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DEVELOPMENT OF PROSPECTIVE SECONDARY SCHOOL MATHEMATICS TEACHERS' SPECIALIZED KNOWLEDGE OF LIMIT CONCEPT THROUGH LESSON STUDY

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

RUYA SAVURAN

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
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Approval of the thesis:

DEVELOPMENT OF PROSPECTIVE SECONDARY SCHOOL MATHEMATICS TEACHERS' SPECIALIZED KNOWLEDGE OF LIMIT CONCEPT THROUGH LESSON STUDY

submitted by **RUYA SAVURAN** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy** i**n Mathematics Education in Mathematics and Science Education, Middle East Technical University** by,

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ABSTRACT

DEVELOPMENT OF PROSPECTIVE SECONDARY SCHOOL MATHEMATICS TEACHERS' SPECIALIZED KNOWLEDGE OF LIMIT CONCEPT THROUGH LESSON STUDY

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Doctor of Philosophy, Mathematics Education in Mathematics and Science Education
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The purpose of this study is to understand the nature and development of prospective mathematics teachers' specialized knowledge for teaching the concept of limit in a broad sense through designed lesson study development model. The teaching experiment methodology was adopted to examine the nature and development of prospective mathematics teachers' specialized knowledge for teaching the concept of limit; in particular, the teacher development experiment was utilized. Accordingly, lesson study with its phases including investigation, planning, research lessons, and reflection was designed and utilized as teacher development experiment in order to provide development of knowledge for teaching the concept of limit. The model of Mathematics Teachers Specialized Knowledge proposed by Carrillo-Yañez and his colleagues (2018) was used as an analytical and theoretical framework to examine the development of knowledge for teaching the concept of limit. Considering the aim of the study, the study focused on the development of specialized knowledge of a prospective mathematics teacher (Mila- as a pseudonym) who was chosen purposefully among the lesson study group members which included three prospective mathematics teachers. The data was collected during the spring semester of 2018-2019 academic year, in particular at the beginning of the semester in the time of the concept of limit. The data were gathered primarily from the clinical interviews conducted before and after the lesson study process, observations of the cycles of lesson study, lesson plans, reflection papers and field notes.

The findings indicated that the prospective mathematics teacher had lack of knowledge for teaching mathematics in some indicators of sub-domains of the model. The designed-lesson study enabled the prospective mathematics teacher to improve her knowledge in a broad view in the concept of limit. By this way, the prospective mathematics teacher took a critically more reflective stance on her teaching and developing her own knowledge by means of thinking on how to teach the concept and how to help students make sense of such an abstract concept in their mind. In addition, the findings revealed a model which includes critical elements such as pre- interviews before the process, rich group discussions, sufficiently long lesson planning, and the nature of the concept, and how knowledge development is observed when these elements are integrated into the process in accordance with the observable characteristics of the lesson study. The study has important implications for teacher preparation programs, mathematics teacher educators and mathematics education researchers in both practical and theortical way.

Keywords: Prospective mathematics teacher education; Mathematics Teachers' Specialized Knowledge; Lesson study; The concept of limit

ÖZ

LİSE MATEMATİK ÖĞRETMEN ADAYLARININ LIMIT KAVRAMINA YÖNELİK UZMANLIK BİLGİLERİNİN DERS İMCESİ YOLUYLA GELİŞTİRİLMESİ

Savuran, Ruya
Doktora, Matematik Eğitimi, Matematik ve Fen Bilimleri Eğitimi
Tez Yöneticisi: Prof. Dr. Mine Işıksal Bostan

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Bu çalışmanın amacı, tasarlanmış ders imecesi geliştirme modeli ile matematik öğretmen adaylarının limit kavramını geniş anlamda öğretmeye yönelik uzmanlık bilgilerinin doğasını ve gelişimini anlamaktır. Matematik öğretmen adaylarının limit kavramını öğretmek için uzmanlık bilgilerinin doğasını ve gelişimini incelemek için öğretim deneyi metodolojisi benimsenmiş; özellikle öğretmen geliştirme deneyinden yararlanılmıştır. Bu doğrultuda, limit kavramının öğretimine yönelik bilgilerin geliştirilmesini sağlamak amacıyla araştırma, planlama, araştırma dersleri ve yansıtma aşamalarını içeren ders imecesi öğretmen geliştirme deneyi olarak tasarlanmış ve kullanılmıştır. Carrillo-Yañez ve meslektaşları (2018) tarafından önerilen Matematik Öğretmenlerinin Uzmanlık Bilgi modeli, limit kavramının öğretimine yönelik bilginin gelişimini incelemek için analitik ve teorik bir çerçeve olarak kullanılmıştır. Araştırmanın amacı dikkate alınarak, üç matematik öğretmeni adayının yer aldığı ders imecesi grup üyeleri arasından amaçlı olarak seçilen bir matematik öğretmeni adayının (Mila-takma isim) uzmanlık bilgisinin geliştirilmesine odaklanılmıştır. Veriler 2018-2019 eğitim-öğretim yılı bahar döneminde, özellikle limit kavramının ele alındığı dönem başında toplanmıştır. Veriler öncelikle ders imecesi süreci öncesi ve sonrasında yapılan klinik görüşmelerden, ders imecesi döngülerine ilişkin gözlemlerden, ders planlarından, yansıtma kağıtlarından ve alan notlarından toplanmıştır.

Bulgular, matematik öğretmeni adayının modelin alt alanlarının bazı göstergelerinde matematik öğretimi konusunda bilgi eksikliğine sahip olduğunu göstermiştir. Tasarlanmış-ders imecesi matematik öğretmeni adayının limit kavramına ilişkin bilgisini geniş bir çerçevede geliştirmesini sağlamıştır. Böylece matematik öğretmeni adayı kavramın nasıl öğretileceğini ve öğrencilerin böyle soyut bir kavramı zihinlerinde anlamlandırmalarına nasıl yardımcı olabileceğini düşünerek öğretimi ve kendi bilgilerini geliştirme konusunda eleştirel olarak daha yansıtıcı bir duruş sergilemiştir. Ayrıca bulgular, süreç öncesi ön görüşmeler, zengin grup tartışmaları, yeterince uzun ders planlaması, ve kavramın doğası gibi kritik unsurları içeren ve bu unsurların ders imecesinin gözlemlenebilir özelliklerine uygun olarak sürece nasıl entegre edildiğinde bilgi gelişimin gözlemlendiğine yönelik bir model ortaya koymuştur. Çalışmanın öğretmen yetiştime programları, matematik öğretmen eğitimcileri ve matematik eğitimi araştırmacıları için hem pratik hem de teorik açıdan önemli çıkarımları bulunmaktadır.

Anahtar Kelimeler: Matematik öğretmeni yetiştirme ve eğitimi; Matematik Öğretmenlerinin Uzmanlık Bilgileri; Ders imecesi; Limit kavramı

To my family

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LIST OF ABBREVIATIONS

ABBREVIATIONS

MoNE	Ministry of National Education
MTSK	Mathematics Teachers' Specialized Knowledge
MK	Mathematical Knowledge
PCK	Pedagogical content knowledge
KoT	Knowledge of Topics
KSM	Knowledge of Structure of Mathematics
KPM	Knowledge of Practices in Mathematics
KFLM	Knowledge of Features of Learning Mathematics
KMT	Knowledge of Mathematics Teaching
KMLS	Knowledge of Mathematics Learning Standards

CHAPTER 1

INTRODUCTION

Mathematics teacher education appears as a significant problem to be solved for helping students improve students' their learning (Morris, Hiebert, & Spitzer, 2009). It is a complex process which includes so many elements (Huling- Austin, 1992; Flores, 2006). Ponte and Chapman (2015) mentioned these elements as prospective teachers', instructors' and other stakeholders' characteristics, elements in the education program, sociocultural factors, educational system and research on prospective teacher education. In their proposed landscape (Ponte & Chapman, 2015, p. 276) they emphasized that those elements are placed around prospective teachers' knowledge for mathematics and teaching mathematics. Therefore, it can be concluded that better prospective teacher education starts with developing their knowledge of mathematics and mathematics teaching.

Significant number of things are expected when it comes to mathematics teaching and there is consensus that "teachers need to know more, and different, mathematics" (Ball et al. 2008, p. 396) than most adults in order to teach effectively. Knowledge for teaching mathematics specific for teachers requires both having the mathematical and pedagogical content knowledge (Shulman, 1986) as well as reflecting this knowledge on practice which includes not only teaching the concepts in the classroom but also planning lessons, understanding the nature of learners, learning and teaching pathways, criticizing teaching processes and revising the defective points in teaching (Leavy & Hourigan, 2016). Kilpatrick and her colleagues (2001) conceptualize mathematics teacher knowledge as

> The kinds of knowledge that make a difference in teaching practice and in students' learning are an elaborated, integrated knowledge of mathematics, a

> knowledge of how students' mathematical understanding develops, and a repertoire of pedagogical practices that take into account the mathematics being taught and how students learn it. (p. 381)

These comprehensive descriptions have been explored in different contexts with different parts in the mathematics education literature for many years. Accordingly, different models have been proposed to understand the nature of knowledge for teaching mathematics based on the attempts of Shulman (1986), who asserts that pedagogical content knowledge is a compulsory type of knowledge for innovative teachers since it includes being aware of how students understand, what kind of problems they might have, and what strategies can be produced to help them grasp the content better. The creation aims of these models differ from each other. For instance, one of the big steps in mathematics teacher knowledge literature is Mathematical Knowledge for Teaching proposed by Ball and her colleagues (2008) which aims to examine mathematics teachers' knowledge based on their classroom practices. On the other hand, Knowledge Quartet (Rowland, Huckstep, & Thwaites, 2005) is based on the idea of providing four types of knowledge content aimed at analyzing the mathematical knowledge used in teaching. Considering the factors about prospective teacher education mentioned by Ponte and Chapman (2015), examining only knowledge in practice can be regarded as insufficient to answer the question of how to develop prospective teachers' knowledge for teaching mathematics. For this reason, prospective mathematics teachers' knowledge could be developed with a more comprehensive model for mathematics teachers' knowledge. As a new attempt to teacher knowledge models, Carrillo-Yañez and his colleagues (2018) proposed the model named Mathematics Teachers' Specialized Knowledge (MTSK) as an analytical and methodological model which is different from the former models in terms of focusing on not only knowledge in practice but also knowledge they had before practice (Carrillo-Yañez et al., 2018). The model has six sub-domains, which represent different knowledge requirements for teaching mathematics, encircled around the belief towards mathematics and learning and teaching mathematics. Three of these sub-domains (knowledge of topics, knowledge of structure of mathematics, knowledge of practices in mathematics) are grouped as mathematical knowledge, and the other three (knowledge of features of learning mathematics, knowledge of mathematics teaching, knowledge of mathematics learning standards) are grouped as pedagogical content knowledge (Carrillo-Yañez et al., 2018). Since the model has indicators in each sub-domain that represent knowledge included in the sub-domain of MTSK, it provides a way for teachers and prospective teachers to understand their knowledge and observe the development of their knowledge.

Another important aspect is to reveal the answer of how different dimensions of learning process which can be called as critical elements promote prospective teachers in developing their necessary knowledge for teaching (Ponte & Chapman, 2015). Though models propose the nature of knowledge for teaching mathematics, the field of prospective teacher education still seeks to answer this question. The results of the studies show that prospective teachers have difficulties in having the necessary knowledge to teach mathematics and developing their existing knowledge (Morris, Hiebert, & Spitzer, 2009) and this has led us to reach the conclusion that what types of knowledge should be determined and emphasized in prospective teacher training in order to support and provide a good mathematics teaching.

Prospective teachers' existing knowledge and knowledge development are directly affected by other mentioned elements such as the context and teaching environment. Therefore, the opportunities offered to prospective teachers are of importance for learning to teach by means of being in a well-designed educational environment before starting the profession (Osmanoğlu, 2010). In Turkey and other similar countries, prospective teachers are offered some opportunities including taking advanced mathematical courses and various mathematics education courses, practicing knowledge gained through these courses during the program and in real classrooms in practice, observing in-service mathematics teachers and their classmates in teaching practice. However, the studies show that these opportunities have limited effects on preparing prospective teachers to their profession’s competencies (e.g., Paker, 2008; Østergaard, 2015; Koponen et al., 2016). To develop these opportunities, the literature suggests that teacher educators and prospective teachers collaborate on inquiry and research-based teacher education in which the development is traced based on a specific target and the joint group rethinks it critically through reflective practice (e.g., Sullivan & Wood, 2008; Jaworski & Huang, 2014). Therefore, there is a consensus on the need for a new approach to develop these opportunities in the development of knowledge of prospective teachers.

When the teacher preparation programs are considered in terms of knowledge for teaching, reflecting skills and insufficient teaching practice of prospective mathematics teachers, it is hard to construct such an environment that promotes sharing knowledge of teaching and learning, experience, and different viewpoints collaboratively (Murata et al., 2012). Since a well-trained prospective mathematics teacher is also effective in teaching mathematics and therefore learning mathematics, researchers in mathematics teacher education have proposed different models and interventions to prepare prospective teachers for effective mathematics teaching (Caccavale, 2017). One of the ways that mostly meets the requirements which are not only to deal with mathematics more; instead, to learn mathematics more deeply and meaningfully to enable students to learn better, and to reconstruct mathematical knowledge based on this knowledge (Ponte & Chapman, 2015) is lesson study (Takahashi & Yoshida, 2004). Lesson study is one of the models that has gained much popularity in teacher education programs worldwide (Huang & Shimizu, 2016) to develop not only knowledge but also crucial elements of teaching including creativity, critical thinking, noticing, feelings, and beliefs towards teaching that facilitate the use of knowledge (Stigler & Hiebert, 1999; Lewis, 2009). Lesson study is a cycling process that teachers work collaboratively and includes several steps which can occur over a number of weeks for instructional development based on students' learning (Lewis, 2002). In lesson study, a group of teachers engage in four phases of studying which starts with determining a goal considering a difficulty in students' learning and constructing a lesson plan in which a group of teachers work collaboratively by focusing particularly on students' learning, which is followed by one of the group members' conducting the lesson plan in classroom and other group members' observation of their group mate to collect data about both students' learning and effectiveness of lesson plan. At last, the group comes together to reflect and discuss their observation regarding the research lesson (Stigler & Hiebert, 1999; Lewis, Perry, & Hurd, 2009; Murata et al., 2012). If it is considered necessary, the cycling process starts in the planning phase and continues with the other steps (Stigler & Hiebert, 1999). Lesson study occurs in the center of students' learning including their strategies, misconceptions, ideas and mathematical understanding (Stigler & Hiebert, 1999). Therefore, lesson study provides the needs mentioned above for development of prospective teachers' knowledge for teaching mathematics by constructing a learning community in the collaboration of prospective teachers and experts (e.g., researcher, teacher educators, mentor teachers) and a learning environment which includes research-based and reflective practices requiring critical thinking (Lewis, Perry & Murata, 2006; Murata et al., 2012; Gunnarsdóttir & Pálsdóttir, 2019). By this way, to provide an effective learning environment for prospective mathematics teachers, lesson study can be seen as an effective and a suitable method to develop prospective mathematics teachers' knowledge for teaching mathematics.

The literature on development of prospective teachers' knowledge through lesson study generally focuses on knowledge of students, knowledge of teaching, knowledge of curriculum, specialized content knowledge, and connection between knowledge in practice and knowledge in researching in mostly elementary school level (e.g., Lewis, Perry & Murata, 2006; Lewis, 2009; Murata et al., 2012; Leavy & Hourigan, 2018). So far, however, there has been little discussion about how lesson study contributes to prospective mathematics teachers' development of in-depth mathematical knowledge in relation with pedagogical content knowledge in the context of secondary school level.

Among the mathematical concepts in secondary school level, the concept of limit can be considered as one of the mathematical concepts to be studied in depth for prospective mathematics teachers' knowledge for teaching (Cornu, 1991). The concept of limit forms the basis of many mathematical concepts (Tall & Vinner, 1981; Monaghan, 1991; Cornu, 1991; Moru, 2009; Fernández-Plaza & Simpson, 2016), and prospective teachers' meaningful and deep learning of this concept is one of the important steps for them to be able to relate it to other mathematical concepts and to better teach one of the concepts that students have the most difficulty in and followed by overcoming those difficulties (Cory & Garofalo, 2011). However, as opposed to the number of studies focusing on examining subject matter knowledge and pedagogical content knowledge of mathematics teachers in different content areas of limit (e.g., Huillet, 2005; Kajander & Lovric, 2017; Montes, Carrillo, & Ribeiro, 2014; Montes & Carrillo, 2015), there is a limited number of studies which focus on the development of prospective teachers' knowledge for teaching the concept of limit (e.g., Kolar & Čadež, 2011; Cory & Grafola, 2011; Oktaviyanthi, Herman, & Dahlan, 2018). In particular, though the concept is one of the big ideas in teaching and learning mathematical concepts both in secondary school and transition from secondary school to tertiary level, little is known about prospective mathematics teachers' mathematical practices in constructing an instructional sequence, reflecting their knowledge in implementing and making instructional decisions through teaching. Furthermore, what is not yet clear is how well prospective mathematics teachers are ready to teach such a deep concept. This indicates a need to reveal and develop prospective mathematics teachers' knowledge for teaching the concept of limit in a broad sense.

In the light of all the facts mentioned above, to meet the needs for developing prospective mathematics teachers' knowledge for teaching the concept of limit, lesson study is regarded as an appropriate development model which provides a research-based collaborative learning environment.

1.1 Purpose of the Study and Research Questions

The purpose of this study is to understand the nature and development of their specialized knowledge for teaching the concept of limit in a broad sense through designed lesson study development model. In other words, this study aims to design a development model to improve prospective secondary mathematics teachers' specialized knowledge for teaching in all domains of knowledge in a broad sense in teaching the concept of limit. Based on the aim of the study, the research questions are determined as:

1. How do prospective secondary mathematics teachers develop their specialized knowledge in the concept of limit while planning and enacting the lesson plans during the lesson study?
2. How well can the critical elements of lesson study be regulated so that they become an integral part of a logical chain to improve prospective secondary mathematics teachers’ specialized knowledge in the concept of limit?

1.2 Significance of the Study

This study provides important opportunities to advance the understanding of prospective teacher education in different aspects. In the first aspect, comprehensive analysis of a prospective mathematics teachers’ knowledge showed how these sub-domains and the indicators of each of them were revealed in nurturing prospective teacher’s knowledge holistically. While the framework was used as an analytical framework, lesson study was also used as a methodological framework in designing teacher development experiment. This combination with in-depth analysis enlightened the use of lesson study in prospective teacher education in terms of which and how sub-domains develop in the phases of the lesson study.

In the literature on knowledge development through lesson study, most of the studies focus on knowledge related to students, teaching and content (e.g., Lewis, Perry, & Hurd, 2009; Dudley, 2013; Shuilleabhain, 2016; Leavy and Hourigan, 2016), and some studies examine the nature of mathematical and pedagogical content knowledge (e.g., Suh & Seshaiyer, 2015; Clivaz & Shuilleabhain, 2019). In terms of both focusing on the knowledge development of the prospective teacher and doing this at the secondary level, the study has the potential to be one of the leading studies to undertake an in-depth analysis of knowledge development through lesson study. Therefore, this combination can provide an additional perspective for teaching limit to be shared with teachers and teacher educators. In other words, it can make a major contribution to research and practice on prospective teacher education by demonstrating how the sub-domains of knowledge develop through phases of lesson study.

Another significant aspect is related to the theoretical part of the current study. The framework used in the current study is one of the emerging models in understanding the nature of mathematics teachers' knowledge. The conceptual framework of Mathematics teachers' Specialized Knowledge (MTSK) (Carrillo-Yañez et al., 2018) compose of six sub-domains around the belief towards mathematics and mathematics teaching. Since it is an emerging and open to be developed model, the combination of lesson study and the model may contribute to the model in some contexts to understand the nature of specialized knowledge. In addition, this new approach among the models for knowledge for teaching mathematics has emerged through working with mathematics teachers in Spain (and some other countries where Spanish is the first language). With the findings of the current study, besides verifying this model in a different culture and context (Turkish context in the current study), working with prospective teachers can offer a different approach to the model. Furthermore, the literature related to mathematical and pedagogical content knowledge for teaching limit in Turkey have commonly used other models (e.g., mathematical knowledge for teaching, knowledge quartet) (Rowland, Huckstep, & Thwaites, 2005; Ball, Thames, & Phelps, 2006) to examine prospective mathematics teachers' knowledge. The model of MTSK can bring a new perspective for mathematics education researchers towards research on prospective mathematics teachers and mathematics teachers.

Furthermore, this study contributes to the growing area of research by exploring the critical aspects of lesson study and how they are used in designing the process. During the lesson study process, the lesson study group comes together in meetings to plan a lesson, conduct it and reflect ideas on it towards a lesson goal (Stigler &

Hiebert, 1999). The literature on lesson study indicates that knowledge development is naturally a result of lesson study in terms of its advances including sharing ideas, discussing on a subject and observing-reflection research lessons (e.g., Teyplo & Moss, 2011). While lesson study allows teachers to share and discuss different perspectives, pedagogies, and ideas to connect student thinking and mathematics content through multiple cycles (Lewis, Perry, & Hurd, 2009; Teyplo & Moss, 2011; Clivaz & Shuilleabhain, 2019), new advances in mathematics education necessitate thinking on a development model in more flexible ways rather than rote implementation (Fernandez, 2005; Teyplo & Moss, 2011; Murata et al., 2012). For the model to be employed in prospective teacher education, it is especially important to understand what critical aspects contribute to knowledge development and what types of knowledge develop at different stages. This study draws attention to this point with its findings for the second research question. The critical elements revealed in the study can shed light on the literature on research on lesson study and relevant teacher preparing processes.

What is more significant is that it provides a holistic picture of development of a prospective teacher's specialized knowledge for teaching mathematics and understanding mathematical and pedagogical practices behind this development process. The concept of limit has been studied in different age groups (from high school to senior students at university) and in different contexts for many years. In prospective teacher education, as opposed to the number of studies focusing on examining subject matter knowledge and pedagogical content knowledge of mathematics teachers in different content areas of limit including limits of functions, limits of sequences, the concepts of convergence and continuity, and the concept of infinity (e.g. Huillet, 2005; Kajander & Lovric, 2007; Montes, Carrillo, & Ribeiro, 2014; Montes & Carrillo, 2015), there is a limited number of studies which focus on prospective mathematics teachers' knowledge for teaching of the concept of limit (e.g. Kolar & Čadež, 2011; Cory & Grafola, 2011; Oktaviyanthi, Herman & Dahlan, 2018). Furthermore, the studies on knowledge development through lesson study have commonly focused on the concepts included in middle school level. From this point, examining knowledge for teaching such an important concept for all school levels, in particular at secondary school level, extends the common understanding on the knowledge of the limit concept and its use in the dimensions.

1.3 Definition of Terms

Prospective mathematics teacher is a senior student in the four-year program of Mathematics Education which awards the students a qualification to teach in high-school students (grades 9-12).

Mathematics Teachers' Specialized Knowledge is an analytical and methodological model which specifies mathematics teachers' knowledge to conduct their profession in not only teaching in the classroom but also in lesson planning or communicating with colleagues (Carrillo-Yañez et al., 2018). MTSK includes two sub-domains: Mathematical Knowledge including mathematics content itself (Knowledge of Topics); the interlinking systems which bind the subject (Knowledge of the Structure of Mathematics); and how one proceeds in mathematics (Knowledge of Practices in Mathematics); in addition, Pedagogical Content Knowledge including how mathematical content is taught in a powerful way (Knowledge of Mathematics Teaching); how students learn mathematical content (Knowledge of Features of Learning Mathematics); and being aware of the curriculum specifications (Knowledge of Mathematics Learning Standards).

Lesson study is a teacher development model in which teachers work collaboratively to improve their teaching based on students' learning (Lewis, 2002). It can be described as a cycling process including four successive and repeating stages which can occur over a number of weeks. These stages are (1) determining a learning goal for a concept which students face difficulties in learning, (2) building a lesson plan within the collaboration of teachers in order to create a learning path by taking into account the challenges students experience to learn this concept efficiently, (3) performing the research lesson in an actual class in which one of the group members teaches the concept and the others observe the students and take notes of their learning and thinking processes, and (4) reflecting and discussing on the research lesson in terms of the lesson's and students' learning efficiency. This stage might be followed by discussions on how to improve the lesson plan, and the cycle might be applied again to revise on missing points (Hart, Alston, & Murata, 2011).

1.4 The Structure of Dissertation

This dissertation unfolds in five chapters. The explanation given below is a summary of each chapter.

In Chapter 1, I describe the starting point of the aim of dissertation in my research journey, indicate the research problem based on the literature, reveal the purpose of the research and the research questions, state the significance of the research both for the readers and the future of the relevant literature, and define the important terms used in the research.

Chapter 2 describes the conceptual framework which provides a detailed account of the use of MTSK as an analytical framework in examining the development of specialized knowledge for teaching the concept of limit. This chapter also reveals the literature analysis on teaching, learning and knowledge of the concept of limit to elaborate upon knowledge needed to teach the concept. Lastly, the literature on lesson study and knowledge development with lesson study is presented.

Chapter 3 presents the design of the study. It gives a brief account of participants of the study, and describes the researcher's position in understanding the participant's contribution to the phases of lesson study. Then, it gives a detailed description about how the lesson study process was designed as a teacher development experiment and the data analysis by using the analytical framework (Mathematical Teachers' Specialized Knowledge-MTSK) towards how lesson study promotes the prospective teacher's development of her specialized knowledge for teaching the concept of limit. In addition, the data analysis towards which critical elements of lesson study nurture the prospective teacher's development of her specialized knowledge is presented. Towards the end, the limitations and delimitations are mentioned.

Chapter 4 is constructed based on the research questions; such as introduction, body and conclusion. First, the existing knowledge of the prospective mathematics teacher is presented to understand her development journey explicitly. Second, the development of her specialized knowledge is revealed within six sub-domains and their indicators. To present the data in-detail, each indicator in each sub-domain is exemplified with excerpts which include contributions of the prospective mathematics teacher to the lesson study process. Though the lesson study process presents the development of almost all indicators with data-based evidence, the analysis of the data gathered from the post-interview is presented to reveal the indicators of knowledge whose development has not been observed sufficiently during the lesson study process. At the last step, the analysis of data gathered to answer the second research question is presented by means of conjectures.

Chapter 5 summarizes the findings from the previous chapter, which can be regarded as a response to the research questions. The chapter also discusses what the study concludes and what the literature has to say related to the findings. The dissertation ends with suggestions and implications of the study for prospective mathematics teacher education research and the models for mathematics teacher knowledge.

CHAPTER 2

LITERATURE REVIEW

This study aims to understand the nature and development of secondary school prospective mathematics teachers' knowledge about the concept of limit through a teaching experiment designed within a lesson study development model by planning and enacting lesson plans. Bearing the purpose of the study in mind, the relevant literature is presented under three parts. The first part gives information about types and components of teacher knowledge in the light of different frameworks. In addition, the model of Mathematics Teachers Specialized Knowledge (MTSK) as the scope of the current study is explained in detail. In the second part, mathematical knowledge about the concept of limit and international and local studies investigating preservice and in-service mathematics teachers' knowledge about the concept of limit are presented and discussed in terms of their differing methodological approaches, findings, and theoretical and practical implications as well as the suggestions for researchers and teacher educators. In the last part, the lesson study is introduced as a developmental tool for improving the prospective secondary mathematics teachers' knowledge about the concept of limit.

2.1 Mathematics Teachers' Knowledge

It is a crystal-clear fact that teachers' knowledge has an important factor on the quality of mathematics teaching and students' achievement (Tchoshanov, 2011). Knowledge for teaching any content cannot be taught without the symbiotic relationship between content knowledge and its pedagogy (Goos, 2013). To understand the nature of teaching and learning process, there have been different attempts to describe knowledge for teaching in the literature. Before specializing this knowledge for the content, the seminal work of Shulman which describes knowledge for teaching with its various components is examined below.

The first seminal attempt was Shulman's (1986) framework based on Subject Matter Knowledge (SMK), Pedagogical Content Knowledge (PCK) and Curricular Knowledge (CK) to answer one of the most widely sought questions for years which were; what teachers know and don't know, what teachers should know, and how teachers can use this knowledge. According to Shulman (1986), only naming the content knowledge was not enough for a qualified teacher. He elaborated pedagogical content knowledge as follows:

> (…) The particular form of content knowledge that embodies the aspects of content most germane to its teachability (Shulman, 1987, p. 9) [...] The most powerful analogies, illustrations, examples, explanations, and demonstrations— [...] The most useful ways of representing and formulating the subject that make it comprehensible to others. [...] Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them.(…) (Shulman, 1987, p.7)

In his widely cited article, he claimed major categories of teacher knowledge as: Content knowledge, general pedagogical knowledge, curriculum knowledge, pedagogical content knowledge, knowledge of learners and their characteristics, knowledge of educational contexts and knowledge of educational ends, purposes, values and their philosophical and historical grounds (Shulman, 1987, p.8). In this context, content knowledge represents how much knowledge a teacher has and in which way he/she organizes them (Shulman, 1986). The other forms of knowledge for teaching arise from content knowledge. If we think of content knowledge in terms of teaching and teachability, it refers to pedagogical content knowledge. Otherwise, teaching materials, programs, and teaching a topic at a specific time are included in curricular knowledge (Shulman, 1986). Shulman's work paved the way for content-specific models for knowledge for teaching. Since Shulman defined knowledge for teaching in a general context, there is a need for "theoretical development of analytic clarification and empirical testing to investigate the nature of professionally oriented subject matter knowledge in mathematics" (Ball, Thames, & Phelps, 2008, p. 389). In this way, with respect to Shulman's (1986) seminal work on knowledge for teaching, the mathematics education literature proposed several models to describe knowledge that a mathematics teacher should have for an effective mathematics education (Ball, Thames, & Phelps, 2008; Rowland, Huckstep, & Thwaites, 2005; Carrillo-Yañez et al., 2018).

Referring to Shulman's (1986) seminal work on knowledge for teaching, Ball, Thames and Phelps (2008) proposed "Mathematical Knowledge for Teaching (MKT)" providing a sense about the answers of what mathematics teachers need to know and how effectively they carry out the work of teaching mathematics. The model of Mathematical Knowledge for Teaching can be regarded as the first attempt to analyze and describe the teaching knowledge for mathematics teachers. The model aimed to answer the questions of "what do teachers do in teaching mathematics, and how does what they do demand mathematical reasoning, insight, understanding, and skill?" (Ball, Thames, Bass, Sleep, Lewis, & Phelp, 2009, p. 95) by focusing on the tasks, which showed that the model of MKT is a practice-based theory in mathematics teacher knowledge. MKT has a multidimensional form and takes Shulman's SMK and PCK as a basis. Under the SMK, there are three domains of mathematical knowledge: Common content knowledge, specialized content knowledge and horizon content knowledge. Common content knowledge is the fundamental sub-domain of subject matter knowledge that is common for all persons who know mathematics (specifically related topic). There is no need to be a teacher to have this knowledge. Specialized content knowledge is unique for teaching and includes a kind of mathematical work which is not needed for other persons. It is also called "the teaching presentation of mathematics" (Ball, Thames, & Phelps, 2008). Horizon content knowledge is described as "an orientation to and familiarity with the discipline (or disciplines) that contribute to the teaching of the school subject at hand, providing teachers with a sense for how the content being taught is situated in and connected to the broader disciplinary territory" (Jakobsen, Thames &

Ribeiro, 2013, p. 3128). Another three domains are included in the PCK: Knowledge of content and students, knowledge of content and teaching and knowledge of content and curriculum. These three domains are related to determinants of teaching in the classroom such as students, curriculum and content. Knowledge of content and students is a kind of PCK that requires knowledge of both content and students to understand students' readiness (conceptions and misconceptions) for the concept. Knowledge of content and teaching is another knowledge that provides the teacher a way to design her/his instruction to ensure his/her students' understanding of the concept. Lastly, knowledge of content and curriculum is knowledge that requires knowing the curriculum designed for teaching of a particular concept. Knowing the curriculum means approaches and characteristics of the curriculum regarding that particular concept, knowing how to adapt it in teaching and instructional and practical materials (Ball, Thames, & Phelps, 2008).

Another important step, which was done almost at the same time in a different country, to understand the nature of knowledge for teaching mathematics was Knowledge Quartet in which the dimensions are framed considering events in mathematics lessons in relation with mathematical knowledge taught and knowledge related to mathematics that teachers refer to at that time (Rowland, Huckstep, & Thwaites, 2005). Knowledge Quartet has four dimensions including foundation, transformation, connection and contingency (Rowland, 2014). The first part of the quartet-Foundation covers teachers' "the foundation of the trainees' theoretical background and beliefs" (Rowland, Huckstep, & Thawaites, 2005, p. 260) which can be considered as teachers' knowledge, understanding and ready which are helpful for their learning in the teacher education program and in their training (intentionally or otherwise) for their role as a teacher. The second part of the quartet - Transformation- covers behaviors which are related to students and answers directed to students' activities (Rowland, Huckstep, & Thawaites, 2005). It includes choice and use of examples, demonstration of procedures, and different kinds of guidance (Rowland, 2014). The third part of the quartet-connection is related to coherence between the mathematical content thought during a teaching episode. It also covers the coherence between mathematical tasks in lesson plan and awareness of the relative cognitive demands of different topics (Rowland, Huckstep, & Thawaites, 2005). The last part of the quartet is contingency which is related to unplanned situations during teaching. It comprises two important aspects including readiness to respond to students' ideas during teaching and preparedness for such situations in planning (Rowland, Huckstep, & Thawaites, 2005).

These pioneer models shed light on the literature to conceptualize mathematics teachers' and prospective teachers' knowledge for teaching mathematics. However, the literature presents some limitations about these models. For instance, the model of MKT aimed to answer the questions of what teachers do while teaching mathematics and how it requires mathematical reasoning, insights, understanding, and skills (Ball, Thames, Bass, Sleep, Lewis, & Phelp, 2009) by focusing on the tasks, which showed that the model of MKT is a practice-based theory in mathematics teacher knowledge. Similarly, the Knowledge Quartet revealed knowledge in action within situations in practice (Rowland, Turner, Thwaites, & Huckstep, 2009). However, the mathematical knowledge specific for teachers require both having the mathematical and pedagogical content knowledge (Shulman, 1986) as well as reflecting this knowledge on practice which includes not only teaching the concepts in the classroom but also planning lessons, understanding the nature of learners, learning and teaching pathways, criticizing teaching processes and revising the defective points in the teaching (Leavy & Hourigan, 2016). Considering these two pioneer models whose basis is on practice in class and the focus their attention on practice as carried out in class, it can be said that the knowledge that teachers bring to the classroom and to the classroom while performing teaching and other activities is limited in these models (Carrillo-Yañez et al., 2018). Furthermore, there is not a clear bound between the sub-domains in the model of MKT (Montes et al., 2013). In other words, there is a difficulty in examining where the common content knowledge ends and specialized content knowledge begins in examining teachers' practices in classroom (Carrillo et al., 2013). For instance, Ball, Thames and Phelps (2008) gave an example related to students' answer for the subtraction of

“307-168” with the “borrowing” method (p. 396). In analyzing students’ wrong answers, Ball and her colleagues (2008) mentioned that anyone who knows the subtraction can see the incorrect result, which is called common content knowledge. The researchers of MKT indicated that in analyzing students’ error teachers have different qualities including identifying the sources of students’ error, which is called specialized content knowledge. However, at this point, there are two distinctions that Ball and her colleagues asserted to describe the specialized content knowledge. While the teacher's asking himself/herself what is happening here mathematically includes mathematical knowledge, on the other hand, how the student thinks while making this error includes knowledge that includes the cognitive processes of the student which is beyond mathematical knowledge (Montes et al., 2013); in particular which is closely related to knowledge of content and students. In addition, there is disagreement on the definition and the location of horizon content knowledge in the model, which some researchers asserted that it can be located as an umbrella above the other sub-domains (Montes et al., 2013; Fernánez & Figueiras, 2014; Zhang, Zhang, & Wang, 2017). Therefore, while the models shed light on the way to examine and develop knowledge for teaching and learning mathematics, the models’ limitations led to the development of other mathematical frameworks (Carrillo et al., 2012).

Considering the importance of knowledge of mathematics teachers related to their profession in not only teaching in the classroom but also in lesson planning or communicating with colleagues, the model of Mathematics Teachers’ Specialized Knowledge (MTSK) has been proposed in recent years (Carrillo-Yañez et al., 2018). In the latter section, the model of Mathematics Teachers’ Specialized Knowledge and its subdomains are presented in-detail.

2.1.1 The Model of Mathematics Teachers' Specialized Knowledge

The main aim of the Mathematics Teachers' Specialized Knowledge (MTSK) team is to create a model by looking at the specific nature of mathematics teachers' knowledge from a holistic perspective. The group of researchers consider not only knowledge of mathematics teachers in action but also their knowledge that they bring to class and communicate with their colleagues including (Carrillo-Yañez et al., 2018). In this way, the model of MTSK has been revealed, which provides a methodological and analytical tool feature for examining the knowledge put into practice by the teacher, giving a deep perspective on this special knowledge of the mathematics teacher, the factors that make up this knowledge, and the interactions between them (Montes et al., 2013; Carreño et al., 2013; Carrillo et al., 2013).

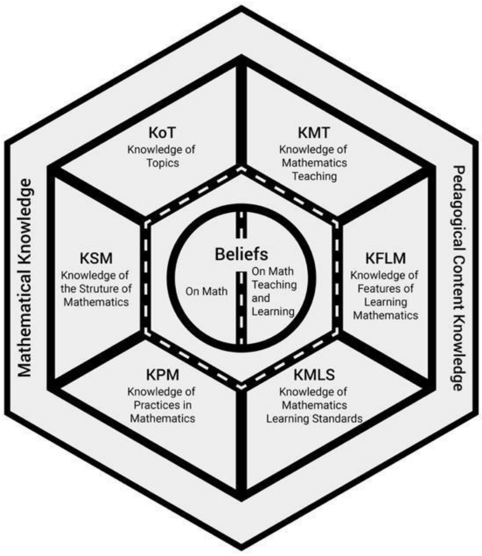

Figure 2.1. The Model of MTSK (Adapted from "The mathematics teachers' specialized knowledge (MTSK) model" by J. Carrillo-Yañez et al., 2018, Research in Mathematics Education, 20(3), p. 241. Copyright © 2018 by Routledge Group: Taylor & Francis

As can be seen in Figure 2.1, the model includes two main dimensions including "Mathematical Knowledge (MK)" and "Pedagogical Content Knowledge (PCK)" in addition to the belief dimension. MK provides teachers an associated teaching across concepts covering all the other information that teachers should have including knowledge of concepts, procedures, historical development of concepts, mathematical language, ways of proceeding and reasoning and noticing of importance of mathematical concepts (Montes et al., 2013; Carrillo-Yañez et.al. 2018). In MK, there are three sub-domains related to content knowledge including Knowledge of Topics (KoT), Knowledge of Structures of Mathematics (KSM) and Knowledge of Practices in Mathematics (KPM). On the other hand, PCK covers the broad range of knowledge specific to teaching mathematics considering the factors of students, learning standards, features of mathematics learning (Carrillo-Yañez et al. 2018). In PCK, there are three sub-domains related to pedagogical content knowledge including knowledge of features of learning (KFLM), knowledge of mathematics teaching (KMT), and knowledge of mathematics learning standards (KFLM). Each sub-domain of both MK and PCK has indicators that mathematics teachers have related to knowledge. Besides, the dimension of belief is grounded in the center of the model such as the belief towards mathematics and the belief towards mathematics teaching and learning (Carrillo-Yañez et al. 2018). "Beliefs on mathematics" is shown at the left side of the figure which is close to MK and "beliefs on mathematics teaching and learning" is shown at the right side of the figure which is close to PCK, as can be seen in Figure 2.1.

As mentioned before, the model of MTSK different from other fundamental knowledge models since it gives importance not only knowledge in class but also knowledge brought to class (Carrillo-Yañez et al., 2018). On the other hand, the model is similar with MKT has six sub-domains of knowledge, which is different from knowledge quartet. Therefore, in the latter sections, the sub-domains are presented in comparison with MKT generally, after the sub-domains are revealed in-detail.

Knowledge of Topics

Knowledge of topics (KoT) can be described as comprehensive understanding towards the fundamental mathematics (Ma, 1999); providing the teacher to specify the different dimensions of a topic such as properties, foundations, definitions, operations, representations and meanings of mathematical concepts; therefore, it covers the complicated nature of mathematical teaching process characterizing aspects of the topic, as well as to understand the disciplinary content presented in texts related to mathematics (Montes et al., 2013; Carillo et al., 2018; Zakaryan & Ribeiro, 2019). In this perspective, this sub-domain comprises everything that forms the basis of the concept, the rules, operations and their execution procedures, as well as the different meanings of the concept within itself (Carreño et al., 2013).

Table 2.1 The sub-domain of KoT and its indicators (Carrillo-Yañez et al., 2018, p. 243)

The Sub-domain	Indicators of the sub-domain
KoT	Procedures (How to do something? When to do something? Why something is done this way? and Characteristics of the result) Definitions, properties and foundations Registers of representation Phenomenology and applications

As can be seen in Table 2.1, the indicators of KoT include the knowledge that the students are expected to learn with a deeper, and maybe more formal and rigorous understanding, including the type of problems the content can be applied to, with their associated contexts and meanings, properties and their underlying principles, definitions and procedures, including connections to items within the same topic, and ways of representing the contents (Carrillo-Yañez et al., 2018, p. 242). For instance, the indicator of knowledge of definition can be exemplified as "in the set Q, with the operations of addition and multiplication, we have a field structure; equations of the type $ax = b$, $a \in Q$ and $b \in Q$ can be solved and a total order can be defined" (Zakaryan & Ribeiro, 2019, p. 29).

Knowledge of Structure of Mathematics

KSM includes mathematical knowledge that reveals how concepts are related to each other and to a mathematical structure (Montes et al., 2013). In general, this sub-domain is related to mathematics teacher knowledge which includes connections between mathematical items. Based on these explanations, KSM seems similar to horizon content knowledge in which the connections with content from other disciplines are included in Ball's model. Both to understand the sub-domain clearly and to reveal the difference from horizon content knowledge, first, the types of connections are presented in this section.

KSM is defined on the connections which consider mathematical connections as an element of mathematical context (Martínez et al., 2011). The connections are: (i) intraconceptual connections (being in the place in the proximity of a concept), (ii) interconceptual connections (mathematical ideas that allow linking different representations of the same concept or different concepts that students face at the same time) and (iii) temporal connections (an upper or lower concept to which a concept is related in different levels of the curriculum) (Martínez et al.,2011). Carrillo-Yañez and his colleagues (2018) dealt with these connections in a way that intraconceptual connections are defined as KoT and the last two connections are included in KSM. Therefore, the indicators of KSM were defined according to these connections which were described in detail below (see Table 2.2).

Table 2.2 The sub-domain of KSM and its indicators (Carrillo-Yañez et al., 2018, p. 244)

The Sub-domain	Indicators of the sub-domain
KSM	Connections based on simplification Connections based on increased complexity Auxiliary connections Transverse connections

Temporal connections are related to as an upper or lower concept to which a concept is related in different levels of the curriculum (Martínez et al., 2011; Montes et al., 2013; Carrillo-Yañez et al., 2018). Therefore, temporal connections are considered in two ways as the indicators of KSM: connections based on simplification (retrospective contextualization) and connections based on complexity (aid to future uptake) (Carrillo-Yañez et al., 2018). For instance, when comparing objects in terms of size, the teacher connects with the idea of scale, initiates the logical classification process and emphasizes that this is a necessary condition for large-small characterization (Carrillo-Yañez et al., 2018). This can be considered as an example of connections which aid to future uptake (complexity).

On the other hand, in inter conceptual connections, connectors are mathematical ideas that allow linking different representations of the same concept or different concepts that students face at the same time (Martínez et al., 2011; Montes et al., 2016). Similar to temporal connections, inter conceptual connections are divided into two ways as other indicators of KSM: Auxiliary connections and transverse connections (Carrillo-Yañez et al., 2018). Auxiliary connections are the connections that are not directly related to the concept while the teacher is teaching the mathematical concept, but enables students to learn this concept in the same context as an auxiliary tool (Montes et al., 2016; Carrillo-Yañez et al., 2018). On the other hand, transverse connection is the connection of different concepts with a common basis, which is related to the nature of the taught concept and confronts the student and the teacher at different stages of education (e.g., infinity) (Montes & Carrillo, 2015) (Montes et al., 2016; Carrillo-Yañez et al., 2018). For instance, in teaching the roots of a function, using equations as an auxiliary element can be an example for auxiliary connections (Carrillo-Yañez et al., 2018).

Knowledge of Practices in Mathematics

KPM is based on the idea of “syntactic knowledge” (Schwab, 1978 as cited in Carrillo-Yañez et al., 2018) and “knowledge about mathematics” (Ball, 1990; Ball & Bass, 2009). In general, we can say that the practice of the systematic operation of mathematics and the knowledge that reveals its underlying logic (Carrillo-Yañez et al., 2018), in other words, KPM includes knowledge of the ways of knowing and creating or producing in mathematics (syntactic knowledge) (Schwab, 1978 as cited in Delgado-Rebolledo & Zakaryan, 2020). The knowledge of the mathematics teacher in this system of practice and operation includes showing, justifying, defining, deduction and induction, and giving direct and counterexamples (Carrillo-Yañez et al., 2018). KPM's perspective is that the teacher knows how to reason mathematically and which way is better than the others, based on the knowledge about mathematics (Ball & McDiarmird, 1990).

While Carrillo-Yañez and his colleagues (2018) mentioned that KPM is an under-developed sub-knowledge of this model and is not divided; yet, Delgado-Rebolledo and Zakaryan (2020) addressed indicators of KPM as "knowledge of ways of proceeding, validating, exploring, and generating knowledge in mathematics, such as knowledge of ways to communicate mathematics" (p. 546). For instance, "the teacher must possess adequate knowledge of the syntactic and semantic meaning of formal symbolisms and mathematical expressions and the role of symbols in different contexts" represents an example for knowledge of ways of communicating (Delgado-Rebolledo & Zakaryan, 2020, p. 572). Likewise, knowledge of ways of proceeding covers selecting elements towards sufficient conditions to define something in mathematics (Delgado-Rebolledo & Zakaryan, 2020). Considering these indicators, García, and her colleagues (2021) detailed the indicators of the sub-domain in six indicators as "(1) knowledge of processes associated with problem-solving as a means of producing mathematics, (2) knowledge of ways of validating and demonstrating, (3) role of symbols and use of formal language, (4) hierarchy and planning as a way of proceeding with the resolution of mathematical problems, (5) particular procedures for mathematical work and (6) necessary and sufficient conditions for generating definitions" (p. 5). (see Table 2.3). As an example, for the detailed indicators, teachers' use of modeling belongs to the indicator of particular procedures for mathematical work (García et al., 2021).

Table 2.3 The sub-domain of KPM and its indicators (Carrillo-Yañez et al., 2018, p. 244)

The Sub-domain	Indicators of the sub-domain
KPM	Knowledge of processes associated with problem-solving as a means of producing mathematics Knowledge of ways of validating and demonstrating Role of symbols and use of formal language Hierarchy and planning as a way of proceeding with the resolution of mathematical problems Particular procedures for mathematical work Necessary and sufficient conditions for generating definitions

Thus far, we have investigated the sub-domains of mathematical knowledge which are placed in the right side of the figure of the model. On the left side of the model, the other fundamental knowledge domain- pedagogical content knowledge (PCK) is placed. Similar to the sub-domains of MK, the three sub-domains included in PCK are presented in order with knowledge of features of learning mathematics (KFLM), knowledge of mathematics teaching (KMT) and knowledge of standards of learning mathematics (KMLS) with their indicators and examples below.

Knowledge of Features of Learning Mathematics

The first sub-domain of PCK is knowledge of features of learning mathematics (KFLM). KFLM relies on the mathematics teacher's knowledge of how students learn mathematics (Montes et al., 2015). This knowledge includes how the student thinks while dealing with a mathematical activity, the difficulties a student encounters, theories about students' cognitive development and models of learning mathematics (e.g., van Hiele geometric thinking levels, SOLO taxonomy) (Carrillo-Yañez et al., 2018). This sub-domain can be matched with knowledge of content and students in Ball's model.

To be more particular, the sub-domain can be divided into four indicators. KFLM includes mathematics learning theories that describe the cognitive development of students towards a specific mathematical content or mathematics. Second, it covers the mathematics teacher's awareness of students' strengths and weaknesses while learning mathematical content (see Table 2.4). In this context, being aware of misconceptions, conceptual difficulties, and training of students for wrong sampling fall into this field of knowledge. It also includes knowledge of the ways in which the learner will interact with a mathematical content when faced with it. Lastly, it comprises mathematics teachers' awareness related to students' feelings towards mathematical content, for instance, mathematics anxiety in learning (Carrillo-Yañez et al., 2018; Delgado-Rebolledo & Zakaryan, 2020; Montes et al., 2015). For instance, a mathematics teacher's awareness about the fact that students make examples to prove an argument and her/his consideration of this awareness in her/his teaching is classified as an indicator of strengths and weaknesses in learning mathematics (Carrillo-Yañez et al., 2018).

Table 2.4 The sub-domain of KFLM and its indicators (Carrillo-Yañez et al., 2018, p. 247)

The Sub-domain	Indicators of the sub-domain
KFLM	Theories of mathematical learning Strengths and weaknesses in learning mathematics Ways pupils interact with mathematical content Emotional aspects of learning mathematics

Knowledge of Mathematics Teaching

Knowledge of mathematics teaching (KMT) is another sub-domain of PCK that includes theories for mathematics education, teachers' personal experiences and practices. In the most general sense, it is a type of footer that allows the mathematics teacher to select the materials and examples that he uses from the textbook to a certain representation for his students to learn a mathematical concept, and to decide which teaching strategies to use (Montes et al., 2015). This knowledge requires awareness of the materials, strategies and techniques required to teach mathematical content for the concept, as well as the difficulties and limitations they bring (Carrillo-Yañez et al., 2018). It also includes having the necessary knowledge to be able to help students grasp the meaning of mathematical items through structured series of examples and knowledge of resources designed in accordance with the mathematical content (Carrillo et al., 2013; Carreño et al., 2013; Delgado-Rebolledo & Zakaryan, 2020). Thus, there are four indicators for KMT including knowledge of theories of mathematics teaching, knowledge of teaching resources (physical and digital) and knowledge of strategies, techniques, tasks and examples (Carrillo-Yañez et al., 2018). First, knowledge of theories of mathematics teaching covers the specific mathematics teaching theories. Second, knowledge of teaching resources requires awareness of the materials, strategies and techniques required to teach mathematical content for the concept, as well as the difficulties and limitations they bring. Third, knowledge of strategies, techniques, tasks and examples can be described as mathematics teachers' awareness about content specific activities, techniques and strategies and their strengths and weaknesses for effective mathematics teaching (Carrillo et al., 2018; Delgado-Rebolledo & Zakaryan, 2020). For instance, teachers' awareness and using "borrowing" metaphor for teaching subtraction can be an example for KMT.

Table 2.5 The sub-domain of KMT and its indicators (Carrillo-Yañez et al., 2018, p. 247)

The Sub-domain	Indicators of the sub-domain
KMT	Theories of mathematics teaching Teaching resources (physical and digital) Strategies, techniques, tasks and examples

As the last indicator, knowledge of strategies can be dealt with as teaching strategies and assessment strategies. While the model has not proposed an additional indicator for knowledge of assessment and assessment strategies, it is included in the indicator of knowledge of strategies. Knowledge of assessment strategies can be contextualized as questions/problems used in lesson plans and questioning strategies during teaching. The questioning strategies are divided into three parts including probing questions, guiding questions and factual questions (Şahin & Kulm, 2008 as cited in Yılmaz, 2019). Probing questions are used to explain/elaborate thinking on prior knowledge to justify/prove an idea. Similarly, guiding questions include specific answers and/or next step of solution, thinking about or recalling heuristics strategies. Lastly, factual questions are used for a specific fact, for an answer to an exercise and to provide the next step in a procedure (Şahin & Kulm, 2008 as cited in Yılmaz, 2019).

Knowledge of Mathematics Learning Standards

The last sub-domain of PCK is knowledge of learning standards (KMLS) which is based on learning standards in mathematics (Carrillo-Yañez et al., 2018). Carrillo-Yañez and his colleagues (2018) mean by 'learning standard' that "any instrument designed to measure students' level of ability in understanding, constructing and using mathematics, and which can be applied at any specific stage of schooling" (p. 248). It can be based on the standards according to curriculum features, standards set by informal but educational institutions and standards emerging from researchers' research (Montes et al., 2015; Carrillo-Yañez et al., 2018). This sub-domain refers to mathematics teachers' awareness of these standards (Delgado-Rebolledo & Zakaryan, 2020).

Table 2.6 The sub-domain of KMLS and its indicators (Carrillo-Yañez et al., 2018, p. 248)

The Sub-domain	Indicators of the sub-domain
KMLS	Expected learning outcomes Expected level of conceptual or procedural development Sequencing of topics

As can be seen in Table 2.6, KMLS includes three indicators as expected learning outcomes, expected level of conceptual or procedural development and sequencing of topics (Carrillo-Yañez et al., 2018). They can be described as the knowledge of the mathematical contents to be taught at any level, how these mathematical contents and topics are ordered according to the student's prior learning and post-learning, and the learning outcomes that the teacher expects from his/her teaching (Montes et al., 2015; Carrillo-Yañez et al., 2018; Delgado-Rebolledo & Zakaryan, 2020). For instance, “the teacher sequencing conceptual and procedural level of multiplication considering that multiplication is dealt with as number of times in grade 1 and 2, and abbreviated addition in grade 3 and 4” is an example for the indicator of sequencing topics (Carrillo-Yañez et al., 2018, p. 248).

The literature presented above shows the three sub-domains of PCK in addition to the sub-domains of MK. Besides the knowledge in those sub-domains, the model also includes “belief” in the center of the model. The belief dimension can be regarded as the conception of mathematics teachers towards mathematics and teaching-learning mathematics (Flores & Carrillo, 2014). The model points out that mathematics teachers’ set of consistent beliefs about mathematics and how mathematics is taught and learned profoundly affects teachers’ classroom practice (Carrillo-Yañez et al., 2018). Different from the sub-domains above, the researcher did not indicate any indicator to explore belief dimension (Carrillo-Yañez et al., 2018). Rather, together with belief dimension, the MTSK model presented a general and precise framework in the light of mathematics teachers' practical knowledge and background knowledge, and the factors affecting them (Carrillo-Yañez et al., 2018).

In consideration of revealing mathematics teachers’ knowledge for teaching mathematics broadly and clearly, there are similarities and differences with other models that shed light on the literature for the same purpose proposed before the model of MTSK. For instance, both the model of MTSK and Ball’s model consist of six sub-domains, grouped in three, under two main areas, which are mathematical knowledge and pedagogical content knowledge (Ball, Thames, & Phelps, 2008; Carrillo-Yañez et al., 2018). However, each sub-domain has some differences between them. For instance, KoT and common content knowledge (in Ball’s model) are dealt with as fundamental knowledge in both models. However, while common content knowledge has been defined as a mathematical knowledge that anyone can have (Ball, Thames, & Phelps, 2008), MTSK explains its sub-domains in addition to KoT without giving references to any other professions (Carrillo-Yañez et al., 2018). For this reason, KoT is different from common content knowledge as its feature is specific for mathematics teachers; it means that any other profession or anyone can have this knowledge (Carrillo-Yañez et al., 2018). Similarly, KSM is closely related to horizon content knowledge in Ball's model, which has mathematical connections in its descriptions. When the mathematical connections are dealt with three titles (intraconceptual, interconceptual and temporal connections), horizon content knowledge covers all these connections (Martínez et al., 2011). However, KSM includes two of these connections including interconceptual and temporal connections (Carrillo-Yañez et al., 2018) and the intraconceptual connections which can be described as being in the place in the proximity of a concept are included in KoT. Since horizon content knowledge includes all connections close or far from a concept, the difference between KSM and horizon content knowledge emerges in this point (Montes et al., 2013; Montes et al., 2016; Carrillo-Yañez et al., 2018). The similarities between the sub-domains are also observed in the sub-domains of PCK. For instance, in MKT, "recognizing which decimals would cause students the most difficulty" (Ball, Thames, & Phelps, 2008, p. 404) is included in knowledge of content and students. This example appears in the examples of "awareness about students' tendency to mistake between prove and exemplify" as the indicator of knowledge of students' strengths and weaknesses in corresponding sub-domain-KFLM in MTSK (Carrillo-Yañez et al., 2018, p. 246). While there are some differences in their detailed descriptions of the sub-domains of both models, in general, the sub-domains can be matched with each other.

While the model of MTSK is adopted in the current study, it is important for teachers to master these areas of knowledge regardless of the models for knowledge when they start their profession, both for their future professional development and for the culture to be created in order to increase the mathematical success of students. However, when the teacher preparation programs are considered in terms of knowledge for teaching, reflecting skills and insufficient teaching practice of prospective teachers, it is hard to construct such an environment that promotes sharing knowledge of teaching and learning, experience, and different viewpoints collaboratively (Murata et al., 2012). Among different models used for prospective mathematics teacher education, lesson study is one of the models that has gained much popularity in teacher education programs worldwide (Huang & Shimizu, 2016) to develop not only knowledge but also crucial elements of teaching including creativity, critical thinking, noticing, feelings, and beliefs towards teaching that facilitate the use of knowledge (Lewis, 2009).

2.2 Mathematics Teachers' Knowledge and Lesson Study

The first attempt of lesson study was a continuous process of school-based professional development of teachers in Japan, which was named as "*Konaikenshu*". *Konaikenshu* included a set of activities, which aimed to improve teachers' professional development and students' success in mathematics. One of its components is "*jugyou kenkyuu*", which means lesson study in English (Yoshida, 1999). After observing the effects of these activities on teacher training and student outcomes (e.g., success in TIMMS and PISA) in Japan, this development model study began to be implemented in US classrooms and then it spread over the world (Stigler & Hiebert, 1999; Yoshida & Jackson, 2011). Nowadays, lesson study is considered as one of the effective development models according to some educational communities including the National Staff Development Council and National Council of Teachers of Mathematics in terms of providing improvement in students' learning and teacher education (Yoshida & Jackson, 2011). In this section, lesson study and the literature on mathematics teachers' knowledge in lesson study are described in detail.

In lesson study (*jugyou kenkyuu*), more than two teachers (optimum 3-6 teachers) meet regularly during the semester or a long time (over weeks) and work together in investigating all aspects of a content, designing, conducting, revising and improving lesson plans (Lewis, 2002; Takahashi, 2005). In other words, lesson study is a cycling process that teachers work collaboratively and includes several steps which can occur over a number of weeks for instructional development based on students' learning (Lewis, 2002). In lesson study, a group of teachers engage in four phases of studying; (1) setting a learning objective for a topic in the curriculum that students have difficulty in understanding, (2) building a research lesson plan in collaboration that envisages how students would react to the concept by paying attention to elements of their gaining understanding such as the materials, content, trajectories and textbooks on that concept, (3) implementing the research lesson in a real classroom where one of the group members teach the concept while others record students' reactions and take notes of their thinking processes, and (4) reflecting and discussing on how effective the lesson was in facilitating acquisition. This phase might be followed by discussions on how to improve the lesson plan, and the cycle might be applied again to revise missing points. (Lewis, Perry, & Hurd, 2009; Murata et al., 2012). The phases can be collapsed as investigation, planning, research lesson and observation, and reflection (Lewis, 2002).

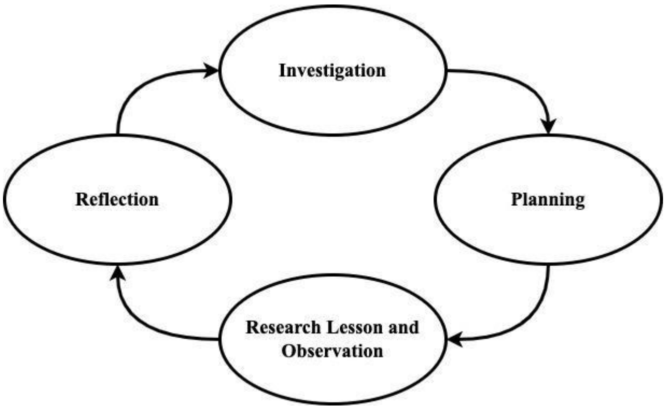

Figure 2.2. The demonstration of cycling process of lesson study (Lewis, Perry, & Murata, 2002)

Lesson study has important factors about not only effective mathematics teaching but also effective professional development. First, lesson study meets the general agreement about effective professional development (Garet et al., 2001) which is a long-term professional development that provides a way for teachers to have rich discussions and development of new ideas (Akiba et al., 2018). The rich discussions during the lesson study focus on students' learning. Accordingly, lesson study may also include raising relevant questions that can present the ways students think and offer new solutions for student learning (Stigler & Hiebert, 1999), which can be described as another important factor for professional development (Borko & Putnam, 1997). These meaningful questions occur through the essential feature of lesson study that all members of the group observe all lessons and students' learning and thinking process (Lewis, 2002). By this way, it enables the teacher to rethink their instruction since planning, teaching and observation of students' learning and thinking are interrelated with each other (Hughes, 2006). In addition, the collaboration in all aspects of the lesson study not only supports teachers' effective instruction but also creates learning opportunities for students by means of their exchange of ideas and experiences (Murata et al., 2016; Akiba et al., 2018).

Considering these benefits of lesson study, the nature of the study and development of the knowledge of mathematics teachers and prospective mathematics teachers through lesson study have existed since the emergence of lesson study. Accordingly, today, it has become one of the key subject areas of researchers as reforms continue to occur in the field of education. In this section, the studies in this research area are reviewed in three sub-sections including the studies conducted with mathematics teachers, prospective mathematics teachers and the studies in Turkish context.

2.2.1 Lesson Study and Knowledge of (In-service) Mathematics Teachers

As mentioned above, lesson study requires 3-6 teachers working collaboratively in four respective steps including identifying instructional goals, investigation on the goal, conducting the research lesson and critical thinking on the research lesson by means of reflection to make the lesson plan better for students' learning (Fernandez, 2002). The studies conducted with mathematics teachers in the literature have usually focused on improving mathematics teachers' instruction and students' thinking and understanding, development of different level of mathematics teachers' knowledge in different sub-domains, observable features of lesson study and constructing theoretical model through lesson study (Yoshida, 1999; Lewis, Perry, & Hurd, 2009; Teyplo & Moss, 2011; Murata et al., 2012; Dudley, 2013; Suh and Seshaiyer, 2015; Widjaja et al., 2017; Clivaz & Shuilleabhain, 2017, 2019; Huang, Gong, & Han; 2019).

In particular, there is a consensus among researchers on the efficacy of lesson study on teachers' instruction and knowledge. The first seminal work related to examination of lesson study's effectiveness on mathematics teachers' instruction can be accepted as Yoshida's dissertation (Yoshida, 1999). The researcher implemented lesson study (called as *Jugyou kenkyu* in his dissertation) in U.S context and gained some insights that the focus of lesson study is directly improvements of teaching, students' thinking and understanding, what good lessons and good teaching are, and its collaborative side (Yoshida, 1999). This work opened a road for improving instruction of teachers not only by implementing a cultural method to another culture but also by directing new research questions to the researchers including how lesson study groups conduct lesson study, how the groups organize, what the activities of teachers in the groups are during the lesson study and how the discussions are conducted. It can be mentioned that the studies examining mathematics teachers' knowledge have been commonly shaped around these questions and the new questions arising from these questions.

To answer these questions asserted above, Lewis, Perry and Hurd (2009) presented a theoretical model demonstrating the efficacy of observable features of lesson study on visibility of various types of knowledge and to strengthen the professional community. In other words, the researchers asserted applicability and benefits of a locally designed model over a North American case. The main idea behind this study is that the way the lesson study can support the development of professional and content knowledge is to develop an effective instructional plan through pathways (Lewis, Perry, & Hurd, 2009).

While Lewis, Perry, and Hurd (2009) did not focus on the types of knowledge (e.g., content knowledge, tacit knowledge, pedagogical content knowledge), Teyplo and Moss (2011) detailed mathematics teachers' change in knowledge for teaching mathematics, in particular teacher content knowledge (the combination of common content knowledge and specialized content knowledge), knowledge of content and students, knowledge of content and teaching, in the domain of fractions through lesson study. The researchers revealed that the changes occurred in each lesson study phase including investigation, planning, research lesson and reflection. While there was not any change in the first phase-investigation, the study revealed that the change in all three knowledge subdomains were observed in all phases of each cycle.

Similarly, Dudley (2013) investigated mathematics teachers' learning in the phases of planning and discussion of research lessons by focusing on how mathematics teachers' interactions reveal teacher learning. Teacher learning was examined under the frame of tacit knowledge which is defined as knowledge about how to do something on doing it. Besides the benefits of lesson study about improving their knowledge of students for seeing and assessing their needs, by means of discourse analysis of teachers' interactions, the study revealed that the path of teachers' learning occurred through doubt, denial, or stepping back, enlightenment, or conversion to a new knowledge.

Clivaz and Shuilleabhain (2019) examined teacher knowledge in a different perspective which is related to what knowledge teachers use in lesson study with levels of teacher activities during the cycling process of lesson study. Similar with Teyplo and Moss (2011) and Dudley (2013), the study indicated that all types of knowledge and all levels of teacher activities were observed in a cycle regardless of how lesson study phases are conducted in a row. Rather, a cycle indicates that where teachers' work converges, a complete cycle also takes place (Clivaz & Shuilleabhain, 2017, 2019).

In other respects, Widjaja and her colleagues (2017) considered the change in professional growth within the change environment by means of personal domain, external domain, domain of practice, salient outcomes and domain of consequence in lesson study. It has been shown that the changes in these areas contribute to the professional development of teachers thanks to the cyclic process of lesson study.

Different from the previous studies, Suh and Seshaiyer (2015) examined teachers' understanding of the mathematical learning progression through a professional learning project which includes lesson study cycles with a vertical team of teachers from multiple grades. Teacher learning related to knowledge of students' mathematical learning was provided through working with a different grade level of mathematics teachers. Similarly, the design-based research revealed that working with vertical teams in the activities during lesson study nurtured teachers to think beyond their teaching level.

While the group consisted of different grade levels of mathematics teachers, Huang, Gong, and Han (2019) constructed a lesson study group with teacher educators and mathematics teachers. They incorporated theory-driven lesson study (learning trajectory and variation pedagogy-based lesson study) to examine how lesson study improves teaching that promotes students' understanding; in particular, how mathematics teachers transferred their knowledge of students' learning by incorporating two notions of teaching including learning trajectory and variation pedagogy in the context of division of fractions. This study demonstrates that by building on the learning trajectory and strategically using the trajectory tasks, the course is improved in students' comprehension, proficiency, and mathematical reasoning.

Up to now, the studies have commonly focused on mathematics teachers' learning through lesson study. Murata and her colleagues (2012) examined the interaction between students' learning, content and teaching by means of teachers' talk paths in the topic of subtraction. The teachers participating in their study were able to develop a new pedagogy for mathematics education by making use of visual representations to connect mathematics content and student thinking. As they clearly state, mathematics teachers collaborated in the whole process to craft this knowledge and it was built upon various interests and experiences of each teacher, which were observed in their participation in discussion activities (Murata et al., 2012).

While the studies presented above mainly focused on mathematics teachers' professional development in lesson study, a growing body of literature has investigated prospective mathematics teachers' development professionally through lesson study. In the next section, the studies related to prospective mathematics teachers' knowledge and development is presented.

2.2.2 Lesson Study and Knowledge of Prospective Mathematics Teachers

Many studies presented above showed that the use of lesson study provided benefits for mathematics teachers' professional development. They also showed successful adaptation of lesson study to different cultures and different contexts. Since the professional development covers not only participating in in-the-moment development models but also being involved in an effective mathematics teacher preparation program. Stigler and Hiebert (1999) described this issue as

> (…) Lesson study is a new concept for teachers entering the profession. If undergraduate methods courses were restructured to introduce students to collaboratively planning and testing lessons, new teachers would be ready to assume leadership roles more quickly (p. 158).

Therefore, recently the mathematics teacher education researchers have started to focus on examining the effectiveness and adaptation of lesson study in prospective teacher education. Lesson study is very important for prospective teachers in that it combines theory and practice in a collaborative and reflective environment, enabling them to look at teaching knowledge from the perspective of the student and to deal with concepts in depth (Ponte, 2017). Since it is hard to engage prospective mathematics teachers to lesson study considering their workload in university, the researchers also focus on the ways in which prospective mathematics teachers can effectively engage in lesson study. As can be seen below, research in prospective teacher education through lesson study has greatly investigated professional identity development, the nature and development of teaching knowledge, and the integration of lesson study into teacher education.

Fernandez (2005) adapted lesson study in prospective teacher education as micro-teaching lesson study to combine theory and practice and develop prospective mathematics teachers' pedagogical content knowledge. From the situative perspective of learning, micro-teaching lesson study provided prospective teachers to think critically on students' learning and mathematical thinking in addition to their own thinking on mathematical content (Fernandez, 2010). The development through micro-teaching lesson study was provided by means of repeated cycles which triggered prospective teachers' curiosity about content and students, meaningful discussions and support from an expert in lesson study group which is called as the knowledgeable other (Yoshida & Jackson, 2011).

Similar to the adaptation of micro-teaching in lesson study, there have been some attempts to adapt and implement lesson study in the courses served in prospective teacher education. For instance, Baldry and Foster (2019) adapted and implemented lesson study in initial teacher education courses to explore opportunities and challenges of lesson study in prospective teacher education. The researchers revealed some key features for implementing lesson study in initial teacher education courses considering some challenges such as typical lesson observation practices and lesson planning approaches: Knowledgeable others, articulating a learning challenge, wider research and qualified resources, coaching to observation and specifying discussion topics.

Similarly, Appova (2018) engaged lesson study in methods courses in prospective teacher education to reveal teacher educators' ways of effective participation of prospective teachers in lesson study. In addition to the key feature asserted by Baldry and Foster (2019), the study drew attention to in-depth and meaningful discussions through repeated cycles in lesson study for strengthening prospective teachers' understanding of students' learning.

Prospective teachers’ understanding of students' learning is directly related to their knowledge of content and students (Ball, Thames, & Phelps, 2008). From broader perspective of the previous studies, Leavy and Hourigan (2016) examined the potential of lesson study to support the development of all aspects of PCK (e.g., knowledge of content and teaching, and knowledge of content and students) of prospective class teachers who taught to 4–5-years-old children in the context of early number topics. The study showed that lesson study revealed that it actually convinced prospective class teachers of situations they had observed before. In addition, the researchers drew attention to how lesson study improved prospective teachers' knowledge. In their later studies, the researchers answered this question by asserting two ways which revealed that engaging lesson study promoted prospective class teachers’ knowledge of content, students and teaching: 1) Gaining students' understanding of mathematical concepts (early number concepts in their study) and awareness of the complex relationships between concepts, and 2) the ability to identify students' mathematical thinking, the nature and source of their mistakes (Leavy & Hourigan, 2018).

Up to now, the studies related to lesson study in prospective teacher education, in particular improvement in knowledge of prospective teachers through lesson study, mentioned some features of lesson study including the knowledgeable other, meaningful discussions, and repeated cycling process. In a different way, Rasmussen (2016) discussed lesson study in prospective mathematics teacher education with anthropological didactic theory under the light of post-lesson reflection. The study showed that practice-related knowledge was developed by means of different sides of discourse in post-lesson reflection. The researcher mentioned that such type of knowledge can shed light on mathematics teacher researchers and educators.

While those studies mentioned above used different frameworks, most of the studies which examined prospective (also, in-service) mathematics teachers’ knowledge worked with the model of Mathematical Knowledge for Teaching (Ball, Thames, & Phelps, 2008). Focusing on only a model for examining knowledge for teaching mathematics through lesson study might narrow the results of the studies. In addition, most of the studies related to lesson study in prospective teacher education focused on more than three participants and often yielded results from an analysis of all participants rather than an individual analysis. Although the analysis of all the participants is a natural consequence of being a model that provides a collaborative environment, the individual developments of the group members can give insights on the application and effectiveness of the lesson study to researchers and teacher educators. One of the studies in the literature, Plummer and Peterson (2009) discusses the experience of a prospective mathematics teacher who was a successful student in undergraduate mathematics during lesson study. The study discussed the lesson study model in terms of the change in the prospective teacher's behaviors in knowing mathematics. The researchers revealed that she was encouraged to re-identify herself about her mathematical knowledge and to rethink herself about her deficiencies thanks to the lesson study, though the participant trusted her mathematical knowledge before the lesson study and tried to protect this image. The study revealed an important finding in terms of gaining awareness, which is an important step for the development of knowledge, in the lesson study journey of the participant.

This section showed that numerous studies have attempted to point out the efficacy of lesson study (and also micro-teaching lesson study) on prospective teacher education in terms of improvement of knowledge for teaching mathematics. While these studies gave answers to the question of "what" and "how", several studies focused on the factors of lesson study in the question of "how". In the next section, the attempts for these factors are presented.

2.2.3 Critical Aspects of Lesson Study in Development of Knowledge

Lesson study has important aspects for teachers' learning. The studies in the literature have mainly concentrated on the effects on teaching knowledge and its nature by applying lesson study as it is. There have been limited studies investigating which aspects of lesson study contribute to mathematics teachers' learning and developing themselves. Furthermore, Takahashi (2011) indicated that lesson study was not as effective as in Japan, though the lesson study was implemented in many countries. Studies have shown that this situation is not sufficient in terms of how the lesson study process mediates the deepening and development of teachers' knowledge and teaching skills (e.g., Meyer & Wilkerson, 2011; Murata, 2011). Bearing this fact in mind, Lewis, Perry and Murata (2009) mentioned three critical research needs for lesson study including expansion of the descriptive knowledge based on Japanese and U.S. lesson study, explication of the innovation mechanism and design-based research cycles (p. 4-5). In this section, the studies which add to the literature on how the lesson study process mediates knowledge development by indicating critical points of lesson study are presented.

Considering the important points asserted by Lewis, Perry and Murata (2009), there have been several studies to meet these needs. For instance, Lewis (2016) presented a theoretical model to explain the mechanism of lesson study with its impact on different aspects of teaching mathematics including instruction, knowledge, beliefs, curriculum and community. The model revealed the pathways of impact on the aspects of teaching mathematics as interaction between teacher knowledge, teacher beliefs and dispositions, teacher learning with community norms-routines and curriculum (Lewis, Perry, & Hurd, 2009).

Similarly, but for the third research need- design-based research cycles-, Akiba, Murata, Howard and Wilkinson (2019) conducted design-based research on a fairly comprehensive region. They revealed the design features of lesson study including duration (time span and amount), facilitator orientations (students thinking and teacher participation) and material quality. Different from the Lewis (2016), design-based research added the time issue to the model.

Unlike Lewis (2016) and Akiba et al. (2019), Parks (2008) pointed out the important points of lesson study from a different perspective which is related to negative aspects of lesson study. For instance, while time span is included in the design features of lesson study which has a positive effect on teacher learning (Akiba et al., 2016), Parks (2008) indicated that workload, time span and spending lots of time cause messy and unintended learning for teachers. Radovic and her colleagues (2014) supported some of his claims in a self-critical study. As they pointed out, the prospective mathematics teachers mentioned that the collaborative lesson planning phase was difficult, too time consuming, and accordingly not effective.

The studies above show that many factors are put forward for the lesson study to contribute to the development of teachers in the intended direction. In contrast to the studies above, Cavey and Berenson (2005) did not focus on the aspects of lesson study; rather, the researchers shaped lesson study based on their aims regarding intended learning. Cavey and Berenson (2005) embedded a sequence of activities which were designed to engage prospective teachers in broadening and deepening their knowledge of teaching and strategies. Since these activities could be embedded in the lesson planning phase, they called it "*lesson plan study*", which provided its participants three opportunities to think about how she would teach the topic, to access teaching resources, and record ideas on paper (p. 187).

To sum up, it can be said that lesson study and its adaptations to prospective mathematics teacher education provide benefits for prospective mathematics teachers in terms of developing their knowledge of content, students and teaching. In addition, the pioneering studies related to critical aspects of lesson study have explained the innovative mechanism of lesson study in different ways.

Although the effectiveness of the lesson study in prospective teacher education has been discussed from many different theoretical frameworks and contexts, there are some pending issues to be discussed for effective adaptation to teacher training. With a literature review on lesson study in prospective teacher education, Ponte (2017) asserted these issues that 1) most of the studies used lesson study with large groups as an intervention, 2) although the concepts covered in the studies were different, the feedback on mathematical content was limited, and 3) most of the studies were done at the senior year level, but it could be done at different levels. (For example, at the initial stage). Furthermore, today's changes in teaching and learning standards show that lesson study needs to go beyond rote implementation (Lewis, Perry, & Murata, 2009; Akiba et al., 2019).

2.2.4 Lesson Study and Knowledge for Teaching Mathematics in Turkey

Like most of the countries apart from Japan, lesson study has gained importance in recent years. Although the lesson study has just started to be popular in our country, mathematics education researchers have brought their studies to the literature in many different ways (e.g., Budak et al., 2011; Baki, 2012; Baki & Arslan, 2015; Özaltun-Çelik & Bukova-Güzel, 2016; Yıldız & Baltacı, 2017; Güner & Akyüz, 2017; Doğan & Özgeldi, 2018; Yılmaz & Yetkin-Özdemir, 2019a). In addition, different from the other countries, most of the studies in lesson study literature chose their participants from prospective mathematics and class teachers. In this section, the studies in Turkey are presented without distinction between prospective teachers and (in-service) teachers.

A general consensus on the efficacy of lesson study on prospective teacher education continues in Turkey as well. For instance, Budak, Budak, Bozkurt and Kaygın (2011) examined the potential of lesson study in prospective mathematics teacher education by means of self-reflection of their participants. The study revealed that the prospective mathematics teachers developed their collaborative learning skills and their teacher profession knowledge through lesson study. Similarly, Baki (2012) investigated the effects of lesson study in improving prospective class teachers' instruction. She found that lesson study improved prospective teachers' mathematical knowledge for teaching and their instruction and asserted that lesson study could be adapted in some courses for better teacher training. In their other study, Baki and Arslan (2015) focused on one phase of lesson study-lesson planning and its effect on knowledge of teaching of prospective class teachers by comparing prospective teachers who were experienced and non-experienced in lesson study. It was found that the experienced prospective class teachers improved themselves on what should be considered in the planning of a lesson, and that this group were better than the non-experienced group in terms of adjusting the number of activities, completing the learning-teaching process within an achievement, and ordering the activities in the appropriate order.

Another respect related to the lesson study is that the nature of lesson study requires thinking critically on students' mathematical learning and teaching. Accordingly, the studies in Turkey, similar to the ones in international literature, have focused on knowledge about students' interaction with mathematical content. For instance, Özaltun-Çelik and Bukova-Güzel (2016) examined the questions asked by a mathematics teacher participating in the lesson study in the context of knowledge of students' mathematical thinking. The study revealed that the teacher's questions encouraged students to think mathematically in terms of socio-mathematical norms during the lesson study process.

While this study did not assert the effect of lesson study, another study which was conducted with prospective mathematics teachers indicated that lesson study plays an active role in their development (Güner & Akyuz, 2017). The researchers investigated what prospective mathematics teachers noticed about students' mathematical thinking which is closely related to knowledge of content and students in the topic of polygons within the scope of the lesson study. The researchers found that while the prospective teachers' awareness about students' mathematical thinking varied in different phases of lesson study, the model provided improvement in meaningful mathematical noticing skills.

In the same vein, Yılmaz and Özdemir-Yetkin (2019a) analyzed how lesson study improved prospective mathematics teachers' skills in the interpretation of students' thinking throughout the lesson study. The study showed that prospective mathematics teachers began to take into consideration students' mathematical thinking during planning, and to determine students' difficulties and guide them, while they had a lack of knowledge about students' mathematical thinking at the beginning of lesson study. Besides, prospective teachers were able to put the thoughts of students in order and connect the significant points to big ideas.

Furthermore, the study asserted some distinctive features of lesson study including planning collaboratively and conducting this plan as the reasons behind the lesson study's positive effects on the improvement of prospective mathematics teachers' knowledge about students' mathematical learning (Yılmaz & Özdemir-Yetkin, 2019b).

In general, the studies above indicated the potential of lesson study in Turkey. Considering the potential of lesson study, Eraslan (2008) questioned whether lesson study can work in Turkey as it works in Japan. The researchers asserted that there are some issues including cultural roles in the classroom, curriculum and its flexibility and role of instructors (university) that can be closely related to applicability of lesson study in Turkey. Stigler and Hiebert (1999) pointed out that teaching, as a part of culture, will develop over time with stable practices. Based on this fact, Eraslan (2008) indicated that lesson study has the potential to improve teacher education programs. In addition, the researcher added that in-service teachers should be supported to participate in the lesson study to make it more applicable in Turkey, because lesson study requires extra time to work on it.

