GENERATING THE MAPPING CLASS GROUP OF A NONORIENTABLE SURFACE BY TWO ELEMENTS OR BY THREE INVOLUTIONS

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Abstract. We prove that, for $g \geq 19$ the mapping class group of a nonorientable surface of genus g, $\mathrm{Mod}(N_g)$, can be generated by two elements, one of which is of order g. We also prove that for $g \geq 26$, $\mathrm{Mod}(N_g)$ can be generated by three involutions if $g \geq 26$.

1. Introduction

The mapping class group $\mathrm{Mod}(N_g)$ of closed connected nonorientable surface N_g is defined to be the group of the isotopy classes of all self-diffeomorphisms of N_g. In this paper, we are interested in finding generating sets for $\mathrm{Mod}(N_g)$ consisting of least possible number of elements. Since this group is not abelian, a generating set must contain at least two elements. Szepietowski [11] proved that $\mathrm{Mod}(N_g)$ is generated by three elements for all $g \geq 3$. Our first result (see Theorem 3.1) answers Problem 3.1(a) in [3, p.91] (cf Problem 5.4 in [6]).

Theorem A. *For $g \geq 19$, the mapping class group $\mathrm{Mod}(N_g)$ is generated by two elements.*

The next aim of the paper is to find an answer Problem 3.1(b) in [3, p.91]. Szepietowski showed that $\mathrm{Mod}(N_g)$ can be generated by involutions [10] and later he showed that $\mathrm{Mod}(N_g)$ can be generated by four involutions if $g \geq 4$ [11]. One can deduce that it can be generated by three involutions by the work of Birman and Chillingworth [2] if $g = 3$. It is known that any group generated by two involutions is isomorphic to a quotient of a dihedral group. Thus the mapping class group $\mathrm{Mod}(N_g)$ cannot be generated by two involutions. This implies that any generating set consisting only involutions must contain at least three elements. In this direction, we get the following result (see Theorem 4.1 and Theorem 4.2):

Theorem B. *For $g \geq 26$, the mapping class group $\mathrm{Mod}(N_g)$ can be generated by three involutions.*

Let us also point out that $\mathrm{Mod}(N_g)$ admits an epimorphism onto the automorphism group of $H_1(N_g;\mathbb{Z}_2)$ preserving the (mod 2) intersection pairing [9] and this group is isomorphic to (see [4] and [12])

$$\begin{cases} Sp(2h;\mathbb{Z}_2) & \text{if } g = 2h+1, \\ Sp(2h;\mathbb{Z}_2) \ltimes \mathbb{Z}_2^{2h+1} & \text{if } g = 2h+2. \end{cases}$$

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Hence, the action of mapping classes on $H_1(N_g;\mathbb{Z}_2)$ induces an epimorphism from $\mathrm{Mod}(N_g)$ to $Sp\big(2\lfloor\frac{g-1}{2}\rfloor;\mathbb{Z}_2\big)$, which immediately implies the following corollary:

Corollary C. *The symplectic group $Sp\big(g-1;\mathbb{Z}_2\big)$ can be generated by two elements for every odd $g\geq 19$ and also by three involutions for every odd $g\geq 27$. Similarly, the group $Sp\big(g-2;\mathbb{Z}_2\big)\ltimes\mathbb{Z}_2^{g-1}$ can be generated by two elements for every even $g\geq 20$ and also by three involutions for every even $g\geq 26$.*

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2. Preliminaries

Let N_g be a closed connected nonorientable surface of genus g. Note that the *genus* for a nonorientable surface is the number of projective planes in a connected sum decomposition. We use the model for the surface N_g as a sphere with g crosscaps represented shaded disks in all figures of this paper. Note that a crosscap is obtained by deleting the interior of such a disk and identifying the antipodal points on the resulting boundary. The *mapping class group* $\mathrm{Mod}(N_g)$ of the surface N_g is the group of the isotopy classes of self-diffeomorphisms of N_g. We use the functional notation for the composition of two diffeomorphisms; if f and g are two diffeomorphisms, the composition fg means that g is applied first.

