ALMOST PERIODIC SOLUTIONS OF RECURRENTLY STRUCTURED IMPULSIVE NEURAL NETWORKS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

GÜLBAHAR TOP

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

MARCH 2022

Approval of the thesis:

ALMOST PERIODIC SOLUTIONS OF RECURRENTLY STRUCTURED IMPULSIVE NEURAL NETWORKS

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ABSTRACT

ALMOST PERIODIC SOLUTIONS OF RECURRENTLY STRUCTURED IMPULSIVE NEURAL NETWORKS

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March 2022, 84 pages

This thesis aims to conduct detailed and precise neural networks research with impulses at nonprescribed moments in terms of periodic and almost periodic solutions. Most of the actions in nature modeled by neural networks involve repetitions. Hence periodic and almost periodic motions become crucial. So in this thesis, the existence, uniqueness, and stability of the periodic and almost periodic motion are served for the neural networks with prescribed and nonprescribed impacts. This impulsive system is a neural network with innovative structured impacts that perfectly match the rates. If one regards the impulses as limits of their continuous counterparts, this makes sense for the application. Thus, the novel system also considers the neural networks' nature in the impulsive part since the sudden noises or impact disturbances can affect the rates or activation functions. New conditions on the coefficients have been designed to be more specific and detailed. The constructive stability conditions are delivered directly related to the system's coefficients. A detailed approach is performed to the systems with variable moments of impulses. For the research, the method of B-equivalence is employed, and the relationship between the original and B-equivalent systems was explicitly established and provided. Furthermore, because the impulsive component of the system is inherent to the neural network, the B-equivalent system

also matches the original structure in terms of differential and impulsive parts. One of the novel aspects of this work is that the possibility of negative capacitance in a neurological system is not neglected. Together with the elimination of the capacitance's positivity requirement, the new structure allows for a more thorough study under optimal conditions. The probability of negative capacitance emphasizes the need for impulses to maintain stability.

Keywords: Recurrent impulsive neural networks, Recurrently structured impacts, Neural networks with negative/positive capacitance, Discontinuous almost periodic motions, B-equivalence method, Asymptotic stability

ÖZ

ÖZYİNELEMELİ YAPIDAKİ İMPALSİF SİNİR AĞLARININ HEMEN HEMEN PERİYODİK ÇÖZÜMLERİ

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Mart 2022 , 84 sayfa

Bu tez, sabit ve değişken impals anlarına sahip sinir ağlarının periyodik ve hemen hemen periyodik çözümleriyle ilgili detaylı ve açık bir araştırma yapmayı amaçlamaktadır.

Sinir ağları tarafından modellenen doğadaki eylemlerin çoğu tekrarlıdır. Bu nedenle periyodik ve hemen hemen periyodik sistemler öne çıkmaktadır. Dolayısıyla bu tezde önceden belirlenmiş ve belirlenmemiş impalslara sahip sinir ağlarının periyodik ve hemen hemen periyodik çözümlerinin varlığı, tekliği ve kararlılığı sunulmaktadır. Bu impalsif sistem, diferansiyel denklem parçasına mükemmel şekilde uyan yenilikçi yapılandırılmış impals denklemine sahip bir sinir ağıdır. İmpalslar, sürekli karşılıklarının limitleri olarak kabul edilirse, bu uygulama için anlamlıdır. Böylece, yeni sistem, ani gürültüler veya darbe bozuklukları hızları veya aktivasyon fonksiyonlarını etkileyebileceğinden, sinir ağlarının doğasını dürtüsel kısımda da dikkate alır. Katsayılar üzerindeki yeni koşullar daha spesifik ve ayrıntılı olacak şekilde tasarlanmıştır. Yapısal kararlılık koşulları, doğrudan sistemin katsayıları ile ilgili olacak şekilde ayarlanmıştır. Değişken impals momentlerine sahip sistemlere ayrıntılı bir yaklaşım

gerçekleştirilmiştir. Araştırmada B-equivalence yöntemi kullanılmış, orijinal ve B-equivalent sistemler arasındaki ilişki açık bir şekilde kurulmuş ve sağlanmıştır. Ayrıca, sistemin impalsif bileşeni sinir ağına benzer olduğundan, B-equivalent sistem, diferansiyel ve impalsif parçalar açısından da orijinal yapıyla eşleşir. Bu çalışmanın yeni yönlerinden biri, nörolojik bir sistemdeki negatif kapasitans olasılığının ihmal edilmemesidir. Yeni yapı, kapasitansın pozitiflik koşulunun ortadan kalkmasıyla birlikte, optimum koşullarda daha kapsamlı bir çalışma yapılmasına olanak tanır. Negatif kapasitans olasılığı, kararlılığı sağlamak için impalsların rolünü vurgular.

Anahtar Kelimeler: Özyinelemeli impalsif sinir ağları, Özyinelemeli yapılandırılmış impalslar, Negatif/pozitif kapasitanslı sinir ağları, Süreksiz hemen hemen periyodik hareketler, B-denklik yöntemi, Asimptotik kararlılık

To my beloved family

ACKNOWLEDGMENTS

First of all, I would like to express my gratitude to my supervisor, Prof. Dr. Marat Akhmet, for accepting me as his student in a time of despair for me and lighting my way like a lighthouse. I also would like to thank him for his patient guidance, motivation, and understanding in preparing this thesis and preparing me for academic life.

I want to thank the members of the examining committee, Prof. Dr. Hasan Taşeli, Assoc. Prof. Dr. Kostyantyn Zheltukhin, Prof. Dr. Mehmet Onur Fen, and especially Assoc. Prof. Dr. Duygu Aruğaslan Çinçin for their valuable comments and feedback on my thesis.

I also would like to thank all my instructors and friends, colleagues, and office mates within the Department of Mathematics at METU. I also wish to thank all my close friends who were essential in completing the thesis and express my apology that I could not mention them personally one by one.

It gives me great pleasure to pay tribute to my wonderful mother, Ummahan Top, and father, Ali top, who, directly or indirectly, made my thesis possible. They have always supported me throughout my life. Without their support and unconditional love, I would not be where I am now. I owe thanks to my beloved brother İbrahim Top for his trust and love in me.

Last but not least, I'd like to express my gratitude to Cem Deniz Erim, my darling husband, for his endless love, encouragement, and motivation since the first time we met. Words are not enough to describe the meaning of his presence in my life.

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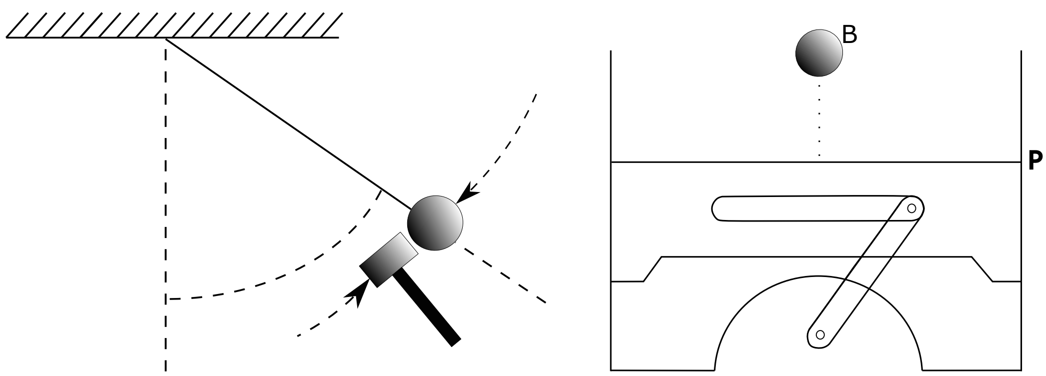
INTRODUCTION AND PRELIMINARIES

1.1 Impulsive Differential Equations

It is well-known that ordinary differential equations are the main mathematical instrument in studying many subjects and disciplines such as physics, chemistry, biology, engineering, and economy applied for predicting motions, the exponential growth or decay for species or diseases, and the investment industry.

In ordinary differential equations, the dynamics are continuous, and on the contrary, not everything is in real world. Indeed, a system can be subject to instantaneous perturbations and abrupt changes at specific instants caused by switching phenomena, frequency change, or some sudden noise. A pendulum, which is simply visualized in Figure 1.1a, hit by a hammer at some point except from its equilibrium point, or a bouncing ball [1] on a sinusoidally vibrating table illustrated in Figure 1.1b serves as examples for sudden velocity change. This arouses the need to study discontinuous dynamics, and the analysis of Krylov and Bogolyubov [2] can be considered one of the first steps in this direction. Inserting piecewise constant arguments and impulses into a continuous system is a path to developing the theory of mathematical models with discontinuities. Pavlidis examined this subject and introduced the term "differential equations with impulses" in his work [3, 4].

Later, Samoilenko and Perestyuk [5] investigated the theory of impulsive differential equations for the existence and uniqueness of solutions, stability, periodicity and almost periodicity properties methodically. In addition to that Lakshmikantham et al. [6] published another book in the theory of impulsive differential equations. For the last several decades, many researchers have been aware of the role



(a) pendulum's velocity suddenly changes after the moment of impact

(b) Mechanical model of a bead bouncing on a vibrating table

Figure 1.1: Two models of discontinuous dynamical systems

of impulsive differential equations in mathematical modeling and solving problems in different areas of research, therefore discontinuous dynamics is of intensively search interest[5, 6, 1, 7, 8]. But systems with variable-moment of impacts are more widespread in practice than fixed-moment impulsive systems. Consequently, Akhmet investigated the differential equations with impulses at non-prescribed moments systematically in his book [1] and proposed the method called B-equivalence that reduces the system with variable moments of impulses to a fixed-moment impulsive differential equation and makes analyzing possible.

Systems of differential equations with fixed moment of impulses consist of two parts: differential equation part and impulse equations [1],

$$\begin{aligned} x' &= f(t,x), \\ \Delta x\big|_{t=\theta_k} &= I_k(x), \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, and the sequence θ_k can be a finite or infinite indexed sequence but throughout this work, θ_k is increasing such that $|\theta_k| \to \infty$ as $k \to \infty$. The jump equations are set by the equation $\Delta x\big|_{t=\theta_k} = x(\theta_k+) - x(\theta_k)$, assuming that the limit $x(\theta_k+) = \lim\limits_{t \to \theta_k^+} x(t)$ exists[1] and $x(t)$ is left-continuous. The left continuity is needed due to mechanical reasons, like just before the hammer hits the pendulum, it has an initial velocity. Theoretically every result in left-continuous case can be applied to right-continuous solution but for the standard notation issues in mathematics and since it is more realistic, the solution is set to be left-continuous. The impact moments $t = \theta_k$ are prescribed before solving the differential equation and does not depend on the solution.

A function $\phi(t)$ is called piecewise continuous function [1, 9] if it is left continuous and continuous except possibly on a set of points θ_k where it has discontinuities of first kind, that is $\lim_{t\to\theta_k^+} \phi(t)$ exists and $\lim_{t\to\theta_k^-} \phi(t) = \phi(\theta_k)$. A piecewise continuous function, $\phi(t)$, with piecewise continuous derivative satisfying $\phi(t_0) = x_0$ is a solution of (1.1) if and only if [1, 9]

$$\phi(t) = \begin{cases} x_0 + \int_{t_0}^{t} f(s, \phi(s))ds + \sum_{t_0\le\theta_k<t} I_k(\phi(\theta_k)), & t \ge t_0, \\ x_0 + \int_{t_0}^{t} f(s, \phi(s))ds - \sum_{t\le\theta_k<t_0} I_k(\phi(\theta_k)), & t < t_0. \end{cases} \tag{1.2}$$

The last equation is called the equivalent integral equation for the impulsive differential equation (1.1). The Gronwall-Bellman Lemma is used in many proofs performed for ordinary differential equations. The following lemma will be used in this study and it is called Gronwall-Belmann Lemma for piecewise continuous functions.

Lemma 1 (Gronwall-Bellmann Lemma for piecewise continuous functions) *[1, 9] Let u(t), v(t) be two nonnegative piecewise continuous functions with discontinuities at the sequence of values θ_k, and let β_k be a nonnegative sequence, $c \in \mathbb{R}$. If the inequality*

$$u(t) \le c + \int_{t_0}^{t} v(s)u(s)ds + \sum_{t_0<\theta_k<t} \beta_k u(\theta_k), \quad t \ge t_0, \tag{1.3}$$

is satisfied, then

$$u(t) \le ce^{\int_{t_0}^{t} v(s)ds} + \prod_{t_0<\theta_k<t} (1 + \beta_k), \quad t \ge t_0. \tag{1.4}$$

Furthermore, if the condition $\beta_k < 1$ is imposed then from the inequality

$$u(t) \le c + \int_{t_0}^{t} v(s)u(s)ds + \sum_{t_0<\theta_k<t} \beta_k u(\theta_k), \quad t < t_0, \tag{1.5}$$

it follows that

$$u(t) \le ce^{\int_{t_0}^{t} v(s)ds} + \prod_{t_0<\theta_k<t} (1 - \beta_k)^{-1}, \quad t < t_0. \tag{1.6}$$

1.1.1 Periodicity of impulsive systems

The system (1.1) is (ω, p)-periodic in t and k uniformly with respect to x if there exists a positive number ω and a natural number p such that

i) impulse moments are (ω, p)-periodic, that is $\theta_{k+p} - \theta_k = \omega$, $k \in \mathbb{Z}$,

ii) function $f(t,x)$ is $\omega-$periodic in t uniformly with respect to x where $t \in \mathbb{R}$, that is $f(t+\omega, x) = f(t,x)$,

iii) sequence of functions for impulses, $I_k(x)$, is $p-$periodic in k uniformly with respect to x where $k \in \mathbb{Z}$, which means $I_{k+p}(x) = I_k(x)$.

When we consider the system with non-fixed moments of impacts, the following will be an additional condition for periodicity of the system

iv) surfaces of discontinuities satisfy $\tau_{k+p}(x) = \tau_k(x)$ for $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$.

1.1.2 Almost periodicity of impulsive systems

In this subsection, the definitions concerning almost periodicity of impulsive differential equations will be introduced. A set of numbers is called *relatively dense* [9], if there is a positive number l, such that any interval of length l contains a member of the set.

A sequence of real or complex n-dimensional vectors a_i, $i \in \mathbb{Z}$, is *almost periodic (a.p.),* if for arbitrary positive ε there exists a relatively dense set $\mathcal{Q}$ of integers $q \in \mathcal{Q}$, that satisfy the inequality $\|a_{i+q} - a_i\| < \varepsilon$ for all $i \in \mathbb{Z}$ [9]. The number q is an ε-almost period of the sequence a_i.

Consider the sequence of real numbers θ_k, $k \in \mathbb{Z}$ in the system (1.1), unbounded in both directions and strictly increasing with respect to the index. Denote $\theta_k^j := \theta_{k+j} - \theta_k$. The sequences θ_k^j, obtained for each integer j, are said to be the *derivative sequences* of θ_k.

The derivative sequences θ_k^j, $k, j \in \mathbb{Z}$, are called uniformly almost periodic in k, if for arbitrary positive ε, there exists a set of ε-almost periods common for all the sequences θ_k^j, $j \in \mathbb{Z}$. If the derivative sequences are uniformly almost periodic then there exist numbers $\underline{\theta} > 0$ and $\overline{\theta} > 0$, satisfying $\underline{\theta} \le \theta_{k+1} - \theta_k \le \overline{\theta}$.

The α-property in the following definition is said to be *conditional uniform continuity* of discontinuous functions, or shortly, *conditional uniform continuity*.

Consider a function $\varphi(t)$ defined on $\mathbb{R}$, that is left-continuous with discontinuities at the moments θ_k, $k \in \mathbb{Z}$, and the discontinuities are of the first kind. The function $\varphi(t)$ is *discontinuous almost periodic* (d.a.p.) [9] if

$\boldsymbol{\alpha}$) for any positive ε, there exists a positive δ such that if $|t_1 - t_2| < \delta$ then $\|\varphi(t_1) - \varphi(t_2)\| < \varepsilon$ where t_1 and t_2 are in the same interval of continuity,

$\boldsymbol{\beta}$) for arbitrary positive ε there exists a respectively dense set $\mathcal{T}$ of ε-almost periods $\tau \in \mathcal{T}$ such that $\|\varphi(t+\tau) - \varphi(t)\| < \varepsilon$ if $t \in \mathbb{R}$ and $|t - \theta_k| > \varepsilon$ for all integer k.

Let $g_k(x)$, $k \in \mathbb{Z}$, be a sequence of functions with the common domain $D \subseteq \mathbb{R}^n$. If for arbitrary $\varepsilon > 0$, there exist relatively dense set $\mathcal{Q}$ of integers, $q \in \mathcal{Q}$, such that $\|g_{k+q}(x) - g_k(x)\| < \varepsilon$ for all $x \in D$, $k \in \mathbb{Z}$, then the sequence $g_k(x)$, is said to be almost periodic uniformly in x.

The system (1.1) is almost periodic if the following holds

i) the derivative sequences θ_k^j, are positive and uniformly almost periodic in k, for all $j \in \mathbb{Z}$,

ii) the function $f(t,x)$ is almost periodic uniformly in x,

iii) the sequence I_k is uniformly periodic in x.

In this thesis, periodic and almost periodic systems are separately investigated in detail but in the title, only almost periodicity is mentioned because if a function or a sequence is periodic, then it is almost periodic, i.e., the set of periodic functions or sequences are subsets of the set of almost periodic functions or sequences respectively. To be more specific, consider a ω-periodic function, $\varphi(t)$, that is $\varphi(t+\ \omega) = \varphi(t)$. For all positive ε choose $\mathcal{T} = \{k\omega\}$, $k \in \mathbb{Z}$, then $\|\varphi(t+\omega) - \varphi(t)\| = 0 < \varepsilon$. Thus, φ is almost periodic.

As an explicit example consider $\varphi(t) = \sin(5t)$ with period $\frac{2\pi}{5}$. Choose $\mathcal{T} = \{\frac{2\pi}{5}k\}$ this set is dense in the set of integers. For all ε, if $q \in \mathcal{T}$ then $q = \frac{2\pi}{5}k$ for some integer k. One can obtain
$|\sin(5(t + \dfrac{2\pi}{5}k)) - \sin(5t)| = |\sin(5t + 2\pi k)) - \sin(5t)| = 0 < \varepsilon$. Then $\sin(5t)$ is almost periodic.

1.2 B-equivalence Method

Systems having variable impact moments are more common in reality than systems with fixed-moment impulses. When we consider sudden changes in the population density of some species due to harvesting or epidemics, it may depend on the number of the species, namely the solution of the system. Consequently, research of the systems with state-dependent impulses becomes essential. The system

$$\begin{aligned} x' &= f(t, x), \\ \Delta x\big|_{t=\theta_k+\tau_k(x)} &= I_k(x). \end{aligned} \tag{1.7}$$

where $x \in G \subset \mathbb{R}^n$ $t \in I \subset \mathbb{R}$, $k \in \mathbb{Z}$, is an impulsive differential system with non-fixed moment of impulses. G is an open connected set, I is an open interval. The functions $f(t, x)$ and $\tau_k(x)$ are continuous on their domains. The impulse moments can be set as $t = \tau_k(x)$, but in this thesis for the theorems and simplicity of proofs the impact moments are set as $t = \theta_k + \tau_k(x)$ which can be arranged easily.

The method of B-equivalence which was introduced by M. Akhmet [1] is an effective instrument that is used in many papers for the analysis of systems with state-dependent impulses [10, 11, 12]. The essence of the method is the reduction of the system with variable moment of impulses to a system with fixed moment of impulses.

Let $x_0(t)$ be the solution of the differential equation $x' = f(t, x)$ satisfying $x_0(\theta_k) = x$, and denote $\xi_k = t = \theta_k + \tau_k(x_0(\xi_k))$ as the meeting moment of the solution with the surface of discontinuity. Consider the system (1.7) together with the following one

$$\begin{aligned} x' &= f(t, x), \\ \Delta x\big|_{t=\theta_k} &= W_k(x). \end{aligned} \tag{1.8}$$

Let $x(t)$ be solution of the system (1.7) and $y(t)$ be solution of (1.8) with the same initial conditions (θ_k, x). The oriented interval $\widehat{(a,b]}$ is denoted as follows: $\widehat{(a,b]} = (a,b]$ if $a < b$ and $(b,a]$ if $b < a$.

We say that the systems (1.7) and (1.8) are B-equivalent in $G \subset \mathbb{R}^n$ if there exists a set $G_1 \subset G$ such that for each solution $x(t)$ of (1.7) defined on an interval U, with discontinuity moments ξ_k and $x(t) \in G_1$, $t \in U$, there is a solution $y(t) : U \to G$ of (1.8) satisfying

$$x(t) = y(t), \quad t \notin \widehat{(\theta_k, \xi_k]}, \quad \text{if } \theta_k \neq \xi_k, \tag{1.9}$$

$$x(\xi_k) = y(\theta_k), \quad \text{if } \theta_k = \xi_k, \tag{1.10}$$

$$x(\theta_k) = y(\theta_k), \quad x(\xi_k+) = y(\xi_k) \quad \text{if } \theta_k < \xi_k, \tag{1.11}$$

$$y(\xi_k) = x(\xi_k), \quad y(\theta_k+) = x(\theta_k) \quad \text{if } \xi_k < \theta_k. \tag{1.12}$$

The solutions may intersect with the surfaces of discontinuities multiple times. To minimize complications in the analysis, one needs to avoid beating against discontinuity surfaces. That is, the solutions must interact with the discontinuity surface exactly once. The conditions for this and the details of obtaining a B-equivalent system, particularly for recurrent neural networks, will be presented in the forthcoming parts of the thesis.

1.3 Neural Networks

Neural networks are inspired from the human nervous system and human brain, accepted as the most powerful computer. Arbib [13] considers the nervous system in three-stages.The receptors, are linked to neurons and convert environmental energy into changes in membrane potential, and it can transmit either actively or passively in the neural network. Once data is collected, the brain, which might be regarded as the head of the neural network and connected to both receptors and effectors, processes the information, and makes the decision. Then the effectors transform the signal from the network into responses as outputs.

Neurons are the structural unit of this neural network called the nervous system. They have different characteristics. They also have basic shared features, for example soma is the body of the neural cell, dendrites branch off the soma, and they are input points of the neurons. In addition, a fiber called the axon whose length might be long or short up to neuron's function, emerges from the cell body. The ends of the axon branches, called nerve terminals, touch other neurons or effectors.

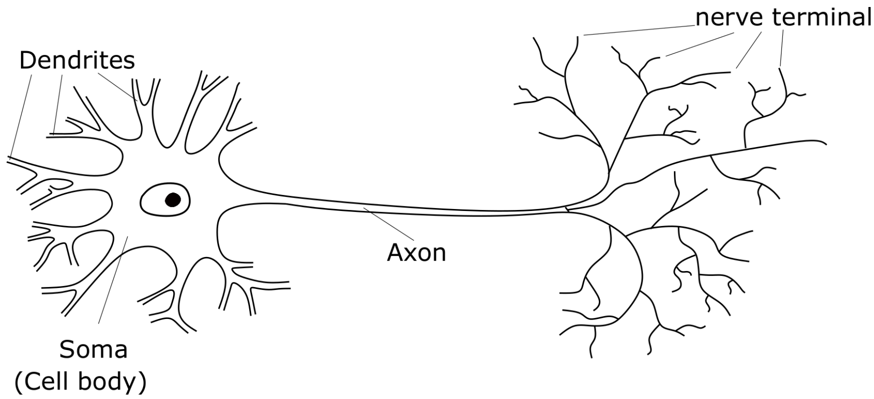


Figure 1.2: A basic neuron

The impulses are received by dendrites and transmitted to soma. Soma processes the signal and a threshold function takes a part here, if the input impulse is big enough an output is generated and delivered to the other neurons by axon through connections named synapses. Shepherd and Koch [14] estimated the number of neurons in the human cortex as approximately 10 billion and the number of synapses or connections as 60 trillion, thus compared to computers brain has a very powerful system.

One of the first mathematical models of neurons was developed by McCulloch and Pitts [15], who considered the characteristic of neural activity as all or none. They revealed how to use excitation, inhibition, and threshold to form various neurons. Each connection between two neurons as output from one to the other as input has an attachment weight. Moreover, if the weight is positive, a synapse is excitatory and inhibitory if the weight is negative. Also, each neuron is associated with a threshold. If the sum of the product of inputs with corresponding synapse weights is larger than the threshold, then the subsequent output is one; otherwise, it is zero. The mathematical neuron model introduced by McCulloch–Pitts, and has been used throughout most neural-network models, is shown in the Figure 1.3, where x_i represents input, w_i stands for weight of the input x_i, and θ represents the threshold function while y stands for the output.