To sum up, the studies in Turkey have mainly focused on improvement of prospective teachers' knowledge of students' learning and knowledge of teaching mathematics through lesson study. As can be seen in the context different from Turkey, lesson study bears the capability to improve prospective teachers' not only pedagogical content knowledge but also mathematical knowledge in different phases. Therefore, there is still a need to examine the improvement of prospective teachers' knowledge for teaching mathematics through lesson study from a comprehensive view in order to ensure the applicability of the lesson study in prospective teacher education with maximum benefit. In addition, while the question of "what" about lesson study has been answered from different perspectives, the answer of the question "how" has not been given a place in the literature in Turkey. The answer to "how" is of importance for providing practicality of lesson study besides eliminating the gap in the literature.

2.3 Teaching and Learning the Concept of Limit

The concept of limit can be described as a cornerstone of fundamental concepts of calculus (Tall & Vinner, 1981; Monaghan, 1991; Cornu, 1991; Beynon & Zollman, 2015; Fernández-Plaza & Simpson 2016). It is one of the central concepts that provides a basis for many mathematical concepts especially the concepts of Calculus such as derivative as approaching a secant line to tangent line and the rate of change, continuity and differentiability, Riemann integral as the limit of sum of infinite small pieces in terms of cumulative rate of change, convergence of sequences (Roh, 2008; Fernández-Plaza & Simpson 2016). In this way, we can say that the foundation of many concepts actually comes from the development of the concept in the history of mathematics. In its mathematical development in history, the concept of limit has emerged in various contents of mathematics including numbers, geometry, derivative, integral from ancient Greeks to modern-era (Burton, 2011).

The concept of limit can be found in either the last years of high school or beginning of undergraduate years in Calculus almost all over the world. In Calculus textbooks, the definition of the concept of limit is dealt with in two ways, in the current study as well. The intuitive and formal definition of limit can be seen below:

> "If $f(x)$ is defined for all x near a, except possibly at a itself, and if we can ensure that $f(x)$ is as close as we want to L by taking x close enough to a, we say that the function f approaches the limit L as x approaches a" (Adams & Essex, 2010).

> "We say that $f(x)$ approaches the limit L as x approaches a if the following condition is satisfied: For every number $\varepsilon > 0$ there exists a number $\delta > 0$, depending on ε, such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \varepsilon$" (Adams & Essex, 2010).

The definitions are included in the curricula of Turkey in a similar way. In Turkey, the concept of limit is dealt with within different scopes at different levels of curriculum. Since there have been lots of attempts for revising the curriculum, the scope of the concept of limit has been reduced in each attempt of high school curriculum reforms from 2005 to 2018. The first curriculum which accepted the constructivism perspective can be considered as the broadest curriculum since its scope varies from the limits of functions and limits of sequences to all indeterminate forms including $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 1^{\infty}, \infty^{0}$ and 0^{0} in limit (MoNE, 2005).

In the latest published high school mathematics curriculum (MoNE, 2018), students in a typical high school (Anatolian high schools) and high schools of science are both introduced the intuitive definition of limit and right-left side limits, operations with limits and the concept of continuity (MoNE, 2018). In addition to these objectives, students of high schools of science are introduced the formal definition of limit, extended real numbers, limits at infinity, infinity limits and indeterminate forms in limits (MoNE, 2018). However, the scope of high stakes exams for university entrance covers only the curriculum prepared for a typical high school. This means that students who are responsible for gaining the objectives of the curriculum for Anatolian high schools would not have an idea about the formal definition of limit until they start their instruction at university. The textbooks designed for high schools introduce the notions of limit and continuity and this is followed by derivative and then the integral (Emin et al, 2020; MoNe, 2018). The change in curricula in years affected the studies related to the concept of limit since it is important to prepare prospective mathematics teachers for all the conditions in their professional years. Therefore, the current study considered the curricula proposed from 2005 to 2018 which indicate the period when a reform movement in national education system was implemented.

In addition to preparing prospective mathematics teachers for all the conditions in their professional years, the fundamental principles of the implemented curriculum lead the direction of their preparation. The fundamental principles of the curriculum aim to (1) develop students' problem-solving skills by examining the problems from different perspectives, (2) gain students' mathematical thinking and practice skills, (3) develop their understandings of mathematics and learning mathematics, the history of mathematics and the use of mathematics in daily life (MoNE, 2018). The principles of assessment and evaluation of the curriculum cover multi-ways of instructional evaluation and assessment and require the active participation of students and teacher in the assessment process (MoNE, 2018).

Considering these principles, prospective mathematics teachers need to know students' conceptions of the concept of limit, the difficulties of learning and teaching the concept of limit and knowledge of mathematics for teaching the concept of limit to achieve the objectives of the curriculum. In addition, it has been pointed out in the literature that not only students but also prospective teachers have these difficulties and conceptions. Therefore, in this section, first of all, the difficulties and conceptions related to the concept of limit will be discussed. Then, the studies on the knowledge of teachers and prospective teachers for the teaching of the concept of limit and finally the studies on the country basis will be presented.

2.3.1 The Students' Conceptions of the Concept of Limit

It can be easily felt that there is a motion in the intuitive definition of limit in "as x goes to a, $f(x)$ goes to L" (Fischbein et al., 1981). In the dynamic conception of limit, students think that the limit is a dynamic process including motion in x-values and y-values (Tall & Vinner, 1981; Davis & Vinner, 1986; Williams, 1991; Szydlik, 2000). On the study of Tall and Vinner (1981), more than half of the university students who participated in the study described the concept of limit as a dynamic definition "the value that $f(x)$ approaches as values of x are taken closer to a is c" (p. 162). Likewise, Williams (1991) found that Calculus class students had a dynamical conception of limit when they worked with functions because of seeing the limit of function as a process including motion feelings in x and y-values. The dynamic conception of limit can be seen as an easy and natural way of thinking for students to develop because of the nature of the concept of limit (Tall & Vinner, 1981). However, in the static-formal conception of limit, students encounter intervals of x-y-values in which they do not feel the motion in intervals (Çetin, 2009). For this reason, some researchers indicate that dynamic-process conceptions prevent students thinking that the limit can reach a number (Tall & Vinner, 1981; Williams, 1991).

As a different perspective, Cotrill and his colleagues (1996) revealed that dynamic conception of limit is not a single process but learning the concept of limit requires a coordinated schema including both dynamic conceptions of limit and formal conception of limit which covers coordination of domain and range process via function to build the formal understanding of limit on the informal (intuitive/dynamic) understanding of limit.

The dynamic conception of the limit in students mind is related to their conceptions of infinity. Since students consider reaching the limit as close as they can in the dynamic conception of limit, it directs students to infinitesimal quantities which can be defined as extremely small numbers and extremely large numbers (Mamona-Downs, 1990). When the concept of infinitesimals is considered, we reach the potential infinity in ancient times as it depends on the number line (Bagni, 2005). Potential infinity is related to the idea of never ending or endless (Tall, 1992; Lakoff & Nunez, 2000). Aristotle (384-322 B.C) considered only potential infinity and kept away from Zeno's paradoxes which represent the actual infinity (Fischbein, 2001). If someone tried to count all of the whole numbers, they would never reach an ending point. Similarly, if one thinks of time progressing through eternity with no end, then time is considered as "inexhaustible" (Jones, 2015, p. 107). These are examples of the idea of potential infinity. Monaghan (2001) indicated potential infinity as a process in students' perceptions. In the same study, students expressed infinity like a repeating process (going on and on). On the other hand, the actual infinity reflects infinity as an existent entity, in other words, the whole body of an infinite set (Tall, 1992). The set of integers, Z, including an infinite number of elements can be considered as an example (Jones, 2015). Considering that Aristotle rejected actual infinity to avoid Zeno's paradoxes, Zeno's paradoxes may be an example for actual infinity from ancient times.

The limit conceptions of students and other factors including the abstract nature of the concept, cognitive obstacles, linguistic problems, prior experiences of students and the role of intuition cause some difficulties in learning and teaching the concept of limit (Tall & Schwarzenbenger, 1978; Tall & Vinner, 1981; Monaghan 1991).

These conceptions are of importance to understand the difficulties in teaching and learning the concept of limit. For instance, the dynamic conception of limit causes to describe the concept as unreachable point in a function (Williams, 1991). Furthermore, prospective teachers also experience similar difficulties they can face during their professional development process. Therefore, these difficulties will be addressed in the next section.

The difficulties related to the language and intuition of limit

The role of language is of importance to understand the concept of limit (Davis & Vinner, 1986; Monaghan, 1991; Cornu, 1991). We encounter this situation in two ways including the role of using the word "limit" in daily life either intuitively or lexically, and the role of words for describing the concept of limit mathematically. Students use the word “limit” in different areas of their daily life; for instance, “speed limit” or “credit card spending limit” (Özmantar & Yeşildere, 2008, p. 186), which cause some misconceptions in learning the concept of limit. For instance, “limit is an unreachable point” is a statement for students with misconception and they usually express that the limit is the point that the function cannot reach (Williams, 1991) or “limit is the boundary point” (Szydlik, 2000, p. 269), which are common misconceptions on limit related to using the words in addition to conceptions of limit and infinity (Davis & Vinner, 1986; Tall & Vinner, 1981; Williams, 1991). As mentioned above, the basis of these misconceptions is confronted in the history of the concept as the epistemological obstacles to understanding the limit (Sierpinska, 1987; Cornu, 1991).

Another issue among linguistic problems is that these misconceptions can be caused by using the phrases such as “tend to”, “approach” and “converge” which can construct a different sense to the term of limit (Monaghan, 1991). While “approaches” and “tends to” have dynamic interpretations of limit, the meaning of “converge” is understood as “line converging”, not as a sequence of numbers that can converge (Monaghan, 1991). In addition, the phrases such as “approaches” and “gets close to” which cover the dynamic conception of limit do not exactly bring the mathematical meanings of the concept of limit (Tall & Vinner 1981; Williams 1991), instead they bring the everyday sense of these words (Roh, 2008). As mentioned in the previous paragraph, using these phrases in teaching and learning of the concept of limit may cause a misconception as reaching the limit point or not reaching, and only approaching the limit point (Cornu, 1991).

The problems derived from the use of language are not only observed in daily life or teaching the concept but also observed in textbooks. Even Calculus books which are used in university courses cannot differentiate the everyday language and mathematical language. For instance; in Adam's Calculus (2001), the analogy used for describing $x \to 3$ is that you can get access infinitely close to a running fan, but obviously you will never reach it because you know what will happen if you reach the running fan (Adams, Thompson, & Hass, 2001). It can be easily seen that the analogy constructed or intensified the misconception which is based on the idea that limit is an unreachable point (Liang, 2016; Szydlik, 2000). When the weight of mathematics courses especially in mathematics teacher preparation programs is considered, such a lack of knowledge of misconceptions of the concept of limit give rise to the problematic issues in both teaching and learning of limit. The word "reach" has also its own difficulties to understand the concept of limit, because it may be understood as "being in the neighborhood of a point" or "landing on a point" (Taback, 1975 as cited in Moru, 2009, p.434). Given that the language used in teaching is important, it is important for mathematics teachers to have both correct knowledge and to pay attention to the language they use while teaching.

The difficulties related to the definition of limit

At the beginning of the mathematics education program, almost all of the curricula of the departments require taking Calculus courses. Contrary to the instruction in high school, students of Calculus course are engaged with the formal definition of limit. The precise, "formal definition of the concept of limit is so complex and counterintuitive that it fails to bring out readily the simple and intuitively obvious ideas which led to it in the first place" (Parameswaran, 2007, p.194). That's why, students know the informal definition of limit given above and they can make calculations and solve problems; however, students have difficulties in the formal definition of limit. Although limit has emerged to solve other difficulties in mathematics and to understand other mathematical concepts, students' inexperienced conceptualization of the concept of limit prevents them from understanding the formal definition (Davis & Vinner, 1986; Cornu, 1991; Williams, 2001).

As can be seen in Figure 2.3 marked in red, there are some aspects in the formal definition of limit including absolute-inequalities, “unknown terms” such as ε and δ, the quantifiers including “for all or for every number” and “there exists” which seems non-understandable and difficult to cover their minds for students Davis & Vinner, 1986; Cornu, 1991; Cottrill et al., 1996; Pinto & Tall, 2002; Kidron & Zehavi, 2002).

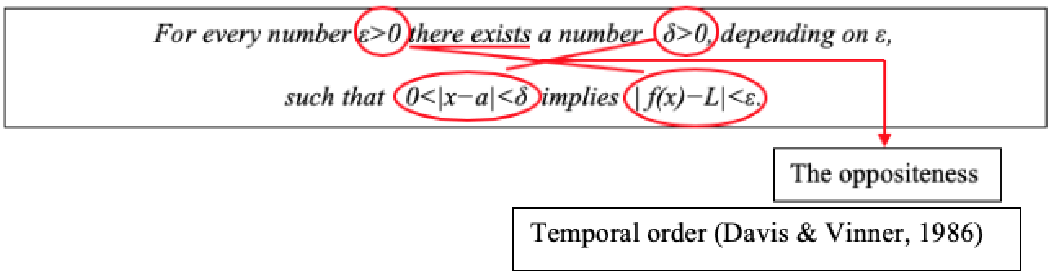

Figure 2.3. The difficulties in the formal definition of limit of function

Though the high school curriculum does not cover the formal definition of limit, it is of importance for teaching the concept, since the formal definition provides the technical tools for demonstrating how a limit works and introduces students to the rigors of calculus (Adiredja, 2020). In this section, the difficulties in learning and teaching the formal definition of limit (Tall & Vinner, 1981; Cottrill et al., 1996; Fernández, 2004; Przenioslo, 2004; Parameswaran, 2007; Swinyard & Larsen, 2012) are presented in line with the related literature.

When students read the formal definition, they questioned ε, δ and where these quantities come from; by this way, the absolute inequality is examined to understand

and interpret algebraically or geometrically (Fernandez, 2004). In addition, students have difficulties in understanding the quantifiers including "*for all*" and "*there exists*" and why these quantifiers and absolute-inequalities are looked in an asymmetry (Davis and Vinner (1986) called it as "temporal order") (Cornu, 1991; Cottrill et al., 1996; Fernandez, 2004). Cornu (1991) indicated that the meaning of these quantifiers in daily life which is different from their own meanings can give rise to noteworthy problems in teaching and learning the concept of limit. In the study of Pinto and Tall (2002), some of the students thought that the quantifiers of definition were not essential to check the convergence of sequences and they continued with only the inner portion of definition, $|a_n - L| < \varepsilon$. However, those who included the quantifiers had confusion about the relation between N on δ (Davis & Vinner, 1986; Kidron & Zehavi, 2002).

Parameswaran (2007) investigated whether students' errors affect their understanding of the concept of limit in the cases where small numbers are used as parameters when students have not been taught the formal definition of limit of function. These students were in tendency to approximate such "small numbers", which represented infinitesimal quantities in this study, to zero and they considered that there should be limit as approximation wherever they see these "small numbers". The erroneous practices of students show that students tend to use the terms "approximation" which we encounter in intuitive definition even in the formal definition. However, the formal definition of a limit provides the technical tools for understanding the concept in an accurate way (Adiredja, 2020).

In numerous high school curricula, there is a passing process from informal understanding of limit to formal understanding of limit. Considering this in mind, Swinyard and Larsen (2012) offered a theory about the learning trajectory for the formal definition. They revealed two challenging steps for students to this passing process: appropriate ordering for ε and δ and encapsulating the arbitrariness of epsilon in the definition. Likewise, Adiredja (2020) indicated that students participating in the study have difficulties in the temporal order (Davis &Vinner,

1986) within the formal definition. In addition, half of these students did not tend to write epsilon first in the temporal order. These challenging steps can be observed in the conceptualization process of students in the formal definition of limit. In the study of Przenioslo (2004), the researcher indicated that the students could not notice or try to notice the contradictory statements between the formal definition and its contents, while students could express the formal definition correctly. The important finding of the study was that the students did not consider the importance of it as an image of the limit unless the formal definition works for them in a problem. In short, the sources of the difficulties can be summarized as the difficulties of infinitesimal quantities and the difficulties related to ordering the quantities and quantifiers.

The difficulties related to infinity concept

If we consider the concept of limit as the skeleton of the structure consisting of Calculus concepts, we can say that infinity and infinite processes are the basis of this structure. The idea of limiting process is based on infinitesimals in the rate of change and infinity such as Cavalieri theorem in the history of the concept (Bagni, 2005). In this section, the concept of infinity and its related difficulties are reviewed towards its relation with the concept of limit.

While infinity and related concepts are used in many mathematical areas including as number configuration, number comparison, and numerical line (Lakoff & Nunez, 2000; Dubinsky et al., 2005), the concept of infinity is an abstruse concept among mathematical concepts (Barahmand, 2017) since we do not have the chance to experience the notion of infinity in daily life due to restriction of our surroundings by finiteness (Güçler, 2013). However, infinity and infinite processes have an important place in teaching and learning the concept of limit. The reason for some misconceptions regarding the concept of limit is students' lack of understanding of infinite processes. Since they confuse limit with the value of the function or a sequence, or an approximation of the limit, they cannot apply finite processes in approximation (Cottrill et al., 1996).

In the abovementioned section, the conceptions of the infinity were indicated as potential infinity and actual infinity (Fischbein, Tirosh, & Hess, 1979; Fischbein, 2001; Monaghan, 2001; Tall, 2001; Dubinsky et al., 2005). The idea of potential infinity causes some misconceptions in limit and in other mathematical concepts. For instance, since many students regard infinite decimals as infinite numbers, this idea causes a misconception towards the thinking that never-ending infinite digits makes the infinite decimals as infinitely large number (Monaghan, 2001). In addition, potential infinity is directly related to the misconception of limit that limit is unreachable point (Williams, 1989). On the other hand, students think of actual infinity as an end point of the infinite process (Fischbein et al., 1979). Monaghan (2001) indicated the actual infinity in students' perspective as an object. In the study of Monaghan (1986), students described infinity as a huge number (as cited in Monaghan, 2001). Students encounter infinity also in repeating decimals as the comparison with a repeating decimal and its closest integer (Tall & Schwarzenberger, 1978). The most known example among them is the conflict between "$0.\overline{9}$ is equal to 1" and "$0.\overline{9}$ is lower than 1" (Sierpinska, 1978; Tall & Schwarzenberger, 1978; Cornu, 1991; Monaghan, 1991). Tall and Schwarzenberger (1978) indicated that most of the students expressed that $0.\overline{9}$ is very close to 1 but not equal to 1 (It is the last number before 1 in the number line). This understanding might stem from understanding limit insufficiently, misinterpreting the symbol $0.\overline{9}$ as a large but finite number of 9s, considering the need of one-to-one correspondence between infinite decimals and real numbers, and the intrusion of infinitesimals. Therefore, this causes conflicts in students' minds between the concept of limit and decimals (Tall & Schwarzenberger, 1978).

Bearing the aforementioned difficulties and obstacles in learning and teaching the concept of limit in mind, mathematics teachers should consider lots of factors in teaching the concept of limit. Tchoshanov (2011) indicated that teachers' mathematical knowledge, in particular, knowledge of concepts and the relations between them, has an important factor on the quality of mathematics teaching and students' achievement. For this reason, development of prospective mathematics teachers’ mathematical knowledge of limit is one of the important steps to overcome these difficulties about the concept of limit (Cory & Garofalo, 2011). It should be emphasized that we need the mathematical basis of teaching the concept of limit to examine and develop prospective mathematics teachers' specialized knowledge both theoretically and practically. For this reason, the conceptions of students and the difficulties related to the concept construct both mathematical and pedagogical content knowledge for prospective mathematics teachers. For the aim of developing prospective mathematics teachers’ knowledge for teaching the concept of limit, it is necessary to understand the knowledge for teaching the concept of limit of prospective mathematics teachers in a broader perspective.

2.3.1.1 In-service Mathematics Teachers’ Knowledge for Teaching and the Concept of Limit

Since students have difficulties to understand the concept of limit as presented above, various researchers from different countries have studied knowledge of in-service and prospective teachers for teaching the concept considering different dimensions of mathematical and pedagogical content knowledge. In general, the studies focused on examining subject matter knowledge and pedagogical content knowledge of in-service mathematics teachers in different content areas of limit including limits of functions, limits of sequences, the concepts of convergence and continuity, and the concept of infinity (Mastorides & Zacharides, 2004; Hulliet, 2005; Kajander & Lovric, 2007; Montes, Carrillo & Ribeiro, 2014; Montes & Carrillo, 2015). The pedagogical content knowledge has been examined in terms of the knowledge of student understanding and the knowledge of instructional strategies, knowledge of representations, and knowledge of students’ difficulties and misconceptions. In addition, the subject matter knowledge has been investigated regarding the knowledge of key concepts and essential features of the concept, knowledge of mathematics, and conceptual knowledge of the limit concept. In this section, the studies on in-service mathematics teachers' knowledge of teaching the limit are presented.

Hulliet (2005) examined the evaluation of six in-service mathematics teachers' knowledge of limits of functions with a holistic approach without separating pedagogical content knowledge and subject matter knowledge. She showed that the in-service mathematics teachers had a strong knowledge of operations in limit. In addition, the in-service mathematics teachers demonstrated their knowledge of students' conceptions and difficulties since they already teach the limit concept at school. However, their knowledge of formal definition was found weak. In addition, they were used to learning rules without demonstrations and they were not able to make the connection between different concepts or between different settings.

In another study, Mastorides and Zacharides (2004) investigated 15 in-service secondary mathematics teachers' understanding and reasoning about the concepts of limit and continuity to provide the extent and sufficiency of subject matter knowledge of teachers. Similar with Hulliet (2005), the participants of the study tended to believe that all the expressions of the general notations of limit are similar to the ones they teach as a result of being already teaching limits at school. However, the participants of the study could not pass from "verbal" representation to symbolic and vice versa. These studies demonstrated that in-service secondary mathematics teachers had a lack of knowledge of representation, subject matter knowledge, knowledge of definitions and conceptual knowledge in the limit concept. In school context, there are a number of factors such as didactic restrictions in teaching the concept and textbooks affecting mathematics teachers' effective teaching. Considering one of these factors, Barbe et al. (2005) which examined how teachers' practices are restricted by mathematical and didactical phenomena under the theoretical framework of Anthropological Didactic Theory (Chevallard, 1980 as cited in Chevallard & Bosch, 2020) in the concept of limit. They revealed that mathematics teachers had some didactic restrictions which determined teaching practices and mathematics taught including institutional restrictions, the nature and epistemological structure of the concept. The study has an important implication for designing a learning environment for mathematics teachers in terms of taking a deeper understanding of the concept into account.

Another factor is textbooks recommended by the curriculum and used by in-service mathematics teachers. Kajander and Lovric (2017) revealed a framework based on MKT to evaluate the quality of mathematics teaching based on specialized and horizon content knowledge by examining textbooks (teacher resources) in the concept of limit. The descriptions of the framework included appropriate use of models and appropriate mathematical reasoning for classroom use (as specialized content knowledge) and opportunities for development of misconceptions, and links to advanced mathematical knowledge (as horizon content knowledge). The study showed that the teaching resources were not adequate to represent the key concepts of the concept of limit. As an implication of the study, there should be a need to develop mathematics teachers' knowledge of accurate intuitive understanding of infinite processes for teaching the concept of limit effectively (Kajander & Lovric, 2017).

As stated in the previous section, infinity is one of the important concepts for teaching the concept of limit. Yopp, Burroughs and Lindaman (2011) examined the reactions of an in-service mathematics teacher to one of the problems about infinity, as mentioned above whether $0.999\ldots$ is equal to 1. The study indicated that the teacher's sense of number and sense of measurement are intertwined, resulting in fragile understanding of repeating decimals. These data present evidence that teachers continue to develop repeated decimal understandings and misunderstandings throughout their careers, and that the curriculum, everyday experience, and perceptions of student learning combine to form or reinforce these understandings. For this reason, the basis of their understanding of infinity is of importance for their further understanding and teaching quality.

In addition to conceptual knowledge, another important issue is the teacher's knowledge in the role of this concept in the classroom. Montes, Carrillo and Ribeiro (2014) investigated the teachers' knowledge of infinity and its role in the classroom from three different theoretical frameworks on mathematics teacher knowledge. While infinity is considered as the specialized content knowledge in MKT, it is included in the sub-domain of *transformation* for knowledge quartet. As the focus of the current study, the infinity is considered as *knowledge of the structure of mathematics* in MTSK framework, since it represents a big idea for the concepts of Calculus. Likewise, Montes and Carrillo (2015) investigated three mathematics teachers' knowledge of infinity regarding the convergence of geometric series with a structured interview. These mathematics teachers performed different knowledge sub-domains such as Knowledge of Topics, Knowledge of Structure of Mathematics, Knowledge of Practices in Mathematics and Knowledge of Features of Learning Mathematics in the same problem.

These studies provide important insights into mathematics teachers' knowledge for teaching mathematics; in particular mathematical knowledge including some important points for the concept and pedagogical content knowledge including knowledge of student understanding and knowledge of instructional strategies and teaching. Considering the former section which is related to difficulties in the concept of limit and mathematics teachers' knowledge for teaching the concept in the same vein, prospective teachers' knowledge is of importance for effective mathematics teaching. In the next section, the literature on prospective mathematics teachers' knowledge for teaching the concept of limit is presented.

2.3.1.2 Prospective Mathematics Teachers' Knowledge for Teaching and the Concept of Limit

Most of the studies with prospective mathematics teachers (from junior to senior) have commonly focused on their conceptions/misconceptions and conceptual/procedural knowledge as shown at the beginning of this section. As opposed to these studies, a limited number of studies have examined prospective mathematics teachers' knowledge for teaching of the concept of limit (Kolar &

Čadež, 2011; Cory & Garfola, 2011; Oktaviyanthi, Herman & Dahlan, 2018; Wasserman et al., 2019).

For instance, Kolar and Čadež (2011) examined prospective primary mathematics teachers' mathematical knowledge of different types of infinity including infinitely large, infinitely many and infinitely close. Considering that these prospective teachers did not receive an in-depth instruction on abstract mathematical content and the foundations of the concept of infinity are actually laid at primary school level, the findings of the study of Kolar and Čadež (2011) are important to understand a few steps before secondary school. They indicated that prospective primary school teachers used their knowledge of actual infinity when they were asked about *infinitely large* and *infinitely many* and their knowledge of potential infinity when they were asked about *infinitely close*.

Similarly, but in different content, Oktaviyanthi, Herman and Dahlan (2018) investigated prospective mathematics teachers' mathematical knowledge through mathematical procedures. In particular, while Oktaviyanthi, Herman and Dahlan (2018) did not directly mention the types of knowledge, they examined the thinking process of prospective mathematics teachers according to their evaluations of a function limit in terms of formal definition. The study pointed out some strategies adopted by prospective teachers. For instance, prospective teachers adopted the strategies including "1) determining delta value by the final statement of formal definition, (2) substituting the given and process, (3) simplifying value in the absolute sign, (4) solving the inequality, and (5) finding the delta value" in the preparation of proof (Oktaviyanthi, Herman, & Dahlan, 2018, p. 209). It can be said that the prospective teachers discuss the strategies they adopt, and it is different from the studies that researched mathematical knowledge mentioned above.

Unlike the previous studies, Cory and Grafola (2011) focused on pedagogical content knowledge. The researchers investigated knowledge of connections between the visual and verbal representations of limits of sequences of prospective secondary school mathematics teachers by interacting with dynamic sketches. The study showed that manipulation and visualization of the formal concept of limit strengthened prospective mathematics teachers' knowledge of connections between the visual and verbal representations. The study supported the findings of other studies conducted on high school students, beginning classes' students of Calculus and students at other departments of universities (Verzosa, Guzon & de Las Peñas, 2014; Jones, 2015).

Many of the studies on knowledge for teaching the concept of limit, continuity and infinity presented the results of in-service or prospective mathematics teachers' knowledge cases in the concept of limit. There are limited studies focusing on the development of knowledge for teaching the concept of limit of mathematics (either prospective or in-service) teachers (e.g., Wasserman et. al., 2019; Yimer & Feza, 2020). For instance, Wasserman and his colleagues (2019) looked at the development of prospective mathematics teachers' knowledge for teaching the concept of limit in a broader angle in which they dealt with the concept of limit in real analysis. In a designed real analysis course, they revealed that at the end of the course, prospective mathematics teachers' mathematical knowledge for teaching (horizon content knowledge in their study) was developed through a series of activities.

Overall, there seems to be some evidence to indicate mathematical knowledge and pedagogical content knowledge of prospective teachers in some topics in the concept of limit. Since the concept of limit covers lots of topics in mathematics (Cornu, 1991), the studies related to prospective mathematics teachers' knowledge for teaching the concept of limit can be extended to a broader sense of the concept in different sub-domains of knowledge.

2.3.1.3 Mathematics Knowledge and the Concept of Limit in Turkey Context

In contrast with the studies conducted in other countries, a great deal of research in Turkey have dealt with knowledge for teaching such as teaching strategies, curriculum, assessment and students' difficulties faced by prospective secondary school mathematics teachers in concepts of limits of functions and continuity (Baştürk & Dönmez, 2011a; 2011b; Kula & Bukova-Güzel, 2015; Turan & Erdoğan, 2017; Kula-Ünver & Bukova-Güzel, 2019). In this section, the studies related to prospective teachers' knowledge of the concept of limit are presented.

In general, the studies conducted in Turkey cover the aims of examining prospective mathematics teachers' conceptual knowledge and pedagogical content knowledge (e.g., knowledge of teaching strategies, knowledge of curriculum) in the concept of limit. For instance, Dönmez and Baştürk (2010) investigated prospective mathematics teachers' knowledge of different teaching methods of the limit and continuity concept. They indicated that the prospective mathematics teachers used only lecturing or question-answer techniques even though they were aware of the importance of using different methods and teaching mathematics to help students make the concept concrete. The researchers also pointed out some evidence about the prospective mathematics teachers' pedagogical content knowledge regarding knowledge of curriculum and knowledge of assessment (Baştürk & Dönmez, 2011a; 2011b). For knowledge of curriculum, the researchers claimed that the participants with wider content knowledge were more willing to stick to the curriculum such as following goals and achievements in the program, paying attention to the order of the concepts, and not covering the concepts removed from the program (Baştürk & Dönmez, 2011a). In their another study which examined the knowledge of assessment in teaching the concept of limit, prospective teachers did not reflect their thoughts in their teaching practices, even though the participants presented thoughts in accordance with the philosophy of alternative measurement and evaluation methods that are desired to be used in the new mathematics teaching program (in this study, the curriculum published in 2005) (Baştürk & Dönmez, 2011b). The researchers explained this situation as lack of knowledge of alternative measurement and evaluation methods or lack of the knowledge of implementation of them in their teaching (Baştürk & Dönmez, 2011b).

Similarly, Kula and Bukova-Güzel (2014, 2015, 2019) investigated prospective mathematics teachers' pedagogical content knowledge in terms of teaching strategies, curriculum, and use of representations in the concept of limit. For instance, Kula and Bukova-Güzel (2014) found that the reflections of the prospective mathematics teachers' knowledge related to the purposes of the mathematics curriculum on their limit teaching included relating the limit concept with real life and different subject areas, providing mathematical thinking and reasoning, improving ability to communicate, using mathematical language, relating mathematics with art, and using technology effectively. While Baştürk and Dönmez (2011a) examined the knowledge of curriculum in terms of comparing participants with less and more pedagogical content knowledge, Kula and Bukova-Güzel (2014) looked at knowledge of curriculum in different perspectives as reflections on the purposes of the curriculum. On the other hand, similar to Dönmez and Baştürk (2010) in the aim of investigating knowledge of teaching strategies, Kula and Bukova-Güzel (2015) revealed different findings. The researchers revealed that the prospective mathematics teachers used activities such as games, examples from daily life, animation, scenarios and analogies in their teaching practices. In their other study related to knowledge of teaching strategies, Kula-Ünver and Bukova-Güzel (2019) investigated the use of representations of preservice secondary mathematics teachers in teaching the concept of limit. The study reported that the prospective secondary mathematics teachers used the six types of representations including number line, tabular, figural, graphical, algebraic and verbal.

For another domain of knowledge-mathematical knowledge, the number of studies examined freshman or sophomore mathematics education students' conceptions of the concept of limit in Turkey (e.g., Bukova, 2006; Çetin, 2009; Biber & Argün, 2015) which were mentioned in the previous sections. However, there are limited studies examining prospective mathematics teachers' mathematical knowledge for teaching the concept of limit (e.g., Turan & Erdoğan, 2017; Baştürk & Dönmez, 2011c; Tuna, Biber, & Korkmaz, 2019). The studies commonly focused on conceptual structures and conceptual knowledge of prospective mathematics teachers in different topics in the concept of limit. For instance, Turan and Erdoğan (2017) investigated conceptual structures of prospective mathematics teachers in the concept of continuity with The Free Word Association Test. The study revealed that the preservice mathematics teachers usually associated continuity with such concepts as "limit, 'function, derivative, discontinuity, ongoing-uninterrupted, definition, right-left limit, convergence, 'neighborhood', integral, limit, extremum dot, infinity, notation, uniform continuity, undefined and value" (Turan & Erdoğan, 2017, p. 405). The most interesting finding of the study is that the prospective mathematics teachers associated the concept with ongoing-uninterrupted with the concept of continuity which can be described as a misconception (Baştürk & Dönmez, 2011c).

In another study which examined prospective mathematics teachers' conceptual knowledge, Tuna, Biber and Korkmaz (2019) examined the concept knowledge about the limits of sequences of prospective mathematics teachers. They revealed that the prospective mathematics teachers did not have knowledge related to the importance of the prerequisite concept, "accumulation point", for the concept of convergence even though they knew the basic elements of the limit of sequences such as "a limit of the general term of the sequence" or "the general term of the sequence, has to converge to a number". When the study is compared with the previous study, the study is supported by Turan and Erdoğan (2017) in which prospective mathematics teachers did not mention "accumulation point", though they indicated the notion of "convergence".

In general, the studies conducted in Turkey cover the aims of examining prospective mathematics teachers' conceptual knowledge and pedagogical content knowledge in the concept of limit. In the next section, the studies are summarized in comparison with overall studies presented in the literature review.

2.4 Summary of Literature Review

The literature review in this section is organized into three main sections including mathematics teachers' knowledge, mathematics teachers' knowledge, lesson study, and teaching and learning of the concept of limit. The studies in this section provide important insights into prospective mathematics teachers' knowledge for teaching, lesson study, and the concept of limit.

Collectively, most of the studies with prospective mathematics teachers (from junior to senior) have commonly focused on their conceptions/misconceptions and conceptual/procedural knowledge in the literature (e.g., Monaghan, 1991; Cottrill et al., 1996; Roh, 2008). However, a limited number of studies have examined prospective mathematics teachers' knowledge for teaching of the concept of limit besides the development of their knowledge. Furthermore, particularly in Turkey, the studies commonly focused on examining prospective mathematics teachers' knowledge of teaching the concept of limit (e.g, Baştürk & Dönmez, 2011a; 2011b; Turan & Erdoğan, 2017; Kula & Bukova-Güzel, 2015; Kula-Ünver & Bukova-Güzel, 2019). Accordingly, there is a lack of studies in the development of knowledge for teaching the concept of limit. Therefore, there exists a gap in the ways in how prospective mathematics teachers' mathematical knowledge, which is specialized for teaching mathematics in Turkey, can be developed.

One of the most appropriate ways to develop prospective mathematics teachers' knowledge for teaching the concept of limit can be considered as lesson study. Together, the studies related to lesson study outline that lesson study enables to reveal and improve different kinds of knowledge of teachers when prospective teachers share and discuss different perspectives, pedagogies, and ideas to connect student thinking and mathematics content by means of multiple cycles (Lewis, 2002). Most of the studies worked with middle school prospective mathematics teachers in the context of the development of knowledge for teaching the concepts in middle school (e.g., Teyplo & Moss, 2011; Dudley, 2013; Clivaz & Shuilleabhain, 2017). In addition, some studies revealed how lesson study contributed to the development of knowledge for teaching by means of some observable features of lesson study such as the knowledgeable other and repeated cycles of lesson study (e.g., Lewis, Perry, & Murata, 2009; Lewis, 2016; Akiba et al., 2019). However, considering the new advances in mathematics education require to think on a development model in more flexible ways rather than rote implementation for the development of new pedagogies of teaching mathematics, there is still room to examine what critical elements contribute to knowledge development in relation with the types of knowledge developing at various stages in prospective teacher education.

Considering all the insights and needs outlined by the literature, the current study aims to design a development model to improve prospective secondary mathematics teachers' mathematical knowledge connecting mathematical concepts in a broad sense and to understand the nature and development of their specialized knowledge for teaching the concept of limit. In the current study, the nature and development of prospective mathematics teachers' knowledge for teaching the concept of limit was examined under the frame of the model of Mathematics Teachers' Specialized Knowledge (MTSK) and its sub-domains. Since the model is relatively new in the mathematics teachers' knowledge literature, the studies related to the model in the context of limit (e.g., Montes, Carrillo, & Ribeiro, 2014; Montes & Carrillo, 2015) presented limited theoretical inferences for researchers. For this reason, the relation between limit literature and the sub-domains of the model was presented, which also constructed a basis for data analysis.

2.4.1 The Relation between the Limit Literature and the Model of MTSK

Considering the aim of the study, the model of MTSK needed to be evaluated in the light of the limit literature. As said before, the sub-domains in the model have its own indicators for knowledge of teaching mathematics. These indicators can be considered as general, not concept-specific indicators. Since the concept of limit was worked in the current study, the indicators were dealt with in the context of the limit concept. In this section, the sub-domains of the model were presented in terms of the concept of limit.

In the light of the literature and the examples related to the indicator, the indicators were matched with knowledge for teaching the concept of limit. Particularly, it was focused on knowledge deemed necessary in terms of difficulties in the learning and teaching process of the concept of limit for both prospective teachers and students. In addition to the literature, the requirements of the curriculum (MoNE, 2018) are also considered. In sum, the concept-based indicators emerged with the following order: First, the indicators were listed, then related concept-based indicators were revealed in the context of their scope and they were organized in accordance with the standards of the country where the study was carried out so that they were context-based. For instance, the indicator of definitions was dealt with intuitive definition, right-left sided limits and formal definition of the concept of limit which proposed by the literature (e.g., Tall & Vinner, 1981; Monaghan, 1991; Cornu, 1991; Adams, 1999; Beynon & Zollman, 2015; Fernández-Plaza & Simpson 2016; MoNE, 2018). While the literature also served the definition of limits of sequences, the curriculum did not include limits of sequences (MoNE, 2018). Therefore, the definition of limits of sequences are not included in the sub-domain as an indicator in the current study (see Table 2.7). Similar steps were conducted for other indicators in other sub-domains. However, some of the sub-domains could not specify in the context of the limit concept, since they can change according to teachers and the context of their students and teaching environment. For instance, the indicator of "ways pupils (students) interact with the content" in knowledge of structure of mathematics needs to be aware of the ways students follow when dealing with the concept and problems related to the concept, such as being aware of the fact that the student is looking for the limit of the function at a point, while trying to find the value of that point in the function. For this reason, it could not specify any specific concept-based indicator.

Table 2.7 An example for the relation between the sub-domains of MTSK and the concept of limit

The Sub-domain	The indicators	The concept-based indicators
Knowledge of Topics	Definitions	Knowledge of intuitive definition Knowledge of right-left sided limits Knowledge of formal definition of limit Temporal order in formal definition Quantifiers (for all, such that, at least) in formal definition Meanings of epsilon-delta in formal definition Transition from intuitive definition to formal definition

As a result, each sub-domain of the model of MTSK was specified with one or more indicator on the basis of the concept of limit throughout the study (Appendix A) and the analytical framework was constructed as concept-specific indicators. As can be seen in the following section, these concept-specific indicators leaded the developmental process of the prospective mathematics teachers' specialized knowledge. In the following section, the methodology of this process was presented considering the research questions.

CHAPTER 3

METHODOLOGY

The purpose of this study is to understand the nature and development of prospective teachers' specialized knowledge for teaching the concept of limit and to design a development model to improve prospective secondary mathematics teachers' specialized knowledge for teaching in all domains of knowledge in a broad sense in teaching the concept of limit. Considering the aim of the study, I firstly mention the design of the current study, reasons why I prefer lesson study as an experiment for prospective teacher development. Throughout this chapter, I also give information about the context and participants, data sources, planning and implementation procedures of the lesson study process as the experiment, data analysis procedures, trustworthiness of the study, and (de)limitations of the study.

3.1 Research Design: Teaching Experiment Methodology

The primary purpose of the study is to understand the nature and development of secondary school prospective mathematics teachers' knowledge about the concept of limit in a broad sense. The aim of the study requires creating an environment which provides opportunities for prospective mathematics teachers to work on the concept of limit in-depth in collaboration of theory and practice. Therefore, the teaching experiment methodology was adopted to examine the nature and development of prospective mathematics teachers' specialized knowledge for teaching the concept of limit; in particular, the teacher development experiment (TDE) was utilized. In this section, the brief information about teaching experiment methodology and TDE, the reason to utilize TDE, and how to adapt the lesson study in TDE are presented.

The origin of the teaching experiment methodology is based on Piaget’s clinical interview. However, it is more comprehensive, exploratory and dynamic than clinical interview (Steffe & Thompson, 2000; Lamb & Geiger, 2012). Teaching experiment can be defined as a dynamic study style that includes a sequence of teaching episodes in order to achieve the aims of researchers to understand the development of students in various fields in a process in the long term (Cobb & Steffe, 1983; Steffe & Thompson, 2000). The teaching episodes included in the teaching experiment involve recording a knowledgeable researcher and one or more students. The teaching episodes are recorded to construct a basis for retrospective analysis to develop later teaching episodes (Cobb & Steffe, 1983). As a result, experimental teaching is a way of research that is centered on conceptually rich environments designed to reveal the intended and observable development of two or more participants, ranging from a few hours to an academic year (Cobb & Steffe, 1983; Cobb, 2000; Steffe & Thompson, 2000).

While teaching experiment methodology has been originally asserted for examining students’ learning process in a progress in mathematics (Lamb & Geiger, 2012; Cobb, 2000), recently, the teaching experiment methodology is combined with teacher professional development model (Lamb & Geiger, 2012). Among different versions of teaching experiment methodology, in the current study, the teacher development experiment (TDE) (Simon, 2000) was adopted to examine the development of prospective mathematics teachers’ specialized knowledge for teaching the concept of limit.

TDE can be described as a method that includes a set of analyses and intervention cycles to examine and support the development of teachers (Simon, 2000). Similar to teaching experiment methodology, in the TDE, there is a researcher that supports the development in progress. In other words, the TDE deals with a community of practice which includes a group of teachers and a teacher-educator or knowledgeable researcher besides students (Simon, 2000). However, there are some attempts to deal with this community as a group of prospective teachers and a teacher-educator or knowledgeable researcher (e.g., Ulusoy, 2016). For both communities, mathematical and pedagogical development are seemed as interrelated notions in TDE methodology. In this way, pedagogical development and development of mathematical knowledge occur concurrently in the mathematics teachers' development experiment. Since the purpose of the current research is to examine the development of prospective mathematics teachers' specialized knowledge including PCK and MK for teaching the concept of limit in a broad sense, the most appropriate design is teacher development experiment design.

While TDE enables the use of both psychological and social perspectives concurrently, from the methodological point, Simon (2000) classified the TDE according to the purposes of the research in two ways: Analysis of collective development (whole class teaching experiment) and analysis of individual development (case study of individual teachers). Considering the aim of the research, it can be said that the research is placed in "case study of individual teachers" in teacher development experiment design (Simon, 2000, p. 352). The requirements of teacher development case study are described below:

> The case study requires making sense of the social context within which individual development occurs, including courses for teachers and the classroom community of the mathematics class taught by the teacher (Simon, 2000, p. 352).

This definition of teacher development case study includes teachers and the classroom community. As said above, in the current study, the design was adapted to prospective mathematics teachers and classroom community. Considering the main focus of the research which is to understand the nature and development of the prospective mathematics teachers' specialized knowledge for teaching the concept of limit in planning and enacting a lesson from start to finish in detail, the lesson study development model was adapted as a TDE to construct a constructivist environment to develop specialized knowledge of prospective mathematics teachers.

There are some reasons to use TDE with lesson study. From the perspective of the purpose of the study, the rote implementation of lesson study can restrict the researcher in investigation of the aim. TDE can provide flexibility the researcher to reach the intended outcome by testing and revising conjectures through cycles (Cobb, Zhao, & Dean, 2009). In addition, this feature can also enable to test the theoretical model (the model of Mathematics Teachers' Specialized Knowledge). From the theoretical perspective, the major components of TDE (e.g., theoretical underpinning of TDE, case study teaching experiments, constructivist teaching experiment) (Simon, 2000) can closely be associated with lesson study. Considering these facts, the structure of lesson study naturally supports taken-as-shared knowledge since the group of prospective mathematics teachers work collaboratively to reveal a common lesson plan. Furthermore, the phases of lesson study are supported with additional resources (e.g., readings, tasks, and examples) and the researcher's interventions (e.g., probing questions, response to misinterpretations of prospective teachers) to support individual contribution according to prospective mathematics teachers' lack of knowledge. The details added in each phase of lesson study was presented in the next section.

In addition to designing the lesson study process, TDE enabled to test the conjectures related to this process which comprised critical elements from lesson study process to the model of MTSK in the teacher development experiment. These elements can be considered as "a means of specifying theoretically salient features of a learning environment design" (Sandoval, 2014, p. 9). In this way, TDE provided to answer the second research question which asked to reveal theoretical and practical features of the learning environment in the lesson study in relation with learning outcomes.

For this aim, these theoretical and practical features, which were called as critical elements during the study, were predicted before the lesson study process to shape the learning environment. The initial critical elements and initial conjectures were pre-interviews, long enough lesson planning, the order of the development (starting from the knowledge of topics (KoT)), and guided reflections. The lesson study process was designed on these critical elements and accordingly conjectures in phase by phase.

As a result, while the lesson study development model has a structured form to conduct in four phases including investigation, planning, research lesson and reflecting (Lewis, 2002), there is flexibility within each of its phases. Thanks to this flexibility that it provides to the researcher, the lesson study development model was designed as a teaching experiment inside each phase considering the predicted critical elements. In the next section, the details of the adaptation of lesson study and teacher development experiment through phases of lesson study development model are presented.

3.1.1 Lesson Study Development Model

Lesson study is a cycling process that teachers work collaboratively and includes several steps which can occur over a number of weeks for instructional development based on students' learning (Lewis, 2002). In each cycle, there are four phases of studying: (1) investigation, (2) planning, (3) research lesson and (4) reflection (Lewis, 2000; Takahashi, 2005). In these four phases, one phase begins when the other one ends and this is repeated in each cycle. Although it has a construction that looks quite structured as it is, the contents of the stages within themselves can be stretched. Considering the aim of the study and TDE, the content of each phase is modified according to the needs of the lesson study group.

There were two lesson study cycles in which the lesson study aimed to conceptualize the concept of limit in students' minds. Different from the usual lesson study processes, the study was designed as a combination of lesson study and micro-teaching lesson study which can be described as a combination of micro-teaching approach and lesson study (Fernandez, 2010) each of which represented one lesson study cycle in the study. In the studies which combined micro-teaching lesson study and lesson study, the research lessons are generally ordered as research lesson, micro-teaching, and second research lesson as teaching in a real classroom. However, researchers suggest that experiences that offer mutual contexts for prospective teachers to explore educational problems and to engage in reflection and criticize the instruction related problems should be sought (Ball & Cohen, 1999; Putnam & Borko, 2000).

On the other side of the coin, when looked at on the basis of the practical reasons, most of the practice teaching courses in the universities follow the same order for teaching practice of prospective mathematics teachers. However, as my own experiences during the methods and practice teaching courses, the micro-teaching does not fully reflect students' learning for enhancing prospective mathematics teachers' learning and the lesson plan. Furthermore, in most of practicing courses in the last year of the universities, prospective mathematics teachers do not have chance to teach 12^{th} grade of students. Since these students are preparing for the university entrance exam, their teachers are reluctant to have prospective teachers teach them. Therefore, prospective teachers are inexperienced in students' mathematical thinking and levels at this grade level and especially in advanced mathematics subjects such as the concept of limit. Considering that this inexperience is effective in prospective teachers' knowledge development and considering the other reasons mentioned above, the lesson study process was designed as an order of teaching in a real classroom and micro-teaching.

Bearing all those abovementioned in mind, in the first cycle, it was aimed that the prospective teachers would observe and experience pedagogical problems in a real classroom by means of lesson study. Then, in the second cycle, I expected them to reflect their observations and experiences to revise the lesson plans by looking at the higher points to the content. Based on this structure, the lesson study phases were implemented in the same vein (see Figure 3.1).

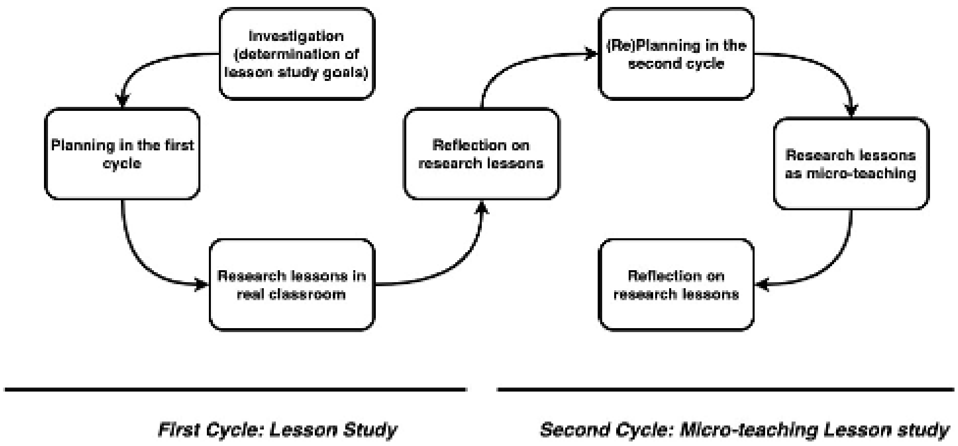

Figure 3.1. The cycling process in lesson study development model

As can be seen in Figure 3.1, the current study is based on the four fundamental phases of lesson study including investigation, planning, research lesson and reflection. In the following four subsections, these phases are presented in detail.

Investigation phase of lesson study

The first phase of lesson study is investigation which includes the lesson study group's determination of a lesson study goal(s) and making an investigation considering the lesson study goal (Stigler & Hiebert, 1999). The lesson study goal requires being aware of problems in educational settings. Being aware of educational problems includes three steps: (1) consideration of interrelated items related to the problem (2) putting forth possible and alternative situations, and (3) establishing the purpose that touches the right points (Ramirez, 2002 as cited in Tan & Caleon, 2015).

In mathematics education, it can be explained as covering students' prerequisite knowledge, critical aspects of the concept that will be taught and determining what is worth learning for students. It is of importance for clear lesson study goals to reflect all these aspects. A mathematics teacher should have the knowledge related to all these aspects for indicating a clear lesson study goal.

In the current study, in the investigation phase, the group determined the lesson study goals. In other words, the lesson study goals are related to a problem about learning and teaching the concept which motivates and triggers the group to work on it (Stigler & Hiebert, 1999). Since the concept of limit has already been a problematic concept for both learning and teaching since the beginning, I aimed to deepen their ideas about problematic issues for the concept. For this reason, they examined the literature, textbooks, curricula and websites related to the topic. They considered the mathematics curricula (from 2005 to 2018), their Calculus course and their own experiences related to this concept. In addition, they considered their experiences in the questions of the pre-interview that was conducted before the lesson study process. Before determining the lesson study goals, I gave them cardboard and stickers and they write the points they deemed necessary to touch on the concept of limit on it.

Figure 3.2. The cardboard that the lesson study group wrote their notes

In this way, it can be understood that they wanted to incorporate the knowledge they saw as incomplete into the lesson study goals. Finally, they determined three lesson study goals that were parallel to the objectives of the curriculum (MoNE, 2018). The lesson study goal can be seen in Table 3.1.

Table 3.1 The Lesson plans and lesson study goals

	Lesson Study Goal	The topics the group want to address	Related objective in the curriculum
Lesson Plan-1	Conceptualization of the concept of limit in students' mind	The intuitive, right-left side limits and formal definition of limit The components of formal definition Historical development of the concept	12.5.1.1. Should be able to explain the concepts of limit of a function at a point, limit on the right side and limit on the left side.
Lesson Plan-2	Applications and mathematical procedures with the concept of limit		12.5.1.2. Should be able to make applications by stating the features of the concept of limit.
Lesson Plan-3	Conceptualization of the concept of continuity, the relation between the other concepts and continuity	The continuity concept The IVT Theorem The relation between derivative and continuity	12.5.1.3. Should be able to explain the continuity of a function at a point.

After they determined the lesson study goals, since brainstorming is one of the problem-solving techniques which trigger the participants' creative thinking with higher order thinking skills, I wanted them to think on and indicate what students think they should know and what they wanted to mention among related mathematical concepts (see Figure 3.3).

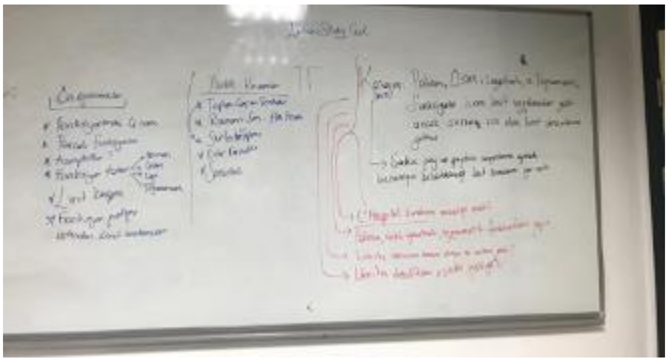

Figure 3.3. The map constructed by the lesson study group

The map shown in Figure 3.3 provided the group to see the big picture and directed them to think on what to look for in investigation. In their investigation, they used the typical resources including the curricula (from 2005 to 2018), the textbooks, their Calculus notes and Calculus books and the textbooks related to the preparation for mathematics teaching field knowledge test. In addition to these typical resources used in lesson study, a comprehensive booklet including literature review and university course notes related to the concept was prepared and given to the group to improve their mathematical knowledge and to help prospective mathematics teachers gain a different viewpoint about teaching the concept of limit. To be more specific, the booklet was prepared in three titles including (1) mathematical knowledge such as definitions, properties, related mathematical concepts, related theorems, its applications, (2) how to teach the concept including literature review, different teaching approaches specific to the concept and (3) how students learn the concept comprised of conceptions, misconceptions, difficulties in learning the concepts of limit.

Planning phase of lesson study

The second phase of lesson study is to make a lesson plan for reaching the goal (Stigler & Hiebert, 1999). In this phase, prospective teachers meet regularly to plan the lesson (Lewis, 2006). Since the aim of this phase is not only to design an effective lesson but also to understand "why and how the lesson works to promote understanding among students" (Stigler & Hiebert, 1999, p. 271), the planning phase has a crucial role in developing the prospective mathematics teachers' knowledge for teaching the concept.

In planning a lesson, teachers have the chance to think about students' expectations and possible actions, to prepare themselves for students' thinking and to develop not only students' mathematical understanding but also their own mathematical knowledge and mathematical thinking (Smith & Stein, 2011). Lesson planning comprises of thinking on the all the aspects of teaching including setting goals for the lesson, formulating appropriate strategies, preparing activities and arranging them with assessment strategies in an appropriate order as well as knowledge for teaching the related topic (Umugiraneza, Bansilal, & North, 2018). Since lesson planning includes some critical issues, such as constructing the teaching process from beginning to end (Rusznyak & Walton, 2011), it can be said that lesson planning is a tool for solving the complex process of teaching (Umugiraneza, Bansilal, & North, 2018).

Considering the requirements of the teacher development experiment, in this phase, the researcher aimed to construct an environment where taken-as-shared knowledge emerged through both individual and social contribution. In this way, the discussions and settings of meetings were guided by the researcher. In the planning phase, the group came together to work collaboratively on the lesson plan and had extensive discussions on both the mathematical background and foreground of the concept of limit and how to teach the concept of limit for reaching the lesson study goal. During the lesson planning process, interim analyses were made regarding the shortcomings of the prospective mathematics teachers and supportive guidance was given accordingly. As a "*knowledgeable other*" (Yoshida & Jackson, 2011, p. 286), I directed the discussions and proposed new resources or new viewpoints. The following figure (see Figure 3.4) showed a short part of the intervention of "*knowledgeable other*".

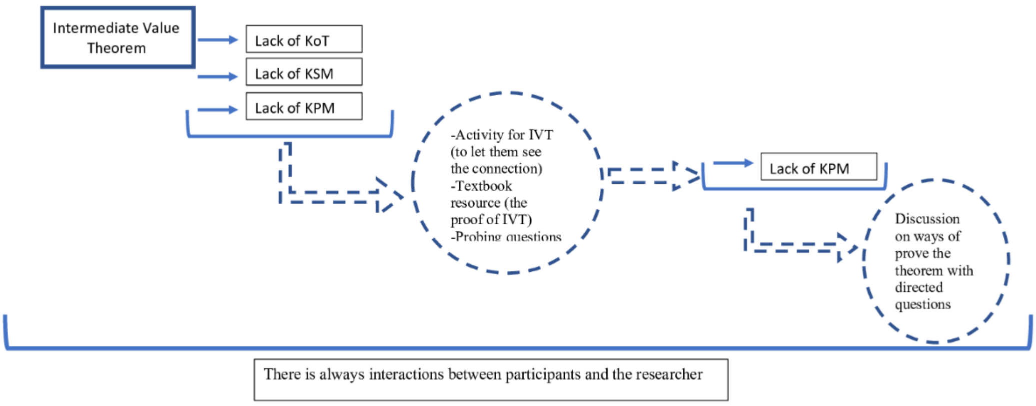

Figure 3.4. A short part of the intervention of "knowledgeable other"

The Figure 3.4 created by the researcher showed an example from the lesson planning process which is related to the intermediate value theorem. The black rectangles show the lack of knowledge of prospective mathematics teachers related to teaching the theorem or using the theorem to answer students' questions, which was determined according to the indicators of the sub-domains. Considering the lack of knowledge, the researcher constructed an environment with rich materials including activity for the theorem, textbook resource, and probing questions during the discussions in planning of the first cycle (the first dotted circle). For instance, it was expected from the prospective mathematics teachers to relate the theorem by finding the roots of polynomials (KSM) and to consider the necessity of continuity for applying the theorem (KoT). However, prospective mathematics teachers had a lack of knowledge on these issues. Therefore, the researcher presented them an activity in which prospective teachers are expected to recognize the assumptions of the theorem considering students' arguments (The Module-6 in the ULTRA project managed by Weber, Wasserman, & Fukawa-Connelly, 2019). After group discussions on both the activity and other resources, there have been some improvements in prospective teachers' lack of knowledge. Then, based on the interim analyses (the second arrow in the figure), the researcher started discussion on ways to prove the theorem with directed questions in the planning of the second cycle (The second dotted circle). By this way, the researcher aimed to overcome their lack of knowledge. In all of the processes shown in Figure 3.4, there is always interaction between the community including the members of the lesson study group and the researcher.

Another feature of TDE is that it includes a set of analyses and intervention cycles to examine and support the development of teachers (Simon, 2000). In the literature, the cycles of lesson study start with the investigation phase and finish with the reflection phase, and it continues in this order (Stigler & Hiebert, 1999; Lewis, 2002). In the current study, the phases, in particular planning phases of lesson study, had its own cycles. The researcher called them "mini-cycles" which included different attempts to overcome the knowledge deficiencies of the prospective teachers based on the interim analyses.

Naturally, these mini-cycles resulted in the planning phase to be longer than the expected time. In other words, to observe the development of the participants' knowledge, the time allocated to the planning phase is one of the important steps during the teaching experiment. For this reason, I kept the planning process long enough for both the first and the second cycle of lesson study. The first lesson planning process took 10 weeks and the second lesson planning process with reflections of the first cycle took 4 weeks.

Research Lesson phase of lesson study

The third phase of lesson study is the research lesson. In the research lesson phase, one of the group members teach the lesson and other group members collect data about the effectiveness of the lesson plan considering the problem determined in the first phase (investigation) by means of observing the lesson and taking notes (Stigler

& Hiebert, 1999; Lewis, Perry, & Hurd, 2009). In terms of TDE, the lesson study process was designed as the combination of lesson study and micro-teaching lesson study, as described above in detail.

In the research lesson phase of the first cycle, one of the members of the group taught the research lesson within two class hours in a school and the other members observed the lesson. To provide the intended learning for prospective teachers, the researcher needs to create the most appropriate classroom environment. Therefore, the classroom in which the prospective teachers teach was observed beforehand by the researcher. In order to create a suitable classroom environment for them, the priority has been to select the class consisting of students who are curious and willing to ask questions. In this way, learning from the students and students' questions was aimed in this phase. In the research lesson phase of the second cycle, the lesson study group implemented their lesson plans in the faculty to their peers. Since the class size is reduced (Fernandez, 2010), prospective teachers get a chance to easily implement their solutions to the problems they experience and observe. In addition, the feedback from peers provides them both to receive feedback on the solution they applied and to look at it from a different perspective.

Reflection phase of lesson study

The last phase of lesson study is the reflection phase. This phase is included in the meeting after the lesson is taught where the group evaluates the research lesson, and shares and discusses their ideas about the effectiveness of the research lesson (Lewis, Perry, & Hurd, 2009). To improve prospective teachers' knowledge for teaching the concept of limit, the reflection phase was extended by writing a reflection paper in addition to the activities in the reflection phase asserted by the literature. In the reflection phase, the group wrote a reflection providing feedback on the lesson with questions about the effectiveness of the lesson plan, whether it worked or not, whether there was a difference between the implemented lesson and the lesson plan.

In order to enable the group to reflect their knowledge on the reflection papers in a clear way, together with their observation, the recorded video was delivered to the prospective teacher to watch both others' teachings and her own teaching repeatedly to look at what they did in the classroom, and think about why they did it and if it worked - a process of self-observation and self-evaluation.

In addition to the evaluation of the lesson study group, the knowledgeable other gives final comments to the group at the end of their evaluation (Stigler & Hiebert, 1999). Since these final comments provide prospective teachers' growth in terms of knowledge for teaching mathematics (Seino & Foster, 2021), the researcher provided final comments in two researchers (a mathematician for mathematical context and a mathematics education researcher for mathematics education) in the first cycle. In the second cycle, there was only the researcher as the knowledgeable other for final comments.

Based on reflections of the group, they discussed the lesson plan to reach a consensus on how to revise the lesson plan. The second cycle started with a re-planning phase which also included the investigation phase again and it continued in the same order of the phases whose details are presented above. In the next section, detailed information will be given on how the lesson study development model took place in a context and with the participants.

3.2 The Context and Participants

The aim of this study is to understand the nature and development of secondary school prospective mathematics teachers' knowledge about the concept of limit through a teaching experiment designed within a lesson study development model. Considering the aim of the study and the fact that the concept of and limit was covered in the spring term according to the curriculum, the current study was conducted in the Spring term of 2018/2019 with a group of senior prospective mathematics teachers. In this way, the study was carried out in the context of lesson study with a lesson study group which consisted of a group of prospective secondary mathematics teachers. This section is structured in two main sections including the participant of the lesson study group and the context of the lesson study.

3.2.1 The Participant-Mila

Lesson study requires a group of prospective mathematics teachers which consists of 3-6 members (Stigler & Hiebert, 1999). In the current study, the lesson study group consisted of 3 senior prospective mathematics teachers. The researcher preferred to work with senior prospective mathematics teachers for some reasons. In particular, in Turkey, the senior prospective teachers should have taken some courses including pure mathematical courses and required educational courses (e.g., Methods of Teaching Mathematics I-II, and School Experience) until they get to the last year (see Appendix B). Since the lesson study process includes researching, lesson planning and making critical decisions about the workings of a lesson plan in the classroom, participants were supposed to be able to do these actions. Therefore, these courses provided them to be able to do the expected actions.

In the semester the study conducted, the participants were selected based on some certain characteristics with a purposive sampling method among non-probability of sampling methods. As described above, lesson study provides a lesson study group a way to learn from each other, knowledgeable other and students' learning (Stigler & Hiebert, 1999; Lewis, 2002). Therefore, it was aimed to build a variation in knowledge and experience among the participants. In other words, the researcher paid attention to whether the participants had different perspectives and different levels of knowledge for teaching the concept of limit. In the term of 2018-2019/Spring, there were eight prospective mathematics teachers who were enrolled in the mathematics education program in a university in Ankara. Considering the pre-interviews implemented with these eight prospective mathematics teachers, four prospective mathematics teachers were offered to participate in the study. However, only three of them volunteered to participate in the study.

The lesson study group consisted of these three prospective mathematics teachers. As mentioned before, the prospective mathematics teachers have completed some courses before the lesson study process. Since the concept of limit is the subject chosen to examine prospective teachers' knowledge development, it was found useful to present participants' grades of some educational courses and pure mathematical courses that involve the content about learning and teaching the topics of Calculus. Table 3.2 shows detailed information about their academic backgrounds.

Table 3.2 The Academic backgrounds of the members of lesson study group

	Calculus I	Calculus II	Advanced Calculus I	Advanced Calculus II	Teaching Methods of Maths I	Teaching Methods of Maths II	Cum. GPA
Alp	CC	DD	DD (2times)	AA	AA	AA	2.99
Fulya	DD	CB (2times)	DD (2times)	BA (2times)	BA	BA	2.57
Mila	CB	CC	DD	CC	AA	BA	2.88

According to the table, prospective teachers completed the required courses of Calculus I-II, Advanced Calculus I-II, Method of Teaching Mathematics I, and Method of Teaching Mathematics II. In the table, I mentioned their number of times in taking the related courses. In the university where the study was carried out, the participants had a chance to take the course from different instructors. It was of importance for the variation in their perspectives about mathematical courses.

The purpose of the study required in-depth analyses to reveal the development of specialized knowledge for teaching the concept of limit and accordingly to design a development model in relation with the development of all domains of knowledge in a broad sense in teaching the concept of limit. For this reason, the current study focused on the lesson study process of one of the prospective mathematics teachers, Mila, who was chosen purposefully. The reasons for this selection and the details about Mila are presented below.

Mila, a 22-year-old female, graduated from an Anatolian High School which was classified as a qualified school in Turkey. With the highest GPA among the participants, Mila was also a well-motivated prospective secondary mathematics teacher. The participant, who took the university exam again to get into the mathematics teaching program, expressed her willingness to teaching as "*My parents didn't want me to be a teacher, but I loved teaching someone math and I wanted to be a teacher*". In addition, she wanted to develop herself in terms of mathematical concepts, teaching mathematics and gaining an identity as a mathematics teacher. Therefore, she showed her interest to participate in the lesson study process to develop herself, since she did not experience such a training about both teaching and learning mathematics different from the undergraduate courses. Since the lesson study process includes lots of discussions on a mathematical concept and how to teach it to keep students learning at the highest level, the participants must be active and like to participate in discussions. Mila ensured this condition with her frankness, talkativeness, and the capacity to express her misunderstanding and to have her friends say the things that she did not agree with when necessary. In addition, she had not taught a lesson about the concept of limit and she had never prepared a lesson plan on this subject before. To get more detailed information about Mila, I conducted an interview with her. Table 3.3 shows the detailed information about Mila.

Table 3.3 The detailed information about Mila

Some questions about Mila	The detailed answers of Mila
Elective courses that she took before the study	Hyperbolic Geometry Partial Differential Equations Number theory The history of mathematics Cryptography Introduction to Mathematical Modelling and Logic Problem solving in mathematics education Research methods for prospective teachers
Her perspective about the knowledge that a secondary school mathematics teacher must have	Knowledge about how to prepare a good exam Knowledge of the methods for teaching mathematics Knowledge of how materials are used Must be a good server Mathematical knowledge which students should learn
Her thoughts about a course utilized explicitly in teaching	Calculus-I-II Fundamentals of mathematics

Table 3.3 presents information about Mila's elective courses, her perspective about the knowledge that a secondary school mathematics teacher must have and her thoughts about a course utilized explicitly in teaching. One of the important pieces of information about Mila is that she took only one elective course (problem solving in mathematics) related to mathematics teaching and learning. Most of her elective courses belong to mathematics itself. Her perspective about the knowledge that a

secondary school mathematics teacher should have focused on the pedagogy of a mathematics teacher. Lastly, she connected only three mathematics courses with the secondary school curriculum, though she had taken more mathematics courses.

3.2.2 The Context of the Lesson Study

Lesson study process as TDE was performed in the term of 2018-2019/Spring with senior prospective mathematics teachers in two different settings including the context where the meetings of lesson study occurred and the context where the research lesson of the first cycle was conducted. By the time the prospective teachers reach their final year, they had completed almost all mathematics and mathematics education courses and were ready for practice teaching. In addition, the concept of limit is included in the 12th grades' analysis subject area in the curriculum. According to the curriculum, 12th graders are required to learn about this subject within the first month of the 2nd semester.

The main part of the study was conducted with the lesson study group which included three senior prospective mathematics teachers and the researcher in the mathematics laboratory of the faculty where the regular meetings occurred. The mathematics laboratory was chosen purposefully, since it had a white board, a projector and materials that should be in any mathematics laboratory. The seating arrangement is in the form of students sitting in U-shaped desks facing each other. In regular meetings, the prospective teachers were introduced to use any of the materials in the laboratory for lesson plans. The prospective teachers usually used smartboard to show their preparations, graphics and other demonstrations, and presentations. Before the research lessons, they had the chance to rehearse for research lessons.

In the context of this study, there were two lesson study cycles, three lesson plans and three participants. For this reason, each participant could teach two lesson plans. Table 3.4 shows the list of each participant's taught lessons.

Table 3.4 The lesson plans taught by prospective mathematics teachers

# of Lesson Plan	The Lesson Study Goal	Cycle-1 (LS- Real Classroom)	Cycle-2 (MLS- Micro-teaching)
Lesson Plan-1	Conceptualization of the concept of limit in students' mind	Alp	Fulya
Lesson Plan-2	Applications and mathematical procedures with the concept of limit	Fulya	Mila
Lesson Plan-3	Conceptualization of the concept of continuity, the relation between the other concepts and continuity	Mila	Alp

The research lesson in the first cycle of lesson study was conducted in an Anatolian High School in Ankara, Turkey. The Anatolian High School can be described to be at a standard level in terms of the high school averages in Turkey. The students' ages were between 17-18. The classroom environment in which the prospective mathematics teachers taught their lessons was designed by the prospective mathematics teacher who was selected to carry out the research lesson to support students' learning during instruction. It should be mentioned that the prospective mathematics teachers tried to remain faithful to the lesson plan while designing the learning environment. The class had a white board and a smart board. Thus, the prospective mathematics teachers were able to present the activity sheets on the smartboard and the students could directly work on the activity presented on it. In addition, the prospective mathematics teachers distributed activity sheets, since the students did not want to take any notes during the lessons. The seating was arranged in a way that the students sat in groups of two, lined up one after the other, facing the board (the smart board and the blackboard were integrated to each other). The second research lesson study was conducted in the mathematics laboratory in the faculty of education. The mathematics laboratory had a white board, a projector and materials. They taught the lessons to their classmates who were in their last year of university.

3.2.2.1 Researcher Role

Lesson study is a collaboration of 3-6 teachers or prospective teachers with a facilitator (or knowledgeable other) to conduct the process from determining lesson goal to research lesson and reflection (Lewis, 2002). In this structure, the researcher is facilitator/knowledgeable other (Yoshida & Jackson, 2011). This means that the researcher actively participated in all of the data collection processes.

In other words, I had two roles throughout the study: Since I had an active role in lesson study process, it can be said that I was a participant-observer during collecting data. The role of participant observer involves the researcher working as an active member of the working group, on an equal status with the participants (Fine, 2001). This role provided me to understand the participant's contribution to the lesson study process clearly. In addition, I had a role who designed the process and supported the members of lesson study group for nurturing their specialized knowledge for teaching. For this reason, I had opportunities to observe and intervene in situations that cannot be carried out during the process or that do not go as planned.

3.3 Data Collection

The study has two main research questions including (1) How do preservice secondary mathematics teachers develop their specialized knowledge in the concept of limit in the lesson study development model? (2) How well can the critical elements of lesson study be regulated so that they become an integral part of a logical chain to improve prospective secondary mathematics teachers' specialized knowledge in the concept of limit? The data collection tools including pre- and post-interviews, observation of lesson study process, reflection papers, lesson plans, and field notes were selected to answer these research questions. The table given below showed the relation between the research questions and the data collection tools.

Table 3.5 The relation between the research questions and the data collection tools

The Research Question	Data Collection Tools
How do preservice secondary mathematics teachers develop their specialized knowledge in the concept of limit in the lesson study development model?	Pre- and post- interviews Observation of lesson study process Reflection papers Initial and revised lesson plans Field notes
How well can the critical elements of lesson study be regulated so that they become an integral part of a logical chain to improve prospective secondary mathematics teachers' specialized knowledge in the concept of limit?	Observation of lesson study process Field notes Pre-interviews Reflection papers

Data collection took four months and included two lesson study cycles which consist of preparation of three lesson plans, three teaching sessions in the 12th grade classrooms and three re-teaching as the micro-teaching sessions, and three reflection processes on these lessons in total. The summary of the data collection process was shown in the table below.

Table 3.6 Summary of the data collection process

		Cycle-1	Cycle-2	
Data Collection Tools	Clinical Individual Pre-interview	Discussions during the lesson planning phases (Research and Planning)	Discussions during the lesson planning phases (Re-research and Re-planning)	Clinical Individual Post-interview
		Teaching in the 12^{th} grade classrooms	Re-teaching (Micro-teaching)	
		Reflection Process	Reflection Process for revised lesson plans	

In this section, the data collection tools from pre- and post- interviews to field notes are presented in detail.

3.3.1 Individual Clinical Pre-and Post-Interview

As it was described in the "social constructivist theory" section, the roots of clinical interviews have been based on Jean Piaget's method of Clinique on child knowledge development (Hunting, 1997). Clinical interviews have been used for various aims including assessment tool, a tool to help students improve their knowledge and research tool (Clement, 2000; McConaughy, 2013). The clinical interview process is flexible in terms of participants' answers and researcher's questions. The researcher asks additional questions according to the answers given by the participant, allowing the participant's views and thoughts to be examined in depth (Jacobs & Empson, 2016). In mathematics education, clinical interviews are one of the assessment methods used to improve students' mathematics learning (Ginsburg, Jacobs, & Lopez, 1993; Hunting, 1997). As a research tool, clinical interviews provide a way for the researcher to investigate students' mathematical understanding thoroughly (Clement, 2000). In this study, individual clinical pre and post-interviews were conducted based on three purposes as (i) to understand the prospective teachers' existing and final state of knowledge for teaching the concept of limit (ii) to prepare the lesson study process and (iii) to observe the development in prospective teachers' knowledge for teaching the concept of limit.

3.3.1.1 Individual Clinical Pre-Interview

The first data collection tool of the whole process is the individual clinical pre-interview. The main aim of the individual clinical pre-interview was to understand the prospective teacher's existing knowledge for teaching the concept of limit before the lesson study process which was used to answer the first research question. In addition, the individual clinical pre-interview was used to prepare her for the lesson study process which made her aware of her lack of knowledge. In this way, it served to answer the second research question with the role of one of the critical elements of the lesson study.

As a part of TDE, the preparation and conducting it were the process that required a long study and revise or rewrite the questions in order to comply with the theoretical model. Therefore, this section was presented in two titles including *preparing individual clinical pre-interview* and *conducting individual clinical pre-interview*, described in detail below.

Preparation individual clinical pre-interview: The first step for the preparation of the pre-interview was to review the literature about teaching and learning the concept of limit. The questions in the literature regardless of the type (open-ended, multiple choice, etc.) and the educational level of the individual (high school, university, teacher candidate, etc.) were collected in a question pool. Later, the questions were categorized considering the model of MTSK. Most of the questions in the literature handled the knowledge of the concept of infinity related to participants' concept image and concept definitions (e.g., Davis & Vinner, 1986; Tall & Vinner, 1981; Cornu, 2001; Mastorides & Zachariades, 2004, Stewart, 2012). Therefore, these questions were regulated for revealing existing knowledge for teaching the concept of limit. In addition, the questions which were not included in the literature but thought to be included in the study were added by the researcher. Furthermore, I prepared some probing questions based on the expected answers of the prospective mathematics teachers for each question. For instance, at the beginning of the pre-interview, the questions were related to the definition of the concept of limit. However, it was not a sufficient knowledge for teaching it. Therefore, I added some probing questions such as "How do you teach it to your students?", "Let's think that I'm your student and I asked what these terms mean mathematically?". Finally, the questions in the question pool were selected and sorted according to the components of the MTSK.

Conducting individual clinical pre-interview: The pre-interview was conducted by ensuring the flexibility for both the researcher to ask probing questions according to prospective teachers' responses and for prospective teachers to express themselves and their lack of knowledge in a comfortable way, as suggested in the literature. A pilot interview was held with a prospective mathematics teacher different from the participants. Then, some probing questions were modified. After that, the interview was held with a prospective mathematics teacher different from the first prospective mathematics teacher to test the new questions and finalize the protocol (see Appendix C).

The pre-interview of the study was conducted before the lesson study process by the researcher and was recorded by an audio and video recorder. The pre-interviews were conducted alone with the researcher in the mathematics laboratory in the faculty building (see Figure 3.4). In the mathematics laboratory, the participant and the researcher sat face to face in which there was a desk between them. The camera is positioned to see the participant's paper, but not to obstruct her, with a tripod assembled to see the paper from the top. The participant had a pre-interview paper in front of her with additional blank papers. The blank papers were given to the participant to scribble something if she felt necessary. In addition, the mathematics laboratory had educational materials, computers, a smartboard and a blackboard. The participant was informed that she could use any of these materials if she wanted to answer questions or to show something about the questions. For this reason, I chose the mathematics laboratory to enhance them for using materials if they needed to use.

Figure 3.5. The photo of clinical interview room

3.3.1.2 Individual Clinical Post-Interview

As said before, the main aim of the clinical post-interview was to support the data gathered from the observations of the lesson study process in order to ensure the final knowledge of the prospective teacher. In this way, the individual clinical interview served to answer the first research question for revealing the development of specialized knowledge for teaching the concept of limit. While it was not directly as a part of the TDE in the current study, it provided an assessment tool for both the development and the lesson study process. Therefore, the preparation and conducting it were of importance for the findings of the study. The section of individual clinical post-interview was presented in two titles, similar to individual clinical pre-interview.

Preparing individual clinical post-interview: The individual clinical post-interview was one of the examination tools for observing the development of prospective mathematics teachers' specialized knowledge for teaching the concept of limit. For this reason, the post-interview was constructed considering the observation of lack of knowledge and knowledge development during the lesson study process. The question was constructed by the researcher according to the participants' answers of pre-interview and participants' knowledge development during the lesson study process. As it will be mentioned in the data analysis section, data analysis informally started during the data collection with the aim of observation of the development. The analysis during the data collection in the lesson study process of both pre-interview and lesson study shed light on the construction of questions. Contrary to the pre-interview, the post-interview could not be conducted by a prospective mathematics teacher different from the lesson study group. However, I asked an expert the validity of the questions. Considering the suggestions of the expert, some questions were added to the post-interview. As a result, the post-interview was finalized to do with the participants (Appendix D).

Conducting individual clinical post-interview: The post-interviews were conducted after the last phase of the second lesson study cycle. The post-interviews were conducted in the same place and in the same context with the pre-interviews and were recorded by an audio and video recorder.

3.3.2 Observation of Lesson Study Process

Observation is a method that enables direct "meaningful" tracking of what is happening in order to examine an event in depth and look at it from different frameworks (Merriam, 2009). The observation method focuses on examining the situations in the natural environment, examining and interpreting the things that the participants can speak comfortably and reveal their knowledge in the process. Therefore, it is one of the oldest and most widely used methods in qualitative research. (Patton, 2002). In this study, observation can be described as the main data collection tool.

Observation of the lesson study served both of the two main research questions. For the first research question, the data gathered from the lesson study process provided an understanding of the setting and context of the lesson study process, change in prospective mathematics teachers' behaviors, knowledge and discourses in a broader sense. For the second research, the data gathered from the observations of lesson study provided the researcher to reveal the critical elements of lesson study which contributed to the participant's development of specialized knowledge for teaching the concept of limit during the process.

To be more detailed, considering that each lesson study cycle included four phases, observations of these phases provided different perspectives of data for the study. The data gathered from observation of determination of lesson study goals was used to understand their knowledge of students, awareness of the difficulties of the concept for them and for students, the conceptual knowledge they saw as incomplete knowledge of curriculum. The data gathered from observation of planning phases included group discussions and each group member's contributions to the lesson plans. By this way, the data showed both ways of the development of knowledge of prospective mathematics teachers and the development of knowledge of prospective mathematics teachers. The data gathered from the research lesson phases was used to examine how they used their knowledge in a real classroom environment and how they responded to students' instant questions. The data obtained from reflection phases was explained in the latter section.

3.3.3 Reflection Papers

In addition to clinical interviews and observations, there were some documentary data collection tools in this study. The first of these documents is the reflection papers. After the research lessons, participants were asked to write a reflection paper for each research lesson. In addition to their observation of the research lesson, the video clips of the research lessons were delivered to them. By this way, they had a chance to watch the research lessons by pausing where they wanted and watching them again and again. The reflection papers provided the researcher to observe how the participant reflected her specialized knowledge by thinking and evaluating the research lessons, which served to answer the first research question with the role of one of the critical elements of the lesson study.

