r(C_{22})) + a(C_{22})(t_{B_2} - t_{C_{22}}) \tag{3.77}$$

where

$$a(C_{22}) = a_0^2 \left(\frac{w_r(C_{22})}{r_{C_{22}}} - \frac{w(C_{22})}{r_{C_{22}}^2} \sin^2 \alpha \right)$$

$$w_r(C_{22}) = w_r(B_1)c_1 + w_r(C_1)c_2$$

$$w_t(C_{22}) = w_t(B_1)c_1 + w_t(C_1)c_2$$

$$w(C_{22}) = w(R)$$

$$c_1 = \left(\frac{r_{C_{22}} - r_{C_1}}{r_{B_1} - r_{C_1}} \right) \left(\frac{t_{C_{22}} - t_{C_1}}{t_{B_1} - t_{C_1}} \right) + \frac{1}{2} cs$$

$$c_2 = \left(\frac{r_{C_{22}} - r_{B_1}}{r_{C_1} - r_{B_1}} \right) \left(\frac{t_{C_{22}} - t_{B_1}}{t_{C_1} - t_{B_1}} \right) + \frac{1}{2} cs$$

$$cs = \left(\frac{r_{C_{22}} - r_{C_1}}{r_{B_1} - r_{C_1}} \right) \left(\frac{t_{C_{22}} - t_{B_1}}{t_{C_1} - t_{B_1}} \right) + \left(\frac{r_{C_{22}} - r_{B_1}}{r_{C_1} - r_{B_1}} \right) \left(\frac{t_{C_{22}} - t_{C_1}}{t_{B_1} - t_{C_1}} \right)$$

We have five equations in five unknowns namely, $w_t(B_2)$, $w_r(B_2)$, $w_t(2,1)$, $w_r(2,1)$, $w_r(C_2)$. The principal determinant is not zero. So the unknown values are uniquely determined. Indeed

$$\Delta_w = \begin{vmatrix} 1 & b_0(B_1) & 0 & 0 & 0 \\ 1 & -a_0 & 0 & 0 & 0 \\ 1 & a_0 & -A_{21} & -A_{22} & 0 \\ 0 & 0 & 1 & -a_0 & 0 \\ 0 & 0 & 1 & a_0 & -A_{23} \end{vmatrix}$$

$$\Delta_w = (a_0 + b_0(B_1))(A_{21}A_{23} + A_{22}A_{23}) \tag{3.78}$$

where

$$A_{21} = a_0 - \frac{a_0^2}{r_{21}^2}(t_{21} - t_{C_1})(t_{B_2} - t_{21})\sin^2\alpha$$

$$A_{22} = a_0 + \frac{a_0}{r_{21}}(t_{B_2} - t_{21}) - \frac{a_0^2}{r_{21}^2}(r_{21} - r_{C_1})(t_{B_2} - t_{21})\sin^2\alpha$$

$$A_{23} = a_0 + \frac{a_0}{r_{C_2}}(t_{21} - t_{C_2})$$

Since $0 < t_{21} - t_{C_1} < 1$, $-1 < t_{B_2} - t_{21} < 0$, $0 \leq \sin^2\alpha \leq 1$, $0 < r_{21} - r_{C_1} < 1$, $-1 < t_{21} - t_{C_2} < 0$ then A_{21}, A_{22} and A_{23} are bounded ($a_0 < A_{21} < 2a_0$, $0 < A_{22} < a_0$, $0 < A_{23} < a_0$). These mean that $A_{21}A_{23} + A_{22}A_{23} > 0$. Also $a_0 < a_0 + b_0(B_1) < 2a_0$. So the principle determinant $\Delta_w > 0$.

The second possible location of B_2 is $r_{B_2} = r_{21}$;

The first equation along B_2A_2 gives

$$x_t(B_2) - x_t(A_2) = -a_0(x_r(B_2) - x_r(A_2)) + a_1(A_2)(t_{B_2} - t_{A_2}) \tag{3.79}$$

The second equation is written at point B_2 as

$$x_t(B_2) + b_0(B_1)x_r(B_2) = v_0 \tan\alpha \tag{3.80}$$

The equations on the left of the transverse wavefront (along B_2C_1 , B_2C_2 and at B_2) have the following difference form

$$w_t(B_2) - w_t(C_1) = a_0(w_r(B_2) - w_r(C_1)) + a(C_1)(t_{B_2} - t_{C_1}) \tag{3.81}$$

$$w_t(B_2) - w_t(C_2) = -a_0(w_r(B_2) - w_r(C_2)) + a(C_2)(t_{B_2} - t_{C_2}) \tag{3.82}$$

$$w_t(B_2) + b_0(B_1)w_r(B_2) = \frac{v_0}{\cos\alpha} - \frac{1+\nu}{1+\sin\alpha}b_0(B_1) \tag{3.83}$$

The equations (3.79)−(3.83) constitute a closed set of five algebraic equations with respect to five unknowns $x_t(B_2)$, $x_r(B_2)$, $w_t(B_2)$, $w_r(B_2)$ and $w_r(C_2)$. The principle determinants are not zero.

$$\Delta_x = \begin{vmatrix} 1 & a_0 \\ 1 & b_0(B_1) \end{vmatrix}$$

$$\Delta_x = b_0(B_1) - a_0 < 0 \tag{3.84}$$

$$\Delta_w = \begin{vmatrix} 1 & b_0(B_1) & 0 \\ 1 & -a_0 & 0 \\ 1 & a_0 & -A_{21} \end{vmatrix}$$

$$\Delta_w = (a_0 + b_0(B_1))A_{21} \tag{3.85}$$

where

$$A_{21} = a_0 + \frac{a_0}{r_{C_2}}(t_{B_2} - t_{C_2})$$

Since $-1 < t_{B_2} - t_{C_2} < 0$ and $0 < A_{21} < a_0$. Then the determinant of coefficients $\Delta_w > 0$.

The third possibility for the location of B_2 is $r_{C_2} < r_{B_2} < r_{21}$; The equations along B_2P_{21} , $P_{21}B_{12}$, $P_{21}A_2$, B_2C_2 , B_2C_{22} and at point B_2 are written as

$$x_t(B_2) - x_t(2,1) = -a_0(x_r(B_2) - x_r(2,1)) + a_1(2,1)(t_{B_2} - t_{21}) \tag{3.86}$$

where

$$a_1(2,1) = a_0^2\left(\frac{x_r(2,1)}{r_{21}} - \frac{x(2,1)}{r_{21}^2}\right)$$

$$x(2,1) = x_t(2,1)(t_{21} - t_{B_{12}}) + x_r(2,1)(r_{21} - r_{B_{12}}) + x(B_{12})$$

$$x(B_{12}) = x(B_1)a_1 + x(B_2)a_2$$

$$x(B_2) = x_t(B_2)(t_{B_2} - t_{B_1}) + x_r(r_{B_2} - r_{B_1}) + x(B_1)$$

$$x_t(2,1) - x_t(B_{12}) = a_0(x_r(2,1) - x_r(B_{12})) + a_1(B_{12})(t_{21} - t_{B_{12}}) \quad (3.87)$$

where

$$x_t(B_{12}) = x_t(B_1)a_1 + x_t(B_2)a_2$$

$$x_r(B_{12}) = x_r(B_1)a_1 + x_r(B_2)a_2$$

$$a_1 = \left(\frac{r_{B_{12}} - r_{B_2}}{r_{B_1} - r_{B_2}}\right)\left(\frac{t_{B_{12}} - t_{B_2}}{t_{B_1} - t_{B_2}}\right) + \frac{1}{2}\left(\frac{r_{B_{12}} - r_{B_2}}{r_{B_1} - r_{B_2}}\right)\left(\frac{t_{B_{12}} - t_{B_1}}{t_{B_2} - t_{B_1}}\right)$$

$$a_2 = \left(\frac{r_{B_{12}} - r_{B_1}}{r_{B_2} - r_{B_1}}\right)\left(\frac{t_{B_{12}} - t_{B_1}}{t_{B_2} - t_{B_1}}\right) + \frac{1}{2}\left(\frac{r_{B_{12}} - r_{B_1}}{r_{B_2} - r_{B_1}}\right)\left(\frac{t_{B_{12}} - t_{B_2}}{t_{B_1} - t_{B_2}}\right)$$

$$x_t(2,1) - x_t(A_2) = -a_0(x_r(2,1) - x_r(A_2)) + a_1(A_2)(t_{21} - t_{A_2}) \quad (3.88)$$

$$x_t(B_2) + b_0(B_1)x_r(B_2) = v_0 \tan\alpha \quad (3.89)$$

There are four equations and four unknowns, namely $x_t(B_2)$, $x_r(B_2)$, $x_t(2,1)$, $x_r(2,1)$. Since determinant of the coefficient matrix is not zero, these unknown values are obtained uniquely.

$$\Delta_x = \begin{vmatrix} 1 & b_0(B_1) & 0 & 0 \\ B_{21} & B_{22} & -B_{23} & -B_{24} \\ B_{25} & B_{26} & 1 & -a_0 \\ 0 & 0 & 1 & a_0 \end{vmatrix}$$

$$\Delta_x = 2(B_{22} - b_0(B_1)B_{21}) + (B_{23} - B_{24})(B_{26} - b_0(B_1)B_{25}) \quad (3.90)$$

where

$$B_{21} = 1 + \frac{a_0^2}{r_{21}^2}(t_{B_2} - t_{B_1})(t_{B_2} - t_{21})a_2$$

$$B_{22} = a_0 + \frac{a_0^2}{r_{21}^2}(r_{B_2} - r_{B_1})(t_{B_2} - t_{21})a_2$$

$$B_{23} = 1 - \frac{a_0^2}{r_{21}^2}(t_{21} - t_{B_{12}})(t_{B_2} - t_{21})$$

$$B_{24} = a_0 + \frac{a_0}{r_{21}}(t_{B_2} - t_{21}) - \frac{a_0^2}{r_{21}^2}(r_{21} - r_{B_{12}})(t_{B_2} - t_{21})$$

$$B_{25} = -a_2 + a_2\frac{a_0^2}{r_{B_{12}}}(t_{B_2} - t_{B_1})(t_{21} - t_{B_{12}})$$

$$B_{26} = a_2a_0 - a_2\frac{a_0}{r_{B_{12}}}(t_{21} - t_{B_{12}}) + a_2\frac{a_0^2}{r_{B_{12}}^2}(r_{B_2} - r_{B_1})(t_{21} - t_{B_{12}})$$

Since $0 < \frac{r_{B_{12}} - r_{B_1}}{r_{B_2} - r_{B_1}} < 1$, $0 < \frac{t_{B_{12}} - t_{B_1}}{t_{B_2} - t_{B_1}} < 1$, $0 < \frac{t_{B_{12}} - t_{B_2}}{t_{B_1} - t_{B_2}} < 1$, $0 < r_{B_2} - r_{B_1} < 1$, $0 < t_{B_2} - t_{B_1} < 1$, $0 < t_{B_2} - t_{21} < 1$, $0 < t_{21} - t_{B_{12}} < 1$, $0 < r_{21} - r_{B_{12}} < 1$, $r_{21} > R$, $r_{B_{12}} > R$ then $B_{22} - b_0B_{21} > 0$, $B_{23} - B_{24} < 0$ and $B_{26} - b_0(B_1)B_{25} < 0$. So $\Delta_w > 0$.

Since $x_t(2,1)$ and $x_r(2,1)$ are evaluated and all nodal values of x_t and x_r are calculated on $j = 1$ recursively, as

$$x_t(k,1) - x_t(k-1,1) = a_0\left(x_r(k,1) - x_r(k-1,1)\right) + a_1(k-1,1)(t_{k,1} - t_{k-1,1}) \tag{3.91}$$

$$x_t(k,1) - x_t(k,0) = -a_0\left(x_r(k,1) - x_r(k,0)\right) + a_1(k,0)(t_{k,1} - t_{k,0}) \tag{3.92}$$

for $k = 3, 4, ..., n$. Besides

$$x(k,1) = x_t(k,1)(t_{k,1} - t_{k-1,1}) + x_r(r_{k,1} - r_{k-1,1}) + x(k-1,1)$$

Since determinant of the coefficients is not zero, the unknowns $x_t(k,1)$ and $x_r(k,1)$ can be determined uniquely.

$$\Delta_x = \begin{vmatrix} 1 & -a_0 \\ 1 & a_0 \end{vmatrix} = 2a_0 > 0$$

The difference equations along B_2C_2, B_2C_{22} and at B_2

$$w_t(B_2) - w_t(C_2) = -a_0(w_r(B_2) - w_r(C_2)) + a(C_2)(t_{B_2} - t_{C_2}) \tag{3.93}$$

$$w_t(B_2) - w_t(C_{22}) = a_0(w_r(B_2) - w_r(C_{22})) + a(C_{22})(t_{B_2} - t_{22}) \tag{3.94}$$

where

$$a(C_{22}) = a_0^2\left(\frac{w_r(C_{22})}{r_{C_{22}}} - \frac{w(C_{22})}{r_{C_{22}}^2}\sin^2\alpha\right)$$

$w_t(C_{22}) = 0$, $w(C_{22}) = w(R)$.

$$w_t(B_2) + b_0(B_1)w_r(B_2) = \frac{v_0}{\cos\alpha} - \frac{1+\nu}{1+\sin\alpha}b_0(B_1) \tag{3.95}$$

The equations (3.93) – (3.95) constitute a closed set of three algebraic equations with respect to three unknowns, $w_t(B_2)$, $w_r(B_2)$ and $w_r(C_2)$. The principle determinant is not zero.

$$\Delta_w = \begin{vmatrix} 1 & b_0(B_1) & 0 \\ 1 & -a_0 & A_{21}\left(\frac{t_{C_{22}}}{h} - 1\right) \\ 1 & a_0 & -A_{22} \end{vmatrix}$$

$$\Delta_w = (a_0 + b_0(B_1))A_{22} - (a_0 - b_0(B_1))A_{21}\left(\frac{t_{C_{22}}}{h} - 1\right) \tag{3.96}$$

where

$$A_{21} = a_0 - \frac{a_0}{r_{C_2}}(t_{B_2} - t_{C_{22}})$$

$$A_{22} = a_0 + \frac{a_0}{r_{22}}(t_{B_2} - t_{C_2})$$

Since $0 < t_{B_2} - t_{C_{22}} < 1$, $-1 < t_{B_2} - t_{C_2} < 0$, $r_{C_2} > R$, $r_{C_{22}} = R$, $-1 < \frac{t_{C_{22}}}{h} - 1 < 0$, then $0 < A_{21} < a_0$ and $0 < A_{22} < a_0$. So $\Delta_w > 0$. Hence we obtain the unique solution.

On continuing this algorithm we can get the numerical solution to the problem at any point of regions I and II as well as on the strong discontinuity wave RB. Indeed suppose that the solution to the problem in regions enclosed into the triangle RC_nA_n including the line C_nA_n is known (see, Fig.3.3).

