COMPLETE b-SYMBOL WEIGHT DISTRIBUTION OF SOME IRREDUCIBLE CYCLIC CODES

HONGWEI ZHU & MINJIA SHI & FERRUH ÖZBUDAK

ABSTRACT. Recently, b-symbol codes are proposed to protect against b-symbol errors in b-symbol read channels. It is an interesting subject of study to consider the complete b-symbol weight distribution of cyclic codes since b-symbol metric is a generalization for Hamming metric. The complete b-symbol Hamming weight distribution of irreducible codes is known in only a few cases. In this paper, we give a complete b-symbol Hamming weight distribution of a class of irreducible codes with two nonzero b-symbol Hamming weights.

Keywords: cyclic code, b-symbol error, b-symbol Hamming weight distribution, irreducible cyclic code

1. INTRODUCTION

In traditional information theory, people often analyze noise channels by dividing information into independent information units. However, with the development of storage technology, one finds that symbols can not always be written or read continuously. In 2011, Cassuto and Blaum [1, 2] first proposed a new coding framework for read channel based on symbol-pair. In this channel, the output of the reading process is the overlapping symbol-pair of the read channel. Symbol-pair codes are designed to correct symbol-pair errors. Later, Cassuto and Litsyn [3] studied the symbol-pair codes corresponding to the cyclic codes and gave the lower bound for symbol-pair distance of cyclic codes through discrete Fourier transform and BCH bound. Next, Chee *et al.* [4, 5] established the Singleton-type bound of codes with symbol-pair metric, and considered the construction of symbol-pair codes reaching this bound. More related results on this topic can be found in [6, 7, 8, 10]. Recently, Yaakobi *et al.* [15] generalized the symbol-pair read channel to the b-symbol read channel (2-symbol is exactly

symbol-pair) and gave a decoding algorithm based on a bounded distance decoder for the cyclic code, where $b \geqslant 1$. When $b = 1$, the 1-symbol weight is exactly Hamming weight. Ding *et al.* [8] established the Singleton-type bound for b-symbol codes, and constructed several infinite families of linear maximum distance separable (MDS) b-symbol codes. In the b-symbol read channel, one tends to consider the cyclic codes. Important motivations for considering cyclic codes include that cyclic shifting does not change the b-symbol weight of the codeword, there are good decoding algorithms and cyclic codes have good algebraic structure. Hence it is natural to consider b-symbol weight distribution of cyclic codes. It is well known that the problem of determining the weight distribution of an irreducible cyclic code is notoriously difficult in general. Similar difficulty holds in the case of arbitrary cyclic codes and also for their b-symbol weights, in general. There are only few cases that the b-symbol weight distribution of cyclic codes are completely determined. Sun *et al.* [14] gave the 2-symbol distance distribution of a class of repeated-root cyclic codes over finite fields. Using geometric approach, Shi it et al. [13] obtain, among other results, tight lower and upper bounds on b-symbol Hamming weight of arbitrary cyclic codes and a class of irreducible cyclic codes with constant b-symbol Hamming weight.

In this paper we consider a natural next step after [13]. We completely determine the b-symbol weight distribution of irreducible cyclic codes of length n over $\mathbb{F}_q$ if $\gcd\left(\frac{q^r-1}{q-1}, \frac{q^r-1}{n}\right) = 2$ with $n \mid (q^r - 1)$ and $r \geqslant 2$. These results correspond to two-weight cyclic codes over the alphabet $\mathbb{F}_q \times \cdots \times \mathbb{F}_q = \mathbb{F}_q^b$, which is not a field (see Remark 3.2 below). It turns out that the complete determination of b-symbol weights of these irreducible cyclic codes are more difficult and we obtain our results up to a new invariant $\mu(b)$ that we introduce in Definition 2.2 below. We find that $\mu(b)$ is a natural arithmetic and geometric invariant and we determine it exactly if $b = r$. We also obtain numerical results for $2 \leqslant b < r$.

Our main results are Theorems 3.1 and 3.3. These results are stronger than the b-symbol weight distributions as we determine the b-symbol weight of any given codeword for the irreducible cyclic codes of this paper. Another important problem in this area is construction of maximum distance separable (MDS) b-symbol codes. There are interesting constructions in the recent literature using various methods, for example [4, 5, 7, 8, 10]. As a consequence of Theorem 3.3 we observe that all of the irreducible cyclic codes we study here are MDS b-symbol if $b = r$.

This paper is organized as follows. In section 2, we introduce basic notations and definitions. In section 3, we show the main results. In section 4, we give the detailed proof of the main results. In the appendix, using exponent sum, we give another proof to show the equality (15) holds.

2. Preliminaries

Throughout this paper we assume and fix the following:

- $\mathbb{F}_q$: finite field with q elements.

- $\mathbb{F}_q^*$: $\mathbb{F}_q \backslash \{0\}$.

- $q = p^e$, $p = \text{char } \mathbb{F}_q$ and p is odd.

- $r \geqslant 2$: an *even* integer.

- $nN = q^r - 1$, where n, N are positive integers.

- $\gcd\left(\frac{q^r - 1}{q - 1}, N\right) = 2$.

- $2 \leqslant b \leqslant n - 1$: an integer.

- $\eta \in \mathbb{F}_{q^r}$: a primitive $(q^r - 1)$-th root of 1, or equivalently a primitive element of $\mathbb{F}_{q^r}$.

- $\text{Tr} : \mathbb{F}_{q^r} \to \mathbb{F}_q$: the trace map defined as $x \mapsto x + x^q + \cdots + x^{q^{r-1}}$.