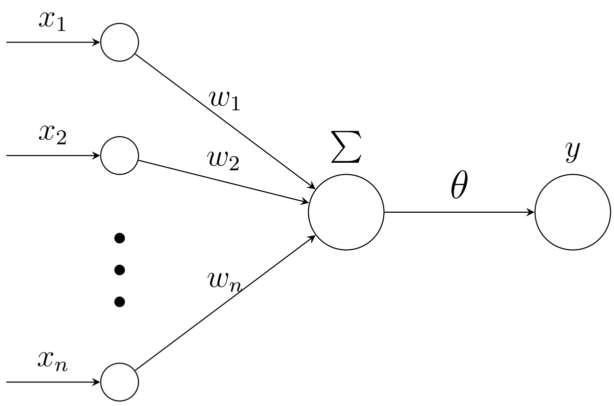


Figure 1.3: The McCulloch–Pitts' mathematical neuron model

Arbib [13] mentions that this model is used in neural computing and played a crucial role in developing automata theory, whereas it is no longer active in computational neuroscience.

Hebb [16] proposed a scenario where the activation of two neurons simultaneously strengthens the connection between them. The predicament with this hypothesis would be that the synapses would become stronger over time until they saturated, removing any connection selectivity. This dilemma was eliminated by the normalization of synapses by Malsburg [17], and to suppress the activity of neighboring cells, he employed lateral inhibition.

In 1958, Rosenblatt [18] pursued a question to form a connection between biological systems and the physical world, leading him to a model he named perceptron. The hypothesis devised was a theoretical nerve system or machine. The perceptron is intended to demonstrate some of the fundamental aspects of intelligent systems in general, rather than becoming too confusing in the specific and often unknown conditions that apply to specific biological organisms.

In time, McCulloch and Pitts' studies were improved by several researchers [19]. Real neurons and the physical devices simulating them have some continuous properties. Thus, Hopfield [19] came up with an elaborated model. Hopfield's model demonstrates that the importance of McCulloch and Pitts' model is preserved while avoiding the simplification caused by all or none features. Neurons have two state, related to the output V_i: 0 or 1. The total input to each neuron is exhibited as

$$\text{input to } i = H_i = \sum_{j \neq i} T_{ij} V_j + I_i, \tag{1.13}$$

where the external inputs I_i and the input from other neurons V_j, with the synaptic connection weight T_{ij} are two type of source of inputs. The continuous Hopfield model is loyal to original model. Continuous and monotone-increasing output V_i which is a function of u_i that is the instantaneous input and the range varies as $V_i^0 \leq V_i \leq V_i^1$. The rate of change of u_i is represented by the equation

$$\begin{aligned} C_i \frac{du_i}{dt} &= \sum_j T_{ij} V_j - \frac{u_i}{R_i} + I_i \\ u_i &= g_i^{-1}(V_i) \end{aligned} \tag{1.14}$$

where $g_i(u_i)$, depicting a nonlinear amplifier's input-output characteristic within a short time, can be used to give inverse output-input relation $g_i^{-1}(V_i)$. The components C_i, R_i and T_{ij} are input capacitance, membrane resistance and impedance between the output V_j and the cell i, respectively. The term I_i is external input to neuron i. Besides, the electrical circuit without capacitance and resistances can be seen in Figure 1.4 [19].

Once the Hopfield network's input signals are analog and the output signals are discrete values, it can be utilized as an effective interface between analog and digital devices [20]. Furthermore, the Hopfield network's ability to minimize energy is used to solve optimization problems.

Another famous neural network model is the Cohen-Grossberg model. It was introduced by Cohen and Grossberg [21] in 1983, and according to Cao [22], Hopfield neural networks are one of the special cases of Cohen-Grossberg models. The corresponding dynamical system for this model is [22]

$$x_i' = a_i(x_i) \left[b_i(x_i) - \sum_{k=1}^{n} c_{ik} d_k(x_k) \right], \quad i = 1, \ldots, n, \tag{1.15}$$

where the components c_{ij} form a symmetric matrix.

Cohen-Grossberg [21] proposes a theorem for determining this particular type of neural network's stability. They claim that the ability of the network to preserve the activity pattern is left invariant by self-organization, by an absolute stability theorem. As a

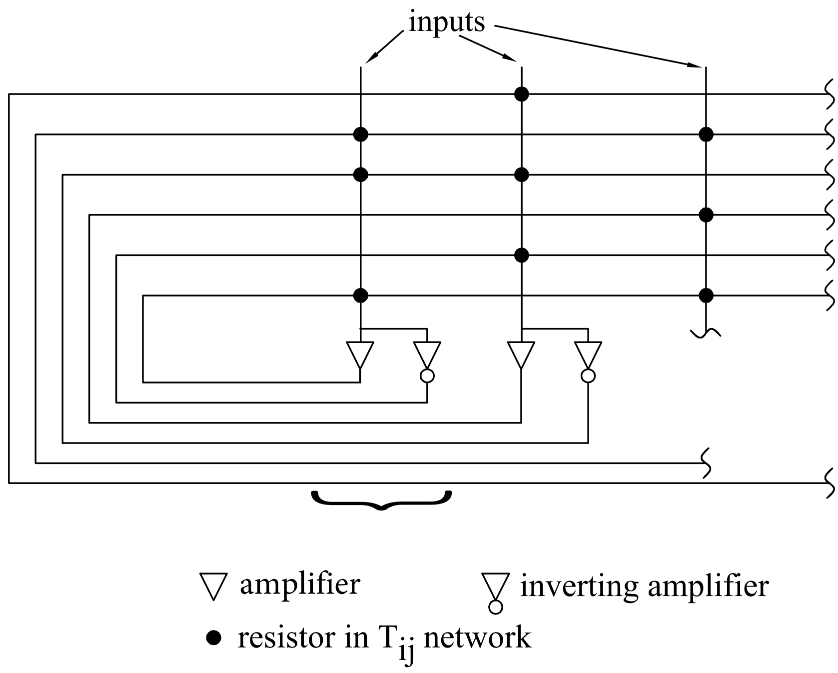


Figure 1.4: Electrical circuit that corresponds to Eq. (1.14) without input capacitance and resistances

result, identifying an absolutely stable class of systems restricts the self-organization processes that a system can interact without getting destabilized in particular input conditions.

In 1988, Chua and Yang [23, 24] introduced a new class of neural networks called cellular neural networks (CNNs). Moreover, they examined the structure of the cellular neural networks in detail. In cellular neural networks, a cell is the fundamental circuit unit. Cellular neural networks are neural networks that are locally interconnected, frequently repeated, and possibly multilayered. Each layer is divided into a two-dimensional array of processing units known as cells $C(i, j)$ with each cell acting as a dynamical subsystem that is connected to its neighbors. For an integer r, the r-neighborhood of a cell $C(i, j)$ is defined [23, 24] as

$$N_r(i, j) = \{C(k, l) : max\{|k - i|, |l - j|\} < r, 1 \leq k \leq M; 1 \leq l \leq N\}. \quad (1.16)$$

Linear capacitors, linear resistors, linear and nonlinear controlled sources, and independent sources are examples of linear and nonlinear circuit parts. Cellular neural networks have a structure such that adjacent cells can communicate with one another directly. Because of the transmission impacts of the continuous-time dynamics of cellular neural networks, cells that are not directly coupled may affect each other indirectly. As information-processing systems, CNNs have offered significant promise.

Bouzerdoum and Pinter [25], recognizing the broad application of shunting inhibition in neural networks [26, 27, 28, 29, 30, 31, 32], described a new class of neural networks. They employed the following ordinary differential equation

$$x'_{ij} = L_{ij}(t) - a_{ij}x_{ij} - \sum_{C(k,l)\in N_r(i,j)} c_{ij}^{k} l f(x_{kl})x_{ij}, \tag{1.17}$$

where x_{ij} is the activity of the cell $C(i,j)$, can be considered as the difference between the resting potential and the membrane voltage; $L_{ij}(t)$ is the external input to the cell; the constant $a_{ij} > 0$ represents rate of cell activity when it is not active; c_{ij} is the connection weight of the activity of $C(k,l)$ l transmitted to $C(i,j)$; $f(x_{kl})$ is a positive continuous function named activation function and represents the cell $C(k,l)$'s output rate. They analyzed the dynamical properties of shunting inhibitory neural networks and analyzed stability results.

Neural networks are being studied for their application in image and signal processing, vision pattern recognition, and optimization [29, 33, 34, 35]. Due to this attraction, numerous articles have been composed since 1980s [23, 24, 33, 34, 36, 37, 38, 39, 40, 41, 42, 43, 44]. In the paper [45], periodicity and global exponential stability of recurrent neural networks with time delays were investigated and it is remarked that recurrent neural networks are important because of their feedback connections, dynamic properties and application in different fields. Recurrent neural networks are used in image, speech and knowledge processing, associative memories, pattern recognition, optimization, cryptography [46, 47, 48].

1.4 Impulsive Neural Networks

The state of the network can be exposed to sudden perturbations and abrupt changes at some points in execution. In other words, the impulses model sudden disturbances and can also be employed as an impact control for cognitive processing. All of this encourages the study of impulsive neural networks. Mathematical models of neural networks were productively developed with insertion of piecewise constant arguments and impulses [27, 49, 50, 51, 52, 53, 54, 55].

Many biological and cognitive activities such as heartbeat, respiration, mastication, and memorization require repetition. Thus periodicity and almost periodicity are among the essential properties to study for the dynamics of neural networks that got increasingly attractive considering periodic and almost periodic solutions [5, 27, 49, 50, 51, 52, 56].

1.4.1 Periodic neural networks

The repetition in nature raises the need for periodic solutions when we study their mathematical models. In [57] the stability properties for periodic solution of Bidirectional Associative Memory (BAM) neural networks with impulses was analyzed. The periodic solutions of Cohen–Grossberg BAM neural networks with impulses on time scales are studied in [58] for the stability and existence.

1.4.2 Almost periodic neural networks

As mentioned in [12], dynamics of neural networks become more realistic if they involve almost periodic environment and motions. Also in [27], it is claimed that when environmental conditions are taken into account, the assumption of almost-periodicity becomes more realistic and broad. As a result, almost periodicity is a fascinating research topic in the study of neural networks. Various aspects of the theory of almost periodic solutions for neural networks are considered in papers [29, 60, 61, 62, 63, 64, 65, 66].

1.5 Stability of Linear Impulsive Equations

In this section, we will be concerned with the asymptotic stability of the system

$$\begin{aligned} &x' = a(t)x, t \neq \theta_k, \\ &\Delta x\big|_{t=\theta_k} = b_k x, \end{aligned} \tag{1.18}$$

where θ_k is increasing and $|\theta_k| \to \infty$ as $k \to \infty$, with constructive conditions depending directly on the coefficients, $a(t)$ and b_k, of the system. One can easily get that the equality

$$x(t,s) = e^{\int_s^t a(u)du} \prod_{s \le \theta_k < t} (1 + b_k), \quad t > s, \tag{1.19}$$

stands for the transition solution [1] of the scalar system (1.18).

If the inequality,

$$|x(t,s)| \le K e^{\gamma(t-s)}, \tag{1.20}$$

is satisfied with negative γ, then the zero solution of the system (1.18) is asymptotically stable [1].

In the forthcoming lemmas, we will examine the conditions for the inequality (1.20), consequently the sufficient conditions for asymptotic stability of the system (1.18) will be explained.

Lemma 2 *Suppose that the following conditions are satisfied:*

(i) *there is a number χ, such that $a(t) \le \chi$ for all $t \in \mathbb{R}$;*

(ii) *the inequality, $\underline{\theta} \le \theta_{k+1} - \theta_k$, holds for arbitrary integer k with a positive number $\underline{\theta}$;*

(iii) *there is a positive number q such that the inequality $1 < |1 + b_k| \le q$ holds for every integer k.*

Then one gets the inequality (1.20) with $\gamma = (\chi + \dfrac{\ln q}{\underline{\theta}})$ and $K = e^{\ln q}$.

Proof. Fix the function $\kappa(t,s)$ as

$$\kappa(t,s) = \int_s^t a(u)du + \sum_{s \le \theta_k < t} \ln|1 + b_k|. \tag{1.21}$$

Then by (1.19) one can get that the following equation is valid:

$$|x(t,s)| = e^{\kappa(t,s)}. \tag{1.22}$$

Define $i([s,t])$ as the number of the discontinuity points, that belong to the interval $[s,t]$, given in the system (1.18). Using the following inequality,

$$i([s,t]) \leq \frac{t-s}{\underline{\theta}} + 1, \quad s \leq t, \tag{1.23}$$

which results from the second condition of the lemma, one can get the inequality that we are looking for,

$$\begin{aligned} \kappa(t,s) &= \int_s^t a(u)du + \sum_{s \leq \theta_k < t} \ln|1+b_k| \\ &= \chi(t-s) + i([s,t]) \ln q \\ &\leq (\chi + \frac{\ln q}{\underline{\theta}})(t-s) + \ln q. \end{aligned} \tag{1.24}$$

Combining the relations (1.24) and (1.22) we obtain

$$|X(t,s)| = e^{\kappa(t,s)} \leq e^{\ln q} e^{(\chi + \frac{\ln q}{\underline{\theta}})(t-s)}. \tag{1.25}$$

That is the inequality (1.20) is correct with $\gamma = (\chi + \dfrac{\ln q}{\underline{\theta}})$ and $K = e^{\ln q}$, and this is precisely the assertion of the lemma. □ The expression $e^{\ln q}$ in previous lemma and e^{-lnq} in the forthcoming lemma can be rewritten as q and $\frac{1}{q}$ respectively. But to keep the exponential form in 1.22 we did not simplify.

Lemma 3 *Assume that the following conditions hold:*

(i) *there exists a real number χ such that $a(t) \leq \chi$ for all $t \in \mathbb{R}$;*

(ii) *there exists a number $\overline{\theta}$ such that $0 < \theta_{k+1} - \theta_k \leq \overline{\theta}$, for each $k \in \mathbb{Z}$;*

(iii) *the inequality, $|1+b_k| \leq q < 1$ is satisfied for all integers k and a number q.*

Then, the inequality (1.20) is satisfied with $\gamma = (\chi + \dfrac{\ln q}{\overline{\theta}})$ and $K = e^{-\ln q} > 0$.

Proof. Let the number of the discontinuity points of the system (1.18) that belong to the interval $[s,t]$, be denoted by $i([s,t])$, as in previous lemma. The condition (ii) implies the inequality

$$i([s,t]) \geq \frac{t-s}{\overline{\theta}} - 1. \tag{1.26}$$

By using the last inequality and the equality (1.21), we can assert that

$$\begin{aligned}\kappa(t,s) &= \int_s^t a(u)du + \sum_{s\le\theta_k<t} \ln|1+b_k| \\ &= \chi(t-s) + i([s,t])\ln q \\ &\le (\chi + \frac{\ln q}{\bar{\theta}})(t-s) - \ln q. \qquad (1.27)\end{aligned}$$

Thus, we obtain the inequality (1.20) with $\gamma = (\chi + \frac{\ln q}{\bar{\theta}})$ and $K = e^{-\ln q}$. □

Throughout the thesis, the norm is fixed as $\|u\| = \max_{1\le i\le m} |u_i|$, $u \in \mathbb{R}^m$, and its corresponding matrix norm as $\|A\| = \max_{1\le i\le m} \sum_{j=1}^{m} |a_{ij}|$, where A is an $m\times m$ real constant matrix.

If we denote $X(t,s) = diag[x_1(t,s), x_2(t,s), \ldots, x_m(t,s)]$, it is the transition matrix for the system (1.18), where $x_i(t,s)$, $i = 1,2,\ldots,m$, are the solutions for each subsystem in (1.18) satisfying $x_i(s,s) = 1$. In order to get asymptotic results, the following condition will be applied throughout the paper.

(C1) $\|X(t,s)\| \le \max_{1\le i\le m} |x_i(t,s)| \le Ke^{\gamma(t-s)}$ with constants $K \ge 1$ and $\gamma < 0$.

Assume that in the system (1.18) for each $i = 1,\ldots,m$, the conditions of either Lemma 2 or Lemma 3 are valid such that the inequalities $|x_i(t,s)| \le k_i e^{\gamma_i(t-s)}$ are correct with $\gamma_i < 0$. Then the condition (C1) is satisfied, where $K = \max_{1\le i\le m} k_i$ and $\gamma = \max_{1\le i\le m} \gamma_i$.

1.6 The Elaboration of the Recurrently Structured Impulsive Neural Network Model

1.6.1 The review of the literature

Significant interest in the theory of recurrent neural networks is attracted to the systems of the form [63, 67],

$$x_i' = a_i(t)x_i + \sum_{j=1}^{m} b_{ij}(t)f_j(x_j) + c_i(t), \qquad (1.28)$$

where $x_i = x_i(t)$ corresponds to the membrane potential of the unit with $i = 1,2,\ldots,m$, $f_{ij}(.)$, $i,j = 1,2,\ldots,m$, denote measure of response or activation to their incoming potentials, b_{ij}, $i, j = 1, 2, \ldots, m$, are the synaptic connection weights of the unit j with the unit i, the functions $c_i(t)$, $i = 1, 2, \ldots, m$, correspond to the external input from outside the network to the unit for $i = 1, 2, \ldots, m$, the coefficients $a_i(t)$, $i = 1, 2, \ldots, m$, are the rates with which the units self-regulate or reset their potentials when isolated from other units and inputs.

In implementations, the state of the network can be subject to instantaneous perturbations and experiences abrupt changes at certain instants, which may be caused by switching phenomenon, frequency change, or other sudden noise. That is, the impulses realize modeling of impact disturbances, and moreover can be used as an impact control for the neural processes. All this stimulates the development of the impulsive neural networks.

In paper [46], the recurrent neural network (1.28) was extended by adding jumps for the membrane potentials, and the following model

$$\begin{aligned} x_i' &= a_i x_i + \sum_{j=1}^{m} b_{ij} f_j(x_j) + c_i, \, t \neq \theta_k, \\ \Delta x_i\big|_{t=\theta_k} &= I_i(x_i), \, i = 1, 2, \ldots, m, \end{aligned} \tag{1.29}$$

with impulses was considered for existence and global stability of equilibrium solutions.

Also in [68], stability and periodicity of the following impulsive neural networks with delays

$$\begin{aligned} & x' = -x + Af(x) + Bf(x(t-\tau)) + u, \, t \neq t_k \, , \, t \geq 0, \\ & \Delta x(t_k) = I_k(x(t_k)), \quad k = 1, 2, \ldots, m, \end{aligned} \tag{1.30}$$

which is represented in matrix-vector form is researched. In the systems 1.29 and 1.30, it is seen that the structure of the impulse action is different than that of the differential equation. If impact actions are considered as limits of continuous dynamics, it is clear that the jump equation must admit the functional structure related to the differential equation of the neural network. For the applications one needs the structure of the jump equation to be developed, more complex, and what is important adequate to the differential equation. Hence, it is of great interest to consider neural networks with the structure of the impulses identical to that of the rate. That is, the components of the impulsive equation must coincide with that one for the differential equation. This will be exemplified in what follows, and one of the first steps in the direction is made in the paper [53], and impulses in the system (1.29) are refined as follows,

$$
\begin{aligned}
x_i' &= a_i(t)x_i + \sum_{j=1}^{m}\{b_{ij}(t)f_j(x_j) + c_{ij}(t)g_j((k_{ij} * x_j)(t))\} + \gamma_i(t), t \neq \theta_k, \\
\Delta x_i\big|_{t=\theta_k} &= \sum_{j=0}^{m} A_{ij}(k)x_j + I_j(x) + \mu_i(k),\ i = 1, 2, \ldots, m,
\end{aligned} \tag{1.31}
$$

and the existence and stability of almost periodic solutions were studied.

Nevertheless, the impulsive equation in the last model still is not adequate completely to the structure in the differential equation. This is also true for the paper [50], where the neural network

$$
\begin{aligned}
x_i' &= \sum_{j=1}^{m} a_{ij}(t)x_j + \sum_{j=1}^{m} \alpha_{ij}(t)f_j(x_j(t-h)) + \gamma_i(t), t \neq \theta_k, \\
\Delta x\big|_{t=\theta_k} &= A_k x + I_k(x) + \gamma_k,\ \ i = 1, 2, \ldots, m,
\end{aligned} \tag{1.32}
$$

was studied, and the existence and uniqueness of almost periodic solutions were considered. In the systems from (1.28) to (1.31), the potential resetting rates $a_i(t)$ with $i = 1, 2, \ldots, m,$ are assumed to be *negative* [46, 53].

In the papers [69, 70] the neural network with variable moment of impulses were considered, and the method of B-equivalence is used, but they have some deficiencies and mistakes about application of the method. Firstly, construction of B-equivalent system is not given explicitly because it is not obtained in the correct order of steps in [69]. The operator that gives the reduced system is not formulated, consequently it is not formed correctly. Lipschitz constant for the nonlinearity term in the reduced system's impulsive part, $k(l)$ is not precise, which is a major deficiency. The deficiency of estimation of the relationship between the original jump operator in the state-dependent impulsive system and the new jump operator in the corresponding fixed-time impulsive system has been fixed in the paper [10]. The reduced system in [69] seems to be given row-by-row, but it includes a matrix added so there is a conceptual mistake about dimensions. The most important part is that the contribution of impulsive terms into stability is not seen. This is why it is crucial to provide a complete research through B-equivalence, and we are motivated for the thorough and careful study of the problem, which is free from mistakes.

1.6.2 The model under research

The research of the present work is motivated by the results in [27, 46, 49, 50, 51, 52, 53, 54, 55, 56, 59, 71]. Since the impulses are the instruments that arise from impacts or disturbances on the continuous dynamics, it is very natural to expect the impacts to imitate the form of the rates. With this in mind, we newly suggest to investigate a system such that the equations for the impulses are of the *same structure* as the differential equations. We also describe the physical perception of the impacts, at first time in literature. Furthermore, we do not neglect the possibility of *negative capacitance* [72, 73, 74], which has not been considered in the literature on neural networks. These novelties open the model for a more thorough analysis and optimized processes. The main tasks of the research are the existence, uniqueness and asymptotic stability of discontinuous almost periodic solutions.

In the present research, we are encouraged by the results in [46, 50, 53] to consider a model with the impulsive part, which is precisely identical in its components to the differential equation. The following neural model is in the focus of our study,

$$\begin{aligned} x_i' &= a_i(t)x_i + \sum_{j=1}^{m} b_{ij}(t)f_j(x_j) + c_i(t), \, t \neq \theta_k + \tau_k(x), \\ \Delta x_i\big|_{t=\theta_k+\tau_k(x)} &= d_{ik}x_i + \sum_{j=1}^{m} e_{ijk}I_i(x_j) + h_{ik}, \quad i = 1, 2, \ldots, m, \end{aligned} \tag{1.33}$$

where $x_i \in \mathbb{R}$ with $i = 1, 2, \ldots, m$, $t \in \mathbb{R}$, and the sequence θ_k is increasing such that $|\theta_k| \to \infty$ as $k \to \infty$. The jump equations are set by $\Delta x_i\big|_{t=\theta_k} = x_i(\theta_k+) - x_i(\theta_k)$, assuming that the limit $x_i(\theta_k+) = \lim_{t\to\theta_k^+} x_i(t)$ exists.

The coefficients $a_i(t)$, $b_{ij}(t)$, $c_i(t)$ and $f_{ij}(\cdot)$ for $i, j = 1, 2, \ldots, m$, are of the type and the role as in (1.28), continuous and bounded functions, defined on the real line. The continuous functions $I_j(\cdot)$ with $j = 1, 2, \ldots, m$, denote a measure of impact response to its incoming potentials, a member of the sequence e_{ijk} denotes the instantaneous synaptic connection weight of the unit j on the unit i, the sequence member h_{ik} corresponds to the external impulsive input from outside the network to the unit i for $i, j = 1, 2, \ldots, m$, the coefficient d_{ik}, $i = 1, 2, \ldots, m$, is the impulsive rate with which the unit self-regulates or resets its potential when isolated from other units and inputs. The function $I_i(\cdot)$ and the sequences d_{ik}, e_{ijk} and h_{ik} are impact analogues of their counterparts in the equation (1.28), that is the functions $f_j(\cdot)$, $a_i(t)$, $b_{ij}(t)$ and $c_i(t)$.

Apart from the previous studies, we do not demand that the rates are necessarily negative. Differently, in our model the coefficients $a_{ij}(t)$ can be positive, negative or zero valued, that is we do not directly assume stability we give constructive conditions on the coefficients of the system, only the condition of almost periodicity is assumed, and it is the most general assumption on electrical properties of the neural networks. The rates presumed to be *positive* as well as negative, issuing from the possibility of the negative capacitance [72, 73, 74] in electrical circuits. This is one of the novelties of the present work, which makes the mathematical analysis more challenging and the model flexible for applications. One of the novel aspects of this research is that the possibility of negative capacitance is not ignored. The benefit of negative capacitance is that even if the differential equation is unstable the impulsive system might be stable by contribution of impulses in stability. As a result, we get a more general problem, which includes impulsive systems with unstable differential equations but stable solutions. This is due to the inclusion of impact elements in the stability equation.