From B_n we draw an element B_nB_{n+1} of the wave RB with the slope $b_0(B_n)$ so that the point B_{n+1} be on the characteristics $A_{n+1}C_{n+1}$, which have a slope $-a_0$ and the distance A_nA_{n+1} is chosen to be equal to the required step of numerical calculation along RA. The solution to the problem in region II is found easily at any point of the characteristic net by knowing the values x, x_t and x_r on the neighboring points just behind and below that point. First we write the characteristic conditions at $A_{n+1,1}$

$$x_t(A_{n+1,1}) - x_t(A_{n,1}) = a_0(x_r(A_{n+1,1}) - x_r(A_{n,1})) + a_1(A_{n,1})(t_{A_{n+1,1}} - t_{A_{n,1}}) \tag{3.97}$$

$$x_t(A_{n+1,1}) - x_t(A_{n+1}) = -a_0(x_r(A_{n+1,1}) - x_r(A_{n+1})) + a_1(A_{n+1})(t_{A_{n+1,1}} - t_{A_{n+1}}) \tag{3.98}$$

Besides we have

$$x(A_{n+1,1}) = x(A_{n,1}) + x_t(A_{n+1,1})(t_{A_{n+1,1}} - t_{A_{n,1}}) + x_r(A_{n+1,1})(r_{A_{n+1,1}} - r_{A_{n,1}}) \tag{3.99}$$

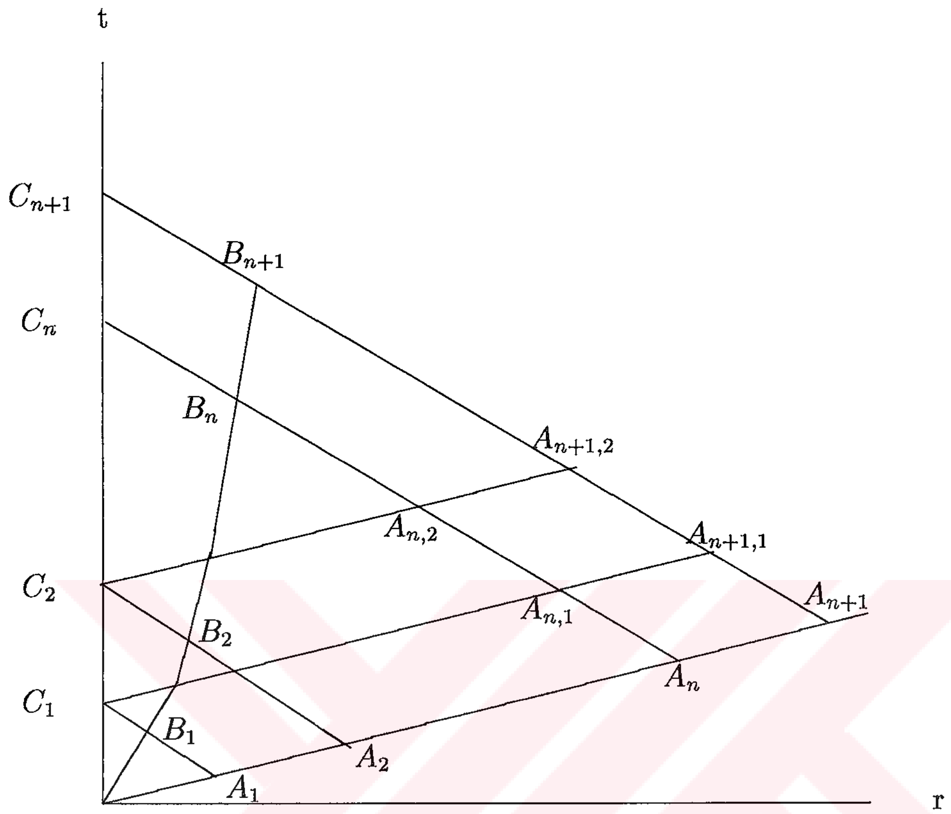

Figure 3.3: Construction of scheme in general

In (3.97) – (3.99) the values of all functions at A_{n+1} is known from the boundary conditions (3.49) – (3.50). Hence, from (3.97) – (3.99) we easily define $x_t(A_{n+1,1})$, $x_r(A_{n+1,1})$ and $x(A_{n+1,1})$ as solution of three consistent linear algebraic equations.

Then upon writing characteristic relations along $A_{n,2}A_{n+1,2}$ and $A_{n+1,1}A_{n+1,2}$ as well as finite differential form of the total differential of $x(r,t)$ at $A_{n+1,2}$ analogous to (3.99) we again get the set of three linear algebraic equations with respect to the unknowns $x_t(A_{n+1,2})$, $x_r(A_{n+1,2})$ and $x(A_{n+1,2})$ and solving these equations we get their values. By this way at each interior point of the region II we get a system of three equations with three unknowns and hence get the solution to the problem in region II.

On the boundary point B_{n+1} there are two different sets of equations depending on the position of $B_n B_{n+1}$. In the first case we have only one relation from the region II, namely the characteristic relation (3.24) along $A_{n+1,k}B_{n+1}$, since the other characteristics with the slope a_0 lies in region I.

The characteristic relation (3.24) along the line $A_{n+1,k}B_{n+1}$ gives

$$x_t(B_{n+1}) - x_t(A_{n+1,k}) = -a_0\left(x_r(B_{n+1}) - x_r(A_{n+1,k})\right) + a_1(A_{n+1,k})(t_{B_{n+1}} - t_{A_{n+1,k}}) \qquad (3.100)$$

At B_{n+1} we have the boundary condition (3.46)

$$x_t(B_{n+1}) = v_0 \tan\alpha - b_0(B_n)x_r(B_{n+1}) \qquad (3.101)$$

Besides

$$x(B_{n+1}) = x(B_n) + x_t(B_{n+1})(t_{B_{n+1}} - t_{B_n}) + x_r(B_{n+1})(r_{B_{n+1}} - r_{B_n}) \qquad (3.102)$$

Since the values of all fuctions at $A_{n+1,k}$ and B_n are known, from (3.100) – (3.102) we easily determine the values $x_t(B_{n+1}), x_r(B_{n+1})$ and $x(B_{n+1})$. Actually the value of $x(B_{n+1})$ is determined. The same is true with respect to the interior points where we have equations analogous to (3.97) – (3.99).

In the second case, we have two relations from the region II, namely condition along $A_{n+1,k}B_{n+1}$ and $A_{n+1,k}$ $A_{n+1,k-1}$. The characteristic relation (3.24) along the line $A_{n+1,k}B_{n+1}$ and $A_{n+1,k}A_{n+1,k-1}$ gives

$$x_t(B_{n+1}) - x_t(A_{n+1,k}) = -a_0\left(x_r(B_{n+1}) - x_r(A_{n+1,k})\right) + a_1(A_{n+1,k})(t_{B_{n+1}} - t_{A_{n+1,k}}) \qquad (3.103)$$

$$x_t(A_{n+1,k}) - x_t(A_{n+1,k-1}) = a_0\left(x_r(A_{n+1,k}) - x_r(A_{n+1,k-1})\right) + a_1(A_{n+1,k-1})(t_{A_{n+1,k}} - t_{A_{n+1,k-1}}) \quad (3.104)$$

At B_{n+1} we have the boundary condition (3.46)

$$x_t(B_{n+1}) = v_0 \tan\alpha - b_0(B_n)x_r(B_{n+1}) \quad (3.105)$$

We have three equations in four unknowns. To obtain unique solution we need another one. This point is chosen as the intersection of B_nB_{n+1} with kth. characteristic having the slope a_0, labelled by $B_{n,n+1}$. Then the characteristic relation (3.24) along $A_{n+1,k}B_{n,n+1}$ is written as

$$x_t(A_{n+1,k}) - x_t(B_{n,n+1}) = a_0\left(x_r(A_{n+1,k}) - x_r(B_{n,n+1})\right) + a_1(B_{n,n+1})(t_{A_{n+1,k}} - t_{B_{n,n+1}}) \quad (3.106)$$

where

$$x_t(B_{n,n+1}) = x_t(B_n)a_1 + x_t(B_{n+1})a_2$$

$$x_r(B_{n,n+1}) = x_r(B_n)a_1 + x_r(B_{n+1})a_2$$

$$x(B_{n,n+1}) = x(B_n)a_1 + x(B_{n+1})a_2$$

$$a_1 = \left(\frac{r_{B_{n,n+1}} - r_{B_n}}{r_{B_{n+1}} - r_{B_n}}\right)\left(\frac{t_{B_{n,n+1}} - t_{B_n}}{t_{B_{n+1}} - t_{B_n}}\right) + \frac{1}{2}\left(\frac{r_{B_{n,n+1}} - r_{B_{n+1}}}{r_{B_n} - r_{B_{n+1}}}\right)\left(\frac{t_{B_{n,n+1}} - t_{B_n}}{t_{B_{n+1}} - t_{B_n}}\right)$$

$$a_2 = \left(\frac{r_{B_{n,n+1}} - r_{B_{n+1}}}{r_{B_n} - r_{B_{n+1}}}\right)\left(\frac{t_{B_{n,n+1}} - t_{B_{n+1}}}{t_{B_n} - t_{B_{n+1}}}\right) + \frac{1}{2}\left(\frac{r_{B_{n,n+1}} - r_{B_{n+1}}}{r_{B_n} - r_{B_{n+1}}}\right)\left(\frac{t_{B_{n,n+1}} - t_{B_n}}{t_{B_{n+1}} - t_{B_n}}\right)$$

Now we have four equations with four unknowns. The values of all functions at B_n and $A_{n+1,k-1}$ are known, so we easily determine the values of $x_t(B_{n+1})$, $x_r(B_{n+1})$, $x_t(A_{n+1,k})$ and $x_r(A_{n+1,k})$.

The principle determinant of the systems in interior points is

$$\Delta_{x_i} = \begin{vmatrix} 1 & -a_0 \\ 1 & a_0 \end{vmatrix} = 2a_0 \neq 0$$

while the principle determinant of (3.92) – (3.94) is

$$\Delta_x = \begin{vmatrix} 1 & a_0 \\ 1 & b_0(B_n) \end{vmatrix} = b_0(B_n) - a_0 < 0$$

On the other hand the principle determinant of (3.103) – (3.106) is

$$\Delta_x = \begin{vmatrix} 1 & b_0(B_n) & 0 & 0 \\ B_{21} & B_{22} & -B_{23} & -B_{24} \\ B_{25} & B_{26} & 1 & -a_0 \\ 0 & 0 & 1 & a_0 \end{vmatrix}$$

$$\Delta_x = 2\left(B_{22} - b_0(B_n)B_{21}\right) + \left(B_{23} - B_{24}\right)\left(B_{26} - b_0(B_n)B_{25}\right)$$

where

$$B_{21} = 1 + \frac{a_0^2}{r_{n+1,n}^2} a_2(t_{B_{n+1}} - t_{B_n})(t_{B_{n+1}} - t_{n+1,n})$$

$$B_{22} = a_0 + \frac{a_0^2}{r_{n+1,n}^2}(r_{B_{n+1}} - r_{B_n})(t_{B_{n+1}} - t_{n+1,n})a_2$$

$$B_{23} = 1 - \frac{a_0^2}{r_{n+1,n}^2} a_2(t_{n+1,n} - t_{B_{n,n+1}})(t_{B_{n+1}} - t_{n+1,n})$$

$$B_{24} = a_0 + \frac{a_0}{r_{n+1,n}}(t_{B_{n+1}} - t_{n+1,n}) - \frac{a_0^2}{r_{n+1,n}^2}(r_{n+1,n} - r_{B_{n,n+1}})(t_{B_{n+1}} - t_{n+1,n})$$

$$B_{25} = -a_2 + a_2\frac{a_0}{r_{B_{n,n+1}}^2}(t_{B_{n+1}} - t_{B_n})(t_{n+1,n} - t_{B_{n,n+1}})$$

$$B_{26} = -a_2 - a_2\frac{a_0}{r_{B_{n,n+1}}}(t_{n+1,n} - t_{B_{n,n+1}}) + a_2\frac{a_0^2}{r_{B_{n,n+1}}^2}(r_{B_{n+1}} - r_{B_n})(t_{n+1,n} - t_{B_{n,n+1}})$$

Since $0 < a_2 < 1, 0 < t_{B_{n+1}} - t_{n+1,n} < 1, 0 < t_{n+1,n} - t_{B_{n,n+1}} < 1, 0 < r_{n+1,n} - r_{B_{n,n+1}} < 1, r_{n+1,n} > R, r_{B_{n,n+1}} > R$ then $B_{22} - b_0(B_n)B_{21} > 0$,

$B_{23} - B_{24} < 0$ and $B_{26} - b_0(B_n)B_{25} < 0$. Hence all these linear algebraic equations possess unique solutions.

Passing to the region I we note that at each interior point of the line $C_{n+1}B_{n+1}$ we have two characteristic relations which yield two linear algebraic equations with respect to two new unknowns $w_t(C_{n+1,i})$ and $w_r(C_{n+1,i})$

$$w_t(C_{n+1,i+1}) - w_t(C_{n+1,i}) = -a_0\left(w_r(C_{n+1,i+1}) - w_r(C_{n+1,i})\right) + a(C_{n+1,i})(t_{C_{n+1,i+1}} - t_{C_{n+1,i}}) \quad (3.107)$$

$$w_t(C_{n+1,i+1}) - w_t(C_{n,i}) = a_0\left(w_r(C_{n+1,i+1}) - w_r(C_{n+1,i})\right) + a(C_{n,i})(t_{C_{n+1,i+1}} - t_{C_{n,i}}) \quad (3.108)$$

Besides

$$w(C_{n+1,i+1}) = w(C_{n,i+1}) + w_t(C_{n+1,i+1})(t_{C_{n+1,i+1}} - t_{C_{n,i+1}}) + w_r(C_{n+1,i+1})(r_{C_{n+1,i+1}} - r_{C_{n,i+1}}) \quad (3.109)$$

At the boundary point B_{n+1} we have three conditions: The same relations (3.107) – (3.109) for $C_{n+1,m}$ (m is the number of interior points of the net in region I) plus the boundary condition (3.47)

$$w_t(B_{n+1}) = \frac{v_0}{\cos\alpha} - b_0(B_n)\left(w_r(B_{n+1}) + \frac{1+\nu}{1+\sin\alpha}\right) \quad (3.110)$$

Besides at the point C_{n+1} on the Rt line we have the boundary condition (3.45)

$$\frac{\partial w}{\partial t}(C_{n+1}) = 0 \quad (3.111)$$

where $w(C_{n+1})$ can be computed as

$$w(C_{n+1}) = \frac{R}{\sin\alpha} - \frac{1+\nu}{1+\sin\alpha}R \qquad (3.112)$$

Thus, in region I at the $(n+1)$st. step we get a system of $2m+3$ linear algebraic equations (3.107), (3.108), and (3.110) with respect to $2m+3$ number of unknowns $w_t(C_{n+1})$, $w_r(C_{n+1})$, $w_t(C_{n+1,i})$, $w_r(C_{n+1,i})$ $(i = 1, 2, ..., m)$, $w_t(B_{n+1})$ and $w_r(B_{n+1})$.