There were two versions of the guidance for the reflection paper which were prepared by the researcher. The first of the guidance was prepared for the prospective mathematics teacher implementing the lesson plan. The second version of the guidance was for the prospective mathematics teachers observing the research lessons. The general theme of the guidance for two versions of the reflection paper includes questions for students' learning, their friend's teaching performance and the effectiveness of the lesson plan. The guided questions of the reflection paper can be seen in Table 3.7.

Table 3.7 The guided questions of the reflection papers

The guidance for the prospective mathematics teacher implementing the lesson plan	The guidance for the prospective mathematics teachers observing the research lessons
Can you share your ideas about the implementation phase of this plan with the following questions? • Which activities / questions in the lesson plan worked for the purpose of the lesson? • Which activities / questions in the lesson plan did not work for the purpose of the lesson / not for student learning? • What difficulties did you encounter as a teacher conducting the lesson? How could these challenges be overcome? • Was there any difference between your real lesson and lesson plan? Can you explain? • When preparing this lesson plan, did you foresee that this would happen? Could you write down your suggestions and justifications for developing this lesson plan?	Can you share your comments about the implementation phase of this plan with the following questions? • According to your observations, which activities / questions in the lesson plan worked for the purpose of the lesson? • According to your observations, which activities / questions in the lesson plan did not work for the purpose of the lesson / not aimed at student learning? • In your opinion, what difficulties did your friend who practiced the lesson encounter? If you were to teach this lesson, how would you overcome these difficulties? What do you think should be changed in this lesson plan and can you write the reasons for these changes as well?

3.3.4 Initial and Revised lesson Plans

Another documentary data is the initial and revised lesson plans which can be considered as the products of the lesson study process. They were also products for examining prospective mathematics teachers' change in specialized knowledge for teaching. Particularly, the data gathered from lesson plans provide one of the ways to consider prospective mathematics teachers' knowledge about teaching materials and teaching strategies, their ways of choosing appropriate materials, assessment techniques and their consideration of students' needs. The initial and revised lesson plans were used to answer the first research question which examined the development of specialized knowledge for teaching the concept of limit.

In the current study, the lesson plan format was given to the prospective mathematics teachers and it was explained in detail at the introductory meeting. The lesson plan format includes questions that guide prospective teachers and has been prepared in such a way that they can freely design lessons. The prospective mathematics teachers prepared two lesson plans for each lesson study goals. The lesson plans were used to observe the change/development in their instructional strategies and knowledge from the first cycle to the second cycle. The group prepared their lesson plans according to the lesson plan template which were given by the researcher.

3.3.5 Field Notes

The last documentary data is field notes, which can be described as a diary of the researcher which includes what the researcher hears, observes, thinks and experiences during the data collection (Fraenkel, Wallen, & Hyun, 2012). The researcher wrote the field notes during the data collection process for each meeting. The field notes were used as a tool which supported or detailed the data gathered from other data collection tools for both of the two research questions.

In this study, two types of field notes including descriptive and reflective ones (Fraenkel, Wallen, & Hyun, 2012) was combined. By means of descriptive field notes, the researcher took notes about the depiction of activities—a detailed description of what happened, along with the order happened during the lesson study. Likewise, reflective field notes provided me a way to take notes related to reflections on analysis—what prospective mathematics teachers are learning, ideas that are developing, patterns or connections seen, and so on. Both of these two types of field notes served the aim of the study. In particular, descriptive field notes supported the data in order to answer the first research question. These notes provided me a way to not to miss any points related to the contribution of the participant to the lesson study process. In addition, reflective field notes supported the data to answer the second research which was related to the mechanism of the lesson study development model. The field notes covered the two cycles of the lesson study process. A form was prepared for the field notes as can be seen in Table 3.8.

Table 3.8 The example column of the filed notes

Weeks	What I observed
1st week	- Prospective teachers were informed about the lesson study and the way it was conducted. A presentation on this (presentation-1) was made and a video was watched. Later, information on how a lesson plan is prepared and what points should be taken into account when preparing a lesson plan for this study were explained (presentation-2). The students were asked to research on the following until the next meeting: · How do we teach the concept of limit considering its relation to other concepts? · How do we teach the limit conceptually? · How do we make use of the history of the limit? · How did they learn the limit concept during the courses they took from the Mathematics Department? They were asked to do research focusing on these questions. The document on the concept of limit prepared by the researcher was shared with the students through 'drive' on 24, February, 2019. This document was prepared based on reviewing the literature by the researcher. The students were given a reference list under each document.

3.4 Data Analysis

In this section, the detailed description of data analysis related to the findings presented in Chapter 4 was given. Data analysis started during the data collection with the aim of discussing any clarifications needed for the participants during the process. In addition, the lesson study process as a teaching experiment was designed based on the points that the participants showed improvement or not. As Stake (1995) indicated "There is no particular moment when data analysis begins. Analysis is a matter of giving meaning to first impressions as well as to final compilations" (p. 71). In order to systematize the analysis process, I analyzed the data following the data collection process.

3.4.1 Analysis of Pre- Post-Interviews

As mentioned in the data collection section, one of the data collection tools were clinical and task-based interviews. One of the appropriate methods for coding the data gathered from the interview were thematic coding (Miles, Huberman & Saldana, 2014) which helps researchers to find out something regarding people's views, opinions, knowledge, experiences or values from a set of qualitative data – for example, interview transcripts, social media profiles, or survey responses. To attain this goal, there were two important requirements: Familiarization of data and familiarization of the related literature. After transcribing all the data gathered from the interview, first, I reviewed the literature related to learning and teaching the concept of infinity examining national and international theoretical models on knowledge for teaching mathematics as well as MTSK. Considering the requirements of thematic coding, it was expected to find what the researcher was looking for. For this purpose, the "pre-codes" and themes (the themes were determined according to the model of MTSK) were determined. These "pre-codes" included the codes proposed by the model and their related descriptions which were determined according to the literature. Table 3.9 showed an example of pre-codes, description, categories, themes.

Table 3.9 The example of pre-codes, description, categories, themes

Theme	Category	Pre-Codes	Description	References
Mathematical Knowledge	Knowledge of topics	Knowledge of Definition	The intuitive definition Right-left sides of definition The formal definition and its ingredients	(Davis & Vinner, 1986) (Cornu, 1991) (Monaghan, 1991) (Moru, 2009)

After the description was provided, it was the time to determine which answers given by students are incomplete, incorrect or at the desired level. The purpose in so-doing was to analyze the prospective teacher's existing knowledge to design the process based on that. Accordingly, all the data were examined according to the pre-codes and categories, and this examination was compared with the notes that I took during the data collection process. After examining how such answers were evaluated in the studies in the literature, the answers were started to be coded as (1) at the desired level, (2) at the desired level but not sufficient for knowledge of teaching, and (3) not at the desired level. Table 3.10 shows an example for this categorization.

Table 3.10 The example categorization for level of knowledge of teaching

Knowledge of formal definition of limit of a function	
Level of knowledge for teaching	Description for related level
Existing Sufficient	Be able to write the definition in correct way and answer the question related to formal definition

Table 3.10 (continued)

Existing but not sufficient	Be able to write the definition in correct way however do not be able to answer the question related to formal definition
Not existing (Not at the desired level)	Do not be able to write the formal definition

By this way, the verbal and written responses in both interviews were utilized to reveal the participants' existing knowledge before participating in the lesson study process and final knowledge which was the product of the process that showed their improvement.

3.4.2 Analysis of the data gathered from lesson study process

Apart from the pre- and post- interviews, other data collection tools emerged during the lesson study process. Therefore, they dealt with the data gathered from the lesson study process in this section. The aim of analysis of the data gathered from the lesson study process was to understand both the development of knowledge of teaching the limit concept and which elements of the mechanism of lesson study provided the participant to develop her knowledge, if there was any development. Therefore, one of the aims of this analysis was to answer the first research question in terms of whether and how the prospective mathematics teacher's knowledge developed for teaching the concept of limit in a broad sense through lesson study. Therefore, the data analysis looked for revealing both whether the prospective teacher had the knowledge and whether an improvement was observed for the first research question. Besides, another aim of this analysis was to reveal an answer for the second research question by means of looking for the critical elements which provided development in the prospective mathematics teacher's knowledge for teaching the concept of limit in a broad sense.

After the data collection process ended, there was a large set of data. Data analysis focused to provide a detailed description of the mathematical knowledge understandings developed by the prospective mathematics teacher throughout the lesson study process. Therefore, naturalistic inquiry was adopted (Lincoln & Guba, 1985) for data analysis. In particular, the multiple level analysis approach was conducted for the data analysis. Before analyzing the data, the videos recorded during the lesson study sessions were divided into four parts including planning (the first two phases of the first cycle of lesson study), teaching (the third phase of the first cycle of lesson study), reflection and re-planning (the fourth phase of the first cycle and the first two phases of the second cycle of lesson study) and micro-teaching (the third phase of the second cycle of lesson study) to understand the development of specialized knowledge of participants. Then, all the videos recorded during the lesson study with their transcriptions were watched with taking screenshots of participants' notes and procedures on the videos. In addition, the specific field notes was taken for each video. After that, the researcher added them to the related places on the transcriptions as memos in MAXQDA. As I said before, to understand the development of specialized knowledge of participants, I read all the transcriptions of the videos and separated the excerpts to make the coding process easier. At this time, I started to analyze the data. First, the data was considered as a whole with a single code "prospective teacher revealed her knowledge here". Thus, considering the main requirement to possess mathematical knowledge, the data was discussed via holistic coding (Miles, Huberman & Saldana, 2014).

In the second level, I enlarged the coded segments in the first level on the indicators of the subdomains of the model with provisional codes which can be revised, modified, or expanded to new codes (Miles, Huberman & Saldana, 2014, p. 83). The first draft of the codebook was constructed by conducting two stages simultaneously. The indicators of the sub-domains of MTSK (Carrillo-Yañez et al., 2018) and the literature about the concepts of limit was used to construct the codebook in a similar way through analyzing the pre- and post- interviews.

Figure 3.6 The demonstration of the data analysis phases

In the second phase of analysis of the data gathered from the lesson study process, I constructed the first draft of the codebook by conducting two stages simultaneously. I used the indicators of the sub-domains of MTSK (Carrillo-Yañez et al., 2018) and the literature about the concepts of limit to construct the codebook in a similar way by analyzing the pre- and post- interviews. After a piece of excerpts was coded initially, I changed some codes in the code list. Finally, the codebook for the pre-post interview has been constructed. Since the codebook was derived from the model of MTSK considering all knowledge dimensions, all codes were used to analyze the data collected for this study without the need for revision again. Each sub-domain of MTSK was considered as a separate theme and the corresponding indicators were treated as codes which were determined through the explanatory specifications regarding each theme.

In the last phase, I tried to answer both of the two research questions. For the first research question, there have been two important points. The first important point of the data analysis was to determine whether or not the participants possess the knowledge. In the light of the literature, in order to observe the individual development of the participants, I took into account their individual contributions in the lesson study process and their performance in the group. Accordingly, for instance, the knowledge that provided the necessary and sufficient condition and effectively reflected this in the lesson plan showed that they possessed this knowledge. One of the examples from the coding example can be seen in the following table (Table 3.11).

Table 3.11 One of the examples from coding

Example Excerpt	Example Coding
M: For instance, $\frac{0}{0}$ is equal to x. Cross multiply it! Then, it can be any real number. For this reason, it is described as an indeterminate form. R: What do you (Alp and Fulya) think about this answer? F: I'm not sure, I may say it as "undefined". A: Where did you get this information? Do all indeterminate forms come out of here? F: It's probably the same for all. M: I found it on the Internet. …	Lack of knowledge of how and why to do something (the case of indeterminate-undefined forms)

Another important and difficult point in data analysis was to reveal the development of the knowledge of participants through lesson study. In order to reveal this claim, which is the main purpose of the study, the pre-interview, lesson study process and the post-interview were evaluated as a whole. Accordingly, it was mentioned that there is an improvement in the preliminary interview and when it was observed that the missing information in the lesson study process developed in the lesson study process and in the last interview. The table below was used as a determinant to demonstrate it. As will be seen in Findings chapter, I used some abbreviations for demonstrating the development including AD (Adequate level of development), NAD (Not Adequate Level of Development or Not Development), AE (Already Existing) and NA (Not Observed). Accordingly, it was argued that there was improvement in those who returned from NE (not existent) or NAD to AD (see Table 3.12).

Table 3.12 The example determinant for development of specialized knowledge for teaching

The categories of KoT	Indicators for the Category	What was seen as an improvement in the prospective mathematics teacher's knowledge of limit	
		From (NE or NAD)	To (AD)
Definitions, properties and its foundations	Knowledge of definition of limit	Expressing the definitions of limit verbally and algebraic sufficiently or insufficiently	Knowing the meaning of quantifiers in the definition

The transition from NAD to AD showed a development, revealing the findings for the first research question. The same analysis also reveals the findings for the second research question. The factors affecting the transition from NAD to AD were the answers to the second research question.

3.5 Trustworthiness of the Study

Trustworthiness of the study is related to the cogency of the research about collecting and analyzing data (Lincoln & Guba, 1985). Considering the elaboration notions of trustworthiness including credibility, transferability, dependability and confirmability, the current study used multiple methods to enhance trustworthiness of the study. In this section, I will present how trustworthiness was ensured while gathering and analyzing the data in these notions.

The first notion is credibility which is related to internal validity of the qualitative study (Lincoln & Guba, 1985; Merriam, 2009). The credibility of the study wants to answer "how the findings are congruent with reality" (Merriam, 2009, p. 213). Credibility demonstrates whether research findings are indeed presented with a correct interpretation of the analysis of data from the original participants and the original data (Korstjens & Moser, 2018). In the current study, three types of technique were used to ensure the credibility of the study. The first technique is prolonged engagement in the study. Issues such as being familiar with the data and interpreting it correctly in this way and feeling comfortable in the answers and discourses that the participants will use are related to prolonged engagement (Fraenkel, Wallen, & Hyun, 2012). Prolonged engagement requires spending extended time with participants in their educational life to ensure an accurate understanding of their behaviors and discourses (Merriam, 2009). In the current study, I was their methods courses' teaching assistants for two years. They were familiar with the researcher for taking feedback for their lesson plans. It also reduced the observer effect which refers "to either the effect the presence of an observer can have on the behavior of the subjects or observer bias in the data reported" (Fraenkel, Wallen, & Hyun, 2012, p. 473). In addition, the prolonged engagement provided to understand the prospective teachers' discourses about what they meant by saying them. However, in the research lesson phase of the first cycle of lesson study, I could not ensure the prolonged engagement of the prospective mathematics teachers in the high school where the research lesson conducted because the high school was not the official practice teaching school of the prospective mathematics teachers. To reduce the possible effects of this issue, the lectures were watched and videotaped before the actual data collection began. Before the research lesson, I was often guest of the school's mathematics classes to get used to both me as the researcher and the presence of the camera, as well as the presence of the prospective mathematics teachers and their teaching.

As described above, prolonged engagement helped about understanding what they meant by saying them. In addition, member check (or respondent/participant validation) is another technique to ensure the credibility of the results. The technique requires to have participants check the data or results for accuracy understanding related to their experiences (Birt et al., 2016). Since I remained in touch with the participants after the data collection process, the participant was asked to validate what she said and how it was interpreted what she did during the whole analysis process. Moreover, the self-assessment questions in the post-interview also helped in this regard.

Triangulation which is the most used strategy to provide credibility was used in this study. Triangulation is the process of using multiple perspectives to deepen and clarify the meaning of the data, as well as to validate the repeatability of interpreting the data (Creswell, 1998; Stake, 2000; Merriam, 2009). The literature serves different types of triangulations including method triangulation, investigator triangulation, theory triangulation, and data source triangulation (Patton, 2014). In the current study, data triangulation was used in which pre- and post- interviews, observations, lesson plans and field notes were triangulated to ensure validating and cross-checking findings. In addition, the professor who was the super-visor of the researcher and followed the observations systematically and gave her valuable opinions about both methodological and analytical issues. In addition, my colleague who worked as a teacher in high school for years shared her valuable ideas related to task-based interviews and interventions during the lesson study about what I had to be careful for evaluating the prospective mathematics teachers' knowledge. In this way, the investigator triangulation was used. It should be said at this point that the investigator triangulation reduced observer bias which refers to the possibility that certain characteristics or ideas of observers may affect what they observe.

The second notion, transferability is related to external validity of the study which is defined as the generalizability of the study (Merriam, 2009). Since the qualitative studies do not aim to reach objective reality, they approach the issue of generalizability as “the extent to which a study’s findings apply to other situations up to the people in those situations.” (Merriam, 2009, p. 226). For enhancing transferability of the qualitative research studies, the most commonly used method and the best way is thick description (Lincoln & Guba, 1985). In this study, it is aimed to reach a rich thick description by describing the study in terms of the context and findings in detail.

Dependability is related to reliability, in other words, it focuses on the consistency and replicability of the study (Lincoln & Guba, 1985). There is a need to control the analysis process about its appropriateness for related design (Korstjens & Moser, 2018). To ensure the reliability in analyzing the data, me and the second separate researcher coded a piece of data gathered from different parts of the study including pre-and post- interviews, the observations of two meetings for each lesson planning phases, and the observations of teaching and reflection phases. Since the aim of the study was to examine the improvement of prospective teachers’ knowledge for teaching the concept of limit, the second researcher should have analyzed the data in a holistic way. Among the techniques for calculating the reliability, the inter reliability approach (Miles & Huberman, 1994) was used. In this regard, the number of agreements and disagreements among the coders were determined. According to Miles and Huberman (2014), the percentage of agreements is expected as equal or higher than 70% to ensure enough reliability. Considering the observation of the development of six sub-domains, there were 47 total number of coded segments. Among the total number of coded segments, there were 12 disagreements and 35 agreements in data coding of the coders. As a result, the inter-rater reliability for the observation of the development was calculated as $35/47 \cong 0.74$ which can be considered as enough reliability (Miles & Huberman, 2014). After that, the coders came together and discuss on the items. By meeting with the other coder, consensus was reached about the items causing the disagreement. In addition, some of the data obtained from the interviews and lesson study process as the researcher of the study were coded twice with an interval of four months and the intra-rater reliability was found 97%.

Likewise, confirmability covers the issues related to neutrality which requires inter-subjectivity in the data. The interpretation should be subjective and be purified researcher's preferences and viewpoints. Since the findings in qualitative studies might change from individual to individual, this issue is regarded as problematic (Merriam, 2009). Triangulation and the role of researcher are among the methods for ensuring consistency, all of which was described above. In addition, audit trail was used to ensure dependability and confirmability. This technique requires providing a complete set of documents regarding the data. In this context, operational and technical detail lesson study regarding the data collection and analysis process were recorded and explained. However, the data analysis process, especially coding categories, was explained in detail both for the second coder and audiences.

3.6 Ethical Issues

Every researcher should consider three ethical principles: "Protecting participants from harm, ensuring confidentiality of research data, and the question of deception of subjects" (Fraenkel, Wallen, & Hyun, 2012, p. 63). In the current study, the study did not include any physical or psychological harm. However, to minimize any kind of risk that the participants may be exposed, I got informed consent from the prospective mathematics teachers (Appendix E). While one dimension of this study is at university, the other one includes a process involving high school students. Since most of the high school students were still underage, their families were informed about the study and allowed their kids to participate in the study (Appendix F). In order to protect the privacy of high school students, their faces or names were not used in any video to reveal their identities. Before conducting the study, the ethical permissions were taken from both the university (Appendix G) and the MoNE (Appendix H).

To ensure the other ethical principle, ensuring confidentiality of research data, it was provided that no one other than the researcher and her supervisor could access the data. In addition, pseudonym names were used for representing the participants. The participants were informed that their names or pictures would never be used in any publications from this dissertation before conducting the study. Furthermore, they were informed that they had a right to withdraw from the study whenever they wanted. The last ethical concern is not in the scope of the current study.

3.7 Limitations and Delimitations of the Study

There are some limitations in this study in terms of: (i) the number of participants, and (ii) the number of lesson study cycles, and (iii) the role of researcher. In this section, I will explain these limitations and how to overcome these limitations in detail.

There were only 8 students who were at the last year in the Secondary Mathematics Teacher Education program in a public university in Ankara. It was a fact that even students in the last year of the program took a lot of lessons in one semester, as the program included very intensive mathematics lessons. Considering the load of the current study, most of the prospective mathematics teachers could not volunteer even if they wanted to. As a result, the number of participants was limited to three prospective mathematics teachers. Since the study aimed to reveal the development of the prospective mathematics teachers' knowledge throughout the lesson study process, the current study did not seek to generalize the results. For this reason, the results of the study stood in local area. However, the number of participants were sufficient to conduct the lesson study, since the lesson study groups consist of three-four participants.

In the current study, the number of lesson study cycles was limited to two cycles, which might cause a limitation in terms of the development of the prospective mathematics teachers' knowledge. It may be surprising that both the reason for this limitation and also the compensation method is the same. I took long the planning of the first cycle not to cut the group discussions of the prospective mathematics teachers and to see the development in their knowledge. For this reason, there did not remain time to conduct an additional lesson study cycle. However, keeping the planning phase long did not necessitate the third cycle as it allowed for deeper discussions and a more in-depth examination of the development.

Lastly, the role of researcher might be a limitation for this study. Though the aim of the study were to observe the development in the prospective teachers' natural setting, the presence of "knowledgeable other" might affect this natural setting and there might be a researcher bias in the setting. As mentioned in trustworthiness of the study section, it was tried to overcome this limitation with prolonged engagement. In addition, before conducting the study, the researcher actively participated in the lesson planning activities of the participants in method courses and got them used to this situation. Then, the researcher made her presence in their natural environment.

There were also delimitations which were my choices that describe the boundaries that I had set for the study. First, the study was delimited with only the prospective mathematics teachers who were in the last year of the secondary mathematics teacher education program. There were some reasons for this delimitation including prospective teachers should have observed a class before and taken all the must courses for mathematics education.

CHAPTER 4

FINDINGS

The purpose of this study is to investigate the nature and development of prospective mathematics teachers' specialized knowledge for teaching the concept of limit in a broad sense through a designed lesson study development model. In line with the purpose of the study, there are two main research questions that investigate how prospective mathematics teachers develop their specialized knowledge in the concept of limit in the lesson study development model and how the critical elements of lesson study can be regulated to improve prospective secondary mathematics teachers' specialized knowledge in the concept of limit. Considering the research questions, this chapter summarizes the findings of the current study in four main sections and related subsections. In the first section, the prospective mathematics teacher's existing mathematical knowledge (MK) and pedagogical content knowledge (PCK) about the concept of limit are presented by using her written and verbal responses in the individual clinical pre-interview that was conducted before starting the lesson study process. In the second section, the development of the prospective mathematics teacher's specialized knowledge for teaching the concept of limit through lesson study in detail is given. In the third section, as a result of the lesson study process, the prospective mathematics teacher's journey of developing MK and PCK after the lesson study process is summarized by using the data gathered from individual post-interviews and lesson plans constructed by the prospective mathematics teacher. So far, these three sections are covered in the first research question. For the second research question, the fourth section presents the findings regarding the role of the critical elements of the lesson study process with related examples from the developmental process.

4.1 The Prospective Mathematics Teacher's Existing Specialized Knowledge in the Concept of Limit

Since the first research question of the current study is about how prospective secondary mathematics teachers develop their specialized knowledge in the concept of limit on planning and enacting the lesson plans in the lesson study development model, the need to learn the prospective teacher's existing knowledge emerged. For this reason, the prospective teacher's existing knowledge of teaching the concept of limit was considered as the sub-research question of the first research question. Thus, the data gathered from the pre-interview was used to reveal her existing specialized knowledge for teaching the concept of limit. The model "Mathematics Teachers' Specialized Knowledge" (MTSK) includes six sub-domains including knowledge of topics, knowledge of the structure of mathematics, knowledge of practices in mathematics, knowledge of features of learning mathematics, knowledge of mathematics teaching, and knowledge of mathematics learning standards (Carrillo-Yañez et al., 2018). Since the model deals with the specialized knowledge in a holistic perspective that includes both theoretical and practical knowledge, some of the sub-domains which require being observed in practice could not be observed in the pre-interview. Therefore, in this section, I present the existing knowledge of Mila that I observed in the pre-interview under the sub-domains of MTSK. The following table shows the summary of Mila's existing specialized knowledge for teaching the limit concept. The table shows Mila's existing knowledge in three categories including *Existing but not sufficient (ENS), Existing-Sufficient (ES), and Non-exist (NE)*[1]. This categorization provided the research to draw a path for developing her lack of knowledge.

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[1] How the categorization was constructed is shown in the data analysis title of the Methodology section.

Table 4.1 The summary of Mila's existing specialized knowledge for teaching limit

Sub-domain	Indicator of the sub-domain	Overall look
KoT	Properties and Its Foundations-Knowledge of Definition	ENS
	Properties and Its Foundations-Knowledge of History	NE
	Properties and Its Foundations-Knowledge of Infinity, Infinitesimal Approach	ENS
	Phenomenology and Application: Applications of the concept	ENS
	Mathematical Procedures: How, when and why to do something	ENS
	Representation Systems: knowledge of the different registers in which a topic can be represented	ES
KFLM	Strengths and weakness in learning mathematics	ENS
	Ways pupils interact with mathematical content	NE
	Emotional aspects for learning mathematics	ENS
KMT	Strategies, techniques, tasks and examples	ENS

Existing but not sufficient (ENS), Existing-Sufficient (ES), and Non-exist (NE)

The following sections deal with the prospective mathematics teacher's existing knowledge in various sub-domains. Among the sub-domains observed in the pre-interview, the sub-domain of KoT includes more than one indicator observed during the pre-interview. Therefore, the indicators of KoT were given as sub-titles in italics under the title of the sub-domain.

4.1.1 The Prospective Mathematics Teacher's Existing Knowledge of Topics in the Concept of Limit

Knowledge of topics (KoT) can be considered as the primary mathematical knowledge for teaching any concept since it includes definitions, properties and foundations, mathematical procedures, phenomenology, and applications and representation systems (Carrillo-Yañez et al., 2018). Among all sub-domains, KoT is one of the best observable sub-domains during pre-interview. In this section, the existing KoT of Mila is presented in the following indicators: definition, history, infinity and infinitesimal approach (as foundations of the concept), phenomenology and applications, mathematical procedures, and representation systems.

Knowledge of Topics: Knowledge of Definition

Knowledge of of definition is one of the critical indicators of KoT, which is closely influential on teachers' instructional decisions, explanations, guidance, and actions in teaching mathematics. In this study, knowledge of definition was dealt with as intuitional definition, limit of right-left sides, formal definition, quantifiers in the formal definition, the temporal order in the formal definition, and the meanings of epsilon-delta in the formal definition determined in light of the literature. In general, the pre-interview showed that Mila lacked knowledge in the formal definition, quantifiers in the formal definition, the temporal order in the formal definition, and the meanings of epsilon-delta in the formal definition.

In the first question of the pre-interview, Mila was asked how she would define the concept of limit verbally and symbolically. It cannot be said that Mila wrote the definition of limit in an incorrect way; however, Mila had some confusion in her mind about the definition of limit. It was observed when the researcher asked her some probing questions, such as how she would teach the definition of the concept and answer the students' questions about the terms in the definition, including epsilon, delta, and absolute value. At first, she answered the question as "*For* *example, when I want to describe the definition of limit, I feel like I need to explain it: As x approaches number a , $f(x)$ approaches number c .".*

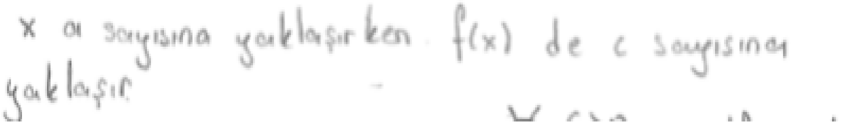


Figure 4.1. The answer of Mila for the definition of limit (I)

I kept on asking questions about the definition of limit and she wrote the formal definition of limit in two phases as following. In her first try, she didn't write the symbol of "for all (∀)". In her second try, she added the symbol of "for all (∀)", however, she didn't add the same symbol for delta (δ) (see Figure 4.2).

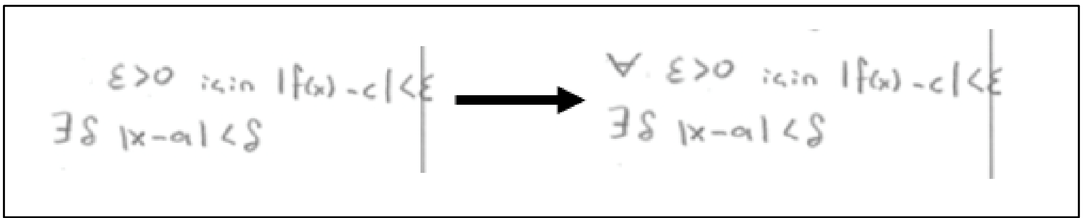


Figure 4.2. The answer of Mila for the definition of limit (II)

To understand the knowledge in-depth, the meanings of epsilon (ε) and delta (δ) were asked as she wrote in the definition. She explained these terms as "very very small numbers; I mean, we cannot see them with our eyes". When she was asked how she would answer a question from any of the students in her class about why we find delta for all epsilon, and why not the opposite one, she could not answer the question. While her answer was not incorrect in the definition of limit, I expected her to answer the question considering that epsilon and delta represent a neighborhood or a distance among the number of " ". For this reason, it can be concluded that Mila had some deficiencies in the definition, actually, in defining the concept of limit. In other words, her knowledge exists but it is not sufficient (*ENS*) to teach the concept of limit.

Knowledge of Topics: History, Infinity and Infinitesimal Approach

KoT can be considered as the fundamental mathematical knowledge for teaching. Therefore, KoT covers the answer of what the foundations of the concept are (Carrillo-Yañez et al., 2018). The foundation of the concept of limit is based on its historical development, which started from Zeno's paradoxes to Weierstrass. Since Mila took a course related to the history of mathematics and was exposed to the history of the concept in the methods course, the expectation of the pre-interview was not to present the history of the limit in terms of years and the names of scientists but to reveal how aware she was about the foundation and what enabled this mathematical development to take place. In her answer to the question related to how she would describe the developmental process of the concept in the history of mathematics, it can be understood that she had a lack of knowledge of history since she expressed that "I do not know anything about the history of the concept of limit, but I think it came from operations such as division by zero, I saw a video related to division by zero about Brahmagupta". As can be seen in Table 4.1, her lack of knowledge was analyzed as *NE* (non-exist) since she expressed herself that she did not have any idea about the history of the concept.

While I interpreted that she had a lack of knowledge of the history of the concept, it did not mean that she did not know anything. Instead, she had an idea of the foundation of the concept, which is the related indicator of this sub-domain; infinity and infinitesimal approach. Infinity and infinite processes are the basis of the concept of limit as the skeleton of the structure consisting of Calculus concepts. Therefore, in some way, infinity can be regarded as a part of "transverse connections," which can be defined as the connection resulting from different content items that have features in common (Carrillo-Yañez et al., 2018) in the knowledge of the structure of mathematics. However, infinity was regarded as the foundation of the concept of limit in this section.

It was expected that Mila had sufficient knowledge about the notion of infinity and infinitesimal approach based on her mathematical background. In the pre-interview, the expected answer for infinity and infinitesimal approach included the answer in relation to the concept of limit. For instance, in the answer to the first question, her explanation of terms (ε,δ) as "*very very small numbers; I mean we cannot see them with our eyes*" can be linked as infinitesimal approach; but it was not considered to be in relation with the concept of limit. Since it is based on the historical development of the concept, it is different from this expression[2]. In the pre-interview, there was not any question regarding the notion of infinity directly. Therefore, I examined her knowledge of infinity considering the other related questions and answers. When I asked about the misconceptions about the concept of limit in the fourth question, Mila referred to her own difficulties in relation to the indeterminate form of $\frac{\infty}{\infty}$. Considering her answer, I asked her some probing questions related to the notion of infinity. One of these questions was how the notion of infinity could be explained to students. Since she described it as a continuous process, her answer was related to potential infinity. However, the systematic description of infinity is related to set theory and infinite sets.

> (…)Researcher: Let's start with infinity. How can the concept of infinity be explained to the students?
>
> Mila: The concept of infinity… something came to my mind before the concept of infinity, can I say that? I watched something about this division by zero. This resets Indians when it comes to trading. Maybe it can be reconciled with that place, so I'm telling you. Here I don't remember his name right now. One of the Indian mathematicians is trying to find zero operations, so when I multiply zero, it becomes zero. When I add it, it's like the number itself. Then it comes to zero compartment, which is very difficult. Then he takes an apple and starts to divide the apple. It divides it into two and divides it into 3. It is trying to get closer to something. He tries to divide it with zero, that is, nothingness, but he realizes that no matter how much the apple shrinks, he cannot do the nothingness process but he realizes that the more he tries to get closer to nothing, the more the number of apple slices he gets. So, he says that if I try to divide something by zero, I will gradually divide it and divide it. I can never divide with zero because I always have an apple. But the number

[2] The idea of limiting process is based on infinitesimals in the rate of change and infinity such as Cavalieri theorem in the history of the concept (Bagni, 2005).

of apples grows and becomes infinite. I think it can be explained by connecting with that.

The first thing she described was the regular division of an apple, which represents a continuous process. As described in the literature review section, potential infinity represents a "never-ending process, continuous process"; therefore, her expression referred to potential infinity.

This finding related to potential infinity can be regarded as an expected finding since it is closely related to the phenomenology of the concept - *approaching*. In addition, infinity has two meanings: potential and actual. However, Mila considered only one of them. Therefore, her knowledge was categorized as existing but not sufficient (ENS).

Knowledge of Topics: Phenomenology and Applications of the concept

Another indicator of KoT is knowledge of phenomenology and applications of the concept. First, knowledge of phenomenology of the concept was dealt within this title. The phenomenology of the concept was considered as the mathematical meaning of the concept. In the current study, the phenomenology of the concept of limit was considered as approaching and behavior of the function based on the literature. In the pre-interview, the questions directly related to the phenomenology of the concept were not asked; however, the probing questions based on her answers gave hints about her knowledge of the phenomenology of the concept. At the beginning of the pre-interview, when the foundations of the definition of the concept were asked, Mila explained that approaching to 'c' ($f(x) = c$) means the behavior of the function at this point. When she was asked whether she meant that limit represents the behavior of the function around 'c', she went back on her idea and denied it.
Mila: We are approaching gradually; we don't have to be that value, we follow it like we are examining a microscopic being, and we are looking at how our function behaves in this interval. And, this is such a short interval that our 'L' value shows us what value our function should be while it is approaching to 'c'.

Researcher: Since we are trying to look at how our function behaves, can we say that limit is the behavior of the function?

Mila: No. I think we don't need to say that. Well, limit is not like the behavior, but it is like the behavior at some point. It's not exactly the behavior of it; it shows us what we observe at each value. It's like what our function shows while approaching each value. Questions I have never thought about…

In the excerpt given above, it can easily be observed that the participant was confused about this issue. This confusion means that she had knowledge about the phenomenology of the concept, but she could not make sense of this knowledge in her mind. I recognized this when we continued other questions. In the middle of the pre-interview, she was asked a question to check her knowledge about the difference between "convergence" and "approach". She revealed her confusion in this question as

> Mila: For example, does limit necessarily mean approaching? I use the word constantly approaching the limit, so it has attracted my attention. I really don't know why I use it like that. But I always speak like this. I wonder if there is any other meaning or how I can express it in another way; I have thought of it for a long time, though. I didn't think of another word.

This perplexity leads to confusion about the concept and its applications. In fact, those indicators have an intertwined relationship. Therefore, the question in the pre-interview for the applications of the concept of limit, emerged from the history of the concept because one of the reasons for the concept of limit to emerge was the need to explain beyond the concept of derivative (Burton, 2007).

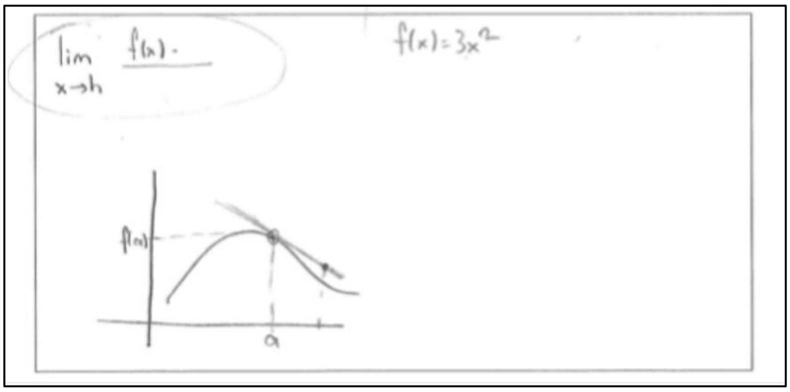


Figure 4.3. The drawing of Mila for the question related to the history of the concept

The question related to the history of the concept included an approach related to French mathematics textbooks as "*In the first half of the twentieth century, French mathematics texts used the notion of limit in an intuitive manner without a formal definition to introduce the definition of the derivative. Later in the same text, a definition would be given which is more in the manner of an "explanation" in a note at the foot of the page*". Since, as it is known, the order of topics in today's textbooks goes as the limit, derivative and integral. First of all, it was questioned how this approach makes the prospective teacher think in terms of whether it is right or not. Then, by means of probing questions, I tried to learn her knowledge about the relationship between the derivative and the limit. The following excerpt shows Mila's answer to the question.

> Mila: I don't think that's quite right.
>
> Researcher: Why?
>
> Mila: Actually, like this; if the student is able to derive this without knowing it, that is how much we have used it operationally until now ... But there is also something like this. There is something we do not know what we have, and we use it. I think it creates confusion in mind. So, we define the derivative. Actually, we use another concept to describe the derivative. But I do not know the concept I use, but on the other hand, I need that concept.
>
> Researcher: How would you explain the relationship between derivative and limit at this point? You say we need the limit to describe the derivative.
>
> Mila: Well… Yes, we need… I can't remember the definition of the derivative right now. There is a definition of the derivative with the limit, but I can't remember exactly. When h goes to 0 ... No, was it on the way to x h? I could not remember it right now. In this way, we use the limit of the derivative in two definitions. I can't really remember right now.
>
> Researcher: Did we need the limit for the creation of the concept of derivative?
>
> Mila: We always see it as relevant, even if it has no limit; we also say it cannot be differentiated.

The expectation for this question was that she could answer it as derivative is one of the applications of the concept by connecting them with their common feature (the common feature is infinity-infinitesimal approach which was indicated in the section on foundations of the concept). She had an idea about its application; however, she could not understand how the concept of limit is applied in the derivative concept. Furthermore, the excerpt showed that she could not make sense of making a connection between the concept of derivative and the concept of limit. She expressed it; "*Actually, I realize that I did not make a connection between the two in my head*" when we asked about the relation between these concepts. Therefore, it can be interpreted that her knowledge existed but was not sufficient (*ENS*) at this point. A similar lack of knowledge was observed in the knowledge of applications, as can be seen below.

Before deciding about the knowledge of applications of the concept, other applications should be mentioned in this section. Based on the literature, I have collected them all in four categories: the applications of limit as derivative, integral, real numbers, and a part of the iterative process (Abbot & Wardle, 1992; Allen, Chui, & Perry, 1989; Gowar, 1979; Larson, 2002; Silverman, 1989). Since the iterative process can be observed in all other three concepts, the pre-interview covered it in other elements of applications. The application of the limit on real numbers can be explained as the correspondence between rational numbers and real numbers (Silverman & Richard, 1989). It is defined as "*The correspondence between functions of rational and real numbers is based on the same idea used to show that the limit of the Bisection Algorithm is* $\sqrt{2}$ *. Namely, for a real number, which is the limit of the sequence of rational numbers* $\{x_i\}$*, it is defined as* $f(x) = f(x_i)$" (Estep, 2002, p. 135). In the current study, I examined this knowledge by asking the halving method (bisection method) to find the place of the 8th question. First, she answered this question by finding it with the compass. After that, I asked whether there might be another way to find (or approximate) it. She thought about this question for a few minutes.

> Now, I know that $\sqrt{2}$ is between 1 and 2. I knowl it's 1.40 odd. So maybe we can do that. We can do this by reducing it to a more precise number and measuring it with something like a ruler.

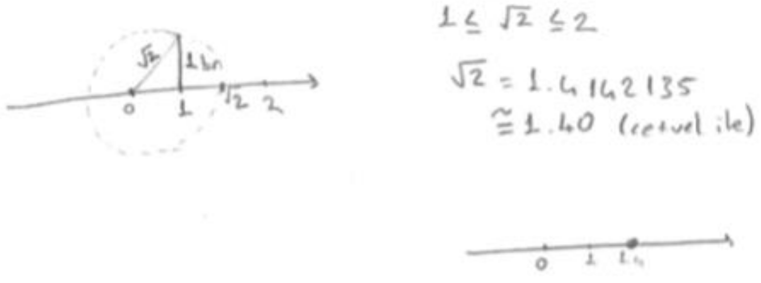


Figure 4.4. The drawing of Mila for the question related to the history of the concept

The answers were not expected to be right; rather, I wanted to see her awareness of the relation between this question and the concept of limit. The question aimed to examine her awareness of the estimation method used in middle school mathematics, which involves saying an estimate arbitrarily close to a value and the iterative approximation method used in high school mathematics. Therefore, I asked her the relation between this question and the concept of limit. She expressed herself as "*well, I don't know, is it possibly related to approaching?"*. She did not state and explain an iterative process used to construct decimal approximations. Instead, she gave one answer only, referring to what she knew. Her answer could not be regarded as an acceptably adequate explanation. Therefore, she had a lack of knowledge in the application of the concept of limit.

Knowledge of Topics: Mathematical Procedures

Another indicator of KoT is knowledge of the procedures involved in a topic. This includes knowledge of how to do something (e.g., algorithms, both conventional and alternative), when to do something (the sufficient and necessary conditions to apply an algorithm), why something is done (the principles underlying algorithms), and the characteristics of the resulting object. The knowledge of mathematical procedures of Mila can be described as at an adequate level. In the pre-interview question that asked the result and the meaning, she simplified the function when she wanted to examine the limit of a given function; then, she reached the correct result.

> Researcher: Let's say the student asked a question like this: this function is not defined in 1. You wrote when x values approached to 1 or how does x values approach to 1?
> Mila: We can actually think of it as accumulation. Like the accumulation of x points somewhere. For the limit to be 3, it does not necessarily have to be defined in 1. For example, my function could behave like this; it could be like this: (see Figure 4.5)
> (…)
> Mila: Now it doesn't have to be 1 here, as we said at first, but I'm very close to 1. I get so close that my function from the right and left, that is, the result I get is gradually approaching to 3. I mean, I don't have to be at that point.

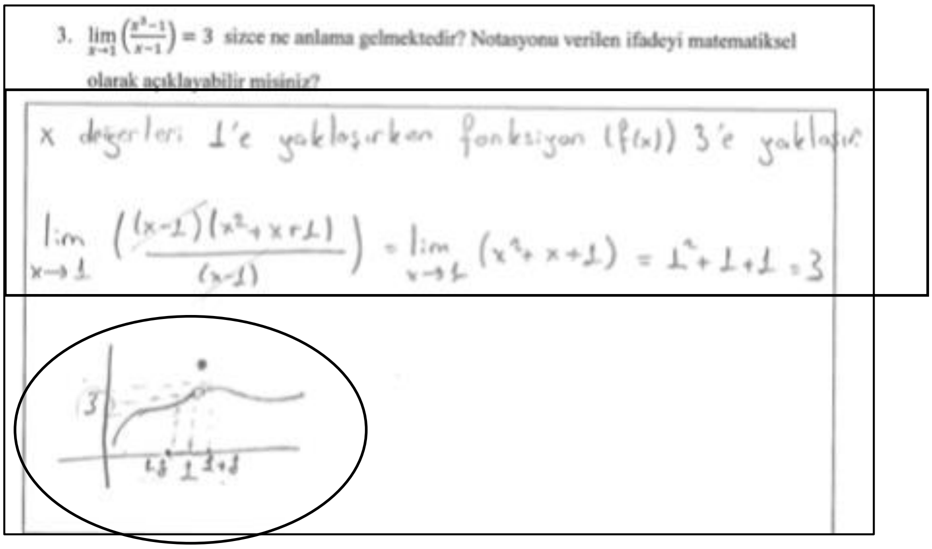


Figure 4.5. The drawing of Mila for the question related to the history of the concept

When she was asked how and why she would simplify in the function, she approached the answer in a correct way as "I'm not talking about being at that value, like equations, I'm talking about approaching that value. So, I can simplify to eliminate the indeterminate form of $\frac{0}{0}$". The evidence showed that Mila had sufficient knowledge for operating mathematical procedures.

Though Mila's knowledge of mathematical procedures seemed at an adequate level, she had some deficiencies when answering the question of why to do something. For instance, when she answered the question related to misconceptions about the concept of limit, she mentioned her own difficulties with the concept.

> Mila: I am thinking about the difficulties I experienced myself, but the limit would have been challenging for me on its own. When I first learned about it, I had a lot of difficulties, but I had trouble in the direct matter, in other words, in the concept. Because there is something; there is something like an operation, we write the numbers into it, something else comes out, but there are uncertainties, for example.
> Researcher: What is the reason for you to have such difficulties?
> Mila: I mean, it's like airborne, something else altogether. It seems like that. For example, I used to have trouble very often in indeterminate forms. For example, the indeterminate form: infinity divided by infinity. When calculating, we would take the leading coefficients. For example, I could never understand that, so what happens to the other; why don't we take it?

She always indicated that "*we do lots of calculations but I see it now, I don't know why we do them*". Her expression showed that she had a lack of knowledge of the reason behind mathematical procedures. When the researcher asked Mila why she did that step, she usually said she did not know. Therefore, this indication showed her awareness about her lack of knowledge on why to do something. As a result, the interpretation was two-fold: She had sufficient knowledge of how and when to do something during mathematical procedures; however, she had a lack of knowledge about why to do something. Therefore, I coded the situation of her knowledge as existing but not sufficient (*ENS*). This finding was crucial for the researcher to determine the pathway of the lesson study process. Based on this fact, the learning outcome was shaped and focused on the knowledge of why to do something in the concept of limit.

Knowledge of Topics: Representation systems

KoT is about what the teacher/prospective teacher knows about a concept, how and in what way he/she defines it. In this context, how he/she defines it is closely related to how he/she represents it. Therefore, the last indicator appears as representation systems. In the current study, knowledge of how to represent the concept in teaching of limit was examined through knowledge of graphical, tabular, figural, number line, verbal and algebraic representations of the limit based on the literature. During the pre-interview (also, post-interview), Mila had paper (on which the questions are written) and a pencil not only for writing the answers but also for using in cases where she could not express herself verbally. In the pre-interview, she used only graphical, verbal, and algebraic representations of her KoT in the context of limit. Figure 4.6 below shows some examples from Mila's pre-interview.

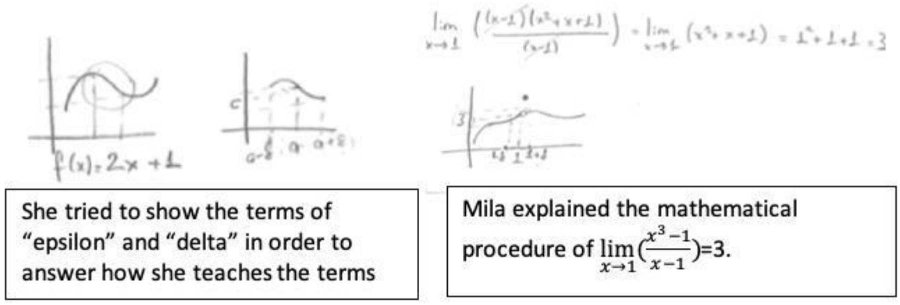


Figure 4.6. Some examples from Mila's pre-interview for the knowledge of representation

During the pre-interview, Mila had the sheet of pre-interview, a blank sheet so that she could write whatever she wanted as well as the resources available at the place of interview. Furthermore, there were not any representation types at the beginning of the pre-interview. Therefore, she was free to use any representation system, and I expected her during the pre-interview to represent the concept in different ways. However, she used only three ways, including algebraic representation (to write the procedures in questions), graphical representation (to show what she meant in her algebraic representation), and naturally verbal representation to show her ways to represent the answers related to the concept. Since she did not use other representation types, it can be interpreted as she could have a lack of knowledge about in what ways she represents the concept for teaching (*existing but not sufficient-ENS*). I said, "she could have", since she did not have a chance to prepare a lesson plan in the pre-interview.

The findings of the pre-interview mentioned above present strong evidence related to Mila's existing KoT. While it was not directly observed in the pre-interview, the data also had implicit evidence related to other sub-domains of mathematical knowledge in the model of MTSK. For instance, Mila's lack of knowledge of applications and inability to consider infinity as a common feature of other Calculus concepts were also considered as proof of her knowledge of the structure of mathematics as existing but not sufficient. Furthermore, in the indicator knowledge of mathematical procedures, while she could efficiently conduct the mathematical procedures at the right time and correctly, she had difficulty answering the question of why to do something. Considering that she did not try to validate her answer using her existing knowledge, her knowledge of practices in mathematics was dealt with but not sufficient. Since those did not have enough evidence in the pre-interview, her existing knowledge of both KSM and KPM was revealed at the beginning of the lesson study process.

4.1.2 The Prospective Mathematics Teacher's Existing Knowledge of Features of Learning Mathematics in the Concept of Limit

Another examined sub-domain of MTSK in the pre-interview was knowledge of features of learning mathematics (KFLM) which is related to specialized knowledge for students in relation to the content. KFLM can be described as the intersection of knowledge of content and knowledge of students (Carrillo-Yañez et al., 2018). For this reason, it includes different indicators which cover students' strengths and weaknesses, how they interact with the content, and their emotional aspects in learning the concept. In the pre-interview, I examined her knowledge related to students' weaknesses (misconceptions in this section) for the topics under the concept of limit.

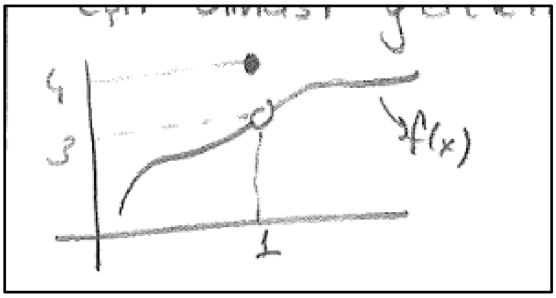
As described in the method section, Mila took mathematics education courses that included the concept of limit. This course was taken one semester ago. It can be said that her PCK is still fresh. Therefore, I expected her to quickly answer the questions related to students' learning. However, she was confused when the misconceptions about the concept of limit were asked. In the following excerpt, Mila could say only one misconception towards the topic. In the fourth question of the pre-interview, I asked her to write at least two misconceptions in learning the concept of limit.

Researcher: Would you write two misconceptions about the limit?

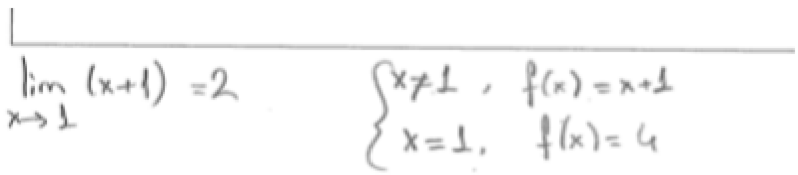
Mila: I still can't tell if some things are precisely errors or misconceptions. For example, I think what I am doing here is a misconception. I think the students think that the function should also take that value.

Researcher: How do you deal with this misconception?

Mila: It's like… I think it's about approaching the limit, so I think it's nice to explain it this way so that it is necessary to get as close as possible. As we approach here, for example, by giving an example like this;



I am also looking at how my function… approach in my function; I'm looking at what value my function approaches as it approaches the value of x. Looking at this, it doesn't have to be equal to that point either, because I don't think how my function behaves when it's 1. So, I think I will explain it this way.



But here, I think there is a problem like this; I mean, we explain it through an example. I think this is not a good thing either.

Researcher: What do you think is the reason the student has such a misconception?

Mila: I had this obviously, and I think mine was due to this reason: when we were a student at first, for example at the point of x=1, when it was asked about $(x + 1)$, or in similar situation like this, we say that, substitute 1 for x and the result is the limit. The result of this limit is 2, for example. But now there is something like this; if my function is like this; $\{x \neq 1, f(x) = x + 1 \; x = 1, f(x) = 4$. When it happens now, for example, it gets different. When I put 1 here, I will find something different again, but I can't actually replace x with 1. Here I was always getting mixed up, perplexed, as if it should be equal. So, what happens when we substitute this value in the function? But I can't replace it either.

> I think all of it comes from here. So that's how we learned, and I was always confused here.

Mila expressed that "I don't know any other; it may be indeterminate forms". This takes us to two points: First, she did not know what the misconception is, as she said in the interview. Second, she did not have any idea about other misconceptions. In addition, she showed one of the misconceptions: "If you are graphing a function without raising your hand, that function is continuous." in a pre-interview question. While Mila was making interpretations about the continuities of graphs of functions, she usually said, "*If you can draw the function without raising your hand, that function is continuous*".

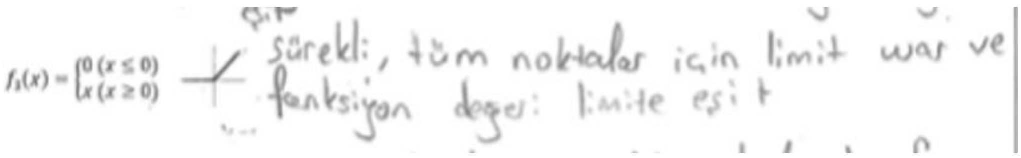


Figure 4.7. Some examples from Mila's pre-interview for the knowledge of representation

> 3rd function (see Figure 4.7)-Mila: I think this is always happening. I have value for every point and I can draw my function without raising my hand. So, I don't have a point that remains undefined. (…) And $x = 0$ is an important point for me. At this important point, it is defined in both parts and the thing takes the same value for both.
> 2nd function ($f(x) = \frac{1}{x}, x \neq 0$): I just want to express where I was stuck. We always say something, or if you can draw the function without raising your hand, that function is continuous. And here I have to raise my hand to draw the function.
> Researcher: Do you think not raising our hand is right?
> Mila: Well, it doesn't seem right now. I mean that this rule matched with the graphs of all functions that I looked at; here... and here…I don't know!

In the excerpt given above, the 2nd and 3rd functions given in the pre-interview were continuous functions on their domains. In the question, she always expressed "*without raising hand in drawing*" when she considered a function as continuous. However, I expected her to give an answer which she deduced from the definition of continuity $(f: R \rightarrow R,\ a \in R,\ f\ is\ continious\ at\ the\ point\ a, if\ f(x) = f(a),$

$f(a) \in R$). Her expression is regarded as a misconception for the continuity in the literature. The insistence on this rule showed that Mila had this misconception. Perhaps the reason she can only say one misconception is because she herself has multiple misconceptions. In this excerpt, Mila's knowledge of students' weaknesses was not observed. On the contrary, a misconception of Mila herself was observed. In order for a teacher to have knowledge of the weaknesses and misconceptions that her/his students have, she/he must first be aware of what they are and what they might be. In this quote, we see that Mila herself has misconception rather than having any awareness of it.

Another indicator for knowledge of features of learning mathematics was the combination of content and students during interaction - ways pupils interact with the mathematical content. During the pre-interview, the prospective teacher was asked how students interact with the mathematical notions related to the concept of limit. Those were not the central questions in the pre-interview, instead they were probing questions; such as "what could be the reason why students experience this difficulty?", "how do you think students would react when faced with such a question?", and "how would you react when one of your students asks such a question?". The excerpt given above, which is about the misconceptions of students, was also an example of this indicator. She showed how students interact with the content with an example to show misconceptions.

The way how pupils interact with the content is affected by students' interests and their emotions towards the content. In this way, another indicator - emotional aspect of learning mathematics is related to awareness of students' interests in brief. The pre-interview was not a suitable environment to evaluate this knowledge. However, I observed the reflection of it in some questions and also observed both its lack and development during the lesson study process. One of the main shortcomings I observed during the pre-interview was that she did not take into account the interests and feelings of students while answering the question I asked about how to teach the concept with her conceptual knowledge. Moreover, she reflected her knowledge based on her own feelings while learning the concept. For instance, when the reason for students' conflict in learning the concept of limit was asked, she explained it as not intending to satisfy students' curiosity but to solve questions.

> Researcher: Later, were the math lessons you took enough to clear this confusion? Because you actually took pretty advanced math?
>
> Mila: Of course, it's enough for now, but one of the reasons for this is that we are constantly solving questions. Since we see it in classses all the time, it turns into memorization after a while. At first, I was confused about this, but then we had to solve so many questions as we were in the 12th grade, but we said that this was not the case. Do we say "no" by understanding the logic of it? But that's how we learn this way.

The prospective teacher made this inference by evaluating her own learning process. She did not express her emotions directly in the excerpt. However, she indicated that she solved problems or exercises without understanding the concept in her learning process. It can be interpreted in this excerpt with her gestures; she was mad about her learning process and learning the concept. Another example can be given from the last question of the pre-interview. When asked to prepare a rough lesson plan on how to explain the concept of limit, she only mentioned the choice of representation as "visualization through graphs". However, she did not think about the students' interests and feelings toward the concept. Therefore, for this dimension, it can be said that Mila did not have adequate knowledge about considering emotional aspects of learning mathematics. Knowledge of emotional aspects and knowledge of ways pupils interact with mathematical content can be considered as interrelated, and her lack of knowledge on these items was interpreted as existing but not sufficient (*ENS*).

4.1.3 The Prospective Mathematics Teacher's Existing Knowledge of Mathematics Teaching in the Concept of Limit

Knowledge of Mathematics Teaching is related to how and in what ways teachers/prospective teachers teach the concept. Similar to the KFLM, the sub-domain is the combination of the knowledge of content and knowledge of teaching (Carrillo-Yañez et al, 2018). Therefore, it covers teaching strategies, tasks, examples, and teaching resources. This knowledge can be observed during planning more explicitly than during the pre-interview. Since I did not have a chance to get the participants to teach the concept or make a lesson plan before they participated in the study, I tried to understand her lack of knowledge through clinical interviews. To examine knowledge of mathematics teaching through clinical interviews, I asked the participant such prompt questions as "how would you teach this issue to your students" to eliminate this limitation. As can be seen in the following examples, the examples were the sub-answers for the questions of other sub-domains.

> Researcher: I just moved on to the second question. They said that siblings or students could not understand the definition of epsilon and branch, or they questioned what these concepts were. How would you describe this concept to them?
>
> Mila: Maybe I could do something like this. Suppose we now draw this. Let's pretend we have such a function and we are working in this range. But this range does not give us enough information. Then I said let's get a little closer to here.
>
> Researcher: Why doesn't he give us good enough information?
>
> Mila: Because the closer we get somewhere, the clearer we see. For example, let's think of microscopic organisms, we cannot see them in points, but they exist. If we can see them with a microscope, we can see them in more detail. I say let's get a little closer. Then I assume I got that ... I get here (in the function, he took the section of the graph he wanted to approach and put it on another coordinate plane) I said a minus delta, a plus delta. Likewise, let's consider these places as epsilon. For example, let there be c. Now I say we got closer to here, but let's always think like this. 0.9 or 0.99 closer to 1? I can make it smaller and smaller forever, but it actually allows us to see this in the epsilon with the delta.

This is the first example from Mila's pre-interview. As described in the description of participants, both Mila and other group members had taken the methods courses in two semesters before participating in the current study. For this reason, they should have known theories of teaching, teaching methods, teaching resources, and how to teach the concept. However, the example given above showed that Mila thought of teaching the concept of limit by using an analogy that was "looking at microscopic organisms from the microscope to see them in detail.” This is the first indicator for Mila's KMT. Using analogies to make the concept understandable for students can be considered that she had this element of the indicator of using analogies in mathematics teaching. However, this was insufficient evidence to interpret her knowledge of strategies, tasks, and examples as existing and sufficient (*ES*).

The second example is related to the concept of infinity. In the fourth question, while Mila was explaining the students' misconceptions about the concept of limit, she talked about indeterminates. Moreover, accordingly at this point, she started to talk about the concept of infinity in her answer. When the question "how you would teach the concept of infinity to your students" was asked, she referred to an analogy again. The related excerpt is given under the title of phenomenology and foundations.

Furthermore, the pre-interview was conducted in a mathematics laboratory to provide participants with an environment where participants easily access all kinds of resources. Then, Mila could reach different kinds of resources in the laboratory. However, she used only paper-pencil during her pre-interview. Bearing all these in mind, she had knowledge of strategies, techniques, tasks, and examples but not sufficient for teaching the concept of limit since the content-specific strategies, techniques, tasks, and examples were not observed during the pre-interview.

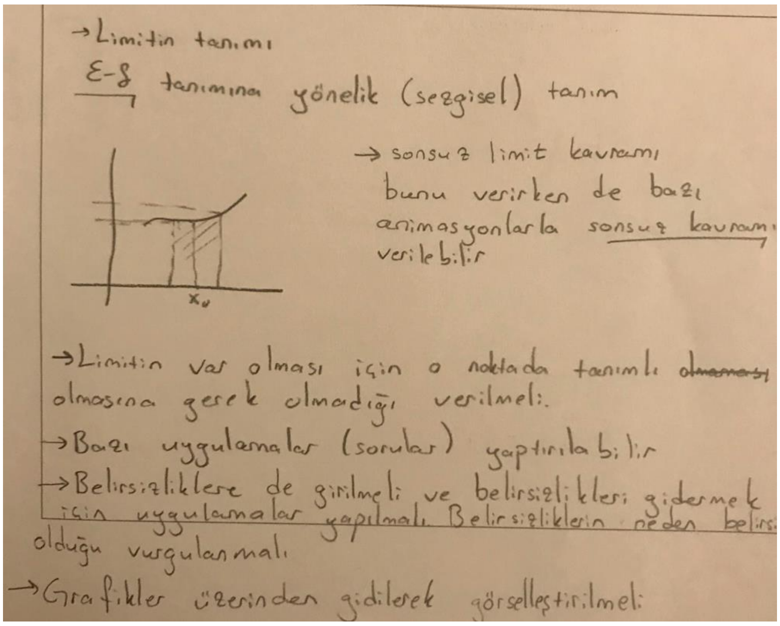


Figure 4.8. The answer of Mila to how prospective mathematics teachers teach the concept of limit in classroom

Collectively, the last question of the pre-interview asked how the prospective mathematics teacher taught the concept of limit in the classroom. In this question, the aim was to see the micro lesson plan related to the concept of limit. The Figure 4.8 shows Mila's answer to this question.

The answer of Mila contradicted her previous answers to the questions related to the concept of limit. As indicated in the methodology section, the pre-interviews were conducted in two parts because some questions took too much time. For this reason, there was a week between those two parts of the pre-interviews. It should be indicated that Mila might do research on the concept of limit. However, there were still gaps in Mila's micro-lesson plan. For instance, she still did not consider students' previous learning (lack of KFLM) since she directly started with the concept's definition. In addition, the fourth item she indicated with an arrow, "some applications (questions) can be made" showed that she did not have sufficient knowledge related to tasks, strategies, and examples in teaching the concept (lack of KMT) since she could not describe the item. Furthermore, she did not refer to the concept of continuity, the relation of the concept of limit and continuity with other mathematical concepts (lack of KoT). In addition, she had a lack of knowledge of the mathematical language. It cannot be observed that Mila was careful about using mathematical language, which is of importance for teaching the content (lack of KPM). For instance, some statements such as "(intuitional) definition of epsilon-delta definition" could not be understood. Last but not least, teaching resources and teaching methods (lack of KMT) had not been indicated in her answer.

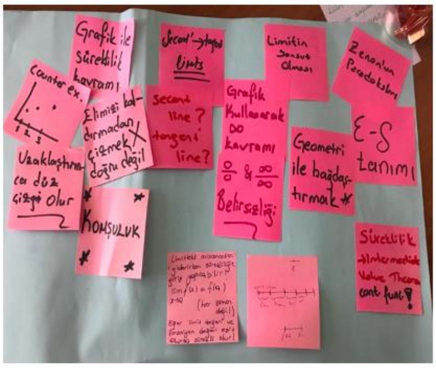
As a result of the pre-interview, in general, it was observed that Mila had knowledge of teaching the concept of limit, but it was considered as not sufficient since she did not show some indicators of the sub-domains of MTSK. The pre-interview was not only used for preparation to answer the first research question of how prospective mathematics teachers develop their specialized knowledge in the concept of limit in the lesson study, it also enabled Mila to do her self-assessment and willingly participate in the learning environment. Second, as the researcher, it provided a way for me to organize the lesson study process based on Mila's lack of knowledge. During the lesson study process, I went over to improve her lack of knowledge.

In the latter section, I present the development of specialized knowledge of Mila for teaching the concept of limit. First, I start with the sub-domains of the mathematical knowledge of the model in order with KoT, KSM, and KPM in telling the prospective teacher's journey during the lesson study process. Then, the development of PCK is presented in order with KFLM, KMT, and KMLS.

4.2 Developments in the Prospective Mathematics Teacher’s Specialized Knowledge in the Concept of Limit

This section presented the answer to the first research question of how the prospective secondary mathematics teacher developed her specialized knowledge for teaching the concept of limit on planning and enacting parts of the lesson study development model. The researcher designed the lesson study process considering the existing knowledge of the prospective teacher and her lack of knowledge and other group members’ lack of knowledge for teaching the limit concept.

Before the group started to discuss planning the lessons, I gave the participants a big cardboard and post-it notes and asked them to write on these post-it notes and stick them on the cardboard, considering all the lesson study goals. This cardboard served two purposes: first, I had the chance to observe whether the participants were aware of the group’s lack of knowledge and existing knowledge for teaching the concept. Second, this cardboard was kept suspended from the board throughout the process so that they would not miss the points they wanted to make throughout the process (see Figure 4.9).



Mila	Fulya	Alp
• Continuity concept with graph • Statement by giving counter example for functions that are not continuous • It is not right to draw without raising our hands • When you zoom out it becomes a straight line • Continuity and Intermediate Value Theorem	• Secant line? Tangent line? • The indeterminate forms of $\frac{0}{0}$ and $\frac{\infty}{\infty}$ • Harmonizing the concept with geometry • To give the concept of infinity using graphs • Limit at infinity and infinite limit • Zeno's paradoxes	• $\varepsilon - \delta$ definition • The relation between real numbers and the concept of limit • Neighborhood

Figure 4.9. The cardboard constructed by the lesson study group

Accordingly, the lesson study group determined the lesson study goals (see Table 3.1), and the lesson study cycles started. There was no clear cut between cycles during the lesson study process in presenting the data below. Therefore, the prospective teacher's knowledge developments were considered as a whole through the lesson study process; thus, the findings were not separated cycle by cycle. Accordingly, in this section, the lesson study process is presented on the basis of sub-domains. In each sub-domain, the findings related to the development of the related sub-domain are presented in two parts of the lesson study, including planning and enacting. The researcher asserted such a presentation; in *planning*, the lesson study group determined a lesson goal and planned the lesson (the first two phases of lesson study), and in *enacting*, the lesson study group conducted the research lesson of the lesson plan and reflected their ideas about the research lesson in terms of students' learning. In this way, the findings started with the development of sub-domains of mathematical knowledge. Mathematical knowledge includes three sub-domains: Knowledge of Topics, Knowledge of Structure of Mathematics and Knowledge of Practices in Mathematics. In each sub-domain, there are indicators that light the way for understanding the nature and development of specialized knowledge. Therefore, the data were presented through indicators under the title of the related sub-domain.

The road map to not to get lost in the findings of this large data can be seen in Figure 4.10.

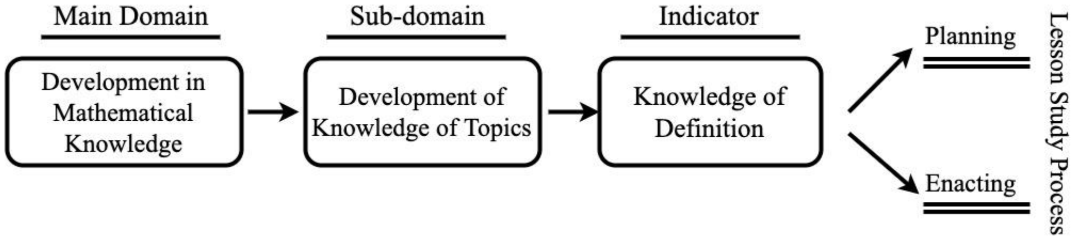


Figure 4.10. The road map about the presentation of development in knowledge of the prospective mathematics teacher cardboard constructed by the lesson study group

4.2.1 Development of the Prospective Mathematics Teacher's Knowledge of Topics in the Concept of Limit

For the first research question, the first sub-domain of the model is Knowledge of Topics (KoT). KoT covers the fundamental knowledge about what and how much the prospective teacher knows about the concept of limit (Carrillo-Yañez et al., 2018). Therefore, KoT can be described as a basis and/or starting point for the development of specialized knowledge for teaching the concept. Based on the pre-interview, the lesson study process was designed to construct a basis with KoT and accordingly to develop other sub-domains.

The development was provided in more than one way. First, the individual pre-interviewing process made her aware of her own lack of knowledge and eager to develop herself in KoT. Furthermore, in the individual pre-interviewing process with Mila, she tried to answer the questions according to her existing knowledge about the concept of limit. However, in most of the answers, she had lack of both MK and PCK. Moreover, as noted in the previous sections, examining the possibility of the other subdomains of MTSK was hard through pre-interview. The lack of knowledge of these sub-domains emerged during the lesson study process's lesson planning phases. When Mila and other group members began to discuss teaching the concept and prepare lesson plans for three lesson study goals, all participants could realize their lack of knowledge about the concept of limit. Second, discussions during planning by means of the learning kit given by the researcher were another factor for this noteworthy development since the rich group discussions were not only related to theoretical parts of MK, but also to the reflection of theoretical knowledge in practice for students' learning. All developments in the sub-domains of MTSK for Mila regarding the concept of limit are explained in the following by referencing definitions, foundations, applications, and representations as noteworthy development.

4.2.1.1 Development of Knowledge of Topics: Knowledge of Definition

In this study, the prospective mathematics teacher' specialized knowledge of teaching the limit concept was examined based on all the indicators of each sub-domain. In KoT, there were five indicators that led the researcher to examine the development of knowledge of the prospective teacher. In this section, I focused on the first indicator as knowledge of definition. In this study, knowledge of definition was dealt with as knowledge of intuitive definition, knowledge of right-left sided limits, knowledge of the formal definition of limit, the temporal order in the formal definition, quantifiers (for all, such that, at least) in the formal definition, meanings of epsilon-delta in the formal definition and transition from intuitive definition to the formal definition[3] (Davis & Vinner, 1986; Cornu, 1991).

The elements for the development of the indicators were embedded in the whole lesson study process, which meant that knowledge of definition was confronted in many stages of the lesson study process. Particularly, I was confronted with this indicator in both of the two cycles of Lesson Plan-1, which aimed to conceptualize the concept of limit in students' minds during the lesson study process. The definition of the limit is confusing both for Mila and the other participants, and it contains too

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[3] The ingredients mentioned in the indicators are shown in Figure 2.3.

many elements. As mentioned in the literature review, it is of importance to define the concept in an accurate way for an effective teaching and learning process, specifically for the effective teaching and learning of the concept of limit. The knowledge of definition is a sub-domain for knowledge of topics (KoT) as definitions, properties, and foundations, and knowledge of practices in mathematics (KPM) as knowledge of how to define something and its elements. In the title of knowledge of definition, these two sub-domains were considered together since they could not be differentiated in the context of the lesson study process.

In addition, when this study was carried out, the formal definition of limit was not included in the curriculum for secondary school students. However, the lesson study group decided to add the formal definition of the limit in Lesson Plan-1, since they wanted to improve their lack of knowledge. In addition, I triggered them by adding the formal definition of limit as a gearing-up activity in Lesson plan-1.

To provide a comprehensive understanding in the knowledge of the definition of the limit, the lesson study process was handled with seven sub-indicators, including knowledge of intuitive definition, knowledge of right-left sided limits, knowledge of the formal definition of limit, the temporal order in the formal definition, quantifiers (for all, such that, at least) in the formal definition, meanings of epsilon-delta in the formal definition and transition from intuitive definition to the formal definition. In the table given below, the development of knowledge of definition through the lesson study process is shown based on the different parts of the lesson study utilizing abbreviations including *AD: Adequate level of development, NAD: Not Adequate Level of Development or Not Development, AE: Already Existing and NA: Not Observed*. The main aim of the development process was to help the participant reach AD in the table.

In addition, in the table given below, some columns include asterisks with NA. They indicate the topics that could not be taught during the research lessons since there was no time or they were not in the curriculum. The expected *adequate level of development* could be described as using the knowledge in her actions and/or suggestions in which she showed the indicator. *Not adequate level of development* could be described as that the participant did not show the indicators of the knowledge or did not use the indicators of the knowledge in her actions even if she verbally expressed it.

Table 4.2 Overall look the development of knowlede of definition the concept in KoT of Mila across phases of lesson study

	Lesson Study Cycle 1		Lesson Study Cycle 2	
The sub-indicators of knowledge of definition	Planning	Enacting *	Planning	Enacting
Knowledge of intuitive definition	AE	AE	AE	AE
Knowledge of right-left sided limits	AE	AE	AE	AE
Knowledge of formal definition of limit	AD	NA	AD	NA
Temporal order in formal definition	AD	NA	AD	NA
Quantifiers (for all, such that, at least) in formal definition				
Meanings of epsilon-delta in formal definition	AD	NA	AD	NA
Transition from intuitive definition to formal definition	NAD	NA	AD	NA

*AD: Adequate level of development, NAD: Not Adequate Level of Development or Not Development, AE: Already Existing and NA: Not Observed *The research lesson of this enacting phase was conducted in real classroom with real curriculum*

Planning Phases of Lesson Study

In the pre-interviewing process, Mila recognized her lack of knowledge about the formal definition of limit. Before the lesson study process started, she had already researched the definition. As the first step for developing her lack of knowledge, the researcher prepared a learning kit for the concept of limit, including mathematical notes in a broad sense considering Mila's as well as the other group members' lack of knowledge. In the first meeting of the Lesson Plan-1, the learning kit was given to them as an assignment to read and have a discussion on it in the upcoming meetings. In the second meeting, we discussed some titles in the kit: the historical development of the concept, the definition of the concept, and its components in the light of how they can conceptualize the concept in students' minds. The group considered the formal definition and the components of the formal definition as a whole. Thus, they had the chance to look at the definition of the concept holistically. They decided to start the lesson plan with a dynamic view of the limit as *approaching*. For this starting point, she prepared an example activity which was named "*finding the exact place of e*" for the beginning of the lesson plan in the third meeting of the first cycle. Mila showed her knowledge about the intuitional definition and the right-left sides limit on the activity in the excerpt below. The steps of the activity included approaching the number of (e) from 1, 2, and so on. Even though it was started from a point too far away to express as neighborhood, it can be said that this approach has intuitive definition knowledge, which is said to exist before, and uses this knowledge to explain the phenomenology of limit to students.

> Mila: I did it like this: I gave 1 and 2, and in the last one I gave 3. I gave the rest rationally so that they could calculate a little more easily, and I also stated their values approximately or clearly. Here, too, has the number e. I scratch it so that they don't write anything. I wrote the answers here, too. "What is it here?", I asked. I think maybe we can get an answer like we can determine some approximate values. Here are the answers to that. How does the function behave as we bring the number x closer to the value e ? Can you give an approximate value for? … We tried to find the closest thing possible by giving big and small values. Actually, I'm trying to make you say that there. After this, I did not know how to do some transitions. Then I wrote this again because I will go over what we had done to find a value close to the value of e. Here we tried to get closer to "e" by giving big and small values. After that, we can show this on GeoGebra on the graph (see Figure 4.10), as

> I try to give an approach from the right to the left, emphasizing that we give both big and small values.

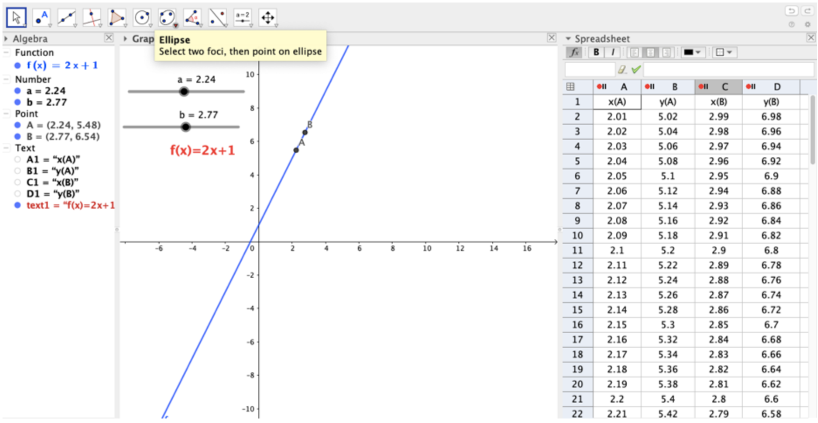


Figure 4.11. The demonstration of the group's activity related to "*e*"

Up to now, she showed her knowledge of definition in terms of intuitive definition and right-left sides of limit, which were at a sufficient level. Other group members accepted the activity; however, both Mila and other group members had difficulty transitioning from the intuitive definition and the right-left sides of the limit to the formal definition. The following excerpt shows Mila's expression about this issue.

> Mila: We need to emphasize that it is not a one-way approach, and we can take big, small, or equal values. You know, if we stand on this and think about the number line, we get something like this. So, when we give smaller values, we approach from the left, and when we have larger values, we approach from the right. Suppose we think as a number line. I thought we were describing this approach called as the right-to-left limit. Or, I don't know, if anyone has another idea, we can state that when we get close enough, we get the limit value of the function. I don't know how we can make the transition from here to the epsilon delta. I just thought of it; I think we should state that this approach is with epsilon and delta, that is, those values.

Her expression was revealed after some discussions about the activity given in the former excerpt. Her expression shows that she was aware of her lack of knowledge when she said, "*I don't know how we can make the transition*". At this point, the vital thing was which elements of the formal definition Mila lacked. Her last expression showed that her lack of knowledge could arise from the notions of epsilon and delta.

After other group members accepted the activity with some revisions, the problem with the transition to the formal definition and the exponents of the formal definition were discussed. For the construction of this discussion, first, they read the learning kit for the concept of limit. Later, they started to discuss the concepts, and their notes on the post-it notes (see Figure 4.9), and the data gathered from these discussions showed the development in knowledge of these elements.

For the epsilon-delta in the formal definition, it was expected that she would have knowledge about these notions representing an interval to define the concept to consider it an adequate level. In the following excerpt, the starting point for improvement in her mathematical knowledge for teaching the formal definition occurred by means of her reading and the discussion initiated by another member of the group.

> Mila: Can I tell you something I found about epsilon-delta? I found these in Thomas Calculus. Now Thomas first said in Calculus; that is, he explained: To show that $f(x)$ is equal to the number L as x goes to x_0, we have to show that when x is held close enough to x_0, the gap from f(x) is also small enough that I can choose: He said let's think of Epsilon and delta like a skeptic and a scientist. The skeptic is skeptical, that is, he constantly presents us with epsilon objections that the limit does not exist or that the limit is something that cannot be doubted. The scientist also says that for every objection, I find that there is a delta equivalent around x_0. And I show that in this range, the function values will keep L around the epsilon. If I showed that there is at least one delta for every epsilon, I would have defeated the skeptic. He turned it into such a game; in fact, he showed the epsilon delta in this way. I found something like this, so maybe we can use it.
>
> Alp: How are the objections of epsilon; What exactly is it objecting to?
>
> Mila: For example… offers you the epsilon objection every time. What if the epsilon was 1, if it were 3, like if it were $\frac{1}{2}$. He constantly presents you with epsilon objections, and then you show a delta value for every value he says every time. This delta value indicates that the limit is in that range around the epsilon. But this goes on forever. Say 1, say 10, say a million or something. You say, what do I do for this? You say that if I find at least one delta for each epsilon, you'll end this discussion. After that, we used to write in terms of epsilon delta in the definition of epsilon delta, which we always say, or we kind of switched to it.

When they (the group) discussed how to integrate the formal definition into lesson plan-1 to conceptualize the concept in a broader sense in students' minds, first, Mila attempted to explain what she gained from the readings. Her explanation of the story of negativists and scientists looked like a superficial expression of formal definition. While the prospective teachers discussed how to make the definition meaningful for students, the formal definition was already on the board. The issue was not related to correctly writing the definition. Instead, they were lacking in its ingredients' in-depth meanings. Therefore, Mila's definition remained a bit superficial to give more profound meanings. The discussion atmosphere, initiated with Alp's question, created a setting to reach deep meanings of the temporal order in the definition through the example. In this way, Mila had to answer his question in detail. During the lesson study process, the group members should convince each other to construct a joint lesson plan. Thus, it provided them a taken-as-shared meaning of the taught concepts. However, it was not considered as a sufficient answer for the temporal order in the definition.

The other element for knowledge for teaching the definition of the concept was the notions of neighborhood- accumulation point-ball. It was expected to show their knowledge related to epsilon-delta as neighborhood and connect their knowledge to notions that have the same meaning in the definition of limit in different contexts. For instance, it was expected to relate the notion of the neighborhood with the notion of the ball in topology. The following excerpt shows how the discussion setting nurtured the prospective teacher in knowledge about these notions for teaching the definition.