One can easily see that the network (1.33) is the most developed, if one compares it with the equations (1.29)-(1.32), since it is recurrently structured, and it is clear that system (1.33) covers all previous models and many other conservative networks. That is, we have established the coherence of the differential and impulsive parts of the neural network. The recurrence is present in both, the differential and the impulsive equation, this is why, we call the model under discussion as the *recurrent impulsive neural network* (RINN).

1.6.3 Matrix-vector presentation

Consider the RINN (1.33) introduced in Section 1.6.2. For simplicity of notation, we examine the system (1.33) in the matrix form as follows

$$\begin{aligned} &x' = \mathcal{A}(t)x + \mathcal{B}(t)F(x) + \mathcal{C}(t),\ t \neq \theta_k + \tau_k(x), \\ &\Delta x\big|_{t=\theta_k+\tau_k(x)} = \mathcal{D}_k x + E_k I(x) + H_k, \end{aligned} \tag{1.34}$$

where $x = [x_1, \ldots, x_m]^T \in \mathbb{R}^m$, $t \in \mathbb{R}$, $k \in \mathbb{Z}$, and T is the notation for the transpose operation of a vector, the $m \times m$ matrix-functions $\mathcal{A}(t)$, $\mathcal{B}(t)$ defined on real numbers are,

$$\mathcal{A}(t) = \begin{bmatrix} a_1(t) & 0 & \ldots & 0 \\ 0 & a_2(t) & \ldots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \ldots & 0 & a_m(t) \end{bmatrix}, \quad \mathcal{B}(t) = \begin{bmatrix} b_{11}(t) & b_{12}(t) & \ldots & b_{1m}(t) \\ b_{21}(t) & b_{22}(t) & \ldots & b_{2m}(t) \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1}(t) & b_{m2}(t) & \ldots & b_{mm}(t) \end{bmatrix}.$$

The $m \times m$ matrix sequences D_k are set by

$$D_k = \begin{bmatrix} d_{1k} & 0 & \ldots & 0 \\ 0 & d_{2k} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & d_{mk} \end{bmatrix}, \quad E_k = \begin{bmatrix} e_{11k} & \ldots & e_{1mk} \\ e_{21k} & \ldots & e_{2mk} \\ \vdots & & \\ e_{m1k} & \ldots & e_{mmk} \end{bmatrix}, \; k \in \mathbb{Z}.$$

Lastly, $m \times 1$ vector-functions $F(x)$, $\mathcal{C}(t)$, and $I(x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^m$, and $m \times 1$ vector-sequence H_k, $k \in \mathbb{Z}$, stand for

$$F(x) = \begin{bmatrix} f_1(x_1) \\ f_2(x_2) \\ \vdots \\ f_m(x_m) \end{bmatrix}, \quad \mathcal{C}(t) = \begin{bmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_m(t) \end{bmatrix}, \quad I(x) = \begin{bmatrix} I_1(x_1) \\ I_2(x_2) \\ \vdots \\ I_m(x_m) \end{bmatrix}, \quad H_k = \begin{bmatrix} h_{1k} \\ h_{2k} \\ \vdots \\ h_{mk} \end{bmatrix}.$$

Fix a positive H, from now on we will use the following notations:

$$\alpha = \sum_{i=1}^{m} \max_{1 \le i \le m} \sup_{t \in \mathbb{R}} |a_i(t)| < \infty, \qquad \beta = \max_{1 \le i \le m} \sup_{t \in \mathbb{R}} \sum_{j=1}^{m} |b_{ij}(t)|,$$

$$\sigma = \sum_{i=1}^{m} \sup_{t \in \mathbb{R}} |c_i(t)| < \infty, \qquad m_f = \max_{1 \le i \le m} \sup_{\|u\| < H} |f_i(u)|,$$

$$m_I = \max_{1 \le i \le m} \sup_{\|u\| < H} |I_i(u)|, \qquad m_h = \max_{1 \le i \le m} \sup_{k \in \mathbb{Z}} |h_{ik}|,$$

$$m_\tau = \sup_{k \in \mathbb{Z}} \sup_{\|x\| < H} = |\tau_k(x)|, \qquad d = \max_{1 \le i \le m} \sup_{k \in \mathbb{Z}} |1 + d_{ik}|.$$

$$E = \max_{1 \le i \le m} \sup_{k \in \mathbb{Z}} \sum_{j=1}^{m} |e_{ijk}|.$$

1.7 The Outline of the Thesis

The remaining part of this dissertation is organized as follows. In Chapter 2, the B-equivalent system of our main model is reduced in a detailed way such that it will be utilized in Chapter 3 and 5.

In Chapter 3, conditions for the periodicity of the impulsive neural network is introduced in detail. Then the B-equivalent system is studied for the proof of stability and periodicity of its solution. Since the solutions of the B-equivalent systems are equal to each other, the solution of the system with variable moments of impacts has the same property on some certain set. Thus what is left is to prove that the main model is satisfying stability and periodicity on the intersection of exterior of that set and the domain of solution.

Chapter 4 can be considered as a preparation for Chapter 5 and it is concerned with almost periodicity and stability of the recurrent neural network (RNN) with fixed moment of impulses.

In Chapter 5, the almost periodicty conditions are explained. Then the B-equivalent system is studied for the proof of stability and periodicity of its solution. The impulsive neural network's periodicity criteria are thoroughly discussed. If systems are B-equivalent, the solution of the system with varying moments of impulses has the same characteristic on a particular set. The only thing remaining is to show that our model is stable and periodic on the intersection of that set's exterior and the domain of solution.

Lastly, in Chapter 6, we summarize the outcomes of this thesis and provide some conclusion comments.

CHAPTER 2

B-EQUIVALENCE METHOD

In this part, we will summarize B-equivalence and give the necessary conditions for the absence of beating in the impulsive differential equtions.

2.1 The Absence of Beating

The study of the systems with variable moments of impulses is more complicated than those with the fixed impulses, not only because of the unpredictability of the impact moments but also the possibility of beating. That is, the solutions might intersect with the surfaces of discontinuities multiple times. In order to avoid difficulties in the analysis of stability and almost periodicity, one should prevent beatings against the surfaces of discontinuities. In other words the solutions must meet the surface of discontinuities exactly once. In this subsection, we propose the conditions that guarantee the meeting of solutions with the surfaces of discontinuity exactly once. The following assumptions, (C2)-(C6), are needed throughout the paper.

(C2) $|f_i(u) - f_i(v)| \leq \ell_f|u - v|$, for $|u| < H$, $|v| < H$, $i = 1, \ldots, m$;
(C3) $|I_{ij}(u) - I_{ij}(v)| \leq \ell_I|u - v|$, for $|u| < H$, $|v| < H$, $i, j = 1, \ldots, m$;
(C4) $|\tau_k(x) - \tau_k(y)| \leq \ell_\tau\|x - y\|$, for $\|x\| < H$, $\|y\| < H$, $k \in \mathbb{Z}$;
(C5) $\ell_\tau(\alpha h + \beta m_f + \sigma) < 1$;
(C6) $\tau_k\left((I + D_k)x + E_kI(x) + H_k\right) \leq \tau_k(x)$, for all $k \in \mathbb{Z}$, and $x \in \mathbb{R}^n$;
(C7) $m_\tau(\alpha H + \beta m_f + C) < H.$

The conditions (C1)-(C7) are the basic ones in our paper and they provide settings for existence and uniqueness of asymptotically stable almost periodic solution of the system (1.34). Moreover, the conditions (C5) and (C6) guarantee that any solution $x(t)$ does not meet any surface of discontinuity more than once. This phenomenon is called *absence of beating* [1].

2.2 Continuity in the Initial Data

Continuous dependence of the solutions on the initial data is one of the most useful properties throughout the proofs in this study. Therefore, it is necessary to give the inequalities related to this property. Let $t_0,\ t_1$ belong to the same interval of discontinuity, such that $t_0 < t_1$, and $x_0(t) = x(t, t_0, x)$, $x_1(t) = x(t, t_1, y)$ are the solutions of the system (1.34), where t is also in the same interval of continuity with t_0 and t_1; that is, the solutions are continuous on the interval $[t_0, t]$.

Lemma 4 *The solutions $x_0(t) = x(t, t_0, x)$, $x_1(t) = x(t, t_1, y)$ of the differential equation (1.34) satisfies the following inequality*

$$\begin{aligned} &\|x(t, t_0, x) - x(t, t_1, y)\| \\ &\le \Big(\|x - y\| + (t_1 - t_0)(\alpha h + \beta m_f + \sigma)\Big) e^{(\alpha+\beta l_f)(t-t_1)}. \end{aligned} \tag{2.1}$$

In addition to that if $t_0 = t_1$, that is the initial times are the same then one can get

$$\|x(t, t_0, x) - x(t, t_0, y)\| \le \|x - y\| e^{(\alpha+\beta l_f)(t-t_0)}. \tag{2.2}$$

Let $t = \xi_k$ and η_k be the instants when the solutions $x(t, t_0, x)$ and $x(t, t_0, y)$ respectively intersect with the surface of discontinuities of the system (1.34), formulated by the equation $t = \theta_k + \tau_k(x(t))$. Then ξ_k and η_k satisfy the equations $\xi_k = \theta_k + \tau_k(x(\xi_k))$ and $\eta_k = \theta_k + \tau_k(y(\eta_k))$. Without loss of generality assume that $\theta_k \le \xi_k \le \eta_k$.

Lemma 5 *The impulsive instants $\theta_k \le \xi_k \le \eta_k$ satisfy the following*

$$|\eta_k - \xi_k| \le \frac{l_\tau e^{(\alpha+\beta l_f)m_\tau} \|x - y\|}{1 - l_\tau(\alpha h + \beta m_f + \sigma)} \tag{2.3}$$

Proof. By the definition and the previous lemma one can easily obtain the following inequalities

$$
\begin{aligned}
|\eta_k - \xi_k| &= |\tau_k(y(\eta_k)) - \tau_k(x(\xi_k))| \\
&\le l_\tau \|y(\eta_k) - x(\xi_k)\| \\
&\le l_\tau \|y(\xi_k) + \int_{\xi_k}^{\eta_k} (\mathcal{A}(s)y(s) + \mathcal{B}(s)F(y(s)) + \mathcal{C}(s))ds - x(\xi_k)\| \\
&\le l_\tau \Bigg(\|x - y\| e^{(\alpha+\beta l_f)m_\tau} + |\eta_k - \xi_k|(\alpha h + \beta m_f + \sigma) \Bigg).
\end{aligned}
$$

Solving the last inequality for $|\eta_k - \xi_k|$, we obtain

$$
|\eta_k - \xi_k|(1 - l_\tau(\alpha h + \beta m_f + \sigma)) \le l_\tau e^{(\alpha+\beta l_f)m_\tau} \|x - y\|
$$

$$
|\eta_k - \xi_k| \le \frac{l_\tau e^{(a+\beta l_f)m_\tau} \|x - y\|}{1 - l_\tau(\alpha h + \beta m_f + \sigma)},
$$

which is the expected inequality and the proof is complete.

2.3 The B-equivalent Model

In this section, we will reduce the system (1.33) to its B-equivalent counterpart which admits the form,

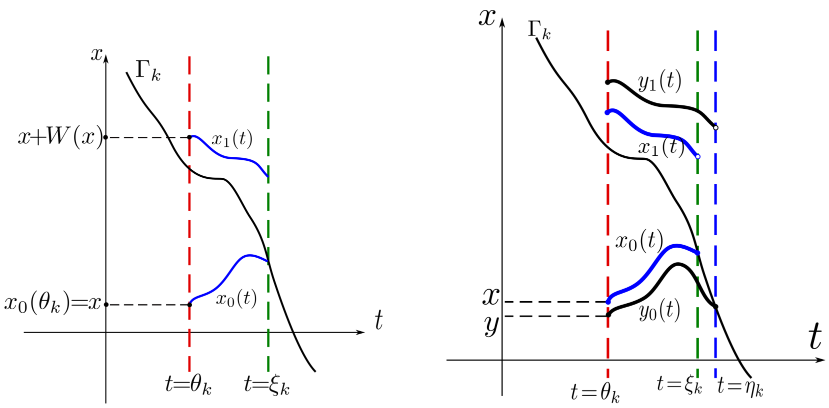
$$
\begin{aligned}
y_i' &= a_i(t)y_i(t) + \sum_{j=1}^{m} b_{ij}(t)f_j(y_j(t)) + c_i(t),\ t \ne \theta_k, \\
\Delta y\Big|_{t=\theta_k} &= d_{ik}y_i(\theta_k) + \sum_{j=1}^{m} e_{ijk}J_{jk}(y) + g_{ik}(y),
\end{aligned}
\tag{2.4}
$$

where it has common coefficients with the system (1.33) except for J_{jk} and g_{ik}, they will be defined later. The system (2.4) can be written in the matrix-vector form as follows,

$$
\begin{aligned}
y' &= \mathcal{A}(t)y(t) + \mathcal{B}(t)F(y(t)) + \mathcal{C}(t), \\
\Delta y\Big|_{t=\theta_k} &= \mathcal{D}_k y + E_k J_k(y) + G_k(y),
\end{aligned}
\tag{2.5}
$$

where $J_k(y) = [J_{1k}(y)\ J_{2k}(y)\ \dots\ J_{mk}(y)]^T$, $G_k(y) = [g_{1k}(y)\ g_{2k}(y)\ \dots\ g_{mk}(y)]^T$ are m-dimensional vector functions with coordinates determined from (2.4), and the other coefficients are as described in (1.34).

The systems (2.5) and (1.34) are *B-equivalent*, if there exists a number $h = h(H) < H$, such that for each solution $x(t) : \mathbb{R} \to \mathbb{R}^m$ of the system (1.34), satisfying $\|x(t)\| < h$, there exists a solution $y(t)$ of the system (2.5), that satisfies $y(t) = x(t)$ for t values that are not between η_k and ξ_k, and $\|y(t)\| < H$ for all t from $\mathbb{R}$. The instants η_k and ξ_k, $k \in \mathbb{Z}$ are discontinuity points of the solutions $y(t)$ and $x(t)$, respectively. Moreover, if $y(t)$ is any solution of the system (2.5), satisfying $\|y(t)\| < h$, then there exists a solution $x(t)$ of the system (1.34) such that $y(t) = x(t)$, if t is not an element of the interval with endpoints η_k and ξ_k, where $\|x(t)\| < H$ for all t that belongs to $\mathbb{R}$.



(a) the graphs of the solutions mentioned in B-Equivalence on $t{-}x$ coordinate system

(b) the graphs of the solutions $x_0(t)$, $y_0(t)$, $x_1(t)$, $y_1(t)$ of the system 1.33 on $t - x$ coordinates

Figure 2.1: Construction of the reduced B-equivalent system and continuous dependence, where $W(x) = D_k x + E_k J_k(x) + G_k(x)$, and Γ_k is the surface of discontinuity given by the equation $\Gamma_k : t = \theta_k + \tau_k(x)$.

To construct $J(x)$ and $G(x)$ in the system (2.5), fix k, and let $x_0(t)$ and $x_1(t)$ be solutions of differential equation in (1.34) with the initial conditions $x_0(\theta_k) = x$ and $x_1(\xi_k) = (I + D_k)x_0(\xi_k) + E_k I(x_0(\xi_k))$ as in Figure (2.1a). Then these solutions are written as

$$\begin{aligned}
x_0(t) &= x + \int_{\theta_k}^{t} (\mathcal{A}(s)x_0(s) + \mathcal{B}(s)F(x_0(s)) + \mathcal{C}(s))ds \\
x_0(\xi_k) &= x + \int_{\theta_k}^{\xi_k} (\mathcal{A}(s)x_0(s) + \mathcal{B}(s)F(x_0(s)) + \mathcal{C}(s))ds.
\end{aligned}$$

Taking the last equation as initial value and solving backwards direction one can obtain the solution:

$$
\begin{aligned}
x_1(t) &= x_0(\xi_k+) + \int_{\xi_k}^{t} (\mathcal{A}(s)x_1(s) + \mathcal{B}(s)F(x_1(s)) + \mathcal{C}(s))ds \\
&= (I + D_k)x + (I + D_k)\Big(\int_{t}^{\xi_k} (\mathcal{A}(s)x_0(s) + \mathcal{B}(s)F(x_0(s)) + \mathcal{C}(s))ds\Big) \\
&\quad + E_k I(x_0(\xi_k)) + H_k + \int_{\xi_k}^{t} (\mathcal{A}(s)x_1(s) + \mathcal{B}(s)F(x_1(s)) + \mathcal{C}(s))ds.
\end{aligned}
$$

Substituting $t = \theta_k$ in the last equation will help us getting impulsive part of B-equivalent system as follows

$$
\begin{aligned}
x_1(\theta_k) &= x_0(\xi_k+) + \int_{\xi_k}^{\theta_k} (\mathcal{A}(s)x_1(s) + \mathcal{B}(s)F(x_1(s)) + \mathcal{C}(s))ds \\
&= (I + D_k)x + (I + D_k)\Big(\int_{\theta_k}^{\xi_k} (\mathcal{A}(s)x_0(s) + \mathcal{B}(s)F(x_0(s)) + \mathcal{C}(s))ds\Big) \\
&\quad + E_k I(x_0(\xi_k)) + H_k + \int_{\xi_k}^{\theta_k} (\mathcal{A}(s)x_1(s) + \mathcal{B}(s)F(x_1(s)) + \mathcal{C}(s))ds.
\end{aligned} \tag{2.6}
$$

For B-equivalence of two systems, the equation

$$
x_1(\theta_k) - x_0(\theta_k) = x_1(\theta_k) - x = D_k x + E_k J_k(x) + G_k(x)
$$

must be valid. Using this equality and substituting $x_1(\theta_k)$ from (2.6), one gets

$$
\begin{aligned}
&(I + D_k)x + E_k J_k(x) + G_k(x) \\
&= (I + D_k)x + (I + D_k)\Big(\int_{\theta_k}^{\xi_k} (\mathcal{A}(s)x_0(s) + \mathcal{B}(s)F(x_0(s)) + \mathcal{C}(s))ds\Big) \\
&\quad + E_k I(x_0(\xi_k)) + H_k + \int_{\xi_k}^{\theta_k} (\mathcal{A}(s)x_1(s) + \mathcal{B}(s)F(x_1(s)) + \mathcal{C}(s))ds - x.
\end{aligned} \tag{2.7}
$$

After cancellations in (2.7), we can equate the terms with similar roles to deduce that

$$
J_k(x) = I(x_0(\xi_k)) = I(x + \int_{\theta_k}^{\xi_k} (\mathcal{A}(s)x_0(s) + \mathcal{B}(s)F(x_0(s)) + \mathcal{C}(s))ds). \tag{2.8}
$$

$$
\begin{aligned}
G_k(x) &= (I + D_k)\Big(\int_{\theta_k}^{\xi_k} (\mathcal{A}(s)x_0(s) + \mathcal{B}(s)F(x_0(s)) + \mathcal{C}(s))ds\Big) \\
&\quad + \int_{\xi_k}^{\theta_k} (\mathcal{A}(s)x_1(s) + \mathcal{B}(s)F(x_1(s)) + \mathcal{C}(s))ds + H_k.
\end{aligned} \tag{2.9}
$$

Let us denote

$$
\begin{aligned}
k(m_{\tau_k}, \ell_f, m_f) &= 1 + m_\tau(\alpha+\beta\ell_f)e^{(\alpha+\beta\ell_f)m_\tau} + \frac{\ell_\tau e^{(\alpha+\beta\ell_f)m_\tau}(\alpha h + \beta m_f + \sigma)}{1-\ell_\tau(\alpha h + \beta m_f + \sigma)}, \\
\bar{k}(m_{\tau_k}, \ell_f, m_f) &= m_\tau(\alpha+\beta\ell_f)\Bigg((1+d_k)e^{(\alpha+\beta\ell_f)m_\tau} \\
&+ \frac{2 + d_k + E_k\ell_I + \ell_\tau(\alpha h + \beta m_f + \sigma)}{1-(\alpha h + \beta m_f + \sigma)} e^{2(\alpha+\beta\ell_f)m_\tau}\Bigg) \\
&+ \frac{(2+d_k)(\alpha h + \beta m_f + \sigma)\ell_\tau e^{(\alpha+\beta\ell_f)m_\tau}}{1-\ell_\tau(\alpha h + \beta m_f + \sigma)}.
\end{aligned}
$$

Lemma 6 *If the conditions (C2)-(C7) are satisfied then for every $x, y \in \mathcal{D}_h = \{x \in \mathbb{R}^m : \|x - x_0\| < H\}$ and for any index k, the functions $G_k(x)$ and $J_k(x)$ are Lipschitzian such that*

$$
\begin{aligned}
\|J_k(x) - J_k(y)\| &< \ell_I k(m_\tau, \ell_f, m_f)\|x-y\|, \\
\|G_k(x) - G_k(y)\| &< \bar{k}(m_\tau, \ell_f, m_f)\|x-y\|.
\end{aligned}
$$

Proof. Let $x_0(t) = x(t, \theta_k, x)$, $y_0(t) = y(t, \theta_k, y)$ be solutions of the system (1.34) with corresponding initial values, and let ξ_k and η_k be the meeting points of these solutions with the surfaces of discontinuities respectively, i.e., $\xi_k = \theta_k + \tau_k(x_0(\xi_k))$ and $\eta_k = \theta_k + \tau_k(y_0(\eta_k))$. Moreover, let $x_1(t) = x(t, \xi_k, x_0(\xi_k+))$, $y_1(t) = y(t, \eta_k, y_0(\eta_k+))$ be solutions of the system (1.34). Firstly, we will show that J_k is Lipschitzian. Consider the following inequalities

$$
\begin{aligned}
&\|J_k(x) - J_k(y)\| \\
&= \|I(x + \int_{\theta_k}^{\xi_k} (\mathcal{A}(s)x_0(s) + \mathcal{B}(s)F(x_0(s)) + \mathcal{C}(s))ds) \\
&\quad -I(y + \int_{\theta_k}^{\xi_k} (\mathcal{A}(s)y_0(s) + \mathcal{B}(s)F(y_0(s)) + \mathcal{C}(s))ds)\| \\
&\leq \ell_I(\|x-y\| + \int_{\theta_k}^{\xi_k} (\|\mathcal{A}(s)\|\|x_0(s) - y_0(s)\| + \|\mathcal{B}(s)\|\|F(x_0(s)) - F(y_0(s))\| \\
&\quad + \int_{\xi_k}^{\eta_k} (\|\mathcal{A}(s)\|\|y_0(s)\| + \|\mathcal{B}(s)\|\|F(y_0(s))\| + \|\mathcal{C}(s)\|)ds) \\
&\leq \ell_I\Big(\|x-y\| + m_\tau(\alpha+\beta\ell_f)\|x-y\|e^{(\alpha+\beta\ell_f)m_\tau} + |\eta_k - \xi_k|(\alpha h + \beta m_f + \sigma)\Big) \\
&\leq \ell_I\left(1 + m_\tau(\alpha+\beta\ell_f)e^{(\alpha+\beta\ell_f)m_\tau} + \frac{\ell_\tau e^{(\alpha+\beta\ell_f)m_{\tau_k}}(\alpha h + \beta m_f + \sigma)}{1-\ell_\tau(\alpha h + \beta m_f + \sigma)}\right)\|x-y\|.
\end{aligned}
$$