The principle determinant as follows from the mentioned equations has a form (written in the order (3.110), (3.107) and (3.108)).

$$\Delta_w = \begin{vmatrix} 1 & b_0(B_n) & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 \\ 1 & -a_0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 \\ 1 & a_0 & -A_{n+1,1} & -A_{n+1,2} & 0 & 0 & . & . & . & 0 & 0 & 0 \\ 0 & 0 & 1 & -a_0 & 0 & 0 & . & . & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 1 & -a_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 1 & a_0 & -A_{n+1,2n+1} \end{vmatrix}$$

$$\begin{aligned} \Delta_w = (a_0 + b_0(B_n))\,(&A_{n+1,1}A_{n+1,3}A_{n+1,5}\cdots \\ &A_{n+1,2n-1}A_{n+1,2n+1} + A_{n+1,1}A_{n+1,3}A_{n+1,5}\cdots \\ &A_{n+1,2n-1}A_{n+1,2n}A_{n+1,2n+1} + \dots + A_{n+1,1}A_{n+1,4}A_{n+1,6}\cdots \\ &A_{n+1,2n}A_{n+1,2n+1} + \dots + A_{n+1,2}A_{n+1,3}A_{n+1,5}\cdots \\ &A_{n+1,2n-1}A_{n+1,2n+1} + \dots + A_{n+1,2}A_{n+1,4}A_{n+1,6}\cdots \\ &A_{n+1,2n}A_{n+1,2n+1}) \end{aligned} \qquad (3.113)$$

where

$$A_{n+1,1} = a_0 - \frac{a_0^2}{r_{n+1,1}^2}(t_{n+1,1} - t_{n,1})(t_{B_{n+1}} - t_{n+1,1})\sin^2\alpha$$

$$A_{n+1,2} = a_0 + \frac{a_0}{r_{n+1,1}}(t_{B_{n+1}} - t_{n+1,1}) - \frac{a_0^2}{r_{n+1,1}^2}(r_{n+1,1} - r_{n,1})(t_{B_{n+1}} - t_{n+1,1})\sin^2\alpha$$

for $l = 2, 3, ..., n$

$$A_{n+1,2l-1} = a_0 - \frac{a_0^2}{r_{n+1,l}^2}(t_{n+1,l} - t_{n,l})(t_{n+1,l-1} - t_{n+1,l})\sin^2\alpha$$

$$A_{n+1,2l} = a_0 + \frac{a_0}{r_{n+1,l}}(t_{n+1,l-1} - t_{n+1,l}) - \frac{a_0^2}{r_{n+1,l}^2}(r_{n+1,l} - r_{n,l})(t_{n+1,l-1} - t_{n+1,l})\sin^2\alpha$$

$$A_{n+1,2n+1} = a_0 + \frac{a_0}{r_{C_{n+1}}}(t_{n+1,n} - t_{C_{n+1}})$$

Since $0 < t_{n+1,1} - t_{n,1} < 1$, $-1 < t_{B_{n+1}} - t_{n+1,1} < 0$, $R \leq r_{n+1,j}$ $(j = 1, 2, ...n + 1)$, $0 < r_{n+1,j} - r_{n,j} < 1$ $(j = 1, 2, ..., n + 1)$, $0 \leq sin^2\alpha \leq 1$, $0 < t_{n+1,l} - t_{n,l} < 1$ $(l = 2, 3, ..., n)$, $-1 < t_{n+1,l-1} - t_{n+1,l} < 0$ $(l = 2, 3, ..., n)$, $0 < r_{n+1,l} - r_{n,l} < 1$ $(l = 2, 3, ..., n)$, and $-1 < t_{n+1,n} - t_{C_{n+1}} < 0$ then $a_0 < A_{n+1,1} < 2a_0$, $0 < A_{n+1,2} < a_0$, for $l = 2, 3, ..., n$ $a_0 < A_{n+1,2l-1} < 2a_0$, $0 < A_{n+1,2l} < a_0$ and $0 < A_{n+1,2n+1} < a_0$. So the principle determinant $\Delta_w > 0$.

Hence the mentioned system possesses a unique solution.

Thus, according to the method of mathematical induction the algorithm gives the unique solution to the problem in the both regions I and II.

No restrictive conditions to ensure stability and convergence are to be expected for this method since it is based on a network which approximates that of characteristic curves.

3.5 Convergence of the Numerical Algorithm

Let us study the convergence of the suggested algorithm. In the continuum mechanics the displacement vector and hence x and w are continuous everywhere. But their partial derivatives may have jump discontinuities across the characteristics of the equations since it is well known that discontinuities in any quantity described by the hyperbolic type equations propagate along the characteristic lines. These discontinuity lines degenerate into the wavefronts-shocks. In our problem the both wavefronts are strong discontinuity waves. On the radial wavefront where $x = R + a_0 t$, $\dfrac{\partial x}{\partial r}$ and $\dfrac{\partial x}{\partial t}$ have jump discontinuity, while across the transverse wavefront ε_r, σ_r, σ_θ, $\dfrac{\partial w}{\partial r}$, $\dfrac{\partial w}{\partial t}$ as well as γ undergo jump discontinuities. Hence in the suggested algorithm the wavefronts are considered as boundaries possessing two shores. Hence by the closed regions we mean the regions plus the corresponding shore of the wavefronts.

On RA, $|x_r| < 1 + C_0$, $|x_t| < a_0 C_0$, $\forall t$ from the equations (3.49) and (3.50). On Rt-line w is constant and $w_t \equiv 0$ (see, equations (3.43), (3.45)). Immediately after impact all unknown functions are constants (see, (3.57), (3.58) and (3.60)) and hence absolutely bounded.

Let D be a closed domain defined as

$$D : [R, r_{\max}] \times [0, t_{\max}]$$

For any finite $r_{\max}$, $t_{\max}$ D consists of two compacts (in the above sense) and hence is the compact. In the problem solution D is divided into two subregions, region I and region II with transverse wavefront. The unknown functions w_t, w_r, x_t and x_r and their first derivatives are continuous in region I and in region II respectively. But they have jump discontinuities across the transverse wavefront. So they are uniformly bounded in each region. Besides the discontinuities are of first order. In other words, all functions occuring in the problem are uniformly bounded.

The determinant of the coefficients of the equations (3.20) and (3.24) is

$$\Delta = \begin{vmatrix} 1 & -a_0 \\ 1 & a_0 \end{vmatrix} = 2a_0 > a_0 > 0$$

Since unknown functions w_t, w_r, x_t, x_r and their first derivatives are uniformly bounded in the open neighborhood of $(R,0)$ so the difference forms are uniformly bounded in this neighborhood. These bounds don't depend on h but depend only on the measure of the domain. Then the differences of unknown functions form uniformly bounded sequences. By using the fact that every uniformly bounded sequences have some uniformly convergent subsequence in a compact domain, and introducing the subsequences

$$w_{t_h}, \quad w_{r_h}, \quad x_{t_h}, \quad x_{r_h}$$

$$w_{t_h}^{[1]}, \quad w_{r_h}^{[1]}, \quad x_{t_h}^{[1]}, \quad x_{r_h}^{[1]}$$

where

$$w_{t_h} = \frac{w(r,t+h)-w(r,t)}{h}, \quad w_{r_h} = \frac{w(r+h,t)-w(r,t)}{h}$$

$$x_{t_h} = \frac{x(r,t+h)-x(r,t)}{h}, \quad x_{r_h} = \frac{x(r+h,t)-x(r,t)}{h}$$

$$w^{[1]}_{t_h} = \frac{w(r,t+h) - 2w(r,t) + w(r,t-h)}{h}$$
$$w^{[1]}_{r_h} = \frac{w(r+h,t) - 2w(r,h) + w(r-h,t)}{h}$$
$$x^{[1]}_{t_h} = \frac{x(r,t+h) - 2x(r,t) + x(r,t-h)}{h}$$
$$x^{[1]}_{r_h} = \frac{x(r+h,t) - 2x(r,t) + x(r-h,t)}{h}$$

we see that because of the continuity of the unknown functions these subsequences uniformly converge to definite values which we denote by

$$w_{t_h} \to F_1, \quad w_{r_h} \to F_2, \quad x_{t_h} \to F_3, \quad x_{r_h} \to F_4$$

Because of the uniquness of the limit point we have

$$F_1 = \lim_{h\to 0} w_{t_{[h]}}, \quad F_2 = \lim_{h\to 0} w_{r_{[h]}}$$

$$F_3 = \lim_{h\to 0} x_{t_{[h]}}, \quad F_4 = \lim_{h\to 0} x_{r_{[h]}}$$

Similarly the subsequences of the second differrence quotients converge uniformly to the second partial derivatives.

We also have

$$\sum_h h w_{t_h} \to \int w_t dt, \quad \sum_h h w_{r_h} \to \int w_r dr$$
$$\sum_h h x_{t_h} \to \int x_t dt, \quad \sum_h h x_{r_h} \to \int x_r dr$$

Similarly multiplication of the sum of second differences with h converges uniformly to the integral of the second derivatives.

Now, we will obtain some bounds for the diagonals B_1C_1, B_2C_2 ,..., B_nC_n, A_1B_1, A_2B_2, ..., A_nC_n and A_1A_{n+1}, $A_{2,1}A_{n+1,1}$, ..., $A_{n,k}A_{n+1,k}$

Along B_1C_1 we have the following relations

$$w_t(B_1) + b_0(R)w_r(B_1) - \frac{v_0}{\cos\alpha} + \frac{1+\nu}{1+\cos\alpha}b_0(R) \tag{3.114}$$

$$\begin{aligned} w_t(B_1) - w_t(C_1) + a_0(w_r(B_1) - w_r(C_1)) \\ -a(C_1)(t_{B_1} - t_{C_1}) = 0 \end{aligned} \tag{3.115}$$

$$\begin{aligned} w_t(B_1) - w_t(C_{11}) - a_0(w_r(B_1) - w_r(C_{11})) \\ -a(C_{11})(t_{B_1} - t_{C_{11}}) \end{aligned} \tag{3.116}$$

$w_t(B_1)$, $w_r(B_1)$ and $b_0(R)$ are bounded. So the relation (3.114) is bounded. $\left|\frac{w_t(B_1) - w_t(C_1)}{t_{B_1} - t_{C_1}}\right|$ and $\left|\frac{w_r(B_1) - w_r(C_1)}{t_{B_1} - t_{C_1}}\right|$ are bounded by Mean Value Theorem. Similarly, $\left|\frac{w_t(B_1) - w_t(C_{11})}{t_{B_1} - t_{C_{11}}}\right|$ and $\left|\frac{w_r(B_1) - w_r(C_{11})}{t_{B_1} - t_{C_{11}}}\right|$ are bounded.

$$\begin{aligned} |a(C_1)| &= \left|a_0^2\left(\frac{w_r(C_1)}{r_{C_1}} - \frac{w(C_1)}{r_{C_1}^2}\sin^2\alpha\right)\right| \\ &\le a_0^2\left(|w_r(C_1)| + |w(C_1)|\right) \\ &\le a_0^2(K_1 + K_2) \le a_0^2 K \end{aligned}$$

$$\begin{aligned} |a(C_{11})| &= \left|a_0^2\left(\frac{w_r(C_{11})}{r_{C_{11}}} - \frac{w(C_{11})}{r_{C_{11}}^2}\sin^2\alpha\right)\right| \\ &\le a_0^2\left(|w_r(C_{11})| + |w(C_{11})|\right) \\ &\le a_0^2(L_1 + L_2) \le a_0^2 L \end{aligned}$$

Thus (3.115) and (3.116) are bounded. Let M_1 is the maximum of these bounds. So all linear combinations along B_1C_1 is bounded by M_1.

Along A_1A_{n+1}, all unknowns are bounded ($|x_t| < a_0C_0$ and $|x_r| < 1 + C_0$). Let N_1 is the maximum of these bounds.

Along A_1B_1 the relations are

$$x_t(B_1) + b_0(R)x_r(B_1) - v_0 \tan\alpha = 0 \tag{3.117}$$

$$\begin{aligned} x_t(B_1) - x_t(A_1) + a_0(x_r(B_1) - x_r(A_1)) \\ -a_1(A_1)(t_{B_1} - t_{A_1}) = 0 \end{aligned} \tag{3.118}$$

These two are bounded. (The quantities $x_t(B_1)$, $x_r(B_1)$ and $b_0(R)$ are bounded). By mean value theorem $\left|\dfrac{x_t(B_1) - x_t(A_1)}{t_{B_1} - t_{A_1}}\right|$ and $\left|\dfrac{x_r(B_1) - x_r(A_1)}{t_{B_1} - t_{A_1}}\right|$ are bounded.

$$\begin{aligned} |a_1(A_1)| &= \left|a_0^2\left(\frac{x_r(A_1)}{r_{A_1}} - \frac{x(A_1)}{r_{A_1}^2}\right)\right| \\ &\leq a_0^2\left(|x_r(A_1)| + |x(A_1)|\right) \\ &\leq a_0^2K' \end{aligned}$$

Along the characteristic B_2C_2 the relations are

$$w_t(B_2) + b_0(B_1)w_r(B_2) - \frac{v_0}{\cos\alpha} + \frac{1+\nu}{1+\sin\alpha}b_0(B_1) = 0 \tag{3.119}$$

$$\begin{aligned} w_t(B_2) - w_t(2,1) + a_0(w_r(B_2) - w_r(2,1)) \\ -a(2,1)(t_{B_2} - t_{21}) = 0 \end{aligned} \tag{3.120}$$

$$\begin{aligned} w_t(2,1) - w_t(C_1) - a_0(w_r(2,1) - w_r(C_1)) \\ -a(C_1)(t_{21} - t_{C_1}) = 0 \end{aligned} \tag{3.121}$$

$$w_t(2,1) - w_t(C_2) + a_0(w_r(2,1) - w_r(C_2))$$
$$-a(C_2)(t_{21} - t_{C_2}) = 0 \tag{3.122}$$

$$w_t(B_2) - w_t(C_{22}) - a_0(w_r(B_2) - w_r(C_{22}))$$
$$-a(C_{22})(t_{B_2} - t_{C_{22}}) = 0 \tag{3.123}$$

The relation (3.119) is bounded ($w_t(B_2)$, $w_r(B_2)$, $b_0(B_1)$ are bounded). Also (3.123) is bounded by mean value theorem and $|a(C_{22})|$ is bounded.

Addition of (3.120), (3.121) and (3.122) will yield

$$\frac{w_t(B_2) - w_t(2,1)}{h} + \frac{w_t(2,1) - w_t(C_2)}{h} + a_0\frac{w_r(B_2) - w_r(2,1)}{h}$$
$$-a_0\frac{w_r(2,1) - w_r(C_2)}{h} - a(2,1) - a(C_2) =$$
$$-\frac{w_t(2,1) - w_t(C_1)}{h} + a_0\frac{w_r(2,1) - w_r(C_1)}{h} + a(C_1) \tag{3.124}$$

where $h = t_{B_2} - t_{21} = t_{C_2} - t_{21} = t_{21} - t_{C_1}$

$$\left|\frac{w_t(2,1) - w_t(C_1)}{h}\right| \le p_1q_1M_1, \left|\frac{w_r(2,1) - w_r(C_1)}{h}\right| \le \frac{p_2q_2}{a_0}M_1$$

$$|a(C_1)| = \left|a_0^2\left(\frac{w_r(C_1)}{r_{C_1}} - \frac{w(C_1)}{r_{C_1}^2}\sin^2\alpha\right)\right|$$
$$\le a_0^2\left(|w_r(C_1)| + |w(C_1)|\right) \le a_0^2\frac{b}{a_0^2}M_1 = bM_1 \tag{3.125}$$

Then the bound for B_2C_2 can be written as

$$M_2 = cM_1 \tag{3.126}$$

where $c = (p_1q_1 + \frac{p_2q_2}{a_0} + b)$ does not depend on h.