Denote by $w(\mathbf{x})$ or $w_H(\mathbf{x})$ the Hamming weight of $\mathbf{x} \in \mathbb{F}_q^n$. The b-symbol Hamming weight $w_b(\mathbf{x})$ of $\mathbf{x} = (x_0, \ldots, x_{n-1}) \in \mathbb{F}_q^n$ is defined as the Hamming weight of $\pi_b(\mathbf{x})$, where

$$(1) \qquad \pi_b(\mathbf{x}) = ((x_0, \ldots, x_{b-1}), (x_1, \ldots, x_b), \cdots, (x_{n-1}, \ldots, x_{b+n-2 (\text{mod } n)}))$$

is in $(\mathbb{F}_q^b)^n$. When $b = 1$, $w_1(\mathbf{x})$ is exactly the Hamming weight of $\mathbf{x}$. For any $\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^N$, we have $\pi_b(\mathbf{x} + \mathbf{y}) = \pi_b(\mathbf{x}) + \pi_b(\mathbf{y})$, and the b-symbol distance (b-distance for short) $d_b(\mathbf{x}, \mathbf{y})$ between $\mathbf{x}$ and $\mathbf{y}$ is defined as $d_b(\mathbf{x}, \mathbf{y}) = w_b(\mathbf{x} - \mathbf{y})$. Let $A_i^{(b)}$ denote the number of codewords with b-symbol Hamming weight i in a code C of length n. The b-symbol Hamming weight enumerator of C is defined by

$$1 + A_1^{(b)} T + A_2^{(b)} T^2 + \cdots + A_n^{(b)} T^n.$$

Ding *et al.* [8] established a Singleton-type bound for b-symbol codes. Let $q \geqslant 2$ and $b \leqslant d_b(C) \leqslant n$. If C is an $(n, M, d_b(C))_q$ b-symbol code, then we have $M \leqslant q^{n - d_b(C) + b}$. An $(n, M, d_b(C))_q$ b-symbol code C with $M = q^{n - d_b(C) + b}$ is called a maximum distance separable (MDS for short) b-symbol code.

For $a \in \mathbb{F}_{q^r}$, let $c(a) \in \mathbb{F}_q^n$ be the codeword defined as

$$c(a) = \left(\text{Tr}(a\eta^{0 \cdot N}), \text{Tr}(a\eta^{1 \cdot N}), \ldots, \text{Tr}(a\eta^{j \cdot N}), \ldots, \text{Tr}(a\eta^{(n-1) \cdot N})\right),$$

where $0 \leqslant j \leqslant n - 1$.

Let C be the $\mathbb{F}_q$-linear code of length n defined as

$$C = \{c(a) : a \in \mathbb{F}_{q^r}\}.$$

The following simple lemma is useful.

Lemma 2.1. *C is the irreducible code of length n over $\mathbb{F}_q$ and $\dim_{\mathbb{F}_q} C = r$.*

Next we introduce an important invariant $\mu(b)$ of the extension $\mathbb{F}_{q^r}/\mathbb{F}_q$. First we introduce a subset $\mathcal{P}(b)$ of $\mathbb{F}_{q^r}^*$, which we use in the definition of $\mu(b)$ below.

Definition 2.1. Let $2 \leqslant b \leqslant r$ be an integer. It follows from [13, Corollary 4.1] that the set $\left\{1, \eta^N, \eta^{2N}, \ldots, \eta^{(b-1)N}\right\}$ is linearly independent over $\mathbb{F}_q$. Let $\mathcal{P}(b)$ be the subset of cardinality $(q^b - 1)/(q - 1)$ in $\mathbb{F}_{q^r}^*$ defined as

$$\mathcal{P}(b) = \bigcup_{j=1}^{b-1} \left\{\eta^{(j-1)N} + x_1\eta^{jN} + \cdots + x_{b-j}\eta^{(b-1)N} : x_1, \cdots, x_j \in \mathbb{F}_q\right\} \cup \left\{\eta^{(b-1)N}\right\}.$$

Example 2.1. *For $r \geqslant 3$ we have*

$$\mathcal{P}(2) = \left\{1 + x_1\eta^N : x_1 \in \mathbb{F}_q\right\} \cup \left\{\eta^N\right\}$$

and

$$\mathcal{P}(3) = \left\{1 + x_1\eta^N + x_2\eta^{2N} : x_1, x_2 \in \mathbb{F}_q\right\} \cup \left\{\eta^N + x_1\eta^{2N} : x_1 \in \mathbb{F}_q\right\} \cup \left\{\eta^{2N}\right\}.$$

Now we are ready to define the invariant $\mu(b)$.

Definition 2.2. For $2 \leqslant b \leqslant r$, let

$$\mu(b) = \left|\left\{x \in \mathcal{P}(b) : x \text{ is a square in } \mathbb{F}_{q^r}^*\right\}\right|.$$

It seems difficult to determine $\mu(b)$ exactly when $b < r$. We determine it exactly when $b = r$ in the following.