Furthermore, it will be shown that $G_k(x)$ is Lipschitzian, before that, some inequalities on $x_1(t)$ and $y_1(t)$ needs to be proved. One can easily show the following inequalities,

$$\begin{aligned}
&\|x_1(\xi_k) - y_1(\xi_k)\| \\
&\leq \|x_0(\xi_k) - y_0(\xi_k)\| + \|\int_{\xi_k}^{\eta_k} (\mathcal{A}(s)y_0(s) + \mathcal{B}(s)F(y_0(s)) + \mathcal{C}(s))ds\| \\
&\quad + \|D_k(x_0(\xi_k) - y_0(\xi_k) - \int_{\xi_k}^{\eta_k} (\mathcal{A}(s)y_0(s) + \mathcal{B}(s)F(y_0(s)) + \mathcal{C}(s))ds)\| \\
&\quad + \|E_k[I(x_0(\xi_k)) - I(y_0(\xi_k) + \int_{\xi_k}^{\eta_k} (\mathcal{A}(s)y_0(s) + \mathcal{B}(s)F(y_0(s)) + \mathcal{C}(s))ds)]\| \\
&\quad + \|\int_{\eta_k}^{\xi_k} (\mathcal{A}(s)y_1(s) + \mathcal{B}(s)F(y_1(s)) + \mathcal{C}(s))ds\| \\
&\leq \|x_0(\xi_k) - y_0(\xi_k)\| + (\eta_k - \xi_k)(\alpha h + \beta m_f + \sigma) \\
&\quad + d_k(\|x_0(\xi_k) - y_0(\xi_k)\| + (\eta_k - \xi_k)(\alpha h + \beta m_f + \sigma)) \\
&\quad + e_k\ell_I(\|x_0(\xi_k) - y_0(\xi_k)\| + (\eta_k - \xi_k)(\alpha h + \beta m_f + \sigma)) \\
&\quad + (\eta_k - \xi_k)(\alpha h + \beta m_f + \sigma) \\
&= \|x_0(\xi_k) - y_0(\xi_k)\|(1 + d_k + e_k\ell_I) \\
&\quad + \|x - y\| \frac{\ell_\tau e^{(\alpha+\beta\ell_f)m_\tau}(\alpha h + \beta m_f + \sigma)}{1 - \ell_\tau(\alpha h + \beta m_f + \sigma)}(1 + d_k + ek\ell_I + 1) \\
&= \|x - y\| e^{(\alpha+\beta\ell_f)m_\tau} \frac{1 + d_k + e_k\ell_I + \ell_\tau(\alpha h + \beta m_f + \sigma)}{1 - \ell_\tau(\alpha h + \beta m_f + \sigma)}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|x_1(s) - y_1(s)\| &\leq \|x_1(\xi_k) - y_1(\xi_k)\| e^{(\alpha+\beta\ell_f)m_\tau} \\
&\leq \|x - y\| e^{2(\alpha+\beta\ell_f)m_\tau} \frac{1 + d_k + e_k\ell_I + \ell_\tau(\alpha h + \beta m_f + \sigma)}{1 - \ell_\tau(\alpha h + \beta m_f + \sigma)}.
\end{aligned}$$

Now, using the last inequality and similar to $J_k(x)$ one can get that $G_k(x)$ also has Lipschitz property as follows

$$\begin{aligned}
\|G_k(x) - G_k(y)\| &= \Big|(I + D_k)(\int_{\theta_k}^{\xi_k} (A(s)x_0(s) + B(s)F(x_0(s)) + C(s))ds) + H_k \\
&+ \int_{\xi_k}^{\theta_k} (A(s)x_1(s) + B(s)F(x_1(s)) + C(s))ds \\
&- (I + D_k)(\int_{\theta_k}^{\eta_k} (A(s)y_0(s) + B(s)F(y_0(s)) + C(s))ds) - H_k \\
&- \int_{\eta_k}^{\theta_k} (A(s)y_1(s) + B(s)F(y_1(s)) + C(s))ds\Big|
\end{aligned}$$

$$
\begin{aligned}
\le\ & \Big|(I+D_k)\bigg(\int_{\theta_k}^{\xi_k}\Big[A(s)(x_0(s)-y_0(s))+B(s)(F(x_0(s))-F(y_0(s)))\Big]ds \\
& -\int_{\xi_k}^{\eta_k}(A(s)y_0(s)+B(s)F(y_0(s))+C(s))ds\bigg) \\
& +\int_{\xi_k}^{\theta_k}\Big[A(s)(x_1(s)-y_1(s))+B(s)(F(x_1(s))-F(y_1(s)))\Big]ds \\
& -\int_{\eta_k}^{\xi_k}(A(s)y_1(s)+B(s)F(y_1(s))+C(s))ds\Big| \\
\le\ & (1+d_k)\Big(\int_{\theta_k}^{\xi_k}\Big[\alpha\|x_0(s)-y_0(s)\|+\beta\ell_f\|x_0(s)-y_0(s)\|\Big]ds \\
& +\int_{\xi_k}^{\eta_k}(\alpha\|y_0(s)\|+\beta m_f+\sigma)ds\Big) \\
& +\int_{\theta_k}^{\xi_k}\Big[\alpha\|x_1(s)-y_1(s)\|+\beta\ell_f\|x_1(s)-y_1(s)\|\Big]ds \\
& +\int_{\xi_k}^{\eta_k}(\alpha\|y_1(s)\|+\beta m_f+\sigma)ds \\
\le\ & (1+d_k)\Big(m_\tau(\alpha+\beta\ell_f)\|x-y\|e^{(\alpha+\beta\ell_f)m_\tau} \\
+\ & \frac{\ell_\tau e^{(a+\beta\ell_f)m_\tau}\|x-y\|}{1-\ell_\tau(\alpha h+\beta m_f+\sigma)}(\alpha h+\beta m_f+\sigma)\Big) \\
& +m_\tau(\alpha+\beta\ell_f)\|x-y\|\left(\frac{1+d_k+E_k\ell_I+\ell_\tau(\alpha h+\beta m_f+\sigma)}{1-(\alpha h+\beta m_f+\sigma)}\right)e^{2(\alpha+\beta\ell_f)m_\tau} \\
& +\frac{\ell_\tau e^{(a+\beta\ell_f)m_\tau}\|x-y\|}{1-\ell_\tau(\alpha h+\beta m_f+\sigma)}(\alpha h+\beta m_f+\sigma) \\
\le\ & \|x-y\|\Bigg[m_\tau(\alpha+\beta\ell_f)\bigg((1+d_k)e^{(\alpha+\beta\ell_f)m_\tau} \\
& +\frac{2+d_k+E_k\ell_I+\ell_\tau(\alpha h+\beta m_f+\sigma)}{1-(\alpha h+\beta m_f+\sigma)}e^{2(\alpha+\beta\ell_f)m_\tau}\bigg) \\
& +\frac{(2+d_k)(\alpha h+\beta m_f+\sigma)\ell_\tau e^{(a+\beta\ell_f)m_\tau}}{1-\ell_\tau(\alpha h+\beta m_f+\sigma)}\Bigg].
\end{aligned}
$$

Next we will show that $\|x(t)\| < h$ implies $\|y(t)\| < H$. If t is not between ξ_k and η_k, they are equal by the construction, so $\|y(t)\| < h$. Without loss of generality, assume that $\eta_k > \xi_k$, let $t \in [\eta_k, \xi_k]$, since $y(\xi_k) = x(\xi_k)$, $y(t)$ can be expressed by the equation

$$
y(t) = x(\xi_k) + \int_{\xi_k}^{t}(A(s)\varphi(s,\xi_k,x_i(\xi_k)) + B(s)F(\varphi_j(s,\xi_k,x_i(\xi_k))) + C(s))ds.
$$

Consequently, one can obtain the following inequalities

$$\begin{aligned}
\|y(t)\| &\leq \|x(\xi_k)\| + \\
&+ \int_{\xi_k}^{t} (\|A(s)\|\|\varphi(s, \xi_k, x_i(\xi_k))\| + \|B(s)\|\|F(\varphi_j(s, \xi_k, x_i(\xi_k)))\| + \|C(s)\|)ds \\
&\leq \|x(\xi_k)\| + \int_{\xi_k}^{t} (\alpha\|\varphi(s, \xi_k, x_i(\xi_k))\| + \beta m_f + \sigma)ds \\
&= h + (\alpha h + \beta m_f + C)m_\tau < H. \qquad \square
\end{aligned}$$

For the forthcoming chapters some additional notations will be needed. So denote

$$\begin{aligned}
&\ell_J = \ell_I k(m_\tau, \ell_f, m_f), \qquad \|J_k(x)\| \leq m_I = m_j, \\
&\ell_G \ = \ \bar{k}(m_\tau, \ell_f, m_f),
\end{aligned}$$

$$\|G_k(x)\| \leq d_k m_\tau(\alpha H + \beta m_f + \sigma) + m_\tau(\alpha H + \beta m_f + \sigma) + m_H = m_g.$$

CHAPTER 3

PERIODIC RECURRENT NEURAL NETWORKS WITH NON-FIXED MOMENT OF IMPULSES

In this chapter, we will first show the periodicity of the B-equivalent system under some conditions and using B-equivalent system the periodicity of the original system will be proved. The following periodicity conditions will be applied in the present part of the thesis.

There exists a positive number ω and a natural number p such that

(P1) impulse moments are (ω, p)-periodic. That is, $\theta_{k+p} - \theta_k = \omega$, $k \in \mathbb{Z}$,
(P2) functions $a_i(t)$, $b_{ij}(t)$, $c_i(t)$ are $\omega-$periodic in t, for $i, j = 1, 2, \ldots, m$,
(P3) sequences d_{ik}, e_{ijk}, h_{ik} are $p-$periodic in k where $i, j = 1, 2, \ldots, m$, $k \in \mathbb{Z}$,
(P4) sequence of discontinuity surfaces is p-periodic in k, such that $\tau_{k+p}(x) = \tau_k$ for all $k \in \mathbb{Z}$ and $x \in \mathbb{R}^m$.

3.1 Periodicity of the Reduced Model

The main task of the present section is to prove that the sequences $J_k(y)$ and $G_k(y)$, given in equations (2.8) and (2.9) are periodic in k. Hence we will prove that the system (2.5) satisfies the conditions *(P1)-(P4)*, consequently it is (ω, p)-periodic in k uniformly with respect to x.

Let us denote by ξ_k, the impact moments where the solution of the system (2.5) meets the surface of discontinuities, i.e. ξ_k is the real number in the interval $[\theta_k, \theta_k + m_\tau]$ such that $\xi_k = \theta_k + \tau_k(x(\xi_k, \theta_k, x))$.

Consider $x(\xi_k)$ and $x(\xi_{k+p})$ which are the solutions of $\xi_k = \theta_k + \tau_k(x(\xi_k, \theta_k, x))$ and $\xi_{k+p} = \theta_{k+p} + \tau_{k+p}(x(\xi_{k+p}, \theta_{k+p}, x))$ respectively, let $x(t, \theta_k, x) = x_0(t)$ and $x(t, \theta_{k+p}, x) = x^0(t)$, then

$$x_0(t) = x + \int_{\theta_k}^{t} (\mathcal{A}(s)x_0(s) + \mathcal{B}(s)\tilde{f}(x_0(s)) + \mathcal{C}(s))ds, \text{ where } t \in [\theta_k, \theta_k + m_\tau],$$

$$x^0(t) = x + \int_{\theta_{k+p}}^{t} (\mathcal{A}(s)x^0(s) + \mathcal{B}(s)\tilde{f}(x^0(s)) + \mathcal{C}(s))ds, \text{ where } t \in [\theta_{k+p}, \theta_{k+p} + m_\tau].$$

Also assume that the vector space on $t \in [\theta_{k+p}, \theta_{k+p} + m_\tau]$ is shifted ω units left with the substitution $t = u + \omega$ in the second equation to obtain:

$$x^0(u+\omega) = x + \int_{\theta_{k+p}}^{u+\omega} (\mathcal{A}(s)x^0(s) + \mathcal{B}(s)\tilde{f}(x^0(s)) + \mathcal{C}(s))ds \text{ where } u \in [\theta_k, \theta_k + m_\tau].$$

Let $\bar{x}(t) = x^0(t+\omega)$ and use change of variables $s = v+\omega$ inside the integral. Hence, it is true that

$$\bar{x}(t) = x + \int_{\theta_k}^{t} (\mathcal{A}(s+\omega)x^0(s+\omega) + \mathcal{B}(s+\omega)\tilde{f}(x^0(s+\omega)) + \mathcal{C}(s+\omega))ds,$$

where $t \in [\theta_k, \theta_k + m_\tau]$.

Now observe that $\bar{x}(t)$ is the solution of the following differential equation

$$\begin{aligned} \bar{x}'(t) &= \mathcal{A}(t+\omega)\bar{x}(t) + \mathcal{B}(t+\omega)\tilde{f}(\bar{x}(t)) + \mathcal{C}(t+\omega) \\ &= \mathcal{A}(t)\bar{x}(t) + \mathcal{B}(t)\tilde{f}(\bar{x}(t)) + \mathcal{C}(t), \end{aligned}$$

with the initial conditions $\bar{x}(\theta_k) = x$ and recall that $x_0(t)$ is the solution of the differential equation of the same system with the same initial conditions. So, $x^0(t+\omega) = \bar{x}(t) = x_0(t)$.

Lemma 7 *The sequence of moments, ξ_k, where the solution meets the discontinuity surfaces is p-periodic.*

Proof. Consider $\xi_{k+p} = \theta_{k+p} + \tau_{k+p}(x^0(\xi_{k+p}))$

$$\begin{aligned} \xi_{k+p} &= \theta_{k+p} + \tau_{k+p}(x^0(\xi_{k+p})), \quad \text{where } \xi_{k+p} \in [\theta_{k+p}, \theta_{k+p} + m_\tau], \\ \xi_{k+p} - \omega &= \theta_k + \tau_{k+p}(x^0(\xi_{k+p})), \quad \text{where } \xi_{k+p} \in [\theta_{k+p}, \theta_{k+p} + m_\tau], \\ \xi_{k+p} - \omega &= \theta_k + \tau_k(x_0(\xi_{k+p} - \omega)), \quad \text{where } \xi_{k+p} - \omega \in [\theta_k, \theta_k + m_\tau]. \end{aligned} \tag{3.1}$$

By definition of ξ_k, a solution of the equation $t = \theta_k + \tau_k(t)$ on $[\theta_k, \theta_k + m_\tau]$ is ξ_k, and by the absence of beating the solution meets each surface of discontinuity once. then $\xi_{k+p} - \omega = \xi_k$. $\square$

Lemma 8 *Under the conditions (C2)-(C7) and (P1)-(P4), the sequences in the B-equivalent system (2.5), $J_k(x)$, $G_k(x)$, $k \in \mathbb{Z}$ are p-periodic uniformly in x.*

Proof. The method for the proof of periodicity of J_k, and G_k are quite similar, firstly the periodicity of J_k will be shown. As above, denote

$$x_0(t) = x + \int_{\theta_k}^{t} (\mathcal{A}(s)x_0(s) + \mathcal{B}(s)F(x_0(s)) + \mathcal{C}(s))ds \text{ with } t \in [\theta_k, \theta_k + m_\tau],$$

$$x^0(t) = x + \int_{\theta_{k+p}}^{t} (\mathcal{A}(s)x^0(s) + \mathcal{B}(s)F(x^0(s)) + \mathcal{C}(s))ds, \text{ with } t \in [\theta_{k+p}, \theta_{k+p} + m_\tau].$$

To show periodicity of $J_k(x) = I(x_0(\xi_k))$, the substituton $s = u + \theta_{k+p} - \theta_k = u + \omega$ and $u = s - \omega$ on $J_{k+p}(x)$ and writing integrand with variable s result in

$$\begin{aligned}
J_{k+p}(x) &= I(x^0(\xi_{k+p})) = I(x + \int_{\theta_{k+p}}^{\xi_{k+p}} (\mathcal{A}(s)x^0(s) + \mathcal{B}(s)F(x^0(s)) + \mathcal{C}(s))ds) \\
&= I(x + \int_{\theta_k}^{\xi_k} (\mathcal{A}(s+\omega)x^0(s+\omega) + \mathcal{B}(s+\omega)F(x^0(s+\omega)) + \mathcal{C}(s+\omega))ds) \\
&= I(x + \int_{\theta_k}^{\xi_k} (\mathcal{A}(s)\bar{x_0}(s) + \mathcal{B}(s)F(\bar{x_0}(s)) + \mathcal{C}(s))ds) \\
&= I(x + \int_{\theta_k}^{\xi_k} (\mathcal{A}(s)x_0(s) + \mathcal{B}(s)F(x_0) + \mathcal{C}(s))ds) \\
&= J_k(x).
\end{aligned}$$

Using the equation

$$\begin{aligned}
G_k(x) &= (I + D_k)\Big(\int_{\theta_k}^{\xi_k} (\mathcal{A}(s)x_0(s) + \mathcal{B}(s)F(x_0(s)) + \mathcal{C}(s))ds\Big) \\
&+ \int_{\xi_k}^{\theta_k} (\mathcal{A}(s)x_1(s) + \mathcal{B}(s)F(x_1(s)) + \mathcal{C}(s))ds + H_k,
\end{aligned}$$

the proof of the periodicity of $G_k(x)$ involves similar calculations as follows

$$\begin{aligned}
G_{k+p}(x) &= (I + D_{k+p})\Big(\int_{\theta_{k+p}}^{\xi_{k+p}} (\mathcal{A}(s)x^0(s) + \mathcal{B}(s)F(x^0(s)) + \mathcal{C}(s))ds\Big) \\
&+ \int_{\xi_{k+p}}^{\theta_{k+p}} (\mathcal{A}(s)x^1(s) + \mathcal{B}(s)F(x^1(s)) + \mathcal{C}(s))ds + H_{k+p}
\end{aligned}$$

$$
\begin{aligned}
&= (I + D_{k+p})\Big(\int_{\theta_k}^{\xi_k} (\mathcal{A}(s+\omega)x^0(s+\omega) \\
&+ \mathcal{B}(s+\omega)F(x^0(s+\omega)) + \mathcal{C}(s+\omega))ds\Big) \\
&+ \int_{\xi_k}^{\theta_k} (\mathcal{A}(s+\omega)x^1(s+\omega) + \mathcal{B}(s+\omega)F(x^1(s+\omega)) + \mathcal{C}(s+\omega))ds + H_{k+p} \\
&= (I + D_k)\Big(\int_{\theta_k}^{\xi_k} (\mathcal{A}(s)x_0(s) + \mathcal{B}(s)F(x_0(s)) + \mathcal{C}(s))ds\Big) \\
&+ \int_{\xi_k}^{\theta_k} (\mathcal{A}(s)x_1(s) + \mathcal{B}(s)F(x_1(s)) + \mathcal{C}(s))ds + H_k \\
&= G_k(x).
\end{aligned}
$$

Hence both $J_k(x)$ and $G_k(k)$ are p-periodic uniformly in x. $\square$

Next, consider the linear homogeneous counterpart of the B-equivalent system (2.5). Denote by $\mathcal{D}$, the space of periodic functions such that they have common discontinuity points, which are periodic, such that $\|\varphi(t)\|_0 < H$, where the (ω, p)-periodic discontinuity points are θ_k, $k \in \mathbb{Z}$, and the corresponding function norm on the space $\mathcal{D}$ is, $\|\varphi(t)\|_0 = \sup\limits_{t\in(-\infty,\infty)} \|\varphi(t)\|$, for $\varphi(t) \in \mathcal{D}$.

On the space $\mathcal{D}$, set the operator

$$
\begin{aligned}
T\varphi(t) &= (T\varphi)(t) \\
&= \int_{-\infty}^{t} X(t,s)\Big(\mathcal{B}(s)F(\varphi(s)) + \mathcal{C}(s)\Big)ds \\
&+ \sum_{\theta_k<t} X(t,\theta_k+)\Big(E_kJ_k(\varphi(\theta_k)) + G_k(\varphi(\theta_k))\Big).
\end{aligned}
$$

The following assertions are needed through the remaining part of this chapter.

(C8) $K\left(\dfrac{m_fB + c}{-\gamma} + \dfrac{m_JE + m_g}{1 - e^{\gamma\underline{\theta}}}\right) < H,$

(C9) $K\left(\dfrac{\ell_fB}{-\gamma} + \dfrac{\ell_JE + \ell_G}{1 - e^{\gamma\underline{\theta}}}\right) < 1,$

(C10) $\gamma + K\beta\ell_f + \frac{\ln(1+K(\ell_JE+\ell_G))}{\underline{\theta}} < 0.$

Lemma 9 *Let $\varphi(t) \in \mathcal{D}$ and the conditions (C8) and (P1)-(P4) be valid. Then $T\varphi(t)$ belongs to $\mathcal{D}$, that is, T is invariant on $\mathcal{D}$.*

Proof. We need to show that if $\varphi \in \mathcal{D}$ then $T\varphi$ also belongs to $\mathcal{D}$. Let us consider the following inequalities

$$
\begin{aligned}
\|T\varphi(t)\| &\leq \int_{-\infty}^{t} \|X(t,s)\| \|\mathcal{B}(s)F(\varphi(s)) + \mathcal{C}(s)\| \\
&+ \sum_{\theta_k < t} \|X(t,\theta_k)\| \|E_k J_k(\varphi(\theta_k)) + G_k(\varphi(\theta_k))\| \\
&\leq \int_{-\infty}^{t} K e^{\gamma(t-s)} (\|\mathcal{B}(s)\| \|F(\varphi(s))\| + \|\mathcal{C}(s)\| \\
&+ \sum_{\theta_k < t} K e^{\gamma(t-\theta_k)} \|E_k J_k(\varphi(\theta_k)) + G_k(\varphi(\theta_k))\| \\
&\leq K\bigg(\frac{m_f\beta + \sigma}{-\gamma} + \frac{m_I E + m_g}{1 - e^{\gamma\underline{\theta}}}\bigg).
\end{aligned}
$$

The last statement, by the condition *(C8)* implies that $\|T\varphi(t)\| < H$. Next we check the periodicity of the image of T.

$$
\begin{aligned}
T\varphi(t+\omega) &= \int_{-\infty}^{t} X(t+\omega, s+\omega)\bigg(\mathcal{B}(s+\omega)F(\varphi(s+\omega)) + \mathcal{C}(s+\omega)\bigg)ds \\
&+ \sum_{\theta_k < t} X(t+\omega, \theta_{k+q})\bigg(E_{(k+q)} J_{k+q}(\varphi(\theta_{k+q})) + G_{k+q}(\varphi(\theta_{k+q}))\bigg) \\
&= \int_{-\infty}^{t} X(t,s)\bigg(\mathcal{B}(s)F(\varphi(s)) + \mathcal{C}(s)\bigg)ds \\
&+ \sum_{\theta_k < t} X(t,\theta_k)\bigg(E_k J_k(\varphi(\theta_k)) + G_k(\varphi(\theta_k))\bigg) = T\varphi(t).
\end{aligned}
$$

This ends the proof, since $T\varphi(t)$ is in $\mathcal{D}$ and ω-periodic T is invariant on $\mathcal{D}$. $\square$

Lemma 10 *Under the conditions (C8),(C9) and (P1)-(P4), the map T is a contraction in $\mathcal{D}$.*

Proof. Suppose that $\varphi(t)$ and $\psi(t)$ are the functions that belong to $\mathcal{D}$. Then, we can assert the following inequality

$$
\begin{aligned}
&\|T\varphi(t) - T\psi(t)\| \\
&\leq \int_{-\infty}^{t} \|X(t,s)\| \|\mathcal{B}(s)(F(\varphi(s)) - F(\psi(s)))\| ds \\
&+ \sum_{\theta_k < t} \|X(t,\theta_k)\| \|E_k(J(\varphi(\theta_k)) - J(\psi(\theta_k))) + G_k(\varphi(\theta_k)) - G_k(\psi(\theta_k))\| \\
&\leq \int_{-\infty}^{t} K e^{\gamma(t-s)} \|\mathcal{B}(s)\| \ell_f \|\varphi(s) - \psi(s)\| ds \\
&+ \sum_{\theta_k < t} K e^{\gamma(t-\theta_k)} (\ell_J E + \ell_G) \|\varphi(\theta_k) - \psi(\theta_k)\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_{-\infty}^{t} Ke^{\gamma(t-s)}\beta\ell_f\|\varphi(s)-\psi(s)\|_0 ds \\
&+ \sum_{\theta_k<t} Ke^{\gamma(t-\theta_k)}(\ell_J E+\ell_G)\|\varphi(\theta_k)-\psi(\theta_k)\| \\
&\leq \int_{-\infty}^{t} Ke^{\gamma(t-s)}\ \beta\ell_f\|\varphi(t)-\psi(t)\|_0 ds \\
&+ \sum_{\theta_k<t} Ke^{\gamma(t-\theta_k)}(\ell_J E+\ell_G)\|\varphi(t)-\psi(t)\|_0 \\
&\leq K\ell_f\beta\|\varphi(t)-\psi(t)\|_0\left(\frac{1}{-\gamma}\right)+K(\ell_J E+\ell_G)\|\varphi(t)-\psi(t)\|_0\left(\frac{1}{1-e^{\gamma\underline{\theta}}}\right) \\
&\leq K\left(\frac{\ell_f\beta}{-\gamma}+\frac{\ell_J E+\ell_G}{1-e^{\gamma\underline{\theta}}}\right)\|\varphi(t)-\psi(t)\|_0.
\end{aligned}
$$

Then $\|T\varphi(t)-T\psi(t)\|_0 \leq K\left(\frac{\ell_f\beta}{-\gamma}+\frac{\ell_J E+\ell_G}{1-e^{\gamma\underline{\theta}}}\right)\|\varphi(t)-\psi(t)\|_0$. Hence by the condition *(C9)*, the map T is a contraction. □

Theorem 11 *Let (C1)-(C10) and (P1)-(P4) be fulfilled. Then the system (2.5) has a unique asymptotically stable periodic solution.*