So for the characteristic $B_{n+1}C_{n+1}$

$$M_{n+1} = cM_n$$

$$M_{n+1} = c^n M_1 \tag{3.127}$$

For each mesh size, the bound at $(n+1)$st step depends only on the bound at the first step.

The relations for $A_{21}A_{n+1,1}$ are

$$x_t(A_{n+1,1}) - x_t(A_{n+1}) - a_0(x_r(A_{n+1,1}) - x_r(A_{n,1})) \\ -a_1(A_{n,1})(t_{A_{n+1,1}} - t_{n,1}) = 0 \tag{3.128}$$

$$x_t(A_{n+1,1}) - x_t(A_{n+1}) - a_0(x_r(A_{n+1,1}) - x_r(A_{n+1})) \\ -a_1(A_{n+1})(t_{A_{n+1,1}} - t_{A_{n+1}}) = 0 \tag{3.129}$$

Addition these two relations will give

$$\frac{x_t(A_{n+1,1}) - x_t(A_{n,1})}{h} - a_0\frac{x_r(A_{n+1,1}) - x_r(A_{n,1})}{h} - a_1(A_{n,1}) = \\ -\frac{x_t(A_{n+1,1}) - x_t(A_{n+1})}{h} - a_0\frac{x_r(A_{n+1,1}) - x_r(A_{n+1})}{h} + a_1(A_{n+1}) \tag{3.130}$$

By the mean value theorem

$$\left|\frac{x_t(A_{n+1,1}) - x_t(A_{n+1})}{h}\right| < s_1 t_1 N_1, \quad \left|\frac{x_r(A_{n+1,1}) - x_r(A_{n+1})}{h}\right| < s_2 t_2 N_1$$

$$|a_1(A_{n+1})| = \left|a_0^2\left(\frac{x_r(A_{n+1})}{r_{A_{n+1}}} - \frac{x(A_{n+1})}{r^2_{A_{n+1}}}\right)\right|$$

$$\leq a_0^2\left(|x_r(A_{n+1})| + |x(A_{n+1})|\right)$$

$$\leq a_0^2\frac{b_1}{a_0^2}N_1 = b_1 N_1$$

Then the bound for $A_{21}A_{n+1,1}$ is

$$N_2 = dN_1 \tag{3.131}$$

where d does not depend on the step size h.

The relations along A_2B_2 are

$$x_t(B_2) + b_0(B_1)x_r(B_2) - v_0 \tan\alpha = 0 \tag{3.132}$$

$$\begin{aligned} x_t(B_2) - x_t(A_2) + a_0(x_r(B_2) - x_r(A_2)) \\ -a(A_2)(t_{B_2} - t_{A_2}) = 0 \end{aligned} \tag{3.133}$$

The relations (3.132) and (3.133) are bounded. ($x_t(B_2)$, $x_r(B_2)$ and $b_0(B_1)$ are bounded in the given domain).

From the equation (3.131) the bound at the $(k+1)$st step can be written as

$$\begin{aligned} N_{k+1} &= dN_k \\ N_{k+1} &= d^k N_1 \end{aligned} \tag{3.134}$$

By using the equations (3.127) and (3.134) we can conclude that the linear combinations in whole domain are bounded. If the differences of unknown functions are initially bounded then all linear combinations of the difference quotients are bounded at each step.

3.6 Discussion of the Numerical Results

The results of numerical solutions for different values of parameters are brought in Figures 3.4-3.15. The algorithm is comprised by three subalgorithms: Subalgorithms for regions I and II and a subalgorithm for the unknown boundary of regions I-II. At each step the algorithm for the unknown boundary comprise three nonlinear relations and are studied easily. Upon finding the boundary values of x_t and x_r from the mentioned set of equations on the unknown boundary, the main body of the algorithm in region II at each step reduces to the set of three linear relations with respect to three unknowns of region II. And this algorithm allows one to continue the computations for any value of r but along the characteristics $\frac{dr}{dt} = a_0$, taking its origin from the boundary point, where x_t and x_r are known from the previous subalgorithm.

The main computational difficulty arises in computing the nodal values of the unknown functions in region I. The matter is that the number of solving equations in region I is incereasing from step to step. But these relations are linear and as shown in the process of setting the algorithm they are closed set of linear equations with nonzero principal determinant and hence possess a unique solution.

The surface and contour plots of ε_r, ε_θ, σ_r, σ_θ are brought in Figures 3.4-3.6, in Figures 3.7-3.9, in Figures 3.10-3.12, and in Figures 3.13-3.15, respectively for $\nu = 0.33$, for $\alpha = 30^0$ and for $b_0 = 0.8, 0.75, 0.72$.

It is proven rigoriously that the Poisson's ratio satisfies the condition:

$0 < \nu < 0.5$. But for most real materials $0.3 \leq \nu \leq 0.45$. We choose step size h is chosen as 0.0001. Since the problem is sensitive to step size. Here we bring the solution to the problem is for $0.7 < b_0 \leq 0.8$ and $\alpha = 30^0$.

Since the thesis is on mathematics we don't pay much attention to the dependence of the unknown functions on the variations of all of the parameters of the problem. We only note that by increasing impact velocity v_0 the ε_r, σ_r and σ_θ increase in region I. Besides at t increases the stress and strains decrease at fixed points. The programs of computation written in FORTRAN are available at the Mathematics department, METU and any interested person may use it to get the numerical results for some needed values of the parameters.

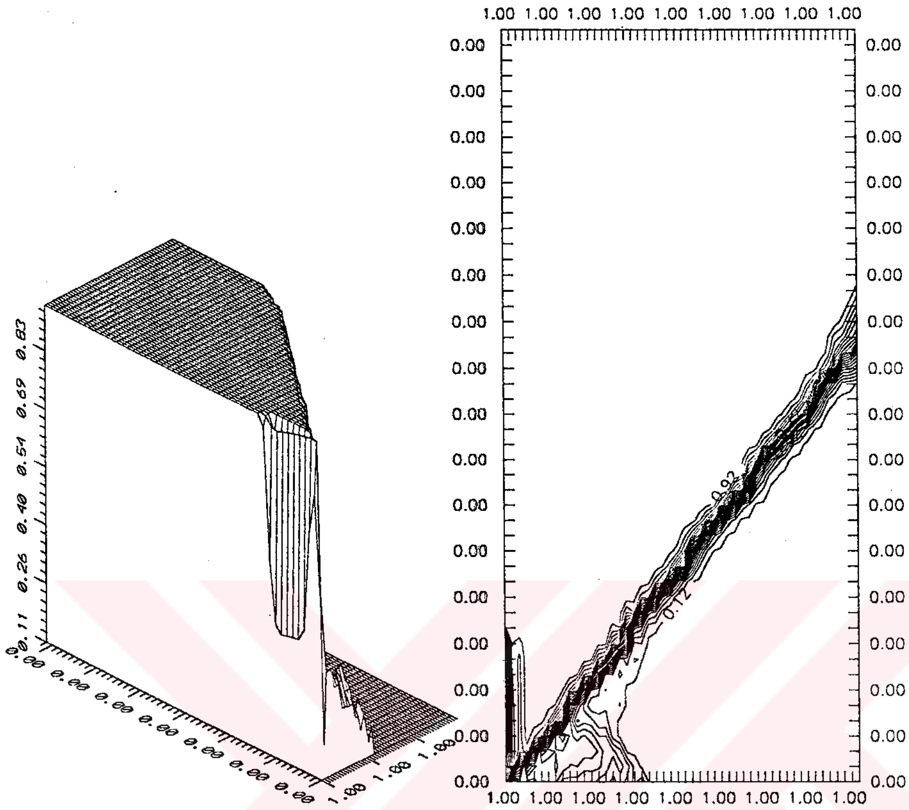

Figure 3.4: Behaviour of ε_r for $\alpha = 30^0$, $b_0 = 0.8$, $\nu = 0.33$.

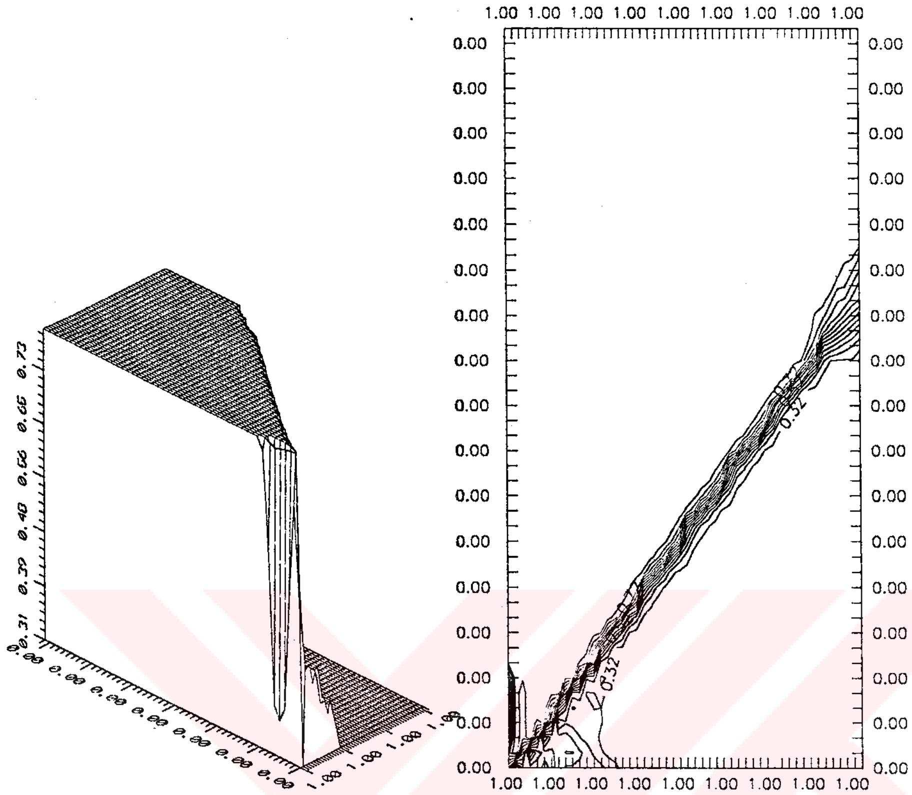

Figure 3.5: Behaviour of ε_r for $\alpha = 30^0$, $b_0 = 0.75$, $\nu = 0.33$.

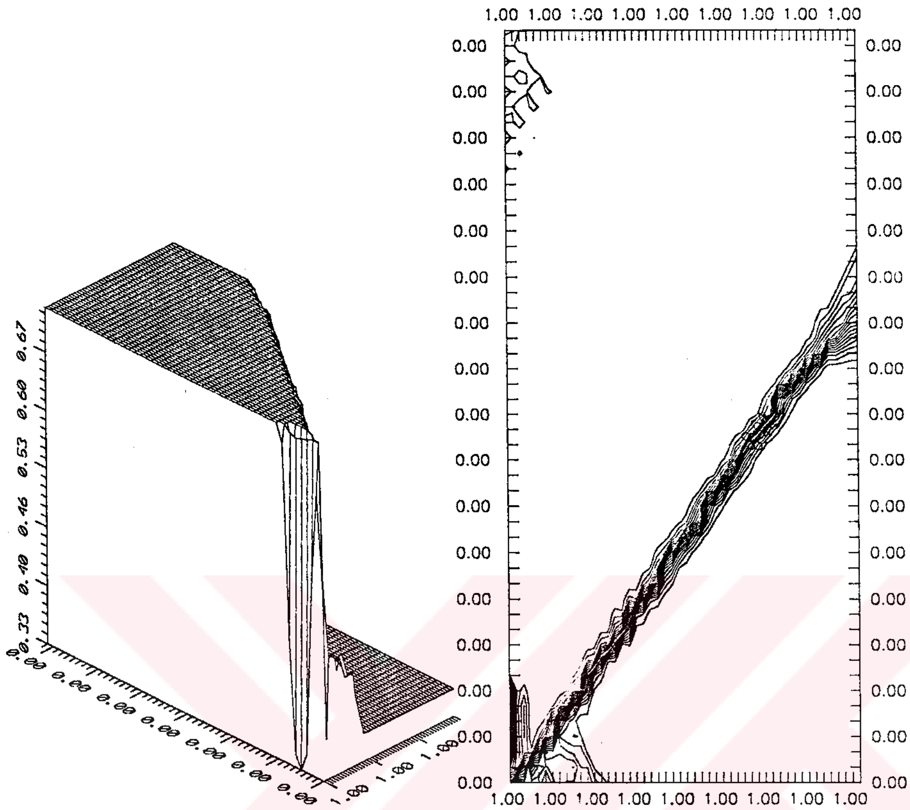

Figure 3.6: Behaviour of ε_r for $\alpha = 30^0$, $b_0 = 0.72$, $\nu = 0.33$.

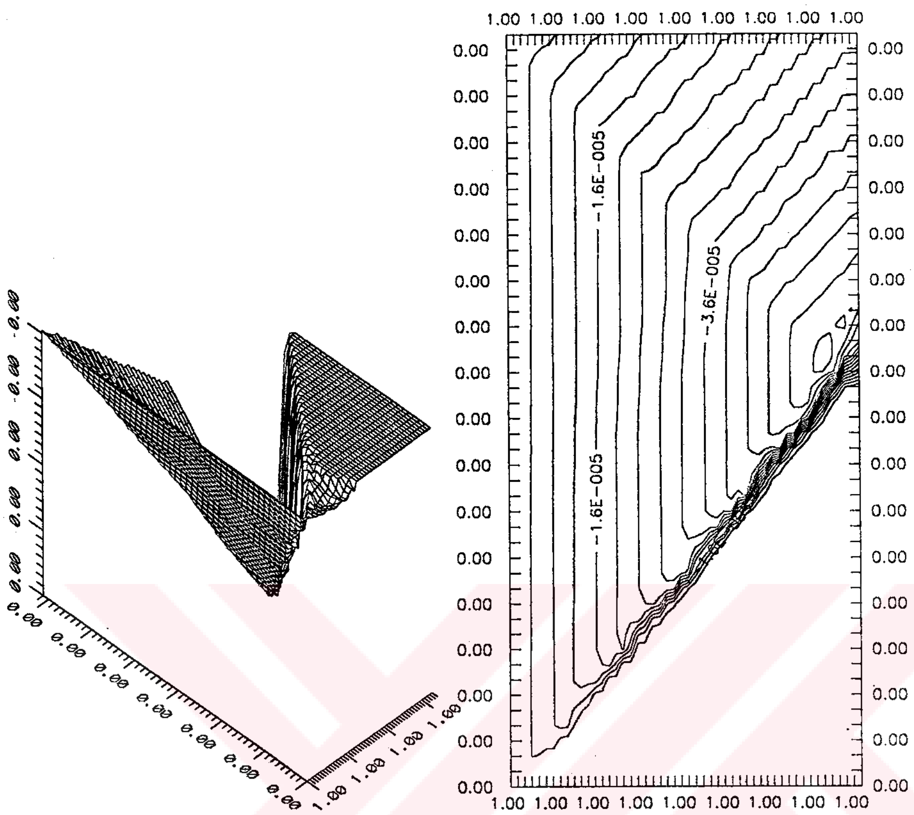

Figure 3.7: Behaviour of ε_θ, for $\alpha = 30^0$, $b_0 = 0.8$, $\nu = 0.33$.