> Alp: We have always given the intuitional definitions, which have been used for centuries. The aim of epsilon delta… as we say in intuitional definition, is that limit means approaching. Epsilon delta, on the other hand, shows how close it approaches. I think, instead of just saying these approaches, we use them to show that interval mathematically. That's why I can't think of something else than interval.

> Mila: I always think of delta as … There's this term *approaching enough*; we use delta to show that we approach enough. Well, I did not use to think that it is such a short interval that it helps us to approach enough until our first

meetings. I'm not sure whether it means the same thing as 'neighborhood', though.

Researcher: Think of the courses you've taken. What have you learned?

Fulya: The concept of ball?

Mila: Yeah, that's right; I've got it now [by demonstrating with her hand]

Researcher: Now why do we subtract c here? If you start from the things you have learned before?

Mila: We thought it was something with Fulya. Neighborhood and approach.

Researcher: Why neighborhood and proximity?

Mila: Hocam, while we were discussing the readings you gave with Fulya, we came to the following conclusion: The limit of a function at point a does not depend on whether the function is defined at that point or not. The aim of the limit is to examine the behavior of the function around this hole, that is, the neighborhood with holes. We must give this to the students.

The researcher wanted to mention this concept in particular, because Mila's description did not go beyond from "*very, very small numbers*". In addition, it was aimed to reconcile them with the advanced mathematics lessons they have taken recently. After they read the kit for the concept of limit, I asked them what they understand from the reading about epsilon and delta ($\varepsilon - \delta$) for their mathematical knowledge for teaching the concept. Mila expressed herself as, "*I always think of delta as ... There's this term 'approaching enough', we use delta to show that we approach enough. Well, I do not use to think that it is such a short interval that it helps us approach enough until our first meetings. I'm not sure whether it means the same thing as 'neighborhood', though*". This can be considered as the correct approach for the notion of delta. Therefore, it was regarded as an adequate level of development on these parts including (epsilon, delta, and what these represent in the definition) for knowledge of the definition in the first cycle.

The other lack of knowledge of Mila was the transition from the intuitional definition to the formal definition. Table 4.3 shows that the first cycle of the lesson study did not provide development. The following excerpt shows the starting point of the development of this knowledge:

> Researcher: How do you connect it to the epsilon delta from here?
> Mila: There was something that Alp told: There was something close to the sound or something similar; it would be very nice if we could feed them.
> Fulya: But we will make different applications in GeoGebra, or how we connect from those applications.
> Alp: Ha, I understand you. You say that before the examples I gave, let's give the analogy roughly to the students.
> Mila: So, I really liked it because you're officially embodying the limit; that is, you're making the epsilon and the delta concrete. The biggest problem is that the concepts are too abstract or concretized in mathematics, so it makes sense to concretize them, so he says that it is something that exists. At least, I think it can be used.
> Alp: The example I gave was as follows while lecturing to a group, the voice came from the middle row. You don't know who he is, but you know he comes from the middle row. Your first epsilon is your middle row. Then you get a little closer; you get closer to the delta. Then you threw out the front row and the back row, for example. You say that you threw the children away; you say they cannot talk.
> Mila: You actually found delta for these.
> Alp: Exactly. Then, since you said that these things could not happen, here we are; epsilon has narrowed a little more. You just got a little bit closer to Delta. Then you eliminated these children and said that it is not them either. So for each interval, you got a little closer and narrowed it down a little bit to find the person speaking.
> Mila: And every time it gets closer, the child stays in the range you approach, you know, we find a delta for each epsilon, or that child stays inside the epsilon.

In the excerpt given above, Mila proposed the idea of Alp, which connected the finding at least a delta (δ) for every epsilon (ε) with a metaphor of finding the person speaking in the classroom. Though Mila considered this metaphor the transition from intuitional to the formal definition, it did not include the transition; rather, it was just related to the simulation of the role of epsilon and delta in the formal definition. She realized it after teaching lesson plan-1. In her reflection, she asserted, "*I didn't like this transition. Yes, it may be a good example, as I said before. But it doesn't seem mathematical at all. In addition, it didn't connect the previous activity to formal definition*". This showed that the development in Mila's knowledge still continued.

The research lesson where the lesson plan (lesson plan-1) was carried out in the lesson study was conducted by another group member. Therefore, Mila observed the research lesson and watched the video-recording of the research lesson. Though Mila did not conduct the research lesson, in both the reflection paper and the reflection meeting for lesson plan-1, Mila evaluated the research lesson. In the reflection process, Mila hit the high spots about the definition of limit in lesson plan-1. The spots were necessary for the lesson study group. In addition, her reflection paper gave me a hint about her specialized knowledge of teaching the concept of limit. For the definition of the concept of limit, she indicated that "*... in addition to these comments, I would like to say that the transition from intuitive and formal definition seemed like hanging in the air. I think it broke the lesson into two different parts. I think we didn't fully understand it, so we couldn't connect them...*". Mila's comment showed that she was aware of the lack of their knowledge of definition at the end of the first cycle.

As mentioned above, at the end of the first cycle, the participants reflected on their ideas about lesson plan-1. Considering not only Mila's reflection but also other group members' reflections, I asked them some probing questions, for instance, "how can we correct the parts you think are missing or not reaching the student here?", "what do you think might be the reason for the "substitution in function" tendency that frequently occurs in students?" and "what are your solutions for this issue?", for the aim of consolidating their knowledge and see if each member of the group with Mila has the same common meaning. Thanks to video recordings of the research lesson, Mila had a chance to watch it again. In this way, Mila noticed both the lack of knowledge in the lesson plan and also lack of students' learning based on the lesson plan. In addition, Mila noticed her colleague's lack of knowledge of teaching mathematics. As an answer to the question of "according to your observations, which activities/questions in the lesson plan worked for the purpose of the lesson?", her comments focused on the first activity related to phenomenology and application.

> Mila: The questions in the activities were excellent and thought-provoking, but since we are talking about *approach*, there are always those who say that

> we will bring the points closer without understanding the exact reason. The reason for these should be questioned.

She realized that the students' learning remained at the same level and that Alp could not reach the intended level. This should not mean that Mila only criticized her friend. At the same time, being asked to comment as if she was telling what should happen at the point of criticism revealed the prospective teacher's own knowledge. On the other hand, she criticized the lesson plan and stated that she was aware of her lack of knowledge. In this way, reflection also made the prospective teacher realize what she needed to focus on and learn in this new cycle, as in the pre-interview.

> Mila: We could not connect the activity of saying the number closest to 1 exactly. We wanted it to create an atmosphere of play, but it was more ridiculous to the students. Second, an additional precaution must be taken if GeoGebra does not work. A screenshot could be placed on the slide as a step-by-step approach. … I think it might be better to explain the definitions rather than writing them down. Because printing the definitions directly is both a waste of time and not very meaningful after students do not understand. … GeoGebra's not working interrupted the lesson, but I guess I would have explained myself like Alp. However, I think we are lacking in this regard, that is, how to act in sudden situations. For example, the students did not fully understand it in that first activity, but I think I would not have known how to put it together.

The reflection showed that Mila and other group members lacked knowledge for the transition from the intuitional definition to the formal definition. In addition, she felt insufficient about how to act in contingency moments (Rowland, Huckstep, & Thwaites, 2005). This awareness provided Mila to be present to answer students' questions and unexpected situations such as technology tools not working. The reflections were considered for revising process of the lesson plan.

In addition to her awareness of herself, one of the most significant contributions of the lesson study process to the prospective teacher was to raise the prospective teachers' awareness of the concept. While the lesson study process was continuing, the courses that the prospective teacher took in the mathematics department were also going on. Awareness of her and other group members' on this and their encounter with other knowledge besides their learning during the lesson study process provides a rich group discussion. Such rich group discussion provided Mila to develop her knowledge. In the following example gathered from the second cycle, Mila and Alp brought an example related to limit:

> Alp: Mila, Meriç Hoca (as pseudonym) showed us something; there was a needle.
> Mila: Teacher, now we have a needle like this, let it be 1 cm long. Its thickness is zero in its zero area. So, it has no area and no thickness. Now we will move this needle in the plane.
> Alp: This needle is like this; we will turn it upside down. Its head will come here.
> Mila: Upside down! Now let's imagine that this is the head of the needle which is moving in the plane—the same point at the same place. We will try to turn it upside down on the same top again itself. What is the smallest area to be scanned; what is the smallest area do you think we know the answer?
> Researcher: circle?
> Fulya: Moving down to be a circle!
> Mila: Here it is not a circle; my teacher even gave us a very insulting word so that we should not call it a circle.
> Fulya: Is it semi-circle?
> Mila: Nope!
> Researcher: We saw that it was a star from there.
> Mila: Hocam, this is epsilon!
> Alp: Smallest greater than 0 in infinitesimal!
> Mila: it's infinitely small because as the area gets smaller, now I'm trying to make the area smaller; for example, I keep moving it like this, I move it like this, the first thing I scan is something like a star, so I can't draw very well, but here you go, I move it like this and turn it gradually.

Before interpreting Mila's contribution in this excerpt, what Mila and Alp talked about should be mentioned. They talked about the trace left on the paper as a result of a needle with ink on its tip being released vertically on the paper. Even if they cannot convey exactly what the professor of the course they are talking about, it is understood that the trace left by the needle tip is the neighborhood of the needle tip. In the excerpt, Mila did not mention *neighborhood*, this example showed not only Mila's awareness of the concept and also of its properties. It is an important step for the development of KoT of the prospective teacher because she always expressed herself as "*I never thought of that before*" in the pre-interview and "*while learning* *all this, I never realized that it was like this*" at the beginning of the lesson study process. For this reason, it is an indication that she had gained awareness when she captured this knowledge for the lesson and brought it excitedly to the lesson study planning process.

Considering all these aspects related to her awareness about both her lack of knowledge and her developed knowledge, she could make meaningful attempts for developing her knowledge. In the second cycle of lesson study, it was observed that she could develop her knowledge of definition. As can be seen in the section of Knowledge of Practices in Mathematics (KPM) in the indicator of "necessary and sufficient conditions", in the second cycle, the clear change was observed for this indicator in the activity in the lesson plan-1. From the first cyle to second cycle, it was observed that she could use her developed knowledge related to temporal order and quantifiers in formal definition appropriately in her proposed activity (see Figure 4.20). Therefore, it can be said that adequate level of development was observed in the indicator of knowledge of definition.

Enacting Phases of Lesson Study

As said before, Mila could not conduct the lesson plan related to knowledge of definition totally. However, in the second cycle of the research lesson, she took over the research lesson from the end of the first lesson plan and continued with the second lesson plan. Therefore, for this sub-domain, I could observe the enactment part at a certain point. Though it can be thought that there must be a reflection of the second research lesson of the first lesson, Mila did not reflect her KoT in the reflection. Rather, she only focused on whether the group completed the mission (lesson plans).

4.2.1.2 Development of Knowledge of Topics: Knowledge of Foundations

Another indicator of KoT is knowledge of foundations which is related to the notion of infinity since it is a foundation of the concept of limit. While the concept of limit is the foundation of the concepts of Calculus, infinity is the foundation of the concept of limit as well as all the concepts in Calculus. Infinity is observed in the logic underlying all the applications (e.g., derivative, integral, real numbers, and a part of the iterative process) as infinite steps, infinite substances, or infinitesimals. Thus, infinity has a connector role between the Calculus concepts as well as the foundation of the concept. Since the connector role is related to the other sub-domain (knowledge of the structure of mathematics) in the model, in this section, as the foundation of the concept of limit, infinity was dealt with its infinitesimals and conceptions of infinity. In this way, knowledge of foundations was presented in *planning* since it was considered as a requirement for knowledge related to its connector role.

The pre-interview was conducted to reveal the prospective teacher's existing knowledge. However, it also provided the prospective teacher's awareness about some notions for the concept of limit, including infinitesimals. The knowledge related to the notion of infinity was revealed implicitly during the pre-interview by means of the researcher's probing questions. She had knowledge of infinity; however, there were some points to be developed so that she could have knowledge of infinity in different perspectives.

The first element for knowledge of foundations is "infinitesimal". Considering the pre-interview, the process was designed on the answer of Mila "*very very small intervals/steps*" in the pre-interview. This development was parallel with the development of knowledge of the application of the concept. In the pre-interview, knowledge of applications and foundations of the concept was examined through the history of the concept. Accordingly, during the lesson study process, I used the history of the concept to improve the prospective teacher's awareness of the foundation of the concept. The *awareness* meant not only having this knowledge but also considering the foundation of the concept in conducting the tasks in the lesson plans. The former was reached in the first cycle of planning the first lesson.

> Fulya: Delta is the range where we call x, and the change in function in epsilon.

> Researcher: Okay! interval a minus delta and a plus delta interval. Delta what?
> Mila: Oh well, isn't it? Delta is such a small number that we have to enter the neighborhood very soon, so here, I think, I just didn't know how to connect them, to be honest, at the time of writing. Delta is such a number that it gets us close enough. So, there's the concept of infinitesimal approximations that we're talking about, that's the infinitesimal approximation thing over there. It's so small that I'm getting close enough to a. On the other hand, I get such epsilons in my L value that it falls into this range, the intervals between the two. So, infinitesimal approximations I think we should connect with that.

In the excerpt given above, it is understood that Mila had the knowledge on infinitesimal approximation. The infinitesimal approach is observed in most subsequent quotations in the first lesson plan. It was observed that she gained this awareness at the knowledge level. On the other hand, she used her knowledge in designing tasks and determining the expected outcome of learning. In the first task of the first lesson plan, she used her knowledge in designing the task. When it was asked why she chose to design such a task, she expressed herself as "*I wrote here that the expected outcome of the task is infinitesimals. We read in the kit you gave us; all of the concepts are based on this approach. So, I thought that this should be the first step for basis*". The expression of Mila showed another expected finding that the development in the journey of the prospective teacher were considered as a whole. In particular, this development can be considered as the result of the nature of the concept.

Enacting Phases of Lesson Study

While the first cycle did not allow me to observe this knowledge in the research lesson phase (where the lesson plan was carried out in the lesson study), in the second cycle, I could observe her knowledge in her teaching. The following excerpt illustrates the example included in this short lecture.

> Mila: What if I got closer to 0.01?
> Student2: Then it would be sequentially like this.
> Teacher: What would it be this time? It would be less than 0.005.

> So, what did Cauchy say in his intuitive definition while I was defining it? So, what did Cauchy say in his intuitive definition while I was defining it?
> Students: ...
> Mila: What do I say about getting closer and closer with infinitely small approaches?
> Students: We are shrinking the range.
> Mila: So, what happens as my gap gets smaller? I find a different behavior, right? Can I say then that I have found a range for the behavior of each of my functions? Here we make the formal definition of it.

She used her knowledge when she transformed the intuitive definition to the formal definition of limit. She directed the students (her classmates for the second cycle) with this knowledge when they did not understand the connection between the intuitive definition and formal definition. By using the foundation of the concept, she referred to the applications of the concept.

Another element for the knowledge of the foundations of the concept is the conceptions towards infinity. Although the notion of infinity is closely related to the theory of sets with its development in history, another close relationship is related to the concept of limit. As said before, the notion of infinity forms the basis of the concept of limit. For this reason, it is important for prospective teachers to be aware of their conceptions and to be aware of the infinitesimal approach that underlies the concept of limit. Another important point is the curiosity of the students related to the notion of infinity. Considering these facts, I have often tried to confront them with the concept of infinity by means of readings, discussions and probing question in the discussions during the lesson study process.

The first confrontation with the notion of infinity was observed at the beginning of the planning phase of Lesson plan-1 with the idea of paradoxes by means of Alp's proposed ideas. The paradox proposed in the group discussion connects the phenomenology of the limit concept (iterative process) and the foundation of the concept (infinitesimals approach). Since I mentioned the notion of infinity in the transverse connections again, it should be indicated that there is not any transverse connection between the mathematical concepts which have the same foundation. In the beginning of the planning phase of the lesson study for the first lesson plan, the participants wanted to start with the aim of constructing the knowledge base for limit.

> Alp: We can start with Zeno. Zeno's paradoxes. There was a small group activity. I had mentioned it before, I guess. It says the number 5, and probably a group says the numbers lower than 5. Another group says the higher ones. They write them in a table. And it goes based on the objectives in the curriculum.
> Alp: It was saying in continuity that if you can draw without moving your finger, it is continuous. They wrote down misconceptions as notes.
> Mila: What is the paradoxes subject? Since I don't think I know them exactly?
> Alp: Well, in the paradox of Achilles and the tortoise, Achilles is in a footrace with the tortoise. Achilles allows the tortoise to start 50 meters ahead of him, for example. They are in such a race that Achilles never catches up with the tortoise, even though the tortoise is halfway through which Achilles takes.
> Mila: Interesting. A start like that would be effective to teach "approach".

The excerpt given above is a part of the discussions on using the paradox. Alp reflected his knowledge of paradoxes as the starting point of limit. As opposed to Alp, Mila had some deficiencies in phenomenological aspects of infinity and making connections between infinity and the limit since she was surprised with this idea, and she claimed that she had not known the paradoxes and the relation between paradoxes and the limit. In this way, it can be said that the learning process was started with rich group discussions through Alp's direction. In the paradox, there is a never-ending process between Achilles and the tortoise since Achille cannot catch tortoise in the paradox and/or cannot reach the target. This represented the potential meaning of infinity. Paradoxes are an important factor for knowledge of teaching the concept of limit. Because the possible conception of infinity can cause misconceptions in students' minds. First, I did not intervene in their discussions to observe their interpretations and contributions to the discussion. However, the expected outcome of the study related to the notion of the study was to gain advanced understanding of it, which included actual infinity as well as potential infinity. After they adopted the idea of using Zeno's "*Achille and the tortoise*" paradox, Fulya prepared the activity about the paradox for the next meeting. However, she made some changes in the activity. She used another paradox of Zeno, *Dichotomy paradox*, in the context of a competition problem in the lesson plan.

> Fulya: I think that it is more useful to start the lesson with a problem. I designed a problem based on real-life: “This week, Survivor will play an arrow shooting game for immunity. For the untouchable game, one contestant from each of the team of celebrities and the team of volunteers is selected, and these contestants cross each other at a distance of x meters and shoot arrows at a specific target. As the arrows move forward, they travel half the current situation each time. I point out here in brackets, regardless of the strength of the passengers and their ability to hit the target. According to this, which team wins the competition and why?” … While preparing the activity, I thought that this might be a misconception on students’ mind.
> Mila: I was just about to ask it to you.
> Fulya: It may cause two misconceptions: One is "the limit value is never reached" and the other is "the limit is always equal to the value of the function at that point".

The activity was related to the repeating process, halving the road repeatedly in this activity. The mathematical foundation of this activity is based on the concept of convergence of a sequence. Both the convergence of sequences and the notion of approach are based on infinity. Therefore, while Mila was not included in the excerpt actively given above, it can be said that the group as well as Mila built a relationship between the concepts. Because Mila was there and the activity was a product of the whole discussion process.

The second lesson plan aimed to construct the knowledge of features of the limit of special functions such as polynomial, trigonometric, exponential, logarithmic in students’ mind. In addition, the goal of the lesson plan was to implement applications within the context of limit except the ones whose result is infinity with the concept of limit. However, the participants thought that the concept of limit cannot be considered without the concept of infinity. On the other hand, the indeterminate forms of $\frac{0}{0}$ and $\frac{\infty}{\infty}$ of the limit are included in the lesson study goal. Therefore, they focused on the concept of infinity directly in Lesson Plan-2. Before designing lesson plan-2, the group determined the topics in which they have faced difficulties throughout their own educational background. Not surprisingly, they all focused on the topics related to the concept of infinity, including limit at infinity, infinite limit, indeterminate and undefined forms. At the beginning of the designing lesson plan-2, they started to discuss how they would refer to the concept of infinity without confusing students. The researcher, as the “knowledgeable other”, directed group discussion on prospective teachers’ conceptions of infinity to deepen their discussions.

> Researcher: What is on your mind? What does “infinity” mean to you? Let’s say one by one.
> Alp: Like the 2001 space adventure movies. I mean, the space shuttle is going towards infinite black. It is like that. The infinity goes on like that (He shows it with his hands)
> Fulya: I mean, the mathematical meaning comes to my mind directly. Well, there is a set, which has a beginning but no end or there is a set, which has an end, but no beginning. Like indetermined, so something indetermined.
> Mila: It is such an endless, far, far away ... I mean, if we think of it as distance, it is a very, very far place. And, we don't know how far it is, but there is such a place, but we also know that. So, I think anything can happen there.

First, Alp revealed what he was thinking about the notion of infinity, which was a mysterious thing like a scene in a space movie similar with Zeno’s paradox given in the Lesson Plan-1. It can be said that his knowledge of teaching infinity is shaped by his conception of infinity as potential infinity. Mila, as well, thought that infinity is an endless distance that cannot be measured. This shows us their KoT in terms of their conceptualization of infinity. On the other hand, in this excerpt, Fulya tried to describe the infinity in accordance with the concept of “set”. As she said, she wanted to describe infinity mathematically. By indicating infinity as a set, she described it as an object or entity. It can be said that her KoT as conception about infinity is actual infinity. I need to indicate their existing knowledge, since the development related to infinity in lesson study process was based on these conceptions. At this point, they were given an assignment related to the notion of infinity such as the reading in the learning kit and the research on infinity.

The second lesson plan was conducted by Fulya and observed by Mila and Alp. In Mila’s reflection, she reflected on her comments related to both Fulya’s teaching and the lesson plan as:

> (Mila’s reflection paper) Fulya's way of expressing infinity was not exactly what I wanted. That's why we need to pay special attention to emphasizing infinity as an adjective. In particular, they are very right in understanding 1 to infinity in that way, so, Fulya couldn't explain it either, and we should work on that as well.

To be able to critically watch and comment on a lecture, she must have that knowledge too. In this reflection, it was observed that Mila had awareness about how the notion of infinity should be expressed during the teaching. She thought it in two ways: both using mathematical language and students’ learning ways. After conducting the research lesson of lesson plan-2, they discussed the concept of infinity again based on Fulya’s (as a teacher of the lesson) expressions for infinity as a number. Based on this claim, they focused on the idea of how they should express the concept of infinity. By questioning each other’s knowledge during the discussion in the revision process, the prospective teachers had a chance to make sense of their knowledge. In the following excerpt, Alp asked his friends whether there can be limited infinity in mathematics. Such a question triggered other participants to think on the notion of infinity.

> Fulya: I think we can talk about the infinity as “constantly increasing”.
> Mila: Yes, I read about that! Infinity is not a quantity; it is a quality. Then, it may be sensible!
> Alp: Well, could there be bounded infinity?
> Mila: What did you mean?
> Alp: Constantly increasing cannot be considered wrong. However, what if bounded infinity? If we say bounded infinity, for example, there is a bounded infinity between 0 and 1. However, there is infinite numbers in this interval.
> Mila: Yes, there could be.

> Alp: However, it is so close. I mean that the place between 1 and 0 is too short.
> Mila: But to whom is it close?
> Alp: It seems this much close to me (showing that there was a very short distance using his thumb and index finger)
> Mila: Too far for me. For example, this distance may be too close for you, but it may be too far for me. I mean, it depends. So, we can think of it as quality from this perspective. I remember that there is a one-to-one correspondence in sets. I think this issue is related to it. This excerpt can be perceived as an important excerpt; as a result of the assignments given in the first cycle, we observe that prospective teachers now consider the infinite from a different side, rather than just thinking of it as a never-ending process in a certain pattern. Since Alp's attempts in the discussion triggered to reveal Mila's knowledge, first Alp's attempts should be explained.

This excerpt can be perceived as an important excerpt, as a result of the assignments given in the first cycle, we observe that prospective teachers now consider the infinite from a different side, rather than just thinking of it as a never-ending process in a certain pattern. Since Alp's attempts in the discussion triggered to reveal Mila's knowledge, first, they should be explained. The excerpt shows Alp's mathematical knowledge for infinity in two sub-domains. First, when Alp asked his friends what about "*bounded infinity*" by relating it, it shows us that he used infinity as a mathematical object. Though he did not indicate Cantor's one-to-one correspondence explicitly, he referred to it by being aware of the existence of infinity. Alp's awareness of actual infinity prompted the others and Mila to reflect on this issue.

It is not wrong to specify infinity as a property in terms of its connection with the limit. On the contrary, it is supported by the literature that this connection is a normal connection due to the nature of the limit. However, it was still expected that Mila would also express infinity in its actual meaning as an infinity. Therefore, the improvement observed in the planning phase was considered as a not adequate level of development.

4.2.1.3 Development of Knowledge of Topics: Knowledge of Phenomenology and Applications

Another indicator of the sub-domain of KoT is phenomenology and application. In this section, I focused on the development of Mila's lack of knowledge of phenomenology and applications of the concept. I dealt with the category of phenomenology and application based on the literature as the applications of limit as derivative, integral, real numbers, and a part of the iterative process (Abbot & Wardle, 1992; Allen, Chui, & Perry, 1989; Gowar, 1979; Larson, 2002; Silverman, 1989). The applications of the concept were handled in each three lesson plans. Mila had already known that the derivative and integral are some applications of the concept. However, both Mila and other members of the group had lack of knowledge in other applications. Since these applications were placed in the curriculum, the derivative and integral were considered in lesson plans through the history of mathematics and the exercises/problems in lesson plans. The others were handled through the tasks (Wasserman et al., 2016) delivered by the researcher and discussion on readings. Similar to the previous sub-domain, the development of phenomenology and application of the concept was observed mainly during the planning and reflection phases of Lesson Plan-1. However, since this sub-domain covers all the lesson study goals different from the previous one, it was observed in other stages of other lesson plans (e.g., research and planning of Lesson Plan-2, the teaching of Lesson Plan-2, teaching of Lesson Plan-3). The Table 4.3 presents the overall view of the development of phenomenology and applications in KoT of Mila across phases of lesson study.

Table 4.3 Overall look the development of phenomenology and application of the concept in KoT of Mila across phases of lesson study

	Lesson Study Cycle 1		Lesson Study Cycle 2	
	Planning	Enacting	Planning	Enacting
Knowledge of derivative as limit of rate of change	AD	NA	AD	AD
Knowledge of integral as limit of series of cumulative change	AD	NA	AD	AD
Knowledge of application in real numbers	NAD	NA	NAD	NA
Knowledge of limit as a part of iterative process	NAD	AD	AD	AD
Knowledge of behavior of function	AD	AD	AD	AD
Knowledge of dynamic view of concept (approaching)	AE	AE	AE	AE
Knowledge that the limit specifies a function	NAD	NA	NAD	NA

AD: Adequate level of development NAD: Not Adequate Level of Development or Not Development AE: Already Existing NA: Not Observed

Table 4.3 shows that Mila developed her knowledge of applications, including derivative, integral, and a part of the iterative process in limit. Among these applications, the knowledge of the relation between real numbers and the concept of limit, and knowledge of limit as a part of the iterative process were not developed during the lesson study process at an adequate level. The expected adequate level of development could be described as using that knowledge in several times and/or suggestions using that knowledge. There was not any evidence observed during the lesson study process.

Planning Phases of Lesson Study

In the planning phases of lesson study, the development of the first two indicators for the applications of the concept was handled with the knowledge of history as the topic of *approaching secant line to tangent line*. The intended outcome for this indicator was to provide the developments in the same direction (see Figure 4.112).

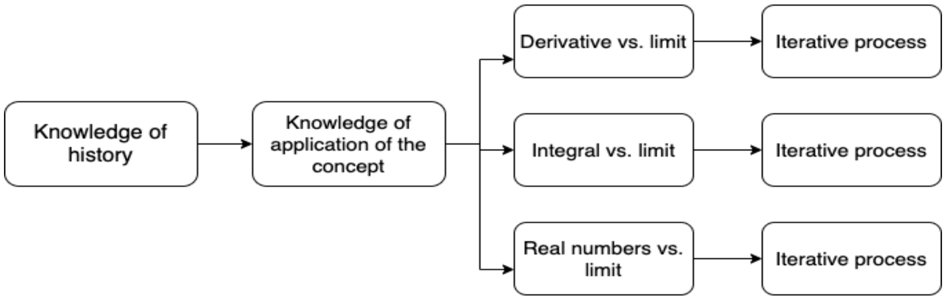


Figure 4.12. How the development occurred during the lesson study process

In other words, it was aimed to hit two targets with one arrow. In fact, the historical development of the concept occurs in advance in geometry (area, volume or length of curve) and astronomy under the light of infinitesimal approach. Before the concept of limit was revealed in the mathematical world, the concepts of derivative and integral had already been in there. The outcome of this attempt in the study was to gain this insight to the prospective teachers for making them aware of the conceptions between mathematical concepts and its applications. To provide the development, first the assignment which contained the following paragraph was given and a discussion environment was constructed with the knowledgeable other's questions.

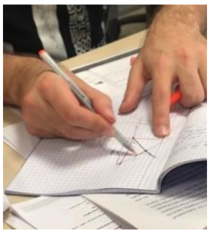
> "In the first half of the twentieth century, the concept of limit was used intuitively to describe the definition of derivative in French mathematics books without a formal definition. Later in the same text, in the note at the bottom of the page, a definition in the form of "description" is given…" (Cornu, 2002, p.153)

The same statement was also given in the pre-interview to examine the prospective teacher's existing knowledge of mathematical connections and the historical development of the concept. It was shown in the section on existing knowledge that Mila had difficulty answering the question. In the discussion of the history of the concept, I threw this statement into the question of the concepts including derivative, integral, and the limit discovered earlier than the others. Except for Alp, Fulya and Mila answered this question similarly as "*you asked it in the pre-interview and it was* *so surprising for us, we have already known that limit explains the derivative, but we don’t know how*”. Since the curriculum presents the order of the topics as limit, derivative and integral, the question was a starting point for the discussion in the first cycle of the lesson plan-1.

> Researcher: Where did the limit come from now? Why was the limit needed? Let's start here!
> Alp: It came out of the ancient Greek mathematicians, or rather, their roughest ideas came from them.
> Researcher: But you talked about this Kaiser's work, for example.
> (…)
> Alp: He was saying something like a rule.
> Fulya: Infinitesimal approximations.
> Researcher: They're talking about infinitely small approaches. Did you know that before, Mila? …No!
> Researcher: Would you like to tell, Fulya?
> Fulya: When it came to analysis in the 19th century, there are the concepts of derivative and integral. So, they know more or less what derivatives and integrals are, but they can't prove it. Because there is no proper function, and there is such a thing as small approximations without. But what is that? They can't do it exactly. Then they arrive at the intuitive definition of the limit from derivative and integral. They make an intuitive definition. It gives the formal definition of Cauchy or something 100 years later.

Learning takes place not only through the individual's self-expression, but also through communication. However, Mila's work with a more experienced group, the groupmates complementing each other's shortcomings, provided the way for knowledge development. The excerpt given above was one of the examples. The reflections of such learning on the development of Mila’s knowledge can be seen in the rest of the planning and other stages in the table below while the rest of the paper shows the findings in two main sections, including planning and enacting. To see the development in the first cycle, the two important observable features of lesson study are revealed in the table below.

Table 4.4 The development observed in the first cycle in planning of Lesson plan-1

	Investigation and Planning
Cycle-1 (Mini-1)	Mila: Now here he said "the limit of the tangent secant". Researcher: What do you think about this? I asked you this in the first meeting, and it was also in the task I gave you. Alp: I couldn't visualize it right now, where was the secant? Where was the tangent? (Here Alp is trying to remember what they are by typing tangent and secant on google on his phone; he showed his friends what he found and everyone started drawing on their own) Fulya: Was the secant passing through the center? Mila: I don't quite understand! Alp: Look now (trying to show the attached drawing here)   Mila: How did this place become secant? Fulya: If we say that, this place becomes secant. From where it intersects with that curve, it becomes tangent there. Alp: If I take the limit of this point towards here, the tangent becomes secant. Mila: It's still not in my head.
Cycle-1 (Mini-2)	Alp: Finally, there is a function drawing that I humbly prepared in GeoGebra. I got one tangent; I specifically took this point to make sure it's tangential. Then I bought a secant. Moving it like this (moving the tangent on the curve) point B approaches point A. Mila: In this we can say: If our secant is tangent, that point is our tangent point. So, what do we do to braid it? What happens if we take your limit? with questions like... Alp: Here's what happens to our apex as point A gets closer to point B? as. Fulya: What is the situation between the point and the line? It may also be a question. Researcher: You are at a very good point now. What does this also refer to? What did you notice in this demonstration? Mila: Aa, hocam, this is the derivative! You know, we used to show this derivative with triangles*, and the triangles were getting smaller and smaller. Ok!

Note. **She meant by saying "triangles" as the demonstration of* $\frac{\Delta x}{\Delta y}$ *on the graph*

In the first stage discussion setting of Cycle-1, which I called mini-1, the fact that they think about tangent and secant only in circles shows that they actually have lack of knowledge about the applications of the limit. In fact, Mila could not think of anything until her other groupmates commented, which is an example of a direct lack of knowledge. Although the prospective teacher is actually someone who can easily say that the slope of a tangent gives a tangent, her inability to think about the limit in the geometric interpretation of the derivative was a situation I could not foresee while planning the lesson study process. I thought that she could easily overcome this situation due to his academic success. In addition, she always expressed herself about this issue "*Frankly, Fulya and I couldn't figure out what we were getting closer to!*". This showed that the idea of "approaching" restricted her from figuring out the derivative and the limit. The term "*approaching*" meant limit for them as well as Mila. However, this term represented the dynamic view of the limit, and the literature indicated that using only the term "*approaching*" caused misconceptions about the limit. For this reason, I intervened in some points that the group fell into misconceptions by asking probing questions such as "*Let's think about students' misconception regarding the concept of limit; how can these misconceptions occur in the student's mind?*". In this way, they could link this knowledge with the knowledge of features of learning mathematics" of pedagogical content knowledge.

At this point where Mila and other group members had difficulty connecting the concepts, I changed the way I planned to follow the indicators. Since the readings and discussions were not enough to develop the knowledge for applications of the concept of limit, the assignments were given to the lesson study group. These assignments included researching and bringing examples about the demonstration of the excerpt. In the second stage of the first cycle, Alp's demonstration provided to reincite the discussion related to the application of the concept. As seen in the third row, when the researcher asked what that demonstration referred to, Mila noticed the derivative on the demonstration. Though she referred to iterative process in her expression, it cannot be observed that she was aware of the iterative process as a phenomenology of the concept.

Different from the derivative, she related the integration process and limit easily. The task was given to the lesson study group which can be seen in Figure 4.13, she could easily make sense of this relation as expressing the Reimann sum. Her expression related to Riemann sum, which was "While calculating this, we actually take sequential steps, that is, we go by shrinking from one outside to the inside" showed the logic she adopted behind this application as an iterative process.

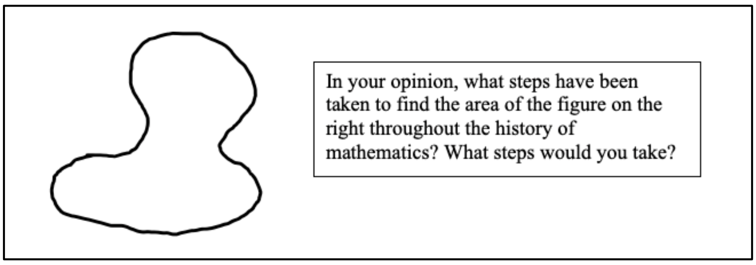


Figure 4.13. The task related to remind the integration process (adapted from Burton, 2011)

Another phenomenology of the concept is that limit means behavior of the function. At the beginning of the planning of lesson plan 1 in the first cycle, only one probing question sufficed to awaken both Mila's and other group members' knowledge related to the phenomenology of the concept. While knowledge of applications of the concept was not observed in her research lessons, her knowledge of the phenomenology of the concept was observed in her research lessons. In particular, she used "behavior of the function" in the answers to students' critical questions.

What is not developed throughout the lesson study process was observed in "limit specifies a function" as not adequate development in knowledge. While this indicator was discussed many times during the lesson planning process, in particular in the second cycle of lesson study, Mila could not find it logical and refused to discuss it.

Enacting Phases of Lesson Study

The enacting phase includes research lessons and reflections of Mila in the lesson study process. In both of her teachings, she used her knowledge of phenomenology explicitly. In the research lesson of the first cycle, she used her knowledge to explain the indeterminate forms. Mila showed this knowledge in the example where she emphasized that two different functions show indeterminate forms because they get different results even though they show the same behavior after she got a question from the students “should two different functions not have two different limits anyway?” while studying those two functions. Similarly, in the second research lesson, while making the formal definition, Mila used her knowledge of phenomenology of the concept while describing the temporal order in the formal definition in her interaction with the students (her classmates).

> (…) Mila: What if I got closer to 0.01?
>
> Student2: Then the interval would be as follows (shows on the board).
>
> Mila: What would that be this time? (Speaking of Delta) It would be less than 0.005. So what did he say in Cauchy's intuitive definition while I was defining it? What do I mean by getting closer and closer with infinitesimal approximations?
>
> Students: We are shrinking the range.
>
> Mila: So, what happens as my gap gets smaller? I find a different behavior, right? Can I say then that I have found a range for the behavior of each of my functions? This is how we make the formal definition.

Similarly, it was observed that Mila used her knowledge in the reflection phase of the lesson study. For instance, she made critical thinking on Alp’s research lesson (the first cycle’s first lesson plan) about Alp’s non emphasizing the applications of the limit sufficiently in the activity related to the secant-tangent line. She indicated that Alp should have made connections between the concept's application and the concept of limit by using their same foundations.

As a result of this sub-section, it can be said that Mila’s knowledge of phenomenology and application of the concept was nurtured through lesson study, in particular through lesson planning. As a result, it can be interpreted that the development in Mila’s knowledge of phenomenology and application was adequate level.

4.2.1.4 Development of Knowledge of Topics: Knowledge of Mathematical Procedures

Another indicator for KoT is Knowledge of mathematical procedures. Knowledge of mathematical procedures includes the answers to the three main questions of the mathematical procedures, including how, when, and why related to the characteristics of the resulting object. As mentioned in the pre-interview, knowledge of how and when to do something already existed in Mila's mathematical knowledge. It can be understood from the pre-interview that she had gained procedural fluency during her mathematical undergraduate courses at university. However, she had difficulty when the questions were about reasoning something. She expressed herself about this issue: "*Maybe if we have a question about the limit, we can solve it, but we have problems conceptually. If they ask 'why' about something, we can't answer*". Therefore, development in the knowledge of the mathematical procedures was constructed in this perspective.

In the light of this perspective, the lesson study process was designed on the question of "*why*." As both knowledgeable other and guide, the researcher asked "*why*" for almost all steps of the development of the prospective teacher. In fact, this question constructs a sociomathematical norm between the researcher and the prospective teachers. When the prospective teacher moved to the research lesson, she revealed her sociomathematical norm[4] (Cobb & Yackel, 1996), which she accepted. In addition, which answer is considered mathematically understandable is essential for the prospective teacher. Since it was hard to answer all the questions of "*why*" during the lesson study, another pillar of such a design process was the steps on where to find the answers to the questions of the prospective teacher and lesson study group.

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[4] The sociomathematical norms can be described as the understanding of what counts as an acceptable mathematical explanation is a sociomathematical norm, the understanding that when discussing a problem students should offer different solutions or mathematical explanations and the understanding of what constitutes mathematical difference (Cobb & Yackel, 1996, p. 461).

For this reason, both Mila and other group members were supported with rich resources and discussions to find their own answers. In addition, Mila and other group members had a chance to make slight differences in the lesson plans they conducted as research lessons. In this way, some group members added some important points that he/she had their own curiosity about the reason behind the procedures. For instance, Mila had her own confusion related to the difference between the continuity for a point and continuity for an interval. She added some important "hashtags" for students' questions about why they consider them different in the procedures.

The indicators of the following table were determined given the fact of Mila's difficulties with mathematical procedures observed in her pre-interview. In addition to the indicator of why something is done, specifically, I determined two essential topics: indeterminate-undefined forms and limit at infinity-infinite limit. The most difficulty she had was knowledge of indeterminate and undefined forms. The pre-interview (9th question) showed that Mila had competency in noticing the indeterminate and undefined forms in questions/exercises/problems and solving the limit-related questions/exercises/problems that contained indeterminate and undefined forms. However, when asked why we call these forms in the limit, she could not answer such questions. On the other hand, during the pre-interview, she could not demonstrate her knowledge related to the procedures related to limit at infinity and infinite limit. In other words, the meanings of limit at infinity and infinite limit in the mathematical procedures were observed as lack of knowledge in the pre-interview of Mila. Since they included the answers to why something is done this way, I considered these forms under the sub-domain of mathematical procedures. The development of this knowledge was observed in all four phases of lesson study cycles. The following table shows the development across the lesson study process.

Table 4.5 Overall look the development of mathematical procedures in KoT of Mila across phases of lesson study

	Lesson Study Cycle 1		Lesson Study Cycle 2	
	Planning	Enacting	Planning	Enacting
Knowledge of how and when to do something	AE	AE	AE	AE
Knowledge of why to do something	AD	AD	AD	AD
Knowledge of meanings of limit at infinity, infinite limit	NAD	NA*	NAD	NAD
Knowledge of indeterminate-undefined forms (how and why)	NAD	NA*	AD	AD

AD: Adequate level of development NAD: Not Adequate Level of Development or Not Development AE: Already Existing NA: Not Observed

In the table given above, there are some columns that include asterisks with NA. They indicate the topics that could not be taught during the research lessons since there was no time or they were not in the curriculum. The table which summarizes the findings related to the development of knowledge of mathematical procedures shows that Mila had an adequate level of development in knowledge of why to do something in the first cycle planning and enacting phases of lesson study. However, it was interesting that the intended level of development could not be observed in knowledge of meanings of limit at infinity and infinite limit. It was observed in both pre-interview and beginning of the lesson study process that although she and other group members had sufficient level for conducting mathematical procedures related to them, she could not the reason behind why the coefficients of polynomials are treated in a procedure involving the division of two polynomial functions at an limit at infinity. As the adequate level of development, it was expected that she could demonstrate and explain the differences between these limits. However, it could not be observed throughout the lesson study process.

In addition, the indicator of why something is done in mathematical procedures is a quite comprehensive. It includes both knowledge of meanings of limit at infinity, infinite limit (why) and knowledge of indeterminate-undefined forms (how and why). Therefore, this section presents knowledge of why to do something in relation with the other indicators.

Planning Phases of Lesson Study

The indeterminate-undefined forms were the topics of lesson plan-2. The high school curriculum comprises only two types of indeterminate forms including $\frac{0}{0}$ and $\frac{\infty}{\infty}$. In addition, the curriculum is not required to indicate the difference between indeterminate and undefined forms. For this reason, in determining the lesson study goal, the group was unsure whether to add the other forms of indetermination to the lesson study goal or not. At first, other group members proposed that the lesson study goal could be the same as the objective of the curriculum. After the remark of the researcher to be careful about the notion of infinity in this objective, Mila mentioned herself as:

> Mila: I think it would never be enough to give one example anyway. In other words, I don't even remember which teacher said, but there was one teacher: it is always sufficient to give an example to prove something is not true, but it is never enough to give one example to show that it is true. So, we can show all these things you said (indeterminates and limit at infinity and infinite limit).

Since Mila had awareness of her own lack of knowledge thanks to the pre-interviews, she could propose such an idea about the lesson study goal. In spite of Mila's idea, the first draft of the lesson study goal did not include the indeterminate forms different from the curriculum. In the continuation of the meeting where the second lesson study goal was determined, there was confusion between the group members about teaching the L'hospital rule. To reveal their (Mila's in this case) own awareness of whether they are competent enough to explain this issue, the researcher asked them the basis of L'Hospital rule, and she answered as "*undefined (form)*". To be sure of the answer Mila, I asked her to check her own answer, and she expressed herself as "*Well, I used to think that they are the same things, aren't they?*". Two situations need to be noticed here: First, the real answer to the question asked is neither undefined nor indeterminate. On the other hand, when trying to say undefined, it is the case of thinking that the two are the same thing and are called undefined.

Since the lesson study process was planned to make them notice these issues the planning lesson phase, I did not intervene in this phase. This indicator was considered in two ways: First, why a form expresses indeterminate, and second, the difference between indeterminate and undefined forms. The development was observed, as seen in the figure below.

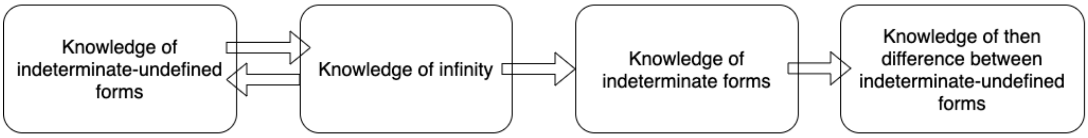


Figure 4.14. The pathway about the development of knowledge of indeterminate and undefined forms

The development of Mila's knowledge occurred through interaction between her groupmates, questions of the researcher, and the learning kit given by the researcher. In the first cycle for lesson plan-2, the first attempt to defined-undefined forms was the question of whether they would explain the notion of infinity or not. Mila mentioned that if the notion of infinity was to be shown, then they had to show number over zero. At this point, I (as a knowledgeable other) started a discussion "Could you please explain what you mean by the division of a number over zero?". The following excerpt shows the rest of the discussion.

> Researcher: Could you please explain what you mean by the division of a number over zero? How do you obtain infinity by division of a number over zero?
> Alp-Mila: Because of apple thing!
> Researcher: What is "apple thing"? Please, explain it to us (me and Fulya) by thinking that we are your students!

Mila: There is a video[5] on YouTube in Turkish, may I find it? But it needs to be explained a little more, and I don't think it is enough! And, it is a bit long to show in class.
(…)
Researcher: Actually, I don't understand the logic in the video! Could all of you tell me what the aim of the video is?
Alp: When I say dividing the apple into two, I understand it like increasing the denominator!
Researcher: What about the "zero"? Does it represent "absence"?
Alp: Well, I think… For example, we cut that apple in half, and the section is four. As a result, we increase the denominator, so I think it is there because it's like this forever here it will be zero!
Researcher: In the video, what did the scientist discover? Did he discover "zero" or "infinity"?
Mila: After your questions, I thought of something like this! By the way, I also had trouble watching this video, and I think it needs improvement. It also comes to my mind; for example, six divided by two is equal to three, or six divided by three is equal to two; maybe there is such a transition.
Researcher: Did you mean "cross multiplication"? Do you think that is possible in reel numbers?
Mila: Yes, I think…
Researcher: Before thinking about "cross multiplication," I would like you to consider how we don't recognize multiplication and the concept of infinity. Let's talk about the research done on this and the pages in the document I gave in the next meeting. In addition, I don't think that the message of the video is related to the notion of infinity!
Alp: I understood what you mean, that is so nice! There was a question related to what you said.

Before interpreting the evidence obtained from the analysis, some issues should be clarified. First, the video that Mila found on YouTube was one of the video series which was related to the history of mathematics. Particularly, Mila's video was related to Brahmagupta and its attempts in the history of mathematics. It can be remembered that Mila mentioned the same video to explain herself the notion of infinity during the pre-interviewing process. This means that the source of the knowledge Mila put forward at the beginning of the lesson study process is about

[5] The link of the video is https://www.youtube.com/watch?v=rfjz0Phv9ps&t=50s in which the time is between 3:07-3:22.

infinity. In the second part of the excerpt, which was associated with the video, she tried to show that the pieces were obtained as a result of dividing a whole into two, then into four. Then the remaining pieces in each step, are innumerable. Since she did not consider the limiting process in this video, it was not regarded as knowledge related to dividing a number by zero. After these discussions, she mentioned a mathematical procedure that she read in an article (this article was a result of an assignment given by the researcher). The mathematical procedure can be symbolized as "$if\ \frac{6}{2} = 3, \frac{6}{3} = 2, then\ \frac{a}{0} = \infty$. This expression cannot be considered as true, since the multiplication of 0 and ∞ is not defined in real numbers. Therefore, they needed to know how to define the "division" in real numbers. It was of importance to understand the undefined and indeterminate forms. Considering Mila's and other group members' lack of knowledge, I gave them a sheet that covered defining the division. In addition, there was an assignment for them to research this issue. However, the discussion at the beginning did not work since Mila insisted on the "cross multiplication" for describing as an indeterminate form.

> Mila: For instance, $\frac{0}{0}$ is equal to x. Cross multiply it! Then, x can be any real number. For this reason, $\frac{0}{0}$ is described as an indeterminate form.
> (…) Fulya: Undefined and indeterminate were different things. Up to this assignment, I did not know that!
> Alp: Please, show it to me, I also would like to see it.
> Mila: For instance, $\frac{0}{0}$ is equal to x. Cross multiply it! (The demonstration that she would like to show: $\frac{0}{0} = x$ then $0 = 0.x$ and x can be any real number in $\mathbb{R}$)
> Alp: Aa, yes!
> (…) Fulya: I realized that undefined had come to my mind when I encountered a mathematical thing that was in indeterminate form. I realized it here! They (indeterminate and undefined forms) are different things.
> Mila: Yes, isn't it?
> Fulya: I was immediately thinking of undefined forms!
> Alp: Yes!

The excerpt above can be seen in two places in this chapter: Development of KoT and development of knowledge for features of learning mathematics. This point of the meeting, where she tried to demonstrate the reason behind the indeterminate forms, could not be regarded as a correct demonstration. The aim of the lesson study process for these forms is to show the different limits in the same limit forms. While it will be mentioned in the second mini-cycle of lesson planning (they planned a rough lesson plan and discussed on it and that discussion was regarded as the second mini-cycle), it can be explained as the difference between $\lim_{x \to 0} \frac{sinx}{x} = 1$ and $\lim_{x \to 0} \frac{x^2}{x} = 0$ which all represent the same form $\frac{0}{0}$. This attempt of Mila can be regarded as incorrect to show why she conducted mathematical procedures about indeterminate forms (for instance, L'Hospital). However, it can be regarded as a step to find the right way. At this point, the researcher considered these difficulties and gave them assignments, including readings about these forms, and the researcher provided them to find the right way. The following excerpt shows the discussion on this topic.

> Researcher: You can also explain it in that way when you are explaining. You will start with indeterminate forms. There was a question the students asked Fulya as "why infinity divided by infinity is not 1". Fulya will ask you if it's a sufficient explanation.
> Fulya: I said that if there is only one number, there is infinity divided by infinity which equals to the number of a, then infinity is equal to a times infinity. Well, do I know this a, but it may be 1, it may be 2, it may be 1000, I asked whether could I say something definite for a, they said no, so I said there would be indeterminate.
> Mila: As if it didn't have an answer, why this is an answer to indeterminate or not.
> Alp: Let's say, here is already undefined! Again, infinity divided by infinity must be one for the inside and outside product.
> ... [Researcher intervened here]
> Fulya: I should have said there is an increase for infinity, it is not an infinite number.
> Mila: We need to be particularly careful about the infinity, a bit as an adjective! So, we can start from there and talk about indeterminates.
> Researcher: So, you can start there!

Fulya: For example, they asked why 1 to the power of infinity and zero times infinity indicated indeterminate.
Mila: They are right about the one to infinity, we should definitely mention it!

In this excerpt, it was observed that Mila had knowledge of the difference between the forms of indeterminate and undefined forms, which constructs a basis for why someone conducted the operations related to indeterminate forms.

What is interesting for this sub-domain is that there was not any development of knowledge of the meanings behind limit at infinity and infinite limit. While the prospective teacher had an adequate level of knowledge about how and when to do something in operations of infinite limit and limit at infinity, the prospective teacher had a lack of knowledge the meanings behind limit at infinity and infinite limit. The intervention during the lesson study process fell short to improve her knowledge about this issue. In addition, there were not any mathematical procedures on this topic in the research lessons of Mila to think on it particularly. For this reason, she was not eager to learn something about this issue when it was compared with other indicators.

Enacting Phases of Lesson Study

The lesson plan in which she would use the mathematical procedures intensely was the second lesson plan. Mila had the chance to conduct the third lesson plan in the first cycle and the second lesson plan in the second cycle. The mentioned development in indeterminate-undefined forms was placed in two lesson plans. The planning phases of all the lesson plans were designed considering all the situations which could occur in the classroom. However, there may be some unplanned situations in the classroom. In planned situations, Mila revealed her knowledge in an accurate way. The following example showed how she reacted to unplanned situations with her knowledge of mathematical procedures.

Mila: So that's why 1 to the power of infinity is indefinite.

Student 1: But that's not 1 to the infinity, right? We said 1 to the infinitely indefinite thing, isn't it something different? e is here. (She talked about $\lim_{n \to 0} \left(1 + \frac{1}{n}\right)^n$)
Mila: Hmm, is it confusing that it's equal to e?
Student1: No, there is a number called n.
Mila: Yes.
Student1: 1 is not infinity, I mean, I don't think they are the same thing!
Mila: There is a number called n, is your question related to it?
Student1: n goes to infinity or exactly 1 to the infinity is not equal to this.
Student 3: He means something (talking about his friend) different in two functions. As if the two functions are different, it's logical that we find different results anyway, isn't it?
Mila: Hmm I got it! But I'm telling you this. So, let's look at the equation I got over here, okay (it shows the resulting limit e)? When I look over there the limit n goes to infinity, and that inner side is equal to 1 for me. Therefore, when I overwrite it here, I get the 1 to the infinity form. Here I got 1 to the infinity, and what happens when I get the same form of other functions? Here I am writing the same thing again (showing the second function). Here, my inner side became 1 and my upper side became infinity. In other words, they seem to be different functions, but since we do not perceive infinity as a number, we say that it is increasing gradually, but we do not know how much it increases, so this is the reason why it creates indeterminate.
Student 1: I get it!
Student 2,5,7: Yeah, I understood perfectly!

Mila mentioned it at the beginning of the third lecture, as the participant who conducted the second lesson plan could not finish it. The students' question, which considered that it is natural for the limits of different functions to be different, was not an expected question for this issue. Using her knowledge of phenomenology with her knowledge of these forms, she could explain it as "these functions show the same behavior even if they are different functions". Such evidence showed that Mila had an adequate level of development for this indicator.

4.2.1.5 Development of Knowledge of Topics: Knowledge of Representation Systems

The final indicator of KoT is the registers of representation which is relayed to how the topic can be represented, including -for the concept of limit- graphical, tabular, geometric, number line, verbal and algebraic representations of limit. The verbal representation includes mathematical language and mathematical vocabulary, as well. In addition, it should be noted that I did not consider representations constructed with technology as a different representation type. Rather, I considered that technology is now everywhere, and all these representations are embedded in technology.

Table 4.6 Overall look at the development of representation systems in KoT of Mila across phases of lesson study

	Lesson Study Cycle 1		Lesson Study Cycle 2	
	Planning	Enacting	Planning	Enacting
Knowledge of graphical, tabular, geometric, number line, verbal and algebraic representations of limit	AD	NAD	AD	NAD
Ways to move between different forms of representations	NAD	AD	AD	AD

AD: Adequate level of development NAD: Not Adequate Level of Development or Not Development AE: Already Existing NA: Not Observed

During the pre-interview and the post-interview, Mila had paper (on which the questions were written) and pencils not only to write down the answers but also to use in cases when she could not express herself verbally. For this reason, the development was observed from the first meeting of the first planning to the last reflection of the second cycle of the lesson study process. For the development of representation, the rich materials showing how many different representations are used in teaching were used. In general, Mila had the chance to experience all registers of representations during the lesson study process, which allowed her to develop her knowledge. The following table shows the overall look at the development of representation systems in KoT of Mila. In the table given below, the abbreviation of NAD did not represent an adequate level of development, rather, it represents that all registers of representations were not used during the research lesson and reflection phases.

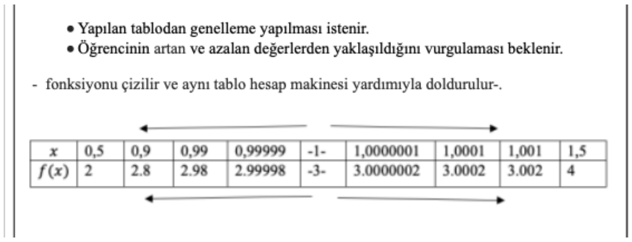
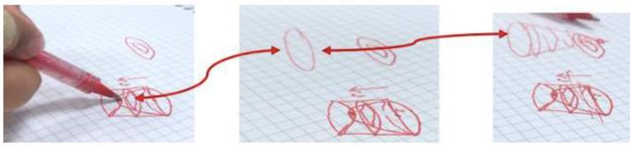
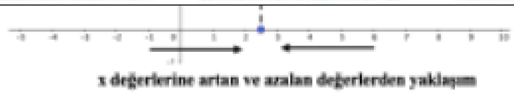
It is one of the interesting findings in Mila's development during the lesson study process that while her knowledge of registers of representation was observed in its contribution to the lesson study process, she did not use all of them during the enacting phase. In other words, she commonly used graphical and algebraic representations to explain something during her teaching. The second thing about knowledge of registers of representation was how to move between different forms of representations. At first, she had difficulty with this transition since she had lack of knowledge about other sub-domains. As others developed, knowledge development for this sub-domain was achieved in the transition between representations.

Planning and Enacting Phases of Lesson Study

During all the planning phases of the lesson study process for both cycles, as I mentioned above, the prospective teachers were triggered to use all representation types. In this section, I focused on directly representation types by which the prospective teacher contributed to the lesson plans during all cycles of the lesson study process for each three lesson plans. In Table 4.6, the registers of representations during all cycles of lesson study are presented with visual examples.

Considering the table and the analysis of data, almost all the development of prospective teacher’s knowledge of registers of representation took place in the first cycle of the first lesson plan. In the planning phases of the first cycle, there were lots of mathematical discussions about which representations should be used to teach the concepts effectively. In this way, they used all representations types proposed by the literature. In her teaching, the expectation of the lesson study process was that she was able to use the representation types which were different from the planned representations. However, Mila did not use any representation types different from the lesson plan to make explanations for students' questions related to the concept. Therefore, it can be said that she gained different perspectives about different representation types; however, an adequate level of development was not observed throughout the lesson study process.

Table 4.7. The registers of representations during all cycles of the lesson study process

<table>
<tr><th>Representation</th><th>Activity/task/assessment</th><th>Cycle</th></tr>
<tr><td>Graphical</td><td>Almost all activities,
Example: The tasks given in the lesson plan of the continuity</td><td>Cycle 1
Cycle 2</td></tr>
<tr><td>Tabular</td><td>The activity names as "approaching 1"

• Yapılan tablodan genelleme yapılması istenir.
• Öğrencinin artan ve azalan değerlerden yaklaşıldığını vurgulaması beklenir.
- fonksiyonu çizilir ve aynı tablo hesap makinesi yardımıyla doldurulur-.
<table><tr><td>x</td><td>0,5</td><td>0,9</td><td>0,99</td><td>0,99999</td><td>-1-</td><td>1,0000001</td><td>1,0001</td><td>1,001</td><td>1,5</td></tr><tr><td>f(x)</td><td>2</td><td>2.8</td><td>2.98</td><td>2.99998</td><td>-3-</td><td>3.0000002</td><td>3.0002</td><td>3.002</td><td>4</td></tr></table></td><td>Cycle 1</td></tr>
<tr><td>Geometric</td><td>Mila's representation proposed in the first lesson plan of the second cycle
</td><td>Cycle 2</td></tr>
<tr><td>Number line</td><td>
</td><td>Cycle 1</td></tr>
<tr><td>Algebraic</td><td>Almost all activities,
Example: The demonstration of the reason of the indeterminate form of 1^{∞}.</td><td>Cycle 1-
Cycle 2</td></tr>
</table>

Another important point for knowledge of representation systems is ways to move between different forms of representations. In the pre-interview of Mila, she usually moved from the algebraic representation to graphical representation or vice versa. The reason of limited usage can be described as her lack of knowledge of various representations, as said before. In planning lesson plans, the transitions between representations were observed as from tabular to number line, graphical and algebraic, and from geometric to tabular and algebraic. For instance, the transition between representations occurred from tabular to graphical representation in the beginning of the lesson plan to provide an understanding of "approaching". In her teaching session, she usually moved from algebraic to verbal, or vice versa, when she acted differently from the lesson plan. Since she had valuable contributions about the transition process, I considered her development as AD in the table (Table 4.7).

4.2.2 Development of the Prospective Mathematics Teacher's Knowledge of Structure of Mathematics in the Concept of Limit

The second sub-domain of mathematical knowledge in the model to answer the first research question is knowledge of the structure of mathematics. Knowledge of the structure of mathematics (KSM) can be described as a mathematics teacher's knowledge related to connections between mathematical concepts (Carrillo-Yañez et al., 2018). While there was not distinct evidence related to Mila's existing KSM, it was understood from interrelated indicators (e.g., phenomenology and applications) that her knowledge was categorized as existing but not sufficient. In addition, the beginning of the lesson study also included evidence about her KSM. The sub-domain consists of four indicators, including connections based on simplification, connections based on complexity, auxiliary connections, and transverse connections.

Based on the first research question, it was aimed to look for developments in these four indicators of KSM. However, interestingly, the development could be provided in two of four indicators of KSM, including transverse connections and auxiliary connections. While there were some evidences related to other indicators of simplification and complexity, these evidences could not be considered as an adequate level of development or as having the knowledge. For other indicators, the development through lesson study is shown in the following table (see Table 4.8).

Table 4.8 Overall look at the development in KSM of Mila across phases of lesson study

Knowledge sub-domain	Lesson Study Cycle-1		Lesson Study Cycle-2	
	Planning	Enacting	Planning	Enacting
Auxiliary connections	NAD	AD	AD	AD
Transverse connections	NAD	NAD	AD	AD

AD: Adequate level of development NAD: Not Adequate Level of Development or Not Development AE: Already Existing NA: Not Observed

Transverse connections can be described as teachers' knowledge about connections of mathematical concepts which have common features in mathematics. In addition, auxiliary connections can be described as teachers' knowledge about connecting mathematical concepts by using them as an auxiliary element for teaching (Carrillo-Yañez et al., 2018). In the curriculum, the concepts after the concept of limit were derivative and integral. Since those have the same basis with the concept of limit, I considered them under the indicator of transverse connections. Auxiliary connections occurred through the lesson study process. I expected that she would use related concepts as auxiliary elements for teaching the concept for an adequate level of development since Mila stated that the limit is a separate concept from other concepts in the pre-interview.

In general, the table shows that the planning phase of the first cycle of the lesson study prepared the participant for the development in relation to other sub-domains for both indicators. Since the first cycle mainly focused on the development of KoT, it was only possible for the participant to connect with other concepts when this fundamental knowledge area (KoT) was developed, and this took place in the second cycle. While the development in auxiliary connections started to be observed in the enacting phase of the first cycle of lesson study, adequate development was provided in the planning phase of the second cycle for knowledge of transverse connection. To be more specific, the change in KSM was observed from the planning phase of Lesson Plan-1 in the first cycle to the reflection process of Lesson Plan-3 in the second cycle.

4.2.2.1 Development of Knowledge of Structure of Mathematics: Knowledge of Transverse Connections

Transverse connections are linked to the nature of mathematical objects that are displayed in various forms in different contexts throughout various stages of education. The nature of the concept of limit is directly related to the notion of infinity. Infinity constructs a basis for other concepts related to the concept of limit as well as the concept of limit. For this reason, the notion of infinity is included in both KoT (foundation of the concept) and KSM in terms of transverse connections in relation to the derivative and integral. In this section, I focus on the notion of infinity in a broad sense with regards to these two functions of infinity in teaching the concept of limit. Therefore, the transverse connections were observed in three lesson plans in each cycle throughout the lesson study process.

In fact, what I expected from the participant for the development of this knowledge was to gain awareness about the transverse connections with the concept of limit. The prospective teacher had already had this awareness about the relation between the concepts, as understood from her discourses- "*(...) that is to say, the concepts related to limit, derivative and integral come to my mind (...)*". However, she was not aware of where this relation comes from, in particular what the underlying concept or feature of this knowledge is. Therefore, the knowledge of infinity is crucial to gain this awareness, which led me to develop her knowledge of infinity. Since the notion of infinity is also considered in fundamental knowledge, the development of knowledge related to the notion of infinity was described in the section of KoT. In this section, the notion of infinity was dealt with the basis for the concepts of transverse connections with the concept of limit. Due to the development of knowledge of infinity and, therefore infinitesimal approximations, the history of the concept is used to bring awareness to which this section was concerned.

Planning Phases of Lesson Study

The historical development of the concept was the first step to raise the prospective teacher and other group members' awareness about the fact that different content items have features in common; in this sense, the notion of infinity connects the concepts of limit, derivative, continuity, and integration with its common feature. Mila and the group needed to understand the logic behind these concepts. In the first cycle of Lesson Plan-1, the learning kit, which included the historical development of the concept, was delivered to the group members. The group studied the statements given in the kit. As described in the section on knowledge of foundation in detail, Mila could consider the relationship between the derivative and integral when she understood what it meant that the secant approaches the tangent. An important point here was to be aware of the infinity and infinitesimal approach in this relation.

In the planning of Lesson plan-1 in the first cycle, she realized the relation shown in Table 4.6. To deepen her understanding, I questioned her understanding as can be seen in the following. Although the following excerpt seems to be the continuation of the except specified in Table 4.6, the prospective teacher came to this level after different discussions and planning activities (For instance, the problems proposed for the lesson plan). Those provided her to see beyond the relationship between the concepts.

> (…) Mila: I think so too, hocam. I thought about it a little. First, we gave the date, you know, we gave it; Then we can say: You see the derivative you use in your physics lessons, which we will see in the future, or the integral that we will study in the future; but actually, the limit was found after these concepts. So even if we see these issues after the limit, a concept was needed to define the derivative and integral, but this concept was named as limit much later. "First of all, Cauchy did it," he said. Did

> you notice the connection with the first event we did? I thought we'd wrap it up with this type of question. I mean, doing something like going into history from here and then coming back here.
> Researcher: Well, good. How would you explain the connection between derivative and limit? So, what makes them common? I expect you to explain as if you were answering your student.
> Mila: Well… Ok … (she thought a bit) Actually, the first thing we talked about is infinity. Look, we talked about infinity and infinitesimal approaches in all of them. For example, since Zeno, infinity and limit seem to have developed together. Now, if we talk about the derivative, actually finding the instantaneous velocity by proportionally smaller and smaller the slope there, for example, the approximation we saw in that Cauchy.

The first expression of Mila indicated that she knew the relationship between the concepts. It was an expected expression for Mila, since she had already revealed her knowledge in the pre-interview. After the researcher asked the probing question, her answer showed that Mila gained her awareness through rich group discussions, since she started with her answer as *the first thing we talked about is infinity*. Then, she caught the relation point between the concepts of infinity and infinitely smaller steps as a transverse connection. Though she did not show her explanation on board or on paper, she tried to show her knowledge by illustrating her hand. Based on the field notes about the prospective teacher's illustration, I could make sense of her illustration as "*getting smaller and smaller of the triangle created while finding the slope*". In this sense, she established a relation between infinity, limit, and derivative. As described earlier (in KoT), her knowledge related to the notion of infinity remained as potential infinity. The reflection of her awareness is consistent with her KoT as the foundation of the concept. While the infinitesimal approach was discussed in the first lesson plan, the development of her knowledge from potential infinity to actual and potential infinity[6] through lesson study cycles. What the interesting finding was that the prospective teacher to be attained this awareness

[6] Monaghan (2001) indicated that the idea of infinity is like the two sides of the coin: infinity as object (actual) and infinity as process (potential).

earlier than others. However, I expected her to show it in selecting or creating a task rather than expressing it verbally. For this reason, the development can be described as a not adequate level of development.

The expectation of the designed lesson study process was yielded in the second cycle. Before moving on to the second cycle, I must point out the preliminary stage of this finding. The literature indicates that the conception of infinity is closely related to the conception of limit. Therefore, in the pre-interview, Mila's existing knowledge related to the notion of infinity was examined. Mila's existing knowledge can be described as 'potential infinity', since she mentioned infinity as 'ever increasing size' in her answers. In addition, when I asked her how infinity could be explained to students, she answered the question by exemplifying her thoughts with a story, including *dividing an apple again and again*. The question was not directly related to KSM; however, it provided me with understanding her knowledge of infinity. She conceptualized infinity in her mind as '*a repeating the process or 'endless move.'* While there was not the concept of limit in the question, this was an expected answer. Such an answer is categorized as potential infinity closely related to the dynamic conception of limit. Therefore, I observed that Mila interrelated the concept of the limit with the potential conception of infinity during the lesson study process. This endless process and infinity-limit connection that Mila and the group established was also seen in the activity they revised in the planning of the second cycle. In the research lesson of lesson plan-1, the participants were not satisfied with the feedback of students they received. The activity they planned did not work effectively in the research lesson. Therefore, they decided to change the activity for the beginning of the lesson plan. Among different ideas for the activity, they decided to make a connection between mathematical concepts including limit, geometry, and infinity in an activity named as "Finding the area of circle". In the activity, students are distributed circles in different sizes for each students' group in the classroom, and teachers want to cut or divide the circles into 4, 5, and 6 pieces. Then, they combine the pieces with one edge adjacent (see Figure 4.15). The following excerpt is a part of the re-planning phase of lesson plan-1.

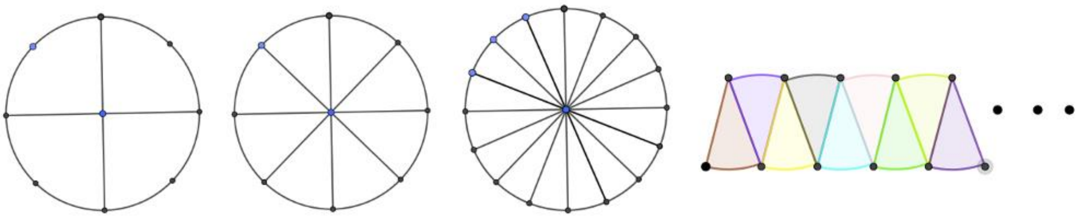


Figure 4.15. The activity named as “Finding the area of circle”

> Mila: We will ask the area of the circle and the students will say πr^2, what if we ask how we can prove it, I mean how can we show it?
> (…) Alp: Yes! They actually use the limit implicitly here. That third shape on the right here, as the area of the triangle in that shape increases, the thing looks like a parallelogram at first. Then, it approaches the rectangle. … It's a rectangle when you get 1 million or so sectors!
> (…) Fulya: Actually, we get "Riemann sum" here, right? Starting from the infinitesimal calculus, we can actually take the limit of the sum of the series in infinity without saying integral!
> Mila: So actually, we can do it starting from this sector, just as we get a parallelogram when we divide the sectors into more and more smaller pieces which we cannot see clearly. The concept of Limit is composed of small approximations in this way, Cauchy said so.

Though the excerpt given above does not include “infinity” explicitly, it shows prospective teachers’ knowledge about infinity. Mila’s last interpretation of the activity, which includes “*more and more smaller pieces which we cannot see clearly”*, indicates a significant property of the remaining item in a series, which is the underlying idea of the infinitesimal. The idea of infinitesimals directs me to potential infinity as phenomenological aspects of infinity. In this way, the interpretation of Mila is a clear example of the relation between infinity and limit, specifically transverse connections in KSM. In addition, based on the notion of infinity and infinitesimal approximations, Mila established a relationship between the concept of Riemann sum (when Fulya mentioned it) and, therefore the integral and its limit. This can be concluded since there was a common sense about the topic in the discussions. Thus, the expectation was met since she could make transverse connections between the related concepts in creating a task. Therefore, an adequate level of development for planning was interpreted considering this finding.

In this way, the lesson study provided her to gain her awareness of these connections between the related concepts. Though she did not reveal her knowledge in her research lessons (in enacting phases), the planning and reflecting stages provided me to observe her awareness. As mentioned at the beginning of this section, the transverse connections are mainly observed in lesson plan-1. For this reason, the reflection on the research lesson of lesson plan-1 was important for observing her awareness.

Enacting Phases of Lesson Study

While the transverse connections were not observed in research lessons of enacting phases, how he reflected her awareness was observed in reflections for the research lessons that were part of enacting phases. In the reflection for the research lesson of lesson plan-1 in the first cycle, the awareness of the transverse connection was observed in the case that Mila drew attention to the issue which Alp (the research lesson of the first cycle conducted by him) could not reach the intended goal of the lesson plan. In the first lesson plan-1, the lesson study group aimed to reinforce students about the relationship between applications of the limit and how they could connect these concepts. The part of her reflection given below shows that Mila was aware of this connection.

> Mila: I guess the secant approach to tangent event is not opened on the smart screen. Of course, you failed to show the relationship between derivative and limit. In the second application, we have to find a solution to this because we can explain this relationship by referring to it in the topics that will come after it.

She did not express her knowledge explicitly. Instead, her last sentence shows that she was aware of how she could use this relation to the topics to be taught. However, she did not mention how these concepts are related to each other. Since the expectation of the lesson study process was to observe her knowledge in enacting phases, there was development but not adequate for the intended level of development. In addition, this reflection also showed that she wanted to use this relation as an additional element for teaching the topics which will come after it.

However, it cannot be considered as evidence for knowledge of the auxiliary connection between the limit concept and other mathematical concepts. In the following section, the findings related to auxiliary connections are presented in detail.

After the development occurred in the planning phases of the second cycle, as given above, she reflected on her knowledge of application. In her research lesson in the second cycle, she specifically indicated the basis of the transverse connections between the Calculus concepts, including derivative and integral.

4.2.2.2 Development of Knowledge of Structure of Mathematics: Knowledge of Auxiliary Connections

Another indicator of KSM observed throughout the lesson study process is knowledge of auxiliary connections. Auxiliary connections are related to mathematical concepts that are not directly linked to the situation being considered in the classroom. It requires the necessary participation of an item in larger processes. Similar to the model's creators (Carrillo-Yañez et al., 2018), I examined this connection by examining the connections built in the problems that the group proposed and used in the lesson plans. In general, they connected the concept of limit with geometric concepts (see Figure 4.15). In this section, these connections are described in detail by associating the other sub-domains of mathematical knowledge in MTSK (e.g., the categories of knowledge of topics).

Planning Phases of Lesson Study

At the beginning of the process, Mila and the group could not relate the concept of limit with other mathematical concepts. Mila expressed herself as "*In fact, I realize that I haven't made much connection between limit and other concepts in my head*." Therefore, I aimed to develop their understanding in relation to other mathematical concepts. Moreover, I sought to enable them to think by making connections between other mathematical concepts and limit, and to reflect this in their teaching.

Considering this aim, I presented them rich materials both for their discussions and their teaching. While the curriculum in Turkey limited the participants in terms of approaching the concept from a wider perspective, the materials presented during the process provided a way for the development of their understanding. These materials were given as needed throughout the lesson study process. In this way, the following table shows the connections in both three lesson plans.

Table 4.9 Auxiliary connections built during the lesson study process

Connection with #	Activity name	Lesson Plan # and Cycle ##
Geometry	Finding area of circular region by mean of parallelogram	Lesson Plan1 – Cycle 2
Geometry	Construction of cylinder	Lesson Plan1- Cycle 2
Equations and Factorization	Exercises using knowledge of equations and factorization	Lesson Plan 2- Cycle 1&2
Algebra	Finding the place of “pi” on the number line (Iterative process)	Lesson Plan1- Cycle 1
Algebra	Function	Lesson Plan2&3- Cycle 1&2

The important point of these connections was Mila’s contribution to these connections. During the planning phase of lesson plan-1, she used the construction of cylinder as an auxiliary element for teaching the phenomenological aspect of the concept as “approaching”. Though this activity did not use in the lesson plan, the contribution of Mila to lesson plan 1 showed that she used the construction of cylinder as an auxiliary element for teaching the concept, as can be seen in the following excerpt.

> (…) Actually, we can use geometry for this activity. At the beginning of the lesson study process, we watched a video related to the sphere. In addition, there is something related to finding the area of unknown shapes.
> Alp: In other words, while we were very creative at the beginning, as we got the information, our creativity seemed to blunt.

Mila: I have said “cylinder”
Fulya: They (students) have already known cylinder. There was something, you were rotating the rectangle, you were creating a cylinder by rotating it 360°, it had phases, you say that, right? You slowly turn...
Mila: Actually, it is what I meant. But I'm not saying rotation; what I'm saying is directly like that (Figure 4.16): We have a triangular prism, and you increase the number of bases from quadrilateral to pentagonal, to hexagonal, you increase it, and it becomes the last cylinder.

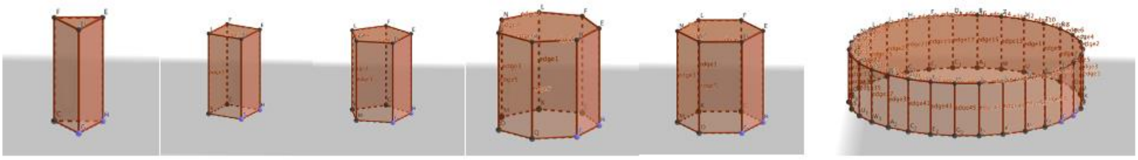


Figure 4.16 The way Mila wants to tell and the way she shows on the screen while telling

In fact, the activity she proposed has been used for finding the volume of cylinder in some materials. She used it as an auxiliary element for providing them to discover the idea of *approaching*. In addition, she connected the concept of integral with limit. However, it cannot be said that just giving this suggestion is sufficient for us to see the connection exactly. Here I expected her to reveal the connection of the concept of integral with the limit underlying and emphasize teaching it. Therefore, it cannot be said that the development was adequate, and I intervened by directing them with related readings at this point.

The same relation was observed between finding the area of the circular region by means of parallelogram and the phenomenology of the concept in the planning phases of the second cycle. The excerpt is given in the transverse connection related to finding the area of the circular region also included auxiliary connection, since finding the area of the circular region was taught in the 10th grade in the subject area of geometry in the curriculum. By means of the rich group discussion, which was held after Mila had done the necessary readings, she revealed the reasoning behind the concept of integral, which we could not observe above. This fact directed the researcher to consider it as an adequate level of development.

Mila: So, I think we can use one of these (he's talking about the cylinder and the circle). These are already directing us to the limit.
Researcher: It's beautiful! In one of these articles, I gave you, they think the limit starts when they look at eighth grade books.
Fulya: So, they say that the concept of limit was introduced there when doing this or creating a Parallelogram from here.
Mila: Maybe we can start with a discussion like this (showing the GeoGebra slider) by asking what they think would happen if we increase it, and then ask them to make an inference based on their observations.
Researcher: And even if they don't know the integral, they can find the area under the curve with the limit.
Mila: Approaching with a margin of error.

As said in transverse connection, it is of importance how this connection would help the prospective teacher to teach which concept of trick point. In the excerpt above, she mentioned it as “approaching with the margin of error”. Since the concept of limit provides eliminating the margin of error, she used geometry to show this important point of the concept. Moreover, this example prepared students for one of the future concepts, integral. This example showed an important finding that the development of knowledge to establish relationships between concepts in mathematics teaching was intertwined during the lesson study process. It can be explained as the nature of the mathematics, in particular the nature of the concept.

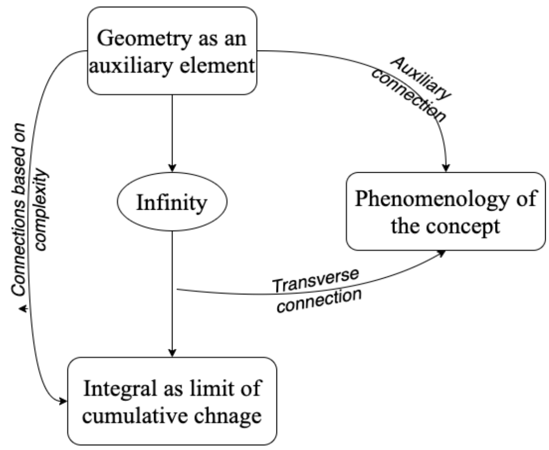


Figure 4.17. The intertwined relation between the developments

Figure 4.17 shows the overall relations between the connection and concepts. While geometry is the main domain in mathematics, which includes more than one concept, I considered it as an element for auxiliary connections. It should be noted that she as well as the group commonly used geometry for auxiliary element to teach the concept of limit. Furthermore, infinity is in the center of the Figure since it is an element included in both phenomenology of the concept (intraconceptual connection) and transverse and auxiliary connections (interconceptual connection). The Figure also includes *connection based on complexity* which I did not mention above as a development. The reason why I did not mention complexity as development was that it was a situation observed individually only, and Mila's contribution cannot be observed sufficiently.

Enacting Phases of Lesson Study

In her pre-interview, Mila could not connect geometry with the concept of limit. When she was asked the reason behind the calculations in indeterminate forms of limit, such as why we only get the coefficients in $\lim_{x \to 0} \frac{sinx}{x}$, she could not answer such a question. Thanks to discussion on the Sandwich theorem in planning phase of lesson study, she understood how to verify the Sandwich theorem. In her teaching, she reflected her knowledge by using geometry as an auxiliary element.

> Mila: We know that limit $\frac{sinx}{x}$ is equal to 1. Yes, this actually indicates an uncertainty, but we actually need a theorem to verify this. We call this theorem the Compression theorem. The compression theorem tells us that when we bring x closer to a, the value of functions $f(x)$ and $g(x)$ get closer to L. If there's a value of $h(x)$ between my functions $f(x)$ and $g(x)$ as I approach this L, I'm tying it with less than equal, and as the limit x approaches a, there's $h(x)$ and $g(x)$ between my functions. We say that the function $h(x)$ has a limit. And from here, we say that we find the limit of the $h(x)$ function in this way. How do we demonstrate this theorem? What mathematical concepts can we use?
> Student 3: May it be L'Hospital? Or, I don't know.
> Mila: [After waiting a second] To demonstrate this theorem, we will use the representations of trigonometric functions on the unit circle.