Proof. The operator T satisfies the differential equation and impulsive part of the system (2.5). Indeed, the derivative of $T\varphi(t)$ with respect to t is

$$
\begin{aligned}
&(T\varphi(t))' \\
&= \int_{-\infty}^{t} X(t,s)\mathcal{A}(t)\Big(\mathcal{B}(s)f(\varphi(s))+\mathcal{C}(s)\Big)ds \\
&+ X(t,t)\Big(\mathcal{B}(t)f(\varphi(t))+\mathcal{C}(t)\Big)+\frac{d}{dt}\sum_{\theta_k<t}X(t,\theta_k)\Big(E_kJ_k(\varphi(\theta_k))+G_k(\varphi(\theta_k))\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \mathcal{A}(t)\int_{-\infty}^{t} x_i(t,s)\Big(\mathcal{B}(s)f(\varphi_j(s))+\mathcal{C}(s)\Big)ds \\
&+ \mathcal{A}(t)\sum_{\theta_k<t}X(t,\theta_k)\Big(E_{ijk}J_{ij}(\varphi(\theta_k))+G_k(x(\theta_k))\Big) \\
&+ \Big(\mathcal{B}(t)f_j(T\varphi_j(t))+\mathcal{C}(t)\Big) \\
&= \mathcal{A}(t)T(\varphi_i(t))+\Big(\mathcal{B}(t)f_j(T\varphi_j(t))+\mathcal{C}(t)\Big).
\end{aligned}
$$

For the impulsive part, we will use $x_i(\theta_q+,s)=x_i(\theta_q,s)(1+d_{iq})$, which leads us to

$x_i(\theta_q+, s) - x_i(\theta_q, s) = d_{iq}$. As a result, it can be seen that

$$
\begin{aligned}
\Delta T\varphi_j(t)\big|_{t=\theta_q} &= T\varphi_j(\theta_q+) - T\varphi_j(\theta_q) \\
&= \int_{-\infty}^{\theta_q+} x_i(\theta_q+, s)\bigg(\sum_{j=1}^{m} b_{ij}(s) f_j(\varphi_j(s)) + c_i(s)\bigg)ds \\
&+ \sum_{\theta_k<\theta_q+} x_i(\theta_q+, \theta_k)\bigg(\sum_{j=1}^{m} e_{ijk} J_{ij}(\varphi(\theta_q)) + g_{ik}(x(\theta_k))\bigg) \\
&- \int_{-\infty}^{\theta_q} x_i(\theta_q, s)\bigg(\sum_{j=1}^{m} b_{ij}(s) f_j(\varphi_j(s)) + c_i(s)\bigg)ds \\
&- \sum_{\theta_k<\theta_q} x_i(\theta_q, \theta_k)\bigg(\sum_{j=1}^{m} e_{ijk} J_{ij}(\varphi(\theta_q)) + g_{ik}(x(\theta_k))\bigg) \\
&= (1 + d_{iq} - 1)\int_{-\infty}^{\theta_q} x_i(\theta_q, s)\bigg(\sum_{j=1}^{m} b_{ij}(s) f_j(\varphi_j(s)) + c_i(s)\bigg)ds \\
&+ x_i(\theta_q+, \theta_q+)\bigg(\sum_{j=1}^{m} e_{ijq} J_{ij}(\varphi(\theta_q)) + g_{ik}(x(\theta_q))\bigg) \\
&+ \sum_{\theta_k<\theta_q+} (x_i(\theta_q+, \theta_k) - x_i(\theta_q, \theta_k))\bigg(\sum_{j=1}^{m} e_{ijk} J_{ij}(\varphi(\theta_q)) + g_{ik}(x(\theta_k))\bigg) \\
&= d_{iq}\int_{-\infty}^{\theta_q} x_i(\theta_q, s)\bigg(\sum_{j=1}^{m} b_{ij}(s) f_j(\varphi_j(s)) + c_i(s)\bigg)ds \\
&+ d_{iq}\sum_{\theta_k<\theta_q+} x_i(\theta_q, \theta_k)\bigg(\sum_{j=1}^{m} e_{ijk} J_{ij}(\varphi(\theta_q+)) + g_{ik}(x(\theta_k))\bigg) \\
&+ \bigg(\sum_{j=1}^{m} e_{ijq} J_{ij}(T\varphi(\theta_q)) + g_{ik}(x(\theta_q))\bigg).
\end{aligned}
$$

By the Lemma 9, the operator T is invariant on the space $\mathcal{D}$. That is, if $\varphi(t) \in \mathcal{D}$, then $T\varphi(t) \in \mathcal{D}$, and by the Lemma 10 the operator is a contraction. Hence, by Banach's fixed point theorem [75], the space $\mathcal{D}$ has a unique element $\omega(t)$, such that $T\omega = \omega$, and it is the unique solution of our system.

Let $z(t) = [z_1(t), \ldots, z_m(t)]$ be a solution of the system (2.5) with the initial data is $z(t_0) = z_0$. Applying the equivalent integral equation for the initial value problem, one can obtain

$$
\begin{aligned}
&\|z(t)-\omega(t)\| \\
&\leq \Big\|X(t,t_0)(z(t_0)-\omega(t_0)) + \int_{t_0}^{t} X(t,s)(\mathcal{B}(F(z(s))-F(\omega(s))))ds \\
&+ \sum_{t_0\leq\theta_k<t} X(t,\theta_k)\Big(E_k(J(z(\theta_k))-J(\omega(\theta_k))) + G_k(z(\theta_k)) - G_k(\omega(\theta_k))\Big)\| \\
&\leq \|x(t,t_0)\|\|(z(t_0)-\omega(t_0))\| + \int_{t_0}^{t} Ke^{\gamma(t-s)}\beta\ell_f\|z(s)-\omega(s)\|ds \\
&+ \sum_{t_0\leq\theta_k<t} Ke^{\gamma(t-\theta_k)}(\ell_J E+\ell_G)\|z(\theta_k)-\omega(\theta_k)\|.
\end{aligned}
$$

Multiplying both sides with $e^{-\gamma t}$ and setting $u(t)=e^{-\gamma t}\|z(t)-\omega(t)\|$ one can obtain

$$
u(t) \leq Ku(t_0) + \int_{t_0}^{t} K\beta\ell_f u(s)ds + \sum_{t_0\leq\theta_k<t} K(\ell_J E+\ell_G)u(\theta_k).
$$

Now, the Gronwall-Bellman Lemma for piecewise continuous functions [1] implies the inequality

$$
\begin{aligned}
u(t) &\leq Ku(t_0)e^{\int_{t_0}^{t} K\beta\ell_f ds} \prod_{t_0\leq\theta_k<t} (1+K(\ell_J E+\ell_G)) \\
&= Ku(t_0)e^{(t-t_0)K\beta\ell_f} \prod_{t_0\leq\theta_k<t} (1+K(\ell_J E+\ell_G)).
\end{aligned}
$$

Finally, back-substituting $u(t)$, multiplicating the inequality by $e^{\gamma t}$ and using (1.23) will result in

$$
\begin{aligned}
&\|z(t)-\omega(t)\| \leq Ke^{(\gamma+K\beta\ell_f)(t-t_0)}\|z(t_0)-\omega(t_0)\|e^{i([t_0,t])\ln(1+K(\ell_J E+\ell_G))} \\
&= Ke^{\Gamma_2(t-t_0)}(1+K(\ell_J E+\ell_G))\|z(t_0)-\omega(t_0)\|, \qquad (3.2)
\end{aligned}
$$

where $\Gamma_2 = \left(\gamma + K\beta\ell_f + \frac{\ln(1+K(\ell_J E+\ell_G))}{\underline{\theta}}\right)$. Hence, the condition *(C10)* implies that the solution $\omega(t)$ is asymptotically stable. □

3.2 The Main Periodicity Result

In the previous section, it is proved that B-equivalent system is (ω,p)-periodic and asymptotically stable. This section is devoted to prove that the solution of the system (1.34), which is a recurrent neural network with structured impulses at non-prescribed moments, is periodic and asymptotically stable. First the periodicity of the impact moments will be proved, then the periodicity of the derivative sequences of impact moments will be shown. Finally the main theorem of the paper will be stated and proved.

Lemma 12 *The discontinuity moments, ζ_k, of the solution of the system (1.34) are periodic.*

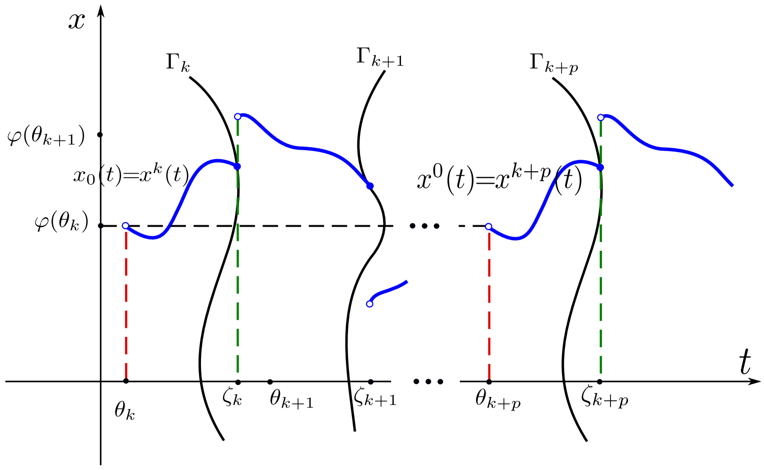


Figure 3.1: The graph of the periodic solution $\psi(t)$ of the system (1.34) that has periodicity conditions.

Proof. Let $\varphi(t)$ be solution of the system (2.5) which is B-equivalent system of (1.34) with initial condition $\varphi(\theta_k) = x_0$. Then the function $\varphi(t)$, $t \in \mathbb{R}$ *is a discontinuous periodic function.* Then the sequence $\varphi(\theta_k)$, $k \in \mathbb{Z}$ is periodic, i.e. $\varphi(\theta_{k+p}) = \varphi(\theta_k)$.

Denote $x^k(t) = x(t, \theta_k, \psi(\theta_k))$ i.e. the continuous solution on the interval $(\theta_k, \zeta_k]$. Also let $\psi(t)$ be solution of the system (1.34) with the same initial conditions $\psi(\theta_k) = x_0$. Then by the definition of B-equivalence, $\psi(t) = \varphi(t)$, for t values that are not in the interval with the endpoints θ_k and ξ_k.

Lastly denote by ζ_k the points where the solution $\psi(t)$ meets the surfaces of discontinuities $t = \theta_k + \tau_k(x(t))$. Thus $\zeta_k = \theta_k + \tau_k(\psi(\zeta_k)) = \theta_k + \tau_k(\varphi(\zeta_k)) = \theta_k + \tau_k(x^k(\zeta_k))$.

If $x_0(t)$ is the solution of the system (1.34) with initial conditions $x(\theta_k) = x = \varphi(\theta_k)$, then for $t \in [\theta_k, \theta_k + m_\tau]$

$$x_0(t) = x + \int_{\theta_k}^{t} (\mathcal{A}(s)x_0(s) + \mathcal{B}(s)F(x_0(s)) + \mathcal{C}(s))ds.$$

As in preparation of Lemma 7, denote $\bar{x}(t) = x^0(t + \omega)$, then $\bar{x}$ is the solution of the differential equation,

$$y' \quad = \quad \mathcal{A}(t + \omega)y(t) + \mathcal{B}(t + \omega)F(y(t)) + \mathcal{C}(t + \omega), \tag{3.3}$$

where $t \in [\theta_k, \theta_k + m_\tau]$ with the initial conditions $\bar{x}(\theta_k) = \psi(\theta_{k+p})$.

Hence $\bar{x}(t) = x_0(t)$ on $t \in [\theta_k, \theta_k + m_\tau]$. Using the equality

$$\begin{aligned}\zeta_{k+p} = \theta_{k+p} + \tau_{k+p}(x^k(\zeta_{k+p})) &= \theta_k + \omega + \tau_k(x^k(\zeta_{k+p})) \\ &= \theta_k + \omega + \tau_k(x_0(\zeta_{k+p} - \omega)),\end{aligned}$$

one can get that

$$\zeta_{k+p} - \omega = \theta_k + \tau_k(x_0(\zeta_{k+p} - \omega)).$$

Since ζ_k is the only solution of $t = \theta_k + \tau_k(x_0(t))$, the discontinuity moments satisfy $\zeta_{k+p} - \omega = \zeta_k$. This means that the sequence ζ_k is periodic. □

Theorem 13 *If the conditions (C1)-(C10) and (P1)-(P3) are valid then the system (1.34) admits a unique periodic and asymptotically stable solution $\psi(t)$.*

Proof. Let $\varphi(t)$ be the periodic solution of the system (2.5). And set $\psi(t)$ as in Lemma 12. Recall by B-equivalence that

$$\begin{aligned}t \le \theta_k; \quad \psi(t) = \varphi(t) \text{ then } \psi(\theta_k) = \varphi(\theta_k), \\ t \ge \zeta_k; \quad \psi(t) = \varphi(t) \text{ then } \psi(\zeta_k) = \varphi(\zeta_k),\end{aligned}$$

and in the previous section, we proved that the periodic solution of the B-equivalent system (2.5), $\varphi(t)$, is asymptotically stable. So whenever $\varphi(t) = \psi(t)$, the solution $\psi(t)$ is periodic. Now we should show that $\psi(t)$ is periodic in the intervals when $\psi(\theta_k) \ne \varphi(\theta_k)$. Observe that $\theta_{k+p} \le t + \omega \le \zeta_{k+p}$ whenever $\theta_k \le t \le \zeta_k$. On the interval $\theta_k \le t \le \zeta_k$, the solution $\psi(t)$ satisfies the following equation:

$$\psi(t) = \varphi(\theta_k) + \int_{\theta_k}^{t} (A(s)\psi(s) + B(s)F(\psi(s)) + C(s))ds. \tag{3.4}$$

Additionally, when $\theta_{k+p} \leq t+\omega \leq \zeta_{k+p}$, one can obtain that the solution of the system (1.34) satisfies

$$\psi(t+\omega) = \varphi(\theta_{k+p}) + \int_{\theta_{k+p}}^{t+\omega} (A(s)\psi(s) + B(s)F(\psi(s)) + C(s))ds. \qquad (3.5)$$

Hence by a simple change of variable and the periodicity of $\varphi(t)$ and the coefficients one can get

$$\begin{aligned} &\psi(t+\omega) \\ &= \varphi(\theta_{k+p}) + \int_{\theta_{k+p}-\omega}^{t} (A(s+\omega)\psi(s+\omega) + B(s+\omega)F(\psi(s+\omega)) + C(s+\omega))ds \\ &= \varphi(\theta_k) + \int_{\theta_k}^{t} (A(s)\psi(s+\omega) + B(s)F(\psi(s+\omega)) + C(s))ds. \end{aligned} \qquad (3.6)$$

Denote $\Psi(t) = \psi(t+\omega)$ where $\theta_k \leq t \leq \zeta_k$. Then

$$\Psi(t) = \varphi(\theta_k) + \int_{\theta_k}^{t} (A(s)\Psi(s) + B(s)F(\Psi(s)) + C(s))ds. \qquad (3.7)$$

The last equation together with the equation (3.4) implies that $\psi(t) = \Psi(t) = \psi(t+\omega)$. Asymptotic stability is proved in the same manner by using $\varphi(t)$'s asymptotic stability. $\square$

CHAPTER 4

ALMOST PERIODIC RECURRENT NEURAL NETWORKS WITH FIXED MOMENT OF IMPULSES

The following neural model with prescribed moments of impulses is in the focus of this chapter,

$$\begin{aligned} x_i' &= a_i(t)x_i + \sum_{j=1}^{m} b_{ij}(t)f_j(x_j) + c_i(t),\, t \neq \theta_k, \\ \Delta x_i\big|_{t=\theta_k} &= d_{ik}x_i + \sum_{j=1}^{m} e_{ijk}I_j(x_j) + h_{ik},\, i = 1, 2, \ldots, m, \end{aligned} \tag{4.1}$$

where $x_i \in \mathbb{R}$ with $i = 1, 2, \ldots, m$, $t \in \mathbb{R}$, and the sequence θ_k is increasing such that $|\theta_k| \to \infty$ as $k \to \infty$. The jump equations are set by $\Delta x_i\big|_{t=\theta_k} = x_i(\theta_k+) - x_i(\theta_k)$, assuming that the limit $x_i(\theta_k+) = \lim_{t\to\theta_k^+} x_i(t)$ exists. The system coefficients are in the same role as system 1.33 except for the impulse moments.

4.1 Almost Periodic Oscillations

Consider the RINN (4.1) and the associated homogeneous linear equations (1.18). We will prove the solution of the system is unique and asymptotically stable that is the main theorem of this chapter. To make it easier, the problem is divided into three parts in what follows. First of all, for the discussion needs we defined a space of discontinuous almost periodic functions and an operator acting on the space. In the next step the invariance of the operator on the specified space and the contraction property of it is proved. Finally, the main theorem will be verified. The following are the conditions needed for almost periodicity.

(A1) The derivative sequences θ_k^j, $k, j \in \mathbb{Z}$ are positive and uniformly almost periodic in k. Consequently, there exist numbers $\underline{\theta} > 0$ and $\overline{\theta} > 0$, with $\underline{\theta} \leq \theta_{k+1} - \theta_k \leq \overline{\theta}$.

(A2) $a_i(t)$, $b_{ij}(t)$, $c_i(t)$ are (uniformly continuous) almost periodic functions for $i = 1, 2, \ldots, m$.

(A3) The sequences d_{ik}, e_{ijk}, h_i are almost periodic in k where $i, j = 1, 2, \ldots, m$, $k \in \mathbb{Z}$.

Lemma 14 *[9] Let $f_i(t)$, $i = 1, 2, \ldots, n$, be discontinuous almost periodic functions with the common sequence of discontinuity moments θ_k, $k \in \mathbb{Z}$, the sequences I_k^i, $k \in \mathbb{Z}$, $i = 1, 2, \ldots, p$, are almost periodic, $g_k^i(x)$, $k \in \mathbb{Z}$, $i = 1, 2, \ldots, l$, are vector functions uniformly almost periodic on their domains. Moreover for the system (4.1) the condition (C1) is valid. Then for arbitrary positive ε and $\nu < \varepsilon$ there exists relatively dense sets of real numbers $\mathcal{R}$ and integers $\mathcal{Q}$ such that*

1. *$\|f_i(t+\tau) - f_i(t)\| < \varepsilon$, for all $t \in \mathbb{R}$, $|t - \theta_k| > \varepsilon$ $i = 1, 2, \ldots, n$,*
2. *$\|g_{k+q}^i(x) - g_k^i(x)\| < \varepsilon$, for all x from the domain, for all $k \in \mathbb{Z}$, $i = 1, 2, \ldots, l$,*
3. *$\|I_{k+q}^i - I_k^i\| < \varepsilon$, for all $k \in \mathbb{Z}$, $i = 1, 2, \ldots, p$,*
4. *$\|\theta_k^q - \tau\| < \nu$, for all $k \in \mathbb{Z}$,*

if $\tau \in \mathcal{R}$ and $q \in \mathcal{Q}$.

This lemma is referred as *the lemma on the hybrid common almost periods* or shortly the lemma on *hybrid almost periods*. By definition, it is helpful in the study of more than one almost periodic discontinuous functions and sequences.

Lemma 15 *[9] Let the conditions (C1) and (A1)-(A2) be valid. Then for any positive ε there exists a relatively dense set of almost periods τ of $A(t)$, such that*

$$\|X(t+\tau, s+\tau) - X(t,s)\| < \varepsilon \mathcal{L} e^{\frac{\gamma}{2}(t-s)} \tag{4.2}$$

where $t \geq s$, $|t - \theta_k| > \varepsilon$, and $\mathcal{L}$ is independent of ε and τ.

Let $\mathcal{D}$ be the space of discontinuous almost periodic (d.a.p.) functions bounded by H, with discontinuity points θ_k, $k \in \mathbb{Z}$. We will utilize in what follows the norm $\|\varphi(t)\|_0 = \sup_{t \in \mathbb{R}} \|\varphi(t)\|$, where $\varphi \in \mathcal{D}$.

On the space $\mathcal{D}$, $\varphi \in \mathcal{D}$, set the operator T such that

$$T\varphi(t) = (T\varphi)(t) = \int_{-\infty}^{t} X(t,s)\Big(\mathcal{B}(s)F(\varphi(s)) + \mathcal{C}(s)\Big)ds$$
$$+\sum_{\theta_k<t} X(t,\theta_k)\Big(E_k I(\varphi(\theta_k)) + H_k\Big).$$

Throughout this chapter the following conditions will be applied:

(C11) $K\left(\dfrac{m_f\beta+\sigma}{-\gamma} + \dfrac{m_I E + m_h}{1-e^{\gamma\underline{\theta}}}\right) < H;$

(C12) $K\left(\dfrac{\ell_f\beta}{-\gamma} + \dfrac{\ell_I E}{1-e^{\gamma\underline{\theta}}}\right) < 1;$

(C13) $\gamma + K\beta\ell_f + \frac{\ln(1+KE\ell_I)}{\underline{\theta}} < 0.$

Lemma 16 *If the conditions (C1),(C11) and (A1)-(A3) are satisfied, then T is invariant on $\mathcal{D}$.*

Proof. For invariance, $T(\mathcal{D}) \subseteq \mathcal{D}$ must be proven.

Let us fix $\varphi \in \mathcal{D}$. To show invariance, estimate first $\|T(\varphi)\|$. We have that

$$\begin{aligned}
\|T\varphi(t)\| &\leq \int_{-\infty}^{t} \|X(t,s)\|\|\mathcal{B}(s)F(\varphi(s)) + \mathcal{C}(s)\| \\
&+ \sum_{\theta_k<t} \|x(t,\theta_k)\|\|E_k I(\varphi(\theta_k)) + H_k\| \\
&\leq \int_{-\infty}^{t} Ke^{\gamma(t-s)}(\|\mathcal{B}(s)\|\|F(\varphi(s))\| + \|\mathcal{C}(s)\|) \\
&+ \sum_{\theta_k<t} Ke^{\gamma(t-\theta_k)}\|E_k I(\varphi(\theta_k)) + H_k\| \\
&\leq K\left(\frac{m_f\beta+\sigma}{-\gamma} + \frac{m_I E + m_h}{1-e^{\gamma\underline{\theta}}}\right).
\end{aligned}$$

The last inequality, by the condition (C11) implies that $\|T\varphi(t)\| < H$.

Consider the numbers $\underline{\theta}$ and $\overline{\theta}$ as constants as mentioned before. Let j be the integer such that $\theta_j < t \leq \theta_{j+1}$. One can obtain the following expression

$$\sum_{\theta_k<t} e^{\gamma(t-\theta_k)} = \sum_{n=0}^{\infty} e^{\gamma(t-\theta_j)}e^{\gamma(\theta_j-\theta_{j-n})} = e^{\gamma(t-\theta_j)}\sum_{n=0}^{\infty} e^{\gamma\underline{\theta}n} \leq \frac{1}{1-e^{\gamma\underline{\theta}}}. \tag{4.3}$$

The inequalities $t-\theta_j > 0$ and $|\theta_{k+q} - \tau - \theta_k| < \varepsilon$ imply that

$$e^{\frac{\gamma}{2}(t-\theta_{j+q}+\tau)} \leq e^{-\varepsilon\frac{\gamma}{2}}. \tag{4.4}$$

By the last inequality one can obtain,

$$
\begin{aligned}
&\sum_{\theta_k<t} \|X(t+\tau,\theta_{k+q}) - X(t,\theta_{k+q}-\tau)\| \\
&\leq \sum_{\theta_k<t} \varepsilon\mathcal{L}e^{\frac{\gamma}{2}(t+\tau-\theta_{k+q})} = \sum_{n=0}^{\infty} \varepsilon\mathcal{L}e^{\frac{\gamma}{2}(t+\tau-\theta_{j+q})}e^{\frac{\gamma}{2}(\theta_{j+q}-\theta_{j+q-n})} = \varepsilon\mathcal{L}e^{-\varepsilon\frac{\gamma}{2}} \sum_{n=0}^{\infty} e^{\frac{\gamma}{2}\underline{\theta}n} \\
&\leq \varepsilon\mathcal{L}e^{-\varepsilon\frac{\gamma}{2}} \frac{1}{1-e^{\frac{\gamma}{2}\underline{\theta}}}. \qquad (4.5)
\end{aligned}
$$

If j is the integer satisfying $\theta_j < \theta_k \leq \theta_{j+1}$, then by using the Lemma 14 one can obtain $\theta_k - \theta_{k+q} + \tau < \eta < \varepsilon < \underline{\theta} < \theta_k - \theta_j$. Then $\theta_j < \theta_{k+q} - \tau$. Similarly, $\theta_{j+1} > \theta_{k+q} - \tau$. Consequently, combining two previous results as $\theta_j < \theta_{k+q} - \tau \leq \theta_{j+1}$, one can obtain that the products $\prod_{\theta_{k+q}-\tau\leq\theta_n<t}(1+d_{in})$ and $\prod_{\theta_k\leq\theta_n<t}(1+d_{in})$, then they are equal to each other.