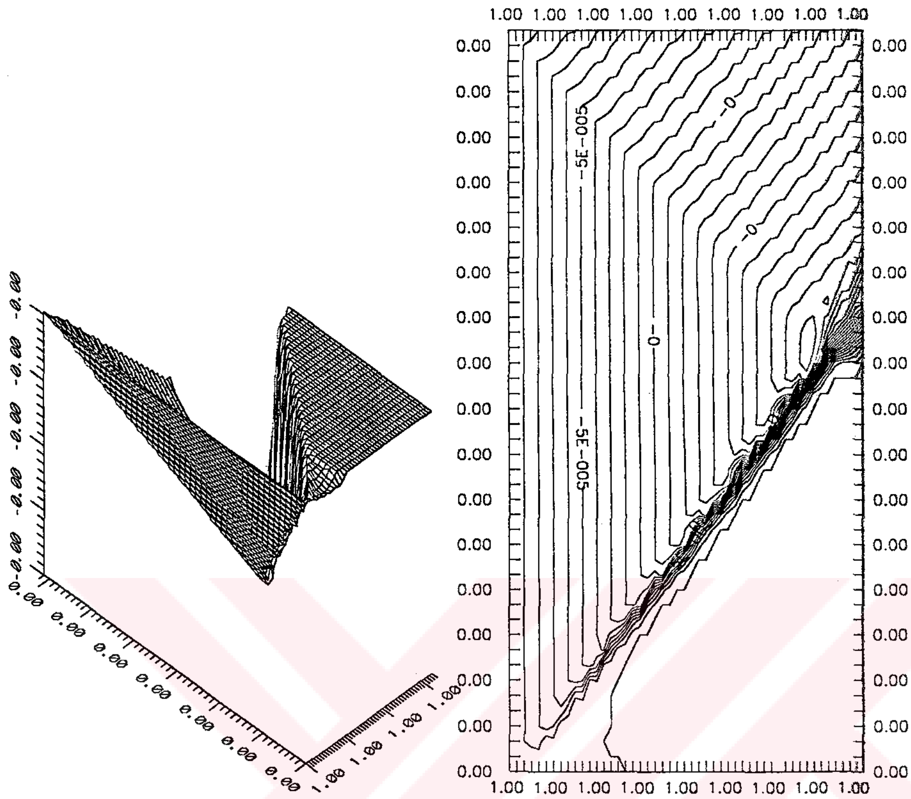

Figure 3.8: Behaviour of ε_θ, for $\alpha = 30^0$, $b_0 = 0.75$, $\nu = 0.33$.

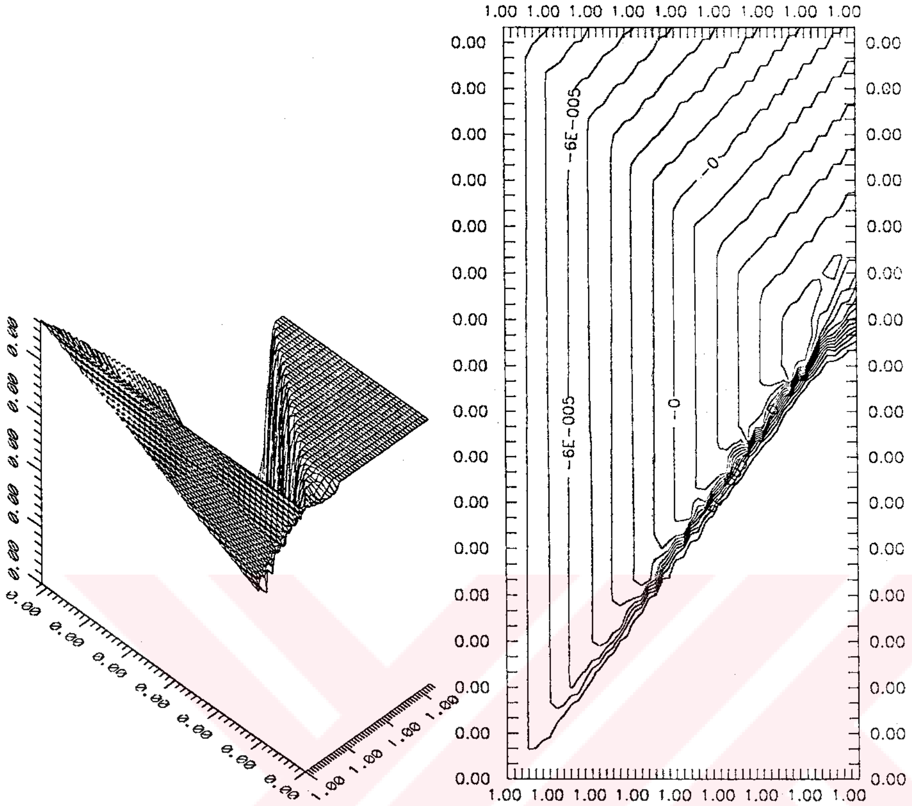

Figure 3.9: Behaviour of ε_θ, for $\alpha = 30^0$, $b_0 = 0.72$, $\nu = 0.33$.

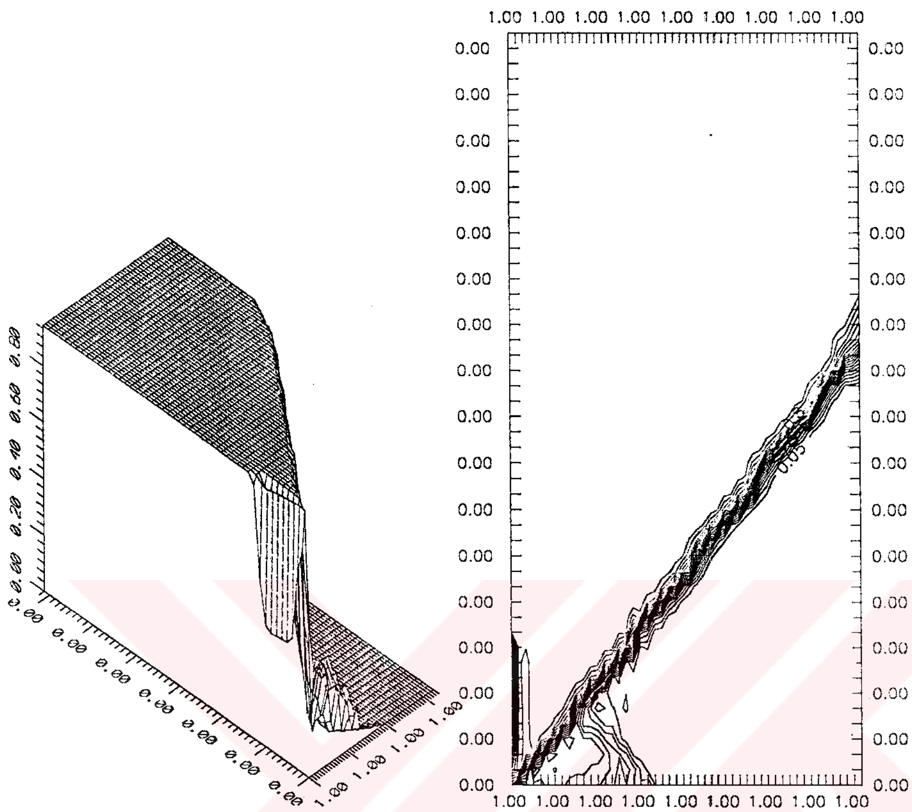

Figure 3.10: Behaviour of σ_r, for $\alpha = 30^0$, $b_0 = 0.8$, $\nu = 0.33$.

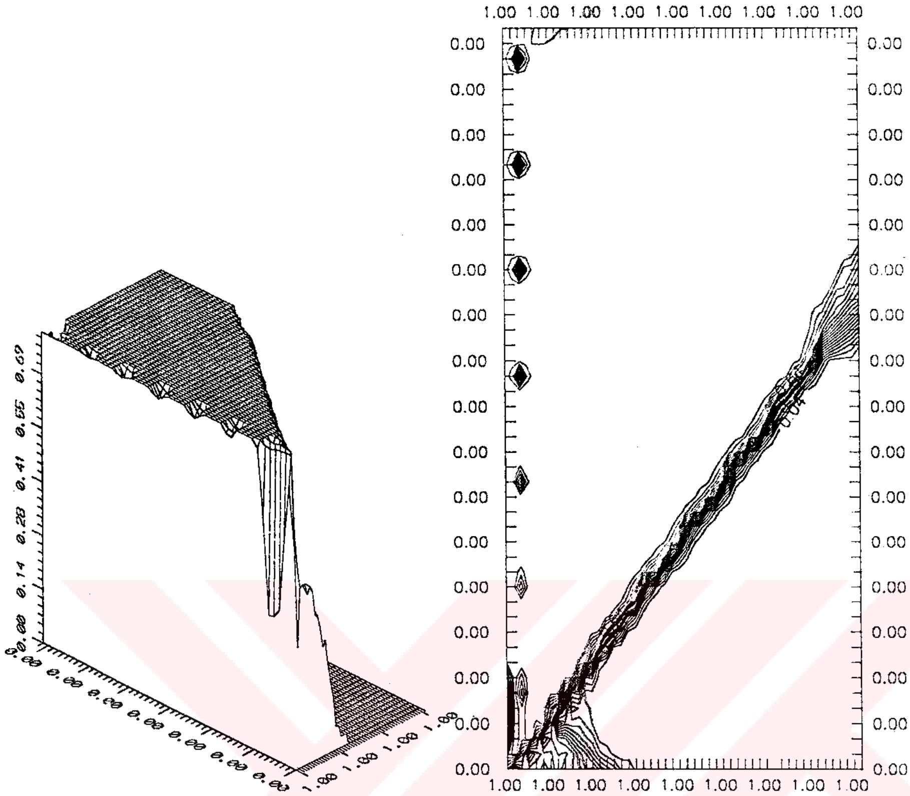

Figure 3.11: Behaviour of σ_r, for $\alpha = 30^0$, $b_0 = 0.75$, $\nu = 0.33$.

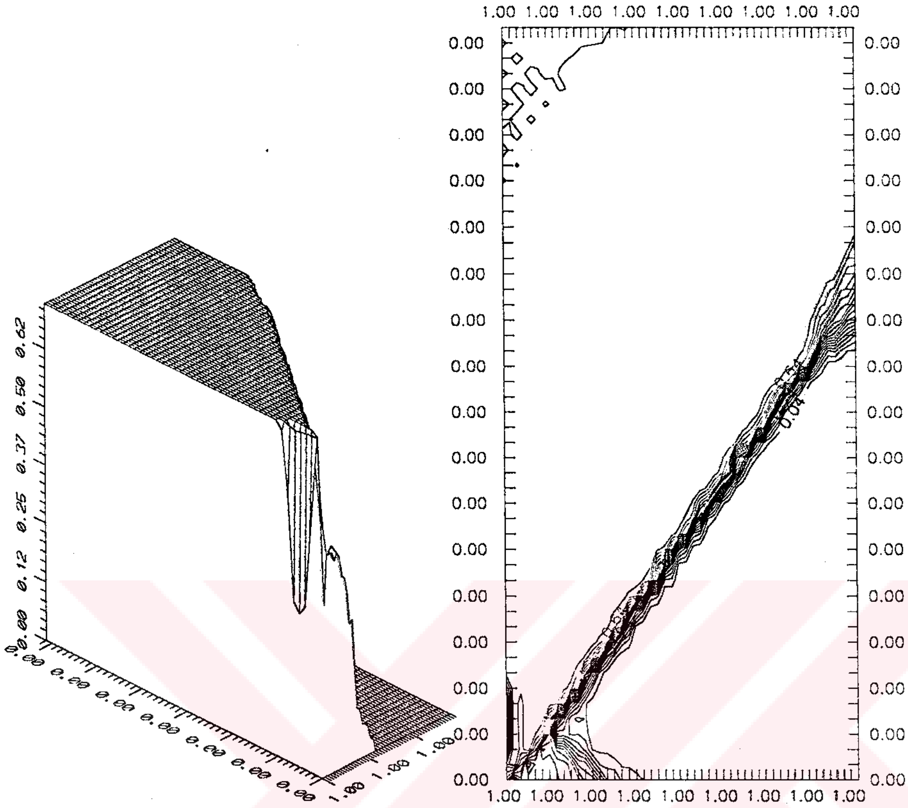

Figure 3.12: Behaviour of σ_r, for $\alpha = 30^0$, $b_0 = 0.72$, $\nu = 0.33$.

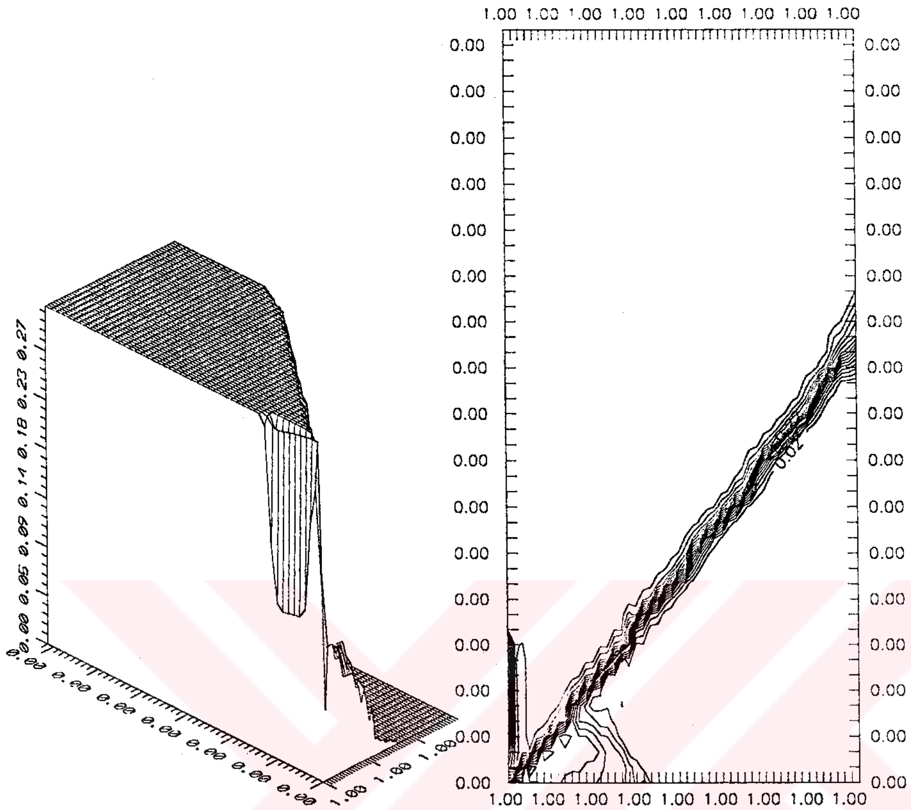

Figure 3.13: Behaviour of σ_θ for $\alpha = 30^0$, $b_0 = 0.8$, $\nu = 0.33$.