The demonstration (Figure 4.18) can be found in any textbook. The finding at this point was her internalization of this subject and her explanations also in unexpected situations. After she demonstrated and verified the Squeeze Theorem on the

$\lim_{x \to 0} \frac{sinx}{x}$ by using sectors and angles (see Figure 4.18), one of the students asked the reason for the sign of inequality in the theorem which is the sign of less than or equal to. At this time, Mila answered the question by thinking on quickly:

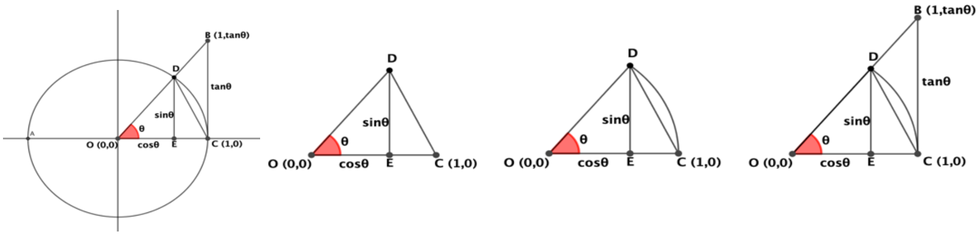


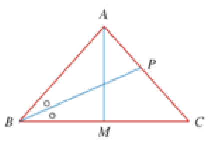
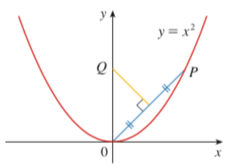
Figure 4.18. The figure used in the proof of Squeeze Theorem

> Mila: When we compare the areas of these three geometric shapes including two triangles and a sector, there is such an inequality $\frac{sin\theta}{\theta} \leq \frac{\theta}{2} \leq \frac{tan\theta}{2}$. (…) Can we get the limit when the θ angle approaches zero for all units of this inequality?
> Students: Yes!
> Mila: Then, the limit of $\frac{\theta}{sin\theta}$ when angle approaches zero is between 1 and 1. For this reason, the limit of $\frac{\theta}{sin\theta}$ became 1 . …
> Student1: I did not understand why we put equality when comparing the fields. The fields may be smaller than each other, but I don't understand why they are equal!
> Mila: Yes, you're right! Because let me tell you, we don't know exactly what this angle is; If I now bring this angle closer to zero, for example, wouldn't the area of the triangle here be equal to the area of the big triangle? I take it because I can change the angle. I change the angle; it gets a different value. So, imagine this point as a playing point. When I play this point, I can get all of them in the same way.
> Student1: Yeah, I understood it! Thank you!

In the excerpt above, Mila answered the student's question by explaining the reason for the sign of inequality in the theorem. Though she could not use the smartboard because of technical problems, she showed what she wanted to teach by using her fingers like a moving point on the circle. When the pre-interview was considered, the current example can be regarded as evidence for the progression of Mila's knowledge.

Another auxiliary element that Mila used for teaching the concept of limit was knowledge of equations and factorization. As said before, she mostly used auxiliary connections in presenting exercises. Using equations and factorization as an auxiliary element might be considered as an expected finding since the current study was conducted in Turkey. In Turkey context, there is a central examination system that specifically effects 12th grade lessons. While the group constructed their lesson study goals on students' active learning, they also aimed to prepare students to high stake exams. Therefore, the exercises in lesson plans are usually linked to the use of equations, factorization, and trigonometry. The following table shows examples from the lesson plans.

Table 4.10 The examples from the lesson plans with auxiliary connections

Examples	Linked to #	Lesson plan #	Cycle #
Write the value of the limit $\lim_{x\to -8}\frac{\sqrt{x^2-15}-7}{\sqrt[3]{x}+2}$.	Equations	Lesson Plan 2	Cycle 1
Find the value of $\lim_{x\to 0}\frac{\left(sinxsin\left(\frac{\pi}{2}-x\right)\right)^2}{xsin4x}$.	Trigonometry	Lesson Plan 2	Cycle 1
 The figure on the right shows the point P on the parabola and the point Q where the mid-perpendicular of OP intersects the y-axis. What would you say about point Q when P approaches the origin along the parabola? Do you think there is a limit operation here? If so, show it.	Geometry	Lesson Plan 1	Cycle 1
 The figure on the right shows the point P on the parabola and the point Q where the mid-perpendicular of OP intersects the y-axis. What would you say about point Q when P approaches the origin along the parabola? Do you think there is a limit operation here? If so, show it.	Geometry	Lesson Plan 1	Cycle 1

4.2.3 Development of the Prospective Mathematics Teacher's Knowledge of Practices in Mathematics in the Concept of Limit

To answer the first research question, the latest sub-domain of mathematical knowledge is knowledge of practices in mathematics (KPM). KPM can be considered how mathematics is developed beyond any particular concept (Carrillo-Yañez et al., 2018). Mainly, KPM covers "knowledge of ways of proceeding, validating, exploring, and generating knowledge in mathematics, such as knowledge of ways to communicate mathematics" (Carrillo-Yañez et al., 2018, p. 245). In this study, the development in KPM was dealt with in three indicators including knowledge of ways of validating and demonstrating, knowledge of the role of symbols and use of formal language, and knowledge of necessary and sufficient conditions for generating definitions (Delgado-Rebolledo & Zakaryan, 2020).

It can be said that the prospective teacher is in good condition when viewed from the point of view of her course grades, regardless of the content of mathematical knowledge. In the pre-interview, while there was not any separate question evaluating the KPM of the prospective teacher, it was understood from answers of Mila that she had existing but not sufficient knowledge about necessary and sufficient conditions for generating definitions, the role of symbols and using formal language and validating-refusing a mathematical statement. Furthermore, at the beginning of the lesson study process, it was observed that the prospective teacher, as well as the other group members tended to memorize the rules or the features of the concept, rather than producing them based on their existing knowledge. Therefore, the primary aim of the lesson study process for KPM was to support them in producing mathematical knowledge.

Considering this aim, the probing questions evoked their knowledge to produce mathematics. At the same time, such an attempt taught the participant how to teach the concept to support students to think with an alternative approach to constructing mathematical knowledge. In addition, the resources given by the researcher provided students to see validating procedures related to the theorems in the limit concept.

However, only resources did not help in the development of this knowledge. Instead, although it was not at the desired level, the development has been achieved thanks to the discussion environments supported by probing questions, as I just mentioned. In this way, the planning phase of lesson study provided to develop their knowledge of mathematics beyond any particular concept.

Table 4.11 Overall look the development in KSM of Mila across phases of lesson study

	Lesson Study Cycle-1		Lesson Study Cycle-2	
	Planning	Enacting	Planning	Enacting
Ways of validating and demonstrating	NAD	NAD	NAD	NAD
Role of symbols and use of formal language	AD	AD	AD	AD
Necessary and sufficient conditions for generating definitions	NAD	NAD	AD	AD

AD: Adequate level of development NAD: Not Adequate Level of Development or Not Development AE: Already Existing NA: Not Observed

Table 4.11 shows that there were three indicators in KPM observed during the lesson study process. In two of them including role of symbols and use of formal language and necessary and sufficient conditions for generating definitions, the researcher observed development at the end of the lesson study process. Knowledge of role of symbols and use of formal language was developed during whole process. Mila was already an aware prospective teacher about using formal language in her teaching. The adequate level of development for this indicator was considered as to be aware and eager of using symbols and formal language in teaching the concept of limit; in other words, expressing herself mathematically. From beginning to end of the process, she gained this awareness in her discourses and her teaching. Therefore, it was considered as adequate level of development during the whole process.

The development of the other indicator-knowledge of necessary and sufficient conditions for generating definitions was developed in the planning phase of the second cycle. In particular, the indicator was considered as another step of knowledge of definition in the sub-domain of knowledge of topics. Therefore, the development in knowledge of definition led to the development of this indicator. However, this development was not examined individually; rather it was observed that the development occurred holistically. Different from these indicators, the development in knowledge of ways of validating and demonstrating was not considered as adequate level of development. It was expected that Mila could generate mathematics, validate or demonstrate a mathematical statement in formal way. However, the adequate level of development could not be observed for this indicator.

4.2.3.1 Development of Knowledge of Practices in Mathematics: Knowledge of Necessary and Sufficient Conditions for Generating Definitions

The first indicator of knowledge of necessary and sufficient conditions can be described as how different characteristics of definitions is provided to develop a mathematical truth (Carrillo-Yañez et al., 2018). As said before, this indicator is closely related to knowledge of definition mentioned in the sub-domain of knowledge of topics (KoT). Therefore, this indicator was not considered as putting forth the definition of the concept. Rather, it was dealt with as constructing a setting to reveal the definition of the concept; the formal definition in this case. In this way, the development was observed in planning phase of lesson plan-1. In this section, the knowledge of necessary and sufficient conditions was presented in the planning phase of lesson study. It could not be examined in the enacting phase, since Mila did not have a chance to conduct the research lesson of lesson plan-1.

Planning Phases of Lesson Study

In the first cycle, Mila and other group members proposed an activity which included a story on the notions of epsilon and delta to link the intuitive definition to formal definition. The idea of activity proposed by another group member (Alp) and Mila was storified into lesson plan-1. As can be seen in Figure 4.18, the story for intuitive definition was related to finding the student talking in the classroom. In the story, the teacher tried to find the student by approaching his/her step by step. The approaching of teacher represented “delta (δ)” and approaching of student represented “epsilon (ε)” in the story. The lesson study group tried to reach the idea of “there can be at least one delta for every epsilon”. However, the activity had an important limitation that it did not include symbols or formal language. Therefore, this attempt was considered as not adequate level of development.

While it was considered as “not adequate level of development”, at the end of the first cycle, as Mila's other knowledge sub-domains were developed, her awareness of the need for a more mathematical way developed. In the reflection for the lesson plan-1, she criticized the activity as “*the lesson plan did not serve our aim about the formal definition, the formal definition came as a crack, as if it had never been connected. So even if the students see it verbally, they cannot understand it because there are no terms*”. It was an important finding for the study that Mila stated the activity she found was logical at first, but later she did not find it mathematically correct and it did not serve well for teaching. The finding showed not only her awareness about using mathematical language but also the importance of development of related indicators (e.g., knowledge of definition) in the mathematical knowledge.

> A teacher hears a noise in her classroom and realizes that the source of the noise is from the desks at the middle of the classroom (this represents our initial epsilon value). The teacher approaches the desks to find the source of the sound (delta approach). When the teacher approaches the desk, he realizes that the sound is not the students in the front or back desks and eliminates them (new epsilon value). The teacher, on the other hand, gets a little closer (a new delta versus a new epsilon).

Figure 4.19. The first activity for the formal definition

In the second cycle, the clear change was observed for this indicator in the activity in lesson plan-1. While there was not any mathematical basis in the first cycle, the second cycle included a flow from the known to the desired to be taught by constructing essential conditions. As can be seen in Mila's proposed activity which she adapted from Stewart Calculus (Stewart, 2008) in Figure 4.19, the lesson plan proposed a function to mathematize what students talked about in the class (this was an assumption). The elements in the former activity to construct the notions of "ε" (epsilon) and "δ" (delta) were used.

A teacher hears a noise in her classroom and realizes that the source of the noise is from the desks at the middle of the classroom (this represents our initial epsilon value). The teacher approaches the desks to find the source of the sound (delta approach). When the teacher approaches the desk, he realizes that the sound is not the students in the front or back desks and eliminates them (new epsilon value). The teacher, on the other hand, gets a little closer (a new delta versus a new epsilon).

$$f(x) = \begin{cases} 2x - 1, x \neq 3 \\ 6, x = 3 \end{cases}$$

Intuitively, we can say that the behavior of the function f(x) at the point $x = 3$ is 5, which is not equal to 3. Then we can say that the limit of the function $f(x)$ at the point $x = 3$ is 5.
Moving from this verbal approach to a slightly more mathematical expression,

How close does x get to 3, such that $f(x)$ is less than 0.1 away from 5?

! It is assumed that the student will show this by drawing on a graph.
! Students are expected to switch from the graph of the function to the absolute value. At this stage, learning needs to apply the guiding steps.
! The reason for choosing 0.1 should be emphasized that the margin of error is 0.1 at first.
! A connection is established between the given sample and the sound sample. (When we get closer to the sound, we reduce the absolute value-epsilon.) That is, when we get closer to the sound, we decrease the distance.

Expected Answer: The distance of the values x can take from 3 is $|x - 3|$ and the distance of the values that $f(x)$ can take from 5 is $|f(x) - 5|$.

Mathematically (symbolically) expected answer:

If $|f(x) - 5| < 0.1$, then $|x - 3| < \delta$. But x should not be 3 ($x \neq 3$).
If $|f(x) - 5| < 0.1$, then $0<|x - 3| < \delta$ (! Students should be mentioned the reason of why the point do not equal 0).

Figure 4.20. The revised activity related to the formal definition of limit

In the former lesson plan in the first cycle, it was hard to construct in students' mind. Because there was not any preparation for the ingredients in the formal definition. By extending the activity by laying the foundation for the definition, she showed her knowledge of necessary and sufficient conditions for definition. Since Mila started to revise the activity with her suggestion, the development of the indicator was considered as adequate level in the second cycle.

The example given in this section also refers to the development of Mila’s knowledge of role of symbols and use of formal language. As said before, from the beginning to the end, there was a clear development in Mila’s knowledge of role of symbols and use of formal language. Considering the evidences similar to the example given above, the development of this indicator was considered as adequate level. However, the development of this indicator could not be observed in another indicator-knowledge of ways of validating and demonstrating. In the following section, this indicator- knowledge of ways of validating and demonstrating is presented.

4.2.3.2 Development of Knowledge of Practices in Mathematics: Knowledge of ways of validation and demonstrating

The rare development was observed in this indicator - ways of validating and demonstrating in KPM of Mila. The indicator includes the ways to verify a mathematical statement, or show it false, and the role of given examples or counterexamples (García et al., 2021). For this indicator, it was expected that Mila could easily validate or demonstrate a mathematical statement, since she took advanced level mathematics courses very recently and in the same period of the lesson study. To reveal her fresh knowledge, the rich mathematical resources including the course notes from different instructors in the same university were provided during the lesson study process. Furthermore, she and other group members were triggered to explain and demonstrate their expressions mathematically. However, the intended development could not be observed in this indicator. In this section, similar with the previous examples, the examples were given in planning and enacting phases of lesson study.

Planning Phases of Lesson Study

Knowledge of ways of validating and demonstrating was observed in planning phases of all three lesson plans with different examples. The elements mentioned above provided her to think on validating and demonstrating a mathematical statement (or a theorem) in planning phases. While her acts could not be considered as adequate level of development, there were two interesting findings about this indicator. First, in the discussions about a possible reason question that a student might ask about a mathematical expression at the planning phases of lesson study, she mostly answered as "*giving counterexamples*". For instance, in the planning phase of Lesson Plan-3, she used a way of "*giving counterexamples*" in her suggestions to provide better learning of continuity. In a similar vein, she suggested the idea of asking them to find a tangent in a discontinuous function in order to enable students to establish the relationship between continuity and derivative.

> (I) Mila: I think we can start like this; they already talk about the behavior of the functions in the previous lesson, we talk about continuity first. Here we can start with the event of drawing the graph of the function without raising our hand, and then continue by giving a counter example.
>
> (II) Researcher: What kind of path would you follow for a student trying to establish the continuity and derivative relationship?
>
> Mila: For example, let's graph a discontinuous function, but let the non-continuous point be empty. And we might want it to be tangent at that point.

Her use of the same ways for validating and demonstrating as "*giving counterexamples*" was not considered as lack of knowledge; instead, it showed her type of knowledge. In other words, this finding showed her style for this indicator.

On the other hand, in the discussions which was not about students' possible questions or students' learning at the planning phases, she often tried to demonstrate a mathematical statement directly by accepting what is given or by accepting the opposite of the statement and move forward with it to demonstrate or prove a mathematical statement. She could not continue in most of these attempts and accepted the suggestion of any of her group mates. Moreover, it did not change from first cycle to second cycle. Therefore, different from the previous example, her attempts were considered as not adequate level of development. The following excerpt shows one of these situations in the discussions which was not about students' possible questions or students' learning at the planning phases.

> Fulya: We can use it to teach the relation between continuity and derivative. Being continuous, it actually has one slope no matter which point I draw on a tangent.
>
> Mila: okay, that's what she provides. I mean, when we accept that what you say is true, we arrive at the proposition "if and only if" ... when we try to show it, we will arrive at a statement like it is continuous if and only if it is continuous.
>
> Fulya: No, it doesn't happen!
>
> Mila: Okay then.

In the excerpt given above, the group discussed the statement of intermediate value theorem if f is a continuous function whose domain contains the interval $[a, b]$, then it takes on any given value between $f(a)$ and $f(b)$ at some point within the interval) about whether it is "if and only if" or "if then" to validate the theorem in relation with derivative. Fulya asserted that the theorem could be used to teach the relation between continuity and derivative. Mila tried to demonstrate that her statement exactly true but it is not necessary. However, she did not continue her demonstration and accepted Fulya's statement. This showed that she did not trust her demonstration and also her knowledge.

Enacting Phases of Lesson Study

Similar attempts were also observed in enacting phases of lesson study. However, these attempts were in the form of explanation based on the question asked by the student to explain the reason behind mathematical procedures to students, rather than generating new knowledge using mathematical knowledge. For instance, in the question of whether there is an equality in the mathematical sign, which indicates inequality, which the student asked for the Sandwich theorem, she tried to eliminate the question mark in the mind of the student rather than producing a mathematical information while showing (see the excerpt given in the section of Development of KSM: Auxiliary Connections). Therefore, no evidence could be found in the enacting phases of lesson study for the indicator of ways of validating and demonstrating.

So far, the chapter has presented development in the sub-domains of mathematical knowledge including Knowledge of Topics (KoT), Knowledge of Structure of Mathematics (KSM), and Knowledge of Practices in Mathematics (KPM) based on their indictors to answer the first research question of how prospective mathematics teachers develop their specialized knowledge in the concept of limit in the lesson study development model. In general, it can be summarized as the development of mathematical knowledge was provided in planning phases of lesson study according to observed indicators of the sub-domains. In particular, the development was provided through different elements of lesson study process including rich group discussions led by the researcher as knowledgable other, readings and tasks given during the lesson study process, and the pre-interviewing process which made her aware of her lack of knowledge.

Another side of the first research question is pedagogical content knowledge (PCK), which is an inseparable whole with mathematical knowledge for teaching a concept. There are three sub-domains including Knowledge of Mathematics Teaching (KMT), Knowledge of Features of Learning Mathematics (KFLM), and Knowledge of Mathematics Learning Standard (KMLS) respectively. Similar with the sub-domains of mathematical knowledge, each sub-domain of PCK includes its own indicators.

The pre-interview showed that Mila had some deficiencies in PCK; particularly in strengths and weaknesses in learning mathematics, ways pupils interact with mathematical content, teaching resources (physical and digital), and strategies, techniques, tasks and examples. To provide the development in these indicators as well as the other indicators which were observed during the process, I constructed rich group discussions supported with rich materials including quotations and common scientific readings in mathematics education literature which are related to teaching and learning the concept of limit enabling students to think on and discuss with each other.

In the following sections, the development in the sub-domains of PCK based on their indicators are presented. Similar with the presentation of sub-domains of mathematical knowledge, the development was dealt with in two main sections of lesson study including planning and enacting. The findings related to development of PCK is started with knowledge of features of learning mathematics in the following section.

4.2.4 Development of the Prospective Mathematics Teacher's Knowledge Features of Learning Mathematics in the Concept of Limit

The first sub-domain of PCK is knowledge of features of learning mathematics (KFLM) which is related to how students learn mathematics and the elements regarding students should be considered by mathematics teachers. The KFLM includes four indicators determined by Carrillo-Yañez et al. (2018): theories of mathematics learning, strengths and weaknesses in learning mathematics, ways pupils interact with mathematical content, and emotional aspects of learning mathematics. In this section, the development of PCK started with KFLM. In general, the main aim of the development of KFLM was to gain the prospective teacher and other members of the lesson study group awareness to consider while teaching the limit concept. For this reason, the lesson study process was designed in all phases of all cycles, considering this aim.

Though the lesson study process was designed to cover all the indicators, the guidance and preferences of the prospective teachers in the process also affected the design process. Therefore, one of the indicators, theories of mathematics learning, was not observed in both the pre-interview and lesson study process. While I encouraged them to consider theories, they avoided addressing this issue in the planning phase. Rather than addressing the concept of theories (e.g., one of the theories for learning the concept of limit-APOS theory) for students' learning of mathematics or planning their lessons accordingly, they focused on developing students' knowledge of how to interact with the concept. As can be seen in the first section of the findings, Mila had a lack of knowledge of strengths and weaknesses in learning mathematics and of the ways pupils interact with mathematical content. The following table shows the development of three indicators of KFLM during the lesson study process.

Table 4.12 Overall look at the development in KFLM of Mila across phases of lesson study

	Lesson Study Cycle 1		Lesson Study Cycle 2	
	Planning	Enacting	Planning	Enacting
Strengths and weaknesses in learning mathematics	NAD	AD	AD	AD
Ways pupils interact with mathematical content	NAD	AD	AD	AD
Emotional aspects of learning mathematics.	AD	AD	AD	AD

AD: Adequate level of development NAD: Not Adequate Level of Development or Not Development AE: Already Existing NA: Not Observed

The abbreviations to show the level of development has the same names and same meanings in the tables in the sub-domains of mathematical knowledge. Since students’ learning is centered in the lesson study process, an adequate level of development was expected so that she could make instructional decisions in planning; for instance, constructing a task in consideration of students’ learning in the related content, and she could act considering students’ learning the related content in enacting.

There are two ways to develop the KFLM using the observable features[7] of the lesson study. In the planning session, rich materials with readings and sheets, classroom dialogues, and group discussions allowed the group to consider the features of learning mathematics from different viewpoints. Another way was the observations and experiences of the group when they met real students. I determined the process in this order in the lesson study. Thus, the development of three indicators was observed in the first cycle of planning. One of them - strengths and weaknesses in learning mathematics - could not be developed at an adequate level in the planning phase of the first cycle since it is related to being aware of where students have difficulties. Since Mila had not planned a lesson or practiced teaching related to the concept of limit, the development was provided at an adequate level in the enacting phase of the first cycle. In the meantime, knowledge of ways how pupils (students) interact with the content was expected to develop in the enacting phase as well. However, during the planning phase, using the researcher's probing questions (e.g., How do you think the student will react when faced with this situation? What are the student's possible answers? How can the teacher respond to these possible answers?), the awareness about students' interaction with content was raised. Lastly, as an expected finding, the development in the knowledge of emotional aspects of learning mathematics was observed since the prospective teacher had the same emotions with students. In this chapter, the KFLM of Mila during the lesson study process is presented through these three sub-indicators.

4.2.4.1 Development of Knowledge Features of Learning Mathematics: Knowledge of Strengths and Weaknesses in Learning Mathematics

The indicator of strengths and weaknesses of learning the concept is not restricted to having knowledge about the misconceptions about the concept of limit. Rather, it is

[7]The observable features of lesson study can be considered as the phases of lesson study including determining lesson goal, planning, research lesson and reflecting.

related to being aware of where students have difficulties and strengths and acting by taking these into consideration in both planning and processing the lesson (Carrillo-Yañez et al., 2018). Therefore, it cannot be said that this is a development, instead, I prefer to say it as *gaining awareness*. For this reason, the lesson study process was designed to make them aware of students while planning and teaching the concept. In this section, raising her awareness related to strengths and weaknesses of learning the concept is presented.

One of the important lacks of knowledge of Mila was knowledge of misconceptions-misunderstandings in learning the concept of limit. Accordingly, she was not aware of students' weaknesses and strengths in learning the concept of limit since she had not had any experience in teaching the concept of limit. Considering all these facts, Mila's first interest was to develop herself towards this indicator. After determining the lesson study goals, her first question was to ask if it was correct when we teach the concept of limit as a bound, which is the dictionary meaning of limit (I). Mila's emphasis on this shows that the students did not know that this constitutes a misconception for the students. Mila's emphasis on limit's meaning as boundary shows that she did not have any idea that it constitutes a misconception for the students. Similarly, in the same conversation, her second question was if it was correct when she drew without raising her hand about continuity (II) in the first meeting of planning phases, as can be seen in the following excerpt. This statement also constitutes a misconception for students about the concept of continuity. It was also Mila's own misconception that I mentioned in the pre-interview.

> (I) Alp: The limit was actually something that could not be reached, but here we are reaching the limit. They have a hard time accepting that a given number is even.
> Mila: I also want to say something. For example, when we look at the dictionary meaning of the limit, it is not the normal mathematical meaning of the word, instead, it is the quantitative aspect of something as the last limit point it can reach. But what we look at in mathematics, it is different, and this is what you just said, isn't it?
> Researcher: There was something you said: When you first were taught the limit ... ?

> Fulya: The concept of limit itself. So, personally, at least I haven't established the concept of limit in my own mind. In fact, the right and left approach is something that underlies that limit concept.
> Mila: During the meeting we had with you, I didn't have it in my mind anyway, so the first thing I looked at was the concept of limit.
> (II) Alp: When describing continuity, it would be better if we show it over the function. On the chart.
> Mila: Visualizing always makes it easier.
> Mila: Oh, the thing about drawing without raising your hand, are we going to mention it?
> Researcher: Do you think that is true?
> Mila: How true is that, I was going to ask exactly that. For example, should we say it or not? Is it true or not? Because the last time, I doubted its accuracy (talking about the interview).

It should be indicated that in the first meeting of the lesson study, the group talked about all the three lesson study goals without any intervention or discussion. Such meetings helped me to draw the way of the lesson study intervention process in planning phases according to the interests and knowledge deficiencies of Mila as well as of the group. In addition, as a part of the intervention process, one of the readings was solely related to misconceptions in learning the topics in the concept of limit and it was always on the desk where the lesson study group worked. For this reason, it was hard to observe the development clearly. As can be seen below, the development of indicators occurred in different ways, which are mentioned in the last section of the findings. Therefore, in this section, the misconceptions are not addressed.

Planning Phases of Lesson Study

When the four phases of the lesson study process are considered as two stages, including planning and enacting, these two stages provided Mila’s development in two different ways. The planning stages (there were three lesson plans which have two planning stages, thus, there were six planning stages in all) were under the control of the researcher. During the planning, the lesson study process was designed on rich materials, including readings, worksheets, and rich group discussions on students’ learning.

As the nature of lesson study, lesson planning mainly focused on students' learning. For this reason, giving different examples from the prospective teacher's journey in developing her knowledge of students' strengths and weaknesses could make this section complicated. Therefore, I gave an example for the planning of indeterminate-undefined forms to present the data as easy to follow. Moreover, the topic of indeterminate and undefined forms was what Mila herself had difficulty in understanding. Therefore, to put it right from the start, Mila actually developed her knowledge of students' strengths and weaknesses based on her own learning.

The researcher observed in the pre-interview that Mila had some confusion about the difference between indeterminate and undefined forms. Therefore, her awareness of students' confusions between these two notions was gained (or developed) in accordance with the awareness of her own lack of knowledge. The following excerpt shows a starting point of the discussion on these two notions in the planning phase:

> Mila: For example, imagine saying $\frac{0}{0}$ equals x. Multiply the ins and outs. x can take more than a value.
> Alp: Ah yes.
> (...) Fulya: When an indeterminate form came to my mind, I realized it, I described it as undefined in my mind. I realized this after reading the article: Undefined and indeterminate are two different things.
> Mila: Honestly, I didn't know either, I think students might get confused, like us!
> (...) Researcher: Then, what do you say about a number divided by zero?
> Fulya: Undefined?
> Alp: Hocam, I think first of all let's start with; what is the difference between undefined and indefinite? Let's start, as Mila said.

The mathematical correctness is discussed in another section of the findings. In this section, I would like to focus on how the prospective teacher considered students' mathematical thinking while constructing the content in planning. In the excerpt given above, Mila's discourse about students' possible confusion was the starting point for gaining awareness about students' strengths and weaknesses. Then, the lesson study group started to work on these notions and their differences and their reflection in this knowledge on the lesson plan. Therefore, the nurturing process was observed through the development of her mathematical knowledge after the researcher gave them an assignment by expecting them to research on these notions. The following excerpt was observed as a continuum of the planning phase for these notions.

> Alp: Undefined, we say it to indicate that something is not defined in mathematics, and this is undefined. In fact, I was teaching a topic to the 11th graders, "2+2 for example equals 4, but 2 +green what?" I asked a question like this. It is something undefined! We cannot add color with a number. Since it is undefined, we call it undefined. We say uncertain because we do not know what that is. I've seen something about it, he said 0=0.x, here we can write a lot of numbers instead of x. So, x here is indeterminate.
> Mila: Yes, hocam, it is the same in all sources, just a more mathematical version!
> Mila: The same thing, hocam, it's a slightly cooler version of what Alp said. I would like to say one more thing; we said to Alp that we could not add it with green, for example, let's show that the same thing happens with a number divided by zero. Let's go where the students know so there is no confusion. Let's show that there is an undefined number divided by zero.
> Researcher: So, where is the number divided by zero undefined?
> Alp: Everywhere. It is undefined with respect to multiplication because in multiplication…
> Fulya: The sets of numbers you say everywhere,
> Researcher: So, in what number sets?
> Researcher: Now I have to point this out. The infinity of the number divided by zero is defined only in the Riemann sphere. That's why I asked you. When we say one divided by zero to infinity, the student perceives infinity as a point.
> Mila: Aaaa, yes!
> Researcher: However, this is not possible in real numbers. Where infinity is perceived as a number, the Riemann set and the Riemann sphere look something like this.

This excerpt is given here to show the process of the development of her knowledge related to students' strengths and weaknesses. In addition, this topic constituted a weakness for both the prospective teacher and her group members. Therefore, this excerpt is of significance to trigger them to think about students' mathematical thinking process considering their own thinking process. It seems that while undefined was considered clearly as a result of the relation between its meaning and name, which refer to the same thing, indeterminate still created confusion in the prospective teacher’s mind. As they focused more on the questions they wrote for lesson plan 2, and no matter how triggered the prospective teachers were, the discussion remained at this level until the first research lesson was conducted. The observation of other group members’ lessons provided her to see beyond her thoughts about students’ strengths and weaknesses in learning the concept of limit. Listening to Fulya's lecture just before Mila and then watching the recording enabled her to make more mathematical contributions to students' learning about mathematics during the planning phase.

> Researcher: You can also explain it in that way when you are explaining. You will start with indeterminate forms. There was a question the students asked Fulya as “why infinity divided by infinity is not.” Fulya will ask you if it's a sufficient explanation.
> Fulya: I said that if there is only one number, there is infinity divided by infinity which equals to the number of a, then infinity equal to a times infinity. Well, do I know this a, but it may be 1, it may be 2, it may be 1000, I asked whether could I say something definite for a, they said no, so I said there would be indeterminate.
> Mila: As if it didn't have an answer, why this is an answer to indeterminate or not.
> Alp: Let’s say, here is already undefined! Again, infinity divided by infinity must be one for the inside and outside product.
> ... [Researcher intervened here]
> Fulya: I should have said there is an increase for infinity, it is not an infinite number.
> Mila: We need to be particularly careful about the infinity; a bit as an adjective! So, we can start from there and talk about indeterminates.
> Researcher: So, you can start there!
>
> Fulya: For example, they asked why 1 to the power of infinity and zero times infinity indicated indeterminate.
> Mila: They are right about the one to infinity, we should definitely mention it!

At first, Fulya explained how she answered to students’ question regarding the reason behind $\frac{\infty}{\infty} \neq 1$. She explained it as $\frac{\infty}{\infty} = a$ (a is an unknown term she described) $\Longrightarrow\ \infty = \infty . a\ \Longrightarrow\ a$ can be any reel numbers.

As I said at the beginning of the section, KFLM is not directly related to knowledge of students; rather, it is the knowledge of students within the awareness of what mathematical content requires. Mila’s reactions to Fulya can be considered as a part of the KFLM. Another part requires combining it with knowledge about students. In this indeterminate form, the important thing is how the notion of infinity is constructed in students’ minds. After the intervention, including reading and discussions, it can be understood that Mila's ability to express more mathematically what aspects of students can be strengthened and which weaknesses can be eliminated shows the change that occurred during the planning phase.

Enacting Phases of Lesson Study

The enacting phases of lesson study included writing reflections and implementing the research lesson. Since Mila implemented the third lesson plan, it is of importance how she reflected her KFLM in the first and second lesson plans. The focus of the first lesson plan was the mathematical meaning of the concept of limit and the importance of its relationship with mathematical concepts. In the reflections, I was looking for what the prospective teacher focused in her colleagues’ teaching session. Though I directed her with questions, for the first research lesson, Mila mainly focused on the effectiveness of activities, actions taken by her friend during implementation, time management and students’ interest with activities. The following excerpt from the first lesson plan shows little evidence about Mila’s awareness of where students showed strengths and weaknesses:

> It is good that it is not defined at point 1 in the first application. The student wanted to replace it, but because it was not defined, he could not replace it and had to approach, which is exactly what we wanted. … We went from very easy to very difficult in problem solving, students had more difficulties than we expected in difficult questions. A smoother transition could have been achieved in question difficulty instead. ... (Mila’s reflection paper for the first lesson plan)

Mila’s reflection showed her awareness related to where students had difficulty during the research lesson. However, it cannot be observed where students showed their strengths and/or how the group used their strengths about the concept of limit to overcome their weaknesses. Since Mila did not gain enough awareness about the first lecture, we kept the reflection meetings of the first lesson plan longer than the others so that there would be more guidance in the second research lesson.

The long enough reflection meetings for the first research lesson developed Mila's awareness of students' learning of the concept of limit. Her reflection on the research lesson of the second lesson plan was more detailed than the first, and her focus was more on the students. The following excerpt shows examples from the second reflection paper of Mila:

> … Step-by-step hints can be determined for questions with higher difficulty levels, from points that students know. Thus, it may be easier for students to be guided to the correct answer. … Expressions such as "zero divided by zero" and "I must take derivatives" were persistently made. Students are not convinced and want to use *L'Hospital all the time. An event can be added for this. When using L'Hospital, lengthy and difficult questions can be added so* that it can always be shown that it does not make sense. … (Mila's reflection paper for the second lesson plan)

The points that Mila indicated in her reflection became more mathematical. It showed that she gained awareness about students' learning for both their strengths and weaknesses. It should be noted that it is not only related to students' strengths and weaknesses; rather, the indicators of KFLM should be considered as a whole. As I wrote in the data collection section, since there were two lesson study cycles, Mila had a chance to implement only two lesson plans. The enacting stages were platforms that members of the group were expected to implement what they learned during the planning stages of the lesson study process.

In implementing the third lesson plan aiming to grasp the concept of continuity by establishing mathematical relations and to use it together with the limit concept in mathematical applications, Mila started her lesson based on her experiences. In other words, she knew that it would happen because she got to know the students from the narrations of her previous friend and told them about their learning only out of curiosity.

> Mila: Now I guess you guys were pretty curious about indeterminates.

Student 1: Yeah!
Mila: Yes, I've already had such a feeling, so I want to examine these uncertainties together with you. First, let's talk about the $\frac{0}{0}$ and $\frac{\infty}{\infty}$ uncertainty. What do you think might be the reason for this uncertainty?
Students: [Grumbling sounds of what they don't know]
Mila: So, I'm going to write you two functions and ask you to find the limit at the point I gave you.

In the excerpt given above, Mila tried to make them notice the different results of the same limit forms, which caused the indeterminateness. By asking them to calculate the limits of the given points of the functions, it can be understood that Mila considered students' strengths in the concept of limit. Moreover, it should be indicated here that this is an example of Mila's awareness of students' expectations about mathematics which is included in the last indicator- emotional aspects of learning mathematics.

This example is an example from a planned situation in implementing the lesson plan. However, my main expectation was to observe the prospective teacher's awareness and actions in unplanned and unexpected situations. It would show how the prospective teacher uses her awareness in her teaching. In the enacting phase of the first cycle, there were two situations she was faced with in this way. In the continuation of the indeterminate form of $\frac{0}{0}$ and $\frac{\infty}{\infty}$, she observed that students tended to make mistakes while exemplifying or demonstrating indeterminates based on the question frequently asked by students in the implementation of the second lesson plan.

Mila: If I ask you a question now (She turned to her presentation but realized that the question she wanted was not there). Yes, I didn't write it here...
Mila: (Closing the presentation screen) Forget about it then. I will write the question myself. Let's say I take $f(x)$ to the $g(x)$ as limit x goes to infinity. What if $f(x)$ was 1 for me as the limit goes to x. If limit x goes to infinity and $g(x)$ is infinity for me. What will the result be for me? So, when I think about this $f(x)$ to the $g(x)$ structure.
Student 1: 1 to infinity (1^{∞}).

Mila: The over 1 becomes infinity. Because I know from the limit rules that I can distribute this limit and the limit $f(x)$ to the $g(x)$ would be 1 to the infinite for me.
Student2: I think it should be 1 too.
Student 3: I've always wondered about that too
Mila: You were wondering, right? I heard it. There is actually a reason for this. What we said is that the main reason for the indeterminates is that we get different results in the same form of the limit. In fact, nothing different from the ones here (she showed what she wrote about $\frac{\infty}{\infty}$) does not appear for 1^{∞}. Now I will write you two examples.

Up to now, the excerpt showed that she tried to use students' strong sides, which were the calculation of the limit, to teach them the logic of indeterminate forms. In addition, it can be said for this introduction that she considered the students' interest (emotional aspects of learning the concept) which was their curiosity about why the forms of 1^{∞}. is not equal to 1. This excerpt showed that Mila started to gain awareness about the aspects related to students.

Mila: So that's why 1 to the power of infinity is indefinite.

Student 1: But that's not 1 to the infinity, right? We said 1 to the infinitely indefinite thing, isn't it something different? E is here.
Mila: Hmm, is it confusing that it's equal to e?
Student1: No, there is a number called n.
Mila: Yes.
Student1: 1 is not infinity, I mean, I don't think they are the same thing!
Mila: There is a number called n, is your question related to it?
Student1: n goes to infinity or exactly 1 to the infinity is not equal to this.
Student 3: He means something (talking about his friend) different in two functions. As if the two functions are different, it's logical that we find different results anyway, isn't it?
Mila: Hmm I got it! But I'm telling you this. So, let's look at the equation I got over here, okay (it shows the resulting limit e)? When I look over there, the limit n goes to infinity and that inner side is equal to 1 for me. Therefore, when I overwrite it here, I get the 1 to the infinity form. Here I got 1 to the infinity, and what happens when I get the same form of other functions. Here I am writing the same thing again (showing the second function). Here, my inner side became 1, and my upper side became infinity. In other words, they seem to be different functions, but since we do not perceive infinity as a number, we say that it is increasing gradually, but we do not know how much it increases, so this is the reason why it creates uncertainty.
Student 1: Now I get it!

Student 2,5,7: Yeah, I understood perfectly!

In the face of this unexpected situation, the prospective teacher drew an unplanned path for herself by using the students' strengths and at the same time establishing mathematical connections. It showed that gaining awareness of students' understanding is closely related to her KPM, which can be explained as awareness of mathematical reasoning on how to explore mathematics by seeing connections. Both in the pre-interview and at the beginning of the lesson study process, Mila commonly preferred to make reasoning by means of counterexample when it was asked how she would show one's correctness or incorrectness. This way showed that Mila put students' learning in the background. There were two important facilitators to develop the KFLM of Mila: group characterization and reflection papers with guiding questions. But there is another important issue that the development of KFLM affects the development of KPM. The fact that the prospective teacher is aware of the example she gives to create mathematical knowledge is closely related to the students' awareness of using calculation, which is one of her strengths.

As a result, the planning phase of the lesson study process provided her to consider students' strengths in learning mathematics as can be seen above. This development can be explained as a result of the nature of the lesson study process. Since the literature indicated that the focus of the lesson study is students' learning, the nature of the lesson study included observing previous participants' lessons and learning about students' understanding. Therefore, this development can be explained as the nature of the lesson study process.

4.2.4.2 Development of Features of Learning Mathematics: Knowledge of Ways Pupils (Students) Interact with the Content

In another indicator of KFLM, as the name suggests, knowledge of ways students interact with the content includes how students interact with the mathematical content, limit in this case. Particularly, it comprises students' procedures, strategies and the terminology they use when they encounter with mathematics (Carrillo-Yañez et al., 2018). As mentioned in the first main title, I observed the prospective teacher's existing knowledge by considering her way of thinking; in other words, I was looking for how much she considered students' learning and students' connection to content in her answers. However, I could not observe this indicator since Mila had not had any experience in teaching the concept of limit.

The development in this indicator was provided by means of triggering the group to think about how students interact with their proposed activity. In particular, one of the main questions while preparing activities in each lesson plan was what the prospective teachers expected from the students. It helped to gain insight for the group, Mila in this case, into how students would interact with the content. In addition, while she did not have any experience in teaching the concept of limit in both her undergraduate process and her own tutoring process, the other group members had experiences in both areas. Therefore, the development process occurred through rich group discussions by means of group members' different backgrounds as well as guidance question. Furthermore, in order to prevent them from making wrong predictions, the development of knowledge of the prospective teachers was supported by various sources from the literature.

In general, the development occurred in the same order as the phases of lesson study cycles. As mentioned above, the predictions made them aware of how they would pay attention to the lesson plan and what they would observe during the research lesson phases. The different predictions of the group members provided them to see the interaction with the content from different angles. In the ongoing process, they had the chance to enact and observe whether their foresight would match with the interaction in the real classroom. In addition, since the second cycle of the lesson study process was a micro-teaching lesson study, the group had the chance to see the interactions of their friends with the same level of education as themselves.

Planning Phases of Lesson Study

In determining the lesson goals, Mila, as well as the group, focused on only the curriculum and their own deficiencies. Particularly in the first planning phase of the first lesson plan, Mila often put forward her predictions about students' mathematical communication based on her own deficiencies. For instance, Mila did not hear paradoxes before she participated in the planning process. Therefore, she thought that the paradoxes would increase the curiosity of the students and that the students would question it. Accordingly, she did not consider students' learning or cognitive level, instead she suggested that the students would answer in the same way as she would answer herself. It can be understood from the similarities between her pre-interview and the first planning phase of the first lesson plan. The following table shows an example.

Table 4.13 An example of the similarities between her pre-interview and the first planning

Time in the lesson study process	Example excerpts to see the similarities
The pre-interview	Mila: What we call this approximation can actually be thought of as a delta. It's very, very small intervals, maybe we're getting closer than we can imagine, but it's like we're not that number.
The first planning phase of the first lesson plan	Mila: Hocam, I said infinitely small approximations for the desired point. Fulya: We will also ask the point to be reached here, to make a generalization. To generalize, let's say how these numbers behave, let's try to impose the word behavior. Mila: I think it's good, too. Researcher: Well, what should the teacher ask first? Mila: I don't know exactly what we will say at that moment, but can this game be a winner? Or how far does this go? Fulya: I think we can start by saying which of us can say the closest number now? Which of you can say the closest number now? Mila: Or how far does this game go? Can it have an end? You know, we tried to give everything in the paradox; there is no winner in the game because I can always go closer. Researcher: You want to ask how far do you think this game will go or will it have a winner? Mila: I hope they can see it; I think. Researcher: Of course, what will be our expected answer here? Mila: It goes on forever; there is no winner.

Her conception related to the notion of infinity reflected the expectations of the students. Her answers to the questions about the notions of infinity in the pre-interview included the answers of "*never-ending process*." In the excerpts above, she regarded students' mathematical thinking as "*it goes on forever*" when they participate in the activity (the activity related to paradoxes). Similar examples were observed until one of the group members conducted the first research lesson. When Mila and other group members showed the interaction in a real classroom, they started to develop their knowledge and awareness of how students interact with the content.

> Mila: I want to say something about the first lesson plan. We were going to make it a bit like a game while we were talking about it. I think we need to give a little more gameplay in the approaching one event. Our aim was to attract attention, but since the approach to one event is an introduction, I think it's a lot of things; since nine overturned numbers are equal to one, I think it has no function in attracting much attention. But I guess the GeoGebra applications of the limit from right to left were in place; I wish they worked. But even though it didn't work, Alp showed it on the board; I think even that was effective.

The excerpt given above was quoted from the meeting which was after the first research lesson. Mila revealed that the students did not interact as they expected. The first research lesson took place halfway through the planning phase of the second lesson plan. While I observed Mila's knowledge of how students interact with the content, the same performance of Mila could not be observed in the remaining process of the second lesson planning until the second research lesson was conducted. After the second research lesson was conducted and the prospective teacher observed and watched her groupmate, the development was started to be observed in real terms. The following table shows Mila's thinking about how students interact with the content at three different times. In this way, the development can be seen more clearly.

Table 4.14 The development in the knowledge of how students interact with content

The phases during the lesson study	Discussions on how students interact with content
The discussions before the first research lesson	Researcher: Well, is there a limit of the division here? Alp: Hocam, students will write the division of the limits at first, then the limit of the divisions to solve this later. Fulya: Will they transform it? Mila: Even if it doesn't transform, it actually becomes $\frac{0}{0}$!
The discussions after the first research lesson	Mila: I had something in mind; what would it be like if we had this lake pollution modeling question done as an activity at the end of the lesson? The purpose of my asking this is what kind of interaction would the students have? Alp: But all the concepts they learned are mentioned in it; I think it would be more reasonable to ask what they learned in order to reinforce what they learned. Fulya: Is there infinity in it? I mean, we just need to know how the students will face it.
The planning phase of the third lesson plan (at the beginning)	(*In the discussion on the question related to the following the continuity of the graph*) Mila: So, I think that right where those two are is square root. Alp: I do not see any reason why it should be discontinuous. Mila: I think that students will answer without thinking what the domain and range of the function is Q). They will say that this function is discontinuous since there is jumping. They will say that they say discontinuous when they see jumping on graphs.

The transition from the first column to the third column showed that the desired improvement in this indicator was revealed after the second research lesson was conducted. In the first column, it cannot be observed any evidence of students' mathematical thinking. In the second column, by posing a question about how the students would think if the order of activities were different, she actually revealed her awareness of this issue. Mila showed her knowledge related to how pupils interact with content which was not sufficient to say there is a development. In the third column, Mila clearly stated her ideas about how students interact with content by revealing how students will answer the question and where and how students will make mistakes.

There were some reasons of this development: First, since there was not enough problem-solving in the first research lesson, the relationship of the students with the tasks was mostly observed. Second, as I said in the previous and will say in the following sections, when the development of mathematical knowledge reached the point where it supported knowledge related to students and teaching, the improvement in this knowledge was observed. For instance, the second lesson plan mainly covered the knowledge of mathematical procedures in KoT and Mila tried to understand the mathematical ground of the problems herself at first in this planning process. The following example shows it just before the excerpt given in the second column.

It can be understood from the table that the development of knowledge about how students interact with the content occurred through the nature of the lesson study process. This can be explained as one of the expected results that will be discussed in the conclusion and discussion section. However, there was another factor in this development, which was the development of mathematical knowledge at the same time. Since Mila thought about how students would interact with the content from herself, as her KoT developed, she was able to reach more accurate approaches with the experience she gained from this development process.

Enacting Phases of Lesson Study

This indicator includes the teacher's knowledge about their students' manner of reasoning and proceeding in mathematics (in particular, their errors, areas of difficulty, and misconceptions), which informs his or her interpretation of their output. In enacting, I considered the prospective teacher's interpretation of the process of students' learning the concept of limit in the reflection process. Actually, the development of this indicator was mainly observed in this reflection phase since she encountered real situations in research lessons and their reflections.

For the first lesson plan, she encountered students' unexpected reactions to the tasks. These reactions were considered as unexpected since the lesson study group did not think that students would react to the activities that way. For instance, the lesson study group believed that the beginning step for the first lesson plan would be effective for conceptualizing *approach* as the curriculum and the textbooks suggested. However, Mila and other group members were mistaken as their knowledge of the level of the students and how they would interact with the steps of the activity was not yet fully developed.

> Researcher: Let's talk about the activities; please begin with the beginning activities:
> Mila: Yes, I would like to start first. We went through continuous tables, and our tables remained very simple for students. You know, we aimed to give different usage areas of the limit; for example, as Alp said at the end, the applications are very geometric, very conceptual, yes, he talks about an approach, but it seemed as if we always emphasized substitution here, the students never behaved as we wanted, they all tended to substitute. In Alp's lecture, most of the students encountered this: For example, he subtracted 1 from r. It wasn't $2x + 1$, if I remember correctly, they asked why did we remove it from someone.

Knowledge of the ways how students interact with content includes the prospective teacher's foresight and awareness about how students will interact with this mathematical concept by prioritizing her mathematical knowledge. As can be seen in the example above, Mila commented on how students did not meet their (the group's) expectations during the research lesson. I expected her and other group members to forecast about students' mathematical thinking and their expectations based on not only their experiences but also literature on the related topic. However, at first, she was mistaken for her knowledge of the level of the students. The guided reflections which were prepared to bear this situation in mind provided her with the ability to interpret how students would interact with content. The guided reflection opened the way to rich group discussions that included mathematical knowledge and students' learning. Therefore, another important factor in this enacting phase was rich group discussion.

As the rich group discussions in the reflection process, which took place after the research lessons passed, I also observed that Mila's awareness of students' mathematical thinking and how they would act when they came across concepts increased during the enacting phase.

> Researcher: Well, if we move on from the example that Alp gave here, he said here, Alp said here we say $\frac{x^2+3}{x}$which represents $\frac{\infty}{\infty}$, we say $\frac{x}{x^2+3}$which also represents $\frac{\infty}{\infty}$ and actually apply the same process.
> Mila: They already understood that this is one-half infinity. Let's say because we did the same approach from the same process.
> Fulya: I could not perceive at all, so my brain was not enough to understand.
> Mila: We actually do the same approach.
> (…) Researcher: Why didn't they ask e? They accepted directly.
> Alp: Just ignore Mila's explanation.
> Mila: This is getting closer and closer to e as I put the numbers here and increase the numbers. If they had asked, I would have said so. Even if you want to try. By saying let's prove it. I had designed it so that it would have a shape like this, but when nobody asked me, I didn't even bother.

It is difficult for a prospective teacher to examine student learning in a classroom discussion and activity. Although these difficulties are observed less in Mila, it is an accepted fact that she does not have every detail. In order to reduce this, a lesson study group was formed with prospective teachers from different backgrounds. Thus, as in the example above, she found the opportunity to see different perspectives on how to teach the concept through the relationship of students with the concept and to pass it through her own mind.

4.2.4.3 Development of Knowledge Features of Learning Mathematics: Knowledge of Emotional Aspects of Students Learning Mathematics

The last indicator of KFLM-Emotional aspects of learning mathematics includes awareness of students' mathematics anxiety, what motivates students, their interests, and expectations. It can be observed in both planning and teaching in the choice of registers of representations and activities that will be used in the classroom when setting a learning environment (Carrillo-Yañez et al., 2018). In this indicator, it cannot be said that the development was observed; rather, she combined her existing knowledge with developing mathematical knowledge. Therefore, in this section, I focused on the nature of the knowledge for this indicator.

The overall examination showed that the nature of the knowledge on emotional aspects of learning the concept varied from what motivates students to what makes students more eligible in the concept of limit according to the development of her mathematical knowledge. For instance, the following table shows an example of this situation.

Table 4.15 An example of the variation in knowledge related to emotional aspects

In which phase	Examples	In which sense
The first planning phase at the beginning	Mila: I think it is very logical to go into paradoxes or something, I read a little, and I liked it very much. This can create a question mark in their minds. Mila: I think that if you create a question mark (in the student's mind), interest can increase.	What motivates students
The first planning phase at the end	Mila: I think that's (GeoGebra application) beautiful, too. Alp: Here, you see both delta and where the epsilon is. Here we will see which one might be better as we get a smaller range each time. I thought it might be better this way. After the first intuitive definition, I think we can show these and give a formal definition of epsilon and delta. Alp: As Mila said, a skeptic is constantly objecting. He/she always objects, we can show a smaller one here. We got epsilon, we got a smaller delta, we got epsilon, we got a smaller delta. Mila: I also say let's combine the two. So, let's give this to students. Let's give the thing. If we	What becomes students more eligible in the concept of limit

make this a story like an application and open GeoGebra at the same time, I think two things will fit together very well.

In the table given above, the first column represents in which phase I got the evidence about the prospective teacher's awareness of students' emotional aspects. The second column represents the evidence of the content in the third column. In the first planning phase at the beginning, Mila thought that the way to attract students' attention and increase their motivation was by making them question themselves. In the first planning phase at the end, her focus changed to what makes students more eligible in the concept of limit. She changed her direction by using her content knowledge and her knowledge of sequencing topics, as can be seen in the third row-second column. It does not mean that this change is a development. Instead, it was just related to the change in the nature of knowledge of emotional aspects.

In enacting, she used the advantage of the lesson study in which she observed the students before she taught the research lesson, and she designed her research lesson by considering the students' expectations. Students were curious about the reason behind the indeterminate forms, in particular the form of 1^{∞}. In her research lesson, Mila emphasized the reason to draw attention to students' motivation. The example was shown in the indeterminate-undefined forms in the above sections.

4.2.5 Development of the Prospective Mathematics Teacher's Knowledge of Mathematics Teaching in the Concept of Limit

To answer the pedagogical content knowledge part of the first research question, another sub-domain of the model is Knowledge of Mathematics Teaching (KMT). KMT is not solely related to knowledge of teaching in pedagogical knowledge, rather it is directly related to knowledge of the concept and teaching, as similar with KFLM. The sub-domain includes three indicators of knowledge; theories of mathematics teaching, teaching resources (physical and digital), and strategies, techniques, tasks and examples (Carrillo-Yañez et al., 2018). In the current study, the data did not give any evidence related to the development in knowledge of theories of mathematics teaching for the group members as well as Mila. For this reason, this section did not include this indicator. As I mentioned in the previous section, despite my efforts, Mila and other prospective teachers did not focus on the theories related to learning and teaching the concept of limit. Mila and other group members focused on their lack of mathematical knowledge in planning phases of lesson study. Furthermore, another dimension for the indicator of teaching resources that appeared in the data but was not included in the model was added in this section. This dimension was named as "knowledge of how to use resources in unexpected situations in the classroom". The knowledge of teaching resources requires a resource that can be used for teaching a particular concept, critical evaluation of how the resource will be used in teaching, and going beyond awareness of how to use these resources. However, knowledge of teaching resources does not include how to use the resources in the classroom when things do not go well during teaching. For instance, while the planned situation was the display of the limit on the graph with GeoGebra, it was necessary to put forward a new indicator about how the prospective teacher handles the existing situation in case the screen does not work. For this reason, the additional dimension was added to the indicator of teaching resources.

Table 4.16 Overall look at the development in KMT of Mila across phases of lesson study

	Lesson Study Cycle 1		Lesson Study Cycle 2	
	Planning	Enacting	Planning	Enacting
Teaching resources (physical and digital)	NAD	NAD	AD	AD
Strategies, techniques, tasks and examples	NAD	NAD	AD	AD

AD: Adequate level of development NAD: Not Adequate Level of Development or Not Development AE: Already Existing NA: Not Observed

In general, Mila had lack of knowledge of teaching resources and strategies, techniques, tasks and examples as can be seen in the pre-interview. In addition, I can say from my own experiences with Mila that she was competent in using teaching resources in micro-teachings conducted in the methods courses. However, the beginning of the planning process showed that she had lack of knowledge in using resources for teaching the concept of limit. Considering the description of the sub-domain, the expectation for the adequate level of development was to implement her knowledge in enacting phase in general. In particular, it is the knowledge for teaching resources, planning to use of different teaching resources in planning and handling the unexpected situations about the teaching resources in the classroom in enacting and for strategies, techniques, tasks and examples, a broad understanding about the concept-specific strategies, techniques, tasks and examples in planning and enacting the lesson.

The development of these indicators occurred in both planning and enacting phases of the second cycle. The development of these indicators occurred through rich materials presented by the researcher, guided activities and rich group discussions. The rich materials enabled them to see different resources and examples for teaching the concept efficiently. For instance, there were different demonstrations of the definition of the concept of limit on graphics in GeoGebra (see Figure 4.15). In order to decide which of these to use, it is necessary for the prospective teacher to consider both the student's learning and look at these resources from a teaching perspective with a critical eye. For this reason, I gave them different resources for the topics during the lesson study process to develop their knowledge. Another factor for development in lesson study was guided activities. The prospective teachers as well as Mila had limited views towards bringing tasks and examples or using different strategies and techniques. Therefore, I presented different activities to set an example for them and these activities were implemented by the participants during the lesson study process. In this way, the prospective teachers had a chance to see and implement different ways and strategies without being limited within their own views (Figure 4.21).

FINDING ZEROS OF POLYNOMIALS
Select one person to be the Teacher, and one person to be the student. Re-enact the following scenario, and consider how you would respond as the teacher.
Classroom Scenario: A high school teacher is discussing and illustrating the use of the Intermediate Value Theorem to discover zeros of functions.

Teacher: So, let's find some specific values for the function, at $x = 2$ *and* $x = 3$ *(showing the function on the board:* $g(x) = x^3 - 3x^2 - 2x + 7$
Student 1: g of 3 *is* 1.
Student 2: g of 2 *is* −1.
Teacher: Thus, since we have $g(2) = -1$ *and* $g(3) = 1$*, we can conclude that* $g(x)$ *has at least one zero between* 2 *and* 3.
Teacher: Ok, for your exit ticket, give me a short summary of the idea we just discussed about finding zeros of a function.
Exit ticket: Okay, so for a function, if a function is less than 0 *somewhere and greater than* 0 *somewhere else then we know there will be a zero somewhere between them. So, like, in general, and if* $f(b) > 0$ *then there is at least one zero somewhere in the interval* (a, b).

Q1. Assess the student's exit ticket response. What sticks out to you?
Q2. Can you think of any assumptions that the student may be making about the situation, and under those assumptions is his/her statement valid? Write down any such assumptions.
Q3. Is it possible to construct a function $f(x)$ that is defined on every point of the interval $[a, b]$, such that $f(a) < 0$, $f(b) > 0$, but $f(x)$ has no zeroes in the interval (a, b),? If it is possible, provide a specific example; if it is impossible, explain why.
Q4. Is it possible to construct a function $h(x): A \rightarrow \mathbb{R}$ that is continuous on its domain A, with $h(a) < 0 < h(b)$but has no zero in the interval (a, b)? If it is possible, provide a specific example; if it is impossible, explain why.
Q5. How would you complete the following statement to make it always true: "Let $f: A \rightarrow \mathbb{R}$, such that ____________________. If $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$ then □ will have a zero in the interval (a, b)."

Figure 4.21. The task for intermediate value theorem adapted from ULTRA project (Weber, Wasserman, & Fukawa-Connelly, 2019)

4.2.5.1 Development of Knowledge of Mathematics Teaching: Knowledge of Strategies, Techniques, Tasks and Examples

This first indicator in KMT is knowledge of strategies, techniques, tasks and examples which covers the prospective teacher's awareness about potential activities, strategies, techniques as well as their possible limitations and obstacles, and different ways of representing specific content (Carrillo-Yañez et al., 2018). As I mentioned in the previous section, Mila's primary focus when making instructional decisions was what motivates students (see the section of Development of KFLM: Emotional aspects of learning mathematics). For this reason, she commonly ignored what mathematical content requires for teaching it at the beginning. Such attempt occurred in her knowledge of strategies, techniques, tasks and examples as well. The development of this indicator was not observed in all elements of the indicator. As the nature of the lesson study, the group constructed lesson plans collaboratively. For this reason, some elements of this indicator could not be observed as individual development.

When the lesson study cycles are considered as a whole, the teaching strategies varied based on the activity in each three lesson plans. In general, the group used questioning and discussion in their lesson plans. In particular, Mila adopted concept motivation and action learning and concept image and definition at the beginning of the first lesson plan, conceptual conflict in the third lesson plan. The related examples were given in other sections.

Likewise, tasks included in the lesson plans were the products of the common sense of the group. The construction process of tasks was used as instruments for developing Mila's and other group members' mathematical knowledge. The structure of tasks in the lesson plan was determined before the planning phases. Therefore, the change in tasks was examined in terms of changing the ways of representing the concept. The following paragraphs show the change in the ways of representing (e.g., analogies, metaphors) the concept.

During the lesson study process, specifically during lesson planning, she commonly preferred using analogies and metaphors to tell other group members what she tried to say. It turned into giving the mathematical background of these analogies or metaphors throughout lesson study process. In addition to analysis of the pre-interview, beginning of the lesson study process showed that Mila had lack of knowledge about content specific examples and ways of representing. It can be understood from the first meeting. Her lack of knowledge was observed as she heard "paradoxes" for the first time in this lesson study process. Considering that both her and other group members' lack of knowledge was observed both in pre-interview and lesson study process, the lesson study process was designed to develop their awareness about content specific examples by means of comprehensive booklet and rich group discussions on the topic.

Just as she did not hear the paradoxes, she tried to describe the phenomenology of the concept "approach", which is dynamic conception of limit, as an analogy which she had heard from one of her classmates different than the group members. The example represented "approach"; however, she could not give the mathematical basis of the concept. For this reason, such an analogy was considered as lack of knowledge for this indicator.

> Researcher: After giving the definition intuitively to the students, what do we expect from them?
> Mila: We expect them to deduce the concept of approach.
> Researcher: If we actually expect the concept of approximation to be inferred, do we want the student to combine it with other mathematical concepts?
> Mila: Yes.
> Researcher: So, after giving the heuristic definition, we need to print it out. Do you need to guide the students to find out where they can apply this definition among the mathematical concepts?
> Mila: I mean, I always think of a circle. It was a story which includes a king who doesn't love one of his sons at all. He says he would give him a piece of land as big as the size of an ox skin. He tells his son to measure and take a piece of land as the size of an ox skin. The son decides to turn the ox skin into a very, very fine thread. Then he makes a triangle first, I don't know what else he does, but when he turns it into a circle, he gets the largest area. He could gradually approach its maximum value. I remember such a thing.
> Researcher: If there is an approximation, for example, what is the reason for subtracting c? How do we observe this here?

The mathematical background that she could not show here was reflected in Mila's other examples as her mathematical background developed. While she did not give any suggestions about teaching the concept at the beginning of the first lesson planning, she began to propose examples about how to teach the formal definition of limit after she developed her mathematical knowledge. It can be said that this indicator was developed through the instrumentality of KoT development. The analogy she uses below has now become content-specific with the development of conceptual and concept teaching knowledge.

> Mila: Can I tell you something that I found about epsilon-delta? I found these in Thomas Calculus. Now Thomas first said in Calculus; that is, he explained: To show that $f(x)$ is equal to the number L as x goes to x_0, we have to show that when x is held close enough to x0, the gap from f(x) is also small enough that I can choose: Let's think of Epsilon and delta like a skeptic and a scientist. The skeptic is skeptical, that is, he constantly presents us with epsilon objections that the limit does not exist or that the limit is something that cannot be doubted as such. The scientist also says that for every objection, I find that there is a delta equivalent around x_0,. And I show that in this range the function values will keep L around the epsilon. If I showed that there is

at least one delta for every epsilon, I would have defeated the skeptic. He turned it into such a game, in fact, he showed the epsilon delta in this way. I found something like this, so maybe we can use it.
(…) Alp: How are the objections of epsilon; What exactly is it objecting to?

Mila: For example... I have to look at the book for a second. I guess I can't answer exactly.

Mila: The scientist offers you the epsilon objection every time. What if the epsilon was 1, if it were 3, like if it were $\frac{1}{2}$. He constantly presents you with epsilon objections, and then you show a delta value each time for every value he says. This delta value shows that the limit is in that range around the epsilon. But this goes on forever. Say 1, say 10, say a million or something. You say what do I do for this? You say that if I find at least one delta for each epsilon, you'll end this discussion. After that, we used to write in terms of epsilon delta in the definition of epsilon delta, which we always say, or we kind of switched to it.

In the excerpt given above, she proposed an analogy which was taken from Calculus textbook. She tried to describe the statement in the formal definition as "for all epsilon there exists at least one delta". The analogy includes a sceptic and a scientist in which the scientist tries to convince sceptic about the existence of delta in existence of epsilon. As mathematically, the analogy could be a good starting point for students to teach epsilon-delta in the formal definition of the concept. From this example, it can be thought that Mila suddenly started giving examples from the zero point. As I mentioned in above sections, Mila had neither experience about the concept of limit nor knowledge of teaching the concept of limit. For this reason, Mila initially remained silent or participated in discussions with her existing knowledge. However, as her KoT improved (this is especially true for the first lesson plan), so did her awareness of the representations, examples, or strategies included in her teaching knowledge. Therefore, she participated in the discussions with her suggestions.

Another important evidence related to development of her awareness regarding content-specific examples, particularly related to the conceptual basis of limit, was Mila's search for the concept of limit in every situation she encountered. Since the prospective teacher was also taking her own advanced mathematics courses while the coursework was going on, she tried to connect her new learnings with her knowledge for teaching the concept of limit. At the end of the first lesson planning, she mentioned another example about the concept of limit to add it the revision of the first lesson plan (the reason of it was that she thought this example was at higher level than students' understanding). The example was presented in the KSM section (4.2.2). This content-specific example was related to a needle and the trace left by the thin end of the needle on the paper when it is left upright on the paper, representing the epsilon. The mathematical side of the example was presented in the related section. What the important thing for this example is that the prospective teacher gave this example, but the question of how to explain the example that even she had difficulty imagining to high school students remained. After the discussions on the example, she revealed her knowledge as "*But I think, we should use it in the revision part, namely micro-teaching. Because I think it is very difficult for high school students to understand this*".

This example showed that Mila went through the examples which built connections between mathematical concepts. In addition, she considered students' learning mathematics and students' cognitive level indirectly. For this reason, this example showed the relation between KFLM and KMT. However, it was not at an adequate level in relation between knowledge of students and teaching strategies, techniques, tasks and examples. For instance, the example of paradoxes can cause misconceptions in students' mind which is "the limit is the point that can never be reached". The prospective teacher should have been aware of this fact. She gained this awareness through discussions with her friends.

4.2.5.1.1 Development of Knowledge of Mathematics Teaching: Knowledge of Assessment Strategies

In the knowledge of strategies, techniques, tasks and examples, I dealt with the 'strategy' in two-fold: Teaching strategy and assessment strategy. While the teaching strategies of the prospective teacher could not be observed individually, assessment strategies of the members of the lesson study group could be observed individually during the lesson study process. The assessment strategies were considered as two parts in different examination: In planning, I kept her assessment strategies in perspective of her awareness of different assessment and evaluation methods and her evaluation attempts of exercises/problems used for evaluation in the lesson plan. In enacting, I considered her questioning types during her teaching, since she applied the lesson plans in her two research lessons without making any changes until she encountered with any unexpected moment.

Planning Phases of Lesson Study

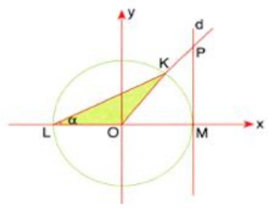
In the pre-interview, she did not consider the assessment strategy when it was asked how to teach the concept. Therefore, the development was examined through the lesson study process. First, in planning, the change of the proposed questions for the assessment of students' learning outcomes was revealed. In enacting, the questioning types were revealed in terms of its change.

Before passing on to the nature and development of assessment strategies of Mila, it should be indicated that this indicator is directly related to knowledge of mathematics learning standards of mathematics (KMLS). The curriculum and high stakes exams of Turkey require thinking on real life problems which covers both reading comprehension and problem solving. For this reason, the prospective teachers should know the assessment requirements along with the objectives of the curriculum.

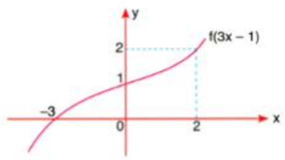
The first planning phase of the first lesson plan was mainly focused on providing a basis for conceptual knowledge of students. The outcomes they expected were determined at the beginning of the planning phase as a comprehensive understanding of the concept of limit which covered right-left-sided limits, definition, and relating the concept with its advanced connections. However, they did not think much about how to assess the learning outcomes of this lesson plan. Particularly, Mila did not propose any assessment tools or exercises/problems to assess students' understanding. Rather, she evaluated the exercises/questions proposed by other group members. For this reason, there was an environment where it was as if the decision maker was Mila and the other group members presented their exercises/problems. Therefore, although Mila did not propose any exercises/problems, the exercises/problems for assessment in the first lesson plan were revealed as product of common sense. The following table (Table 4.17) shows the exercises/problems in the first mini-cycle which Mila confirmed to put in the lesson plan.

Table 4.17 An example towards the variation in knowledge related to emotional aspects

The proposed exercises/problems
What is the result of the $\lim_{x\to\infty}(3x-2).\sin\frac{4}{x}$?
What is the result of $\lim_{x\to\infty}\frac{\sqrt{16x+\sqrt{9x+\sqrt{2x}}}}{\sqrt{x+3}}$?



The line-d is tangent to the unit circle with center O at point M. If $m(\widehat{OLK})$ is equal to α, what is the result of $\lim_{\alpha\to 0}\frac{|PM|}{Area\,(\overset{\Delta}{KOL})}$?



In the figure given above, the graphical representation of the function $y = f(3x-1)$ is given.
Accordingly, what is the result of $\lim_{x\to 1}\frac{f(x+4)}{f^{-1}(x-1)}$?

While the aim of the first lesson plan was to give the basis of the concept, the proposed exercises/problems addressed to the second and indirectly to the third lesson plan. As can be seen in the table given above, the questions examine different knowledge of students. They presented to be found to ask the students difficult questions. The most fundamental point to be said here is that these questions are in the exercise category rather than an open-ended and thought-provoking problem as suggested by the curriculum and examination system. The questions in this exercise style are not exercises for the first lesson plan, but rather exercises for the second or even third lesson plan.

Before moving to another part of the mini-cycle, I would like to refer the change in Mila’s and other group members’ knowledge of resources, since it is directly related to developing knowledge of assessment strategies particularly in the planning phase. In the beginning of the first lesson planning process, Mila commonly preferred to focus on the learning kit given by the researcher and the textbooks used in the curriculum. In addition, she insistently suggested using the Calculus book (Adams, 2017). However, one of the expected outcomes of the lesson study was to develop the prospective teachers’ perspective about using different resources when they plan their lessons. Considering this aim, different teaching resources including the textbooks from different countries, the web-sites, not including the blogs were proposed to the lesson study group. On the other hand, other group members who were experienced in working on the concept of limit brought different resources including KPSS (Public Personnel Selection Exam) specialized content knowledge books, advanced mathematics lecture notes, different high school text books. This provided Mila to raise awareness that one should not rely solely on certain sources for such a concept that students have difficulties and that comes from the nature of the difficulty. In the post-interview, Mila expressed herself about this development as:

> Mila: I looked at the MEB book, I benefited from the sources on the internet apart from the MEB book: I looked at various khan Academy videos, I benefited from different YouTube videos (I can find such interesting examples, such as the ones from daily life), what you gave and my friends provided me contributed a lot to me. Maybe inspired by them, I looked at some articles, I can't name them right now, but I looked at some articles, I looked at the Calculus books by Thomas and Adams, I borrowed a few books from the library. They will allow me to reconcile Calculus in daily life like this, even one book did not work well for me but there are things that I can definitely use in the future, related to such games. I actually read too much.

For the knowledge of assessment, the prospective teacher needed to see different assessment tools and assessment strategies. It can be provided by observing different mathematics teachers from different schools and maybe countries, or by examining different strategies through reading and discussions on them. After this mini-cycle of the first lesson plan, the intervene was performed as giving different resources, questioning them to construct a discussion environment on the combination of assessment strategies and expected level of conceptual or procedural development. In this way, it was aimed to make them use different assessment strategies considering both the content and the learning goals. The second mini-cycle showed that the awareness about assessment strategies started to develop in terms of the content. Table 4.18 shows the differences between the first part and the second part of mini cycle of the first lesson planning process and the ensuing process.

Table 4.18 The differences between the first part and the second part of mini cycle of the first lesson planning

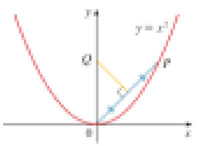
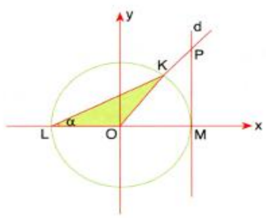
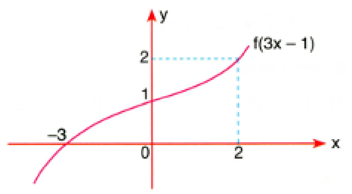
From #	To #
What is the result of the $\lim_{x\to\infty}(3x-2).\sin\frac{4}{x}$?	 The figure on the right shows the point P on the parabola and the point Q where the mid-perpendicular of OP intersects the y-axis. What would you say about point Q when P approaches the origin along the parabola? Do you think there is a limit operation here? If so, show it.
What is the result of $\lim_{x\to\infty}\frac{\sqrt{16x+\sqrt{9x+\sqrt{2x}}}}{\sqrt{x+3}}$	The figure on the right shows the isosceles triangle with equal angles B and C. The bisector of angle B intersects side AC at point P. Suppose the base BC remains constant, but the height of the triangle approaches 0, then A approaches the midpoint M of side BC. What happens to P in this process? Do you think there is a limit operation here? If so, show it.
	A nuclear scientist is working on an experiment. He found a function $f(t)$

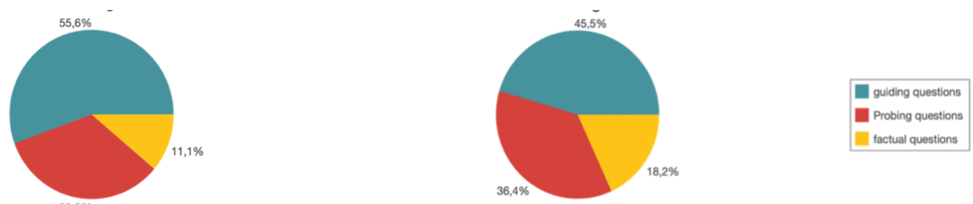
Table 4.18 (continued)

The line-d is tangent to the unit circle with center O at point M. If $m(\widehat{OLK})$ is equal to α, what is the result of $\lim_{\alpha \to 0} \frac{\lvert PM \rvert}{\overset{\Delta}{Area\ (\widehat{KOL})}}$?	representing the molecular number of a radioactive substance as shown below: $f(t) = \frac{at^2 - b}{t-2}$ Here, t represents the time in minutes since the start of the reaction. The scientist who lost his grades in the laboratory does not know the value of a and b. However, he remembers that 2 minutes after the start of the reaction, the number of molecules of the radioactive substance approached 4. In the light of this information, find the values of a and b.
 In the figure given above, the graphical representation of the function $y = f(3x-1)$ is given. Accordingly, what is the result of $\lim_{x \to 1} \frac{f(x+4)}{f^{-1}(x-1)}$? $a)\ -\frac{1}{10}$ b) $-\frac{1}{5}$ c) $\frac{1}{2}$ d) 1 e) 2	A study investigating the driving costs of 1992 small cars found that the average cost (car tax, fuel, insurance, maintenance and repair) in tl/km is approximately "$C(x) = \frac{2010}{x^{2.2}} + 17.80$ (x, times the car's times in 1 year)". It shows the value of the road in TL/km)". a) What would you say about the average cost of a small car driving 5000 km per year? 10000 km/year? 25000 km/year? b) If we consider that the distance traveled by the car in a year increases indefinitely, what would you say about the cost of the car?

In the first part of the mini-cycle, it can be stated that the problems examine students' calculation skills regardless of the difficulty of the problems. In connection with KMLS, the standards in the curriculum require developing the students' skills including reading comprehension, and interpreting on what is understood. In comparison with the first and second column, the table shows that questions/problems changed from the questions which assess students' calculation skills for the questions which assess students' understanding and applying the concept of limit in context-based problems.

Enacting Phases of Lesson Study

In general, the lesson plans were grounded on the questioning and discussion teaching strategy to teach the concept of limit. It can be said that this was one of the appropriate strategies that could be applied in such a concept that students had difficulty in understanding and confusing. The point to be considered was how the teacher asked questions and how well she could meet this in a mathematical context while guiding the students to the right points. In addition, how students change their planned strategies according to their learning and question types is also an important point. While evaluation strategies were evaluated as the types of questions they prepared at the planning stage, how these and the activities of the lesson plan were applied in the lecture were also investigated according to the questioning types. Each of the three question types including probing, guiding and factual question serve different purposes during teaching the concept. For this reason, the study did not aim to put forward one of them considering whether there was a development or not. However, I considered the question types about these purposes. The following figure shows the frequencies of the question types she used during each research lesson of the cycles.



The research lesson of the first

The research lesson of the second

Figure 4.22. The prospective teachers' used question types during the research lessons

The frequencies of the questions during the research lessons presented almost same results. In particular, the percentage of the probing questions remained almost the same. She used probing questions in both two cycles for explaining and elaborating students' thinking by asking "*why*" or "*why not*" based on students' answers. The guiding questions decreased in the research lesson in the second cycle. However, it is not a significant finding for the current study since the second cycle was conducted as micro-teaching research lesson. For this reason, Mila did not need to guide other prospective teachers for a specific answer or a next step for solution. However, the question types in guiding questions changed from the first cycle to second cycle. In the first cycle, she used the questions for specific answer/next step of solution. In the second cycle, she asked students (prospective teaches in this cycle) to think about or recall heuristics strategies in addition to the specific answer (see Figure 4.22).

While the percentage of the probing questions decreased, the percentage of factual questions increased. The factual questions were related to a specific fact, for an answer to an exercise and to provide the next step in a procedure. Therefore, it can be said that the factual questions lead students to produce mathematics for their mathematical knowledge. For this reason, the increase in factual question can be interpreted as the factual questions took students to a higher level. Moreover, the factual questions are closely related to development of the prospective teacher's knowledge of practices in mathematics (KPM). Therefore, this increase can be explained as the development of KPM of Mila. Different from the other questions, there was not any change in the question types of factual question. She commonly posed questions about facts and answers for an exercise. The following table shows examples from the research lessons of each cycle.

In the table given below (Table 4.19), the examples are from both cycles for the questioning strategies. In the example of probing question from the first cycle, Mila asked the reason for the student's answer to reveal assessment of the student's learning through the statement of the student. In the second cycle, she used the probing question by asking the reason behind the students' answer to reveal students' further thinking. In the both examples for guiding question, Mila tried to guide students by using their previous knowledge including trigonometric functions and exponential expressions in the related mathematical procedures of the limit concept.

Table 4.19 An example towards the variation in knowledge related to emotional aspects

	The first cycle	The second cycle
Probing questions	Mila: That's why 1 to the power of infinity is indefinite. Student 1: But that's not 1 to the infinity, right? We said 1 to the infinitely indefinite, isn't it something different? There's an "e" here! Mila: hmm, why do you think that? Confused that it is equal to e?	Mila: Let's not see the 2 over 0 limit operation now. Because what was it to us? Student 3: was undefined.) Mila: so why was it undefined?
Guiding questions	He solved the first question himself. He has the student solve the other question. Mila: My advice to you is to use trigonometric function knowledge while solving this question.	Student 3: Is e to the minus infinity, hocam? I must be doing something wrong, hocam! Mila: Try to remember the exponential expressions there. For example, what were we doing when I said 5 to the minus one?
Factual questions	Mila: If I ask you a question now (he turns to his presentation but realizes that the problem is not there) I did not write it here… Mila: (closing the presentation screen) forget about this. I will write the question myself. Let's say I take $f(x)$ $f(x)$ to the $g(x)$ as limit x goes to infinity. What if $f(x)$ was 1 for me as the limit goes to x?. If limit x goes to infinity and $g(x)$is infinity for me. What will the result be for me? So, when I think about that $f(x)$ to the $g(x)$ structure?	Student1: I took the h from here. I wrote zero instead of h directly. Mila: Ok let's stop here, now you have simplified those h, have you lost root?

In the example of the factual question from the first cycle (Table 4.19), Mila directed her actual questions as asking “what if” structure to reveal a direct answer including a fact. Although the example was given under the factual question, it also had probing question which probed the factual question. Lastly, in the example of factual question from the second cycle, she tried to reveal the possible misconceptions of the students on this subject by asking whether the simplification made also lost its root in finding the limit of a function containing polynomial division at the zero pointIt aimed to reveal the possible misconceptions of the students on this subject by asking whether the simplification made also lost its root.

4.2.5.2 Development of Knowledge of Mathematics Teaching: Knowledge of Teaching Resources

Another indicator of KMT is knowledge of teaching resources. This knowledge includes how to develop teaching resources for teaching a concept beyond just the knowledge of what teaching resources are, how they are used, and evaluating the limitations and benefits of the resource on a concept basis (Carrillo-Yañez et al., 2018). In the pre-interview, it was hard to observe knowledge of teaching resources, since it requires evaluating it in planning or teaching the concept. Based on the researcher’s experience with Mila before the lesson study process, it can be said that Mila was a talented prospective teacher about using different teaching resources. However, she had lack of knowledge to combine her knowledge with mathematical knowledge, which was observed at the beginning meeting of lesson study process. Therefore, the development occurred in accordance with the development of mathematical knowledge throughout the lesson study.