A simple closed curve on a nonorientable surface N_g is *one-sided* if its regular neighbourhood is a Möbius band and *two-sided* if it is an annulus. If a is a two-sided simple closed curve on N_g, to define the Dehn twist t_a about the curve a, we need to choose one of two possible orientations of its regular neighbourhood (as we did for the curves in Figure 1). Throughout the paper, the right-handed Dehn twist t_a about the curve a will be denoted by the corresponding capital letter A. In our notation, both the curves on N_g and self-diffeomorphisms of N_g shall be considered up to isotopy. In the following we shall make repeated use of some basic relations in $\mathrm{Mod}(N_g)$: for two-sided simple closed curves a and b on N_g and for any $f\in\mathrm{Mod}(N_g)$,

- *Commutativity:* If a and b are disjoint, then $AB=BA$.
- *Conjugation:* If $f(a)=b$, then $fAf^{-1}=B^{\varepsilon}$, where $\varepsilon=\pm1$ depending on the orientation of a regular neighbourhood of $f(a)$ with respect to the chosen orientation.

Consider the Klein bottle K with a hole in Figure 2. We define a *crosscap transposition* u as the isotopy classes of a diffeomorphism interchanging two consecutive crosscaps as shown on the left hand side of Figure 2 and equals to the identity outside the Klein bottle with one hole K. The effect of the diffeomorphism $y=Au$ on the interval c as in Figure 2 can be also constructed as sliding a Möbius band once along the core of another one and keeping each point of the boundary of K fixed. This is a *Y-homeomorphism* [8] (also called a *crosscap slide* [5]). Note that $A^{-1}u$ is a Y-homemorphism i.e. the other choice of the orientation for a neighbourhood of the curve a also gives a Y-homeomorphism. We also note that y^2 is a Dehn twist about ∂K.

It is known that $\mathrm{Mod}(N_g)$ is generated by Dehn twists and a Y-homeomorphism (one crosscap slide) [8]. We remark that crosscap transpositions can be used instead

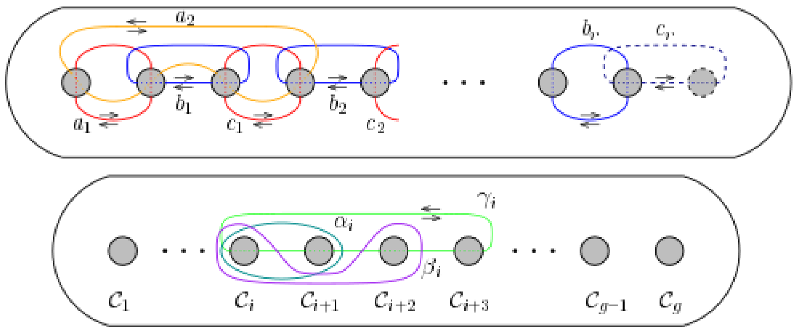

FIGURE 1. The curves $a_1, a_2, b_i, c_i, \alpha_i, \beta_i$ and γ_i on the surface N_g, where $g = 2r$ or $g = 2r + 2$. Note that we do not have the curve c_r when g is odd.

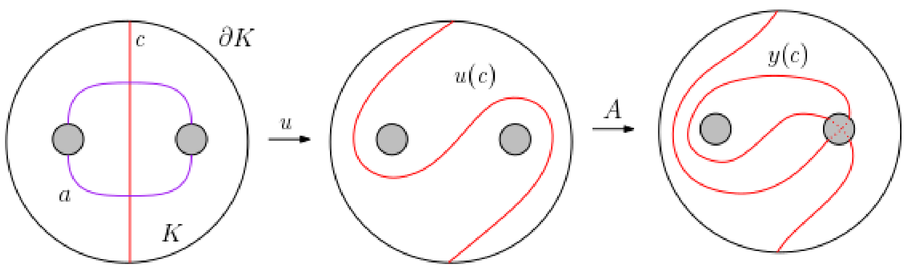

FIGURE 2. The homeomorphisms u and $y = Au$.

of crosscap slides since a crosscap transposition equals to the product of a Dehn twist and a crosscap slide.