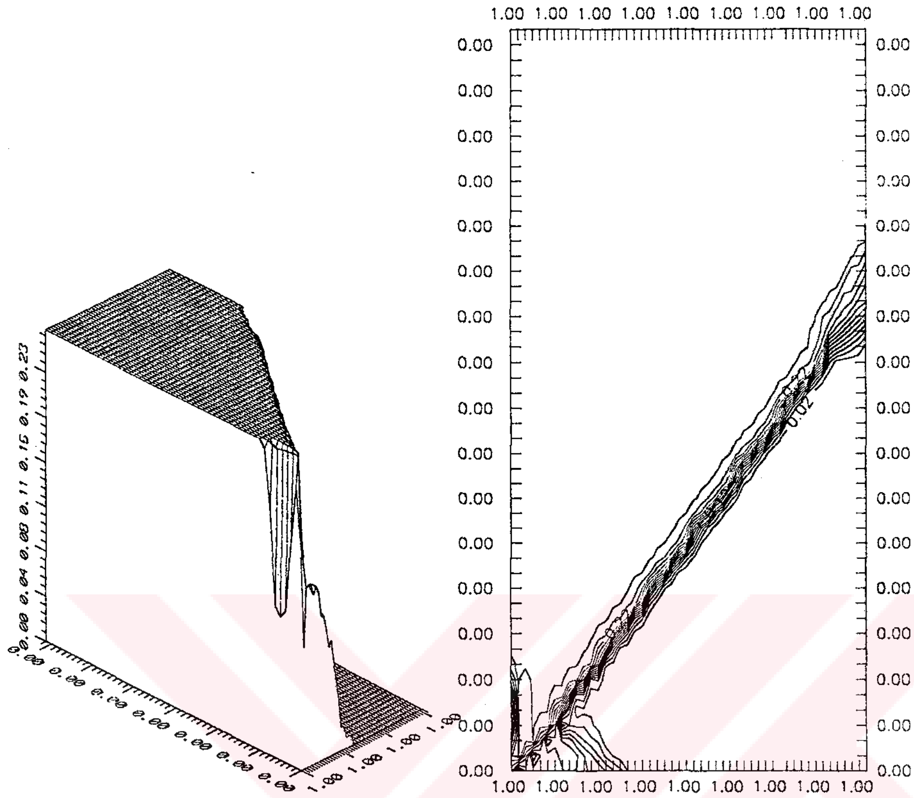

Figure 3.14: Behaviour of σ_θ for $\alpha = 30^0$, $b_0 = 0.75$, $\nu = 0.33$.

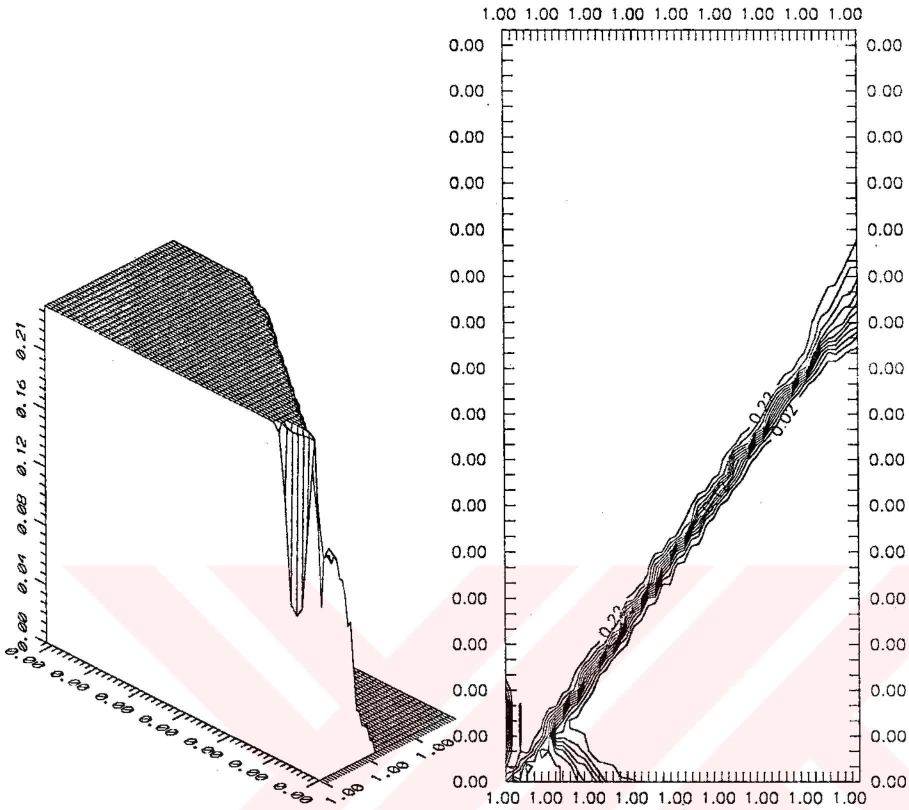

Figure 3.15: Behaviour of σ_θ for $\alpha = 30^0$, $b_0 = 0.72$, $\nu = 0.33$.

CHAPTER 4

DYNAMICS OF A MEMBRANE WITH A FREE HOLE SUBJECT TO A TRANSVERSE IMPACT

In Chapter 3, the motion of a membrane with a rigidly fixed hole is considered. In this chapter the motion of the membranes with a free hole and the possible wave scheme of motion of the membranes are studied. This problem has no similarity solution either. The numerical solution to the problem is obtained by employing the method of characteristics.

4.1 Problem Statement

We consider initially planar membrane with a free hole of infinite extend which is at rest. At time $t = 0$ the membrane is subjected to the transverse impact with constant velocity v_0 by a cone. The motion of the membrane is covered by two different regions as in Section 3.1 (see, Fig 3.1). In region I, there is transverse motion and membrane is contact with the cone. In the region II, motion is pure radial. The transverse wavefront is the strong discontinuity wave.

The equations of transverse motion, in region I, can be written as

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta \sin \alpha}{r} \tag{4.1}$$

$$\varepsilon_r = \frac{\partial u}{\partial r} - 1, \quad \varepsilon_\theta = \frac{u}{r} \sin \alpha - 1 \tag{4.2}$$

$$\sigma_r = \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta), \quad \sigma_\theta = \frac{E}{1-\nu^2}(\varepsilon_\theta + \nu\varepsilon_r) \tag{4.3}$$

If we substitute the equations (4.2) and (4.3) into (4.1) we obtain

$$\frac{1}{a_0^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\sin^2\alpha - \frac{1+\nu}{r}(1-\sin\alpha) \tag{4.4}$$

In region II, the radial motion is described by equations

$$\rho_0\frac{\partial^2 x}{\partial t^2} = \frac{\partial \sigma_{r1}}{\partial r} + \frac{\sigma_{r1} - \sigma_{\theta 1}}{r} \tag{4.5}$$

$$\varepsilon_{r1} = \frac{\partial x}{\partial r} - 1, \quad \varepsilon_{\theta 1} = \frac{x}{r} - 1 \tag{4.6}$$

$$\sigma_r = \frac{E}{1-\nu^2}(\varepsilon_{r1} + \nu\varepsilon_{\theta 1}), \quad \sigma_{\theta 1} = \frac{E}{1-\nu^2}(\varepsilon_{\theta 1} + \nu\varepsilon_{r1}) \tag{4.7}$$

From equations (4.5), (4.6) and (4.7) we can write the following equation

$$\frac{1}{a_0^2}\frac{\partial^2 x}{\partial t^2} = \frac{\partial^2 x}{\partial r^2} + \frac{1}{r}\frac{\partial x}{\partial r} - \frac{x}{r^2} \tag{4.8}$$

Initial conditions at $t = 0$ are

$$\begin{aligned} x(r,0) &= r, & \frac{\partial x}{\partial t}(r,0) &= 0 \\ u(r,0) &= \frac{r}{\sin\alpha}, & \frac{\partial u}{\partial t}(r,0) &= 0 \end{aligned} \tag{4.9}$$

The boundary condition at $r = R$

$$\sigma_r(R,t) = 0 \tag{4.10}$$

If we introduce the new variable

$$w(r,t) = u(r,t) - \frac{1+\nu}{1+\sin\alpha}r \tag{4.11}$$

and substitute into equation (4.4), the equations of motion will yield

In region I

$$\frac{1}{a_0^2}\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{w}{r^2}\sin^2\alpha \tag{4.12}$$

In region II

$$\frac{1}{a_0^2}\frac{\partial^2 x}{\partial t^2} = \frac{\partial^2 x}{\partial r^2} + \frac{1}{r}\frac{\partial x}{\partial r} - \frac{x}{r^2} \tag{4.13}$$

Initial conditions at $t = 0$ are

$$\begin{aligned} x(r,0) = r, \quad & \frac{\partial x}{\partial t}(r,0) = 0 \\ w(r,0) = \frac{r}{\sin\alpha} - \frac{1+\nu}{1+\sin\alpha}r, \quad & \frac{\partial w}{\partial t}(r,0) = 0 \end{aligned} \tag{4.14}$$

The boundary condition at $r = R$

$$\sigma_r(R,t) = 0 \tag{4.15}$$

As shown in Chapter 3, the equations (4.12) and (4.13) are of hyperbolic type with the characteristic velocities

$$\lambda = \pm a_0 \tag{4.16}$$

Characteristic relations have a form

In region I

$$dw_t = \pm a_0 dw_r + a\,dt \quad \text{along} \quad \lambda = \pm a_0 \tag{4.17}$$

where $a = a_0^2\left(\frac{1}{r}w_r - \frac{w}{r^2}\sin^2\alpha\right)$

In region II

$$dx_t = \pm dx_r + a_1 dt \quad \text{along} \quad \lambda = \pm a_0 \tag{4.18}$$

where $a_1 = a_0^2 \left(\frac{1}{r} x_r - \frac{x}{r^2} \right)$

4.2 Solution to the Problem on the Radial Wavefront

Since the displacement $x(r, t)$ is continuous on the radial wavefront RA, we have

$$x(R + a_0 t, t) = R + a_0 t \tag{4.19}$$

Total derivative of equation (4.19) will yield

$$\frac{\partial x}{\partial t} = -a_0 \left(\frac{\partial x}{\partial r} - 1 \right) \tag{4.20}$$

Characteristic relation along $r = R + a_0 t$ is

$$dx_t = a_0 dx_r + a_0^2 \left(\frac{1}{r} x_r - \frac{x}{r^2} \right) dt \tag{4.21}$$

substitution of the equation (4.20) into (4.21) yield

$$\frac{d(x_r - 1)}{x_r - 1} = -\frac{dr}{2r} \tag{4.22}$$

If we solve the equation (4.22) we obtain

$$\frac{\partial x}{\partial r} = 1 + C_0 \sqrt{\frac{R}{R + a_0 t}} \tag{4.23}$$

$$\frac{\partial x}{\partial t} = -a_0 C_0 \sqrt{\frac{R}{R + a_0 t}} \tag{4.24}$$

So, ε_{r1} and $\varepsilon_{\theta 1}$ can easily be obtained from equations (4.19) and (4.23) which depends on C_0 and t.

From the boundary condition (4.10) at $r = R$ we can write

$$\varepsilon_r(R, t) + \nu \varepsilon_\theta(R, t) = 0 \tag{4.25}$$

Equation (4.25) implies that

$$\frac{\partial w}{\partial r}(R,t) = (1-\nu^2)\frac{\sin\alpha}{1+\sin\alpha} - \nu\sin\alpha\frac{w}{R} \tag{4.26}$$

As in the previous case, in this problem transverse wavefront RB, $\frac{dr_\star}{dt} = b_0$, is a strong discontinuity wave. Shock condition on RB is

$$\rho_0(-b_0)[\overline{V}] = [\overline{T}] + \overline{Q} \tag{4.27}$$

solution of the equation (4.27) will give

$$b_0 = \frac{\dot{\omega}(t) - v_0\cos\alpha}{2\left((1+\varepsilon_{r1})\sin\alpha - (1+\varepsilon_r)\right)} + \\ +\sqrt{\left(\frac{\dot{\omega}(t) - v_0\cos\alpha}{2\left((1+\varepsilon_{r1})\sin\alpha - (1+\varepsilon_r)\right)}\right)^2 + \frac{\sigma_{r1}\sin\alpha - \sigma_r}{\rho_0\left((1+\varepsilon_{r1})\sin\alpha - (1+\varepsilon_r)\right)}} \tag{4.28}$$

where $\dot{\omega}(t) = \frac{\partial w}{\partial t}|_{r=R}$

On the boundary RB where $\frac{dr_\star}{dt} = b_0$

$$x(r_\star(t),t) = R + v_0 t\tan\alpha \tag{4.29}$$

$$w(r_\star(t),t) = \frac{R + v_0 t\tan\alpha}{\sin\alpha} - \frac{1+\nu}{1+\sin\alpha}r_\star(t) \tag{4.30}$$

Total derivatives of the equations (4.29) and (4.30) yield, respectively

$$\frac{\partial x}{\partial t} = v_0\tan\alpha - b_0\frac{\partial x}{\partial r} \tag{4.31}$$

$$\frac{\partial w}{\partial t} = \frac{v_0}{\cos\alpha} - b_0\left(\frac{\partial w}{\partial r} + \frac{1+\nu}{1+\sin\alpha}\right) \tag{4.32}$$

Thus, we have the following boundary conditions

On Rt where $\sigma_r(R,t) = 0$ implies that

$$\frac{\partial w}{\partial r} = (1-\nu^2)\frac{\sin\alpha}{1+\sin\alpha} - \nu\sin\alpha\frac{w}{R} \tag{4.33}$$

On RB we have

$$\frac{\partial x}{\partial t} = v_0\tan\alpha - b_0(1+\varepsilon_{r1}) \tag{4.34}$$

$$\frac{\partial w}{\partial t} = \frac{v_0}{\cos\alpha} - b_0\,(1+\varepsilon_r) \tag{4.35}$$

$$\begin{aligned} b_0 = {} & \frac{\dot{\omega}(t) - v_0\cos\alpha}{2\left((1+\varepsilon_{r1})\sin\alpha - (1+\varepsilon_r)\right)} + \\ & + \sqrt{\left(\frac{\dot{\omega}(t) - v_0\cos\alpha}{2\left((1+\varepsilon_{r1})\sin\alpha - (1+\varepsilon_r)\right)}\right)^2 + \frac{\sigma_{r1}\sin\alpha - \sigma_r}{\rho_0\left((1+\varepsilon_{r1})\sin\alpha - (1+\varepsilon_r)\right)}} \end{aligned} \tag{4.36}$$

On RA

$$\frac{\partial x}{\partial r} = 1 + C_0\sqrt{\frac{R}{R+a_0 t}} \tag{4.37}$$

$$\frac{\partial x}{\partial t} = -a_0 C_0\sqrt{\frac{R}{R+a_0 t}} \tag{4.38}$$

Now we define the value of unknown functions at $r = R, t = 0^+$ that is immediately after the impact.