This indicator was dealt with in two parts as planning and enacting similar with other sub-domains in PCK. In addition to the indicator of the model, additional indicator - knowledge of how to use the resources in unexpected situations was added to examine knowledge of teaching resources in enacting phases of lesson study.

Planning Phases of Lesson Study

During the planning phase, the lesson study group mostly discussed on what teaching resources should be used in research lesson. At the beginning of the first planning, Mila and other group members focused on attracting students' attentions with resources. For instance, Mila's first attempt about using the resources was to show an animation constructed in Pawtoon to start with the history of the concept.

> Mila: I looked at its history. There's a lot of stuff about Cauchy. There is information that he lived at the time of the revolution or something. Something occurred to me as well. We said both Cauchy and Zeno's paradoxes with the help of animation in Pawtoon, I think we can do them. (*The second planning meeting of the first lesson plan*)

The commonly used teaching resource was GeoGebra in almost all three lesson plans. GeoGebra was the safe place for Mila and other group members. I looked at both how they planned to use GeoGebra and how they planned to teach mathematical content by using GeoGebra's tools. At the beginning of the lesson study process, Mila did not have a critical view to GeoGebra applications for teaching the concept. The meaning of using GeoGebra was composed of "*displayed via GeoGebra*" as can be seen in the following excerpt. The excerpt is taken from the planning phase of the first lesson plan in which the group discussed on their suggestions.

> Mila: We talked about an activity related to "approaching to the number e". I prepared a worksheet related to it. Coincidentally, we wrote the same functions as Alp.
> Alp: $f(x) = 2x + 1$.
> Mila: The steps of approaching consist of 1, 2, 3 and the close rational numbers so that they can calculate a little more easily. (…) After that, I will ask them the behavior of the function when it approaches to e. I don't know these last two questions; we can combine them a bit more maybe it's a bit shaky question. One of the possible answers I expected: we tried to find the closest thing possible by giving big and small values. Actually, I'm trying to get this to say at the task. After this task, I did not know how to make the transitions in the sequence. After that, we talked about how we can show this on a graph with the help of GeoGebra, since I tried to give an approximation from left to right, by emphasizing that we gave both large and small values. So, we show it over GeoGebra. (The third meeting of planning of the first lesson plan)

When it was asked how she planned to show it on GeoGebra, she answered as “I don’t know but we can show it”. This can be considered as lack of knowledge for teaching resources since she did not consider the effect of using tools and demonstrations on effective teaching. Therefore, different types of demonstrations were presented to the group to develop their awareness about how to use them by considering their effectiveness. One of the demonstrations on GeoGebra included the intersection of the areas that scan the inequality specified by delta and epsilon. Another one also demonstrated the interaction as well; however, it did not include the graph of the function (see Figure 4.23). The difference between these two graphs can be explained from the formal definition of the limit. Considering one of the important difficulties related to the concept of limit, which is how close one can be to a point (in other words, the concept of neighborhood actually), the researcher expected the prospective teacher and lesson study group in evaluating both demonstrations to criticize them considering the definition of limit. Because the definition of limit indicates that “if $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$” which constructs a rectangular region ($[c - \delta, c + \delta] \times [L - \varepsilon, L + \varepsilon]$). The lesson study process was designed to create a group discussion on it to develop their both KoT and KMT.

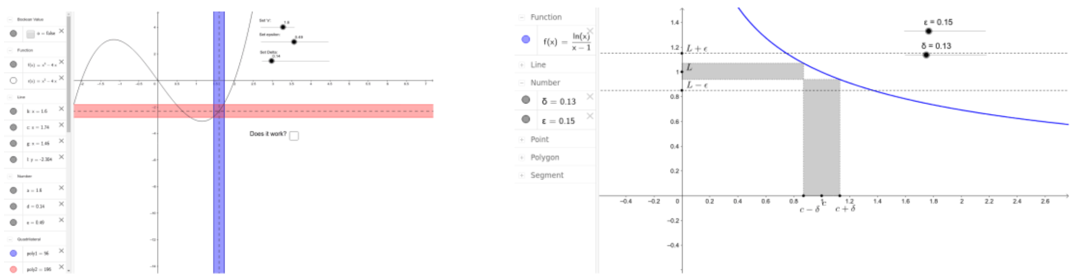


Figure 4.23. The demonstrations of the definition of limit in GeoGebra

The activity provided her to make critical comments related to the content. In this way, she could develop her awareness of which tool will be effective when used and how. After this discussion, the perspective on the use of teaching materials has changed for a correct and effective teaching of the concept, both during the planning phase and while interpreting the research lessons. One of the evidences about this development was her critical comments on Alp's research lesson. In Alp's lesson, the planned activity on GeoGebra did not work and Alp handled this situation by using the board marker and his hands. In the discussion of Alp's research lesson, Mila evaluated his lesson as following:

> Researcher: There were different problems in Alp's lesson, did you notice them?
> Fulya: I wish the apps worked; I think it would be effective.
> Researcher: What do you think about Alp's reactions to these unexpected situations?
> Mila: I think Alp handled this situation but it didn't work in the activity of "approaching secant line to tangent line on the graphics".
> Fulya: I agree with you; I don't think that the activity had the exact effect we wanted on the students.
> Mila: I do, so. But it may be because GeoGebra is not working. If GeoGebra had worked, the effect might have been different.

The excerpt shows that Mila criticized the effectiveness of Alp's actions (simulating the approach with different color board markers) and the planned activity in GeoGebra. The second lesson plan was based on the mathematical procedures with the concept of limit. Therefore, the lesson study group focused on the exercises and problems in the second lesson plan. The process related to KMT for this lesson plan will be considered as "knowledge of assessment" in the further sections.

In the second cycle of the first lesson plan, the lesson study group had confusion on which one of the examples should be used at the beginning of the first lesson plan to form an idea in the minds of students what the limit is. As mentioned above, the first cycle presented the beginning as approaching a number of a function by filling tables. However, Mila as well as other group members expressed that the students got bored in filling the tables. For this reason, Mila remembered them in their first brainstorming on this issue, specifically the geometric approach for the concept of limit as the following example shows:

> Mila: Hocam, now, there is a task related to "approaching 1". I think it was too simple because they knew so much, it lost its impressiveness, I think, let's either replace it with something else or find an alternative entry activity.

Fulya: I saw that. I wrote exactly the same. with the same words.
(…) Mila: Hocam, maybe it will be as simple as a table, but in geometry or something, circles are formed, or we increase the number of corners of the bases to make them cylinders, there are activities related to them, I think we can use them too.
Fulya: You were rotating the rectangle, you were creating a cylinder by rotating it 360° around it, it had phases, you say that, you rotate it slowly, did you even use the moonlight, he rotated the triangle to form the cone...
Mila: I'm not saying rotation, what I'm saying is there is a direct triangular prism, you increase the number of bases from a quadrilateral to a pentagon, you increase it, you increase it and it becomes the last cylinder.
Researcher: Okay, let's think on what will be different in this activity when it is compared with your first activity (constructing table) for teaching the limit?
Mila: Hocam, they are interested in such things. I distributed exit tickets to the students, some of them wrote something in the part of the things I find interesting: It was very interesting that a cylinder was formed from a prism. That sort of thing might be interesting for them too.
Fulya: The connection with integral?
Mila: And… the subject we just talked about (the margin of error), you know, things used in numerical analysis and so on. Even if it has the same purpose as the table, I think we cannot give the exact situation where that limit eliminates the margin of error in the table.

The excerpt where the improvement was revealed occurred by means of several factors. First, the reflection of the first lesson plan provided her to criticize the lesson plan based on its mathematical basis and students' learning. In this way, as the lesson study cycle progresses and this progress improves the prospective teacher's mathematical knowledge, the prospective teacher could give a suggestion about teaching resources for the concept of limit based on the nature of the concept of limit. Therefore, the prospective teacher revealed improvement in this indicator of KMT.

The third lesson plan aimed to conceptualize the concept of continuity and the procedures with continuity. Similar with the first lesson plan, the group aimed to demonstrate their goals on GeoGebra. Thanks to the experiences she gained from the first lesson plan and its reflection, Mila could make regardable criticisms about the activities on GeoGebra. One of the important points the group and Mila considered in the third lesson plan was to overcome the misconception "*If there is no space in the graph (if the graph can be drawn without raising a hand), that function is* *definitely continuous in the domain*" with counterexamples shown on GeoGebra. It should be indicated here that Mila had already had this misconception before she participated the lesson study process. When she overcame her own misconception, she wanted to focus specifically on this issue (As I mentioned in the previous section, Mila commonly thought that students interacted with content in a similar way with her). Their first attempt to teach the concept for the first activity was to use a metaphor which was described as "*in the world that is claimed to be a portal, people who are normally there at that moment explain that they can pass through the bridge in that portal even though there is a gap where the portal is*" (This metaphor was suggested by Alp, and Mila was interested in this metaphor). After they passed through from metaphor to mathematical content, they aimed to construct "cognitive conflict" in students mind by showing an example of a continuous function whose graph does not consist of a single piece. One of them was the graph of $f: \mathbb{Q} \rightarrow \mathbb{Q}, f(x) = \begin{cases} 0, x < 0 \; or \; x^2 < 2 \\ 1, x > 0 \; or \; x^2 > 2 \end{cases}$ and its continuity on the domain of the function. The function was brought from the text which I gave them to discuss on its continuity. At first, they discussed on the graphs (Figure 4.24) about its continuity since they themselves were quite confused about this (see section-Development of KoT). While they discussed on the topic, they also focused on the demonstration of graph. As can be seen in Figure 4.24, the critical point ($x = \sqrt{2}$) is not mentioned as hollow point ($\circ$) or highlighted point. Therefore, the group and Mila focused on this function. However, the graph did not seem as similar with the figure in the book (Özmantar, & Yeşildere, 2008).

> Fulya: Does it happen all the time at 2? Aren't there two critical points in the drawing, hocam, I don't see it wrong.
> Alp: Post slipped.
> Fulya: Slipped?
> Mila: I think there is a problem there too.
> Researcher: I think the drawing refers to the critical point as 2, look at this line.
> Alp: He tried to show it as the square root.
> Fulya: Hocam, that thing. Is that square root two and minus square root?

Researcher: I have no control over this drawing, so how you use it in teaching or not is up to you.
Alp: No! Drawing, I said.
Researcher: I put the drawing as it is, by the way, I couldn't figure it out myself.
Alp: Now, I tried to draw it on GeoGebra but I couldn’t draw it myself.
Mila: I think that split point is like square root.
Alp: Square root, I think.
Fulya: I think so.
Mila: So, I think that right where those two are is square root.
Alp: I don't see any reason why it should be discontinuous. (The first planning phase of the third lesson plan)

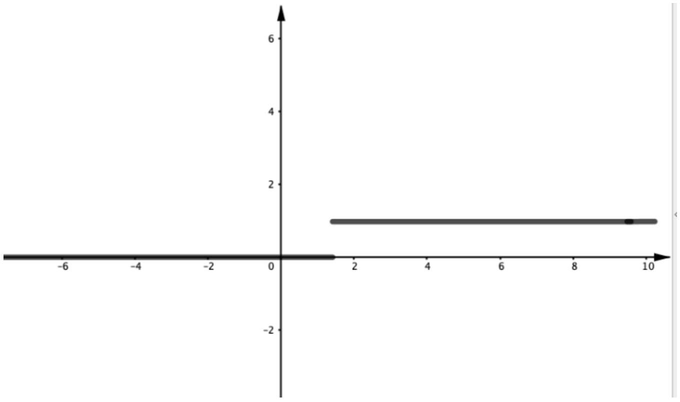


Figure 4.24. An example a continuous function whose graph does not consist of a single piece (Özmantar, & Yeşildere, 2008, p. 205)

The succession of the planning stages of the lesson plans and the fact that the first one passes through the research lesson while planning the other when it is finished, enabled me to see the effect of planning on development.

The second cycle of planning provided them to combine their experiences from their planning phase and the teaching experiences. In addition, with the development of KFLM, it can be said that Mila and her colleagues became more conscious about the awareness of teaching resources which could be effective. In the second cycle of the first lesson plan, the group had awareness of students’ interests and the resources. For this reason, the group discussions have emerged from the bounds of GeoGebra applications only, and have evolved into talking over hands-on materials or combining these materials with GeoGebra. It should be indicated that the idea of using geometric constructions in teaching limit was mentioned in the first cycle of planning of the first lesson plan as well. However, the group and Mila as well did not think of using scissors, compass and ruler in the first cycle. In the reflection of the first research lesson, Mila insistently emphasized that 'approaching with tables' activities did not attract the students and were not processed in accordance with the teaching purpose. Therefore, she proposed the idea related to finding the area of circle. First, she questioned students' previous learning about whether they have known it or not. Second, she proposed to start the activity with questioning the area of circle as how we find it and how we show it. While she thought of using simulations about this demonstration, the group turned to using hands-on materials by means of Alp's suggestion including cutting papers to form a parallelogram-like shape.

> Alp: Let's look at this activity in the curriculum about circle areas. It takes three.. This way they were cut into four. Then they put the cut pieces together and formed a parallelogram-like shape for it. Then they were making eight pieces, then they were making 16 pieces. They were asking the student what would happen if we did more. But the topic is not about the concept of limit, it is about teaching the concept of area.
> Mila: How will they cut circles in the same area?
> Researcher: Now, let's think on the mathematical connection here.
> (They discussed on the mathematical connection here, it can be seen in section of knowledge of structures of mathematics)
> Alp: When they (students) work with papers (cutting and compounding pieces for parallelogram), they encounter with a shape which looks like a parallelogram but a parallelogram whose two opposite sides are wavy.
> Fulya: But I observed that it is more beneficial when they cut out and make their own.
> Mila: Okay! I understood what you mean! So, I have an idea. Not to spend lots of time, we can combine both simulation and hands-on activity.
> Fulya: In this way, we can show the million pieces like approaching limit!
> Mila: That's what I wanted to say!

As can be understood from the excerpt, Mila had aware of the reason of the activity. For this reason, she insistently suggested using simulations in addition to hands-on activity. Mila has shown that she has this awareness by focusing on the purpose of using the resources rather than how they are used. It should be indicated that her awareness was revealed when she saw the mathematical connection between the activity and the concept.

To reflect her nature of KMT-teaching resources, her contribution to the lesson study should be indicated. As I said before, she was a competent prospective teacher during her undergraduate process up to the lesson study process. Therefore, she was prone to use different teaching materials. For instance, she implemented the third research lesson in the first cycle. While the former lesson plans were products of the common sense of the group, the third lesson plan was mostly dominated by Mila in terms of examples and teaching materials. Mila and the group members planned to conduct the modelling problem –"Lake Pollution [Göl Kirliliği]" (Erbaş et al., 2016, p. 60) by dividing students into groups. Since it was really hard to construct heterogeneous groups, she proposed to use a program 'Superteachers' in which students are numbered and then divided by simultaneously. Such programs are not considered as what I meant as "teaching resources" in KMT. However, her awareness and knowledge related to different sources for using in the classroom made her teaching more effective. Therefore, I presented it in this section. Similarly, she proposed to use 'exit tickets' for the evaluation of the all three research lessons, since she taught the last research lesson in the first cycle. She described her aim for using exit ticket as "*First we will see their (students') concept images. I mean…We will teach the concept, we will construct the correct definitions in class, at the end of the lesson we will ask them, for example, what they knew/thought about continuity before the lesson and what they know/think after the lesson, and how did they feel. That's an exit ticket!*". She used it as an evaluation from at the end of the lesson to reveal the students' concept images. It can be also regarded as an assessment tool. None of other group members proposed such resources for effective teaching. Therefore, it can be concluded that her awareness about resources and content was developed during the lesson study process.

Enacting Phases of Lesson Study

During the planning phase, I looked for the development of her awareness about both using different teaching resources and combination of content and using resources. In enacting, I analyzed the indicator from the perspective of how to use resources in unexpected situations in the classroom. There were two reasons for this new indicator: First, both Mila and other group members were prone to conduct the lesson plan without making changes unless they encountered a situation they never expected. Second, they did not have any experience in classroom teaching before they participated the study. Therefore, I considered the examination of how to handle the problems about teaching resources which were planned to be used for revealing this indicator.

The first cycle of research lesson was conducted in a high school with the senior students. The third lesson plan included using GeoGebra and Superteachers integrated in PowerPoint on the smartboard. Since she had already known that GeoGebra gives errors when it is used in the smartboard, she minimized the number of GeoGebra activities. In addition, she prepared the screenshots of GeoGebra activity just in.

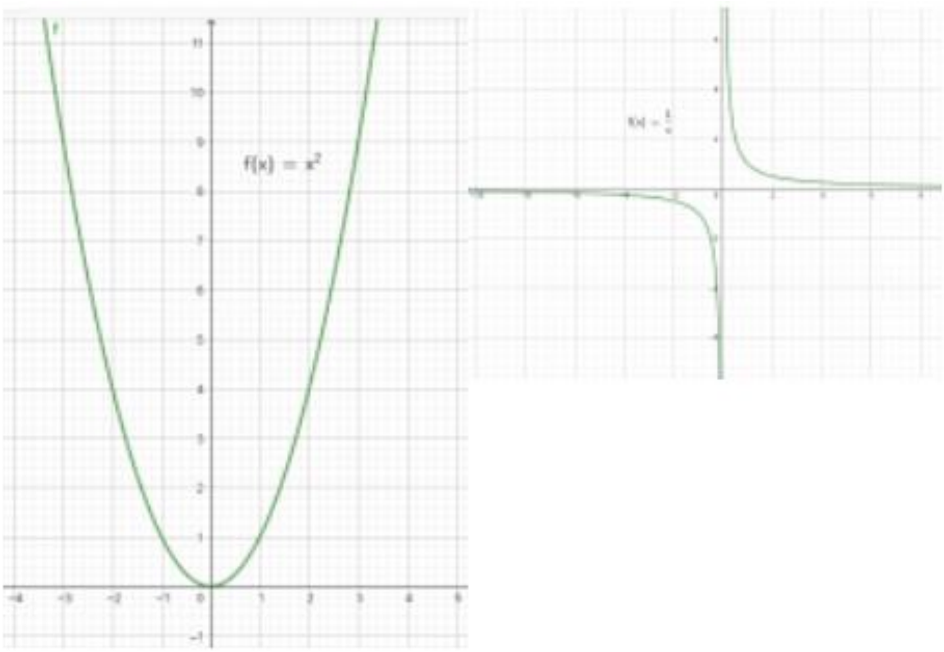


Figure 4.25. The screenshots of GeoGebra activities (an example)

One of the unexpected problems with teaching resources was the questions that the prospective teacher tried to discover the indeterminates disappeared on the smartboard. She handled this situation by writing some limits which made sense towards the logic behind the indeterminate forms in students' minds. This can be explained as the development of KFLM as ways of interacting with students.

Similarly, she wanted to use Instant Classroom in her class to divide students homogenously. However, she could not run the program even though she tried again and again. She tackled with this problem by randomly changing the location of students and class order.

The research lesson of the second cycle was conducted as micro-teaching to her classmates. Since it was conducted in the mathematics laboratory of the university, there were not any technical problems about the applications. However, she had some time management problems since her classmates asked more questions about the activities in lesson plan that they ever expected. She overcame this situation by eliminating some questions from the lesson plans. In addition, while she would show the demonstration of limit of functions types including polynomial, trigonometric, piecewise on GeoGebra, she did not prefer to use GeoGebra, rather she demonstrated them on PowerPoint with screenshots.

4.2.6 Development of the Prospective Mathematics Teacher's Knowledge Mathematics Learning Standards in the Concept of Limit

Knowledge of Mathematics Learning Standards includes the pedagogical knowledge related to the school mathematics curriculum across the grades, knowledge of appropriate instructional materials, evaluation instruments, and standards. As explained in the literature review, the knowledge includes three sub-domains including expected learning outcomes, expected level of conceptual or procedural development, and sequencing of topics.

In general, I did not observe any development in this knowledge sub-domain. In Turkey, there are not any standards which are different from the curricula. For this reason, the prospective teachers had a limited perspective to develop their knowledge about learning standards. Therefore, this limited perspective constructed a barrier for developing the knowledge of learning standards. The same issue appeared in the indicator of sequencing topics. The lesson study group did not get off the sequence of topics in the concept of limit. As a requirement of the lesson study, the prospective teachers should conduct research lessons in real classroom. For this reason, they had a responsibility to the mentor teacher of the school where the lesson study group conducted their research lessons in. In this study, the mentor teacher did not want to go out of the boundaries of the curriculum, except the formal definition of limit (The formal definition of limit is not included in the curriculum). Since the formal definition of limit was considered as a learning outcome of the lesson study process and the prospective teacher thought that it should be taught based on her own lack of knowledge, the formal definition was considered in the lesson plans. Therefore, except that, Mila and other prospective teachers as well could not make a big effort in the sequence of the subjects.

Table 4.20 The expected learning outcomes which Mila and other group member's asserted

Lesson Plans	The expected learning outcomes
Lesson Plan 1	Conceptualization of the concept of limit in students' mind
Lesson Plan 2	Applications and mathematical procedures with the concept of limit
Lesson Plan 3	Conceptualization of the concept of continuity, the relation between the other concepts and continuity

Another indicator was expected learning outcomes which can be considered as an element for the cornerstone of lesson study process which is determining the lesson study goal. This first phase was conducted only once. While the lesson study goals were not determined again, I observed a development in expected learning outcomes in a roundabout way considering the change in question types during the lesson study process. As mentioned in KMT section, Mila and other group members put forward practice-based questions. Considering these questions as assessments tools, it can be understood from such questions that the expected learning outcomes were 'correct calculation of limits in problems. In second cycles of the lesson plans, Mila proposed the context-based problems which required both reading comprehension and mathematical reading comprehension. Therefore, the aim of these context-based problems could be to understand what problem is about, relevant-irrelevant data, and the mathematical procedures required to solve the context-based problems (Wijaya, van den Heuvel-Panhuizen, & Doorman, 2015). The important issue in here was that the prospective teacher did not consider the expectations of curriculum in suggesting these types of questions. The changes occurred by means of the observation of research lessons and the guidance by the researcher. Therefore, it could not be regarded as a development which was a result of the awareness about mathematics learning standards.

4.3 The Prospective Mathematics Teacher's Observed Development of Specialized Knowledge in the Concept of Limit

The final step to answer the first research question was to support the findings with examining the prospective mathematics teacher's knowledge at the end of the lesson study process. The post-interview was designed to obtain the final reflections about lesson study process in terms of specialized knowledge. As I said in the methodology, the post-interview was constructed considering the observation of lack of knowledge and knowledge development during the lesson study process. Knowledge development for sub-domains was observed during the lesson study process in lesson planning and/or enacting lessons. Therefore, the post-interview can be considered as a supportive tool for examining the development. The post-interview consisted of summative and reflective questions, not as comprehensive and detailed as in the first interview. The titles in the post-interview can be summarized as conceptual definition of limit and how to teach this concept, the relations between mathematical concepts and limit, the topics of infinity-undefined-indeterminate, teaching resources, and reflections about the lesson study process. Therefore, I presented the findings of the post-interview in two main titles: Reflection on mathematical knowledge and reflection on pedagogical content knowledge. The findings are presented as in comparison with the pre-interview.

Before passing on the findings related to specialized knowledge, the reflection on the lesson study process is of importance as an examination of the process. First of all, she thought that lesson study process had a positive impact on her professional development. In particular, she focused on noticing their own deficiencies in terms of mathematical knowledge, noticing the concept and teaching, and having interesting (for her) conceptual knowledge that she never knew, as can be seen in her description about her journey in the following excerpt.

> Frankly, we started with a subject that I thought I knew a lot. But then I realized that I didn't know the concept that much. I mean, not knowing or not, I have a lot of shortcomings. Then as I started working on it, I enjoyed learning a lot. Because I found so many interesting things. It's about associating it with daily life and about different things. (…) I saw how difficult it is to prepare a lesson plan. I mean, it's not just saying, "I say this and that," just like that. Everything had to be tied together, that is, both the concepts and the order of instruction.

Mila stated above that she thought she knew a lot at first, but realized that she had deficiencies in the process, and we can say that the process was successful in this regard, by making the teacher candidates question themselves and see it as a professional competence, which is actually one of the aims of the lesson study process. In the continuum of the post-interview, she supported this claim by expressing "*I started to question myself all the time, I wonder if it is true, I wonder if there is another important point*".

At the end of the post-interview, I wanted her to summarize the lesson study process in terms of her development process. She mentioned some items related to lesson study process and its effectiveness. These items can be ordered with the evidence from the post-interview as how to start teaching a concept (*For example, I had an idea about how to start teaching a concept, because I didn't really know it*), integrating the history of the concept into the lesson plan (*For example, a professor was telling us that if you're always making history, combine it with your course, and*

I didn't know how to combine it, how to do something), integrating daily examples into lesson plan (*we associate it with daily life. Of course, I associated basketball with handball and I think it was very effective. I think it's something that not everyone can think of*) and how to research a topic in the literature (*I learned how to research certain things. It contributed a lot to the research, you know, at least it became an idea. I mean, we also learned a lot of things mathematically, I think*). These items provided to reveal the critical elements and the conjectures for constructing lesson study development model which will be presented in the last section of this chapter.

The post-interview did not include the examinations of the sub-domains separately. Rather, the questions in the post-interview were summative and reflective questions, not as comprehensive and detailed. Therefore, this section was structured into two main sections including mathematical knowledge and pedagogical content knowledge.

4.3.1 The Prospective Mathematics Teacher's Observed Development of Mathematical Knowledge

What the lack of knowledge was observed in the pre-interview was the definition of the concept of limit, in particular, how to define the concept. Therefore, the first examination in the post-interview was related to how to define the concept. The answers of Mila supported my claims that the lesson study process nurtured the prospective teacher's knowledge of definition (KoT) as I said in the lesson study process under the title of KoT. When it was asked how you define the concept of limit, she confidentially answered the questions and she wrote the definitions correctly. The first mention of Mila was that limit can be described as behavior of the function at a point. Second, she touched on that the concept of limit emerged to explain the concepts of derivative and integral and that it became a formal definition relatively recently. The following excerpt taken from the post-interview shows her knowledge related to this topic.

> Mila: With the limit we look at how a function behaves at certain points. … We have always said that the concept of limit emerges after the derivative integral, but in the derivative integral, you know, the limit is actually mentioned with infinitesimal approximations, briefly later Cauchy introduces the concept of limit. After Cauchy's definition, it is now slowly progressing towards formal definition.

These answers were related to KoT; in particular phenomenology of the concept and definition of the concept. In the other questions, I could observe KSM when I asked her how they relate the concept with other mathematical concepts. She reacted this question as "*I will not say derivative and integral as everyone expects*". She gave an example about the relation between geometry and limit. As can be seen in the excerpt given below, she connected geometry and the concept of limit, in particular the series and limit in implicit way.

> Mila: Hocam, the limit is everywhere, I learned that at the end of this process. So, I'm surprised to see it like this. I mean, it's here too. Basically, we observe the limit as a behavior and there is the approach with infinitely small moves. Actually, I think we use it too much in different places. While I was preparing the lesson plan during the internship, I saw geometry as well. The base is first triangular and then gradually increases. As we increase the number of lines and the number of edges, we say that the cylinder is obtained as we increase the number of vertices. And from here they even said that prisms are actually cylinders. I was very surprised when I searched the internet. So even here there is a limit. …In the same way, we can use it very similarly, for example. We even used the limit of the secant line, which caught my attention the most. Because for us, the tangent line represents the derivative. So, it has a connection like this.

She built this connection by using the common feature of the concepts; infinity in this example (she expressed it infinitesimal steps). Therefore, it can be said that she used her KSM, specifically transverse connections. Her another example was the limit of secant line, which was one of the most discussed topics in the lesson study process. In this example, she used her knowledge of auxiliary connection, since she used this example as an auxiliary element to describe me the connection between derivative and limit. At this point, it is necessary to mention the applications of the limit. In the above sections, the applications of limit were categorized into four parts: derivative, integral, irrational numbers and iterative process. When she mentioned derivative and integral as applications of the concept, I asked her what about the relation between irrational numbers and the concept. She answered without thinking on it as "*…For example, let's consider the number pi or the number e. Now, for example, we do not know the exact value of it, but we will try to get closer to these or their value in some way. We also use the limit to approximate these and estimate the in value*…". While her answer was not mathematical, as in the pre-interview, I observed that Mila used knowledge of applications where she tried to explain with an example.

Another topic which she had lack of knowledge in the pre-interview and mentioned in the post-interview was infinity and the difference between indetermined and undefined. In the pre-interview, she could not describe the indeterminate forms. As a matter of fact, she was confused about the indetermined forms and infinity in indeterminate forms. The post-interview showed that Mila overcame this confusion in relation with indeterminate-undefined forms, as can be seen in the following excerpt.

> Mila: Indeterminate … We can talk about this already in the limit state. Let's say the results I have are different. For example, we can talk about $\frac{0}{0}$, like for $\frac{sinx}{x}$. Now, if I take two functions in the form of $\frac{0}{0}$ and look at their behavior, I see that they are different. For example, let's say I obtain someone turn out to be 1 and someone to get 4. In other words, since I find the results different, an indeterminate situation arises here. … Undefined means not being defined directly there. How can I say… We gave an example: when I say apples times 2, I don't know what this apple means to me if I'm working with integers. it's not defined in integers, so I can't write something like this.

Indeterminate forms do not include sufficient information about the functions' behavior. In the excerpt given above, she described the indeterminate forms in her own words in which she tried to show the different behaviors of functions. It can be said that she developed her knowledge related to indeterminate and undefined forms, since "undefined" means that state of not being mathematically defined. While I expected her to give an example about her confusing things, she gave an example proposed by Alp in the lesson study process. It can be considered as a correct example. However, this shows us that Mila has not made enough progress in using the mathematical language. As can be seen in this and previous examples, she usually expressed herself to describe a mathematical thing with examples. In addition, some examples like apple times 2 was not an intended mathematical level, since the group and the researcher talked more mathematically during the lesson study process. Therefore, the development of using mathematical language which is under the sub-domain of KPM can be represented as not sufficient development.

Indeterminate forms are closely related to the notion of infinity, since it is a mathematical expression with $0, \infty$ or 1 . Since she had confusion with indeterminate forms and infinity in the pre-interview, another important question related to the notion of infinity. In the question, I did not expect her to explain the concept of infinity entirely. Rather, I expected her to reveal her knowledge of infinity in terms of teaching the concept of limit. The notion of infinity was asked after her expression related to "very very small numbers" When it was asked how she described the notion of infinity, she expressed herself as "*I always thought of infinity as a number throughout my university life and before I prepared these lesson plans. It really was like a number to me*". This expression is of importance about Mila's self-consciousness and the benefit of the lesson study process. The continuum of the answer was given below:

> Mila: If we talk about limit, that is, for teaching; Infinite is just an adjective for us. Infinity means increasing or decreasing for us. ... For example, we took the number divided by infinity in the concept of limit. Let's say $\frac{1}{x}$ as x goes to infinity. For example, here we normally say 1 divided by infinity and operate directly, but what we need to think here is that I am (x is) constantly increasing the denominator of the function of $\frac{1}{x}$. For example, it was 1, then it became 70000, so it gradually increased and increased. I'm actually looking at how this function behaves when it's constantly increasing.

Again, she used an example to reveal her knowledge of infinity according to the question in the post-interview. In the excerpt given above, the most important part was Mila's starting point which was "if we talk about limit". It shows that she was aware of the actual meaning of infinity which can be described in numbers explicitly.

This was one of my aims in the lesson study process. In addition, her answer showed that Mila gained knowledge of infinity in limit, since her answer can be understood as a correct step for teaching limit at infinity.

Lastly, I asked her to what she pay attention when teaching the concept of limit. I expected her to touch on the points that I worked on with the group considering their lack of knowledge. However, it should be emphasized first that Mila did not meet this expectation while the findings showed that Mila put across the lesson study process. The points she mentioned were that limit is not an operation, rather it represents the behavior of the function, and for a point to be a limit, it does not have to be defined at that point. However, my expectation was that she should have expressed previous learnings, related concepts, the ingredients of the formal definition, history of the concept, continuity and so on. She only focused on how to define the concept. Her approach was true but insufficient. To reveal her knowledge, the probing questions were asked her-for instance, what the previous learning of the concept should be for effective teaching. Her answer can be shown below:

> If we're only talking for the concept without operations, I think you're asking for knowledge of function. You know, the ranges where the functions are defined, etc. You know, there are too many differences between the situation of being defined in integers and the situation of being defined in real numbers, I think that this kind of knowledge is needed. Here it should be able to do some operations on functions, for example, given $f(x)$, it should be able to return $f(5)$. Apart from the knowledge of function; in other words, they need to know that in general, everything is not just about the calculations, we have slightly different works. In geometry; triangles and something with a triangle base (she's talking about solids here). Apart from that, the concept of limit, I still consider it (the limit concept) a little bit more separately, it seems like there is no different knowledge to understand it directly at the moment.

She pointed out some concepts including functions, sets, and geometry. These points can be considered as a good approach for describing previous learning of the concept. However, it is not sufficient. I expected her to give a more comprehensive answer including related mathematical concepts. However, besides all this, the main point that should not be overlooked in this answer is that she still regards the concept of limit apart from other mathematical concepts. Since she expressed the same statement in the pre-interview, one of aims of the lesson study process was to gain her broader perspective about both the concept and teaching the concept. While she thought that the lesson study process contributed much more things to her professional development as she mentioned at the beginning of the pre-interview, this statement showed that knowledge on connection between mathematical concepts and the concept of limit is an unsatisfactory level of development for Mila. To learn what she tried to mean as “much more thing”, I continued the post-interview with her pedagogical content knowledge which is presented in the following section.

4.3.2 The Prospective Mathematics Teacher’s Observed Development of Pedagogical Content Knowledge

The pedagogical content knowledge in the model includes knowledge of features of learning mathematics, knowledge of mathematics teaching and knowledge of mathematics learning standards. The lesson study process showed that Mila gained PCK for most of the indicators in the sub-domains of it. Since the idea behind the lesson study is to think on how to teach the concept effectively in the center of students’ learning, the lesson study process naturally provided the participant to develop her pedagogical content knowledge. As can be seen in the above sections (development of KFLM and development of KMT), I explicitly observed the development of PCK and overcoming her lack of knowledge related to KFLM and KMT throughout the lesson study process. For this reason, I focused on more mathematical questions instead of the questions examining pedagogical content knowledge.

For PCK, it was aimed to reveal the participant's PCK with a question that was not directly related to the teaching of the concept, but whose development in the lesson study process I could not observe in evidence. When she was asked what resources she used for the lesson plans and how this affected her teaching, she combined both the answer and her PCK.

> Mila: I'm telling you other than what you gave. I looked at the MoNE books, I benefited from the resources on the internet apart from the MoNE books: I benefited from different YouTube videos or some Khan Academy videos (to see if I can find interesting examples to attract students' attention in this way, if I can find examples from daily life), I read some articles that I won't be able to name them now, and they include students' answers to the problems. The Calculus books, Thomas and Adams… I bought a few books from the library, they allowed me to reconcile Calculus with my daily life … I actually read too much.

First of all, at the beginning of the lesson study process, she only used the forums which are based on questions-answers related to mathematics. These forums, the sources of which are not clear, are not considered desirable for the professional development of a prospective teacher. Therefore, the learning kit was given to the participants and each text in the kit was discussed in detail with its original sources. In the excerpt, Mila mentioned this fact that it provided her to be aware of different teaching resources including articles, books and videos. In addition, she indicated their intended use in lesson planning. Her focus was on students' learning, in particular, students' attraction to learn the concept. It can be considered as both emotional aspects of learning the concept and ways of pupils interact with content. I can say the latter one, since it was very difficult to attract the attention of the students with whom they conducted the research lessons.

On the other hand, finding examples that will attract the attention of students, or examples from daily life, which she put forward as the intended use, shows her KMT, in particular, knowledge of tasks, examples, strategies and techniques. While she showed her knowledge in this question, it was an interesting finding that she did not mention any indicator when she answered the question - how she teaches the concept, which points she pay attention (see the previous section-mathematical knowledge).

Finally, her lack of specialized knowledge that was observed in the pre-interview was overcome and not regarded as lack of knowledge in the post-interview. In other words, the lesson study process provided an atmosphere for Mila (and other group members, as well) to develop her specialized knowledge. At this point, another question - which and how critical elements construct a logical chain to provide this environment may come to mind. Up to now, I tried to reveal the journey of a participant in developing her specialized knowledge for teaching the concept of limit in detail. In the next section, I will explain how the critical elements of the lesson study process that enabled this development construct a logical chain to create a teaching experiment environment.

4.4 The Critical Elements in Lesson Study for Developing the Prospective Mathematics Teacher's Knowledge in the Concept of Limit

The second research question of the study was how well the critical elements of lesson study can be regulated so that they become an integral part of a logical chain to improve prospective mathematics teachers' specialized knowledge in the concept of limit. This section presents the findings to answer the second research question. In this section, the critical elements were presented in the conjectures which were related to the answers for the questions of "why" and "how" to develop specialized knowledge of prospective teachers based on the observable features of lesson study. The critical elements arise through testing the conjectures on the data gathered from the participant who was selected purposefully.

In addition, the model of MTSK allowed to look at the pre- and post- teaching knowledge development of prospective teachers from an analytical perspective. The analytical perspective to knowledge development with an experiment provided to regulate the lesson study development model so that they become an integral part of a logical chain to improve prospective secondary mathematics teachers' specialized knowledge. This model is closely related to which knowledge sub-domain developed at which stage and how. For this reason, the model was developed based on the findings of the first research question.

The general conjecture of the model is "By taking into account certain learning outcomes the well-regulated and designed lesson study process provides a pathway to the development of the prospective mathematics teachers". The critical elements of the model that demonstrate the accuracy of this conjecture and its relation with the observable features of lesson study are of importance for constructing the model. The critical elements of the model can be ordered as the nature of the concept, sufficiently long planning process, rich group discussions, prospective teachers' curiousness and willingness to learn the concept, the intervention of knowledgeable other and accordingly rich tasks and rich reading materials.

Conjecture 1: The regulation of the content in the observable features of lesson study according to the nature of the concept improves prospective teachers' specialized knowledge for teaching the concept.

One of the critical elements for the development of the prospective mathematics teacher was to design the lesson study development model considering "the nature of the concept". In fact, this element constructed a basis for other critical elements. For this reason, it cannot be said that the nature of the concept was regulated to develop prospective mathematics teachers' specialized knowledge. Rather, it was one of the elements to be utilized for others to form a logical chain. As described in the literature review chapter, knowledge for teaching the concept of limit requires examining the concept in broader sense during researching and planning. In this context, the "broader sense" means taking the concept from its foundation and establishing connections between mathematical concepts on how to carry it forward with its reflections throughout the curriculum. Considering the lack of experience, it is very difficult for a prospective teacher to have this awareness without any teaching experience, which is one of the aims of this study. One of the examples for this statement can be seen in the following excerpt which was taken from the first interview which aimed to get to know the participant.

> Researcher: What types of knowledge should a good math teacher have?
>
> Mila: Knowledge of mathematics (within the context of topics in high school curriculum), knowledge of how to prepare a good exam, methods, materials (GeoGebra and concrete materials) and he/she must be a good presenter.
>
> Researcher: You mean math topics that students learn… In this case, can a high school graduate be a math teacher?

> Mila: He/she can't do it very professionally. But, for example, we used to tell each other things when we were all students. If we think of this as teaching, he/she can do it that way. But I'm not sure if he/she would have exactly been a teacher. Because I think this is not just about teaching the lesson, we need to know how much the others (students) understood. In addition to the lesson, it is necessary to deal with things such as children's private life and personalities.
>
> Researcher: So, can someone who takes a short training become a math teacher? I do not think that this competence can be achieved with a short-term formation.
>
> Mila: I think that teaching should also have a proper education. Because here we see how to communicate.

The excerpt shows that Mila could not recognize the depth of knowledge for teaching the concept of limit as well as all the mathematical concepts. She argued that someone who graduated from a high school can only teach by acquiring pedagogical knowledge. During the lesson study process, she usually expressed herself as "*I never thought of it that way, I didn't know there was so much behind it*". Her thoughts finalized as noticing the awareness of the deficiencies and development of her knowledge. The following excerpt shows the beginning of the post-interview.

> Frankly, we started with a subject that I thought I knew a lot, but then I realized that I didn't know that much at first. I mean, not knowing or not, I have a lot of shortcomings. Then, as I started working on it, I liked to learn. Because I found so many interesting things. Let it be both to associate it with daily life and to associate it with different things (…)

The post-interview did not only examine the prospective mathematics teacher's observed development in her specialized knowledge for teaching the concept of limit, but also aimed to get final reflection about the lesson study process and her thoughts on her development. In the excerpt above, it is shown that Mila realized that she did not know when she went into detail to teach the limit concept she thought she knew. It was an expected finding considering her thoughts in the pre-interview mentioned above and the complex structure of limit which triggered her to think multidimensionally. This multidimensional thinking was the result of regulation of the lesson study process according to the abstract nature of the concept.

The abstract nature of the concept of limit is described in the literature review section in detail. Since the nature of the concept is abstract for both students and prospective teachers, the first step to develop their knowledge is to provide the learning of the definition and how to define it. Considering definition represents, perhaps, "more than anything else the conflict between the structure of mathematics, as conceived by professional mathematicians, and the cognitive processes of concept acquisition" (Vinner, 2006, p. 65), the starting point of the lesson study process was to develop the existing knowledge of the definition of the participants, Mila in this case. This statement was verified for some sub-domains and indicators during the lesson study process. However, development of knowledge of definition was not provided with rich materials and group discussions solely. Rather, the knowledge of definition was developed in relation with knowledge of graphical representation. As described in the first section of findings (the findings for the first research question-pre-interview), the prospective mathematics teacher had already known the graphical representation of limit. However, both Mila and the other participant had lack of knowledge about how to represent the formal definition. When they gained the knowledge about how to represent the formal definition which made the concept understandable, Mila developed her knowledge of definition by means of graphical representation accordingly. The following pathway shows the development order in the Lesson Plan-1 of the both cycles.

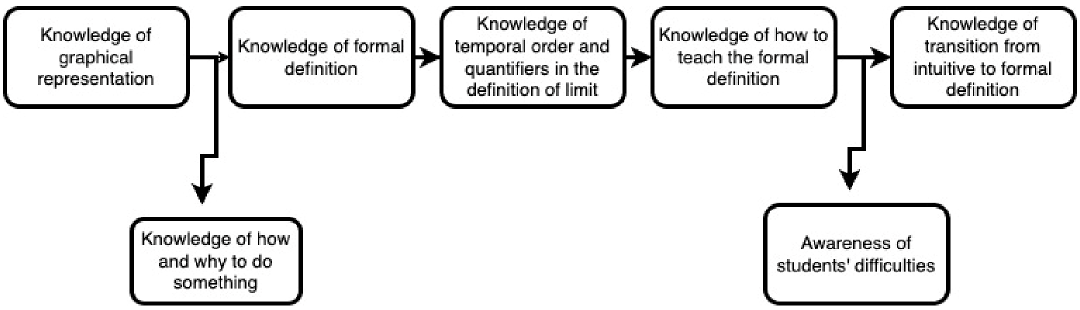


Figure 4.26. The pathway for the knowledge development in the first cycle of the Lesson Plan-1

In the pathway given above, the arrows pointing right shows the order in the development, and the arrows pointing down shows indicators that provided the development between two indicators. For instance, from knowledge of how to teach the formal definition to knowledge of transition from intuitive to formal definition was developed by means of gaining awareness of students' difficulties in formal definition. By means of interim analysis in the lesson study process, it was observed that the nature of the concept required making the lesson study goal concrete in students' mind at the same time with the prospective mathematics teacher's mind. Therefore, the group was directed to design the instructional plans considering the relation between both related mathematical concepts and real-life examples. In this way, the pathway started with knowledge of graphical representation. Since the prospective teacher had difficulty in knowledge of definition and how to teach it effectively, making her difficulties visible led to development of related indicators in the sub-domain of knowledge of topics. The other issue regarding the nature of the concept was to see what kind of development is observed at which stages. While it is described above in detail, the following table summarizes the development of MK considering the cycles and pre-and post- interviews. Each lesson study cycle was divided into two phases: planning which referred to determining a lesson goal and planning instructional plans, and enacting which referred to conducting research lessons and reflecting on the research lessons.

Table 4.21 The development of MK considering the cycles and pre-and post-interviews

Knowledge sub-domain	Lesson Study Cycle-1		Lesson Study Cycle-2	
	Planning-1	Enacting-1	Planning-2	Enacting-2
KoT	NAD	NAD	AD	AD
KSM	NAD	NAD	NAD	NAD
KPM	NAD	NAD	NAD	NAD
KFLM	NAD	AD/NAD	AD	NA
KMT	NAD	NAD*	AD	AD
KMLS	AE/AD	NA	AD	NA

AD: Adequate level of development NAD: Not Adequate Level of Development or Not Development AE: Already Existing NA: Not Observed

The table considered three lesson plans overall under the lesson study cycles. As said before, the development could be regarded when the abbreviations transformed from NAD to AD. Some indicators cannot be observed utilizing post-interview. For these indicators, the last phase of the second cycle was considered as the last step to observe the development of prospective mathematics teacher's knowledge. Therefore, there are some expressions as "there was no opportunity to see the development" in some columns.

In general, lesson study development model provided the development of mathematical knowledge. The table given above shows from which stage the development takes place to which stage. It can be drawn from this table that almost all of the development occurred in the second cycle, particularly in the planning phase of the second cycle. Mila stated this issue in the first meeting of the re-planning phase (Planning-2) as "*I think one of the things we need to change in the first place is the first activities and the transition from intuitive definition to formal definition. We did not dwell on it, but I understood from the students' questions that the concept was actually built on it"*. For this reason, it can be said that the nature of the concept required seeing the concept from students' eyes by means of enacting the lesson plans.

Conjecture 2: The lesson planning process, which is kept long enough until development is observed, provides intended outcome regarding prospective teachers' fundamental knowledge.

Correspondingly, the other critical elements emerged from the nature of the concept. The first element is "sufficiently long planning process" in planning phase of lesson study. The longest meeting in the lesson study belonged to the development of knowledge of definition (See Table 4.22).

As described above, the starting point of the lesson goal for the first lesson plan was knowledge of definition for secondary school students, and accordingly the aim of the planning phase of the first cycle was to develop the existing knowledge of the definition of the participants, Mila in this case. The indicator of the definition of the concept of limit was divided into seven sub- and detailed indicators including knowledge of intuitive definition, knowledge of right-left sided limits, knowledge of formal definition of limit, temporal order in formal definition, quantifiers (for all, such that, at least) in formal definition, meanings of epsilon-delta in formal definition, transition from intuitive definition to formal definition. These indicators were considered as learning outcomes of the lesson study development model. The first two indicators were in the category of "already existing knowledge" of the participants. Therefore, the planning process of the first lesson plan was kept long enough to achieve the other learning outcomes. In addition, the duration of each meeting is also very important, as well as how long the planning process is kept. As can be seen in Table 4.21, the first cycle of the first lesson planning process was kept for almost 5 meetings and 8 hours.

Table 4.22 The planning meetings of the first cycle in Lesson Plan-1

First Cycle	The observed knowledge development	Duration
1st meeting	Knowledge of formal definition of limit	88 min.
2nd meeting	Meanings of epsilon-delta in formal definition	55 min.
3rd meeting	Meanings of epsilon-delta in formal definition	149 min.
4th meeting	Temporal order in formal definition	113 min.
5th meeting	Temporal order in formal definition	80 min.

The first planning process of the first cycle had its own mini-cycles according to the lack of knowledge of the group which was determined by analysis conducted during the lesson planning process. In these meetings, Mila had reached the three of them by means of learning with conversation (Sfard et al., 1998) in group discussions in addition to the iterative process in the mini-cycles.

However, it had negative results for some other indicators. Since the time was limited in an academic year, keeping sufficiently long planning processes for some indicators gave rise to spending less time for some indicators. In the last reflection of the lesson study, Mila mentioned this issue as can be seen in the following excerpt.

> Mila: The lesson planning process was pretty good, I think. You know, maybe it took a long time for us to prepare the first lesson plan. Maybe something can be done to speed this up. Of course, I don't know if it was a predictable thing, but for example, I think it took a long time because it was the first concept. It might work differently. I don't think it would be possible to consider limit and continuity separately. First, I thought, if the continuity had not been processed at all, would it be more comfortable, would it have been progressed faster. But when it worked together like this, it was very nice, so it all came together. That's why I gave up on this suggestion.

When I asked her in the member check session whether she meant that we spent the least time for the notion of continuity, she confirmed me and added that "*Although the lesson plan we worked on the least was continuity, it was the best. Maybe it was effective that we gave the first lesson plan so long because that was the core of the topic*". While Mila had some confusions about this issue, in the last cycle of the study I revised the conjecture as "*The lesson study planning process requires keeping it long considering the envisaged lesson study time until an improvement in prospective teachers' fundamental knowledge is observed*".

Conjecture 3: The development of specialized knowledge for teaching mathematics occurs through rich group discussions.

Lesson study process is based on collaborative learning that is a powerful vehicle to mobilize teacher instructional change and pedagogical practices, and to improve student achievement (Lawrence & Chong, 2010, p. 565). Since collaborative learning requires taken-as-shared mathematical meanings in the group, rich group discussions in three lesson study phases including determining lesson goals, planning, and reflection on the research lessons, can be considered as one of the most important critical elements for the development of specialized knowledge for teaching the concept considering its the abstract nature. This critical element is related to "mathematical conversation in group discussions" in knowledge development. To ensure rich taken-as-shared meanings in mathematical conversation in group discussions, I considered three critical elements including knowledgeable other, rich materials and characteristics of lesson study group and their relations with observable features of lesson study process.

One of the theoretical foundations of the lesson study, particularly collaborative learning, is Vygotsky’s “zone of proximal development” which is a level of competence on a task in which the student cannot yet master the task on his or her own but can perform the task with appropriate guidance and support from a more capable partner (O’donnell & Hmelo-Silver, 2013, p. 22). For this reason, to ensure rich group discussions, the group with different learning levels should be provided. In the current lesson study group, Fulya and Alp had already an experience to study on how to teach the calculus concepts. In their methods courses which they took in the previous semester, they had prepared a plan for teaching the concept of limit. However, they had limited perspectives about the concept. On the other hand, Mila had not thought on the concept of limit before. This provided them to think on the concept of limit from a different perspective. The following excerpt shows an example for this situation:

> Fulya: A game was mentioned in the 12th Grade book by MoNE. We added it to the lesson plan. We define a point in the middle and define a border. We give students something like a small marble in their hand and ask them to throw it away. It is simple; the one who goes to the goal the most wins the game. It's such a game. We tried to do it in class, it didn't work well.
>
> Researcher: Why didn't it work? Can you evaluate this from the perspective of teacher and student?
>
> Fulya: As a teacher, we wanted them to give 3 points a certain value, but we didn't give a limit. Let’s say the values you will give between 0 and 1, such as we did not give. So, there were too many approaches. There were those who said 7 and there were those who said 10.
>
> Mila: It was because we were so few. In other words, I think that the more data, the higher the efficiency.

This excerpt was taken form the second meeting of the first lesson plan. As can be seen in the excerpt, the diversity in the group provided them to examine the activity as a teacher and a student. By this way, the group, Mila in this case, could develop KFLM as well as KMT.

Similarly, the different levels of knowledge provided them to develop their knowledge. For instance, after conducting the research lesson of the lesson plan-2, the participants tried to revise the activities, problems or exercises that did not work in the research lesson. They discussed on the concept of infinity again based on Fulya's (as a teacher of the lesson) expressions for infinity as a number. This was really interesting that she had insistently mentioned in Phase 1 that infinity is not a number. Based on this claim, they focused on the idea of how they should express the concept of infinity.

> Fulya: I think we can talk about the infinity as something which is "constantly increasing".
>
> Mila: Yes, I read about that! Infinity is not a quantity; it is a quality. Then, it may be sensible! It says we use the concept of infinity as an adjective in mathematics. We do not use it as a noun, the article says, like a finite adjective, infinite is an adjective used in mathematics. This means that infinite is the opposite of finite. In other words, things that are not finite in mathematics are called infinite.
>
> Alp: Well, could there be bounded infinity?
>
> Mila: What do you mean?
>
> Alp: Constantly increasing cannot be considered as wrong. However, what about bounded infinity? If we say bounded infinity, for example, there is a bounded infinity between 0 and 1. However, there are infinite numbers in this interval.
>
> Mila: Yes, there could be.
>
> Alp: However, it is so close. I mean that the place between 1 and 0 is too short.
>
> Mila: But, close according to who?
>
> Alp: It seems this much close to me (showing that there was a very short distance using his thumb and index finger).
>
> Mila: Too far for me. For example, this distance may be too close for you, but it may be too far for me. I mean, it depends. So, we can think of it as a quality from this perspective. I remember that there is a one-to-one correspondence in sets. I think this issue is related to it.

By questioning each other's knowledge during the discussion in the revision process in planning, the prospective teachers had a chance to make sense of their knowledge.

In the following excerpt, Alp asked his friends whether there can be limited infinity in mathematics. Such a question triggered other participants to think both on their knowledge and Alp's knowledge.

Besides the benefits of providing members who had different characteristics, their misunderstandings, misconceptions or faulty knowledge can affect others' understanding and knowledge. For this reason, the role of "knowledgeable other" is of importance at this stage. In the current study, one of the missions of knowledgeable other was to direct the group discussions to the intended development. As described in former sections, the intended development was determined considering the indicators of the sub-domains in MTSK. Except from the research lesson phase in which the prospective teachers taught the lesson in a classroom, there were an intervention of knowledgeable other in other phases of the lesson study. For instance, as the following excerpt indicates, in the former meetings of the planning, Mila had a lack of knowledge about how and why to show the indeterminate forms in limit. In the first column (I), she researched this issue and found her answer about "$\frac{0}{0}$" (*Phase 1: Determining the lesson study goal and research on it*). As I planned before conducting the lesson study process, I gave them an assignment including the questions shown in the first column in the table and articles for these terms. In the latter meeting of the planning, she gained her knowledge of indeterminate and undefined forms. In addition, the discussion during the planning phase of the lesson study process developed her knowledge on these terms (*Planning phase*).

> (I) Mila: For instance, 0/0 is equal to x. Cross multiply it! 0=x.0 Then, x can be any real number. For this reason, 0/0 is described as an indeterminate form.
>
> Researcher: What do you (Alp and Fulya) think about this answer?
>
> Fulya: I'm not sure, I may say it as "undefined".
>
> Alp: Where did you get this information? Do all the uncertainties come out of here?
>
> Fulya: It's probably the same for all.
>
> Mila: I found it on the internet.

Researcher: Are these two terms same things? Please, think on these questions: How do you define the “division” in mathematics? In addition, is infinity included in the real numbers?

…

(II) Fulya: Undefined and indeterminate were different things. Up to this assignment, I did not know that!

Mila: Yeah! How ridiculous are the division and cross multiplication things I told you in the previous meeting!

Fulya: When things like this were said, I always thought of undefined.

Alp: May I explain it?

Mila: Yes, please ☺

Alp: (He explained it)

Mila: I'm really happy to learn that!

In the excerpt numbered as (I), Mila had wrong information about this subject and she was trying to present this information by convincing her friends. Therefore, an intervention was needed so that knowledge did not develop in the wrong direction. At this stage, as knowledgeable other, I diverted the discussion to direct them to the desired knowledge. Thus, the group discussion became richer by becoming research-based and took shape in the right direction.

On the other hand, in some situations, the development of knowledge of the group should be promoted utilizing external resources given by the researcher including tasks, scenarios, and additional materials for their lesson plans. For instance, in the planning phase of lesson plan 3, the group thought about the applications of the notion of continuity and its relation with other mathematical topics. One of the applications of continuity is the Intermediate Value Theorem (IVT). First, the group tried to write the definition of IVT that they confused with MVT. However, it is not sufficient to say that they had knowledge of IVT. For this reason, I posed the usual question of the lesson study process, which was why there is such a theorem and what its function is. Because no progress was made in group discussions, I presented them a task which was adapted from the ULTRA project[8] (Weber, Wasserman, & Fukawa-Connelly, 2019). The task served the relation between IVT and polynomials (see Figure 4.27).

> So, let's find some specific values of the function ($g(x) = x^3 - 3x^2 - 3x + 7$), at $x = 2$ and $x = 3$.
> Student1: g of 2 is -1.
> Student2: g of 3 is 1.
> Teacher: Thus, we have $g(2) = -1$ and $g(3) = 1$, we can conclude that g(x) has at least one zero between 2 and 3.
> Teacher: Oki for your exit ticket, give me a short summary of the idea we just discussed finding zeros of a function.
>
> **Question 1: What do you think of these students' comments? How do you think this issue might have something to do with IVT?**
>
> One of the students wrote that:
> Okay, so for a function, if a function is less than 0 somewhere and greater than 0 somewhere else that we know there will be a zero between them. So like, I in general, and if $f(a) < 0$ and f$(b) > 0$ then there is at least one zero somewhere in the interval (a, b).
>
> **Question 2: What do you think of this student's answer? What do you think the student is talking about? Do you think the student's answer is always correct?**

Figure 4.27 The task for showing the relation between IVT and other mathematical concepts (Weber, Wasserman, & Fukawa-Connelly. (2019).

Such tasks and questions helped to develop their MK including KoT, KSM and KPM. It has been tried to provide a development by presenting not only the task that they would implement, but also the resources that they could use in their teaching. As mentioned earlier, a learning kit for the concept of limit was prepared and given to the group and they discussed on it. While it provided a broader perspective to them, they had lack of knowledge in KMT, particularly knowledge of strategies, techniques, tasks and examples. To develop the knowledge, I didn't give them a fish; instead, taught them how to fish. I mean that, during the planning phase, both the resources including tasks and examples, and how they acquired these resources were given to the group. A particular attention was given to Mila, since she had not taught the concept before while others had a knowledge about this as existing but not sufficient level.

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[8] The task was taken from the web site of the Project that can be considered as open source. The Project can be found in https://sites.google.com/view/ultranalysis/home?authuser=0

Table 4.23 The development presentation before and after the activity

Before the activity	After the activity
Researcher: When you talk about the application of continuity or the situations where we benefit from continuity, what do you talk about? Mila: We can say derivative. Researcher: Okay, you're right! Let's think on the theorems. Fulya: Mean Value…Extreme value… (…) Researcher: Okay, I'll send you an activity related to finding zero in polynomials. Fulya: Yes, I have already thought that finding roots, right?	Mila: Hocam, can we use the intermediate value theorem to show continuity in the interval? Researcher: How do you reach this statement? Mila: I did a lot of research, so it's used for different purposes, it's really a lot, but when I looked, this intermediate value theorem also tells us that the function is continuous in the range we are looking at, so we can say it directly. Fulya was saying that no matter where we take a point, it has an exact value. Mila: You know, because you can always find a value in the interval, I say that it is constant. So, I thought, we need to give it for a purpose somehow, or maybe we can use the intermediate value theorem to show continuity in the interval, I thought we can prove it like this.

For instance, Mila had usually proposed the same type of questions for the lesson plan. In general, these same types of questions were thought as exercises addressing students' operational skills. However, the aim of lesson study process was to enable not only prospective mathematics teachers but also students to learn the concept from a broad perspective. For this purpose, the tasks given to the students and the questions asked were also very important. The following excerpt shows how Mila encountered with the modelling question and adapted it in the lesson plan considering students' learning. She encountered with the modelling question at different times during the lesson study process.

(I)Researcher: Have you read this modeling question?

> Alp: I tried to solve it, but I made a comma-related error … If I knew the value of n in liters per cubic meter, I would have done it right.
>
> Mila: I've never read such a problem before. It takes effort even to find what is given and what is wanted. Fulya, shall we do it together?
>
> (II)Mila: If I'm not mistaken in the problem, they had to write a function. That's why I think it can be shown after the polynomials are shown here, at least they seem to remember the polynomials a little more because if I'm not mistaken. It's a bit of a difficult question, or a question that requires more thought, I think we can give it a little towards the end.

At first (I), she just read the question and made comments on it considering what is given and what is asked. Most of the time when there was not the knowledgeable other, Mila tried to solve the question with her group friends. However, it would be wrong to deny the contribution of Mila in this process. Another contribution of Mila to the discussion (II) before she conducted the related research lesson led to broaden their perspective related to knowledge of mathematics teaching and knowledge of features of learning mathematics. She thought out loud about the difficulty of the problems related to students' mathematical level and order of instruction.

The group and Mila had limited perspective, such as asking questions related to only procedural fluency, or about how to provide formative assessment during their teaching. Directing her to different sources for teaching the concept, developing her own mathematical knowledge and discussion on the proposed assessment techniques provided Mila to develop her knowledge of real-life applications of the concept, in addition to the other. She indicated that she felt that limit was irrelevant from other mathematical concepts and it was so abstract to find it in real-life applications. The following excerpt shows Mila's reflection on her development about this issue.

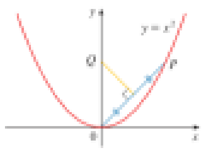
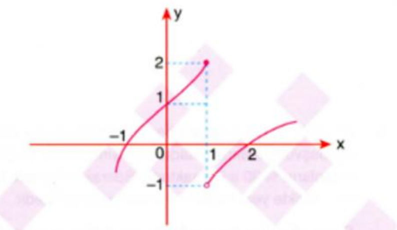
> Mila: Well, the limit is everywhere. We really use it in so many places that it's the limit. So, I'm surprised to see it like this. I mean, it's here too. I simply mean we observe the limit as a behavior and there is the approach with infinitesimal moves. Actually, I think we use it in too many different places. While I was preparing the lesson plan during my internship, I saw geometry as well, the base is first triangular and then it gradually increases. As we increase the number of bases and the number of sides, we get a cylinder as we increase the number of vertices. And they even said from here that prisms

are actually cylinders. I was very surprised when I searched the internet. So even here there is a limit.

(…) For example, it was easier to emphasize this issue that limit can be observed everywhere with problems, since we saw it in many, many different places along with those problems.

These resources given to prospective teachers not only enriched the teaching material created, but also improved the prospective teachers' KMT. In the table below, you can see the change in some example questions asked during the mini-cycles in the first lesson planning stage.

Table 4.24 Some example questions asked during the mini-cycles in the first lesson planning

1st versions in the mini-cycle	2nd versions in the mini-cycle
Find the result of $\lim_{x \to -6} \frac{\sqrt{x^2-15}-7}{\sqrt[3]{x}+2}$.	The figure on the right shows the point P on the parabola and the point Q where the mid-perpendicular of OP intersects the y-axis.  What would you say about point Q when P approaches the origin along the parabola? Do you think there is a limit operation here? If so, show it.
 The graphical representation of y=f(x) is shown above. According to the graphic, find the result of $\lim_{x \to 1^+} (fof)(x) + \lim_{x \to 1^-} (fofof)(x)$.	A nuclear scientist is working on an experiment. He found a function $f(t)$ representing the molecular number of a radioactive substance as shown: $f(t) = \frac{at^2-b}{t-2}$. Here, t represents the time in minutes since the start of the reaction. The scientist who lost his grades in the laboratory does not know the value of a and b. However, he remembers that 2 minutes after the start of the reaction, the number of molecules of the radioactive substance approached 4. In the light of this information, find the values of a and b.

It did not mean that the first column was unnecessary for the lesson plan. As the literature supported, there were five competencies including conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition for mathematical proficiency (Kilpatrick, Swafford, & Findell, 2001, p. 5). For this reason, their proposed questions were also supported. The outcomes of this intervention were observed after she saw the students' reaction to the questions when she taught the research lessons.

Conjecture 4: Pre-interviews which conduct before the lesson study process supports pre-service teachers to realize their lack of specialized knowledge and draw learning routes.

While the aim of the pre-interviews was to observe the existing knowledge of the prospective mathematics teachers throughout the lesson study process, there was another aim, which was to increase the curiosity of prospective teachers about the concept they would learn and to make them aware of their own knowledge. I conjectured on that it triggered prospective teachers to participate in the group discussions and to be more curious for the concept.

In the first meeting of the first cycle, I gave them a cardboard and stickers and wanted them to write what they thought they needed to teach about the concept of limit. It was observed that the topics on the stickers of each participant matched with the topics that they became aware of their lack of knowledge. The Table 4.25 shows the match between Mila's lack of knowledge and her thoughts about what she needed to teach about the concept of limit.

Table 4.25 Some example questions asked during the mini-cycles in the first lesson planning

Mila's stickers on the cardboard	Mila's lack of knowledge
Infinity concept using graphics	When limit at infinity and infinite limit were asked, she tried to show it on graphics, but she could not.

Table 4.25 (continued)

The indetermination of zero divided by zero and infinity over infinity	When Mila was asked about her misconceptions, she expressed herself as follows, although it was not actually related to the misconception: "*I used to have a lot of trouble with uncertainty. For example, in the indetermination of infinity over infinity, there would be something I throw away. We would take the leading coefficients. For example, I could never understand this, so what happens to the rest of them; why not take*?"
Secant line? Tangent line?	She could not make a connection between the limit and secant-tangent lines.
It's not right to draw without raising your hand	In the pre-interview, the participant described continuity with the statement that we can say that it is continuous when we do not raise our hand while drawing the graph.

This critical element played a role in the planning phase of lesson study. While the clinical pre-interview was conducted before the study, the reflection papers and discussions on the reflection papers served the same role with the pre-interview for the second cycle of lesson study. In this way, the usefulness of the pre-interview for the model was clearly demonstrated.

Conjecture 5: The guided reflection on research lesson adopted in lesson study provides improvement in prospective mathematics teacher's awareness on students' learning.

As mentioned in the methods section, the last phase of the lesson study requires presenting and discussing data from research lesson and drawing out implications for the latter version of lesson study. In the classic lesson study process, the group observes a research lesson and collects data to reveal students' learning and effectiveness of lesson plan in the classes. Then, they present and discuss these data. When the group consists of prospective mathematics teachers, it may be hard to notice what they will observe. Therefore, guided reflection process is of importance for reaching the intended development for specialized knowledge for teaching.

“Guided reflection” means that the researcher directed the group with questions given before the observation of research lesson. It can be said that the conjecture was verified, in particular for knowledge of mathematics teaching and knowledge of features in learning mathematics. Furthermore, it also provided to improve Mila’s observation and noticing skills for mathematics teaching. The table below (Table 4.26) shows two reflections, one of which is Mila's reflection for her own lesson and one for one of the other lessons.

The guided reflection was examined in two parts: Self-reflection of the prospective teacher: Evaluation of the lesson plan by herself and reflection on the group members’ research lessons. The guidance was given by the researcher by means of asking ‘to the point’ questions to the prospective teacher. Since she was the third prospective teacher who conducted the research lesson, she had an opportunity to observe other research lessons and write reflections on them. For this reason, she was very careful in her teaching and in particular about her self-reflection. It should be remembered that the questions for self-reflection and reflection were differentiated from each other to reveal different kinds of knowledge. In the second row, Mila made reflection on the other group member’s research lessons. In the excerpt, she touched on two points: First, she indicated students’ learning during the activities and whether their reactions meet the expectations of the lesson plan or not. Since the points mentioned by the prospective teacher show the ways of interaction of students with mathematical content, this situation refers to KFLM.

Table 4.26 The example reflections for critical element of “guided reflection”

What the reflection is about	The reflection	What knowledge did it reveal?
To reflect the situation on the expressions of the friends of the prospective teachers	It is good that it is not defined at point 1 in the first application. The student wanted to replace it, but because it was not defined, he could not replace it and had to approach, which is exactly what we wanted. The questions were good (application) and thought provoking, but since we are talking about approaching, there are always those who say that we will bring the points closer without understanding the reason. The reason for this should be questioned.	KFLM – ways pupils interact with content KMT- teaching techniques
Self-reflection of the prospective teacher: Evaluation of the lesson plan by herself	Since I was the last member in conducting lesson plan and I observed the others’ lessons, it provided me to react as how I should teach the topics. I think the activities worked well. (…) So, for example, I wasn't ready for the questions that students would suddenly ask. At that moment, I tried to handle it immediately, but it was quite difficult. I tried to go there from the things students know. For example, they know how to calculate limits, or I immediately gave a function or something, when that slide did not work. I hope I was able to explain.	KFLM-Strengths and weaknesses of students

Similarly, she mentioned that her group mates should have questioned the students' reactions to the steps of the activity; in which she revealed her awareness about strategies of mathematics teaching in terms of students' learning mathematics (KMT). Mila's reflection on her friends' research lessons allowed Mila to work on different domains of knowledge with guided questions. In the second row, the excerpt was a part of her self-reflection. Different from her reflection to the group mate's research lessons, she mentioned on students' learning and their mathematical thinking. In particular, she mentioned how she used students' strengths to overcome at-the-moment difficulties, which refers to KFLM. Though the questions in the guided-self-reflection include thinking on own teaching strategies, tasks and examples, this might be considered as understandable because it may be hard to commented on her own teaching.

Conjecture 6: The development of specialized knowledge occurs in order from KoT to other sub-domains.

the previous conjectures were about the observable features of the lesson study, this conjecture was put forward to reveal the relationship of the lesson study with MTSK. Using the model of MTSK as both methodological and analytical tool for research requires thinking on prospective mathematics teachers' knowledge to conduct their profession in not only teaching in the classroom but also in lesson planning or communicating with colleagues. That was described above referring to the reason of using this model. However, the model does not serve on how to connect the model with any development model. Therefore, this conjecture is of importance for latter sections.

The first version of this conjecture was "The development of specialized knowledge occurs in order from KMT to other sub-domains", since the group usually started with a question as "How do you think we should show (teach) this (subject) to students? Where do we start?". However, lack of knowledge of Mila and other group members did not allow to answer these questions. Therefore, the group discussion on how to represent/teach the concept in planning passed through the fundamentals of the concept. After the first meetings, the conjecture was revised in the mini-cycle in the planning process.

This new conjecture emerged also from the pre-interviews of the prospective mathematics teachers. Mila, in this case, had a lack of knowledge regarding both mathematical knowledge (MK) and pedagogical content knowledge (PCK) of the concept of limit. Particularly, fundamental knowledge for the concept of limit and related notions (knowledge of topics) and knowledge about learning the concept of limit (knowledge of features of learning mathematics) was observed as a lack of knowledge for Mila. This meant that first of all, it was thought that the basics of the subject would be learned as if they were a student and that students would discover the characteristics of their learning through this process. Thus, the lesson study development model was designed considering this issue.

The main vision of MTSK is to examine mathematics teachers' knowledge from holistic perspective. When we strip it down to the concept of limit, each sub-domain can be observed in all three lesson plans. For instance, the lesson planning processes, of course, required KMT and KFLM regarding the lesson goals, because the origin of lesson study is based on how students learn more effectively. More specifically, the relation between sub-domains of MTSK and the lesson goals are shown in the Table 4.27.

In general, it can be said that the conjecture was verified by using two cycles of the study. This has been demonstrated in different examples in the previous section. Below is the development of Mila via an object showing how this conjecture has been validated. By saying "object", I mean an element for teaching the concept of limit. As a related notion, the concept of continuity is addressed to show this conjecture. The concept of continuity was included in the third lesson plan. Teaching the concept of continuity required having knowledge of definition (KoT), its relation with other mathematical concepts (KSM), ways of generating knowledge in mathematics (KPM), misconceptions regarding the concept of continuity (KFLM), tasks and examples to teach the concept (KMT) and being aware of the expected learning outcomes of the curriculum (KMLS).

Table 4.27 The three lesson goals which could be associated with related sub-domain

	Lesson Study Goal	The topics the group want to address	The sub-domains in MTSK
Lesson Plan-1	Conceptualization of the concept of limit in students' mind	The intuitive, right-left side limits and formal definition of limit The components of formal definition Historical development of the concept	KoT KFLM KMT
Lesson Plan-2	Applications and mathematical procedures with the concept of limit	The relation with the previous learning of students (for example; functions, sets) To help them understand what comes from where Using mathematical calculations in limit in the context-based problems	KoT KSM KMT
Lesson Plan-3	Conceptualization of the concept of continuity, the relation between the other concepts and continuity	The continuity concepts The IVT Theorem The relation between derivative and continuity	KoT KPM KFLM

Mila had lack of knowledge of the concept of continuity including knowledge of definition, relation between continuity and other mathematical concepts, application of the concept, misconception (especially she had own misconception which is "continuity requires to draw the graph of function without raising hand") and how to teach the concept (specifically, how to include continuity into lesson plan related to the concept of limit). In the pre-interview, she had existing knowledge related to expected learning outcomes of the curriculum. It can be said that Mila developed her lack of specialized knowledge through the lesson study.

Since the first phase of lesson study process, determining the lesson goals, was not repeated again during the process, this conjecture did not include this phase. Mostly, the starting point of the lesson plans was a question as “How do you think we should show (teach) this (subject) to students? Where do we start?”. For this reason, it was observed that the starting point of development was KMT regarding the concept of continuity. However, their lack of knowledge did not allow to answer these questions. Therefore, the group discussion on how to represent/teach the concept in planning passed through the fundamentals of the concept. After the first meetings, the conjecture was revised in the mini-cycle in the planning process.

> Alp: When describing continuity, it would be better if we show it over the function on the chart.
>
> Mila: Visualizing always makes it easier. … Oh, but that conversation without raising your hand, you will draw without raising your hand.
>
> Researcher: Do you think that is true?
>
> Mila: How true is that, I was going to ask that exactly. For example, should we say it or not? Is it true or not? Because, the last time I doubted this veracity (she's talking about the interview)
>
> Researcher: So, if we're going to start here, how would you define continuity to the student?
>
> (…) Researcher: Now, there is something like this, at the point where you all confuse, continuity and continuity in interval, for example, I think we need to establish it well with students. I also think that while you are doing it, you generally take continuity and interval as the same thing, for example, based on your questions, continuity is not directly related to definition. Continuity is directly related to the limit, something that is given as an additional condition, we can already find its limit. So, when we say that it should be defined absolutely, it gives up the limit from the right to the left, it gives up taking the limit, it just looks at the definiteness and says it's okay if it is defined at that point.
>
> Mila-Fulya: Of course, we need to talk about right-to-left continuity here.
>
> (…) Mila: I don't know why when someone learns something wrong, it is easier to remember, but that's how the human brain learns wrong more easily.

> For example, I said that if you draw without raising your hand, it is continuous.

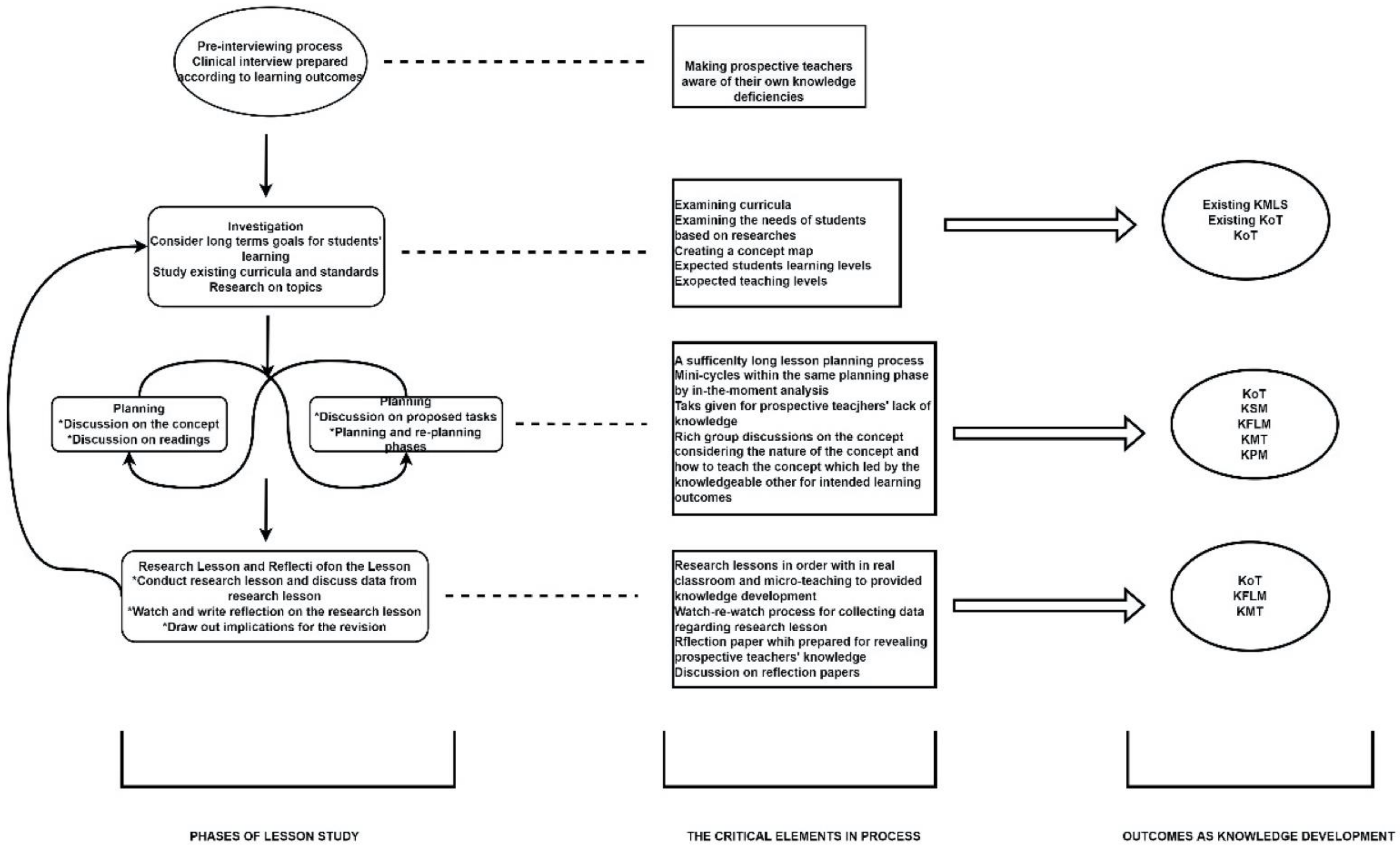
KoT includes knowledge of definition, phenomenology, applications, foundations and representations of the concept. In the most part of the lesson study process, the starting point of the lesson plans were the definition of the concept to be taught. The same process was conducted as shown in the example excerpt given above which was taken from the planning the third lesson plan. The aim of the third lesson plan was to conceptualize the concept of continuity in relation with other mathematical concepts in students' mind. While Alp started the discussion with the statement about how to teach the concept, he turned it to the definition of the concept. As an effect of the pre-interview (Conjecture 4), Mila had come prepared by doing a study to fill her own lack of knowledge including definition of continuity for a point and for an interval or a set. By means of knowledge of definition, she could overcome her misconception as "continuity requires drawing the graphs without raising hand".

Considering the whole journey of Mila, it has been observed that the development of different sub-domains of specialized knowledge for teaching the concept of limit promoted different elements of lesson study when they were regulated according to expected outcomes (the indicators of the sub-domains in this case). The following section shows the regulation of these elements and its outcomes within the lesson study development model.

4.4.1 The Lesson Study Development Model

Considering the conjectures given above and the findings related to the development of specialized knowledge, the model was developed to show how well the critical elements given in the conjectures can be regulated so that they become an integral part of a logical chain. This logical chain included the critical elements that occurred during the lesson study process for the development of specialized knowledge, the observable features of the lesson study process including the phases of it and its relation with the critical elements, and the outcomes of this process.

Figure 4.28. The lesson study development model for specialized knowledge for teaching the concept of limit



The figure is composed of three main parts including phases which represent the phases of lesson study investigation, planning, research lesson and reflection, the critical elements in the process which are listed in the previous section, and outcomes as knowledge development which are determined according to the observed indicators of the sub-domains of the model of Mathematics Teachers' Specialized Knowledge (MTSK). The rectangles in the middle column of the figure show the actions in the critical elements in the related phases which are connected with each other with dotted segments.

In this re-interpretation of lesson study according to the development of specialized knowledge for teaching the concept of limit, the planning phases were separated and connected with each other with winding arrow. There were three lesson study goals and accordingly three lesson plans. In the classic lesson study process, the lessons are planned separately. In this model, the planning of these three lesson plans were intertwined with each other. Thus, the winding arrow represented both mini-cycles and the cycles between lesson plans. In particular, the lesson plan-1 and lesson plan-2 were intertwined (see Figure 4.29).