Before we finish Preliminaries, let us state a theorem which is used in the proofs of following theorems. We work with the model in Figure 3 in such a way that the surface is obtained from the 2-sphere by deleting the interiors of g disjoint disks which are in a circular position and identifying the antipodal points on the boundary. Moreover, note that the rotation T by $\frac{2\pi}{g}$ about the x-axis maps the crosscap $\mathcal{C}_i$ to $\mathcal{C}_{i+1}$ for $i = 1, \ldots, g-1$ and $\mathcal{C}_g$ to $\mathcal{C}_1$.

Theorem 2.1. *For $g \geq 7$, the mapping class group* $\text{Mod}(N_g)$ *can be generated by the elements T, $A_1A_2^{-1}$, $B_1B_2^{-1}$, and a Y-homeomorphism (or a crosscap transposition).*

Proof. Let G be the subgroup of $\text{Mod}(N_g)$ generated by the set $\{T, A_1A_2^{-1}, B_1B_2^{-1}\}$. Szepietowski [11, Theorem 3] showed that A_1, A_2, B_i and C_i as shown in Figure 1, together with a Y-homeomorphism generate $\text{Mod}(N_g)$. Therefore, it is enough to prove that the elements A_1, A_2, B_i and C_i are contained in G for $i = 1, \ldots, r$.

Let $\mathcal{S}$ denote the finite set of isotopy classes of two-sided non-separating simple closed curves appearing throughout the paper with chosen orientations of neighborhoods. Define a subset $\mathcal{G}$ of $\mathcal{S} \times \mathcal{S}$ as

$$\mathcal{G} = \{(a, b) : AB^{-1} \in G\}.$$

Using the similar arguments in the proof of [7, Theorem 5], the set $\mathcal{G}$ satisfies

- if $(a,b) \in \mathcal{G}$, then $(b,a) \in \mathcal{G}$ (symmetry),
- if (a,b) and $(b,c) \in \mathcal{G}$, then $(a,c) \in \mathcal{G}$ (transitivity) and
- if $(a,b) \in \mathcal{G}$ and $H \in G$ then $(H(a), H(b)) \in \mathcal{G}$ (G-invariance).

Thus, $\mathcal{G}$ defines an equivalence relation on $\mathcal{S}$.
We begin by showing that $B_iC_j^{-1}$ is contained in G for all i, j. It follows from the definition of G and from the fact that $T(b_1, b_2) = (c_1, c_2)$, we have $C_1C_2^{-1} \in G$ (here, we use the notation $f(a,b)$ to denote $(f(a), f(b))$). Also, by conjugating $C_1C_2^{-1}$ with powers of T, one can conclude that G contains the elements $B_iB_{i+1}^{-1}$ and $C_iC_{i+1}^{-1}$. Moreover, the transitivity implies that the elements $B_iB_j^{-1}$ and $C_iC_j^{-1}$ are in G. To start with, since $B_2B_3^{-1} \in G$ and it is easy to verify that

$$B_2B_3^{-1}A_2A_1^{-1}(b_2, b_3) = (a_2, b_3),$$

so that $A_2B_3^{-1} \in G$. Then, we have

$$(A_1A_2^{-1})(A_2B_3^{-1})(B_3B_2^{-1}) = A_1B_2^{-1} \in G,$$

since G contains each of the factors. Thus, $T(a_1, b_2) = (b_1, c_2)$ implies that $B_1C_2^{-1}$ is also in G. Moreover, G contains the element

$$B_1C_1^{-1} = (B_1C_2^{-1})(C_2C_1^{-1}).$$

Thus, $B_iC_i^{-1} \in G$ by conjugating with powers of T for all $i = 1, \dots, r-1$. Again,the transitivity implies that $B_iC_j^{-1} \in G$. Note that, we have

- $(A_1B_2^{-1})(B_2C_1^{-1}) = A_1C_1^{-1} \in G$,
- $(C_1A_1^{-1})(A_1A_2^{-1}) = C_1A_2^{-1} \in G$ and
- $(C_2C_1^{-1})(C_1A_1^{-1}) = C_2A_1^{-1} \in G$

from which it follows that the elements $A_1C_1^{-1}$, $C_1A_2^{-1}$ and $C_2A_1^{-1}$ are all in G.