Lemma 2.2. *Under notation as above, we have*

$$\mu(r) = \frac{1}{2}\left|\mathcal{P}(b)\right| = \frac{q^r - 1}{2(q - 1)}.$$

Proof. First note that $(q^r - 1)/(q - 1)$ is an even integer as r is even. Moreover any element of $\mathbb{F}_q^*$ is a square in $\mathbb{F}_{q^r}^*$ as r is even. It follows from Definition 2.1 that

$$\mathbb{F}_{q^r}^* = \bigsqcup_{x \in \mathcal{P}(b)} x\mathbb{F}_q^*,$$

where $\bigsqcup$ is a disjoint union. Hence $x \in \mathcal{P}(b)$ is a square in $\mathbb{F}_{q^r}^*$ if and only if each element of $x\mathbb{F}_q^*$ is a square in $\mathbb{F}_{q^r}^*$. This implies that

$$\mu(b)(q - 1) = \frac{q^r - 1}{2},$$

which is the number of squares in $\mathbb{F}_{q^r}^*$. This completes the proof. $\square$

Example 2.2. *Here are some numerical examples about $\mu(b)$ which computed by Magma.*

p	q	r	N	b	$\mu(b)$
3	3	2	2	2	$2 = \frac{q^r-1}{2(q-1)}$
3	3	4	2	2	3
3	3	4	2	3	8
3	3	4	2	4	$20 = \frac{q^r-1}{2(q-1)}$
5	5	2	2	2	$3 = \frac{q^r-1}{2(q-1)}$
5	5	4	2	2	4
5	5	4	2	3	18
5	5	4	2	4	$78 = \frac{q^r-1}{2(q-1)}$
3	9	2	2	2	$5 = \frac{q^r-1}{2(q-1)}$
3	9	4	2	2	4
3	9	4	2	3	50
3	9	4	2	4	$410 = \frac{q^r-1}{2(q-1)}$
3	9	6	2	2	4
3	9	6	2	3	51
3	9	6	2	4	401
3	9	6	2	5	3728
3	9	6	2	6	$33215 = \frac{q^r-1}{2(q-1)}$
5	25	2	2	2	$13 = \frac{q^r-1}{2(q-1)}$
5	25	4	2	2	11
5	25	4	2	3	338
5	25	4	2	4	$8138 = \frac{q^r-1}{2(q-1)}$

According to Example 2.2, when $2 \leqslant b < r$, we notice that the value of $\mu(b)$ is close to, but not equal to, $\frac{q^b-1}{2(q-1)}$. It is natural for us to arise such an open question as follows.

Open Problem 2.3. Determine the invariant $\mu(b)$ when $2 \leqslant b < r$ or give *good* lower and upper bounds to $\mu(b)$.

3. The main results

Next we state our first main result. We completely determine the b-symbol Hamming weight of a given arbitrary nonzero codeword of C for $2 \leqslant b < r$. Recall that $q = p^e$, $p = \text{char } \mathbb{F}_q$ and p is odd.

Theorem 3.1. *Let $a \in \mathbb{F}_{q^r}^*$. Assume that $2 \leqslant b < r$. Then we determine $w_b(c(a))$ explicitly as follows:*

- If $p \equiv 1 \mod 4$ and a is a square in $\mathbb{F}_{q^r}^*$, then

$$w_b(c(a)) \quad = \quad \frac{q^b - 1}{N(q-1)q^{b-1}} \left(q^r - \frac{q^r + (q-1)q^{r/2}}{q} \right) + \frac{2\mu(b)(q-1)q^{r/2}}{Nq^b}.$$

- If $p \equiv 1 \mod 4$ and a is a non-square in $\mathbb{F}_{q^r}^*$, then

$$w_b(c(a)) \quad = \quad \frac{q^b - 1}{N(q-1)q^{b-1}} \left(q^r - \frac{q^r - (q-1)q^{r/2}}{q} \right) - \frac{2\mu(b)(q-1)q^{r/2}}{Nq^b}.$$

- If $p \equiv 3 \mod 4$ and a is a square in $\mathbb{F}_{q^r}^*$, then

$$w_b(c(a)) \quad = \quad \frac{q^b - 1}{N(q-1)q^{b-1}} \left(q^r - \frac{q^r + (-1)^{er/2}(q-1)q^{r/2}}{q} \right) + \frac{2\mu(b)(-1)^{er/2}(q-1)q^{r/2}}{Nq^b}.$$

- If $p \equiv 3 \mod 4$ and a is a non-square in $\mathbb{F}_{q^r}^*$, then

$$w_b(c(a)) \quad = \quad \frac{q^b - 1}{N(q-1)q^{b-1}} \left(q^r - \frac{q^r - (-1)^{er/2}(q-1)q^{r/2}}{q} \right) - \frac{2\mu(b)(-1)^{er/2}(q-1)q^{r/2}}{Nq^b}.$$

Remark 3.2. The Hamming weight distribution of the above cyclic codes has been considered in [9], and C is a two-weight code under the Hamming metric. In this paper we consider b-symbol weight distribution of such codes. Using the map π_b in (1), the problem becomes Hamming weight distribution of some 2-weight cyclic codes over the alphabet $\mathbb{F}_q \times \cdots \mathbb{F}_q = \mathbb{F}_q^b$, which is not a field. We remark that two-weight irreducible cyclic codes *over finite fields* were characterized in [12], and it would be interesting to obtain such a characterization over the alphabet $\mathbb{F}_q^b$. We think that this would be related to Open Problem 2.3 above.

It remains to consider $r \leqslant b < n$ for C. Using [13, Theorem 4.2] and Lemma 2.2 we obtain a nice formula in the following.

Theorem 3.3. *Let $a \in \mathbb{F}_{q^r}^*$. For $r \leqslant b < n$ we have*

$$w_b(c(a)) = n.$$

Combining Theorem 3.1 and Theorem 3.3 we obtain the weight b-symbol Hamming weight enumerators of C completely.