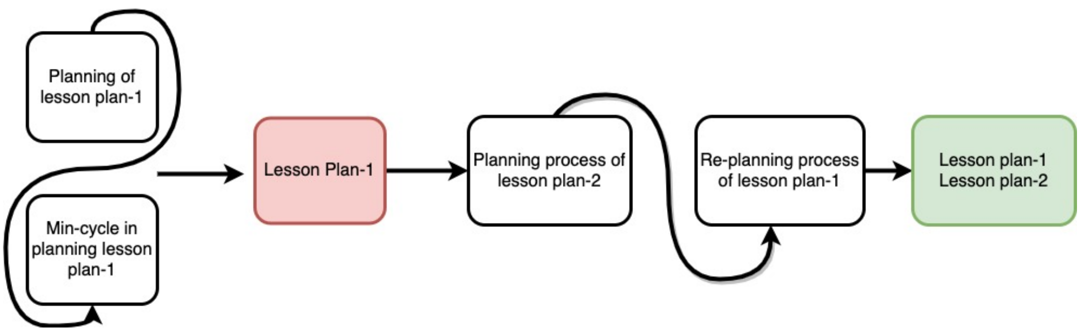


Figure 4.29. The cycling process between lesson plan-1 and lesson plan-2

After lesson plan-1 was constructed, the planning process of lesson plan-2 started. Then, re-planning process for lesson plan-1 was placed in the lesson planning process of lesson plan-2. All these processes were conducted before research lessons of these plans. So, what did this provide to prospective mathematics teachers? One of the important achievements was to develop sub-domain of knowledge as reaching learning outcomes. For instance, the indicators of KoT were covered in both lesson plan-1 (as knowledge of definition, properties and foundations, phenomenology and representations) and lesson plan-2 (as knowledge of application, mathematical procedures and representations). Thus, the development of indicators of KoT supported each other by means of holistic perspective.

The second column named as “the critical elements in the process” shows the critical elements in relation with the phases of lesson study process. Although in the figure these elements seem to be related to the lesson study, they have a logical sequence in themselves, which is the answer to the second research question. In addition, this chain shows how prospective teachers, Mila in this case, developed her knowledge during the process. The figure given below showed the lensed state of the second column in the model (see Figure 4.30).

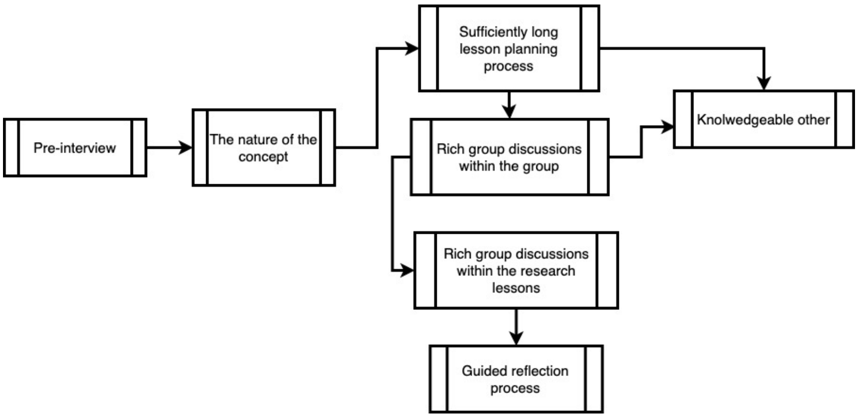


Figure 4.30 The cycling process between lesson plan-1 and lesson plan-2

The third column shows the outcomes as knowledge development. In other words, the third column is the findings related to the answers to the first research question. As said in the findings to answer the first research question, the main support of lesson study to knowledge development was observed in planning phases of lesson study; in particular, the sub-domains of mathematical knowledge were mainly developed in the planning phases of lesson study. However, considering the critical elements such as rich group discussions on the research lesson, it did not mean that planning phases only provided the development. Instead, it meant that the adequate level of evidence was observed in planning phases.

CHAPTER 5

CONCLUSION AND DISCUSSION

The purpose of this study is to understand the nature and development of prospective mathematics teachers' specialized knowledge for teaching the concept of limit in a broad sense through a designed lesson study development model. There were three prospective mathematics teachers with the researcher in the lesson study group. The findings were presented to one of the prospective mathematics teachers as a journey for development of specialized knowledge to reveal the development model with in-depth analysis. Drawing on the results provided for the aim of the study, this chapter addresses summary and conclusions of the main findings, and evaluations and discussions about results that emerged in the study relating to the fundamental research questions considering the related and current literature. In addition, it also discusses elaboration and critical evaluation of the MTSK framework and lesson study development model which was constructed through the findings of the study. It is followed by a discussion on the limitations of the study. Finally, recommendations for future research are presented as well as implications of the study on prospective teacher education concerning the overall conclusions of the study.

5.1 The development in the Prospective Mathematics Teacher's Specialized Knowledge of the Concept of Limit through Lesson Study

The first research question examined how prospective secondary mathematics teachers develop their specialized knowledge in the concept of limit in the lesson study development model. The question was two-fold: first, the existing knowledge of the prospective mathematics teacher, and second, the development process of specialized knowledge of prospective mathematics teachers throughout the lesson study development model were examined. Except for some indicators and some sub-domains, it can be said that the lesson study process, designed for a holistic development, improves the prospective teacher's specialized knowledge for teaching. In this section, I presented the summary of the findings related to the existing knowledge and the development process of specialized knowledge.

Knowledge of Topics

The first sub-domain of the model- knowledge of topics (KoT) can be defined as a sub-domain of mathematical knowledge, which is approached from an in-depth perspective on fundamental mathematical knowledge and contains comprehensive knowledge from mathematical definitions to procedures (Carrillo-Yañez et al., 2018). In the current study, KoT was examined within four indicators including knowledge of definition, history, infinity and infinitesimal approach, phenomenology and applications of the concept, mathematical procedures and representation systems. Based on these indicators, the pre-interview showed that Mila's KoT was at an insufficient level. For instance, the findings for knowledge of definition of the concept showed that the participant had already had knowledge of the intuitional definition of limit and applied it in the mathematical procedures and in the epistemologies of limit as "approaching". "Approaching" can be considered as one of the right terms in defining the concept of limit intuitively (Stewart, 2008), which is also proposed as an intuitive definition of the term in the curriculum. The literature indicates that students at higher level education (e.g., students in engineering, mathematics, and mathematics education) tend to define the concept as "approaching", since it represents the dynamic conception of limit (Tall & Vinner, 1981; Cornu, 1991). However, this constructs a gap between university and school since mathematics teachers learn the formal definition of the limit at university but need to teach only an intuitive limit concept at schools, when studying the derivative, integral or introduction of real numbers, which is framed as "double discontinuity" by Felix Klein (Klein, 1925 as cited in Kilpatrick, 2019). The findings supported Klein's claims that Mila had some confusions about the formal definition and its components. In particular, the pre-interview showed that she had lived this “double-discontinuity” which means teacher students realize a first gap when they enter the university studies, and then a second time when they enter the school in their profession after their studies (Klein, 1925 as cited in Kilpatrick, 2019), since she had not thought about the formal definition and did not expect to be asked about it. From this perspective, this finding is of importance in terms of the Calculus education in prospective teacher education. In other words, it is important that the content of the Calculus course is created by associating the prospective teachers with how these concepts can be taught, so that the prospective teachers do not fall between this duality (which definition should be known for teaching).

On the other hand, the lack of knowledge related to the formal definition of limit can be considered as an expected finding when compared with the ones in the literature. In the literature, knowledge of the formal definition of limit has been examined as a conceptual knowledge for different levels including high school students, prospective mathematics teachers, Calculus students, engineering students, which revealed that participants have struggle with the formal definition of limit (Cottrill et al., 1996; Swinyard & Larsen, 2012; Beynon & Zollman, 2015; Oktaviyanthi, Herman, & Dahlan, 2018; Adiredja, 2020).

Similar to the indicator of knowledge of definition, the pre-interview had important findings related to other indicators of knowledge of topics (KoT). The lack of knowledge was also observed in other indicators including phenomenology and applications of the concept, foundations of the concept and representation systems except for the mathematical procedures. Knowledge of mathematical procedures was considered as not desired but sufficient level in the findings, since she had knowledge of how and when to do something. Since the education system in Turkey is exam-oriented and procedure-based, it is a rather expected finding that she could easily do the limit exercises and follow the process-related steps. On the other hand, in another part of the indicator which is the knowledge of why to do something, the pre-interview showed that she had a lack of knowledge. In particular, the pre-interview showed that she had difficulty when she was asked the reason behind the mathematical procedures she conducted. One of the reasons behind this lack of knowledge can be explained as the lack of knowledge of other indicators. In particular, lack of knowledge of definition might cause difficulty in explaining why to do something, because, knowledge of definition is not only a language used to express mathematics, but also enables teachers to organize the mathematical concepts and mathematical thoughts they want to express, and to create their own expressions (Shield, 2004). In addition, knowledge of definition is of importance for teachers' instructional decisions and their performance in mathematics discussions (Zazkis & Leikin, 2008; Ginting's, Mawengkang & Syahril, 2018). Therefore, it can be interpreted as that lack of knowledge of definition might cause a lack of other knowledge indicators. Another reason might be the education system since it does not direct them to think critically on the reason behind the mathematical procedures because of its exam-oriented and practice-based features.

The findings showed that in general the planning phases of lesson study promoted the development in knowledge of topics (KoT). In particular, the different elements of planning phases nurtured the different indicators of KoT. For instance, the findings showed that the effect of the pre-interviewing process in planning phases, in particular in the first cycle, provided the participant focusing more on developing knowledge of definition and knowledge of history. Furthermore, the collaboration of the effect of the pre-interviewing process on the planning phase and the intervention of the researcher (as knowledgeable other) with rich materials constructed a learning environment to develop the other indicators of KoT besides knowledge of definition through the discussions on the concept. In the planning phase of the second cycle, the development was supported through learning with conversation. In contrast with planning phases of lesson study, the enacting phases did not have effect on most of the indicators of KoT including knowledge of definition, phenomenology and application, and representation systems. On the other hand, the enacting phases of lesson study had an effect on knowledge of mathematical procedures, which was nurtured by the students in the research lessons of lesson study.

Since the different indicators of KoT are intertwined, the lesson study process showed that the development of indicators of KoT affected each others' development. Bearing this fact in mind, Figure 5.1 shows how the development of indicators affected each other. Therefore, an overall summary of findings in Figure 5.1 is given above to depict the relationship between indicators.

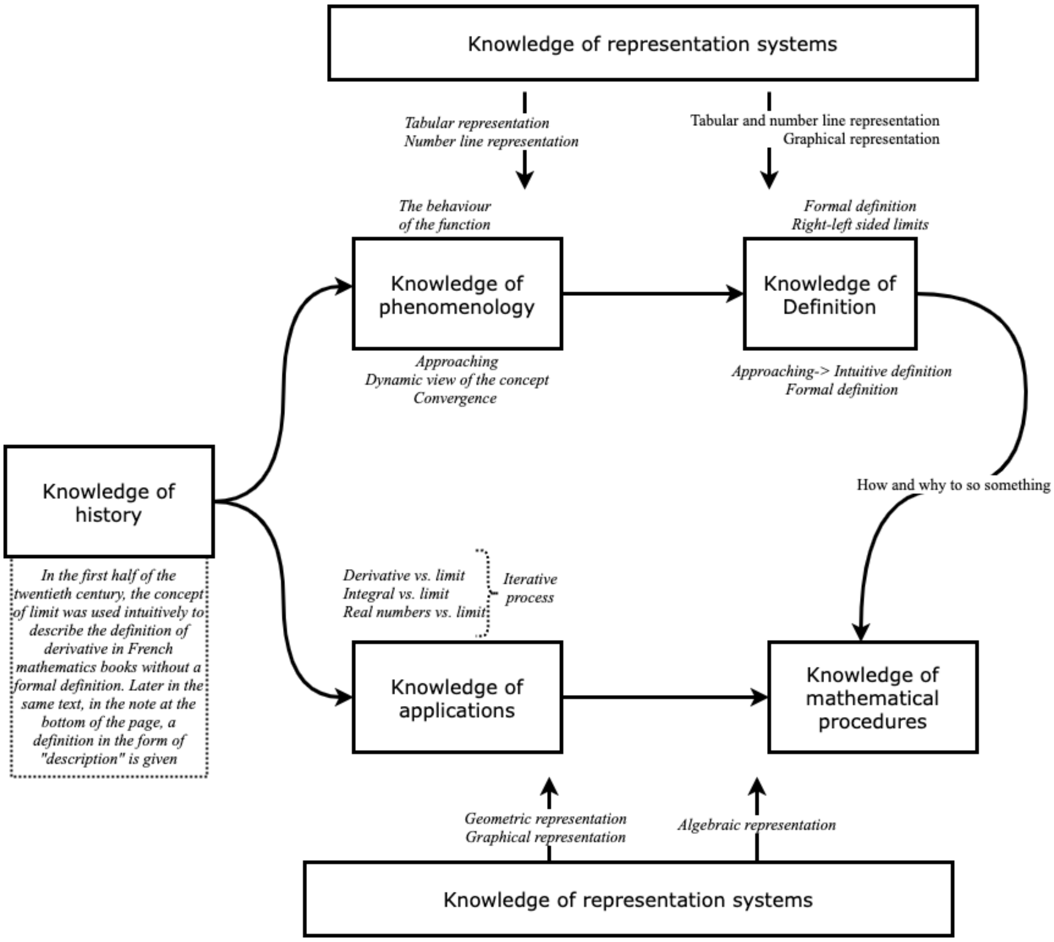


Figure 5.1 The overall summary of knowledge of topics development of Mila

In the figure, the rectangles show the indicators and the direction of the arrows is from the affecting indicators to the affected indicators. In addition, the text on the arrows depicts the relation between the indicators of knowledge of topics (KoT). For instance, the improvement in the knowledge of history of the concept paves the way for the improvement of both knowledge of application and knowledge of phenomenology in accordance with the knowledge of representation since Mila focused on how to show her mathematical knowledge. Likewise, the improvement of the knowledge of meaning of the concept requires thinking beyond the curriculum. It can be concluded that this finding is related to the nature of the concept since the historical development of the concept of limit is included in the relation of it with other mathematical concepts (e.g., derivative, integral, numbers, sets) (Stewart, 2008). Considering the nature of the concept in designing the lesson study process, the researchers gave appropriate readings as long as they observed the deficiencies of the lesson study group and they opened discussions on these readings in the planning phase of lesson study. As the literature indicated (e.g., Murata et al., 2012; Clivaz & Shuilleabhain, 2019), by means of collaborative learning and sharing interests and knowledge in lesson study, the sub-domains of KoT supported each other in the development process.

Unlike the most of the indicators in knowledge of topics (KoT), knowledge of definition has its own indicators including intuitive, right-left sided limit, formal definition and its components (such as quantifiers in the formal definition ($\forall$ $(for\ all), \exists (at\ least)$) and temporal order in the formal definition) which students have difficulty in learning the concept of limit (Tall & Vinner, 1981), and transition from intuitive definition to formal definition. Apart from the latest indicator-knowledge of transition from intuitive definition to formal definition which is dealt with in the knowledge of practices in mathematics (KPM), the development in formal definition and its components was provided long enough lesson planning process in which the prospective mathematics teacher had time and rich materials to think on the notions and related discussions in terms of how to teach them to students. The findings supported the claim about teacher learning that one of the most effective ways for teacher learning is to collaborate with colleagues for thinking on teaching and solving problems related to student learning (Rock & Wilson, 2005). This finding also showed the importance of lesson planning process for construction of prospective mathematics teachers' knowledge in terms of iterative process by means of prompts of a facilitator and feedbacks from the tasks included in lesson plan (Zavlavsky, 2008). Accordingly, in the development of knowledge of definition, one of the important findings was that while some indicators were affected from the prospective mathematics teacher's preferences which she thought that she had lack of knowledge in the pre-interview, knowledge in quantifiers ($(\forall\ (for\ all), \exists(at\ least)$) of formal definition was observed as the researcher's expectations as she should have given her focus on these notions.

What is not developed throughout the lesson study process was also observed in knowledge related to "limit specifies a function" as a not adequate development in knowledge. Knowledge of "limit specifies a function" can be described "if the value L (must be a number) exist as $x \to c$ we say that the limit exist at $x = c$". While the lesson study promoted the prospective teacher as well as other group members to think of the statement as "limit is a function", the prospective teacher rejected the statement or did not prefer to think about it. There may be some reasons for this not adequate level of development. First of all, the prospective teacher worked with limits of functions since the curriculum permitted only this side of the concept. While the group was triggered to think on limits of series (to understand the basis of integration process and iterative process, for instance), particularly Mila, among the group members had negative emotions towards this topic. For this reason, Mila avoided discussing this issue. Therefore, she had confusion about the question "when taking the limit of the function at a point, how can it be a function?". Second, the lack of knowledge of ways of validating and proceeding (knowledge of practices in mathematics) might cause her not to figure out the statement in her mind. She did not use both of her knowledge of definition of function and definition of limit to try the statement, though she had knowledge of definitions. Third, the lesson study process requires thinking about the concept to present for teaching. She could not use this knowledge anywhere since she did not internalize this knowledge.

Knowledge of topics (KoT) is considered as fundamental mathematical knowledge in the model of MTSK (Carrillo-Yañez et al., 2018). While it is not in the center of the model, the findings supported the claim that KoT constructs basis for other sub-domains in both mathematical knowledge and pedagogical content knowledge in the context of the limit concept. For instance, when the prospective teacher had indicators of KoT (e.g., phenomenology of the concept), she was able to make connections with other mathematical concepts in her instructional decisions. In the following section, the development of knowledge related to mathematical connections are summarized and discussed.

Knowledge of Structures of Mathematics

Another sub-domain of the model- knowledge of structures of mathematics (KSM) is one of the sub-domains which could not be observed in the pre-interview because of its structure. KSM includes mathematical knowledge that reveals how concepts are related to each other and a related mathematical structure (Montes et al., 2013). In the studies related to this model, the researchers have worked on problems and their interactions with students to reveal how mathematics teachers relate the topic with other mathematical concepts. In the current study, I examined the development of the prospective mathematics teacher in the lens of MTSK by means of the interaction with her colleagues and her students. However, in the pre-interview, it was hard to examine this sub-domain with open-ended questions. Rather, I examined them during early planning phases to design the lesson study process to provide and reveal the development of the sub-domain.

The findings of the observation at the beginning of the planning phases of the first cycle showed that the prospective teacher had already had the awareness about the relation between the concepts; however, she was not aware where the relation comes from; in particular, what the underlying concept or feature of this knowledge is. This can be also understood from her pre-interview that she could indicate that there is a relation between the Calculus concepts but she could not describe the relations. Therefore, it was considered as a lack of knowledge and the lesson study process was designed to provide the prospective teacher with this awareness and make instructional decisions according to this awareness.

Knowledge of structure of mathematics (KSM) includes two types of connections with four indicators: (1) temporal connections with connections based on simplification and connections based on complexity, (2) interconceptual connections with transverse connections and auxiliary connections. As said before, the development was not observed in some indicators; in this sub-domain, there was not any development in temporal connections. Mila did not connect the concept of limit with the mathematical concepts which were studied before the limit concept (*connections based on simplification*) and what will be studied after the limit concept (*connections based on complexity*) during her journey, the findings focused on the development of interconceptual connections including transverse connections and auxiliary connections. In fact, I observed some evidence about temporal connections during the lesson study process. However, those were not continuous enough that the development could not be observed in a process. Therefore, such situations were not given as a finding.

As mentioned in the Literature Review, knowledge of structure of mathematics (KSM) can be described as a re-interpretation of horizon content knowledge in the model of mathematical knowledge for teaching which is defined as "an awareness of how mathematical topics are related over the span of mathematics" (Ball, Thames, & Phelps, 2008, p. 403) in terms of interconceptual and temporal connections. In the literature, there are not enough studies related to the level of horizon content knowledge of mathematics teachers and/or prospective mathematics teachers. Rather, the studies commonly focused on defining horizon content knowledge in terms of teacher practices, textbooks, and solving problems (e.g., Wasserman & Stockton, 2013; Zhang, Zhang, & Wang, 2017; Jauchen, 2019). Therefore, even though there is not enough evidence related to development of temporal connection, this finding is of importance for the literature. An important part of the contribution of this finding could be to show the relation between the nature of the concept and the development. While the concept of limit is a cornerstone for many mathematical concepts (Tall & Vinner, 1981; Monaghan, 1991; Cornu, 1991), it has a complex structure on its own (Parameswaran, 2006). Therefore, it might be hard to overcome its complex structure to make connections between more complex mathematical concepts. In particular, due to both the structure of the concept and the level of education, it may not have provided an opportunity to think further and complex, or to reduce it to a simpler concept.

The other indicators including transverse and auxiliary connections are related to interconceptual connections. Unlike the temporal connections, the lesson study process supported the development of interconceptual connections, even though an indicator was not observed in all phases of lesson study. For instance, although the reflection of the knowledge of transverse connection in teaching is not observed much, it has been observed that each indicator actually supports the development of each other. Although it is a delimitation not to compare the development of another concept in the context of transverse connection in this study, the nature of the concept is one of the reasons for the development of this knowledge of connection. In particular, the fact that the concept of limit actually forms the basis of many concepts such as derivative, integral, real numbers and has common features with all other concepts, including infinity, has enabled the same type of knowledge to be observed under different sub-domains. Akkoç, Yeşildere and Özmantar (2007) indicated that they could not make this connection mathematically, though the prospective teachers knew that the limit for integral is a prerequisite knowledge. Furthermore, the inappropriate and weak mental connections between knowledge of the concept of limit and knowledge of other Calculus concepts lead to students' misinterpretations and non-meaningful thoughts about the basis of these connections (Bezuidenthout, 2001). Therefore, the knowledge of transverse connections with the concept of limit could be an important part of the development of specialized knowledge.

Another indicator of interconceptual connections, auxiliary connections, was observed through examining the connections built in the problems. While Mila could not associate the limit concept with any other mathematical concept and she could not exemplify the limit concept by means of another mathematical concept, resources given in the investigation phase and development in other sub-domains (e.g., phenomenology and history of the concept) provided a way for the prospective teacher to develop her knowledge of auxiliary connections. She, as well as the lesson study group, usually used "geometrical concept" as auxiliary elements to teach the limit concept and logic behind the mathematical procedures. This was an expected finding, since the history of the concept includes such connections (Cornu, 1991).

The study examined the development of the prospective teacher's specialized knowledge for teaching the concept holistically. Therefore, the answer of the question of which one of the indicators developed first and which one triggered the other forms important findings was sought. In the light of the findings for knowledge of structure of mathematics (KSM), it can be said that in my participant's journey of knowledge development, the application of the concept developed first and led to the development of others. Based on the development of the participant's knowledge of transverse connection, it can be concluded that the development occurred when geometry is used as an auxiliary element on the basis of infinity (see Figure 4.15). In addition, the relation shows that the development of knowledge of interconceptual connections is closely related to knowledge of phenomenology of the concept.

Carrillo-Yañez et al. (2018) indicated that such relation between the concepts has not been considered in the former theoretical models for teacher knowledge. For instance, Kajander and Lovric (2017) asserted a framework based on horizon content knowledge of the concept of limit which categorized it as “a use of higher-order abstractions or more complex ideas, misconceptions and authentic connections to advanced mathematical concepts” (p. 1031). The study has considered “geometrical examples”, which was also studied in the current study, as authentic connections which connect the geometrical examples to advanced mathematical concepts (limit). In the current study, the same example was considered in interconceptual connection, since the prospective teacher used geometry as an auxiliary element to teach the concept of limit. While this situation is due to the use of two different models, the detailed connection between mathematical concepts has made it possible to clearly observe how and in which context the development took place.

Knowledge of Practices in Mathematics

To answer the first research question examining how prospective mathematics teachers develop their specialized knowledge in the concept of limit in the lesson study development model, another sub-domain of this specialized knowledge, knowledge of practices in mathematics (KPM), was examined. KPM is an emerging sub-domain in models of teacher education (Carrillo-Yañez et al., 2018). The term “mathematical practices” in the name of the sub-domain are based on two fundamental aspects including proving and refuting (Komatsu & Jones, 2022). These aspects can be observed in mathematical knowledge of teachers towards ways of generating mathematics. KPM covers “knowledge of ways of proceeding, validating, exploring, and generating knowledge in mathematics, such as knowledge of ways to communicate mathematics” (Carrillo-Yañez et al., 2018, p. 245). Considering the level of prospective teachers and the concept, KPM dealt with knowledge of ways of validating and demonstrating necessary and sufficient conditions for generating definition and role of symbols and use of formal language.

In the pre-interview, similar with some other sub-domains, there was not enough evidence related to each indicator knowledge of practices in mathematics (KPM). However, when it was dealt with holistically in relation to the beginning of the lesson study process, it can be said that Mila had a lack of KPM. In particular, Mila had lack of knowledge about how to define the concept and the concepts related to limits, in particular, the characteristics of the formal definition of limit, and how to generate and reason on the new knowledge as well as the response to students' mathematical reasoning for verifying them or pointing out that they are wrong by justifying it in their minds. To be clearer, she could not generate the definitions of "continuity in interval" by means of using her knowledge related to "continuity at a point". In addition, her ways of validating or refuting any mathematical statement related to procedures related to limit concept was commonly to use counterexamples without a mathematical language (e.g., using the mathematical terms related to limit concept). This was both an expected and an unexpected finding in terms of different sides of coin. First, this was an expected finding for her existing knowledge, since she had a lack of fundamental knowledge, knowledge of topics (KoT), in this case. For instance, the lack of knowledge of definition led us to have knowledge of validating the rules in limits. On the other side of the coin, this was an unexpected finding since the prospective teacher was in a good condition considering together with her grades of the courses of which contents include proving and refuting.

In general, the sufficient level of development in this sub-domain could not be observed through lesson study. While there was a development in knowledge of the role of symbols and use of formal language, there was not enough evidence to mention other indicators as sufficient levels of development. For the indicator of knowledge of necessary and sufficient conditions for generating definition was considered as constructing an environment where students easily defined the concept in a formal way. As mentioned in the Findings section, the lesson study group mediated upon this issue in most of the planning meetings for Lesson Plan-1. Therefore, the development occurred among the group holistically. In the literature, the formal definition was discussed in terms of the conceptual knowledge related to notions, ε, δ, and the temporal order in the definition (e.g., Davis & Vinner, 1986; Cornu, 1991; Williams, 2001). However, there is limited research on knowledge of how to teach the formal definition and knowledge of how to construct an environment to transfer from intuitive definition to formal definition (e.g., Swinyard, 2011; Oktaviyanthi & Dahlan, 2018). For this reason, this finding is of importance for the literature.

The development could not be provided individually; rather the development occurred in group through rich group discussions in planning phases of lesson study. Unlike the planning phases, in the enacting phases, the prospective teacher was alone with the students, although she adhered to the lesson plan. However, for knowledge of practices in mathematics (KPM), enacting phases did not contribute to the development. While it did not nurture the development, this finding can yield important inferences, since there is not any study related to the contribution of the enacting phase (research lesson and reflection) on mathematical knowledge in the accessible literature. To be more specific, this finding showed the importance of curriculum and the characteristics of students, because when the teacher enters the classroom, students' expectations and questions are one of the important factors in revealing the teacher's knowledge (Çopur-Gençturk, Plowman, & Bai, 2019).

In this study, it was observed that since the expectations of the students were more practice-based, their expectations might not trigger her to reveal and improve this knowledge. Furthermore, there was a relation between the development of knowledge sub-domains; in particular, knowledge of necessary and sufficient conditions to generate definition and knowledge of definition. One of the indicators of knowledge of topics (KoT) is “knowledge of definition” which provides a basis for the indicator of KPM-Knowledge of necessary and sufficient conditions for generating definition. In the current study, in order for the participants to create the necessary and sufficient conditions for the students to define the concept formally, they had to have the indicators under the KoT sub-domain first. Only after these indicators developed, they were able to develop KPM's indicator. The relations between subdomains were discussed in terms of KPM and the sub-domains of pedagogical content knowledge (PCK) with university mathematics instructors (e.g., Zakaryan & Ribeiro, 2016; Delgado-Rebolledo & Zakaryan, 2020). Such a relation which can be considered as an expected relation has not been indicated in the literature before. It was observed for development of an indicator of KoT and KPM in the current study. It can be explained as the nature of the concept, because, in order to teach the concept of limit, it is necessary to create an environment that teaches necessary and sufficient conditions, but to have sufficient knowledge of definition.

For the indicator of knowledge of ways of validating and demonstrating, Mila usually employed “giving counterexamples” to verify a mathematical truth in planning. In addition to the role of counter examples, while teaching the concept of limit, it was expected that she would gain the knowledge and skills about where and by which method the mathematical truth was validated. However, this development could not be observed during the lesson study process. In other words, she usually used the same ways for validating and demonstrating in mathematics. Her use of the same ways for validating and demonstrating did not show that she had a lack of knowledge; instead, it showed her type of knowledge.

On the other hand, it was an interesting finding that she could not use proving to demonstrate the reason behind the mathematical procedures (e.g., the sum of limit) by using the formal definition of the limit, even though she developed her knowledge of definition. Instead, she preferred to describe it verbally and informally by using some notations of formal definition (e.g., delta and epsilon). This finding also contradicts the study of Oktaviyanthi, Herman and Dahlan (2018) in which the prospective teachers adopted a formal strategy to evaluate mathematical procedures with the formal definition. It can be explained with two reasons: First, the lack of support for the issue of proving in lesson study. Since both Mila and other group members had lack of fundamental knowledge, after providing the development of fundamental knowledge, sufficient support may not be provided to the subject of the mathematical procedures. Second, though the lesson study was designed considering intended development of knowledge for teaching the concept of limit, the choice of the members of the lesson study group shaped the process. Since both Mila and other group members thought that this knowledge was not necessary for teaching and was above the student level, they stayed away from this subject, although it was mentioned many times during the lesson study process. It can be concluded that personal affections are of importance for development of knowledge.

The findings of the post-interview supported the findings gathered through the lesson study process. While there was not any specific question to examine any indicator of knowledge of practices in mathematics (KPM), it can be understood from her answer related to the limit of indeterminate forms that she could answer the questions by giving examples and counterexamples. In addition, the post interview supported the findings related to insufficient development in knowledge of using formal language, which can be understood from her examples.

So far, the section has presented a prospective mathematics teacher's journey of development of her mathematical knowledge through lesson study. In general, the lesson study development model nurtured the prospective mathematics teacher's mathematical knowledge through rich group discussions with the support of resources and facilitator (knowledgeable other). In addition, it can be said that the development of mathematical knowledge takes place holistically with the development of sub-domains that support each other. Similar support was observed in both sub-domains of pedagogical content knowledge (PCK). Carrillo-Yañez and his colleagues (2018) stated that the focus of PCK is not restricted with the intersection of mathematics and pedagogy; rather it is “specific type of knowledge of pedagogy which derives chiefly from mathematics” (p. 246). Therefore, the development of the sub-domains of PCK is closely supported by the development of mathematical knowledge as can be seen in the following sections.

Knowledge of Features of Learning Mathematics

The specialized knowledge for teaching the concept of limit mentioned in the first research question has two pillars including mathematical knowledge (MK), which were discussed above, and pedagogical content knowledge (PCK). The first sub-domain-Knowledge of features of learning mathematics (KFLM) considers knowledge of students’ learning of and interaction with the mathematical content (Carrillo-Yañez et al., 2018). KFLM includes four indicators determined by Carrillo-Yañez et al. (2018): theories of mathematics learning, strengths and weaknesses in learning mathematics, ways pupils interact with mathematical content and emotional aspects of learning mathematics (Carrillo-Yañez et al., 2018). Since the pre-interview was a clinical interview and did not include a practice session for the prospective teacher, the pre-interview examined the indicator of knowledge of strengths and weaknesses in learning mathematics among the four indicators mentioned above. The pre-interview showed that Mila had a lack of knowledge related to students’ strengths and weaknesses; in particular, students’ conceptions and misconceptions related to the limit concept. There might be some reasons for this issue: First, the prospective teacher did not have a chance to work on the concept of limit until participating in this study, and accordingly she did not experience interacting with students. Thus, she might not have thought about students' strengths and weaknesses on the limit concept before. Second, as I just said, she had a lack of knowledge of topics (KoT) and accordingly she had her own misconceptions about the topics in the concept of limit. To be more specific, she had her own misconception related that the graph of the function must be able to be drawn without lifting one's hand for continuity (Tall & Vinner, 1981). Therefore, it can be considered as an expected finding that she was not aware of the strengths and weaknesses of students since she was not aware of her own strengths, weaknesses and misconceptions. This finding showed that it would be better to start the development of specialized knowledge for teaching by nurturing the fundamental knowledge, which is knowledge of topics in this model. This issue is presented and discussed in the following paragraphs and sections.

Though the lesson study process was designed to cover four indicators of knowledge of features of learning mathematics (KFLM), the guidance and preferences of the prospective teacher and her group mates in the process also affected the design process. Therefore, one of the indicators, theories of mathematics learning, was not observed in both pre-interview and lesson study process. The reason why learning theories were not mentioned during the lesson study process might be the preferences of the lesson study group. Before starting the process, the learning theories were embedded in the lesson planning phases, in particular at the beginning of the lesson planning process. They were given articles related to learning theories; however, both Mila and other group members did not pay attention to theories. Rather, they were interested in how students interact with content and their weaknesses and/or misconceptions. Such a situation progresses in line with the expectations of the lesson study group, even though the outcomes of the lesson study process are designed to be clear from the beginning. In this case, I can say that the *knowledgeable other* is more effective in the development of knowledge towards the expectations of other participants.

The findings showed that the lesson study process promoted the prospective teacher's knowledge for gaining awareness of students' strengths and weaknesses in learning the topics of the concept of limit. In planning, at the beginning of the lesson study process, Mila used her own experiences related to learning the concept. However, such views can be considered as limited, in other words, lack of awareness, for effective mathematics teaching, because attentions of both prospective and in-service teachers to students' mathematical thinking and learning process in interaction with mathematical content provide them to see beyond what they think about mathematical knowledge and enable them to make instructional decisions (Sherin, Jacobs, & Philips, 2011). The indicator of KFLM does not only include knowing where students have difficulties or strengths; rather it covers how to enact this knowledge combining with content knowledge in making instructional decisions (Carrillo-Yañez et al., 2018). Considering that Mila took instructional decisions to help students overcome difficulties by leveraging their strengths, it can be concluded that the enacting phase of lesson study presented a great opportunity to develop this side of knowledge of students' learning mathematics. In addition, the examples given in this part of the findings contributed to the limited literature about the prospective teacher's knowledge of indeterminate-undefined forms in the context of limit concept. Therefore, the findings related to this topic can shed light on the literature within this respect. Lastly, as a part of the nature of the limit concept, there is a powerful relation between the development of mathematical knowledge (Knowledge of practices in mathematics-awareness of mathematical reasoning on how to explore and generate new knowledge in mathematics) and increasing awareness of students' learning mathematics.

Similar to the previous indicator, the lesson study development model provided the development of knowledge of the ways that students interact with content. In particular, the development was supported by means of triggering the group to think and predict on how students interact with their proposed activity. Such predictions made her aware of how she would pay attention to the lesson plan and what she would observe during the research lesson phases. In addition, the different predictions of the group members allowed her to see the interaction with the content from different viewpoints. In the ongoing process, the enactment process (both teaching and observing) gave her a chance to see the interactions of both students and her friends with the same level of education.

The idea of lesson study is originally based on centralized students' learning for instructional decisions (Stigler & Hiebert, 1999). For this reason, the findings related to promoting the prospective teacher to be aware of students' mathematical thinking can be considered as a result of the nature of the lesson study process. Specifically, as the literature supported (e.g., Leavy & Hourigan, 2016; Guner & Akyuz, 2020), working collaboratively on students' learning in planning might promote opening their eyes to observe students' strengths and weaknesses in interaction with mathematical content. Accordingly, the development of Mila's awareness enabled her to make the instructional decision to create her own path by being aware of the strengths of the students rather than applying the lesson plan as it is in the enacting phase.

Knowledge of Mathematics Teaching

Another sub-domain of pedagogical content knowledge (PCK) is the knowledge of mathematics teaching (KMT). KMT is not solely related to knowledge of teaching in pedagogical knowledge, rather it is directly related to knowledge of the concept and teaching, similar to knowledge of features of learning mathematics (KFLM) (Carrillo-Yañez et al., 2018). The sub-domain includes three indicators of knowledge; theories of mathematics teaching, teaching resources (physical and digital), and strategies, techniques, tasks and examples (Carrillo-Yañez et al., 2018). Since other indicators could not be examined in the pre-interview, the pre-interview had evidence related to her knowledge of strategies, techniques, tasks and examples.

Mila was one of the high achieving students in her semester and her grades, in particular the ones for the method courses, were one of the highest grades in the class. Thus, it was expected that she had knowledge of strategies and techniques for teaching mathematics. However, she had a lack of knowledge of strategies, techniques, tasks and examples when she was asked for content-specific strategies, techniques, tasks and examples. KMT requires using pedagogical knowledge in concept-based contexts. Since she had a lack of mathematical knowledge (e.g., lack of knowledge of topics), she had difficulty combining mathematical knowledge and knowledge for teaching. Specifically, she could not give specific tasks or not indicate any strategy in answering a question related to how to teach the limit concept. The findings showed that she only focused on visualizing the content in teaching, which can be considered as a general statement to describe a strategy for teaching. Therefore, for KMT, it was described as existing but not sufficient for the indicator of tasks, examples, strategies and techniques. Considering this indicator and other indicators of KMT, different tasks, examples, strategies and techniques were presented to the lesson study group to make them gain a mathematical point of view for teaching.

Similar with the other subdomains of the model, the lesson study development model was designed to provide the development of all three indicators. Among three indicators, there was not enough evidence to interpret this as sufficient level of development in knowledge of theories of mathematics teaching. It can be considered as an interesting finding that the indicators related to theories (learning and teaching) did not have sufficient level of development both in KFLM and KMT. One of the reasons might be the prospective teachers' expectations and demands, as said in knowledge of learning theories. Considering the findings related to both learning theories and theories of mathematics teaching, the finding can be interpreted that Mila and other group members focused on the practical way, instead of working on theories. In addition, the group mainly focused on developing their fundamental mathematical knowledge.

The findings showed that the lesson study process promoted the development of knowledge of strategies, techniques, tasks and examples from the holistic view. To be more specific, the teaching strategies varied based on the activity in each of the three lesson plans. In general, the group used questioning and discussion techniques in relation to games, examples from daily life which were indicated through problems, animations in GeoGebra, and analogies in their lesson plans. In particular, Mila adopted concept motivation and action learning and concept image and definition at the beginning of the first lesson plan, and conceptual conflict in the third lesson plan. In general, the findings related to knowledge of strategies, tasks and examples are supported by the literature (Dönmez & Baştürk, 2010; Kula & Bukova-Güzel, 2015).

It should be mentioned that the development of this indicator was not observed in all contents. As a nature of the lesson study, the group constructed lesson plans collaboratively. For this reason, some indicators similar to this could not be observed as individual development. The studies related to knowledge development in lesson study literature have commonly focused on the development of knowledge for teaching for the group of mathematics teachers of prospective mathematics teachers (e.g., Tepylo & Moss, 2011; Cajkler et al., 2013; Leavy & Hourigan, 2018). Since lesson study is a process including designing a lesson collaboratively, such studies have shed light on examining the overall development of knowledge in the lesson study process. At this point, the current study is important to extend the understanding of the literature towards the methodological issues about how to observe individual development. Similar to the current study, it might be revealed through each lesson study members' narratives in discussions during the lesson study process. Clemente and Ramirez (2007) used this narrative approach to reveal fundamental knowledge about educational practice. Besides, discourse analysis is another way to observe both collective and individual development (e.g., Dudley, 2013; Cajkler et al., 2014).

In the model of MTSK, the knowledge of assessment is not mentioned explicitly. However, mathematics teachers' knowledge of how to assess what they teach is of importance in their teaching knowledge, in addition to their knowledge of teaching strategies. Therefore, knowledge of assessment strategies was added into knowledge of mathematics teaching (KMT), under the indicator of knowledge of teaching strategies. While the teaching strategies of the prospective teacher could not be observed individually, assessment strategies of the members of the lesson study group could be observed individually during the lesson study process. The findings showed that development in knowledge of assessment was observed in different ways for two titles. In planning, the findings showed that the proposed questions/problems changed from knowledge and computational exercise-type questions to problems that require context-based understanding and practice. In enacting, the findings showed that there is not a significant change in the questioning type of the prospective teacher. Therefore, it might not be considered as a development in the prospective teacher’s knowledge. However, there are some issues that might be discussed in this chapter. For instance, while she used probing questions at the same level in research lessons of each cycle, the guiding questions decreased in the research lesson in the second cycle. The strongest possible reason for this situation might be that the second research lesson was conducted as micro-teaching in which the prospective teacher taught her lesson to her classmates. For this reason, Mila may not have needed to guide them as much as real classroom students. Another finding related to questioning is the increase in factual questions which lead students to produce mathematics for their mathematical knowledge. This increase can be explained with the development of mathematical knowledge, in particular knowledge of practices in mathematics (KPM), since the prospective teacher who does not question and produce mathematics herself cannot question students about mathematical facts.

Furthermore, the findings showed that lesson study nurtured the participant to use alternative assessment techniques that she did not think of using before the meetings of lesson study. In this point of view, the study extends the limited literature about knowledge of assessment. For instance, by demonstrating knowledge development with lesson study, the current study provides a way for the knowledge development related to alternative methods which was associated with lack of knowledge of teaching asserted by Baştürk and Dönmez (2011c).

The rare development in knowledge of mathematics teaching (KMT) was observed in the indicator of teaching resources. Mila was a talented and competent prospective teacher who used teaching resources in physical settings and technological platforms. However, it can be understood from the findings that lack of mathematical knowledge led her to use her knowledge in content-based applications. The requirements of KMT include to use the knowledge of resources with content knowledge. As long as her mathematical knowledge developed, in particular knowledge of topics (KoT) and knowledge of structure of mathematics (KSM) to see mathematical connection between the concepts related to the concept of limit, she could combine her competency with content knowledge. In addition, she was more attentive in her research lessons about using different resources for both teaching and assessment; for instance, exit tickets, instant classroom for grouping students. One of the valuable reasons for this issue might be the fact that one of the group members conducted the research lesson. The group members were selected randomly; however, this made them extra aware of their own research lessons. This situation was observed more seriously in Mila. To reduce this possibility, the selection was conducted some time before the research lesson (in the planning phase of the first lesson plan) and just before the research lessons.

In planning and enacting lesson plans in lesson study, the group considered some expected outcomes asserted by the curricula (they were given curricula from 2005 to 2018). Therefore, these developments are closely related to knowledge of mathematics learning standards (KMLS). For instance, knowledge of assessment strategies might be included in knowledge of mathematics learning standards. But assessment cannot be considered only as an expected outcome, rather it can be considered as a strategy that is observed from beginning to end. For this reason, the current study examined its development under the sub-domain of knowledge of mathematics teaching (KMT). While it was not examined under KMLS, the fact that this indicator is directly related to knowledge of mathematics learning standards of mathematics was considered in observing it. In the next section, the summary of findings of KMLS are presented in detail.

Knowledge of Mathematics Learning Standards

The last sub-domain of pedagogical content knowledge (PCK) in the model is knowledge of mathematics learning standards, which can be defined as an instrument that sets the standards for students to understand, construct and use mathematics and is not specific to a certain level (Carrillo-Yañez et al., 2018). In other words, knowledge of mathematics learning standards (KMLS) is knowledge of the standards set by professional mathematics education associations or research groups and of the official educational program in any country at a specific time (Liñán-García et al., 2021). The sub-domain includes three indicators as expected learning outcomes, expected level of conceptual or procedural development, and sequencing of topics (Carrillo-Yañez et al, 2018). Though the knowledge was not examined in the pre-interview, at the beginning of the lesson study process, the findings showed that Mila had sufficient knowledge related to expected learning outcomes (in other words, objectives) and the sequence of the topics in the curriculum. In particular, she could easily mention the topics before and after the topic including the concept of limit and its objectives. It was an expected finding, since the current study was conducted immediately after the courses related to curricula in Turkey and School Experience. Therefore, she was already familiar with expected outcomes and sequence of the topics in relation with the content.

The lesson study development model expected the participant to extend their knowledge to global standards to think more critically about expected outcomes, learning levels and the sequence of topics. However, the findings showed that there was not any development observed in KMLS. Furthermore, the non-development was observed for all participants, though other participants were not presented in the current study.

The lesson study process began with determining lesson goals as learning outcomes and the group should sequence the topics according to these lesson goals. The lesson study group were free to change the sequence of topics and to write their own objectives within lesson study goals. However, the participant and her group mates stayed within the borders of the curriculum. There might be several reasons to explain this finding. The first reason might be the expectations of the school where the first research lesson was conducted. Though they wanted to add something which is not included in the curriculum, for instance formal definition, the school where the research lessons of the first cycle were conducted did not permit getting out of the standards of the curriculum. Another point might also explain the reason why the participant and her group mates extended their viewpoints in the second cycle in which there were no expectations of the school. This might be a deficiency and a limitation of the designed lesson study process. Because the researcher did not give the lesson study group (Mila in this study) some other resources related to learning standards, such as other countries' curricula or some associations' learning standards, extra resources might be given to the lesson study group to develop her knowledge of mathematics learning standards (KMLS) in the concept of limit. This finding might contradict with Baştürk and Dönmez (2011a) who revealed that the prospective mathematics teachers with wider content knowledge were more willing to stick to the curriculum. However, the current study showed that there was not any change in the prospective mathematics teacher's dependence to the curriculum in planning in the second cycle, even her mathematical knowledge became wider than the first cycle of lesson study.

While it might not be considered as development, different from the other indicators, there was a change in expected level of conceptual or procedural development from the beginning to end of the planning of the first cycle. The findings showed that the participant's expectations from students' learning level went from more procedural development to conceptual development through the intervention of the knowledgeable other and discussion on expected level of conceptual or procedural development. To be more specific, her proposed questions in the planning changed from "the question of what is the answer to a mathematical operation of limit" to "context-based problem that can be solved by using knowledge of limit". It can be interpreted that the development of knowledge of learning standards might be an important factor for other subdomains' development. Similar to the example given above related to knowledge of expected level of conceptual or procedural development, the development of learning standards is directly related to knowledge of teaching and assessment strategies. In retrospect, the reason for non-development in teaching strategies, as a matter-of fact of non-changing in teaching strategies, might be a conclusion of this issue.

Collectively, in general it can be said for the first research question that the findings indicated that the first cycle of the lesson study provided more benefits for developing mathematical knowledge of participants than the second cycle. On the other hand, implementing lesson plans provided them a way to see the mathematical topics from students' eyes. By this way, the second cycle was more effective than the first cycle about the development of pedagogical content knowledge. In Turkey, the practice teaching courses often follow the order as first micro-teaching and later practice teaching. However, in the current study, the opposite way showed a more powerful way than that.

Moreover, not observing as well as observing the improvement in knowledge indicators has important consequences for the first research question. The development in both mathematical knowledge (MK) and pedagogical content knowledge (PCK) occurred as an intended way by means of the interventions throughout the lesson study. As can be seen in the findings, some indicators were not observed as a development. There were some reasons for non-development. While I mentioned that the different backgrounds of the lesson study group members provided rich group discussions, first, the group members had different backgrounds about the concept of limit but for some indicators, it revealed unintended consequences. In particular, Mila being the most inexperienced prospective teacher of the group with her distrust of her own knowledge created a social norm among them, and a cycle was formed in which other experienced participants dominated some indicators. Although there were guided and intervened phases, they were able to advance group discussions among themselves as they wished. Therefore, in some cases, individual needs and development could not be observed, just as with some indicators of the knowledge of mathematics teaching (KMT). This finding can be considered as a finding that can be improved. The findings presented in this dissertation are based on the data which were collected by paying attention to the fact that there was not a social norm between the students and the researcher that would affect the findings of the study. However, this study was based on the 'social constructivist theory' revealing the idea that social interactions play a major role in constructing understanding and language forms thought, and mathematics is not a static body of knowledge, but a socially constructed and evolving way of thinking. While the social interaction with the researcher and the participant(s) could be controlled, social interaction between the participants could have included more careful attempts to control the intended learning outcomes of the prospective(s).

5.2 The Lesson Study Development Model

The critical elements were shown for the findings of the second research question. The second research question aimed to reveal how the critical elements of lesson study can be regulated so that they become an integral part of a logical chain to improve prospective secondary mathematics teachers' specialized knowledge in the concept of limit. The findings pointed out the critical elements of lesson study including conducting pre-interview before the lesson study process, rich group discussions, long enough lesson planning and the nature of the concept and how they promoted the knowledge development of the prospective mathematics teacher. Considering the findings related to how they promoted the development of the prospective teachers' specialized knowledge in the context of limits, I asserted a logical chain of these critical elements in line with the observable features of the lesson study (as can be seen in Chapter 4) to show the implementation of the critical elements.

The first critical element I proposed was "conducting pre-interview before the lesson study process". It is a well-known situation in the literature that clinical interviews contribute not only to assess mathematics teachers' development but also to support their development (Ambrose et al., 2004; Taylan, 2018). However, in this study, I did not directly observe the development of the prospective teacher but revealed how the prospective teacher contributed to the design of the lesson study process through the development of knowledge. The findings revealed that the pre-interview and the lesson study process had a mutual relationship in which the pre-interview promoted the design of the process, and the group, Mila in this case, often referred to the pre-interview during the lesson study process. One of the reasons for this relationship was the construction of the questions in the pre-interview in accordance with the lesson study stages. For instance, since prospective teachers had to set lesson goals at the first phase of lesson study, the priority of lesson study was to enable them to focus on students' learning through their lack of knowledge and to focus on a comprehensive purpose rather than a superficial objective. At the same time, I aimed to get them to think and question before moving on to the second stage, which is the planning stage.

The logical chain presented in Chapter 4 showed how these critical elements interact with each other for the development of prospective mathematics teachers' knowledge. One of the important findings was built on these relations which can be seen as a contribution of the sufficiently long lesson planning process. There were two important points for this element: First, as Tepylo and Moss (2011) indicated, the superficial planning process resulted in little evidence of teachers' knowledge development. Second, considering the outcomes of the process, the nature of the concept, which has a complex structure as the concept of limit, became more important for prospective teacher's development. As mentioned in the literature, the concept of limit is one of the difficult concepts for teaching that requires knowledge of broader sense about its phenomenological aspects and basis, its features and its position in mathematical concepts (Cornu, 1991; Kajander & Lovric, 2017). When a prospective teacher has lack of these knowledge, it is hard to reveal students' learning correctly. As Smith and Stein (2011) indicated, the sufficiently long lesson planning process enabled prospective teachers, Mila in this case, to prepare herself by thinking on what to expect from students, which led to the development of both students' mathematical understanding and her own mathematical knowledge. In this way, it provided knowledge development in both MK and PCK. This critical element also supported the critical features of professional development including content focus, duration, collective participation, active learning and policy reflects

(Desimone, 2009). Moreover, it can be said that sufficiently long lesson planning can be described as the combination of them through lesson study in our design.

Another important element was “rich group discussions” which were implemented in planning phases particularly, since the nature of the concept requires examining it in-depth with different viewpoints (Cottrill et al., 1996). Lesson study is based on collaborative learning and correspondingly its theoretical underpinning is the social constructivist theory (Vygotsky, 1978) in which learning occurs within a group in cultural contexts by means of social interactions (Richardson, 1997). Thus, rich group discussions, which required discussing both teaching and learning the concept and the concept itself in-depth in the current study, was an important step for knowledge development. Rich group discussions were divided into three elements including knowledgeable other, rich materials, and characteristics of the lesson study group. The rich group discussions were integrated mostly in planning phases by means of readings, assignments and tasks which were given based on the observation of their lack of knowledge throughout the lesson study process. When I said ‘rich group discussions’, I was not only talking about integrating the prospective teachers into the planning phase where their discussions took place, but also about the prospective teacher's discussion with the students in unexpected moments (Doğan-Coşkun, Isıksal-Bostan, & Rowland, 2021) during the research lesson phase. As can be seen in the findings, the discussion between the prospective teacher and the students, which started with a small question, enabled them to see a point, which the prospective teacher and her group mates had never considered before, and to improve their knowledge. The integration of the knowledgeable other in group discussion in both planning and reflection of the discussions of the research lesson guided them for meaningful discussions and learning. The previous studies, for instance Horn and Little (2010), promoted the asserted claim of this study that richer learning opportunities are provided when the discussion is directed in the desired direction by a guidance (knowledgeable other in lesson study), not when the prospective teachers are expected to discuss activities of each phase by themselves. Moreover, the current study showed that the implementation of these three sub-elements collaboratively can minimize the unintended and problematic learnings (Parks, 2008).

Though the critical elements in this study are usually emphasized in the literature which is related to lesson study and prospective teacher education, the model extends our knowledge of how the critical elements are implemented during the lesson study to develop prospective teachers' knowledge for teaching mathematics. Focusing particularly on the concept of limit revealed an important element: the nature of the concept. In this way, the current findings related to the "nature of the concept" and how it nurtures other critical elements during the lesson study process add to the growing body of literature on lesson study.

While conducting only qualitative methods for data collection can be considered as a limitation and/or a delimitation of the study, it is encouraging to compare the model with what was found by Akiba and her colleagues (2019) who revealed the relation between design features of lesson study and teacher learning outcomes with path analysis. For instance, while they addressed the duration of lesson study which was closely related to teacher preparation, I detailed the "duration" as indicating a sufficiently long lesson planning process.

In this designed as a teacher development experiment, the current study presented examples on one participant, Mila. The study has some theoretical and methodological contributions to the literature, and implications and suggestions in terms of development of knowledge for teaching mathematics through lesson study and the implementation of lesson study in prospective teacher education for future research. In the next section, the implications and recommendations for future research are presented.

5.3 Theoretical and Methodological Contributions of the Study

In the current study, it was examined the development of prospective mathematics teachers' specialized knowledge for teaching the concept of limit in a broad sense through lesson study. To reach this aim, the lesson study process was designed within the frame of teacher development experiment. With this aspect of the study, the current study can provide theoretical and methodological contributions to the literature, which can be grouped as the combination of teacher development experiment and lesson study (methodological), how this combination provided change in knowledge (theoretical), and the contribution of this combination to the model utilized in the study in the context of limit concept (theoretical).

One of the theoretical and methodological contributions of the study can be using lesson study as a teacher development experiment. The reason of using this design was the similarities between lesson study and teacher development experiment which includes a set of analyses, a presence of a facilitator with a community of practice including a group of teachers and students and interrelated development of mathematical knowledge and pedagogical content knowledge (Simon, 2000). Since the origin of lesson study is centralized students' learning, it can be considered as the natural results of the literature on knowledge development through lesson study that most of the studies focused on the development of prospective mathematics teachers' knowledge for students, students' learning, and teaching mathematics. Teacher development experiment made it possible to develop not only specialized knowledge under the pedagogical content knowledge, but also mathematical knowledge within the mutually supportive development. For instance, teacher development experiment provided a flexibility to conduct the lesson planning phases by reversing the three lesson plans (see Figure 4.26), which enabled the development of mathematical knowledge in the context of the nature of the concept. Therefore, it can be said that these findings can enhance our understanding related to designing lesson study through teacher development experiment. Furthermore, there is a limited number of studies combining such design-based research with lesson study in the literature (e.g., Presmeg & Barrett, 2003; Jagals & Van der Walt, 2018), such studies commonly focused on integration of prospective teachers' design process of an instructional sequences through lesson study. Therefore, the methodological approach of the study can extend our understanding regarding the methodological literature on lesson study.

While the methodological frame of the study was lesson study, the theoretical and analytical framework of the study was the model of MTSK which can be considered as relatively new in the mathematics teachers knowledge literature. Therefore, it is of importance to validate the model in the context of a mathematical concept. In the current study, this validation was provided through the combination of TDE and lesson study in the light of the data for almost all indicators. First, the phases of lesson study (investigation, planning, research lesson and reflection) provided to observe almost all indicators and sub-domains of the model. Furthermore, consideration of the indicators of this model as a learning outcome enabled the theory to be validated, since the lesson study process was designed as a TDE. In other words, there is a mutual relation between the indicators and TDE, since the indicators shaped the designing process of lesson study as TDE. In this way, it can be said that the study brings theoretical and methodological perspective for both lesson study and the model of MTSK.

Accordingly, the study contributed the literature on models for mathematics teachers' knowledge in terms of concept-specific indicators. While the theoretical models for knowledge for teaching mathematics were commonly worked in the context of a mathematical concept (Scheiner, 2015), there are limited studies which revealed concept-specific indicators in one of the models for mathematics teachers' knowledge (e.g., Kula & Bukova-Güzel, 2014). In addition to the fact that the findings of the study supported the concept-specific indicators related to knowledge for teaching the concept of limit, the study revealed additional indicators for the model (e.g., knowledge of assessment strategies). Thus, it can be said that the study offers a way for researchers who want to work on the development of prospective mathematics teachers' knowledge of the limit concept.

5.4 Implications of the Study

In the light of the findings and conclusions, the study revealed some implications for mathematics education researchers, teacher educators and curriculum developers who might make use of the process carried out in the current research and improve the research on the development of mathematics teachers and mathematics education.

The findings of the current research indicate that a well-designed lesson study which is combined with micro-teaching lesson study has the potential to facilitate development of prospective mathematics teachers' specialized knowledge for teaching the concept of limit. In this way, considering the fact that the role of teacher education programs is to prepare future mathematics teachers, adaptation of an effective professional development model such as lesson study is crucial in terms of developing prospective teachers' knowledge for teaching mathematics holistically. Since lesson study requires both theoretical and practical knowledge in addition to knowledge to communicate with colleagues during its phases, the results of the study support the idea that lesson study provides a holistic development in prospective mathematics teachers by examining all the aspects of specialized knowledge for teaching the concept. In this way, the study can suggest that taking part in lesson study should be encouraged throughout various phases of the teacher education programs.

Another significant finding of the study was that prospective mathematics teachers can conceptualize their mathematical knowledge. Bearing the fact that mathematics education courses and teaching practice courses of teacher education programs are carried out in separate contexts and in the last years of the program, the lesson study model in collaboration of mathematics education courses and teaching practice courses might be implemented. This will create an environment in teacher education programs which help prospective teachers experience realities of classroom settings and teaching (Butler et al., 2006), and learn how to apply what they have learned theoretically, thus, improve their specialized knowledge for teaching mathematics.

On the other hand, this study sought to create an effective way to avoid rote implementation of the lesson study and offer a more effective professional development by meeting the research needs that Lewis (2006) referred to. Thus, both mathematics education researchers and mathematics teacher educators can benefit from this implication. For mathematics teacher educator, it can be said that given the complex nature of the prospective teacher training process (Ponte & Chapman, 2015), a way to train teachers who offer a more effective instruction can be devised by considering these critical elements including pre-interview, the nature of the concept, rich group discussions, guided reflection, and the knowledgeable other. Besides, while some of them including the knowledgeable other and rich group discussions have been mentioned in different contexts in the literature, the current study asserted new elements (e.g., the nature of the concept and pre-interview) for the lesson study development model. Thus, working on these elements can extend the understanding of lesson study for mathematics education researchers.

At last, the current study examined the development of prospective mathematics teachers' knowledge for teaching the concept of limit in the light of the model of MTSK. The model used in this study (Carrillo-Yañez et al., 2018) is based on the collaborative research group including 12 members whose aim is to reveal learning opportunities created by teachers in the course of their work. The members who were included in the study were "pre-school, primary and secondary teachers, trainee teachers and researchers into Mathematics Education" (Carrillo-Yañez et al, 2018, p. 237). However, the model is based on the practices of primary and secondary school teachers. Therefore, using the model for analyzing prospective teachers' practices can bring a new perspective to this new model. Furthermore, in the current study, the model was implemented in a different cultural context. Since the model can be considered as relatively new in mathematics teacher knowledge literature, working on it in a different context can be regarded as the validation of the model. Thus, implementing the model on different topics in other contexts can extend the understanding of the model as well.

5.5 Limitations and Recommendations for Further Research

The current study aims to examine the nature and development of prospective teachers' specialized knowledge for teaching the concept of limit in a broad sense through a designed lesson study development model. In addition to the implications driven from the findings of the study, the study has some limitations and recommendations for the future studies.

First, the study focused on a prospective mathematics teacher's knowledge development through lesson study development model. During the academic year in which the study was conducted, there were eight prospective mathematics teachers enrolled in the teacher education program, and they did not want to get into the workload of the lesson study alongside their intense program. Therefore, the lesson study group included three participants. While it can be seen as a limitation for the study, focusing on one of these three participants allowed the researcher to make an in-depth analysis, allowing the findings to be examined in a broader perspective. Further studies can extend this research by working with more than a lesson study group and more than a prospective mathematics teacher to reveal more generalizable results.

Second, this study is limited to a specific mathematical domain of the concept of limit and its applications. It can be conducted by employing distinct mathematics domains and subjects in order to understand how the development process of prospective mathematics teachers’ knowledge for teaching mathematics through lesson study is influenced by the change. It can be considered as a limitation for the critical element- the nature of the concept of the lesson study development model. Thus, it can be further investigated to validate the critical element of lesson study.

In addition to the concept of limit, the current study is limited to the prospective mathematics teachers' knowledge development through lesson study in secondary school level. Considering the fact that most of the studies related to knowledge development in lesson study involved middle school prospective mathematics teachers (e.g., Leavy & Hourigan, 2016; Clivaz & Shuilleabhain, 2019), by means of this limitation, the current study might lead the literature in terms of working with prospective secondary mathematics teachers. Since the prospective mathematics teacher education in different levels have different features practically, more research is required to determine the efficacy of lesson study in prospective mathematics teacher education in secondary school level.

At last, lesson study relied heavily on the context where the study is conducted and was limited to two cycles. Considering the aim of the study, the limitation of two cycles of lesson study also worked for the benefit of the study because the planning phase of the first cycle was kept long to provide sufficient level of development in their knowledge in the context where the study was conducted. Therefore, the study did not require an additional cycle of lesson study. However, longer cycles should not be considered as a necessity. Further studies can focus on the development with more cycles which take almost equal time in different contexts.

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APPENDICES

A. The Concept-specific Indicators based on the Model of MTSK

The Sub-domains	The indicators	The concept-based indicators
Knowledge of Topics	Definitions	Knowledge of intuitive definition Knowledge of right-left sided limits Knowledge of formal definition of limit Temporal order in formal definition Quantifiers (for all, such that, at least) in formal definition Meanings of epsilon-delta in formal definition Transition from intuitive definition to formal definition
	Properties and foundations	Knowledge of infinity, infinitesimal approach
	Phenomenology and applications	Knowledge of derivative as limit of rate of change Knowledge of integral as limit of series of cumulative change Knowledge of application in real numbers Knowledge of limit as a part of iterative process Knowledge of behavior of function Knowledge of dynamic view of concept (approaching) Knowledge that the limit specifies a function
	Procedures (How, when, why something is done and characteristics of the result)	Knowledge of how and when to do something Knowledge of why to do something Knowledge of meanings of limit at infinity, infinite limit Knowledge of indeterminate-undefined forms (how and why)
	Registers of representation	Knowledge of graphical, tabular, geometric, number line, verbal and algebraic representations of limit Knowledge of ways to move between different forms of representations
Knowledge of Structure of Mathematics	Auxiliary connections	Using geometrical and algebraic concepts as auxiliary elements for the limit concept
	Transverse connections	Knowledge of the basis of the relation between the concepts of derivative and integral-the notion of infinity
Knowledge of Practices in Mathematics	Ways of validating and demonstrating	Demonstration (proving or validating) of mathematical procedures of limit concept by using the formal definition of limit Proving/validating of limit related theorems,
	Role of symbols and use of formal language	Using the symbols related to limit concept (ε, δ, $\exists$, $\forall$) in appropriate places and using formal language in teaching the concept
	Necessary and sufficient conditions for generating definitions	Necessary and sufficient conditions including necessary prerequisites such as temporal order for generating definitions.
Knowledge of Features of Learning Mathematics	Theories of mathematical learning	Knowledge of the concept-based theories (e.g., APOS theory, concept image-concept definition)
	Strengths and weaknesses in learning mathematics	Knowledge of conceptions-misconceptions of the concept of limit Being aware of teachers' own students' strengths and weaknesses during learning the concept
	Ways pupils interact with mathematical content	Being aware of the ways students follow when dealing with the concept and problems related to the concept (e.g., the fact that the student is looking for the limit of the function at a point, while trying to find the value of that point in the function)
Knowledge of Mathematics Teaching	Theories of mathematics teaching	Knowledge of the concept-based teaching theories (e.g., conceptual conflict)
	Teaching resources (physical and digital)	Knowledge of graphing calculator, GeoGebra for teaching the limit concept
	Strategies, techniques, tasks and examples	Knowledge of game-based activities, daily life example, animations, scenario supported by images, analogy, Escher's pictures, different fields of science, using paradoxes for teaching the concept of limit Knowledge of assessment strategies
Knowledge of Mathematics Learning Standards	Expected learning outcomes Expected level of conceptual or procedural development Sequencing of topics	Knowledge related to the place of the limit concept in the curriculum, the learning outcomes of the students for the learning of the limit concept, the development that the prospective teacher expects from the student at the end of the learning of the limit concept.

B. The Undergraduate Curriculum of The Secondary Teacher Education Program

SSME - MATHEMATICS EDUCATION

1st Year

First Semester			Second Semester		
MATH111	Fundamentals of Mathematics	(3-0) 3	MATH112	Introductory Discrete Mathematics	(3-0) 3
MATH115	Analytic Geometry	(3-0) 3	MATH116	Basic Algebraic Structures	(3-0) 3
MATH153	Calculus for Mathematics Students I	(4-2) 5	MATH154	Calculus for Mathematics students II	(4-2) 5
ENG101	English for Academic Purposes I	(4-0) 4	ENG102	English for Academic Purposes II	(4-0) 4
PHYS111	Physics I (Mechanics)	(4-2) 5	PHYS112	Physics II (Electricity and Magnetism)	(4-2) 5
			IS100	Introduction to Information Technology and Applications	NC

2nd Year

Third Semester			Fourth Semester		
EDS200	Introduction to Education	(3-0) 3	SSME 210	Theories and Approaches in Teaching and Learning of Science/Mathematics	(3-0) 3
MATH251	Advanced Calculus I	(4-0) 4	MATH252	Advanced Calculus II	(3-2) 4
MATH261	Linear Algebra I	(4-0) 4	MATH262	Linear Algebra 2	(4-0) 4
ENG211	Academic Oral Presentation	(3-0) 3	MATH254	Differential Equations	(4-0) 4
HIST2201	Principles of Kemal Atatürk I	NC	HIST2202	Principles of Kemal Atatürk II	NC
CENG230	Introduction to C Programming	(2-2) 3	Elective I*		(3-0) 3

3rd Year

Fifth Semester			Sixth Semester		
SSME301	Curriculum Development & Instruction in Science/Mathematics Education	(3-0) 3	SSME302	Measurement and Evaluation in Science/Mathematics Education	(3-0) 3
ENG311	Advanced Communication Skills	(3-0) 3	SSME310	Methods of Science/Mathematics Teaching I	(2-2) 3
TURK305	Oral Communication	(2-0) 2	MATH201	Elementary Geometry	(3-0) 3
Restricted Elective I**		(3-0) 3	TURK306	Written Expression	(2-0) 2
Restricted Elective II		(3-0) 3	Restricted Elective III		(3-0) 3
Departmental Elective I***		(3-0) 3			

4th Year

Seventh Semester			Eighth Semester		
SSME411	Methods of Science/Mathematics Teaching II	(2-2) 3	SSME446	Practice Teaching in Science / Mathematics Education	(2-6) 5
SSME417	Instructional Technology and Material Development	(2-2) 3	EDS424	Guidance	(3-0) 3
SSME433	School Experience in Science / Mathematics Education	(1-4) 3	Restricted Elective V		(3-0) 3
EDS304	Classroom Management	(3-0) 3	Departmental Elective II		(3-0) 3
Restricted Elective IV		(3-0) 3	Elective II		(3-0) 3

* Elective: Courses offered by any department.
** Restricted Elective: 3xxx and 4xxx levels mathematics courses except Math 223, Math 321, Math 387, Math 388, Math 395, Math 396, Math470, Math 486, Math373
*** Departmental Elective: Courses offered by Department of Secondary Science and Mathematics Education.

C. The Pre-interview Conducted before Lesson Study

1. Write the definition of limits? (I mean ε, δ definition)

   (If he/she doesn't remember the definition, give it.)

   Let f be a function defined on an open interval containing c (except possibly at c) and let L be a real number. The statement $\lim_{x \to c} f(x) = L$ means that

   $$\text{for each } \overbrace{\epsilon > 0}^{?}, \text{ there exists } \overbrace{\delta > 0}^{?} \text{ such that if}$$

   $$\overbrace{\underbrace{0 < |x - c| < \delta}_{\textbf{absolutevalue?}} \text{ then } \underbrace{|f(x) - L| < \epsilon}_{\textbf{absolutevalue?}}.}^{\textbf{Asymmetry:why?}}$$

   Figure 1. Students' Questions on ϵ-δ Definition. Unbolded print represents text given in book; bolded text represents students' questions.

   a. Write down 2 things about the Epsilon-Delta Definition of a Limit that you now understand that you didn't understand when you first saw it. What happened to you to understand later?
   b. When students questioned what ε and δ are, or, as one student put it, "Where do ε and δ come from?", how would you answer their question?
   c. which mathematical knowledge did you give the answer to this question? or ask, what knowledge / course did you help in answering this question?

2. Write two students' misconceptions and explain how you would deal with them.

3. "In the first half of the twentieth century, French mathematics texts used be notion of limit in an intuitive manner without a formal definition to introduce the definition of the derivative. Later in the same text a definition would be given which is more in the manner of an "explanation" in a note at the foot of the page."

   What do you think about this approach? How do you explain the relationship between limits and derivative?

4. What does $\lim_{x \to 1} \left(\frac{x^3 - 1}{x - 1}\right) = 3$ mean? Could you please explain it mathematically?

5. "Which of the following functions are continuous? If possible, give your reason for your answer?

$f_1(x) = x^2$

$f_2(x) = 1/x \ (x \neq 0)$

$f_3(x) = \begin{cases} 0 & (x \leqslant 0) \\ x & (x \geqslant 0) \end{cases}$

$f_4(x) = \begin{cases} 0 & (x \leqslant 0) \\ 1 & (x > 0) \end{cases}$

$f_5(x) = \begin{cases} 0 & (x \text{ rational}) \\ 1 & (x \text{ irrational}) \end{cases}$

6. Write down a definition of $\lim_{x \to a} f(x) = c$. What does it mean? How can you describe it to your students?
7. Is 0.999 . . . (nought point nine recurring) equal to one, or just less than one?
   a. If one of your students asks a question like this, how do you answer the question?

   7.In another way to ask this question;
   Imagine that you showed the sequence 0.9, 0.99, 0.999, 0.9999, … on the board and ask your students to make comments about this sequence. Which of the following answers is true of this sequence?
   i) It tends to $0.\overline{9}$
   ii) It tends to 1.
   iii) It approaches $0.\overline{9}$
   iv) It approaches 1
   v) It converges to 1
   vi) Its limit is $0.\overline{9}$
   vii) Its limit is 1

8. √2=1.4142135..., how can we locate its position on a number line? Please, demonstrate it by at least two ways.
9. Let $A \subseteq \mathbb{R}$ and $: A \to \mathbb{R}$. Give the definitions of continuity of f at at point $x_0 \in A$ and on A. How would you explain these concepts?
10. Write one students' misconception about the continuity and explain how you would deal with.

D. The Post-interview Conducted afte Lesson Study

POST INTERVIEW QUESTIONS

In this whole lesson study (lesson study), we became very involved with the limit. As I did at the beginning, I will ask you some questions. I would appreciate it a lot if you can answer my questions in detail.

1. How did the limit develop conceptually in the history of mathematics?
a. How do you reflect its historical development to teaching?
b. Which of these things-that you mentioned- did you learn through the work we did?

2. How can you define the limit?
a. You first explain the limit verbally in your own words and then write it down mathematically with explaining me (please think that I'm your students).
b. What do you think is the point or points that should be emphasized in the teaching of the definition of the limit?
c. What mathematical concepts must a high school student have learned to understand this definition? Why? Can you explain a bit?
d. How can you define the limit with topological concepts and metric space concepts?

3. What would you say about the relation of the limit with other mathematical concepts? Can you tell me the concepts associated with limit by telling me how they relate?

4. I will ask you about some concepts that you did and did not mention in the previous question. Can you explain how each concept below is related to the limit mathematically? Think yourself as a teacher in constructing a lesson plan.

Epsilon, delta (How do you explain the reason behind the temporal order in definition of limit to your students?) Absolute value-Inequalities Secant and Tangent Lines of a graph of a function and Derivative Limit at infinity and infinite limit Indeterminate -Undefined forms (The difference between them) Area calculation Irrational numbers Neighborhood Function Ball concept (in topological space)	Derivative (Why is derivative required for continuity)- (What do you think about the relation of limit and continuity with advanced mathematics?) Function types (special functions) L'hospital Rule

5. How would you answer the students' questions about the properties of the limit? What kind of teaching path do you follow on this subject?

6. Please think on your teaching and your colleagues teaching. How do students learn the concept? What did you notice in this process? How did you benefit this knowledge of students' learning? How did your expectation from students change?

6. What resources did you use for this lesson planning and teaching?
a. How did you benefit from the notes about the lessons you took from the mathematics department during the lesson study? Can you talk about the limit sources that the researcher did not lead you but that you used?

7. If you think about the lessons you taught twice, what would you like to change if you had the chance to teach them again? What kind of a teaching process would you prefer? What kind of assessment techniques would you prefer?

a. If you go back again, which lesson would you like to give more importance to in order to be able to teach "limit" better?

7. What were the positive and negative aspects of this study for you? How did you develop yourself, if you believe you have done so?
a. In conclusion, if you evaluate the course of this lesson study in terms of your mathematical knowledge and your pedagogy, can you explain how it contributed to you?

8. In this process, when you think about the feedbacks of your friends, both in our meetings at the lesson plan stage and after the lesson plan, which sources that you got feedback contributed to your development? Can you give a specific example?

9. If you were to evaluate this whole process, how did you see yourself in the beginning and how do you see yourself now?

10. Since I know that you will be pursuing a master's and doctorate, what else would you do for the teaching of the limit if you had done this study?
a. Were there any things you wanted to add or remove in this study? If so, can you explain?

E. The Sample Informed Consent Form

Şubat 2019

ARAŞTIRMAYA GÖNÜLLÜ KATILIM FORMU

Bu araştırma, ODTÜ araştırma görevlilerinden Rüya Savuran tarafından, Prof. Dr. Mine IŞIKSAL-BOSTAN danışmanlığında doktora tezi kapsamında yürütülmektedir. Bu form sizi araştırma koşulları hakkında bilgilendirmek için hazırlanmıştır.

Çalışmanın Amacı Nedir?

Araştırmanın amacı, ortaöğretim matematik öğretmen adaylarının kapsamlı alan bilgilerinin ders imecesi profesyonel gelişim modeli bağlamında incelemektir.

Bize Nasıl Yardımcı Olmanızı İsteyeceğiz?

Araştırmaya katılmayı kabul ederseniz, sizden her hafta yapılacak bir saatlik çalıştay tarzında geçecek bir toplantıya katılmanız ve bu katılımda bilgi birikimlerinizi paylaşmanız beklenmektedir. Toplantılar video kaydına alınacak olup, Bu toplantılarda, tüm katılımcılar ve araştırmacı ile birlikte limit kavramına yönelik bir ders planı dizgisi oluşturması hedeflenmektedir. Bu toplantılardan önce ve sonra sizinle yarı-yapılandırılmış görüşmeler yapılarak çalışmanın amacına yönelik sorular sorulacaktır.

Sizden Topladığımız Bilgileri Nasıl Kullanacağız?

Araştırmaya katılımınız tamamen gönüllülük temelinde olmalıdır. Çalışma boyunca sizden kimlik veya belirleyici herhangi bir bilgi istenmemektedir. Görüşmelerde verdiğiniz cevaplarınız tamamıyla gizli tutulacak, sadece araştırmacılar tarafından değerlendirilecektir. Katılımcılardan elde edilecek bilgiler toplu halde değerlendirilecek ve bilimsel yayımlarda kullanılacaktır. Sağladığınız veriler gönüllü katılım formlarında toplanan kimlik bilgileri ile eşleştirilmeyecektir.

Katılımınızla ilgili bilmeniz gerekenler:

Çalışma, genel olarak kişisel rahatsızlık verecek sorular içermemektedir. Ancak, katılım sırasında sorulardan ya da herhangi başka bir nedenden ötürü kendinizi rahatsız hissederseniz cevaplama işini yarıda bırakıp çıkmakta serbestsiniz. Böyle bir durumda, çalışmadan çıkmak istediğinizi söylemek yeterli olacaktır.

Araştırmayla ilgili daha fazla bilgi almak isterseniz:

Bu çalışmaya katıldığınız için şimdiden teşekkür ederiz. Çalışma hakkında daha fazla bilgi almak için araştırma görevlisi Rüya Savuran (E-posta: ruyasay@metu.edu.tr) ile iletişim kurabilirsiniz.

Yukarıdaki bilgileri okudum ve bu çalışmaya tamamen gönüllü olarak katılıyorum.
(Formu doldurup imzaladıktan sonra uygulayıcıya geri veriniz).

İsim Soyad   Tarih   İmza

---/----/-----

F. The Parent Approval Letter

Veli Onay Mektubu

Sayın Veliler, Sevgili Anne-Babalar,

Bu araştırma, ODTÜ araştırma görevlilerinden Rüya Savuran tarafından, Prof. Dr. Mine IŞIKSAL-BOSTAN danışmanlığında doktora tezi kapsamında yürütülmektedir. Bu form sizi araştırma koşulları hakkında bilgilendirmek için hazırlanmıştır.

Bu çalışmanın amacı nedir?: Araştırmanın amacı, ortaöğretim matematik öğretmen adaylarının kapsamlı alan bilgilerinin ders imecesi profesyonel gelişim modeli bağlamında incelemektir.

Sizin ve çocuğunuzun katılımcı olarak ne yapmasını istiyoruz?: Çalışmanın amacını gerçekleştirebilmek için ODTÜ matematik öğretmenliği 4. Sınıf öğrencileri çocuklarınızın sınıflarında hazırladıkları ders planlarını uygulayacaklardır. Bu uygulama çocuklarınızın matematik bilgisini arttırmaya ve daha ileri seviyede bir matematik dersi öğretimine yöneliktir. Bu uygulama esnasında öğretmen adaylarını sınıf ortamında video kaydına alınacaktır. Video kaydı sırasında çocuklarınızın kayda alınmamaya dikkat edilecek, herhangi bir şekilde kayda girmesi durumunda yüzleri kayıttan silinecektir. Sizden çocuğunuzun katılımcı olmasıyla ilgili izin istediğimiz gibi, çalışmaya başlamadan çocuğunuzdan da sözlü olarak katılımıyla ilgili rızası mutlaka alınacaktır.

Çocuğunuzdan alınan bilgiler ne amaçla ve nasıl kullanılacak?: Bu araştırmadaki ders kayıtları sadece bilimsel araştırma amacıyla kullanılacaktır. Çocuğunuzun ya da sizin ismi ve kimlik bilgileriniz, hiçbir şekilde kimseyle paylaşılmayacaktır. Araştırma sonuçlarının özeti tarafımızdan okula ulaştırılacaktır.

Çocuğunuz ya da siz çalışmayı yarıda kesmek isterseniz ne yapmalısınız?: Bu uygulamanın çocuğunuzun psikolojik gelişimine olumsuz etkisi olmayacağından emin olabilirsiniz. Yine de, bu formu imzaladıktan sonra hem siz hem de çocuğunuz katılımcılıktan ayrılma hakkına sahipsiniz. Katılım sırasında sorulan sorulardan ya da herhangi bir uygulama ile ilgili başka bir nedenden ötürü çocuğunuz kendisini rahatsız hissettiğini belirtirse, ya da kendi belirtmese de araştırmacı çocuğun rahatsız olduğunu öngörürse, çalışmaya uygulama tamamlanmadan ve derhal son verilecektir. Şayet siz çocuğunuzun rahatsız olduğunu hissederseniz, böyle bir durumda çalışmadan sorumlu kişiye çocuğunuzun çalışmadan ayrılmasını istediğinizi söylemeniz yeterli olacaktır.

Bu çalışmayla ilgili daha fazla bilgi almak isterseniz: Araştırmayla ilgili sorularınızı aşağıdaki e-posta adresini kullanarak bize yöneltebilirsiniz.

Saygılarımızla,

DANIŞMAN: Prof. Dr. Mine IŞIKSAL-BOSTAN ARAŞTIRMACI: Arş. Gör. Rüya SAVURAN

Matematik ve Fen Alanlar Eğitimi Bölümü
Orta Doğu Teknik Üniversitesi, Ankara
e-posta: ruyasay@metu.edu.tr / sayruya@gmail.com

Lütfen bu araştırmaya katılmak konusundaki tercihinizi aşağıdaki seçeneklerden size en uygun gelenin altına imzanızı atarak belirtiniz ve bu formu çocuğunuzla okula geri gönderiniz.

A) Bu araştırmaya tamamen gönüllü olarak katılıyorum ve çocuğum ....'nın da katılımcı olmasına izin veriyorum. Çalışmayı istediğim zaman yarıda kesip bırakabileceğimi biliyorum ve verdiğim bilgilerin bilimsel amaçlı olarak kullanılmasını kabul ediyorum.

Baba Adı-Soyadı................................... Anne Adı-Soyadı.......................................

İmza .... İmza ....

B) Bu çalışmaya katılmayı kabul etmiyorum ve çocuğumun ....'nın da katılımcı olmasına izin vermiyorum.

Baba Adı-Soyadı................................... Anne Adı-Soyadı.......................................

İmza .... İmza ....