Displacements and hence $\varepsilon_\theta(r,t)$ are assumed to be continuous at every point r for any time t (the main assumption of continuum mechanics). Thus, we have

$$\lim_{t\to 0^+} \varepsilon_\theta(r,t) = 0 \tag{4.39}$$

in particular

$$\varepsilon_\theta(R, 0^+) = 0 \tag{4.40}$$

and

$$\varepsilon_{\theta 1}(R, 0^+) = 0 \tag{4.41}$$

as well the boundary condition (4.10) on Rt leads

$$\varepsilon_r(R, 0^+) = 0 \tag{4.42}$$

Since boundary is free of stresses then

$$\varepsilon_{r1}(R, 0^+) = C_0 = 0 \tag{4.43}$$

In relations (4.34), (4.37) and (4.38) passing to the limit $r \to R, t \to 0^+$ we get

$$\frac{\partial x}{\partial t}(R, 0^+) = v_0 \tan\alpha - b_0 \frac{\partial x}{\partial r}(R, 0^+) \tag{4.44}$$

$$\frac{\partial x}{\partial r}(R, 0^+) = 1 \tag{4.45}$$

$$\frac{\partial x}{\partial t}(R, 0^+) = 0 \tag{4.46}$$

equations (4.44) – (4.46) yield

$$b_0 = v_0 \tan\alpha \tag{4.47}$$

Now approaching $r \to R$ and $t \to 0^+$ in (4.35) and taking into account (4.42) we get

$$\frac{\partial w}{\partial t}(R, 0^+) = \frac{v_0(1 - \sin\alpha)}{\cos\alpha} \tag{4.48}$$

For given semivertex angle α and constant initial velocity v_0, b_0 can be easily evaluated by using the equation (4.47). Then from (4.48) $w_t(R, 0^+)$ is calculated.

It is interest to note that for this problem the radial wavefront RA is not a strong discontinuity wave. The derivatives of unknown functions are continuous thereon.

4.3 Algorithm of Numerical Solution to the Problem

The algorithm given in Section 3.4 is used to obtain numerical solution to the problem of free hole defined by the equations (4.1) – (4.10). There are some differences between two algorithms. These differences are due to the different boundary conditions defined on Rt line. For the problem of rigid hole given by the equations (3.1) – (3.10), the boundary condition on Rt is

$$\varepsilon_\theta(R, t) = 0 \tag{4.49}$$

The equation (4.49) implies that w is independent on t.

On Rt

$$\frac{\partial w}{\partial t}(R, t) = 0 \tag{4.50}$$

so

$$w(R, t) = \frac{R}{\sin \alpha} - \frac{1 + \nu}{1 + \sin \alpha} R \tag{4.51}$$

But for this problem the boundary condition on Rt is

$$\sigma_r(R, t) = 0 \tag{4.52}$$

By using (4.52) we obtain

$$w_r(R,t) = (1-\nu^2)\frac{\sin\alpha}{1+\sin\alpha} - \nu\sin\alpha\frac{w(R,t)}{R} \tag{4.53}$$

Thus the differences in this algorithm are only in region I. In region II, we use the same relations given in Section 3.4. We showed that the unknowns can be uniquely determined in region II.

The initial slope $b_0(R,0^+)$ of the unknown transverse wavefront RB can be easily calculated from (4.47) for given v_0 and α. Since the boundary is free of stresses then $C_0 \equiv 0$.

In this section we will show that this scheme has unique solution in region I.

The difference equation along B_1C_1, (see, Figure 3.2) can be written by using the characteristic relation (4.17)

$$\begin{aligned} w_t(B_1) - w_t(C_1) = -a_0(w_r(B_1) - w_r(C_1)) &+ \\ a(C_1)(t_{B_1} - t_{C_1}) & \end{aligned} \tag{4.54}$$

where

$$\begin{aligned} w_r(C_1) &= (1-\nu^2)\frac{\sin\alpha}{1+\sin\alpha} - \nu\sin\alpha\frac{w(C_1)}{r_{C_1}} \\ w(C_1) &= w(R) + w_t(C_1)t_{C_1} \\ a(C_1) &= a_0^2\left(\frac{w_r(C_1)}{r_{C_1}} - \frac{w(C_1)}{r_{C_1}^2}\sin^2\alpha\right) \end{aligned}$$

We write the second equation at point B_1 by using the equation (4.35)

$$w_t(B_1) + b_0(R)w_r(B_1) = \frac{v_0}{\cos\alpha} - \frac{1+\nu}{1+\sin\alpha}b_0(R) \tag{4.55}$$

We have two equations in three unknowns, namely in $w_t(B_1)$, $w_r(B_1)$ and $w_t(C_1)$. Since system is totally hyperbolic we need another equation to obtain the unique solution. We draw a characteristic line with slope a_0 which passes through B_1 and parallel to RA. Its intersection with t-axis is labelled by C_{11}. The characteristic relation along B_1C_{11} is

$$w_t(B_1) - w_t(C_{11}) = a_0(w_r(B_1) - w_r(C_{11})) + a(C_{11})(t_{B_1} - t_{C_{11}}) \tag{4.56}$$

where

$$w_r(C_{11}) = (1-\nu^2)\frac{\sin\alpha}{1+\sin\alpha} - \nu\sin\alpha\frac{w_{C_{11}}}{r_{C_1}}$$

$$w_t(C_{11}) = w_t(R)\frac{t_{C_{11}} - t_{C_1}}{t_{C_1}} + w_t(C_1)\frac{t_{C_{11}}}{t_{C_1}}$$

$$w(C_{11}) = w(R)\frac{t_{C_{11}} - t_{C_1}}{t_{C_1}} + w(C_1)\frac{t_{C_{11}}}{t_{C_1}}$$

.

Now we have three equations in three unknowns. The principal determinant is not zero.

$$\Delta_w = \begin{vmatrix} 1 & b_0(R) & 0 \\ 1 & -a_0 & A_{11} \\ 1 & a_0 & A_{12} \end{vmatrix}$$

$$\Delta_w = (b_0(R) - a_0)A_{11} - (a_0 + b_0(R))A_{12} \tag{4.57}$$

where

$$A_{11} = -d_2 - \frac{a_0}{R}\nu\sin\alpha t_{C_1}d_2 + \frac{a_0^2}{R^2}\sin\alpha(\nu + \sin\alpha)t_{C_1}d_2$$

$$A_{12} = -1 + \frac{a_0}{R}\nu \sin \alpha t_{C_1} + \frac{a_0^2}{R^2}(\nu + \sin \alpha) t_{C_1}$$
$$d_2 = \frac{t_{C_1} - t_{C_{11}}}{t_{C_1}}$$

Since $0 < d_2 < 1$, $-1 \leq \sin \alpha \leq 1$, $0 < t_{C_1}$, $0 < \nu + \sin \alpha$ then $A_{11} < 0$ and $A_{12} < 0$. Also $a_0 + b_0(R) > 0$ and $b_0(R) - a_0 < 0$. All these implay that $\Delta_w > 0$. So we determine the unknowns $w_t(B_1)$, $w_r(B_1)$ and $w_t(C_1)$ uniquely.

At the second step, there are three possibilities for the position of B_2: The first posssibility for the position of B_2 is $r_{21} < r_{B_2} < r_{20}$; By using the equations (4.17) and (4.35), we write the equations along $B_2 P_{21}$, $P_{21} C_1$, $P_{21} C_2$ and at B_2 as

$$w_t(B_2) - w_t(2,1) = -a_0(w_r(B_2) - w_r(2,1)) + a(2,1)(t_{B_2} - t_{21}) \tag{4.58}$$

where

$$a(2,1) = a_0^2 \left(\frac{w_r(2,1)}{r_{21}} - \frac{w(2,1)}{r_{21}^2} \sin^2 \alpha \right)$$
$$w(2,1) = w_t(2,1)(t_{21} - t_{C_1}) + w_r(2,1)(r_{21} - r_{C_1}) + w(C_1)$$

$$w_t(2,1) - w_t(C_1) = a_0(w_r(2,1) - w_r(C_1)) + a(C_1)(t_{21} - t_{C_1}) \tag{4.59}$$

$$w_t(2,1) - w_t(C_2) = -a_0(w_r(2,1) - w_r(C_2)) + a(C_2)(t_{21} - t_{C_2}) \tag{4.60}$$

where

$$a(C_2) = a_0^2 \left(\frac{w_r(C_2)}{r_{C_2}} - \frac{w(C_2)}{r_{C_2}^2} \sin^2 \alpha \right)$$

$$w_r(C_2) = (1-\nu^2)\frac{\sin\alpha}{1+\sin\alpha} - \nu \sin\alpha \frac{w(C_2)}{r_{C_2}}$$

$$w(C_2) = w(C_1) + w_t(C_2)(t_{C_2} - t_{C_1})$$

$$w_t(B_2) + b_0(B_1)w_r(B_2) = \frac{v_0}{\cos\alpha} - \frac{1+\nu}{1+\sin\alpha} b_0(B_1) \tag{4.61}$$

$$w_t(B_2) - w_t(C_{22}) = a_0(w_r(B_2) - w_r(C_{22})) + a(C_{22})(t_{B_2} - t_{C_{22}}) \tag{4.62}$$

where

$$a(C_{22}) = a_0^2 \left(\frac{w_r(C_{22})}{r_{C_{22}}} - \frac{w(C_{22})}{r_{C_{22}}^2} \sin^2 \alpha \right)$$

$$w_r(C_{22}) = w_r(B_1)c_1 + w_r(C_1)c_2$$

$$w_t(C_{22}) = w_t(B_1)c_1 + w_t(C_1)c_2$$

$$w(C_{22}) = w(B_1)c_1 + w(C_1)c_2$$

$$c_1 = \left(\frac{r_{C_2} - r_{C_1}}{r_{B_1} - r_{C_1}}\right)\left(\frac{t_{C_2} - t_{C_1}}{t_{B_1} - t_{C_1}}\right) + \frac{1}{2}cs$$

$$c_2 = \left(\frac{r_{C_2} - r_{B_1}}{r_{C_1} - r_{B_1}}\right)\left(\frac{t_{C_2} - t_{B_1}}{t_{C_1} - t_{B_1}}\right) + \frac{1}{2}cs$$

$$cs = \left(\frac{r_{C_2} - r_{C_1}}{r_{B_1} - r_{C_1}}\right)\left(\frac{t_{C_2} - t_{B_1}}{t_{C_1} - t_{B_1}}\right) + \left(\frac{r_{C_2} - r_{B_1}}{r_{C_1} - r_{B_1}}\right)\left(\frac{t_{C_2} - t_{C_1}}{t_{B_1} - t_{C_1}}\right)$$

We have five equations in five unknowns, namely $w_t(B_2)$, $w_r(B_2)$, $w_t(2,1)$, $w_r(2,1)$ and $w_t(C_2)$. The principle determinant is not zero. So the unknown values are uniquely determined.

$$\Delta_w = \begin{vmatrix} 1 & b_0(B_1) & 0 & 0 & 0 \\ 1 & -a_0 & 0 & 0 & 0 \\ 1 & a_0 & -A_{21} & -A_{22} & 0 \\ 0 & 0 & 1 & -a_0 & 0 \\ 0 & 0 & 1 & a_0 & -A_{23} \end{vmatrix}$$

$$\Delta_w = (a_0 + b_0(B_1))(A_{21}A_{23} + A_{22}A_{23}) \tag{4.63}$$

where

$$A_{21} = 1 - \frac{a_0^2}{r_{21}^2}(t_{21} - t_{C_1})(t_{B_2} - t_{21}) \sin^2 \alpha$$

$$A_{22} = a_0 + \frac{a_0}{r_{21}}(t_{B_2} - t_{21}) - \frac{a_0^2}{r_{21}^2}(r_{21} - r_{C_1})(t_{B_2} - t_{21}) \sin^2 \alpha$$

$$A_{23} = 1 - \frac{a_0}{r_{C_2}}\nu(t_{C_2} - t_{C_1}) \sin \alpha - \frac{a_0^2}{r_{21}^2} \sin \alpha(\nu + \sin \alpha)(t_{C_2} - t_{C_1})(t_{21} - t_{C_2})$$

Since $0 < t_{21} - t_{C_1} < 1$, $-1 < t_{B_2} - t_{21} < 0$, $0 < r_{21} - r_{C_1} < 1$, $0 \leq \sin^2 \alpha \leq 1$, $0 < t_{C_2} - t_{C_1} < 1$, $-1 < t_{21} - t_{C_2} < 0$ then $0 < A_{21}$, $0 < A_{22}$, $0 < A_{23}$. These mean that $A_{21}A_{23} + A_{22}A_{23} > 0$. Also $0 < a_0 + b_0(B_1)$. So the principal determinant $\Delta_w > 0$ and we determine the unknowns $w_t(B_2)$, $w_r(B_2)$, $w_t(2,1)$, $w_r(2,1)$ and $w_t(C_2)$ uniquely.

The second possible location of B_2 is $r_{B_2} = r_{21}$; The difference equations along B_2C_1, B_2C_2 and at point B_2 have the following form

$$w_t(B_2) - w_t(C_1) = a_0(w_r(B_2) - w_r(C_1)) + a(C_1)(t_{B_2} - t_{C_1}) \tag{4.64}$$

$$w(B_2) - w_t(C_2) = a_0(w_r(B_2) - w_r(C_2)) + a(C_2)(t_{B_2} - t_{C_2}) \tag{4.65}$$

$$w_t(B_2) + b_0(B_1)w_r(B_2) = \frac{v_0}{\cos\alpha} - \frac{1+\nu}{1+\sin\alpha}b_0(B_1) \tag{4.66}$$

These equations constitute a closed set of three algebraic equations with respect to three unknowns $w_t(B_2)$, $w_r(B_2)$ and $w_t(C_2)$. The principal determinant is not zero.

$$\Delta_w = \begin{vmatrix} 1 & b_0(B_1) & 0 \\ 1 & -a_0 & 0 \\ 1 & a_0 & -A_{21} \end{vmatrix}$$

$$\Delta_w = (a_0 + b_0(B_1))A_{21} \tag{4.67}$$

where

$$A_{21} = a_0 - \frac{a_0}{r_{C_2}}\nu\sin\alpha(t_{C_2} - t_{C_1}) - \frac{a_0^2}{r_{C_2}^2}\sin\alpha(\nu + \sin\alpha)(t_{C_2} - t_{C_1})(t_{21} - t_{C_2})$$

Since $0 < t_{C_2} - t_{C_1} < 1$ and $-1 < t_{21} - t_{C_2} < 0$ then $0 < A_{21}$. So the determinant of the coefficients $\Delta_w > 0$ and the unknowns $w_t(B_2)$, $w_r(B_2)$, and $w_t(C_2)$ are determined uniquely.