Corollary 3.1. *For $2 \leqslant b \leqslant r - 1$, the b-symbol Hamming weight enumerator of C is*

$$A(T) = 1 + \frac{q^r - 1}{2} \left(T^{u_1} + T^{u_2} \right),$$

where

$$u_1 = \begin{cases} \frac{q^b-1}{N(q-1)q^{b-1}} \left(q^r - \frac{q^r + (q-1)q^{r/2}}{q} \right) + \frac{2\mu(b)(q-1)q^{r/2}}{Nq^b} & \text{if } p \equiv 1 \mod 4, \\[3mm] \frac{q^b-1}{N(q-1)q^{b-1}} \left(q^r - \frac{q^r + (-1)^{er/2}(q-1)q^{r/2}}{q} \right) + \frac{2\mu(b)(-1)^{er/2}(q-1)q^{r/2}}{Nq^b} & \text{if } p \equiv 3 \mod 4, \end{cases}$$

and

$$u_2 = \begin{cases} \frac{q^b-1}{N(q-1)q^{b-1}}\left(q^r - \frac{q^r-(q-1)q^{r/2}}{q}\right) - \frac{2\mu(b)(q-1)q^{r/2}}{Nq^b} & \text{if } p \equiv 1 \mod 4, \\[3mm] \frac{q^b-1}{N(q-1)q^{b-1}}\left(q^r - \frac{q^r-(-1)^{er/2}(q-1)q^{r/2}}{q}\right) - \frac{2\mu(b)(-1)^{er/2}(q-1)q^{r/2}}{Nq^b} & \text{if } p \equiv 3 \mod 4. \end{cases}$$

For $r \leqslant b < n - 1$, the b-symbol Hamming weight enumerator of C is

$$A(T) = 1 + (q^r - 1)T^n.$$

Moreover, C is an MDS b-symbol code when $b = r$.

4. Proofs of the main results

4.1. **Proof of Theorem 3.1.** For $a \in \mathbb{F}_{q^r}$, let $\hat{c}(a) \in \mathbb{F}_q^{q^r-1}$ be the extended codeword of length $q^r - 1$ defined as

$$\hat{c}(a) = \left(\text{Tr}(a\eta^{0\cdot N}), \text{Tr}(a\eta^{1\cdot N}), \ldots, \text{Tr}(a\eta^{j\cdot N}), \ldots, \text{Tr}(a\eta^{(q^r-2)\cdot N})\right),$$

where $0 \leqslant j \leqslant q^r - 2$. As $\eta^{nN} = 1$, we observe that

(2) $$w(c(a)) = \frac{1}{N}w(\hat{c}(a)) \quad \text{and} \quad w_b(c(a)) = \frac{1}{N}w_b(\hat{c}(a))$$

for each $2 \leqslant b \leqslant n - 1$.

For $a \in \mathbb{F}_{q^r}$, let $Z(a) \in \mathbb{C}$ be defined as

(3) $$Z(a) = \frac{1}{q}\sum_{y\in\mathbb{F}_q}\sum_{x\in\mathbb{F}_{q^r}} e^{\frac{2\pi\sqrt{-1}}{p}\text{Tr}_{q/p}\left(y\text{Tr}_{q^r/q}(ax^N)\right)}.$$

For each $x \in \mathbb{F}_{q^r}^*$, we observe that

$$\sum_{y\in\mathbb{F}_q} e^{\frac{2\pi\sqrt{-1}}{p}\text{Tr}_{q/p}\left(y\text{Tr}_{q^r/q}(ax^N)\right)} = \begin{cases} q & \text{if } \text{Tr}_{q^r/q}(ax^N) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that

(4) $$w(\hat{c}(a)) = q^r - Z(a).$$

Let $H \leqslant \mathbb{F}_{q^r}^*$ be the multiplicative subgroup of order $\frac{q^r-1}{N}$. Using (3) we obtain that

$$
\begin{aligned}
Z(a) \;=\; & \frac{1}{q}\left\{q^r + \sum_{y\in\mathbb{F}_q^*}\sum_{x\in\mathbb{F}_{q^r}} e^{\frac{2\pi\sqrt{-1}}{p}\mathrm{Tr}_{q^r/q}\left(y\mathrm{Tr}_{q^r/q}(ax^N)\right)}\right\} \\[2mm]
=\; & \frac{1}{q}\left\{q^r + \sum_{y\in\mathbb{F}_q^*}\left[1 + \sum_{x\in\mathbb{F}_{q^r}^*} e^{\frac{2\pi\sqrt{-1}}{p}\mathrm{Tr}_{q^r/p}\left(ayx^N\right)}\right]\right\} \\[2mm]
=\; & \frac{1}{q}\left\{q^r + q - 1 + \sum_{y\in\mathbb{F}_q^*}\sum_{x\in\mathbb{F}_{q^r}^*} e^{\frac{2\pi\sqrt{-1}}{p}\mathrm{Tr}_{q^r/p}\left(ayx^N\right)}\right\} \\[2mm]
=\; & \frac{1}{q}\left\{q^r + q - 1 + N\sum_{y\in\mathbb{F}_q^*}\sum_{z\in H} e^{\frac{2\pi\sqrt{-1}}{p}\mathrm{Tr}_{q^r/p}(ayz)}\right\}
\end{aligned}
$$

(5)

It is well known (see, for example, [11, Theorem 5.15]) that

(6)
$$
\sum_{\substack{y\in\mathbb{F}_{q^r}^* \\ y \text{ is square}}} e^{\frac{2\pi\sqrt{-1}}{p}\mathrm{Tr}_{q^r/p}(y)} = \begin{cases} \frac{-q^{r/2}-1}{2} & \text{if } p \equiv 1 \mod 4, \\ \frac{-(-1)^{er/2}q^{r/2}-1}{2} & \text{if } p \equiv 3 \mod 4, \end{cases}
$$

and

(7)
$$
\sum_{\substack{y\in\mathbb{F}_{q^r}^* \\ y \text{ is non-square}}} e^{\frac{2\pi\sqrt{-1}}{p}\mathrm{Tr}_{q^r/p}(y)} = \begin{cases} \frac{q^{r/2}-1}{2} & \text{if } p \equiv 1 \mod 4, \\ \frac{(-1)^{er/2}q^{r/2}-1}{2} & \text{if } p \equiv 3 \mod 4, \end{cases}
$$

The following lemma is also useful.