G. The Approval of the Ethics Committee of METU Research Center for Applied Ethics

UYGULAMALI ETİK ARAŞTIRMA MERKEZİ
APPLIED ETHICS RESEARCH CENTER

ORTA DOĞU TEKNİK ÜNİVERSİTESİ
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Sayı: 28620816 / 72

20 Şubat 2019

Konu: Değerlendirme Sonucu

Gönderen: ODTÜ İnsan Araştırmaları Etik Kurulu (İAEK)

İlgi: İnsan Araştırmaları Etik Kurulu Başvurusu

Sayın Prof.Dr. Mine Işıksal BOSTAN

Danışmanlığını yaptığınız ***Rüya SAVURAN' ın*** **"Ders İmecesi Yoluyla Lise Matematik Öğretmen Adaylarının Kapsamlı Alan Bilgilerinin İncelenmesi"** başlıklı araştırması İnsan Araştırmaları Etik Kurulu tarafından uygun görülmüş ve **063-ODTÜ-2019** protokol numarası ile onaylanmıştır.

Saygılarımızla bilgilerinize sunarız.

Prof. Dr. Tülin GENÇÖZ
Başkan

Prof. Dr. Ayhan SOL
Üye

Prof. Dr. Ayhan Gürbüz DEMİR
Üye

Prof. Dr. Yaşar KONDAKÇI (4.)
Üye

Doç. Dr. Emre SELÇUK
Üye

Doç. Dr. Pınar KAYGAN
Üye

Dr. Öğr. Üyesi Ali Emre TURGUT
Üye

H. The Approval of the Ethics Committee of MoNE

T.C.
ANKARA VALİLİĞİ
Milli Eğitim Müdürlüğü

Sayı : 14588481-605.99-E.5164716 — 11.03.2019
Konu : Araştırma izni

ORTA DOĞU TEKNİK ÜNİVERSİTESİNE
(Öğrenci İşleri Daire Başkanlığı)

İlgi: a) MEB Yenilik ve Eğitim Teknolojileri Genel Müdürlüğünün 2017/25 nolu Genelgesi.
b) 06.03.2019 tarihli ve 101 sayılı yazınız.

Üniversiteniz Fen Bilimleri Enstitüsü Doktora Öğrencisi Rüya SAVURAN' ın **"Ders imecesi yoluyla lise matematik öğretmen adaylarının kapsamlı alan bilgilerinin incelenmesi"** konulu tezi kapsamında uygulama yapma talebi Müdürlüğümüzce uygun görülmüş ve uygulamanın yapılacağı İlçe Milli Eğitim Müdürlüğüne bilgi verilmiştir.

Uygulama formunun (4 sayfa) araştırmacı tarafından uygulama yapılacak sayıda çoğaltılması ve çalışmanın bitiminde bir örneğinin (cd ortamında) Müdürlüğümüz Strateji Geliştirme Şubesine gönderilmesini rica ederim.

Turan AKPINAR
Vali a.
Milli Eğitim Müdürü

Güvenli Elektronik İmzalı
Aslı İle Aynıdır.
(.(.....10.3./2019...

Adres: Emniyet Mah. Alparslan Türkeş Cad. 4/A Yenimahalle
Elektronik Ağ: ankara.meb.gov.tr
e-posta: istatistik06@meb.gov.tr

Bilgi için: Emine KONUK
Tel: 0 (312) 212 36 00
Faks: 0 ( ) ___

Bu evrak güvenli elektronik imza ile imzalanmıştır. https://evraksorgu.meb.gov.tr adresinden aee3-1c0b-3c0c-838b-e2c9 kodu ile teyit edilebilir.

I. The Lesson Plans produced through Lesson Study Process

Lesson Plan-1

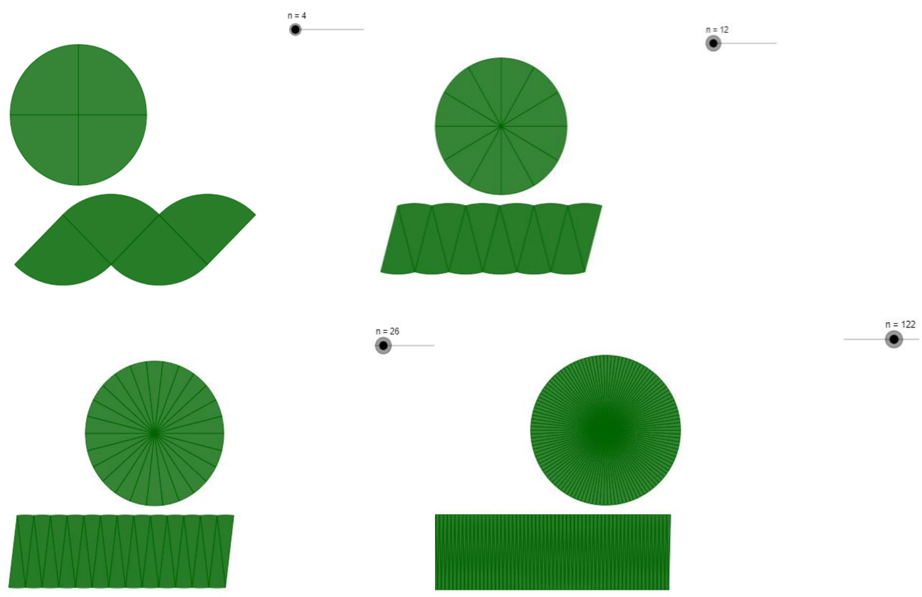
Öğretmenin İsmi-Soysimi	Lesson Study Group
Ders Konusu:	Limit Zamanlama: 90 dk
Ders İmecesi Amacı:	Öğrencilerin limitin formal tanımı olan epsilon-delta kavramını kullanarak, sağ-sol limiti kavramsal olarak anlamalarını sağlamak.
İlgili Kazanımlar:	12.5.1.1. Bir fonksiyonun bir noktadaki limiti, soldan limit ve sağdan limit kavramlarını açıklar. a) Limit kavramı bir bağımsız değişkenin verilen bir sayıya yaklaşmasından hareketle, tablo ve grafikler yardımıyla açıklanır. b) Bilgi ve iletişim teknolojilerinden yararlanılır. c) Cauchy'nin çalışmalarına yer verilir.
Önkoşul bilgi ve / veya beceriler:	Fonksiyon Mutlak değerli eşitsizlikler ↔ Uzaklık kavramı Aralık ↔ Komşuluk Sayılar → Devirli Ondalık Sayılar ↔ Sonsuzluk Geometri → Secant eğrisinin tanjanta yaklaşmasının limit almak olduğunu İrrasyonel Sayılar
İlişkili Kavramlar	Katı Cisimlerin hacimleri (Kürenin hacmi) Analitik Geometri → Eğim ↔ Türev Alan Hesabı ↔ İntegral Dizi-Seri İrrasyonel sayıların yerini belirleme↔Sonsuzluk İleri düzey olasılık (olasılık dağılımı) İstatistik↔ Merkezi Limit Teoremi
Materyaller:	GeoGebra, Aktivite kâğıdı, Excel **Matematiksel Görev / Aktivitenin Kısa Açıklaması:** Yaklaşım kavramının sezgisel olarak kavranması için daire etkinliği yaptırılır. Bununla Cauchy limit tanımı arasında bağlantı kurulur. Yaklaşım kavramı da sezdirilerek limitin sezgisel tanımı verilir. e sayısına yaklaşma etkinliği ile

	GeoGebra uygulaması kullanılarak sağdan-soldan yaklaşım-limit kavramı verilir. Günlük hayattan örnekler ve uygulamalar yapıldıktan sonra öğrenci seviyesine göre; delta ve epsilonun aralık (mesafe) belirttiğini belirtilerek limitin formal tanımı verilir. Öğrenci seviyesine göre, epsilon delta uygulaması yapılır.

<u>Giriş</u>

Etkinlik-1: Yaklaşım kavramını öğrencilerin sezgisel olarak kavraması (Daire etkinliği-Kepler yöntemi)

! Öğrencilere karton, makas, pergel, renkli kalemler, bant (yapıştırıcı) dağıtılır.



Kartondan yarı çap uzunlukları r birim olan 3 daire kesilir.

• Birinci daire, 4 eş dilime ayrılarak ikisi boyanması istenir.

• Bu dairenin dilimleri, yarısı üst, yarısı alt taban olacak şekilde yerleştirilir.

• Benzer şekilde diğer daireler sırasıyla, 8 ve 16 eş dilime ayrılarak yarısının boyanması istenir.

• Dilim sayısı arttıkça şeklin paralelkenarsal bölgeye dönüştüğü farkettirilir.

• Paralel kenarsal bölgenin alanından yararlanılarak dairenin alan bağıntısı bulunması istenir.

! Burada öğrencilerin oluşan şeklin aslında tam bir paralelkenarsal bölge olmadığını ve kenarların en fazla parçaya bölündüğünde bile düz bir doğru parçası olamayacağını söylediklerini düşünerek, bu işlemde limit alarak aslında hata payını minimum seviyeye indirildiği vurgulanır. Bunlar aşağıdaki GeoGebra aktivitesi üzerinde gösterilir.

https://www.geogebra.org/m/qq7pd4rx

Varılmak istenen noktada öğretmenin açıklaması: Kepler yöntemi kullanılarak üçgenlerin paralelkenarsal bölgeye yaklaştığı vurgulanır.

↓

Daire dilimlerini git gide küçülttüğümüzde paralelkenarsal bölgeye yaklaştığımızı gözlemledik. Aslında limit de benzer bir açıklamaya sahiptir. Bu açıklamayı Cauchy yapmıştır ve Cauchy'nin limit tanımı şu şekildedir:

Limitin Tarihsel Gelişimindeki Cauchy Tanımı

Bir değişken sabit bir değere peş peşe sonsuz hamle ile yeterince yaklaştığında (aralarındaki uzaklık istenildiği kadar küçük olduğunda) bu değere diğerlerinin limiti denir."

- Bu yapılan işlemin aslında matematik tarihi ve sonsuz küçük hesabı ile Cauchy'nin çalışmaları ile bağlantısı üzerinde tartışılır.

  ! Bu tanım ile Kepler aktivitesi arasında öğrencinin bağlantı kurması sağlanır.
  ? Bu tanım ile yaptığımız etkinlik arasında nasıl bir bağlantı kurabiliriz?

Beklenen cevap: Biz daire dilimlerini git gide küçülterek paralelkenarsal bölgeye yaklaştık.

Orta

Etkinlik-2: Yaklaşım kavramının uygulamasının yapılması (Bir bağımsız değişkenin verilen bir sayıya yaklaşmasını örneklerle açıklar.)

$f: R \rightarrow R\ f(x) = x^2 + 1$ fonksiyonu veriliyor. x=2.5 olmak üzere; f(x) değerini nasıl buluruz?

Yönlendirici Adımlar

- f(x) Değerinin yaklaşık değerini bulmak için aşağıdaki tabloyu doldurunuz.

2.43	2.47	2.48	2.49	2.499	x	2.501	2.51	2.52	2.53	[illegible]
6.9049	7.1009	7.1504	7.2001	7.245001	f(x)	7.255001	7.3001	7.3504	7.4009	[illegible]

- x değerlerini 2.5 sayısına yaklaştırdıkça f(x) fonksiyonu nasıl davranmaktadır? f(x) için yaklaşık bir değer belirtebilir misiniz?
- Yaklaşık bir $f(x)$ değeri bulmak için ne yaptınız?

! Limit değerinin fonksiyonda yerine konularak elde edilmediği vurgulanır.

! f(x) için net bir değer belirlenemediği, yaklaşık değerler ile belirlenebilir. x değerleri x sayısına yaklaştıkça fonksiyon değeri de 2.499 ile 2.511 arasında değişmektedir. f(x) yaklaşık olarak 7.24 ile 7.30 civarındadır.

! Yapılan tablodan genelleme yapılması istenir.

! Öğrencinin artan ve azalan değerlerden yaklaşıldığını vurgulaması beklenir.

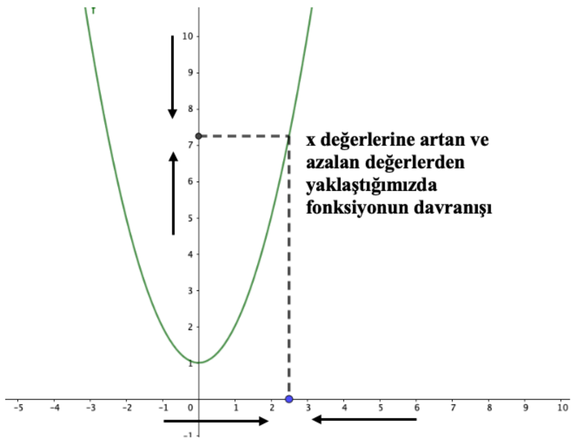
! Sağdan-soldan limitin gösterimi vurgulanır.

Yönlendirici Adımlar

- $f(x)$ değerine yakın bir değer bulabilmek için yani $f(x)$ değerine yaklaşmak için ne yapmıştık?

Beklenen cevap: x sayısından büyük ve küçük olmak üzere x yerine değerler vererek $f(x)$ değerlerini bulduk. Böylece $f(x)$ için yaklaşık bir değer bulduk.

! Hem büyük hem küçük değerler verildiğinin altı çizilerek grafik üzerinden bu yaklaşım gösterilir.



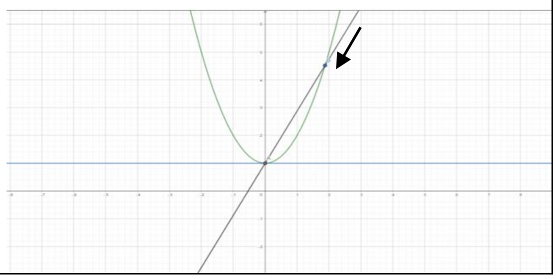
x değerlerine artan ve azalan değerlerden yaklaşım

! x’in 2.5 değerine yaklaşmasının tek yönlü bir yaklaşım olmadığı belirtilir.

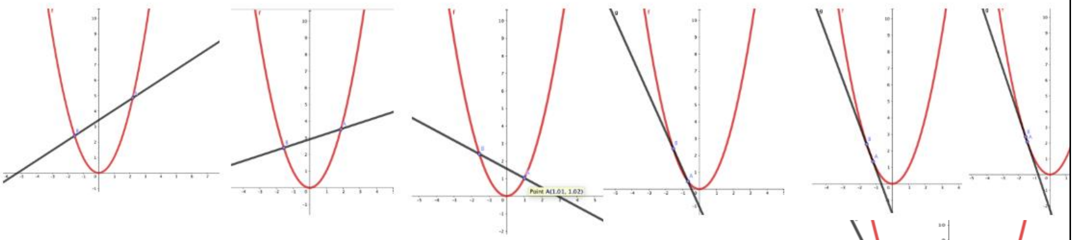
! x’in 2.5’den büyük, küçük veya 2.5’ye eşit değerler alabileceği belirtilir.

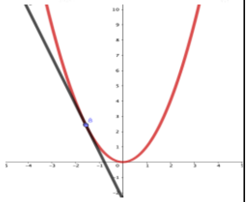
! Sağdan-soldan limitin gösterimi vurgulanır.

Etkinlik-3: Biraz önce söylediğimiz gibi, limit, türev ve integral kavramlarından çok daha sonra ve onları açıklamak için sonsuz küçük kavramından yola çıkılarak ortaya çıkmıştır. Peki, türev ile limitin bağlantısı nasıl ortaya konmuştur?



 Bir parabolü iki noktada kesen doğrunun parabol üzerinde kestiği noktalar bir diğerine ya da birbirlerine yaklaştıklarında doğrunun son konumu için ne söylersiniz?





Eğriyi kesen doğrunun her iki noktası birbirine yaklaştığında bu doğru eğriye teğet olur. Bu limitin tarihinde, sekant doğrusunun tanjant doğrusuna yaklaşması anlamına gelir.

Sağdan-Soldan (Noktaya artan ve azalan değerlerden yaklaşarak) Limit Kavramları

$A \subseteq \mathbb{R}$, ve $f: A \to \mathbb{R}$ olmak üzere,

f fonksiyonunun a noktasındaki davranışı incelenirken, x değişkeni a ‘ya a’dan küçük değerlerle yaklaşıyorsa, bu tür yaklaşmaya soldan yaklaşma denir. x değişkeni a’ya a’dan büyük değerlerle yaklaşıyorsa bu tür yaklaşmaya sağdan yaklaşma denir. a noktasına artan ya da azalan değerlerden yaklaşarak aradığımız limite yaklaşma ile aynı isimle f fonksiyonun sağdan/soldan limiti denir.

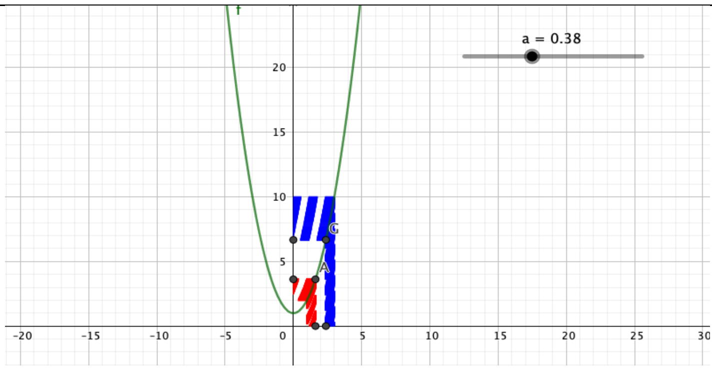
Uygulama-1: Sağdan-soldan limit kavramı uygulamaları (Uygulama-1)

- Limitin sezgisel tanımı verilir.

Sezgisel Limit Tanımı: A ve B ve bir fonksiyon olmak üzere, olmak üzere herhangi bir noktası noktasına (a) sağdan ve soldan yaklaşırken fonksiyonu bir L sayısına yaklaşıyorsa fonksiyonunun bu noktasında limiti vardır ve $\lim_{x \to a} f(x) = L$ ile gösterilir.

Öğretmen Açıklaması: Öğretmen herhangi bir f fonksiyonu üzerinden tanımdaki kavramları pekiştirir.

! Öğrencinin GeoGebra kullanılarak bir noktadaki limiti ile soldan, sağdan limitleri arasındaki ilişkiyi belirtmesi beklenir.



Uygulama-2: Limitin disiplinler arası bağlantısında hazırlanmış sorular (Ek-xx)

Etkinlik-3: Bir fonksiyonun bir noktadaki limiti, soldan limiti ve sağdan limiti kavramlarını örneklerle açıklar ve bir noktadaki limiti ile soldan, sağdan limitleri arasındaki ilişkiyi belirtir. ($\varepsilon - \delta$ (gearing up) tanımına değinilir)

Gearing Up Activity: Bir öğretmen ses duyuyor sınıfında, sesin kaynağının orta sıra olduğunu anlıyor (ilk epsilonumuz) ve sesin kaynağını bulmak için öğretmen sıraya yaklaşıyor (delta yaklaşımı). Öğretmen sıraya yaklaştığında sesin ön ve arka sıradaki öğrenciler olmadığını anlıyor ve onları eliyor (yeni epsilon) ve öğretmen biraz daha yaklaşıyor buna karşılık (yeni epsilona karşı yeni bir delta)

Limitin yukarıda yaklaşmak olduğundan bahsetmiştik, peki ne kadar yaklaştık? Ya da bizim istenen sonucu elde etmek için ne kadar yaklaşmamız gerekir? Aşağıdaki örnekte bakalım.

$$f(x) = \begin{cases} 2x - 1; x \neq 3 \\ 6; x = 3 \end{cases}$$

Sezgisel olarak, 3'e eşit olmayacak şekilde x 3'e yaklaştığında f(x) fonksiyonu da 5'e yaklaşır. O zaman limit x3'e giderken f (x) eşittir 5.

Bu sözel yaklaşımdan biraz daha matematiksel ifadeye geçersek,

x 3'e ne kadar yakın olursa, f(x) 5'e olan uzaklığı 0.1'den daha az olacak şekilde yaklaşır?

! Öğrencilerin bunu çizerek gösterecekleri varsayılır.

! Öğrencilerin fonksiyonun grafiğinden mutlak değere geçiş yapmaları beklenir. Bu aşamada öğretmenin yönlendirme adımlarını uygulaması gerekir.

! 0.1 seçilmesinin sebebi ilk aşamada hata payının 0.1 olduğu vurgulanmalıdır.

! Verilen örnek ile ses örneği arasında bağlantı kurulur. (Sese yaklaştığımızda mutlak değerin alabileceği değeri-epsilonu küçültüyoruz.-) Yani ses yaklaştığımızda uzaklığı azaltıyoruz.

Beklenen cevap: x'in alabileceği değerlerin 3'e olan uzaklığı $|x - 3|$ ve f(x)'in alabileceği değerlerin 5'e olan uzaklığı da $|f(x) - 5|$ olur.

Matematiksel (sembolik) olarak beklenen cevap:

$|f(x) - 5| < 0.1$ ise $|x - 3| < \delta$ama x≠ 3olmalıdır.

$|f(x) - 5| < 0.1$ ise $0 < |x - 3| < \delta$olmalıdır. (Öğrencilerin neden 0'a eşit olmayacağını vurgulaması beklenir).

Öğretmen açıklaması: Düşündüğümüz 0,1, 0,01 ve 0,001 sayıları, fonksiyonun davranışını gözlemlerken göze aldığımız hata paylarıdır. 5, x'in 3'e yaklaştıkça f (x) 'in limiti olması için, f (x) ile 5 arasındaki farkın sadece bu üç sayı ile sınırlandırılamaz. Herhangi bir pozitif sayı için ε (yunanca epsilon harfi) yazarsak,

...denklem…(1)

denklemi ile δ(Yunanca delta) değerini buluruz.

Bu, x'in 3'e yaklaştığında f (x) fonksiyonun davranışının 5'e yakın olduğunu söylemenin kesin bir yoludur, çünkü (1) ifadesi bize 5'ten ε mesafesi ile f(x) değerlerini 3 noktasından $\varepsilon/2$ mesafesinden yaklaşarak bulunduğunu söyler.

Genel olarak; limitin formal tanımı şu şekildedir:

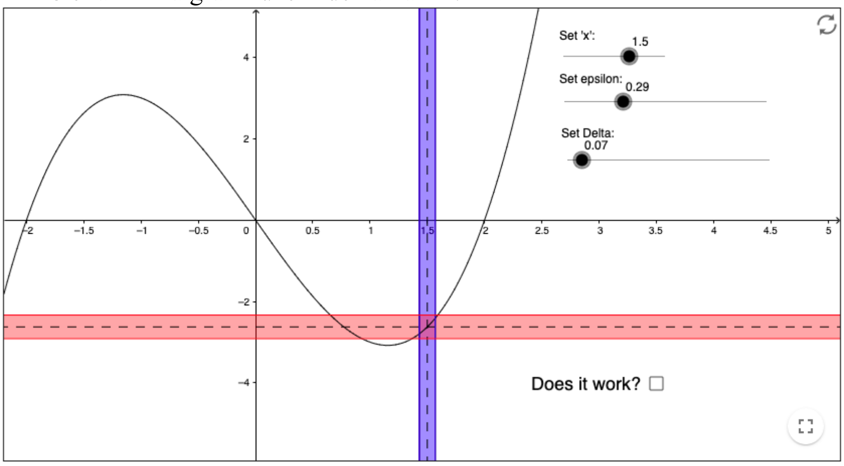
Limitin Formal Tanımı

$A \subseteq \mathbb{R}$bir aralıkolmak üzere, $f: A \to \mathbb{R}$ bir fonksiyon olsun.

$c \in \mathbb{R}$ olmak üzere, eğer her ε reel sayısına karşılık $|f(x) - L| < \varepsilon$ eşitsizliğini sağlayan x reel sayıları için $0 < |x - c| < \delta$ olacak şekilde bir δ reel sayısı varsa o zaman L sayısına f fonksiyonunun c noktasındaki limiti denir.

! Neden her epsilon diyoruz? → Öğretmen açıklaması: Fonksiyonun herhangi bir x değerinde nasıl davrandığını bilmiyoruz ama elimizdeki x değerini biliyoruz. O belirli bir sayı. bu sebeple de bilinmeyen olan fonksiyon davranışı için bir değer bulmalıyız. Bu sebeple her epsilon için en az bir delta bulmaya çalışıyoruz.

! ε-δ tanımına grafik üzerinden anlatılır.



! Her seferinde yeni bir epsilon ile yaklaştık ve bu epsilonları genelleyebilmeliyiz ki buna karşılık bir delta bulalım. Elimizdeki bir L değerinden epsilon kadar uzaklıktaki mesafe aldığımızda fonksiyonun davranışı olmuş oluyor ya bu

Fonksiyonumuzun davranışına karşılık gelen bir aralık bulmuş olduk epsilonla, eğer bu aralığa düşen delta bulabiliyorsak limitimizin bu aralıkta var olduğunu kanıtlamış oluruz. Deltaları epsilonla genellediğimizde her epsilona karşılık bir delta bulmuş oluruz.

Soru: her epsilon için bir delta ifadesi ile fonksiyonun tanımı arasında sizce nasıl bir benzerlik ya da farklılık var?

Delta, epsilon cinsinden bir fonksiyon olacağı için aynı fonksiyondaki gibi bütün epislonlar için bir delta değeri bulmalıyız. Böylelikle aynı fonksiyonlardaki gibi tanım kümesindeki tüm elemanları değer kümesindeki bir elemanla eşlemiş olacağız. Bu da bize aslında limitinde bir fonksiyon olduğunu gösterir.

Uygulama-3: $f(x) = \{2x \quad ,x \neq 5\ x \quad ,x = 5$ fonksiyonun limitinin $f(x) = 10$ olduğunu limitin formal tanımını kullanarak gösteriniz.

! Öğrencinin epsilon-delta tanımını uygulaması beklenir.

! Öğrencinin ne öğrenmesi bekleniyor? Burada öğrencinin epsilon cinsinden deltayı elde edeceği bir fonksiyon bulmasını istiyoruz.

$$|x - 5| < \delta$$

$$2|x - 5| < 2\delta$$

$$|2x - 10| < 2\delta$$

$$|f(x) - L| < 2\delta$$

$$2\delta = \epsilon$$

$$\delta = \frac{\epsilon}{2} \qquad |f(x) - L| < 2\delta = 2\frac{\epsilon}{2} = \epsilon$$

Bitiş: Limitin formal tanımından yola çıkarak, limit kavramı hakkında ne söylersiniz ve kendi kelimelerinizle nasıl tanımlarsınız diye sorularak öğrencilerin ne anlayıp anlamadığı sorgulanır.

Değerlendirme Kağıdı: Öğrencilerin öğrenip öğrenmediğini değerlendirecek şekilde öğrencilere exit card verilir ve bu derste ne öğrendiklerini yazılı olarak vermeleri istenir.

EKLER-

UYGULAMA 1

Aşağıdaki soruları dikkatli okuyunuz. Cevapları boş bırakılan kısma açıklayıcı bir şekilde yazınız.

1. Aşağıdaki tabloda verilen değerleri hesap makinesi kullanarak istenen limit değerlerini hesaplayınız.

a)

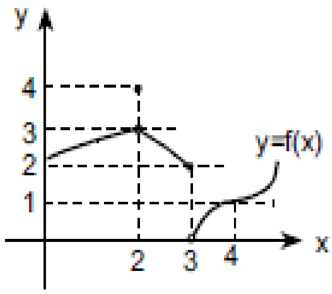
			→0←		
$f(x) = \frac{x+3}{2x-1}$			**→f(0)←**		

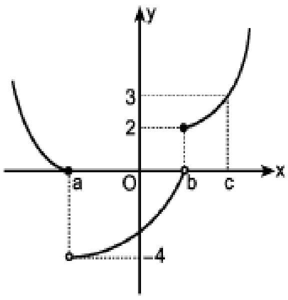
			→5←		
$f(x) = \frac{x^2+5}{x-5}$			→f(5)←		

i) Yaptığınız işlemi matematiksel olarak gösteriniz.
ii) Yaptığınız işlemin matematiksel olarak ne anlama geldiğini yazınız.

2. Yanda verilen grafiğe göre aşağıda istenenleri bulunuz.

$\lim_{x\to 3^+} f(x)$ $\lim_{x\to 2^-} f(x)$	$\lim_{x\to 4^+} f(x)$ $\lim_{x\to 3^-} f(x)$





3. Yanda f fonksiyonun grafiği verilmiştir.

Buna göre $\lim_{x\to a^+} f(x) + \lim_{x\to b^-} f(x) + \lim_{x\to c^+} f(x)$ toplamı kaçtır?

4. $f(x) = \{\frac{|x|}{x} \quad , x \neq 0 \; 3 \qquad , x = 0$

Fonksiyonu için; $f(x) \; = a \; f(x) \; = b$

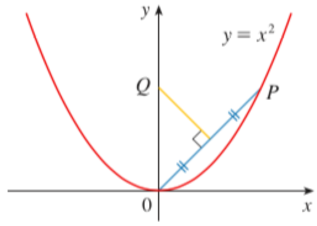
olduğuna göre $a.b$ kaçtır?

UYGULAMA-2

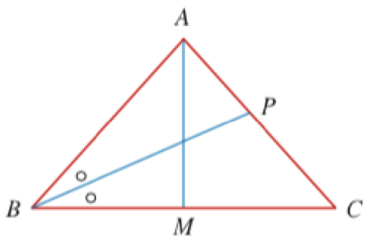
Aşağıdaki soruları dikkatli okuyunuz. Cevapları boş bırakılan kısma açıklayıcı bir şekilde yazınız.

1) Bir borudan akan bir sıvı molekülünün hızı, molekülün borunun merkezinden uzaklığına bağlıdır. Bir molekülün saniyedeki hızı inç cinsinden, r molekülün borunun merkezine olan uzaklığı, R borunun inç cinsinden yarıçapı ve k bir sabit olmak üzere $v(r) = k(R^2 - r^2)$ fonksiyonu ile bulunur. k=0.65 ve R=0.5 olduğunda; molekülün borunun duvarına oldukça yakın olduğunda molekülün limit hızı ne olur?

2) Yandaki şekil parabol üzerindeki P noktasını ve OP' nin orta dikmesinin y eksenini kestiği Q noktasını gösterir. P, parabol boyunca orijine yaklaştığında, Q noktası hakkında ne söylersiniz? Sizce, burada bir limit işlemi var mıdır? Eğer varsa, bunu gösteriniz.



3)



Yandaki şekil ∠B ve ∠C açıları eşit olan ikizkenar ABC üçgenini göstermektedir. B açısının açıortayı, AC kenarını P noktasından kesmektedir. BC tabanının sabit kaldığını, ancak üçgenin $|AM|$ yüksekliğinin 0'a yaklaştığını varsayalım, bu durumda A, BC kenarına ait M orta noktasına yaklaşır. Bu süreçte P'ye ne olur? Sizce, burada bir limit işlemi var mıdır? Eğer varsa, bunu gösteriniz.

Lesson Plan-2

Öğretmenin İsmi-Soysimi	Lesson Study Group
Ders Konusu:	Limit / Zamanlama: 90 dk
Ders İmecesi Amacı:	Bu ders sonunda, öğrencilerin limitin özelliklerini farklı alanlardan örneklerle grafikler yardımıyla öğrenmesi ve uygulamaları yapması beklenir. ✔ Sonsuzluk kavramı ✔ Belirsizlik durumları ✔ Belirsizlik ve tanımsızlık arasındaki fark
İlgili Kazanımlar:	12.5.1.2. Limit ile ilgili özellikleri belirterek uygulamalar yapar. a) Polinom, üstel, logaritmik ve trigonometrik fonksiyonlar içeren limit uygulamaları yapılır ancak sonucu $\mp\infty$ olan limit durumlarına girilmez. b) Sadece pay, paydası çarpanlarına ayrılarak belirsizliği kaldırılabileceği limit örnekleri verilir.

Önkoşul bilgi ve / veya beceriler:	❖ Polinom, üstel, logaritmik ve trigonometrik fonksiyonların yapısı bilinir. ❖ Polinom, üstel, logaritmik ve trigonometrik fonksiyonlarla işlem yapılır. ❖ Çarpanlara ayırma yöntemi bilinir. ❖ Polinom, üstel, logaritmik ve trigonometrik fonksiyonların türevini alma bilinir. ❖ Limit kavramı bilinir. ❖ Grafiği verilen fonksiyonun istenen noktadaki limitini bulabilir. ❖ Sonsuz kavramı bilinir.
İlişkili Kavramlar	Katı Cisimlerin hacimleri (Kürenin hacmi) Analitik Geometri → Eğim ↔ Türev Alan Hesabı ↔ İntegral Dizi-Seri İrrasyonel sayıların yerini belirleme↔Sonsuzluk İleri düzey olasılık (olasılık dağılımı)- İstatistik↔ Merkezi Limit Teorem
Materyaller:	GeoGebra, Aktivite kâğıdı, Excel Matematiksel Görev / Aktivitenin Kısa Açıklaması: Bir önceki dersin tekrarı ile derse başlanır. Öğrencilere limite yönelik işlemler, limitin farklı alanlarda uygulaması ile gösterilir. Limitin özellikleri anlatılarak, uygulamalar yaptırılır.

Etkinlikler (Ders Sürecinin Açıklanması)

Giriş

! Bir önceki dersin tekrarına yönelik açıklamalar yapılır.
Yönlendirici Soru: Bir önceki derste limiti nasıl tanımlamıştınız?

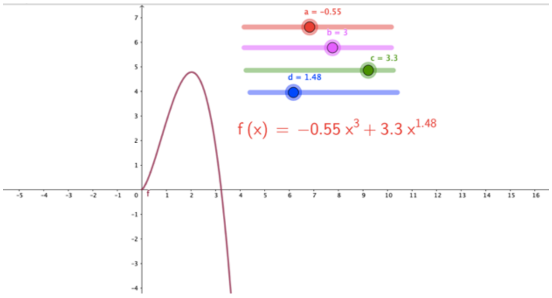
Beklenen cevap: Yaklaşım ve fonksiyonun o noktadaki davranışı olduğunu vurgulaması beklenir.

Orta

! Limitin özellikleri hatırlatılır (Aşağıda gösterilmiştir). Bu özelliklerin nedenleri sorgulatılır ve öğrencilerden cevaplar beklenir. Örnek soru: Sabit sayı (c) ile çarpma işlemine giren bir fonksiyonun limiti, fonksiyonunun kendisinin neden c katı olur? Sezgisel ya da formal olarak gösteriniz.

! Aşağıda özellikleri verilen özel fonksiyonların özellikleri hatırlatılarak, bu fonksiyonların davranışları üzerinde durularak uygulamalara geçilir.

Polinom Fonksiyonların Limiti

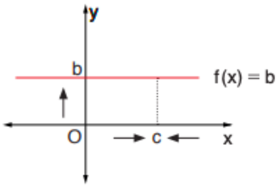


P (x) polinom fonksiyonu x=c noktasındaki limiti P (c) olur.

! Nedeni öğrencilerle birlikte bir tartışma ortamı yaratılarak sorgulanır.

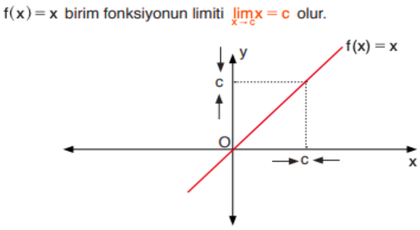
Sabit bir Gerçek sayının Limiti

Sabit bir gerçek sayının (f(x)=c)limiti yine kendisidir.



Birim fonksiyonun Limiti

F(x)=x birim fonksiyonunun c noktasındaki limiti kendisi olur.



- ! Polinom fonksiyonunun davranışı ile ilişkisi gösterilir.
- ! Uygulamalarda sadece x, y, x gibi bilinen değişken isimlerinin yanı sıra h, t, a… gibi değişkenlere de yer verilir.

İki Fonksiyonun Toplamının, Farkının, Çarpımının ve Bölümünün Limiti

$A \subseteq R, c \in R$ ve $f, g\colon A \to R$ fonksiyonları c noktasında limitlidir. Buna göre,

$$\lim_{x \to c}(f(x)+g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$$

$$\lim_{x \to c}(f(x)-g(x)) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$$

$$\lim_{x \to c}(f(x) \cdot g(x)) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$$

$$\lim_{x \to c}(b \cdot f(x)) = b \cdot \lim_{x \to c} f(x)$$

$$\lim_{x \to c}(f(x))^n = \left(\lim_{x \to c} f(x)\right)^n \text{ olur.}$$

$n \in Z$

$$g(x) \neq 0 \text{ ve } \lim_{x \to c} g(x) \neq 0 \text{ olmak üzere } \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} \text{ olur.}$$

Uygulama-1:

! $\lim_{t\to 1} e^{i\pi} + 1$ limitini bulunuz.

Euler eşitliği: Bu denklemin matematik bilen herkes için dayanılmaz bir cazibesi vardır. Çünkü o, matematiğin beş önemli nesnesini içerir: 0, 1, e, i , π. Minimal tamlık ilkesine uyar, çünkü içinde gereksiz hiçbir şey yoktur. Maksimal yarar ilkesine uyar, çünkü bu basit bağıntı bir çok yerde kullanılabilir. Bu yalın formül, içerdiği zengin ve yararlı anlam yanında, uygarlıklarımızın yarattığı altı önemli nesneyi içeriyor ve onlar arasında bağ kuruyor. Matematiği bir dil olarak görürsek, hiç bir şair, bir dilin altı sözcüğünü bu kadar yalın, bu kadar anlamlı, bu kadar genel, bu kadar yararlı biçimde bir araya getirememiştir. İşte matematiksel zarafet budur.

! $\lim_{h\to 0} \frac{h^2+5h}{h}$ limitinin değerini hesaplayınız.

a. Yaptığınız işlemi kısaca açıklayınız.

! $\frac{P(x)}{Q(x)}$, Q(a)= 0 ve $\frac{P(x)}{Q(x)} = H(x)$ olduğu durumda (x) ‘in $\frac{P(x)}{Q(x)}$ ‘e eşit olduğu: her iki fonksiyonun a noktası komşuluğundaki davranışların eşit olduğu vurgulanarak anlatılır. GeoGebra üzerinden uygulama gösterilerek, kök kaybının söz konusu olmadığından bahsedilir.

! $\lim_{x\to 1} \frac{(x^3-1)}{x-1}$ limitini hesaplayınız.

a. Yaptığınız işlemi kısaca açıklayınız.

! $f(t) + f(t-1) = 3$ olduğuna göre, $(f(2t-1))$ limitinin değerini bulunuz.

! Bir bisiklet üreticisi, aylık bir bisiklet üretme maliyetinin işlev tarafından verildiğini öğrenir:

$$C(x) = \frac{200}{x^{2.5}} + 25$$

Üretici bu ay 50 bisiklet alırsa maliyeti ne olur?

! $\lim_{x\to 4}(x^2 + 4)(x-3)$ işlemini yapınız.

! Bir nükleer bilim uzmanı bir deney üzerinde çalışmaktadır. Bir radyo aktif maddenin molekül sayısını temsil eden bir f (t) aşağıda gösterildiği gibi bulmuştur:

$$f(t) = \frac{at^2 - b}{t-2}$$

Burada, t, reaksiyonun başlamasından bu yana geçen süreyi dakika cinsinden temsil eder. Laboratuvarda notlarını kaybeden bilim uzmanı, a ve b’nin değerini bilmemektedir.

Ancak, reaksiyonun başlamasından 2 dakika sonra radyo aktif maddenin molekül sayısının 4'e yaklaştığını hatırlamaktadır. Bu bilgiler ışığında, a ve b değerlerini bulunuz.

! $\lim_{x \to 1} \frac{x^2-x+1}{x-1}$ işlemini yapınız .

<u>Yönlendirici soru</u>: Limit işleminde $\frac{1}{0}$' a neden sonsuz diyoruz? Bunu hiç düşündünüz mü?

! Bu işlemin sonucunun $\frac{1}{0}$ çıkacağı ve bu sonucun ∞ olduğundan bahsedilir. Bu işleminin sonucunun neden ∞ olduğu önce öğrencilere tartışma ortamı yaratılarak sorgulatılır.

! Tanımsız ve belirsiz arasındaki fark vurgulanır;

Öğretmen Açıklaması: 2+2=4 diyebiliriz lakin 2+yeşil diyemeyiz çünkü 2+yeşil toplama işlemine göre tanımlanmamıştır bu yüzden tanımsızdır.

Öğretmeninin varmak istediği nokta: Limit işleminde $\frac{1}{0} = \infty$ işleminin öğrencilere gösterimi

x > 0 olmak üzere;

$$\frac{1}{x} < \frac{1}{\left(\frac{x}{2}\right)} < \frac{1}{\left(\frac{x}{4}\right)} < \frac{1}{\left(\frac{x}{8}\right)} < \frac{1}{\left(\frac{x}{16}\right)} < \frac{1}{\left(\frac{x}{32}\right)} < \cdots < \infty$$

! Sonsuzluk kavramının öğrencinin kafasındaki anlamı sorgulatılır:

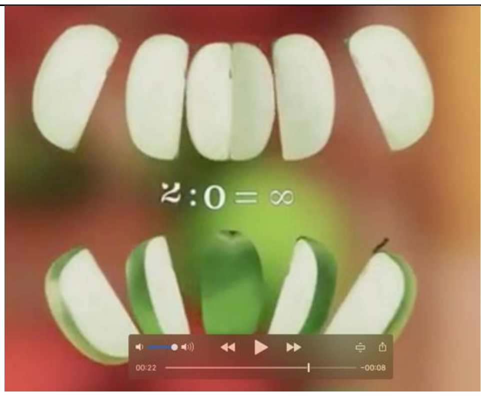
10) $\lim_{x \to \infty} \frac{1}{x^2+x+1}$ şeklinde verilen limiti işlemini yorumlayınız.

- **Yönlendirici soru**: Bu işlemin sonucunun ne olduğunu düşünüyorsunuz?

<u>Yönlendirici soru</u>: Bu işlemin sonucunun neden 0 olduğunu açıklar mısınız?

! Bu işlemin sonucunun $\frac{1}{\infty}$ çıkacağı ve bu sonucun 0 olduğundan bahsedilir. Bu işleminin sonucunun neden 0 olduğu önce öğrencilere tartışma ortamı yaratılarak sorgulatılır.

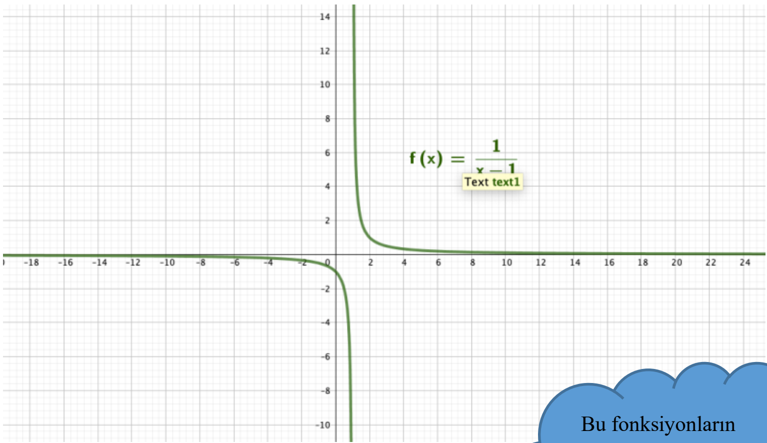
Öğretmenin varmak istediği nokta: $\frac{1}{\infty} = 0$ yani bir bütünü sonsuz kez parçalara ayırdığımızda elimizde kalan parçalar oldukça küçülerek; sıfıra yaklaşırız.



Asimptot:

! Belirli bir f fonksiyonun ait olan grafikte bir noktaya sağdan ve soldan yaklaşırken fonksiyonun sonsuz artan ya da sonsuz azalan bir davranış sergilemesi, fonksiyonun o noktada bir dikey asimptota sahip olduğunu söyler.

! Belirli bir f fonksiyonun ait olan grafiğe sonsuz artan ya da sonsuz azalan değerlerden yaklaştığımızda, fonksiyonun davranışının herhangi bir a sayısına yaklaşması, fonksiyonun bir yatay asimptota sahip olduğunu gösterir.



Bu fonksiyonların davranışları üzerinde durulur.

Köklü ifadelerin Limiti

$f: R \to R, c \in R, n \in Z^+$ ve n çift sayı ise f(x)≥ 0 olmak üzere $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$ olur.

! Neden f(x)≥ 0 olması gerektiği vurgulanır.

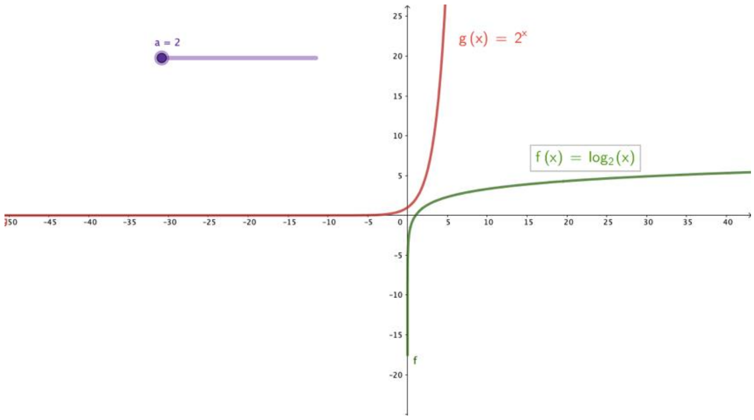
Üslü ve Logaritmik İfadelerin limiti

$f: R \to R,\ a \in R^{+}$ ve $a \neq 1$ olmak üzere $a^{f(x)}$ fonksiyonun x=c noktasındaki limiti

$\lim_{x \to c} a^{f(x)} = a^{\lim_{x \to c} f(x)}$

$f: R \to R^{+},\ a \in R^{+}$, a≠ 1 ve f(x)>0 olmak üzere $\log_a f(x)$ fonksiyonunun x=c noktasındaki limiti

$\lim_{x \to c} \log_a f(x) = \log_a \lim_{x \to c} f(x)$



- ! Bu fonksiyonların davranışları üzerinde durulur.
- ! Üstel fonksiyonda tabanın neden her zaman pozitifi olacağı örnekler verilerek üstünde durulur. (Üstün ½ olma durumunda kökün içinin negatif olamayacağı köklü fonksiyonlara değinilerek vurgulanır.)
- ! Üstel fonksiyonun logaritmik fonksiyonun tersi olduğunu, bu fonksiyonların davranışlarının y=x fonksiyonuna göre simetrik olduğu vurgulanır. Bu durumda logaritmik fonksiyonun tabanının her zaman büyük-eşit 1 olması gerektiği gösterilir.

! Günlük hayat örnekleri verilir.

Uygulama-2:

- ! 2^{lnx} limitini bulunuz.
- ! f(x)=ln($\frac{e}{x^2+10x+25)}$) ve g(x)= ln (x+5) olmak üzere, $\lim_{x \to e}(f(x) + 2g(x))$ limitini bulunuz.
- ! $\lim_{x \to 2\sqrt{2}} \log_3(x^2 + 1)$ ifadesinin değeri kaçtır?
- ! $f(x) = 4^{x-1}$

  $g(x) = 2^{x+1}$ olmak üzere $\frac{\lim_{x \to 3} f(x)}{\lim_{x \to -2} g(x)}$ ifadesinin değeri kaçtır?
- ! $\lim_{t \to 2} \sqrt{t\sqrt{t\sqrt{t\sqrt{2t}}}}$ limitinin değerini bulunuz.

<u>Trigonometrik fonskiyonların limiti</u>

$A \subseteq R, c \in R$ ve $f: A \to R$

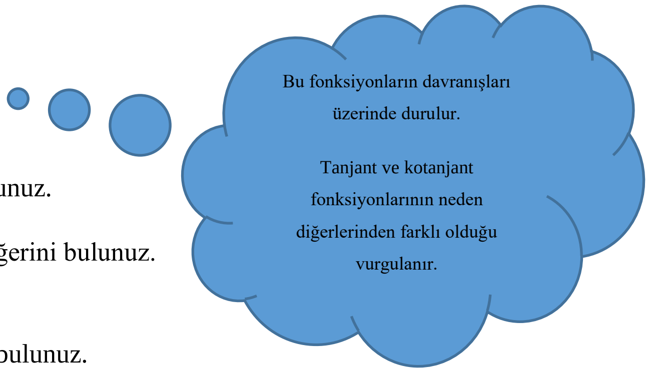
$\lim_{x\to c}(sinf(x)) = \sin[\lim_{x\to c} f(x)]$

$\lim_{x\to c}(cosf(x)) = \cos[\lim_{x\to c} f(x)]$

$\lim_{x\to c}(tanf(x)) = \tan[\lim_{x\to c} f(x)]$

$\lim_{x\to c}(cotf(x)) = \cot[\lim_{x\to c} f(x)]$

! Bu fonksiyonların davranışları üzerinde durulur. (Salınım fonksiyon)
! Tanjant ve kotanjant fonksiyonlarının neden diğerlerinden farklı olduğu vurgulanır.
! Asimptot kavramından bahsedilir.



Uygulama-3:

! $\lim_{x\to\pi} sin(cosx)$ limitini bulunuz.

! $\lim_{x\to\frac{\pi}{4}} \frac{sin2x-1}{tanx+cotx}$ ifadesinin değerini bulunuz.

! $\lim_{x\to 0} \frac{\left(sinxsin\left(\frac{\pi}{2}-x\right)\right)^2}{xsin4x}$ limitini bulunuz.

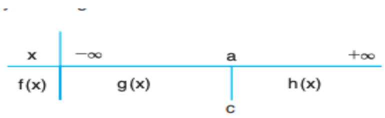
! $\lim_{x\to 30^o}[sin\,(2x-15^o)]$ limitini bulunuz.

<u>Parçalı Fonksiyonların Limiti</u>

$A, B \subseteq R$ ve $f: A \to B$

f(x)= $\begin{cases} g(x), x < a \\ c, x = a \\ h(x), x > a \end{cases}$ parçalı tanımlı

fonskiyonun x=a noktasındaki davranışı hakkında ne söylersiniz?



! x=a noktasının kritik nokta olduğu üzerinde durulur.

! $\lim_{x\to a^-} f(x) = \lim_{x\to a} g(x) = L_1$

$\lim_{x\to a^-} f(x) = \lim_{x\to a} h(x) = L_2$

$L_1 = L_2 = L$ ise x=a noktasındaki davranışı hakkında konuşabiliriz.

$L_1 \neq L_2$ ise limiti yoktur deriz.

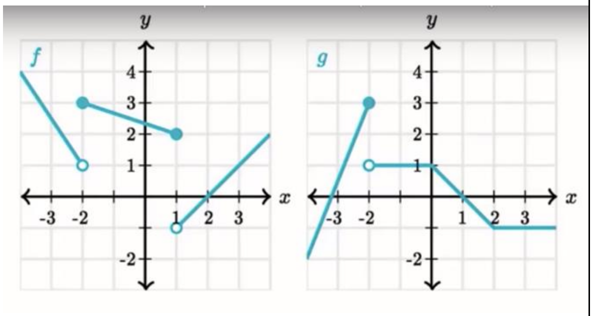
Uygulama-5:

! g(m) $= \begin{cases} 5m-7, m \leq 4 \\ \frac{8m-6}{2}, m > 4 \end{cases}$

$f(m) = |2m-8|$ olduğuna göre, $\lim_{m\to 4} \frac{2f(m)+g(m)}{mf(m)g(m)}$ limit işleminin sonucunu bulunuz.

Varılmak istenen nokta: burada ise diğer sorunun aksine fonksiyonlarımız 4 noktasında limiti olduğu için limitleri ayırarak işlem yapabiliyoruz.

2) $f: R \to R$ ve $g: R \to R$ olmak üzere, yanda grafikleri verilen f ve g fonksiyonları ile ilgili aşağıdaki soruları cevaplandırınız.



! $\lim_{x\to -2}\left(f(x)+g(x)\right)$

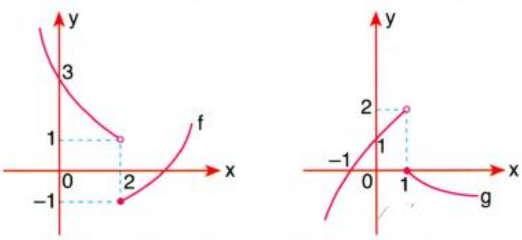
! $\lim_{x\to 1}\left(f(x)+g(x)\right)$

! $\lim_{x\to 1}\left(f(x).g(x)\right)$

Vurgulanmak istenen nokta:

$\lim_{x\to a}\left(f(x)+g(x)\right) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ kuralının uygulanabilmesi için $f(x)$ ve $g(x)$ fonksiyonlarının a noktasında limiti olmalıdır. Bu soruda -2 noktasında limit olmadığından bu kuralı uygulayamadık. Fakat sağ ve soldan $f(x)+g(x)$ limitine baktığımızda ikisinin eşit olduğunu görüyoruz. Bu da bize $\lim_{x\to a} f(x)+g(x)$ in var olduğunu söyleyebiliriz. Bu durum aynı şekilde diğer limit ile dört işlem uygulamalarında da geçerlidir.

! Pekiştirme sorusu:



Şekilde f ve g fonksiyonlarının grafikleri verilmiştir.

Buna göre, $\lim_{x\to 1^+}$ (fog)(x – 2) değeri kaçtır?

A) –1 B) 0 C) 1 D) 2 E) 3

$\frac{0}{0}$ **Belirsizliği ve** $\frac{\infty}{\infty}$ **Belirsizliği**

$\lim_{x\to 0} f(x) = 0$ $\lim_{x\to 0} g(x) = 0$	$\lim_{x\to\infty} h(x) = \infty$ $\lim_{x\to\infty} t(x) = \infty$

$\lim_{x\to 0}\frac{f(x)}{g(x)} = \frac{0}{0}$ ve $\lim_{x\to\infty}\frac{h(x)}{t(x)} = \frac{\infty}{\infty}$ durumları “belirsizlik” belirtir.

! $\frac{0}{0}$'ın neden belirsizlik olduğu üzerinde tartışılır.

Yönlendirici Soru: Yukarıda sayı bölü sıfıra tanımsız demiştik ve o sayının sıfırdan farklı olacağını söylemiştik, peki o sayı sıfır olsaydı durum ne olurdu?

Yönlendirici Soru: Yukarıda sayı bölü sıfıra tanımsız demiştik ve o sayının sıfırdan farklı olacağını söylemiştik, peki o sayı sıfır olsaydı durum ne olurdu?

! $\lim\limits_{x\to 0}\frac{sinx}{x} = 1$

$\lim\limits_{x\to 0}\frac{x^2}{x} = 0$

Her iki fonksiyonda 0 noktasında aynı davranışı (0/0) gösterirken, sonuçlar birbirinden farklı çıkmaktadır. Bu durumda matematikte belirsiz bir durum ortaya çıktığı söylenir. Bu nedenle, 0/0 belirsizliktir.

! $\frac{\infty}{\infty}$ **için de aynı durum geçerlidir.** Sonsuzun bir nitelik olduğunun, bir sayı olmadığının dolayısıyla bunun bir sayı gibi bölme işlemine tabi tutulamayacağı söylenir.

! Nitelediğimiz çokluğun nasıl arttığını bilemediğimiz için sonsuz bölü sonuz matematikte bir belirsizlik oluşturur. Bu sebepten dolayı, limit alırken sonuçlar farklı çıkabilir ve limitteki belirsizliklerin temel sebebi budur.

Yönlendirici Soru: 1^∞ hakkında ne söylersiniz?

! Eğer $\lim\limits_{n\to\infty} 1^n = 1$, tabiki hepinizin tahmin ettiği gibidir. Peki, ya şöyle olsaydı?

! Elimizde f ve g fonksiyonları olsun ve bunların n→ ∞ durumunda limitleri için $\lim\limits_{n\to\infty} f(n) = 1$ ve $\lim\limits_{n\to\infty} g(n) = \infty$ diyelim. Bu durumda, $\lim\limits_{n\to\infty} f(n)^{g(n)} = 1^\infty$ için 1 mi dersiniz?

Örneğin; $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx e \approx 2.7182 ...$ 'dir.

Diğer taraftan, $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{4n} = e^4$ ya da $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n^2} = \infty$ olur.

Bu durumda şunu söyleyebiliriz: Bu formun bir limiti her zaman kendi değerlerine göre değerlendirilmelidir; f ve g'nin sınırları kendi başına değerini belirlemez. Bu da belirsizlik durumunu oluşturur.

Uygulama-6:

! $\lim\limits_{x\to-\infty}\frac{2x^3+4x+1}{x^4+2}$ limitinin değerini L'hospital kuralını kullanmadan bulunuz.

! $\lim\limits_{x\to\pi/4}\frac{cos2x}{sinx-cosx}$ limitin değerini L'hospital kuralını kullanmadan bulunuz.

! 1992 model küçük arabaların sürüş maliyetlerini araştıran bir çalışma, ortalama maliyetin (araba vergisi, yakıt, sigorta, bakım ve onarım) tl/km cinsinden yaklaşık olarak,

$C(x) = \frac{2010}{x^{2.2}} + 17.80$ (x, arabanın1 senede kat ettiği yolun km cinsinden değerini göstermektedir)

olduğunu ortaya koymuştur.

! Küçük bir arabanın yılda 5000km yaptığında ortalama maliyeti hakkında ne söylersiniz? 10000km/yıl? 25000km/yıl?

! Arabanın bir yılda kat ettiği yolun sınırsız bir şekilde arttığını düşünürsek, arabanın maliyeti hakkında ne söylersiniz?

! $\lim_{x \to 0} \frac{sinx}{x}$ işleminin sonucunu bulunuz.

! İşleminin sonucunu bulmak için sıkıştırma teoremine ihtiyaç duyulduğu belirtilir.

Sıkıştırma Teoremi

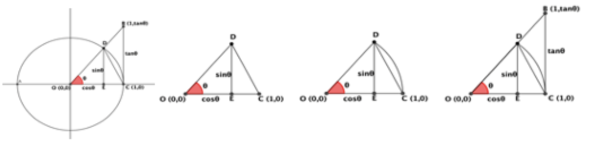
$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = L$ olsun. $\forall x \in R$ için $f(x) \leq h(x) \leq g(x)$ ise

$\lim_{x \to a} f(x) \leq \lim_{x \to a} h(x) \leq \lim_{x \to a} g(x)$ olur. Buradan da $\lim_{x \to a} h(x) = L$ elde edilir.

! Sıkıştırma teoremi günlük hayattan ilişkilendirilerek öğrencinin aklında bir temsil oluşturulması sağlanır.

! Sıkıştırma teoremini matematikte nerede kullanıyor olabiliriz? Bir tartışma ortamı oluşturularak öğrencilerin akıl yürütmesi sağlanır.

Yandaki POA üçgeninin alanı BOA daire diliminin alanından büyüktür. BOA daire diliminin alanı da BOA üçgeninin alanından büyüktür.



BOA üçgeninin alanı ≤ BOA daire diliminin alanı ≤ POA üçgeninin alanı

Buradan, $\frac{1.sinx}{2} \leq \pi.1^2.\frac{x}{2\pi} \leq \frac{1.tanx}{2}$

$sinx \leq x \leq tanx$

$\frac{sinx}{sinx} \leq \frac{x}{sinx} \leq \frac{1}{cosx} \Rightarrow 1 \leq \frac{x}{sinx} \leq \frac{1}{cosx}$ bulunur.

Her tarafın limiti alınarak;

$\lim_{x \to 0} 1 \leq \lim_{x \to 0} \frac{x}{sinx} \leq \lim_{x \to 0} \frac{1}{cosx} \Rightarrow 1 \leq \lim_{x \to 0} \frac{x}{sinx} \leq 1 \Rightarrow \lim_{x \to 0} \frac{x}{sinx} = 1$ bulunur.

Benzer şekilde, $1 \leq \frac{x}{sinx} \leq \frac{1}{cosx}$ eşitsizliği $1 \leq \frac{sinx}{x} \leq \cos x$ biçiminde yazılır ve

$\lim_{x \to 0} 1 \geq \lim_{x \to 0} \frac{sinx}{x} \geq \lim_{x \to 0} \cos x \Rightarrow \lim_{x \to 0} \frac{sinx}{x} = 1$ bulunur.

Uygulama ödevi: $\lim_{x \to 0} \frac{tanx}{x}$ işleminin sonucunun ne olduğunu trigonometrik fonksiyonlara yönelik bilginizi kullanarak bulunuz.

! Aşağıda verilen limit işlemlerinin sonucunu trigonometrik fonksiyonlar ve limit ile ilgili bilgilerinizi kullanarak bulunuz.

! $\lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right) cosh}{h}$

! $\lim_{\theta \to 0} \frac{\sin\theta cos4\theta}{\theta}$ **<u>Bitiş:</u>** Bu derste anlatılan konular üzerinden tekrar yapılır. Öğrencilere ne öğrendiklerini yazdıkları bir bitiş kartı yazmaları istenir.

Lesson Plan-3

Öğretmenin İsmi-Soysimi	Lesson Study Group
Ders Konusu:	Limit / Zamanlama: 90 dk
Ders İmecesi Amacı:	Süreklilik kavramını grafikler üzerinden anlar.
İlgili Kazanımlar:	12.5.1.5. Bir fonksiyonun bir noktadaki sürekliliğini açıklar. a) Fonksiyonun grafiği üzerinde sürekli ve süreksiz olduğu noktalar buldurulur. b) Ara değer teoremi verilerek uygulamalar yaptırılır. c) Limitin tarihsel gelişiminden ve Salih Zeki'nin bu alana katkılarından bahsedilir. ç) Bilgi ve iletişim teknolojileri yardımıyla süreklilik uygulamaları yaptırılır.
Önkoşul bilgi ve / veya beceriler:	Limit kavramı Fonksiyon, özel fonksiyonlar ve grafikleri
İlişkili Kavramlar	Orta değer teoremi Analitik Geometri → Eğim ↔ Türev Alan Hesabı ↔ İntegral Dizi-Seri İrrasyonel sayıların yerini belirleme↔Sonsuzluk İleri düzey olasılık (olasılık dağılımı) İstatistik↔ Merkezi Limit Teoremi
Materyaller:	GeoGebra, Aktivite kâğıdı, Excel **Matematiksel Görev / Aktivitenin Kısa Açıklaması:** Bir önceki dersten kalan bölüm ile devam edilir (Lesson Plan-2). Günlük hayattan örnek ile süreklilik kavramına giriş yapılır ve noktada ve aralıkta süreklilik tanımları üzerinde durularak, öğrencilerin parçalı fonksiyon üzerinden süreklilik ile ilgili bilgilerini uygulamaları istenir. Uygulanan tüm üç ders planının değerlendirmesi olarak modelleme etkinliği yaptırılır.

Giriş

Her iki spor dalında da steps kuralı vardır. Her ikisinde de steps olduğunda oyun durur ve top rakip takıma geçer. Basketbolda en fazla 2 adım hakkımız varken, hentbolda 3 adım hakkımız vardır. Adım sayısı aşıldığında steps olur.

Eğer hentbolde 3 adım atarsak oyunun sürekliliği bozulmaz ve gole gidebiliriz. Çünkü steps hentbolde farklı tanımlanmıştır ve basketteki 2 adım kuralı hentboldeki steps tanımında yoktur. Ancak basketbolda 3 adım attığımızda oyunun sürekliliği bozulur ve baskete gidemeyiz.

Orta

! **Öğretmen açıklaması (konuya bağlama):** Matematikteki fonksiyonlarda süreklilikte bunun gibidir. Fonksiyonun tanım kümesi bize fonksiyonun sürekli olup olmadığına dair yönlendirir.

! **Vurgulanmak istenen nokta:** Tanım kümesi, tanımlılık ve limitin süreklilik için 3 temel taş olduğunun vurgulanmasıdır.

Noktada Süreklilik

A$\subseteq \mathbb{R}$ ve $f: A \to \mathbb{R}$ bir fonksiyon olsun. a $\in A$ olmak üzere, $\lim_{x \to a} f(x) = f(a)$ is f fonksiyonu **x=a noktasında sürekli̇dir** denir .

Buna göre;

a) f fonksiyonu x = a noktasında tanımlı olmalıdır.

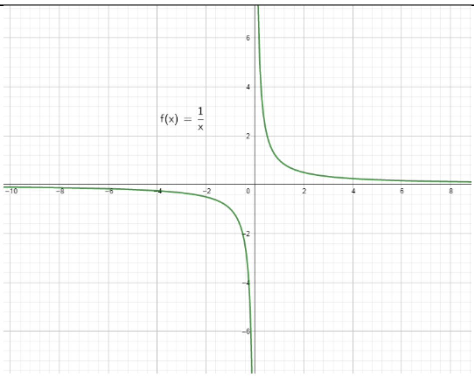
b) f fonksiyonunun a noktasında limiti olmalıdır.

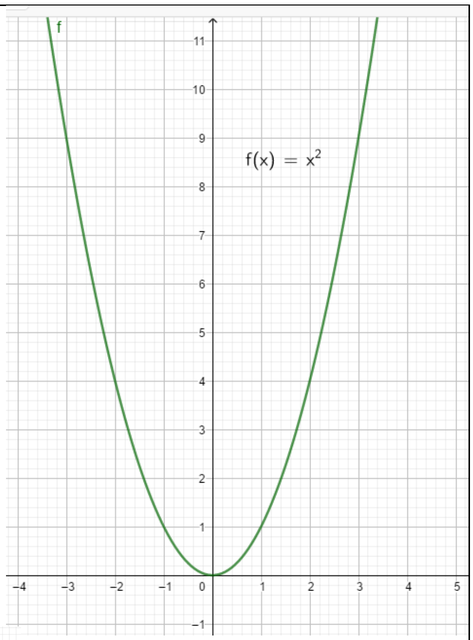
c) Fonksiyonun a noktasındaki limiti a noktasındaki fonksiyon değerine eşit olmalıdır.

UYGULAMA-1:

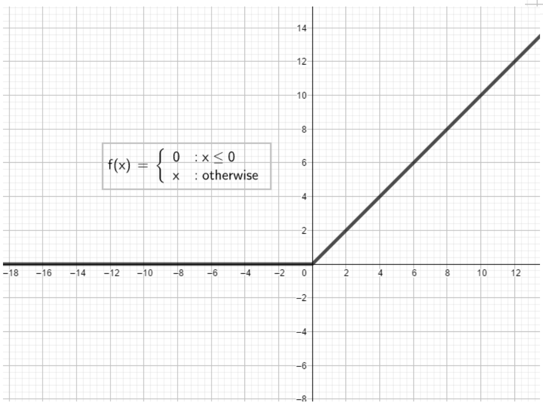
Aşağıdaki fonksiyonların varsa süreksiz olduğunu noktaları bulunuz.

1)

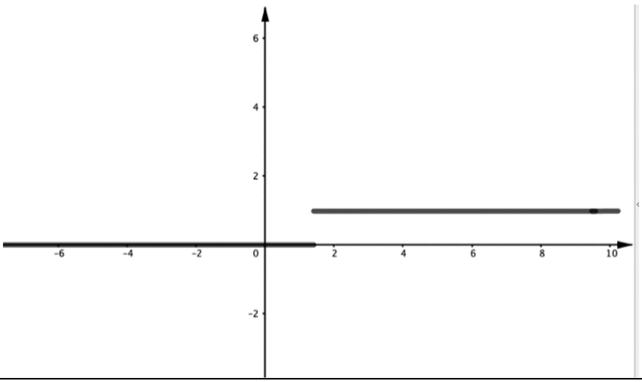




f(x)=$\frac{1}{x}$; $x \neq 0$



! İlk verilen örnekte tanım kümesinden 0 noktasını çıkardığımız için, fonksiyonun sürekli olduğunu söyleyebiliriz.



2) $f: \mathbb{Q} \to \mathbb{Q}\ f(x) = \begin{cases} 0, x < 0\ veya\ x^2 < 2 \\ 1, x > 0\ ve\ x^2 > 2 \end{cases}$ fonksiyonun varsa süreksiz olduğunu nokta ya da noktaları bulunuz.

! Bu fonksiyonda kritik noktamız $\sqrt{2}$ olduğu için ve $\sqrt{2}$ tanım kümesinde olmadığı için bu fonksiyon tüm noktalarında süreklidir.

! Bu fonksiyon grafiği tek parçadan oluşmayan sürekli fonksiyon örneğidir. Bu fonksiyonda tanım ve değer kümesi $\mathbb{Q} \to \mathbb{Q}$ seçilerek grafiği tek parçadan oluşmayan, sürekli f(x) fonksiyonu tanımlanmıştır. F(x) fonksiyonu süreklidir ama grafiğinde boşluk bulunmaktadır.

3) Matematikte, işaret fonksiyonu (öteki adıyla signum fonksiyonu) gerçel sayının işaretini bulmamızı sağlar. İşaret fonksiyonu, tanımlanan değerin işaretine göre, -1, 0 ve +1 sonuçlarını veren bir fonksiyondur. Tanımlanacak değer 0'dan küçükse: -1, 0'a eşitse: 0 ve 0'dan büyükse: +1 sonucunu verir. İşaret fonksiyonu genel olarak **sgn** olarak tanımlanır ve:

$$f(x) = \begin{cases} -1; x < 0 \\ 0; x = 0 \\ 1; x > 0 \end{cases}$$

şeklinde tanımlanır. Buna göre aşağıda fonksiyonların süreksiz olduğu noktaları bulunuz.

a. $Sgn_1: \mathbb{R} \to \mathbb{R}\ sgn[(x^2 - 6x + 9)]$

b. $Sgn_2: \mathbb{R} \to \mathbb{R}\ sgn[(x - 2)(x + 3]$

4) Bir reel sayıdan büyük olmayan bir başka deyişle küçük veya eşit olan en büyük tamsayıya o sayının tam değeri denir. Tam değer $[|x|]$ işareti ile gösterilir. Reel sayıları tam değeri ile eşleyen $f: \mathbb{R} \to \mathbb{R}\ f(x) = [|x|]$ fonksiyonuna **tam değer fonksiyonu** denir. Örneğin; $[|3.4|] = 3$ $[|-4.6|]$ =-5'e eşittir. Bu bilgiye göre, aşağıdaki fonksiyonların

a. $f: \mathbb{R} \to \mathbb{R}\ f(x) = [|2x + 1|]$ fonksiyonun varsa süreksiz olduğu noktaları bulunuz.

b. $f: \mathbb{R} \to \mathbb{R}\ f(x) = [|\cos^2 x|]$ fonksiyonun x=$\frac{\pi}{2}$ noktasında sürekli olup olmadığını yorumlayınız.

! Bu işlemlerde kritik noktalar belirlenerek işlem yapılır. 2x+1 için kritik noktalar -0.5$\leq 2x + 1 \leq 0.5$ yapılarak bulunur.

! Aynı şekilde $\cos^2 x$ fonksiyonu için kritik nokta olarak $\frac{\pi}{3}$, $\frac{2\pi}{3}$ belirlenir ve bunun üzerinden karesi alındığında hem $\frac{\pi}{3}$ hem de $\frac{2\pi}{3}$ değerinin pozitif olduğu ve sıfıra yaklaştığı görülür.

<u>Varılmak istenen nokta</u>: Öğrencilerin farklı fonksiyon türlerinde çalıştıklarında süreklilik ve süreksizlik konusunda yorum yapmalarını görmektir.

! Tam değer fonksiyonunda içini tam sayı yapan değerlerin süreksiz olduğunu düşünmeleri beklenir. Bu nedenle, karşıt bir örnek verilmiştir.

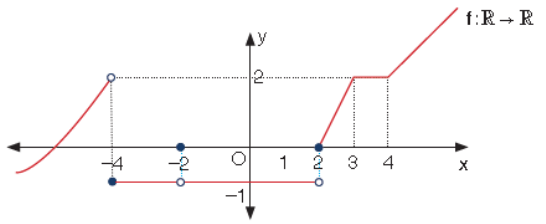
! İşaret fonksiyonu ile öğrencilerin denklem çözme becerileri, okuduğunu anlama ve süreklilik kavramını anlama becerileri ölçülmüştür.

Aralıkta Süreklilik

A⊆ ℝ ve $f: A \to \mathbb{R}$ bir fonksiyon olsun. Tüm a∈ A elemanları için,

$\lim_{x \to a} f(x) = f(a)$ oluyorsa, f fonksiyonu **A kümesi üzerinde süreklidir** denir.

UYGULAMA-2

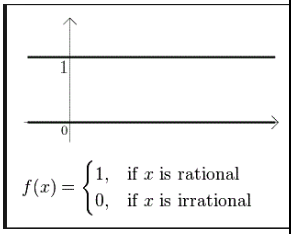


1) Yanda grafiği verilen y = f(x) fonksiyonunun sürekli olduğunu aralıkları bulunuz.

2) $f: \mathbb{R} \to \mathbb{R}\ f(x) = \log(x^2 + 6x - k + 12)$ fonksiyonunun reel sayılarda sürekli olması için k ne olmalıdır?

3) $f: \mathbb{R} \to \mathbb{R}\ f(x) = \begin{cases} \frac{x+3}{x^2-25}; x < 1 \\ 4; x = 1 \\ \frac{x}{x^2-x-6}; x > 1 \end{cases}$ olmak üzere, f fonksiyonunun sürekli olduğu en geniş kümeyi bulunuz.

4) $f: \mathbb{R} \to \mathbb{R}\ f(x) = \begin{cases} 0; x\ rasyonel \\ 1; x\ irrasyonel \end{cases}$ fonksiyonu tanım kümesi üzerinde sürekli olup-olmadığını yorumlayınız.



SÜREKLİ FONKSİYONLARIN ÖZELLİKLERİ

$A \subseteq R, f: A \to R$ ve $g: A \to R$ fonksiyonları $a \in A$ noktasında sürekli fonksiyonlar olsun. Bu durumda,

- $f + g$, $f - g$ ve $f.g$ fonksiyonlarının a noktasındaki sürekliliği hakkında ne söylersiniz?
- $k \in R$ olmak üzere k.f fonskiyonun $x = a$ noktasında sürekliliği hakkında ne söylersiniz?
- $\frac{f}{g}$ fonksiyonun $x = a$ noktasında sürekli olması için ne olmalıdır?

- $f + g$, $f - g$ ve $f.g$ fonksiyonlarının x=a noktasındaki süreklilidir.
- $k \in R$ olmak üzere k.f fonskiyonun $x = a$ noktasında süreklidir.

- $g(a) \neq 0 olmak\ üzere\ \frac{f}{g}$ fonksiyonun $x = a$ noktasında süreklidir.

Bitiş: Bu zamana kadar tüm konuların bir özeti geçilir.

Tüm kavramın (üç ders planını içerecek şekilde) anlaşılıp anlaşılmadığını görmek için göl kirliliği etkinliği yapılır.

GÖL KİRLİLİĞİ

Dünya Kuşları Koruma Kurumu'nca geliştirilen kriterlere göre Mogan Gölü Türkiye'deki 184 önemli kuş alanından biridir.

Dünya Kuşları Koruma Derneği'nin raporunda Ankara'nın Gölbaşı ilçesinde yer alan Mogan Gölü Tükiye'deki kirlilik riski taşıyan biri olarak gösteriliyor. Raporu dikkat alan Çevre ve Orman Bakanlığı yetkilileri yaptıkları incelemede 13.34 Milyon m^3 su hacmine sahip Mogan Gölü'ne çok yakın kurulmuş olan yeni bir fabrikanın atıklarının göle karıştığını tespit ediyorlar. Bakanlık yetkilileri, göldeki cıva oranının (g/L) %0.00001 olduğunu ve fabrikanın göle günde 100 litre sıvı atık karıştığını ve bu atıktaki cıva konsantrasyonunun 0.04 g/L olduğunu tespit ediyorlar. Çevre ve Orman Bakanlığı'nın yayınladığı tehlikeli maddelerin su ve çevresinde neden olduğu kirliliğin kontrolü yönetmeliğine göre, sudan içen kuşlara ve doğan yaşama zarar vermemesi için "iç yüzeysel sularda toplam cıva konsantrasyonun 10^4 g/L'yi aşmaması gerekmektedir.

Sizin göreviniz Mogan Gölü'nün kirlilik durumu ile ilgili aşağıdaki konuları içeren bir rapor hazırlanmasında yetkililere yardımcı olmaktır. Raporu hazırlarken göldeki su miktarının fabrika atığı dışında herhangi bir sebepten dolayı değişmediğini varsayabiliriz:

* Gölde biriken cıvanın kuşlara ve çevreye bir zararının olup olmayacağı olacaksa ne zaman tehlike sınırına ulaşacaktır?

* Herhangi bir müdahale olmazsa çok uzun zaman sonra göldeki cıva konsantrasyonun ulaşacağı oran nedir?

! İkinci 45dklık dilime girmeden önce tüm dersin hızlıca tekrar yapılır (10dk).

! Öğrenciler 3er kişilik gruplara ayrılarak etkinlik kağıdı verilir.

! Öğrencilere 20 dakikalık süre verilerek problem üzerinde çalışmaları beklenir.

! Öğretmen grupların arasında dolaşarak rehberlik etme rolündedir.

CURRICULUM VITAE

Surname, Name: Savuran, Ruya

EDUCATION

Degree	Institution	Year of Graduation
MS	Marmara University Secondary Mathematics Education	2014
BS	Dokuz Eylül University Secondary Mathematics Teacher Education	2012
High School	Ödemiş Anadolu Öğretmen High School, Izmir	2007

WORK EXPERIENCE

Enrollment	Place	Years
Mathematics Teacher	İstanbul AEK High School	2013-2015
Research Assistant	İstanbul Aydın University	2015-2016
Research Assistant	Middle East Technical University	2016-2021

PUBLICATIONS

1. Savuran, R. & Akkoç, H. (2021). Examining pre-service mathematics teachers' use of technology from a sociomathematical norm perspective, *International Journal of Mathematical Education in Science and Technology*. https://doi.org/10.1080/0020739X.2021.1966529

2. Savuran, R., & Isiksal-Bostan, M. (2022). Revealing implicit knowledge of pre-service mathematics teachers in lesson planning: Knowledge of Infinity. *European Journal of Science and Mathematics Education, 10*(3), 269-283. https://doi.org/10.30935/scimath/11838

3. Savuran, R. & Isıksal-Bostan, M. (In review). Developing knowledge of topics through lesson study: The concept of limit.

4. Savuran, R. & Isıksal-Bostan, M. (2021). A preservice secondary mathematics teacher's specialized knowledge: the case of limit, *Paper accepted as long paper for 14th International Congress on Mathematical Education (ICME 14)*, 12-19 July 2021, Shanghai, China.

5. Savuran, R. & Akkoç, H. (2020). Pre-service Mathematics Teachers' Endorsed Socio-Mathematical Norms in Technology-Enhanced Lessons. In *Proceedings of SITE Interactive Online 2020 Conference* (pp. 548-555). Association for the Advancement of Computing in Education (AACE). Retrieved November 2, 2020 from http s://www.learntechlib.org/primary/p/218212/.

6. Savuran, R. & Isıksal-Bostan, M. (2020). Investigation of pre-service mathematics teachers' mathematical knowledge of infinity in lesson planning. *Paper accepted for 7th International Eurasian Educational Research Congress*, 10-13 September 2020, Eskisehir, Turkey

7. Şay, R. & Akkoç, H. (2016). Mediating role of technology: Prospective upper secondary mathematics teachers' practice, *British Society of Research in Mathematics Learning (BSRLM)*, 36 (1), 88 -93, 27 February 2016, Manchester Metropolitan University, UK.

8. Şay, R. & Akkoç, H. (2015). Beyond orchestration: Norm perspective in technology integration, *Ninth Congress of European Research in Mathematics Education* (*CERME 9)*, 4-8 February 2015, Prague, Czech Republic.

9. Şay, R. & Akkoç, H. (2014). Effective use of technology during instruction: the role of mediation, *The North American Conference on Education*, Rhode Island, United States of America, 25-28 September 2014.

10. Kozaklı, T., Şay, R., & Akkoç, H. (2014). Prospective Mathematics Teachers' Preferences for Instrumental Orchestration Types and Endorsed Norms, *International Conference on Education in Mathematics, Science & Technology*. 16-18 May 2014, Konya, Turkey.

11. Şay, R., Kozaklı, T., & Akkoç, H. (2013). Instrumental Orchestration Types Planned by Pre-Service Mathematics Teachers, *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education*, Vol. V (pp. 162-163). Kiel, Germany.

INTERNATIONAL SCIENTIFIC TRAININGS AND MEETINGS

Utrecht Summer School-2020, 26-27 August 2020, Utrecht University, Holland
Presenting paper: Development of knowledge of features of learning mathematics in planning through lesson study

YESS 9 - Young Erme Summer School, 20-25 August 2018, Montpellier, France
Presenting Paper: Investigation of Mathematics Teachers' Use of Horizon Content Knowledge in Practice

HONOURS, SCHOLARSHIP, AWARDS AND GRANTS

Dokuz Eylul University, Mathematics Education, 3rd Highest Ranked Student In Class Of 2012

The Scientific And Technological Research Council Of Turkey (TUBİTAK)-2228 Senior Undergraduate Students Domestic Graduate Scholarship Program

Pme 37-The Richard Skemp Memorial Support Fund – The International Group for The Psychology of Mathematics Education

CERME 9-The Graham Litter Fund – Congress of The European Society for Research in Mathematics Education

ICME 14-Solidarity Fund – International Congress On Mathematical Education

CERME 12-The Graham Litter Fund – Congress of The European Society for Research in Mathematics Education