Using the state transition solution of (1.18) and (C1) the following result is obtained

$$
\begin{aligned}
&\sum_{\theta_k<t} \|X(t,\theta_{k+q}-\tau) - X(t,\theta_k)\| \\
&= \sum_{\theta_k<t} \|e^{\int_{\theta_{k+q}-\tau}^{t}\mathcal{A}(u)du} \prod_{\theta_{k+q}-\tau\leq\theta_n<t}(I+D_n) - e^{\int_{\theta_k}^{t}\mathcal{A}(u)du} \prod_{\theta_k\leq\theta_n<t}(I+D_n)\| \\
&\leq \sum_{\theta_k<t} Ke^{\gamma(t-\theta_k)}|e^{\varepsilon\alpha}-1| \leq |1-e^{\varepsilon\alpha}|K\sum_{\theta_k<t} e^{\gamma(t-\theta_k)} \leq |1-e^{\varepsilon\alpha}|\frac{K}{1-e^{\gamma\underline{\theta}}}.
\end{aligned}
$$

By the Lemma 3.9 in [9], $\varphi(\theta_k)$ is an almost periodic sequence in k.

Applying Lemma 15 and lemma on hybrid almost periods, which is Lemma 14,, one can show that

$$
\begin{aligned}
&\|T\varphi(t+\tau) - T\varphi(t)\| \\
&\leq \int_{-\infty}^{t} \|X(t+\tau,s+\tau)(\mathcal{B}(s+\tau)F(\varphi(s+\tau)) + \mathcal{C}(s+\tau)) \\
&- X(t,s)(\mathcal{B}(s)F(\varphi(s)) + C(s))\|ds \\
&+ \sum_{\theta_k<t} \|X(t+\tau,\theta_{k+q})(E_{(k+q)}I(\varphi(\theta_{k+q})) + H_{k+q}) - X(t,\theta_k)(E_kI(\varphi(\theta_k)) + H_k)\| \\
&\leq \int_{-\infty}^{t} \|X(t+\tau,s+\tau) - X(t,s)\| \big\|\mathcal{B}(s+\tau)F(\varphi_j(s+\tau)) + \mathcal{C}(s+\tau)\big\|ds \\
&+ \int_{-\infty}^{t} \|X(t,s)\|\|\mathcal{B}(s+\tau) - \mathcal{B}(s)\|\|F(\varphi(s+\tau))\|ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_{-\infty}^{t} \|X(t,s)\| \|\mathcal{B}(s)\| \|F(\varphi(s+\tau)) - F(\varphi(s))\| ds \\
&+ \int_{-\infty}^{t} \|X(t,s)\| \|\mathcal{C}(s+\tau) - \mathcal{C}(s)\| ds \\
&+ \sum_{\theta_k<t} \Big(\|X(t+\tau, \theta_{k+q}) - X(t, \theta_{k+q} - \tau)\| + \|X(t, \theta_{k+q} - \tau) - X(t, \theta_k)\| \Big) \times \\
&\times \big\| E_{k+q} I(\varphi(\theta_{k+q})) + H_{k+q} \big\| \\
&+ \sum_{\theta_k<t} \|X(t,\theta_k)\| \|E_{k+q} - E_k\| \big\| I(\varphi(\theta_{k+q})) \big\| \\
&+ \sum_{\theta_k<t} \|X(t,\theta_k)\| \|E_k\| \big\| I(\varphi(\theta_{k+q})) - I(\varphi(\theta_k)) \big\| \\
&+ \sum_{\theta_k<t} \|X(t,\theta_k)\| \|H_{k+q} - H_k\| \\
&\leq \int_{-\infty}^{t} \varepsilon \mathcal{L} e^{\frac{\gamma}{2}(t-s)} (\beta m_f + \sigma) ds + \int_{-\infty}^{t} K e^{\gamma(t-s)} \varepsilon m_f ds \\
&+ \int_{-\infty}^{t} K e^{\gamma(t-s)} \ell_f B \underline{\varepsilon} ds + \int_{-\infty}^{t} K e^{\gamma(t-s)} \varepsilon ds \\
&+ \sum_{\theta_k<t} (\varepsilon \mathcal{L} e^{\frac{\gamma}{2}(t-(\theta_{k+q}-\tau))} + K(1-e^{\varepsilon\alpha}) e^{\gamma(t-\theta_k)})(m_I E + m_H) \\
&+ \sum_{\theta_k<t} K e^{\gamma(t-\theta_k)} \varepsilon m_I + \sum_{\theta_k<t} K e^{\gamma(t-\theta_k)} \ell_I E \underline{\varepsilon} + \sum_{\theta_k<t} K e^{\gamma(t-\theta_k)} \varepsilon \\
&\leq \Big(2\varepsilon \mathcal{L} (\beta m_f + \sigma) + K \varepsilon m_f + K \ell_f B \varepsilon + K \varepsilon \Big) \frac{1}{-\gamma} \\
&+ \Big(\varepsilon \mathcal{L} e^{-\varepsilon \frac{\gamma}{2}} \frac{1}{1 - e^{\frac{\gamma}{2}\underline{\theta}}} + (1 - e^{\varepsilon\alpha}) \frac{K}{1 - e^{\gamma \underline{\theta}}} \Big) (m_I E + m_H) \\
&+ \Big(K \varepsilon m_I + K \ell_I E \varepsilon + K \varepsilon \Big) \frac{1}{1 - e^{\gamma \underline{\theta}}} = \Gamma_1(\varepsilon) \varepsilon,
\end{aligned}
$$

where $\Gamma_1(\varepsilon)$ is a bounded function of ε. □

Lemma 17 *If the conditions (C1), (C11)-(C12) and (A1)-(A3) are valid, then T is a contractive map.*

Proof. Let $\varphi(t)$ and $\psi(t)$ be elements of $\mathcal{D}$. One can obtain the following inequality

$$
\begin{aligned}
&\|T\varphi(t) - T\psi(t)\| \\
&\quad \leq \int_{-\infty}^{t} \|X(t,s)\| \|\mathcal{B}(s)(F(\varphi(s)) - F(\psi(s)))\| ds \\
&\quad + \sum_{\theta_k<t} \|X(t,\theta_k)\| \|E_k(I(\varphi(\theta_k)) - I(\psi(\theta_k)))\| \qquad (4.6)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_{-\infty}^{t} Ke^{\gamma(t-s)}\|\mathcal{B}(s)\|\ell_f\|\varphi(s)-\psi(s)\|ds + \sum_{\theta_k<t} Ke^{\gamma(t-\theta_k)}E_k\ell_I\|\varphi(\theta_k)-\psi(\theta_k)\| \\
&\leq \int_{-\infty}^{t} Ke^{\gamma(t-s)}\beta\ell_f\|\varphi(s)-\psi(s)\|_0 ds + \sum_{\theta_k<t} Ke^{\gamma(t-\theta_k)}E\ell_I\|\varphi(\theta_k)-\psi(\theta_k)\| \\
&\leq \int_{-\infty}^{t} Ke^{\gamma(t-s)}\ \beta\ell_f\|\varphi(t)-\psi(t)\|_0 ds + \sum_{\theta_k<t} Ke^{\gamma(t-\theta_k)}E\ell_I\|\varphi(t)-\psi(t)\|_0 \\
&\leq K\ell_f\beta\|\varphi(t)-\psi(t)\|_0\left(\frac{1}{-\gamma}\right) + K\ell_I E\|\varphi(t)-\psi(t)\|_0\left(\frac{1}{1-e^{\gamma\underline{\theta}}}\right) \\
&\leq K\left(\frac{\ell_f\beta}{-\gamma} + \frac{\ell_I E}{1-e^{\gamma\underline{\theta}}}\right)\|\varphi(t)-\psi(t)\|_0. \qquad (4.7)
\end{aligned}
$$

Then $\|T\varphi(t)-T\psi(t)\|_0 \leq K\left(\frac{\ell_f\beta}{-\gamma} + \frac{\ell_I E}{1-e^{\gamma\underline{\theta}}}\right)\|\varphi(t)-\psi(t)\|_0$. Hence by the condition (C12), T is a contraction. $\square$

Theorem 18 *Assume that (C1), (C11)-(C13) and (A1)-(A3) are fulfilled. Then the RINN (4.1) has a unique asymptotically stable discontinuous almost periodic solution.*

Proof. By the Lemmas 16 and 17, the operator T is invariant on the space $\mathcal{D}$. That is, if $\varphi(t) \in \mathcal{D}$, then $T\varphi(t) \in \mathcal{D}$, and the operator is a contraction. Hence, $\mathcal{D}$ has a unique element $\omega(t)$, such that $T\varphi = \varphi$, and it is the unique solution of our system.

Let $z(t) = [z_1(t), \ldots, z_m(t)]$ be the solution of the system (4.1) with the initial conditions $z(t_0) = z_0$. Applying the equivalent integral equation for the initial value problem one can obtain

$$
\begin{aligned}
\|z(t)-\omega(t)\| &\leq \Big\|X(t,t_0)(z(t_0)-\omega(t_0)) + \int_{t_0}^{t} X(t,s)(\mathcal{B}(F(z(s))-F(\omega(s))))ds \\
&+ \sum_{t_0\leq\theta_k<t} X(t,\theta_k)\big(E_k(I(z(\theta_k))-I(\omega(\theta_k)))\big)\Big\| \\
&\leq \|x(t,t_0)\|\|(z(t_0)-\omega(t_0))\| + \int_{t_0}^{t} Ke^{\gamma(t-s)}\beta\ell_f\|z(s)-\omega(s)\|ds \\
&+ \sum_{t_0\leq\theta_k<t} Ke^{\gamma(t-\theta_k)}E\ell_I\|z(\theta_k)-\omega(\theta_k)\|.
\end{aligned}
$$

Multiplying both sides with $e^{-\gamma t}$ and setting $u(t) = e^{-\gamma t}\|z(t)-\omega(t)\|$ one can obtain

$$
u(t) \leq Ku(t_0) + \int_{t_0}^{t} K\beta\ell_f u(s)ds + \sum_{t_0\leq\theta_k<t} KE\ell_I u(\theta_k).
$$

Now, the Gronwall-Bellman Lemma for piecewise continuous functions [1] implies the inequality

$$u(t) \;\leq\; Ku(t_0)e^{\int_{t_0}^{t} K\beta\ell_f ds} \prod_{t_0\leq\theta_k<t}(1+KE\ell_I) = Ku(t_0)e^{(t-t_0)K\beta\ell_f} \prod_{t_0\leq\theta_k<t}(1+KE\ell_I).$$

Lastly, back substitution of $u(t)$, and multiplying the inequality by $e^{\gamma t}$ and using (1.23) will result with

$$\begin{aligned}\|z(t)-\omega(t)\| &\leq Ke^{(\gamma+K\beta\ell_f)(t-t_0)}\|z(t_0)-\omega(t_0)\|e^{i([t_0,t])\ln(1+KE\ell_I)}\\ &= Ke^{\Gamma_3(t-t_0)}(1+KE\ell_I)\|z(t_0)-\omega(t_0)\|,\end{aligned} \tag{4.8}$$

where $\Gamma_3 = \left(\gamma + K\beta\ell_f + \frac{\ln(1+KE\ell_I)}{\underline{\theta}}\right)$. Hence, the condition (C13) implies that the solution $\omega(t)$ is an asymptotically stable solution. □

4.2 An Example

Let us consider the following system, which consists of two neurons,

$$\begin{aligned}
x_1' &= a_1(t)x_1 + b_{11}(t)\arctan(x_1) + b_{12}(t)\arctan(\frac{x_2}{2}) + c_1(t),\, t\neq\theta_k, && (4.9)\\
x_2' &= a_2(t)x_2 + b_{21}(t)\arctan(x_1) + b_{22}(t)\arctan(\frac{x_2}{2}) + c_2(t),\, t\neq\theta_k,\\
\Delta x_1\big|_{t=\theta_k} &= d_{1k}x_1 + e_{11k}\arctan(x_1) + e_{12k}\frac{1}{1+e^{x_2}} + h_{1k},\\
\Delta x_2\big|_{t=\theta_k} &= d_{2k}x_2 + e_{21k}\arctan(x_1) + e_{22k}\frac{1}{1+e^{x_2}} + h_{2k}.
\end{aligned}$$

The fixed moments of impulses are given by the sequence $\theta_k = k + \frac{1}{8}(\sin(k) + \cos(\sqrt{2}k))$, $k\in\mathbb{Z}$. One can check that the inequalities $\frac{3}{4}\leq\theta_{k+1}-\theta_k\leq\frac{5}{4}$ are correct for all k, and the condition (A1) is satisfied, that is the sequences θ_k^j, $j\in\mathbb{Z}$, are positive and equipotentially almost periodic in k.

The self regulation rates, $a_1(t) = -2+\frac{1}{10}|\sin(t)+\cos(\sqrt{7}t)|$, $a_2(t) = \frac{1}{2}+\frac{1}{10}|\sin(t)+\cos(\sqrt{11}t)|$, $t\in\mathbb{R}$, the synaptic connection weights of unit j on the unit i for $i,j = 1,2$ $b_{11}(t) = \frac{1}{100}(\sin(\sqrt{2}t)+\cos(t))$, $b_{12}(t) = \frac{1}{100}(\sin(\sqrt{3}t)+\cos(t))$, $b_{21}(t) = \frac{1}{100}(\sin(\sqrt{5}t)+\cos(t))$, $b_{22}(t) = \frac{1}{100}(\sin(\sqrt{7}t)+\cos(t))$, the external inputs $c_1(t) = \frac{1}{10}(\sin(\sqrt{2}t)+\cos(\sqrt{5}t))$ and $c_2(t) = \frac{1}{10}(\sin(\sqrt{3}t)+\cos(\sqrt{7}t))$ are almost periodic functions [76].

The impulsive rates are equal to $d_{1k} = 0,8 - \dfrac{1}{10}|\sin(k) + \cos(\sqrt{3}k)|$, $d_{2k} = -1 - \dfrac{1}{10}|\sin(k)+\cos(\sqrt{5}k)|$ and the instantaneous synaptic connection weights are $e_{11k} = \dfrac{1}{100}(\sin(\sqrt{2}k)+\cos(\sqrt{3}k))$, $e_{12k} = \dfrac{1}{100}(\sin(\sqrt{3}k)+\cos(2k))$, $e_{21k} = \dfrac{1}{100}(sin(2k)+cos(\sqrt{5}k))$, $e_{22k} = \dfrac{1}{100}(\sin(\sqrt{5}k) + \cos(\sqrt{7}k))$. The members of the sequence h_{ik} which corresponds to the external impulsive inputs are equal to $h_{1k} = \dfrac{1}{4}(\sin(\sqrt{2}k) + \cos(\sqrt{7}k))$ and $h_{2k} = \dfrac{1}{4}(\sin(\sqrt{2}k) + \cos(\sqrt{7}k))$. The sequences d_{ik}, e_{ijk}, h_i, $i, j = 1, 2$, are almost periodic in $k \in \mathbb{Z}$ [9]. Thus, the conditions (A2) and (A3) are satisfied.

The sigmoid activation functions, are presented by $f_1(u) = \arctan u$, $f_2(u) = \arctan \frac{u}{2}$ and the functions $I_1(u) = \arctan u$, $I_2(u) = \dfrac{1}{1+e^{-u}}$, $u \in \mathbb{R}$, denote measure of impact responses or impact activations.

Moreover, the functions f_i and I_i, $i = 1, 2$, satisfy Lipschitz condition with $\ell_f = 1$ and $\ell_I = 1$. Consider the initial conditions selected as $(x_1(0), x_2(0)) = (4, 4)$, and $H = 20$. One can verify that the coefficients of the system satisfy $-2 \leq a_1(t) \leq -1.8$ and $1 < 1 + d_{1k} \leq 1.8 = q_1$, $\gamma_1 = -1.016284$, $k_1 = 1.8$, $0.5 \leq a_2(t) \leq 0.7$ and $0 \leq |1 + d_{2k}| \leq 0.2 = q_2$, $\gamma_2 = -0.58755$, $k_2 = 5$. Hence, $|x_i(t, s)| \leq k_i e^{\gamma_i(t-s)}$ for $i = 1, 2$. Letting $K = \max\limits_{1\leq i\leq 2} k_i = 5$ and $\gamma = \max\limits_{1\leq i\leq 2} \gamma_i = -0.58755$, we verify that (C1) is satisfied, that is $\|X(t, s)\| \leq Ke^{\gamma(t-s)}$. One can check that $K\left(\dfrac{m_f\beta + \sigma}{-\gamma} + \dfrac{m_I E + m_h}{1 - e^{\gamma\underline{\theta}}}\right) = 11.834$ and $K\left(\dfrac{\ell_f\beta}{-\gamma} + \dfrac{\ell_I E}{1 - e^{\gamma\underline{\theta}}}\right) = 0.901$, that is (C11) and (C12) are valid. It is verified that $\gamma + K\beta\ell_f + \frac{\ln(1+KE\ell_I)}{\underline{\theta}} = -0.144 < 0$. This implies that the condition (C13) holds.

Thus, by the Theorem 18 there exists a unique asymptotically stable almost periodic solution of the system (4.9).

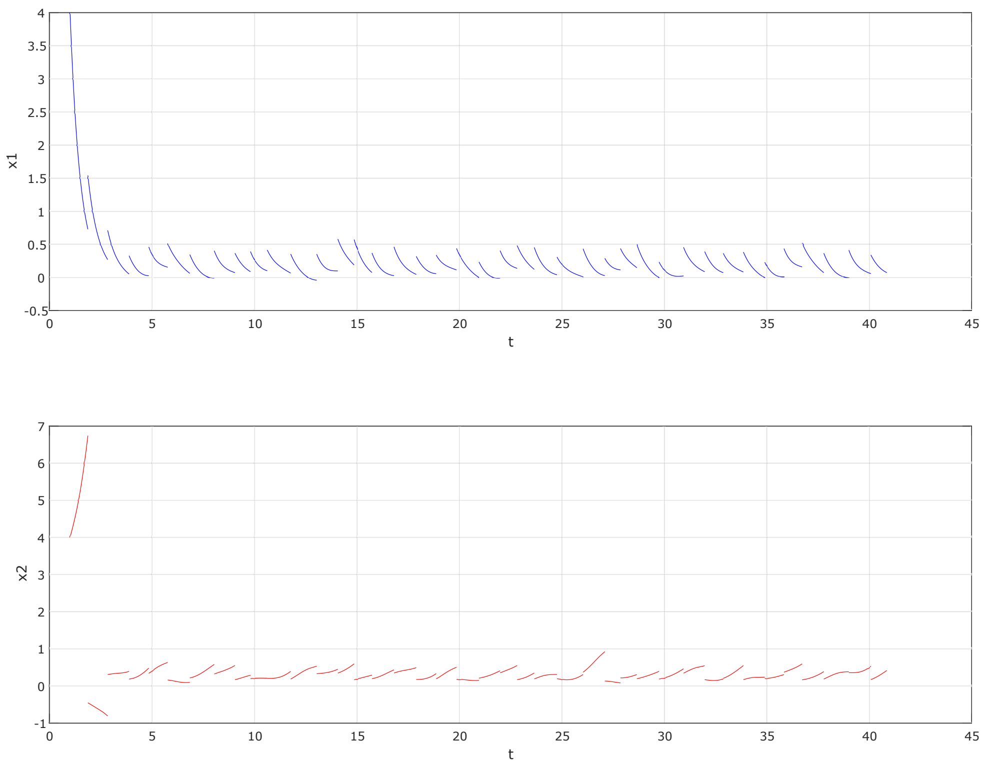


Figure 4.1: The simulation of the solution of the system 4.9 on $t - x_1$ and $t - x_2$ axes

CHAPTER 5

ALMOST PERIODIC RECURRENT NEURAL NETWORKS WITH VARIABLE MOMENT OF IMPULSES

In this chapter, we study the system (1.33), in its matrix form (1.34) which is a recurrently structured impulsive neural network with non-prescribed moment of impulses, under almost periodicity conditions. The system is studied for existence, uniqueness, stability and almost periodicity of a solution. The Lemma 14 which is on the hybrid almost periods and Lemma 15 from Chapter 4 will be employed in this chapter also.

5.1 Almost Periodicity of the Reduced Recurrent Neural Network

The B-equivalent system, (2.5), of (1.34) is the focus of this section. The almost periodicity assumptions that are used in this chapter are the following.

(A1) The derivative sequences θ_k^j, $k, j \in \mathbb{Z}$ are positive and uniformly almost periodic in k. That is, for arbitrary positive ε, there exists a set of ε-almost periods common for all the sequences θ_k^j, $j \in \mathbb{Z}$.

As a consequence of this condition there exist numbers $\underline{\theta} > 0$ and $\overline{\theta} > 0$, satisfying $\underline{\theta} \leq \theta_{k+1} - \theta_k \leq \overline{\theta}$.

(A2) $a_i(t)$, $b_{ij}(t)$, $c_i(t)$ are almost periodic functions for $i = 1, 2, \ldots, m$.

(A3) The sequences d_{ik}, e_{ijk}, h_i are almost periodic in k, $k \in \mathbb{Z}$, $i, j = 1, 2, \ldots, m$.

Let us denote by ξ_k, the impact moments where the solution of the system (2.5) meets the surface of discontinuities, i.e. ξ_k is the real number in the interval $[\theta_k, \theta_k + m_\tau]$ such that $\xi_k = \theta_k + \tau_k(x(\xi_k, \theta_k, x))$.

Lemma 19 *If the conditions (C2)-(C7) and (A1)-(A3) are satisfied, then the sequence $\xi_k - \theta_k$, $k \in \mathbb{Z}$, is almost periodic.*

Proof. We have that $\xi_k = \theta_k + \tau_k(x_0(\xi_k))$ and $\xi_{k+q} = \theta_{k+q} + \tau_{k+q}(x_0(\xi_{k+q}))$, where $x_0(t) = x(t, \theta_k, x)$ and $x^0(t) = x(t, \theta_{k+q}, x)$ are solutions of the differential equation

$$x' = \mathcal{A}(t)x(t) + \mathcal{B}(t)F(x(t)) + \mathcal{C}(t)$$

with initial conditions $x_0(\theta_k) = x$ and $x^0(\theta_{k+q}) = x$, respectively. Then the solutions satisfy the following,

$$x_0(t) = x + \int_{\theta_k}^{t} (\mathcal{A}(s)x_0(s) + \mathcal{B}(s)F(x_0(s)) + \mathcal{C}(s))ds, \quad t \in [\theta_k, \theta_k + m_\tau],$$

$$x^0(t) = x + \int_{\theta_{k+q}}^{t} (\mathcal{A}(s)x^0(s) + \mathcal{B}(s)F(x^0(s)) + \mathcal{C}(s))ds, \quad t \in [\theta_{k+q}, \theta_{k+q} + m_\tau].$$

Let us substitute $t = u + (\theta_{k+q} - \theta_k)$ to the last equation to obtain

$x^0(u+(\theta_{k+q}-\theta_k)) = x + \int_{\theta_{k+q}}^{u+(\theta_{k+q}-\theta_k)} (\mathcal{A}(s)x^0(s)+\mathcal{B}(s)F(x^0(s))+\mathcal{C}(s))ds$, where $u \in [\theta_k, \theta_k + m_\tau]$.

Denote $\bar{x}(t) = x^0(t + (\theta_{k+q} - \theta_k))$, from the last inequality $\bar{x}$ is the solution of the differential equation,

$$\begin{aligned} y' &= \mathcal{A}(t + (\theta_{k+q} - \theta_k))y(t) + \mathcal{B}(t + (\theta_{k+q} - \theta_k))F(y(t)) \\ &+ \mathcal{C}(t + (\theta_{k+q} - \theta_k)) \end{aligned} \tag{5.1}$$

where $t \in [\theta_k, \theta_k + m_\tau]$ with the initial conditions, $\bar{x}(t) = y(t, \theta_k, x)$.

Recall that $x_0(t) = x(t, \theta_k, x)$, where $x(t, \theta_k, x)$ is the solution of the system (2.5). Observe that the coefficients of the right hand side of the differential equation satisfy the inequality

$$\|\mathcal{A}(t+(\theta_{k+q}-\theta_k))-\mathcal{A}(t)\| \leq \|\mathcal{A}(t+(\theta_{k+q}-\theta_k))-\mathcal{A}(t+\tau)\|+\|\mathcal{A}(t+\tau)-\mathcal{A}(t)\| \leq \kappa_{\mathcal{A}}(\varepsilon)+\varepsilon,$$

$$\|\mathcal{B}(t+(\theta_{k+q}-\theta_k))-\mathcal{B}(t)\| \leq \|\mathcal{B}(t+(\theta_{k+q}-\theta_k))-\mathcal{B}(t+\tau)\|+\|\mathcal{B}(t+\tau)-\mathcal{B}(t)\| \leq \kappa_{\mathcal{B}}(\varepsilon)+\varepsilon,$$

$$\|\mathcal{C}(t+(\theta_{k+q}-\theta_k))-\mathcal{C}(t)\| \leq \|\mathcal{C}(t+(\theta_{k+q}-\theta_k))-\mathcal{C}(t+\tau)\|+\|\mathcal{C}(t+\tau)-\mathcal{C}(t)\| \leq \kappa_{\mathcal{C}}(\varepsilon)+\varepsilon,$$

where $|\theta_{k+q} - \theta_k - \tau| < \nu$ are the cause for the presence of the $\kappa_{\mathcal{A}}(\varepsilon)$, $\kappa_{\mathcal{B}}(\varepsilon)$, $\kappa_{\mathcal{C}}(\varepsilon)$, and $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are uniformly continuous, and will be used in continuous dependence of the solution on the right hand side of the differential equation. Observe that as $\varepsilon \to 0$, $\kappa_{\mathcal{A}}(\varepsilon) \to 0$, $\kappa_{\mathcal{B}}(\varepsilon) \to 0$, $\kappa_{\mathcal{C}}(\varepsilon) \to 0$.