Lemma 4.1. *Consider $\mathbb{F}_q^* \times H$ as a group with componentwise multiplicative operations. Let ϕ be the map defined as*

$$
\begin{aligned}
\phi: \mathbb{F}_q^* \times H \;&\rightarrow\; \mathbb{F}_{q^r}^* \\
(y,z) \;&\mapsto\; yz
\end{aligned}
$$

Then ϕ is a group homomorphism and $\mathrm{Ker}\,\phi = \{(h,h^{-1}) : h \in H_0\}$, where $H_0 \leqslant \mathbb{F}_{q^r}^*$ is the multiplicative subgroup of order $\frac{2(q-1)}{N}$.

Proof. Note that $(y,z) \in \mathrm{Ker}\,\phi$ if and only if $yz = 1$. This implies that if $(y,z) \in \mathrm{Ker}\,\phi$, then $y \in \mathbb{F}_q^* \cap H$, $z \in \mathbb{F}_q^* \cap H$ and $z = y^{-1}$. Conversely if $h \in \mathbb{F}_q^* \cap H$, then $(h,h^{-1}) \in \mathrm{Ker}\,\phi$. It remains to prove that $\mathbb{F}_q^* \cap H = H_0$. This is equivalent to the statement that

(8)
$$
\gcd\left(q-1, \frac{q^r-1}{N}\right) = \frac{2(q-1)}{N}.
$$

Note that $\gcd\left(\frac{q^r-1}{q-1}, N\right) = 2$ by one of the main assumptions of this paper in Section 2. Hence we have

$$(9) \qquad \gcd\left(\frac{q^r-1}{2(q-1)}, \frac{N}{2}\right) = 1.$$

As $\frac{N}{2} \mid \frac{q^r-1}{2}$ and $\frac{q^r-1}{2} = \frac{q^r-1}{2(q-1)} \cdot (q-1)$, using (9) we conclude that

$$(10) \qquad \frac{N}{2}\bigg|(q-1).$$

Moreover $2\left|\frac{q^r-1}{q-1}\right.$ as r is even and q is odd. Using (10) we obtain that

$$(11) \qquad \frac{2(q-1)}{N}\bigg|\gcd\left(q-1, \frac{q^r-1}{N}\right).$$

Combining (9) and (11) we conclude that the statement in (8) holds. This completes the proof. $\qquad\square$

In the following lemma we use the multiset notation $\{* \cdots *\}$ and the multiplicities are denoted as multiplication coefficients. For example we have $\{*x_1, x_1, x_2*\} = \{*2 \cdot x_1, x_2*\}$.

Lemma 4.2. *Let $a \in \mathbb{F}_{q^r}^*$. Using the multiset notation as above we have the followings:*

- *If a is a square, then*

$$\{*ayz : y \in \mathbb{F}_q^*, \ z \in H*\} = \left\{*\frac{2(q-1)}{N} \cdot x : x \in \mathbb{F}_{q^r}^*, \ x \text{ is a square}*\right\}.$$

- *If a is a non-square, then*

$$\{*ayz : y \in \mathbb{F}_q^*, \ z \in H*\} = \left\{*\frac{2(q-1)}{N} \cdot x : x \in \mathbb{F}_{q^r}^*, \ x \text{ is a non-square}*\right\}.$$

Proof. First we assume that $a = 1$. The homomorphism $\phi : \mathbb{F}_q^* \times H \to \mathbb{F}_{q^r}^*$ defined in Lemma 4.1 has a kernel of cardinality $\frac{2(q-1)}{N}$ by Lemma 4.1. This implies that the image of ϕ has cardinality

$$(12) \qquad \frac{(q-1)\frac{q^r-1}{N}}{\frac{2(q-1)}{N}} = \frac{q^r-1}{2}.$$

Moreover $\phi(y, z)$ is a square in $\mathbb{F}_{q^r}^*$ for any $y \in \mathbb{F}_q^*$ and $z \in H$ as r is even and N is even. Hence using (12) we obtain that $\text{Im}\phi$ is exactly the set $\{x \in \mathbb{F}_{q^r}^* : x \text{ is a square}\}$. Counting the multiplicities we complete the proof if $a = 1$.

If $a \in \mathbb{F}_{q^r}^* \setminus \{1\}$ is a square, then it is clear that the multiset $\{*ayz : y \in \mathbb{F}_q^*, \ z \in H*\}$ is the same as the multiset $\{*yz : y \in \mathbb{F}_q^*, \ z \in H*\}$.