The third possible location for B_2 is $r_{C_2} < r_{B_2} < r_{21}$; The equations along B_2C_2, B_2C_{22} and at B_2 are

$$w_t(B_2) - w_t(C_2) = a_0(w_r(B_2) - w_t(C_2)) + a(C_2)(t_{B_2} - t_{C_2}) \tag{4.68}$$

$$w_t(B_2) - w_t(C_{22}) = a_0(w_r(B_2) - w_r(C_{22})) + a(C_{22})(t_{B_2} - t_{C_{22}}) \tag{4.69}$$

where

$$a(C_{22}) = a_0^2 \left(\frac{w_r(C_{22})}{r_{C_{22}}} - \frac{w(C_{22})}{r_{C_{22}}} \sin^2 \alpha \right)$$

$$w_r(C_{22}) = (1-\nu)\frac{\sin\alpha}{1+\sin\alpha} - \nu \sin\alpha \frac{w(C_{22})}{r_{C_{22}}}$$

$$w(C_{22}) = w(C_1) + w_t(C_{22})(t_{C_{22}} - t_{C_1})$$

$$w_t(C_{22}) = w_t(C_1)\left(\frac{t_{C_{22}} - t_{C_2}}{t_{C_1} - t_{C_2}}\right) + w_t(C_2)\left(\frac{t_{C_{22}} - t_{C_1}}{t_{C_2} - t_{C_1}}\right)$$

$$w_t(B_2) + b_0(B_1)w_r(B_2) = \frac{v_0}{\cos\alpha} - \frac{1+\nu}{1+\sin\alpha} b_0(B_1) \tag{4.70}$$

These three equations constitute a closed set of three algebraic equations with respect to the three unknowns, $w_t(B_2)$, $w_r(B_2)$ and $w_t(C_2)$. The principal determinant is not zero.

$$\Delta_w = \begin{vmatrix} 1 & b_0(B_1) & 0 \\ 1 & -a_0 & A_{21} \\ 1 & a_0 & A_{22} \end{vmatrix}$$

$$\Delta_w = -(a_0 - b_0(B_1))A_{21} - (a_0 + b_0(B_1))A_{22} \tag{4.71}$$

where

$$A_{21} = -d_2 - \frac{a_0}{R}\sin\alpha(t_{C_2} - t_{C_1})d_2 + \frac{a_0^2}{R^2}\sin\alpha(\nu + \sin\alpha)d_2(t_{C_2} - t_{C_1})$$

$$A_{22} = -a_0 + \frac{a_0}{R}\sin\alpha(t_{C_2} - t_{C_1}) + \frac{a_0^2}{R^2}\sin\alpha(t_{C_2} - t_{C_1})(t_{B_2} - t_{C_2})$$

$$d_2 = \frac{t_{C_{22}} - t_{C_2}}{t_{C_1} - t_{C_2}}$$

Since $0 < d_2 < 1$, $0 < t_{C_2} - t_{C_1} < 1$, $-1 < t_{B_2} - t_{C_2} < 0$ then $0 < A_{21}$ and $0 < A_{22}$. So $\Delta_w > 0$. Hence we obtain the unique solution.

On continuing this algorithm we can get the numerical solution to the problem at any point of regions I as well as on the strong discontinuity wave RB. The algorithm gives the unique solution to the problem according to the mathematical induction.

At each interior point of the line $C_{n+1}B_{n+1}$ we have two characteristic relations which yield two linear algebraic equations with respect to the unknowns $w_t(C_{n+1,i})$ and $w_r(C_{n+1,i})$ (see Figure 3.3).

$$w_t(C_{n+1,i+1}) - w_t(C_{n+1,i}) = a_0(w_r(C_{n+1,i+1}) - w_r(C_{n+1,i})) + a(C_{n+1,i})(t_{C_{n+1,i+1}} - t_{C_{n+1,i}}) \tag{4.72}$$

$$w_t(C_{n+1,i+1}) - w_t(C_{n,i}) = a_0(w_r(C_{n+1,i+1}) - w_r(C_{n,i})) + a(C_{n,i})(t_{C_{n+1,i+1}} - t_{C_{n,i}}) \tag{4.73}$$

$$w(C_{n+1,i+1}) = w(C_{n,i+1}) + w_t(C_{n+1,i+1})(t_{n+1,i+1} - t_{n,i+1}) + w_r(C_{n+1,i+1})(r_{n+1,i+1} - r_{n,i+1}) \tag{4.74}$$

At the boundary point B_{n+1} we have three conditions: The same relations (4.72) – (4.74) for $C_{n+1,m}$ (m is the number of interior points of the net in region I) plus the boundary condition (4.35)

$$w_t(B_{n+1}) + b_0(B_n)w_r(B_{n+1}) = \frac{v_0}{\cos\alpha} - \frac{1+\nu}{1+\sin\alpha}b_0(B_n) \tag{4.75}$$

Besides at the point C_{n+1} on the Rt line we have the boundary condition,

$$\frac{\partial w}{\partial r}(C_{n+1}) = (1-\nu^2)\frac{\sin\alpha}{1+\sin\alpha} - \nu\sin\alpha\frac{w(C_{n+1})}{R^2} \tag{4.76}$$

Thus, at the $(n+1)$st. step we get a system of $2m+3$ linear algebraic equations (4.72), (4.73) and (4.74) with respect to $2m+3$ number of unknowns $w_t(C_{n+1})$, $w_r(C_{n+1})$, $w_t(C_{n+1,i})$, $w_r(C_{n+1,i})$ $i=1,2,...,m$, $w_t(B_{m+1})$ and $w_r(B_{m+1})$

The principal determinant of these equations has a form

$$\Delta_w = \begin{vmatrix} 1 & b_0(B_n) & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -a_0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & a_0 & -A_{n+1,1} & -A_{n+1,2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -a_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ . & . & . & . & . & . & \cdots & . & . & . \\ . & . & . & . & . & . & \cdots & . & . & . \\ . & . & . & . & . & . & \cdots & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -a_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & a_0 & -A_{n+1,2n+1} \end{vmatrix}$$

$$\begin{aligned} \Delta_w = (a_0 + b_0(B_n))(A_{n+1,1}A_{n+1,3}A_{n+1,5}\cdots \\ A_{n+1,2n-1}A_{n+1,2n+1} + A_{n+1,1}A_{n+1,3}A_{n+1,5}\cdots \\ A_{n+1,2n-1}A_{n+1,2n}A_{n+1,2n+1} + ... + A_{n+1,1}A_{n+1,4}A_{n+1,6}\cdots \\ A_{n+1,2n}A_{n+1,2n+1} + ... + A_{n+1,2}A_{n+1,3}A_{n+1,5}\cdots \\ A_{n+1,2n-1}A_{n+1,2n+1} + ... + A_{n+1,2}A_{n+1,4}A_{n+1,6}\cdots \\ A_{n+1,2n}A_{n+1,2n+1}) \end{aligned} \quad (4.77)$$

where

$$A_{n+1,1} = 1 - \frac{a_0^2}{r_{n+1,1}^2}(t_{n+1,1} - t_{n,1})(t_{B_{n+1}} - t_{n+1,1})\sin^2\alpha$$

$$A_{n+1,2} = a_0 + \frac{a_0}{r_{n+1,1}}(t_{B_{n+1}} - t_{n+1,1}) - \frac{a_0^2}{r_{n+1,1}^2}(r_{n+1,1} - r_{n,1})(t_{B_{n+1}} - t_{n+1,1})$$

for $l = 2, 3, ..., n$

$$A_{n+1,2l-1} = 1 - \frac{a_0^2}{r_{n+1,l}^2}(t_{n+1,l} - t_{n,l})(t_{n+1,l-1} - t_{n+1,l})(t_{n+1,l-1} - t_{n+1,l}) \sin^2 \alpha$$

$$A_{n+1,2l} = a_0 + \frac{a_0}{r_{n+1,l}}(t_{n+1,l-1} - t_{n+1,l}) - \frac{a_0^2}{r_{n+1,l}^2}(r_{n+1,l} - r_{n,l})(t_{n+1,l-1} - t_{n+1,l}) \sin^2 \alpha$$

$$A_{n+1,2n+1} = a_0 - \frac{a_0}{r_{C_{n+1}}}(t_{C_{n+1}} - t_{C_n}) - \frac{a_0^2}{r_{C_{n+1}}} \sin \alpha(\nu + \sin \alpha)(t_{C_{n+1}} - t_{C_n})(t_{n+1,n} - t_{C_{n+1}})$$

Since $0 < t_{n+1,1} - t_{n,1} < 1$, $-1 < t_{B_{n+1}} - t_{n+1,1} < 0$, $R \leq r_{n+1,j}(j = 1, 2, ...n + 1)$, $0 < r_{n+1,j} - r_{n,j} < 1(j = 1, 2, ..., n + 1)$, $0 < t_{n+1,l} - t_{n,l}, 1(l = 2, 3...n)$, $-1 < t_{n+1,l-1} - t_{n+1,l} < 0(l = 2, 3, ..., n)$, $0 < r_{n+1,l} - r_{n,l} < 1(l = 2, 3, ..., n)$ and $-1 < t_{n+1,n} - t_{C_{n+1}} < 0$ then $0 < A_{n+1,1}$, $0 < A_{n+1,2}$ for $l = 2, 3, ..., n$, $0 < A_{n+1,2l-1}$, $0 < A_{n+1,2l}$ and $0 < A_{n+1,2n+1}$. So the principal determinant $\Delta_w > 0$. Hence the unknowns are determined uniquely.

The convergence of the numerical algorithm can be proven analogous to the Section 3.5. The only difference from the point of view of convergence is in continuity of unknown functions on the radial wavefront which smoothness the problem.

Because of the boundness of the volume of the thesis we don't bring here the results of numerical calculations for $\sigma_r|_{r=R} = 0$.

REFERENCES

[1] Agalarov,D.G. and Nuriyev,B.R. and Rakhmatulin,Kh.A., *Impact by a Cone to the Deformable Threads*, Jour. Prikladnaya Mathematika i Mekhanika ,V.45, No.2, 1981, pp.389-394.

[2] Ames,W.F.,Numerical Methods for Partial Differential Equations, Academic Press, New York, 1977.

[3] Busch,W., Esser,R., Hackbusch,W. and Herrmann, U., *Extrapolation Applied to the Method of Characteristics for a First Order System of Two Partial Differential Equations*, Numer. Math., V.24, 1975, pp.331-353.

[4] Crank,J., *Two Methods for the Numerical Solution of Moving Boundary Problems in Diffusion and Heat Flow*, Quart. Jour. Mech. and Applied Math., V.10, Pt.2, 1957, pp.220-229.

[5] Cristescu,N., *Dynamic Plasticity*, Amsterdam, 1967, pp.288.

[6] Dittrich,S. and Hackbush,W., *A Method of Characteristics Solving the Initial Value Problems of a Hyperbolic Differential Equation of Second Order*, Numer. Math., V.34, 1980, pp.217-234

[7] Fox,L., Numerical Solution of Ordinary and Partial Differential Equations ,1962, Addison-Wesley Publishing Company Inc.

[8] Galin,M.P., *Impact to the Flexible Plate*, Trudi Instituta Mechanici AN SSSR, 1949, pp.93-98.

[9] George,A. and Liu,J.W., Computer Solution Of Large Sparse Positive Definite System, 1981, Prentice-Hall Inc.

[10] Grigoryan,D.M., *Normal Impact to an Unbounded Thin Membrane*, Jour. Prikladnaya Mathematika i Mekhanika V.13, No.3, 1949, pp.227-283.

[11] Grigoryan,S.S. and Grigoryan,D.M., *On the Impact on a Thin Elastic Membrane*, Jour. Prikladnaya Mathematika and Mekhanika V.30, No.6, 1966, pp.1035-1041.

[12] Hackbush,W., *On a Method of Characteristics for Solving a Hyperbolic Equation of Second Order*, Computing, V.20, 1978, pp.47-60.

[13] Higdon,L.R., *Boundary Conditions for Suppressing Rapidly Moving Component in Hyperbolic Systems*, Siam. J. Numer. Anal., V.21, No.3, 1984.

[14] James,M.L., Smith,G.M. and Wolford,S.C., Applied Numerical Methods for Digital Computation with Fortran, 1968, International Textbook Company.

[15] Magamedov,K.M. and Kholodov,A.S., *The Construction of Difference Schemes for Hyperbolic Equations*, USSR, Comput. Maths. Math. Phys., V.9, No.2, 1969, pp.373-386.

[16] Meek,C.P. and Norbury,J. *Nonlinear Moving Boundary Problems and a Keller Box Schemes*, Siam. J. Numer. Anal., V.21, No.5, 1984.

[17] Nuriyev,B.R., *Impact by a Cone to the Flexible Threads in the Presence of a Region of Free Transverse Motion*, Jour. Izvestiya AN Azerbaijan, No.2, 1976, pp.83-87.

[18] Nuriyev,B.R., *Propagation of Nonlinear Waves in the Flexible Threads due to Impact by a Cone*, Jour. Doklady AN Azerbaijan, No.2, 1980, pp.18-25.

[19] Nuriyev,B.R., *Transverse Impact by a Cone to the Elasto-Plastic Membrane*, Jour. Doklady AN AZerbaijan, No.12, 1981, pp.15-19.

[20] Nuriyev,B.R., *Propagation of Stress Waves in a Membrane due to Transverse Impact*, Proceedings of the VII National Congress on Applied Mathematics, USA, Ithaca, N.Y., 1982, p.51.

[21] Nuriyev,B.R., *The Solution to the Problem of Transverse Impact to the Elastic Membrane in the Presence of a Region of Free Transverse Motions*, Jour. Izvestiya AN Azerbaijan, No.6, 1985,pp.24-30.

[22] Nuriyev,B.R., *Propagation of Plastic Deformations in Membranes Under Transverse Impact*, Jour Prikladnaya Mekhanika, V.27, No.12, 1991, pp.72-78.

[23] Nuriyev,B.R. and Hayashi,I., *Transverse Impact to the Elastic Plastic Membranes*, Bulletin of Science and Engineering Research, Waseda University, No.133, 1991, pp.13-26.

[24] Pavlenko,A.L., *Normal Impact to the Flexible Membrane*, Moscow University, 1952, pp.118.

[25] Peradze,D.G., *Difference Scheme fo a Nonlinear Problem on the Bending Plate*, USSR, Comput. Maths. Math. Phys., V.26, No.1, 1986, pp.157-162.

[26] Rakhmatulin,Kh.A., *On the High Velocity Oblique Impact to the Flexible Thread in the Presence of Friction*, Jour. Prikladnaya Mathematika i Mekhanika, V.9, No.6, 1945, pp.449-462.

[27] Rakhmatulin,Kh.A., *On the Impact to the Flexible Threads*, Jour. Prikladnaya Mathematika i Mekhanika, V.11, No.3, 1947, pp.379-382

[28] Rakhmatulin,Kh.A. and Demyanov,Y.A., *Strengh under High Transient Loads*, Moscow, 1961, English translation, Jerusalem, 1966.

[29] Rakhmatulin,Kh.A. and Kerimov,R.A., *On the Transverse Impact to the Membranes*, Jour. Doklady AN SSSR, V.209, 1973, pp.1043-1045.

[30] Rakhmatulin,Kh.A. and Agolarov,D.G., *On the non Onedimensional Automodel Problems in Dynamics of Membranes*, Jour. Izvestiya AN Azerbaijan, No.5, 1977, pp.136-138.

[31] Tanaka,K. and Kurukowa,T., *Ideal Plastic Deformations in Membranes*, Inst. Phys. Ser., V.70, 1984, pp.503-509.

[32] Tikhonov,A.N. and Samarski,A.A., Partial Differential Equations of Mathematical Physics, V.1, 1963, Pergamon Press, New York, London.

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