Hence, by the continuous dependence on the right hand side of the differential equation [77],

$$\|\bar{x}(t) - x_0(t)\| < \kappa(\varepsilon) \tag{5.2}$$

on $t \in [\theta_k, \theta_k + m_\tau]$, and as $\varepsilon \to 0$, $\kappa(\varepsilon) \to 0$

By definition, $\xi_k = \theta_k + \tau_k(x_0(\xi_k))$ and $\xi_{k+q} = \theta_{k+q} + \tau_{k+q}(x_0(\xi_{k+q}))$. Denoting $\xi'_k = \xi_{k+q} - (\theta_{k+q} - \theta_k)$, then $\xi'_k = \theta_k + \tau_{k+q}(\bar{x}(\xi'_k))$ holds. Without loss of generality, assume that $\xi'_k > \xi_k$ then one can get,

$$\begin{aligned}
|\xi'_k - \xi_k| &= |\tau_{k+q}(\bar{x}(\xi'_k)) - \tau_k(x_0(\xi_k))| \\
&\leq |\tau_{k+q}(\bar{x}(\xi'_k)) - \tau_k(\bar{x}(\xi'_k)) + \tau_k(\bar{x}(\xi'_k)) - \tau_k(x_0(\xi_k))| \\
&\leq \varepsilon + \ell_\tau \|\bar{x}(\xi'_k) - \bar{x}(\xi_k) + \bar{x}(\xi_k) - x_0(\xi_k)\| \\
&\leq \varepsilon + \ell_\tau \kappa(\varepsilon) + \ell_\tau \| \int_{\xi_k}^{\xi'_k} (\mathcal{A}(s + (\theta_{k+q} - \theta_k))\bar{x}(s) \\
&\quad + \mathcal{B}(s + (\theta_{k+q} - \theta_k) F(\bar{(s)}) + \mathcal{C}(s + (\theta_{k+q} - \theta_k))ds\| \\
&\leq \varepsilon + \ell_\tau \kappa(\varepsilon) + \ell_\tau \|(\xi'_{k+q} - \xi_k)(\alpha h + \beta m_f + \sigma)\|.
\end{aligned}$$

Hence

$$|\xi'_{k+q} - \xi_k| \leq \frac{\varepsilon + \ell_\tau \kappa(\varepsilon)}{1 - \ell_\tau(\alpha h + \beta m_f + \sigma)}.$$

Since $\varepsilon > 0$ is arbitrarily small, it follows that the sequence $\xi'_{k+q} - \xi_k$ is almost periodic. On the other hand $|\xi'_{k+q} - \xi_k| = |\xi_{k+q} - (\theta_{k+q} - \theta_k) - \xi_k|$ and this implies

$$|(\xi_{k+q} - \theta_{k+q}) - (\xi_k - \theta_k)| \leq \frac{\varepsilon + \ell_\tau \kappa(\varepsilon)}{1 - \ell_\tau(\alpha h + \beta m_f + \sigma)}. \tag{5.3}$$

Hence the sequence $\xi_k - \theta_k$ is almost periodic. □

Lemma 20 *The derivative sequences ξ_k^j, $j \in \mathbb{Z}$, are uniformly almost periodic in k.*

Proof. Fix an integer j and fix a positive number ε. Since the derivative sequences, θ_k^j, are uniformly almost periodic and the sequence $\xi_k - \theta_k$ is almost periodic, by the lemma on hybrid almost periods there is relatively dense set of common $\varepsilon/3$-almost periods. Then one can obtain the following inequalities

$$\begin{aligned}
|\xi^j_{k+q} - \xi^j_k| &= |\xi_{k+j+q} - \xi_{k+j} - \xi_{k+j} + \xi_k - \theta^j_{k+q} + \theta^j_k + \theta^j_{k+q} - \theta^j_k| \\
&\le |(\xi_{k+j+q} - \theta_{k+j+q}) - (\xi_{k+j} - \theta_{k+j}) - (\xi_{k+q} - \theta_{k+q}) + (\xi_k - \theta_k)| \\
&+ |\theta^j_{k+q} - \theta^j_k| \\
&\le |(\xi_{k+j+q} - \theta_{k+j+q}) - (\xi_{k+j} - \theta_{k+j})| + |(\xi_{k+q} - \theta_{k+q}) - (\xi_k - \theta_k)| \\
&+ |\theta^j_{k+q} - \theta^j_k| \\
&\le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\end{aligned}$$

Consequently ξ^j_k, $j \in \mathbb{Z}$, are uniformly almost periodic. $\square$

Lemma 21 *Under the conditions (C2)-(C7) and (A1)-(A3), the coefficients in the B-equivalent system (2.5) $J_k(x)$, $G_k(x)$ are uniformly almost periodic.*

Proof. The method for the proof of almost periodicity of J_k, and G_k are quite similar, hence only almost periodicity of J_k will be shown. As above, denote

$x_0(t) = x + \int_{\theta_k}^{t} (\mathcal{A}(s)x_0(s) + \mathcal{B}(s)F(x_0(s)) + \mathcal{C}(s))ds$, where $t \in [\theta_k, \theta_k + m_\tau]$,

$x^0(t) = x + \int_{\theta_{k+q}}^{t} (\mathcal{A}(s)x^0(s) + \mathcal{B}(s)F(x^0(s)) + \mathcal{C}(s))ds$, where $t \in [\theta_{k+q}, \theta_{k+q} + m_\tau]$.

$$\begin{aligned}
&\|J_{k+q}(x) - J_k(x)\| = \|I(x^0(\xi_{k+q})) - I(x_0(\xi_k))\| \\
&= \|I(x + \int_{\theta_{k+q}}^{\xi_{k+q}} (\mathcal{A}(s)x^0(s) + \mathcal{B}(s)F(x^0(s)) + \mathcal{C}(s))ds) \\
&\quad -I(x + \int_{\theta_k}^{\xi_k} (\mathcal{A}(s)x_0(s) + \mathcal{B}(s)F(x_0(s)) + \mathcal{C}(s))ds)\| \\
&\le \ell_I \| \int_{\theta_{k+q}}^{\xi_{k+q}} (\mathcal{A}(s)x^0(s) + \mathcal{B}(s)F(x^0(s)) + \mathcal{C}(s))ds \\
&\quad - \int_{\theta_k}^{\xi_k} (\mathcal{A}(s)x_0(s) + \mathcal{B}(s)F(x_0(s)) + \mathcal{C}(s))ds\|
\end{aligned}$$

(substitute $s = u + \theta_{k+q} - \theta_k$ in the first integral and write integrand with variable s, and use notation $\xi_k - (\theta_{k+q} - \theta_k)) = \xi'_k$

$$\begin{aligned}
&= \ell_I \| \int_{\theta_k}^{\xi'_k} (\mathcal{A}(s + \theta_{k+q} - \theta_k)x^0(s + \theta_{k+q} - \theta_k) \\
&\quad + \mathcal{B}(s + \theta_{k+q} - \theta_k)F(x^0(s + \theta_{k+q} - \theta_k)) + \mathcal{C}(s + \theta_{k+q} - \theta_k))ds \\
&\quad - \int_{\theta_k}^{\xi_k} (\mathcal{A}(s)x_0(s) + \mathcal{B}(s)F(x_0(s)) + \mathcal{C}(s))ds\|
\end{aligned}$$

$$
\begin{aligned}
&= \ell_I \| \int_{\xi_k}^{\xi_k'} (\mathcal{A}(s+\theta_{k+q}-\theta_k)x^0(s+\theta_{k+q}-\theta_k) \\
&\quad +\mathcal{B}(s+\theta_{k+q}-\theta_k)F(x^0(s+\theta_{k+q}-\theta_k)) + \mathcal{C}(s+\theta_{k+q}-\theta_k))ds\| \\
&\quad +\ell_I \| \int_{\theta_k}^{\xi_k} (\mathcal{A}(s+\theta_{k+q}-\theta_k)\bar{x}_0(s) - \mathcal{A}(s)x_0(s) \\
&\quad +\mathcal{B}(s+\theta_{k+q}-\theta_k)F(\bar{x}_0(s)) - \mathcal{B}(s)F(x_0(s)) \\
&\quad +\mathcal{C}(s+\theta_{k+q}-\theta_k) - \mathcal{C}(s))ds\| \\
&= \ell_I \| \int_{\xi_k}^{\xi_k'} (\alpha h + \beta m_f + \sigma)ds\| \\
&\quad +\ell_I \| \int_{\theta_k}^{\xi_k} (\mathcal{A}(s+\theta_{k+q}-\theta_k)(\bar{x}_0(s) - x_0(s)) + (\mathcal{A}(s+\theta_{k+q}-\theta_k) - \mathcal{A}(s))x_0(s) \\
&\quad +\mathcal{B}(s+\theta_{k+q}-\theta_k)(F(\bar{x}_0(s)) - F(x_0(s))) + (\mathcal{B}(s+\theta_{k+q}-\theta_k) - \mathcal{B}(s))F(x_0(s)) \\
&\quad +\mathcal{C}(s+\theta_{k+q}-\theta_k) - \mathcal{C}(s))ds\| \\
&\le \ell_I |\xi_k' - \xi_k|(\alpha h + \beta m_f + \sigma) + \ell_I \| \int_{\theta_k}^{\xi_k} (\alpha(\bar{x}_0(s) - x_0(s)) + \varepsilon_{\mathcal{A}} x_0(s) \\
&\quad +\beta\ell_f(\bar{x}_0(s) - x_0(s)) + \varepsilon_{\mathcal{B}} F(x_0(s)) + \varepsilon_{\mathcal{C}})ds\| \\
&\le \ell_I \frac{\varepsilon + \ell_\tau \kappa(\varepsilon)}{1 - \ell_\tau(\alpha h + \beta m_f + \sigma)}(\alpha h + \beta m_f + \sigma) + \ell_I m_\tau((\alpha + \beta\ell_f)\kappa(\varepsilon) \\
&\quad +(\kappa_{\mathcal{A}}(\varepsilon) + \varepsilon)h + (\kappa_{\mathcal{B}}(\varepsilon) + \varepsilon)m_f + \kappa_{\mathcal{C}}(\varepsilon) + \varepsilon) \\
&= M(\varepsilon).
\end{aligned}
$$

Note that $M(\varepsilon) \to 0$ as $\varepsilon \to 0$. This ends the proof of the first part. As mentioned before, the proof of the almost periodicity of G_k involves similar calculations. $\square$

In what follows, almost periodicity of the solution of the system (2.5) which is the B-equivalent system of (1.34) will be studied and proved. Consider the linear homogenous counterpart of the B-equivalent system (2.5)

$$
\begin{aligned}
y_i' &= a_i(t)y_i, t \neq \theta_k, \\
\Delta y\big|_{t=\theta_k} &= d_{ik}y_i(\theta_k).
\end{aligned} \tag{5.4}
$$

where $a_i(t)$, $b_{ij}(t)$, $c_i(t)$, $f_j(u)$ are real valued continuous functions defined on real numbers, also d_{ik}, e_{ijk} are sequences in integer k.

Denote by $\mathcal{D}$, the space of discontinuous almost periodic (d.a.p.) functions bounded by H, where the discontinuity points are θ_k, $k \in \mathbb{Z}$, and the corresponding norm is $\|\varphi(t)\|_0 = \sup\limits_{t\in\mathbb{R}} \|\varphi(t)\|$, for $\varphi(t) \in \mathcal{D}$.

On the space $\mathcal{D}$, set the operator

$$
\begin{aligned}
T\varphi(t) &= (T\varphi)(t) \\
&= \int_{-\infty}^{t} X(t,s)\Big(\mathcal{B}(s)F(\varphi(s)) + \mathcal{C}(s)\Big)ds \\
&+ \sum_{\theta_k<t} X(t,\theta_k)\Big(E_kJ_k(\varphi(\theta_k)) + G_k(\varphi(\theta_k))\Big).
\end{aligned}
$$

Recall the following conditions which are needed through the remaining part of this chapter.

(C8) $K\left(\dfrac{m_fB+c}{-\gamma} + \dfrac{m_JE+m_g}{1-e^{\gamma\underline{\theta}}}\right) < H,$

(C9) $K\left(\dfrac{\ell_fB}{-\gamma} + \dfrac{\ell_JE+\ell_G}{1-e^{\gamma\underline{\theta}}}\right) < 1,$

(C10) $\gamma + K\beta\ell_f + \frac{\ln(1+K(\ell_JE+\ell_G))}{\underline{\theta}} < 0.$

Lemma 22 *Let $\varphi(t) \in \mathcal{D}$. If the conditions (C8) and (A1)-(A3) are valid, then $\varphi(t)$ belongs to $T(\mathcal{D})$, that is, T is invariant on $\mathcal{D}$.*

Proof. We need to show that if $\varphi \in \mathcal{D}$ then $T\varphi$ also belongs to $\mathcal{D}$. Let us consider the following inequalities

$$
\begin{aligned}
\|T\varphi(t)\| &\leq \int_{-\infty}^{t} \|X(t,s)\|\|\mathcal{B}(s)F(\varphi(s)) + \mathcal{C}(s)\| \\
&\quad + \sum_{\theta_k<t} \|X(t,\theta_k)\|\|E_kJ_k(\varphi(\theta_k)) + G_k(\varphi(\theta_k))\| \\
&\leq \int_{-\infty}^{t} Ke^{\gamma(t-s)}(\|\mathcal{B}(s)\|\|F(\varphi(s))\| + \|\mathcal{C}(s)\|) \\
&\quad + \sum_{\theta_k<t} Ke^{\gamma(t-\theta_k)}\|E_kJ_k(\varphi(\theta_k)) + G_k(\varphi(\theta_k))\| \\
&\leq K\left(\frac{m_f\beta+\sigma}{-\gamma} + \frac{m_IE+m_g}{1-e^{\gamma\underline{\theta}}}\right).
\end{aligned}
$$

The last statement, by the condition (C8) implies that $\|T\varphi(t)\| < H$.

Let $\underline{\theta}$ and $\overline{\theta}$ be the real numbers that satisfy $\underline{\theta} \leq \theta_{k+1} - \theta_k \leq \overline{\theta}$, as mentioned before. Let j be the integer, that gives the consecutive members of the sequence θ_k such that $\theta_j < t \leq \theta_{j+1}$. Then one can get the following

$$
\sum_{\theta_k<t} e^{\gamma(t-\theta_k)} = \sum_{n=0}^{\infty} e^{\gamma(t-\theta_j)}e^{\gamma(\theta_j-\theta_{j-n})} = e^{\gamma(t-\theta_j)}\sum_{n=0}^{\infty} e^{\gamma\underline{\theta}n} \leq \frac{1}{1-e^{\gamma\underline{\theta}}}. \quad (5.5)
$$

Using the inequalities $t - \theta_j > 0$ and $|\theta_{k+q} - \tau - \theta_k| < \varepsilon$, we reach that

$$e^{\frac{\gamma}{2}(t-\theta_{j+q}+\tau)} \leq e^{-\varepsilon\frac{\gamma}{2}}. \tag{5.6}$$

Moreover, the last inequality implies that

$$\begin{aligned}
&\sum_{\theta_k<t} \|X(t+\tau, \theta_{k+q}) - X(t, \theta_{k+q} - \tau)\| \\
&\quad \leq \sum_{\theta_k<t} \varepsilon \mathcal{L} e^{\frac{\gamma}{2}(t+\tau-\theta_{k+q})} = \sum_{n=0}^{\infty} \varepsilon \mathcal{L} e^{\frac{\gamma}{2}(t+\tau-\theta_{j+q})} e^{\frac{\gamma}{2}(\theta_{j+q}-\theta_{j+q-n})} \\
&\quad = \varepsilon \mathcal{L} e^{-\varepsilon\frac{\gamma}{2}} \sum_{n=0}^{\infty} e^{\frac{\gamma}{2}\underline{\theta} n} \\
&\quad \leq \varepsilon \mathcal{L} e^{-\varepsilon\frac{\gamma}{2}} \frac{1}{1-e^{\frac{\gamma}{2}\underline{\theta}}}.
\end{aligned} \tag{5.7}$$

Let j be the integer satisfying $\theta_j < \theta_k \leq \theta_{j+1}$. By Lemma 14 one can obtain $\theta_k - \theta_{k+q} + \tau < \eta < \varepsilon < \underline{\theta} < \theta_k - \theta_j$, and $\theta_j < \theta_{k+q} - \tau$. Similarly, $\theta_{j+1} > \theta_{k+q} - \tau$. Combining these two results as $\theta_j < \theta_{k+q} - \tau \leq \theta_{j+1}$, one can state that the products $\prod_{\theta_{k+q}-\tau\leq\theta_n<t} (1 + d_{in})$ and $\prod_{\theta_k\leq\theta_n<t} (1 + d_{in})$, are equal to each other.

Using the state transition solution of (1.18) and the condition (C1), one can obtain

$$\begin{aligned}
&\sum_{\theta_k<t} \|X(t, \theta_{k+q} - \tau) - X(t, \theta_k)\| \\
&= \sum_{\theta_k<t} \|e^{\int_{\theta_{k+q}-\tau}^{t} \mathcal{A}(u)du} \prod_{\theta_{k+q}-\tau\leq\theta_n<t} (I + D_n) - e^{\int_{\theta_k}^{t} \mathcal{A}(u)du} \prod_{\theta_k\leq\theta_n<t} (I + D_n)\| \\
&\leq \sum_{\theta_k<t} K e^{\gamma(t-\theta_k)} |e^{\varepsilon\alpha} - 1| \leq |1 - e^{\varepsilon\alpha}| K \sum_{\theta_k<t} e^{\gamma(t-\theta_k)} \leq |1 - e^{\varepsilon\alpha}| \frac{K}{1 - e^{\gamma\underline{\theta}}}.
\end{aligned}$$

Furthermore, $\varphi(\theta_k)$ is an almost periodic sequence in k by the Lemma 3.9 in [9]. The Lemma 14 which is on hybrid almost periods and Lemma 15 can be used to show that

$$\begin{aligned}
&\|T\varphi(t+\tau) - T\varphi(t)\| \\
&\leq \int_{-\infty}^{t} \|X(t+\tau, s+\tau)\Big(\mathcal{B}(s+\tau)F(\varphi(s+\tau)) + \mathcal{C}(s+\tau)\Big) \\
&\qquad -X(t,s)\Big(\mathcal{B}(s)F(\varphi(s)) + C(s)\Big)\|ds \\
&+ \sum_{\theta_k<t} \Big\| X(t+\tau, \theta_{k+q})\Big(E_{(k+q)} J_{k+q}(\varphi(\theta_{k+q})) + G_{k+q}(\varphi(\theta_{k+q}))\Big) \\
&\qquad -X(t, \theta_k)\Big(E_k J_k(\varphi(\theta_k)) + G_k(\varphi(\theta_k))\Big)\Big\|
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \int_{-\infty}^{t} \|X(t+\tau,s+\tau)-X(t,s)\|\big\|\mathcal{B}(s+\tau)F(\varphi(s+\tau))+\mathcal{C}(s+\tau)\big\|ds \\
+\ & \int_{-\infty}^{t} \|X(t,s)\|\|\mathcal{B}(s+\tau)-\mathcal{B}(s)\|\|F(\varphi(s+\tau))\|ds \\
+\ & \int_{-\infty}^{t} \|X(t,s)\|\|\mathcal{B}(s)\|\|F(\varphi(s+\tau))-F(\varphi(s))\|ds \\
+\ & \int_{-\infty}^{t} \|X(t,s)\|\|\mathcal{C}(s+\tau)-\mathcal{C}(s)\|ds \\
+\ & \sum_{\theta_k<t}\Big(\|X(t+\tau,\theta_{k+q})-X(t,\theta_{k+q}-\tau)\|+\|X(t,\theta_{k+q}-\tau)-X(t,\theta_k)\|\Big)\times \\
& \times\big\|E_{k+q}J_{k+q}(\varphi(\theta_{k+q}))+G_{k+q}(\varphi(\theta_{k+q}))\big\| \\
+\ & \sum_{\theta_k<t}\|X(t,\theta_k)\|\|E_{k+q}-E_k\|\big\|J_k(\varphi(\theta_{k+q}))\big\| \\
+\ & \sum_{\theta_k<t}\|X(t,\theta_k)\|\|E_k\|\big\|J_{k+q}(\varphi(\theta_{k+q}))-J_k(\varphi(\theta_k))\big\| \\
+\ & \sum_{\theta_k<t}\|X(t,\theta_k)\|\Big(\|G_{k+q}(\varphi(\theta_{k+q}))-G_k(\varphi(\theta_{k+q}))\|+\|G_k(\varphi(\theta_{k+q}))-G_k(\varphi(\theta_k))\|\Big) \\
\leq\ & \int_{-\infty}^{t}\varepsilon\mathcal{L}e^{\frac{\gamma}{2}(t-s)}(\beta m_f+\sigma)ds+\int_{-\infty}^{t}Ke^{\gamma(t-s)}\varepsilon m_f ds+\int_{-\infty}^{t}Ke^{\gamma(t-s)}\ell_f B\underline{\varepsilon}ds \\
+\ & \int_{-\infty}^{t}Ke^{\gamma(t-s)}\varepsilon ds+\sum_{\theta_k<t}\Big(\varepsilon\mathcal{L}e^{\frac{\gamma}{2}(t-(\theta_{k+q}-\tau))}+K(1-e^{\varepsilon\alpha})e^{\gamma(t-\theta_k)}\Big)(m_J E+m_G) \\
+\ & \sum_{\theta_k<t}Ke^{\gamma(t-\theta_k)}\varepsilon m_J+\sum_{\theta_k<t}Ke^{\gamma(t-\theta_k)}\ell_J E\varepsilon+\sum_{\theta_k<t}Ke^{\gamma(t-\theta_k)}\varepsilon+\sum_{\theta_k<t}Ke^{\gamma(t-\theta_k)}\ell_G\varepsilon \\
\leq\ & \Big(2\varepsilon\mathcal{L}(\beta m_f+\sigma)+K\varepsilon m_f+K\ell_f B\varepsilon+K\varepsilon\Big)\frac{1}{-\gamma} \\
+\ & \Big(\varepsilon\mathcal{L}e^{-\varepsilon\frac{\gamma}{2}}\frac{1}{1-e^{\frac{\gamma}{2}\underline{\theta}}}+(1-e^{\varepsilon\alpha})\frac{K}{1-e^{\gamma\underline{\theta}}}\Big)(m_J E+m_G) \\
+\ & \Big(K\varepsilon m_I+K\ell_J E\varepsilon+K\varepsilon+K\ell_G\varepsilon\Big)\frac{1}{1-e^{\gamma\underline{\theta}}}=\Gamma_1(\varepsilon)\varepsilon.
\end{aligned}
$$

Here $\Gamma_1(\varepsilon)$ is a bounded function of ε. Hence $T\varphi(t)$ is almost periodic. Consequently, T is invariant on $\mathcal{D}$. □

Lemma 23 *Under the conditions (C9) and (A1)-(A3), T is a contraction.*

Proof. Suppose that $\varphi(t)$ and $\psi(t)$ are the functions that belong to $\mathcal{D}$. If we consider $T\varphi(t)-T\psi(t)$ we can assert the following inequality

$$
\begin{aligned}
&\|T\varphi(t) - T\psi(t)\| \\
&\le \int_{-\infty}^{t} \|X(t,s)\| \|\mathcal{B}(s)(F(\varphi(s)) - F(\psi(s)))\| ds \\
&\quad + \sum_{\theta_k < t} \|X(t,\theta_k)\| \|E_k(J(\varphi(\theta_k)) - J(\psi(\theta_k))) + G_k(\varphi(\theta_k)) - G_k(\psi(\theta_k))\| \\
&\le \int_{-\infty}^{t} K e^{\gamma(t-s)} \|\mathcal{B}(s)\| \ell_f \|\varphi(s) - \psi(s)\| ds \\
&\quad + \sum_{\theta_k < t} K e^{\gamma(t-\theta_k)} (\ell_J E + \ell_G) \|\varphi(\theta_k) - \psi(\theta_k)\| \\
&\le \int_{-\infty}^{t} K e^{\gamma(t-s)} \beta \ell_f \|\varphi(s) - \psi(s)\|_0 ds \\
&\quad + \sum_{\theta_k < t} K e^{\gamma(t-\theta_k)} (\ell_J E + \ell_G) \|\varphi(\theta_k) - \psi(\theta_k)\| \\
&\le \int_{-\infty}^{t} K e^{\gamma(t-s)}\, \beta \ell_f \|\varphi(t) - \psi(t)\|_0 ds + \sum_{\theta_k < t} K e^{\gamma(t-\theta_k)} (\ell_J E + \ell_G) \|\varphi(t) - \psi(t)\|_0 \\
&\le K \ell_f \beta \|\varphi(t) - \psi(t)\|_0 \left(\frac{1}{-\gamma} \right) + K(\ell_J E + \ell_G) \|\varphi(t) - \psi(t)\|_0 \left(\frac{1}{1 - e^{\gamma \underline{\theta}}} \right) \\
&\le K \left(\frac{\ell_f \beta}{-\gamma} + \frac{\ell_J E + \ell_G}{1 - e^{\gamma \underline{\theta}}} \right) \|\varphi(t) - \psi(t)\|_0. \qquad (5.8)
\end{aligned}
$$

Then $\|T\varphi(t) - T\psi(t)\|_0 \le K\left(\frac{\ell_f \beta}{-\gamma} + \frac{\ell_J E + \ell_G}{1 - e^{\gamma \underline{\theta}}} \right) \|\varphi(t) - \psi(t)\|_0$. Hence by the condition (C9), T is a contraction. □

Theorem 24 *Let (C1)-(C10) and (A1)-(A3) be fulfilled. Then the system (2.5) has a unique asymptotically stable discontinuous almost periodic solution.*

Proof. T is the integral operator of (2.5) so it satisfies the differential equation and impulsive part of the system as shown before. By the Lemma 22, the operator T is invariant on the space $\mathcal{D}$. That is, if $\varphi(t) \in \mathcal{D}$, then $T\varphi(t) \in \mathcal{D}$. Moreover by Lemma 23 the operator is a contraction. Hence, $\mathcal{D}$ has a unique element $\omega(t)$ such that $T\omega = \omega$, and it is the unique solution of our system.