If $a \in \mathbb{F}_{q^r}^*$ is a non-square, then we obtain that the set $\{ayz : y \in \mathbb{F}_q^*, \ z \in H\}$ is equal to the set $\{x : x \in \mathbb{F}_{q^r}^*, \ x \text{ is a non-square}\}$. Moreover the multiplicities are the same. This completes the proof. $\qquad\square$

Assume that $a \in \mathbb{F}_{q^r}^*$ is a square. Using (5), (6) and Lemma 4.2 we obtain that

$$Z(a) = \frac{1}{q} \left\{ q^r + q - 1 + N \frac{2(q-1)}{N} \sum_{\substack{y \in \mathbb{F}_{q^r}^* \\ y \text{ is square}}} e^{\frac{2\pi\sqrt{-1}}{p} \operatorname{Tr}_{q^r/p}(y)} \right\}$$

$$(13) \qquad = \begin{cases} \frac{1}{q}\left(q^r + q - 1 + 2(q-1)\frac{-q^{r/2}-1}{2}\right) & \text{if } p \equiv 1 \mod 4, \\[3mm] \frac{1}{q}\left(q^r + q - 1 + 2(q-1)\frac{q^{r/2}-1}{2}\right) & \text{if } p \equiv 3 \mod 4, \end{cases}$$

$$= \begin{cases} \frac{q^r - (q-1)q^{r/2}}{q} & \text{if } p \equiv 1 \mod 4, \\[3mm] \frac{q^r - (-1)^{er/2}(q-1)q^{r/2}}{q} & \text{if } p \equiv 3 \mod 4, \end{cases}$$

Next we assume that $a \in \mathbb{F}_{q^r}^*$ is a non-square. Using (5), (7) and Lemma 4.2 similarly we obtain that

$$(14) \qquad Z(a) = \begin{cases} \frac{q^r + (q-1)q^{r/2}}{q} & \text{if } p \equiv 1 \mod 4, \\[3mm] \frac{q^r + (-1)^{er/2}(q-1)q^{r/2}}{q} & \text{if } p \equiv 3 \mod 4, \end{cases}$$

Recall that the set $\mathcal{P}(b)$ is defined in Definition 2.1. An important result obtained in [13] using algebraic curves over finite fields is that

$$(15) \qquad w_b(\hat{c}(a)) = \frac{1}{q^{b-1}} \sum_{\theta \in \mathcal{P}(b)} w\left(\hat{c}(\theta a)\right).$$

Recall that $\mu(b)$ is the integer defined in Definition 2.2.

Now assume that $p \equiv 1 \mod 4$ and a is a square in $\mathbb{F}_{q^r}^*$. Combining (4), (13), (14) and (15) we obtain that

$$w_b\left(\hat{c}(a)\right) = \frac{1}{q^{b-1}}\left(\mu(b)\left[q^r - \frac{q^r - (q-1)q^{r/2}}{q}\right]\right)$$

$$(16) \qquad\qquad + \frac{1}{q^{b-1}}\left(\left(\frac{q^b-1}{q-1} - \mu(b)\right)\left[q^r - \frac{q^r + (q-1)q^{r/2}}{q}\right]\right)$$

$$= \frac{q^b-1}{(q-1)q^{b-1}}\left(q^r - \frac{q^r + (q-1)q^{r/2}}{q}\right) + \frac{2\mu(b)(q-1)q^{r/2}}{q^b}.$$

Next assume that $p \equiv 1 \mod 4$ and a is a non-square in $\mathbb{F}_{q^r}^*$. Combining (4), (13), (14) and (15) we obtain that

$$
\begin{aligned}
w_b\left(\hat{c}(a)\right) &= \frac{1}{q^{b-1}}\left(\mu(b)\left[q^r - \frac{q^r + (q-1)q^{r/2}}{q}\right]\right) \\
(17) \qquad &+ \frac{1}{q^{b-1}}\left(\left(\frac{q^b-1}{q-1} - \mu(b)\right)\left[q^r - \frac{q^r - (q-1)q^{r/2}}{q}\right]\right) \\
&= \frac{q^b-1}{(q-1)q^{b-1}}\left(q^r - \frac{q^r - (q-1)q^{r/2}}{q}\right) - \frac{2\mu(b)(q-1)q^{r/2}}{q^b}.
\end{aligned}
$$

Again next assume that $p \equiv 3 \mod 4$ and a is a square in $\mathbb{F}_{q^r}^*$. Combining (4), (13), (14) and (15) we obtain that

$$
\begin{aligned}
w_b\left(\hat{c}(a)\right) &= \frac{1}{q^{b-1}}\left(\mu(b)\left[q^r - \frac{q^r - (-1)^{er/2}(q-1)q^{r/2}}{q}\right]\right) \\
(18) \qquad &+ \frac{1}{q^{b-1}}\left(\left(\frac{q^b-1}{q-1} - \mu(b)\right)\left[q^r - \frac{q^r + (-1)^{er/2}(q-1)q^{r/2}}{q}\right]\right) \\
&= \frac{q^b-1}{(q-1)q^{b-1}}\left(q^r - \frac{q^r + (-1)^{er/2}(q-1)q^{r/2}}{q}\right) + \frac{2\mu(b)(-1)^{er/2}(q-1)q^{r/2}}{q^b}.
\end{aligned}
$$

Finally assume that $p \equiv 3 \mod 4$ and a is a non-square in $\mathbb{F}_{q^r}^*$. Combining (4), (13), (14) and (15) we obtain that

$$
\begin{aligned}
w_b\left(\hat{c}(a)\right) &= \frac{1}{q^{b-1}}\left(\mu(b)\left[q^r - \frac{q^r + (-1)^{er/2}(q-1)q^{r/2}}{q}\right]\right) \\
(19) \qquad &+ \frac{1}{q^{b-1}}\left(\left(\frac{q^b-1}{q-1} - \mu(b)\right)\left[q^r - \frac{q^r - (-1)^{er/2}(q-1)q^{r/2}}{q}\right]\right) \\
&= \frac{q^b-1}{(q-1)q^{b-1}}\left(q^r - \frac{q^r - (-1)^{er/2}(q-1)q^{r/2}}{q}\right) - \frac{2\mu(b)(-1)^{er/2}(q-1)q^{r/2}}{q^b}.
\end{aligned}
$$

Using (2), (16), (17), (18) and (19) we complete the proof of Theorem 3.1.

4.2. **Proof of Theorem 3.3.** If $r < b \leqslant n - 1$, then using the proof of [13, Theorem 4.2] we obtain that $w_b(c(a)) = w_r(c(a))$. It remains to prove Theorem 3.3 for $b = r$.

Assume that $p \equiv 1 \mod 4$ and a is a square in $\mathbb{F}_{q^r}^*$. Then using Lemma 2.2 and Theorem 3.1 we obtain that $\mu(r) = \frac{q^r-1}{2(q-1)}$ and hence

$$
\begin{aligned}
w_r(c(a)) &= \frac{q^r-1}{N(q-1)q^{r-1}} \left(q^r - \frac{q^r + (q-1)q^{r/2}}{q} \right) + \frac{(q^r-1)q^{r/2}}{Nq^r} \\
&= \frac{q^r-1}{N(q-1)q^{r-1}} \left(q^r - q^{r-1} \right) + \frac{q^r-1}{Nq^r} \left(-q^{r/2} + q^{r/2} \right) \\
&= \frac{(q^r-1)\,q^{r-1}}{Nq^{r-1}} = \frac{q^r-1}{N} = n.
\end{aligned}
$$

The proof is similar for the case that $p \equiv 1 \mod 4$ and a is a non-square.

Next assume that $p \equiv 3 \mod 4$ and a is a square in $\mathbb{F}_{q^r}^*$. Then similarly using Lemma 2.2 and Theorem 3.1 we obtain that

$$
\begin{aligned}
w_r(c(a)) &= \frac{q^r-1}{N(q-1)q^{r-1}} \left(q^r - \frac{q^r + (-1)^{er/2}(q-1)q^{r/2}}{q} \right) + \frac{(-1)^{er/2}(q^r-1)q^{r/2}}{Nq^r} \\
&= \frac{q^r-1}{N(q-1)q^{r-1}} \left(q^r - q^{r-1} \right) + \frac{(-1)^{er/2}q^r-1}{Nq^r} \left(-q^{r/2} + q^{r/2} \right) \\
&= \frac{(q^r-1)\,q^{r-1}}{Nq^{r-1}} = \frac{q^r-1}{N} = n.
\end{aligned}
$$

The proof is similar for the case that $p \equiv 3 \mod 4$ and a is a non-square. From the definition of the MDS b-symbol codes, it is easy to check that C is MDS when $b = r$. This completes the proof of Theorem 3.3.

Example 4.1. *Let γ be a primitive element of $\mathbb{F}_{81}$ with $\gamma^4 + 2\gamma^3 + 2 = 0$.*

According to Table 1 of [9], the Hamming weight enumerator of C is $1 + A_{24}^{(1)}T^{24} + A_{30}^{(1)}T^{30} = 1 + 40T^{24} + 40T^{30}$ when $q = 3, r = 4, N = 2$ and $\gcd(\frac{q^r-1}{q-1}, N) = 2$. When $b = 2$, $|\mathcal{P}(2)| = q + 1 = 4$. Let $\mathcal{P}(2) = \{1, \gamma^2, 1 + \gamma^2, 1 + 2\gamma^2\}$. If $a \in C_i^{(2,r)}$, then $\gamma^2 a, (1+\gamma^2)a = \gamma^{58}a \in C_i^{(2,r)}, (1+2\gamma^2)a = \gamma^{65}a \in C_{i+1(\mathrm{mod}\ 2)}^{(2,r)}$, where $i = 0, 1$. According to the equality (15), we have

$$
\begin{aligned}
w_2(c(a)) &= \frac{1}{3}[w_1(c(a)) + w_1(c(\gamma^2 a)) + w_1(c((1+\gamma^2)a)) + w_1(c((1+2\gamma^2)a))] \\
&= \begin{cases} \frac{1}{3}(24 + 24 + 24 + 30) = 34 & a \in C_0^{(2,r)}, \\ \frac{1}{3}(30 + 30 + 30 + 24) = 38 & a \in C_1^{(2,r)}. \end{cases}
\end{aligned}
$$

Therefore the 2-symbol Hamming weight enumerator of C is $1 + A_{34}^{(2)}T^{34} + A_{38}^{(2)}T^{38} = 1 + |C_0^{(2,r)}|T^{34} + |C_1^{(2,r)}|T^{38} = 1 + 40T^{34} + 40T^{38}$.

Combining Example 2.2 and Corollary 3.1, we have $\mu(2) = 3$, $u_1 = 38$ and $u_2 = 34$, which is the same as above result. The above result also verified by Magma experiment.

Acknowledgement

References

[1] Y. Cassuto and M. Blaum: *Codes for symbol-pair read channels*, in Proc. Int. Symp. Inf. Theory, Austin, TX, USA, Jun. (2010), pp. 988–992.

[2] Y. Cassuto and M. Blaum: *Codes for symbol-pair read channels*, IEEE Trans. Inform. Theory **57** (2011), no. 12, pp. 8011–8020.

[3] Y. Cassuto and S. Litsyn: *Symbol-pair codes: Algebraic constructions and asymptotic bounds*, in Proc. Int. Symp. Inf. Theory, St. Petersburg, Russia, Jul./Aug. (2011), pp. 2348–2352.

[4] Y.M. Chee, L. Ji, H.M. Kiah, C. Wang and J. Yin: *Maximum distance separable codes for symbol pair read channels, IEEE Trans. Inform. Theory*, **59** (2013), no. 11, pp. 7259–7267.