Let $z(t) = [z_1(t), \ldots, z_m(t)]$ be the solution of the system (2.5) with the initial condition $z(t_0) = z_0$. Applying the equivalent integral equation for the initial value problem one can obtain

$$
\begin{aligned}
&\|z(t)-\omega(t)\| \\
&\leq \ \Big\|X(t,t_0)(z(t_0)-\omega(t_0))+\int_{t_0}^{t}X(t,s)(\mathcal{B}(F(z(s))-F(\omega(s))))ds \\
&+ \sum_{t_0\leq\theta_k<t}X(t,\theta_k)\Big(E_k(J(z(\theta_k))-J(\omega(\theta_k)))+G_k(z(\theta_k))-G_k(\omega(\theta_k))\Big)\Big\| \\
&\leq \ \|x(t,t_0)\|\|(z(t_0)-\omega(t_0))\|+\int_{t_0}^{t}Ke^{\gamma(t-s)}\beta\ell_f\|z(s)-\omega(s)\|ds \\
&+ \sum_{t_0\leq\theta_k<t}Ke^{\gamma(t-\theta_k)}(\ell_JE+\ell_G)\|z(\theta_k)-\omega(\theta_k)\|.
\end{aligned}
$$

Multiplying both sides with $e^{-\gamma t}$ and setting $u(t)=e^{-\gamma t}\|z(t)-\omega(t)\|$ one can obtain

$$
u(t) \ \leq \ Ku(t_0)+\int_{t_0}^{t}K\beta\ell_f u(s)ds+\sum_{t_0\leq\theta_k<t}K(\ell_JE+\ell_G)u(\theta_k).
$$

Now, the Gronwall-Bellman Lemma for piecewise continuous functions [1] implies the inequality

$$
\begin{aligned}
u(t) \ &\leq \ Ku(t_0)e^{\int_{t_0}^{t}K\beta\ell_f ds}\prod_{t_0\leq\theta_k<t}(1+K(\ell_JE+\ell_G)) \\
&= \ Ku(t_0)e^{(t-t_0)K\beta\ell_f}\prod_{t_0\leq\theta_k<t}(1+K(\ell_JE+\ell_G)).
\end{aligned}
$$

Finally, multiplicating the inequality by $e^{\gamma t}$ after back-substituting $u(t)$, and using (1.23) will result with

$$
\begin{aligned}
\|z(t)-\omega(t)\| \ &\leq \ Ke^{(\gamma+K\beta\ell_f)(t-t_0)}\|z(t_0)-\omega(t_0)\|e^{i([t_0,t])\ln(1+K(\ell_JE+\ell_G))} \\
&= \ Ke^{\Gamma_4(t-t_0)}(1+K(\ell_JE+\ell_G))\|z(t_0)-\omega(t_0)\|, \qquad (5.9)
\end{aligned}
$$

where $\Gamma_4=\left(\gamma+K\beta\ell_f+\dfrac{\ln(1+K(\ell_JE+\ell_G))}{\underline{\theta}}\right)$. Hence, the condition (C10) implies that the solution $\omega(t)$ is an asymptotically stable solution. $\square$

5.2 The Main Almost Periodicity Result

This section is devoted to prove that the solution of the system (1.34), which is a recurrent neural network with structured impulses at non-prescribed moments, is almost periodic and asymptotically stable. First the almost periodicity of the impact moments will be proved, then the uniform almost periodicity of the derivative sequences of impact moments will be shown. Finally the main theorem of the paper will be stated and proved.

Lemma 25 *Let $x(t)$ be solution of the system (1.34) and let the sequence $\{\zeta_k\}$ be the meeting points of the solution with the surfaces of discontinuities, then the sequence $\zeta_k - \theta_k$ is almost periodic.*

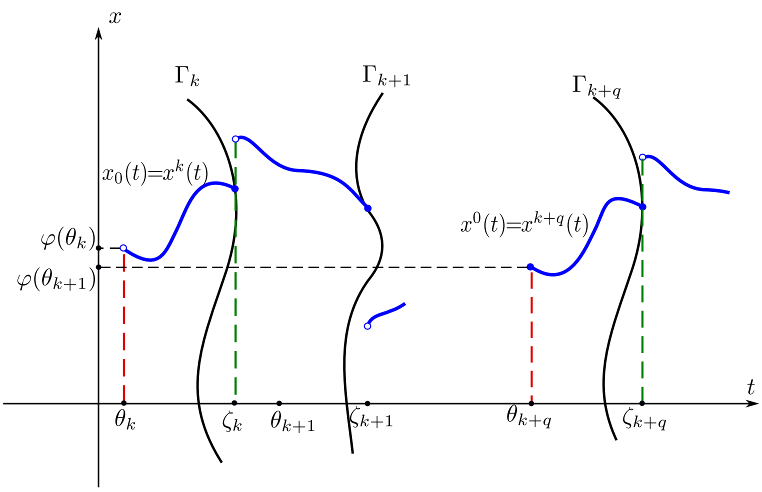


Figure 5.1: The graph of the almost periodic solution $\psi(t)$ of the system (1.34) which is subject to almost periodicity conditions.

Proof. Let $\varphi(t)$ be the solution of the system (2.5) which is B-equivalent system of (1.34) with initial condition $\varphi(\theta_k) = x_0$. *Then the function $\varphi(t)$, $t \in \mathbb{R}$ is a discontinuous almost periodic function.* Then by Lemma 3.9 in [9] the sequence $\varphi(\theta_k)$, $k \in \mathbb{Z}$, is almost periodic i.e., for all $\varepsilon > 0$ there exists a relatively dense set of integers q satisfying $\|\varphi(\theta_{k+q}) - \varphi(\theta_k)\| < \varepsilon$.

Denote $x^k(t) = x(t, \theta_k, \psi(\theta_k))$ i.e., the solution on the subintervals of piece-wise continuity. Also let $\psi(t)$ be solution of the system (1.34) with the same initial condition $\psi(\theta_k) = x_0$ Then by the definition of B-equivalence, $\psi(t) = \varphi(t)$, for t values that are not in the interval with the endpoints θ_k and ξ_k.

Lastly denote by ζ_k the points where the solution $\psi(t)$ meets the surfaces of discontinuities $t = \theta_k + \tau_k(x(t))$. Thus $\zeta_k = \theta_k + \tau_k(\psi(\zeta_k)) = \theta_k + \tau_k(\varphi(\zeta_k)) = \theta_k + \tau_k(x^k(\zeta_k))$.

If $x_0(t)$ is the solution of the system (2.5) with initial condition $x(\theta_k) = x = \varphi(\theta_k)$ then for $t \in [\theta_k, \theta_k + m_\tau]$

$$x_0(t) = x + \int_{\theta_k}^{t} (\mathcal{A}(s)x_0(s) + \mathcal{B}(s)F(x_0(s)) + \mathcal{C}(s))ds.$$

As in Lemma 19 denote $\bar{x}(t) = x^0(t + (\theta_{k+q} - \theta_k))$, then $\bar{x}$ is the solution of the differential equation (5.1), where $t \in [\theta_k, \theta_k + m_\tau]$ with the initial conditions $\bar{x}(\theta_k) = \psi(\theta_{k+q})$.

Hence, by the continuous dependence on the initial data and on the right hand side of the differential equation, [78],

$$\|\bar{x}(t) - x_0(t)\| < \kappa(\varepsilon) \tag{5.10}$$

on $t \in [\theta_k, \theta_k + m_\tau]$.

Also denoting $\zeta_k' = \zeta_{k+q} - (\theta_{k+q} - \theta_k)$, the following is obtained $\zeta_{k+q} = \theta_{k+q} + \tau_{k+q}(\varphi(\zeta_{k+q})) = \tau_{k+q} + \theta_{k+q}(\psi(\zeta_{k+q})) = \theta_{k+q} + \tau_{k+q}(x^{k+q}(\zeta_{k+q})) = \theta_{k+q} + \tau_{k+q}(\bar{x}(\zeta_k'))$. Using $\zeta_k' = \theta_k + \tau_{k+q}(\bar{x}(\zeta_k'))$ we will prove an inequality on $|\zeta_k' - \zeta_k|$

$$\begin{aligned}
|\zeta_k' - \zeta_k| &= |\theta_k + \tau_{k+q}(\bar{x}(\zeta_k')) - (\theta_k + \tau_k(x(\zeta_k)))| \\
&= |\tau_{k+q}(\bar{x}(\zeta_k')) - \tau_k(x_0(\zeta_k))| \\
&\le |\tau_{k+q}(\bar{x}(\zeta_k')) - \tau_k(\bar{x}(\zeta_k')) + \tau_k(\bar{x}(\zeta_k')) - \tau_k(x_0(\zeta_k))| \\
&\le \varepsilon + \ell_\tau \|\bar{x}(\zeta_k') - \bar{x}(\zeta_k) + \bar{x}(\zeta_k) - x_0(\zeta_k)\| \\
&\le \varepsilon + \ell_\tau \kappa(\varepsilon) + \ell_\tau \| \int_{\zeta_k}^{\zeta_k'} (\mathcal{A}(s - (\theta_{k+q} - \theta_k))\bar{x}(s) \\
&\quad + \mathcal{B}(s - (\theta_{k+q} - \theta_k)F(\bar{(s)}) + \mathcal{C}(s - (\theta_{k+q} - \theta_k))ds\| \\
&\le \varepsilon + \ell_\tau \kappa(\varepsilon) + \ell_\tau \|(\zeta_k' - \zeta_k)(\alpha h + \beta m_f + \sigma).\|
\end{aligned}$$

One can solve this inequality for $|\zeta_k' - \zeta_k|$ to obtain

$$|\zeta_k' - \zeta_k| \le \frac{\varepsilon + \ell_\tau \kappa(\varepsilon)}{1 - \ell_\tau(\alpha h + \beta m_f + \sigma)}$$

Since $\varepsilon > 0$ is arbitrarily small, it follows that the sequence $\zeta_k' - \zeta_k$ is almost periodic. On the other hand $|\zeta_k' - \zeta_k| = |\zeta_{k+q} - (\theta_{k+q} - \theta_k) - \zeta_k|$ implies

$$|(\zeta_{k+q} - \theta_{k+q}) - (\zeta_k - \theta_k)| \le \frac{\varepsilon + \ell_\tau \kappa(\varepsilon)}{1 - \ell_\tau(\alpha h + \beta m_f + \sigma)}, \tag{5.11}$$

Hence the sequence $\zeta_k - \theta_k$ is almost periodic. □

Remark 1 *Using the Lemma 14 and Lemma 25 we obtain that*

$$\begin{aligned}
|\zeta_{k+q}-\zeta_k-\tau| &= |\zeta_{k+q}-\zeta_k-\tau+\theta_{k+q}-\theta_k-\theta_{k+q}+\theta_k| \\
&= |(\zeta_{k+q}-\theta_{k+q})-(\zeta_k-\theta_k)+\theta_{k+q}-\theta_k-\tau| \\
&\leq |(\zeta_{k+q}-\theta_{k+q})-(\zeta_k-\theta_k)|+|\theta_{k+q}-\theta_k-\tau| \\
&\leq \frac{\varepsilon+\ell_\tau\kappa(\varepsilon)}{1-\ell_\tau(\alpha h+\beta m_f+\sigma)}+\varepsilon. \qquad (5.12)
\end{aligned}$$

Lemma 26 *The derivative sequences ζ_k^j, $j,k\in\mathbb{Z}$, are uniformly almost periodic.*

Proof of this assertion is similar with the one given for Lemma 20.

Theorem 27 *Let (C1)-(C10) and (A1)-(A3) be fulfilled. Then solution $\psi(t)$ of the system (1.34) is unique and asymptotically stable, discontinuous almost periodic.*

Proof. Let $\varphi(t)$ be the discontinuous almost periodic solution of the system (2.5). Thus $\|\varphi(t)\|<h$ and $t\in\mathbb{R}$ given $\varepsilon>0$, there exists a relatively dense set $\mathcal{T}$, $\mathcal{Q}$, and there exists $(\tau,q)\in\mathcal{T}\times\mathcal{Q}$, such that

(α) $\|\varphi(t+\tau)-\varphi(t)\|<\eta<\varepsilon$,

(β) the derivative sequences θ_k^j are uniformly almost periodic.

Let $\psi(t)$ be as in Lemma 25. Recall by B-equivalence that

$$\begin{aligned}
t\leq\theta_k;\quad \psi(t)=\varphi(t) \text{ then } \psi(\theta_k)=\varphi(\theta_k)\\
t\geq\zeta_k;\quad \psi(t)=\varphi(t) \text{ then } \psi(\zeta_k)=\varphi(\zeta_k).
\end{aligned}$$

In the previous section, we proved that the almost periodic solution $\varphi(t)$ of the B-equivalent system (2.5) is asymptotically stable. So whenever $\varphi(t)=\psi(t)$ the solution $\psi(t)$ is almost periodic.

Let $|t-\zeta_k|>\varepsilon$, and without loss of generality assume $\tau_k\geq 0$, we have two possibilities:

(i) Consider $\zeta_k+\varepsilon\leq t$, by lemma on hybrid almost periods $|\zeta_{k+q}-\zeta_k-\tau|<\varepsilon$ is satisfied, then $-\varepsilon<\zeta_k-\zeta_{k+q}+\tau<\varepsilon$ hence one obtains $0<\zeta_k-\zeta_{k+q}+\tau+\varepsilon<2\varepsilon$. Also $\zeta_k+\varepsilon-\zeta_{k+q}+\tau<t-\zeta_{k+q}+\tau$, consequently $\zeta_{k+q}<t+\tau$, parenthetically $\psi(t+\tau)=\varphi(t+\tau)$.

Thus, $\|\psi(t+\tau)-\psi(t)\| = \|\varphi(t+\tau)-\varphi(t)\| < \eta$.

(ii) Consider $t \le \zeta_k - \varepsilon$. In that case $\psi(t)$ is given by the equation

$$\psi(t) = \varphi(\theta_k) + \int_{\theta_k}^{t} (A(s)\psi(s) + B(s)F(\psi(s)) + C(s))ds. \tag{5.13}$$

Again using $|\zeta_{k+q} - \zeta_k - \tau| < \varepsilon$ in the other direction one obtains $-2\varepsilon < \zeta_k - \zeta_{k+q} + \tau - \varepsilon < 0$.

Additionally, $t - \zeta_{k+q} + \tau < \zeta_k - \zeta_{k+q} + \tau + \varepsilon$ so we get $t + \tau < \zeta_{k+q}$, then $\psi(t+\tau) \neq \varphi(t+\tau)$, but $\psi(t+\tau)$ satisfies

$$\psi(t+\tau) = \varphi(\theta_{k+q}) + \int_{\theta_{k+q}}^{t+\tau} (A(s)\psi(s) + B(s)F(\psi(s)) + C(s))ds. \tag{5.14}$$

Hence by a simple change of variable one can get

$$\psi(t+\tau) = \varphi(\theta_{k+q}) + \int_{\theta_{k+q}-\tau}^{t} (A(s+\tau)\psi(s+\tau) + B(s+\tau)F(\psi(s+\tau)) + C(s+\tau))ds.$$

Without loss of generality assume $\theta_{k+q} - \tau < \theta_k$ and make use of the equation (5.13) together with (5.14), then it follows that

$$\begin{aligned}
&\|\psi(t+\tau)-\psi(t)\| \\
&= \|\varphi(\theta_{k+q}) + \int_{\theta_{k+q}-\tau}^{t} (A(s+\tau)\psi(s+\tau) + B(s+\tau)F(\psi(s+\tau)) + C(s+\tau))ds \\
&- \varphi(\theta_k) - \int_{\theta_k}^{t} (A(s)\psi(s) + B(s)F(\psi(s)) + C(s))ds\| \\
&\le \|\varphi(\theta_{k+q}) - \varphi(\theta_k)\| + \|\int_{\theta_{k+q}-\tau}^{\theta_k} (A(s+\tau)\psi(s+\tau) \\
&\quad +B(s+\tau)F(\psi(s+\tau)) + C(s+\tau))ds\| \\
&+ \|\int_{\theta_k}^{t} (A(s+\tau)\psi(s+\tau) - A(s)\psi(s) + B(s+\tau)F(\psi(s+\tau)) \\
&\quad -B(s)F(\psi(s)) + C(s+\tau) - C(s))ds\| \\
&\le \|\varphi(\theta_{k+q}) - \varphi(\theta_k)\| \\
&+ \|\int_{\theta_{k+q}-\tau}^{\theta_k} (A(s+\tau)\psi(s+\tau) + B(s+\tau)F(\psi(s+\tau)) + C(s+\tau))ds\| \\
&+ \|\int_{\theta_k}^{t} (A(s+\tau)\Big(\psi(s+\tau) - \psi(s)\Big) + \big(A(s+\tau) - A(s)\big)\psi(s) \\
&+ B(s+\tau)\Big(F(\psi(s+\tau)) - F(\psi(s))\Big) + \Big(B(s+\tau) - B(s)\Big)F(\psi(s)) \\
&\quad +C(s+\tau) - C(s))ds\|
\end{aligned}$$

$$
\begin{aligned}
&\le \nu + \nu(\alpha H + \beta m_f + \sigma) + \| \int_{\theta_k}^{t} ((\alpha + \beta \ell_f)(\psi(s+\tau) - \psi(s)) + \nu(H + m_f + 1))ds \| \\
&\le \nu + \nu(\alpha H + \beta m_f + \sigma) + m_\tau \nu(H + m_f + 1) + \int_{\theta_k}^{t} (\alpha + \beta \ell_f) \|\psi(s+\tau) - \psi(s)\| ds,
\end{aligned}
$$

By Gronwall-Bellman's Lemma

$$
\begin{aligned}
&\|\psi(t+\tau) - \psi(t)\| \\
&\quad \le (\nu + \nu(\alpha H + \beta m_f + \sigma) + m_\tau \nu(H + m_f + 1)) e^{\int_{\theta_k}^{t} (\alpha + \beta \ell_f) ds} \\
&\quad \le (\nu + \nu(\alpha H + \beta m_f + \sigma) + m_\tau \nu(H + m_f + 1)) e^{m_\tau(\alpha + \beta \ell_f)}. \qquad (5.15)
\end{aligned}
$$

This finishes the proof as the method of proof carries over into all other possible cases since each of them can be considered as (i) or (ii). Also, asymptotic stability is proved in the same manner by using $\varphi(t)$'s asymptotic stability. $\square$

CHAPTER 6

CONCLUSION

The purpose of this thesis is to conduct detailed and precise research of neural networks with non-fixed impulse moments. To this aim, we suggested an impulsive neural network with novel structured impacts that perfectly match the rates, that is differential equation part. This has a sense for application if one considers the impacts as limits of their continuous counterparts.

The new system also considers the neural networks' nature in the impulsive part since the sudden noises or impact disturbances can affect the rates or activation functions. As impulsive actions are compatible with the model's differential equation, the structured system covers all similar impulsive neural networks previously considered.

This is the first time in the literature that the recurrent neural networks with variable moments of impulses are investigated in such a detailed approach. Specific and refined conditions on the coefficients have been newly developed. The constructive conditions for the stability are provided, directly related to the system's coefficients. Hence, one can see that we particularly did not assume the differential equation's stability, as some of our predecessors did.

The method of B-equivalence became widely used for theoretical and practical research of systems with state-dependent impulses. We provide the most detailed description and proof concerning B-equivalence in the present work. We arranged and offered the relation between the original system and the B-equivalent one explicitly by the equations (2.7-2.9). This usually has been avoided in literature. Consequently, the results are either wrong, unclear, or not sufficiently general as a modeling process requires. Thus a strongly systematic interpretation was needed for the challenging problem of B-equivalent system reduction. Moreover, one can observe that; since the impulsive part of our system is in the neural network's nature, the B-equivalent system we studied also replicates the original structure in terms of differential and impulsive parts.

Furthermore, we presented the physical aspects of the impacts for the first time. And we offer a property that was not considered before for the neural networks. One of the innovative parts of this study is that the possibility of negative capacitance in a neural system is not ignored since the advantages of negative capacitance are revealed in electric circuit studies. The new structure and the elimination of the capacitance's positivity condition allow a more thorough analysis with optimized conditions. The possibility of negative capacitance increases the importance of the contribution of impulses into stability. Hence we have the most general system that additionally covers the systems with unstable differential equation parts yet with stable solutions.

All these novelties provide more constructive and challenging circumstances for the analysis. To make the presentation more transparent, we have analyzed the system in the matrix-vector form, which improves the clarity of the work done.

We wanted to serve a detailed approach to the systems with variable moments of impulses. Thus we first examined the B-equivalence method for the neural network system in Chapter 2. While doing this, we defined a non-standard impulsive part. The B-equivalent system's impulse equation also mimics the neural network structure.

We studied the existence, uniqueness, and stability of the periodic and almost periodic motion for the reduced system, which is B-equivalent of the initial system and is a neural network with fixed moments of structured impulses. Then we applied those results to the systems with state-dependent and structured impulses.

In chapter three, we dealt with the reduced system with periodicity properties. We then proved the existence, uniqueness, and stability of a periodic solution of the RINN with variable impulse moments.

The proofs concerning almost periodicity were more sophisticated than periodicity proofs; thus, the case of almost periodicity is divided into two parts. Chapter 4 is devoted to almost periodicity properties for a system with fixed moments of structured impulses. Then, a numerical example is offered to illustrate the results. Finally, detailed proofs for the stability, uniqueness, and existence of almost periodic solutions of the systems with non-fixed moments of structured impacts are served in Chapter 5.

The novelties in this study increase neural systems' opportunities and improve the power of neural networks studies in the application in different fields. These methods can be improved to apply in different types of neural networks, such as shunting inhibitory cellular neural networks and cellular neural networks.

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PUBLICATIONS

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2. M. Akhmet, G. Erim, "Almost Periodic Solutions of Recurrent Neural Networks With State-Dependent Structured Impulses," Discontinuity, *Nonlinearity, and Complexity*, (accepted).

International Conference Publications

1. M. Akhmet, G. Erim, M. Tleubergenova "Oscillations in Recurrent Neural Networks with Structured and Variable Impulses," accepted to Springer Lecture Notes on Data Engineering and Communications Technologies, Proceeding of the 2nd International Conference on Artificial Intelligence and Applied Mathematics in Engineering ICAIAME-2021.

CONFERENCES ATTENDED

1. 2nd International Conference on Artificial Intelligence and Applied Mathematics in Engineering, ICAIAME, Antalya, Turkey (2021)

   Talk: Oscillations in Recurrent Neural Networks with Structured and Variable Impulses.