[5] Y.M. Chee, H.M. Kiah, C. Wang and J. Yin: *Maximum distance separable symbol-pair codes*, In: Proc. IEEE Int. Symp. Inf. Theory, Cambridge, MA, USA, (2012), 2886–2890.

[6] B. Chen, L. Lin and H. Liu: *Constacyclic symbol-pair codes: lower bounds and optimal constructions*, IEEE Trans. Inform. Theory **63** (2017) no. 12, pp. 7661–7666.

[7] B. Ding, G. Ge, J. Zhang, T. Zhang and Y. Zhang: *New constructions of MDS symbol-pair codes. Des. Codes Cryptogr.*, **86** (2018) no. 4, pp. 841–859.

[8] B. Ding, T. Zhang and G. Ge: *Maximum distance separable codes for b-symbol read channels, Finite Fields Appl.*, **49** (2018), pp. 180–197.

[9] C. Ding and J. Yang: *Hamming weights in irreducible cyclic codes*, Discrete Math. **313** (2013), no. 4, pp. 434–446.

[10] X. Kai, S. Zhu, and P. Li: *A construction of new MDS symbol-pair codes*, IEEE Trans. Inform. Theory **61** (2015), no. 11, pp. 5828–5834.

[11] R. Lidl and H. Niederreiter: *Finite Fields*, Second Edition, Encyclopedia of Mathematics and its Applications, 20, Cambridge University Press, Cambridge, 1997.

[12] B. Schmidt and C. White, *All two-weight irreducible cyclic codes?*, Finite Fields Appl. **8** (2002), no. 1, pp. 1–17.

[13] M. Shi, F. Özbudak and P. Solé : *Geometric Approach to b-Symbol Hamming Weights of Cyclic Codes*, IEEE Trans. Inform. Theory, accepted, 2021.

[14] Z. Sun, S. Zhu and L. Wang: *The symbol-pair distance distribution of a class of repeated-root cyclic codes over $\mathbb{F}_{p^m}$*, Cryptogr. Commun.,**10** (2018), no. 4, pp. 643–653.

[15] E. Yaakobi, J. Bruck and P. H. Siegel: *Constructions and Decoding of Cyclic Codes Over b-Symbol Read Channels*, IEEE Trans. Inform. Theory **62** (2016), no. 4, pp. 1541–1551.

5. appendix

Using algebraic curves, Shi *et al.* [13] proved that the equality (15) holds. Here we give another proof which using exponent sum.

Let C be a cyclic irreducible code of period n and dimension k with zero $\eta = \gamma^N$ where γ^N is a primitive element in $\mathbb{F}_{q^r}$ and $N = \frac{q^k-1}{n}$. Let $I = \{1, \eta, \ldots, \eta^{n-1}\}$. Recall that $C = \{c(a) = \left(\text{Tr}(a\eta^{0 \cdot N}), \text{Tr}(a\eta^{1 \cdot N}), \ldots, \text{Tr}(a\eta^{j \cdot N}), \ldots, \text{Tr}(a\eta^{(n-1) \cdot N})\right)\}$ and the Hamming weight of a codeword $w_1(c(a))$ is

$$n - w_1(c(a)) = \sum_{x \in I} \frac{1}{q} \sum_{y \in \mathbb{F}_q} \chi(yax),$$

where χ is an additive character of $\mathbb{F}_{q^r}$. To find the b-symbol Hamming weight of $c(a)$, then one needs to count how often b consecutive positions $(\text{Tr}(ax), \text{Tr}(a\eta x), \cdots, \text{Tr}(s\eta^b x))$ are all 0. This implies

$$
\begin{aligned}
n - w_b(c(a)) &= \sum_{x \in I} \frac{1}{q^b} \sum_{y_1, \ldots, y_b \in \mathbb{F}_q} \chi((y_1 + y_2\eta + \cdots + y_{b-1}\eta^{b-1})ax) \\
&= \frac{1}{q^{b-1}} \sum_{x \in I} \frac{1}{q} \sum_{u \in V} \chi(uax),
\end{aligned}
$$

where $V = <1, \eta, \cdots, \eta^{b-1}>$. Let $\bigcup_{i=0}^{\frac{q^b-1}{q-1}} \theta_i \mathbb{F}_q^* \cup 0$ then

$$
\begin{aligned}
n - w_b(c(a)) &= \frac{1}{q^{b-1}} \sum_{i=1}^{\frac{q^b-1}{q-1}} \sum_{x \in I} \frac{1}{q} \sum_{y \in \mathbb{F}_q^*} \chi(y\theta_i ax) + \frac{n}{q^b} \\
&= \frac{1}{q^{b-1}} \sum_{i=1}^{\frac{q^b-1}{q-1}} \left(n - w_1(c(\theta_i a))\right) - \frac{n(q^b-1)}{q^b(q-1)} + \frac{n}{q^b} \\
&= \frac{n(q^b-1)}{q^{b-1}(q-1)} - \frac{1}{q^{b-1}} \sum_{i=1}^{\frac{q^b-1}{q-1}} w_1(c(\theta_i a)) - \frac{n(q^b-q)}{q^b(q-1)} \\
&= n - \frac{1}{q^{b-1}} \sum_{i=1}^{\frac{q^b-1}{q-1}} w_1(c(\theta_i a)).
\end{aligned}
$$

Hence, $w_b(c(a)) = \frac{1}{q^{b-1}} \sum_{i=1}^{\frac{q^b-1}{q-1}} w_1(c(\theta_i a))$.