It can also be verified that

$$(A_1B_2^{-1})(A_1C_1^{-1})(A_1C_2^{-1})(A_1B_2^{-1})(a_2, a_1) = (d_2, a_1)$$

so that $D_2A_1^{-1} \in G$. Also, the element $D_2C_2^{-1} = (D_2A_1^{-1})(A_1C_2^{-1})$ is in G. It can also be shown that

$$(C_2B_1^{-1})(C_2A_1^{-1})(C_2C_1^{-1})(C_2B_1^{-1})(d_2, c_2) = (d_1, c_2),$$

which implies that G contains $D_1C_2^{-1}$. Thus, G contains the element

$$D_1A_1^{-1} = (D_1C_2^{-1})(C_2A_1^{-1})$$

(here, the curves d_1 and d_2 are shown in [1, Figure 4]). By similar arguments as in the proof of [1, Lemma 5], for $g \geq 7$ the lantern relation implies that

$$A_3 = (A_2C_1^{-1})(D_1C_2^{-1})(D_2A_1^{-1}).$$

Since G contains each factor on the right hand side, $A_3 \in G$. It follows from the diffeomorphism $A_3(B_3B_1^{-1})$ maps the curve a_3 to b_3 that

$$B_3 = A_3(B_3B_1^{-1})A_3(B_1B_3^{-1})A_3^{-1} \in G.$$

By conjugating B_3 with the powers of T, we conclude that $A_1, B_1, C_1, \dots B_{r-1}, C_{r-1}$ and B_r are all in G. Moreover,

$$A_2 = (A_2A_1^{-1})A_1 \in G.$$

Therefore, the Dehn twist generators are contained in G. This finishes the proof. $\square$

3. A generating set for $\mathrm{Mod}(N_g)$

In this section, we work with the model in Figure 3. Let us denote by u_i the crosscap transposition supported on the one holed Klein bottle whose boundary is the curve α_i shown in Figure 1. Note that the rotation T takes α_i to α_{i+1} and the crosscap $\mathcal{C}_i$ to $\mathcal{C}_{i+1}$, which implies that $Tu_iT^{-1} = u_{i+1}$.

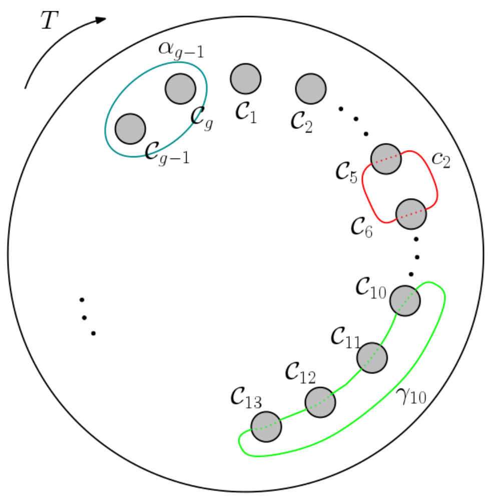

FIGURE 3. The rotation T and the curves c_2, γ_{10} and α_{g-1}.

Theorem 3.1. *For $g \geq 19$, the mapping class group $\mathrm{Mod}(N_g)$ is generated by $\{T, u_{g-1}\Gamma_{10}C_2^{-1}\}$.*

Proof. Let $F_1 = u_{g-1}\Gamma_{10}C_2^{-1}$ and let us denote by G the subgroup of $\mathrm{Mod}(N_g)$ generated by T and F_1. It follows from Theorem 2.1 that it suffices to prove that the subgroup G contains the elements $A_1A_2^{-1}$, $B_1B_2^{-1}$ and u_{g-1} to prove that $G = \mathrm{Mod}(N_g)$.

Let F_2 denote the conjugation of F_1 by T^{-4}. It follows from T^{-4} maps the curves $(\alpha_{g-1}, \gamma_{10}, c_2)$ to $(\alpha_{g-5}, \gamma_6, a_1)$ that

$$F_2 = T^{-4}F_1T^4 = u_{g-5}\Gamma_6A_1^{-1}$$

is contained in G. Let F_3 denote the element $(F_2F_1^{-1})F_2(F_2F_1^{-1})^{-1}$ that is contained in G. Hence

$$F_3 = (F_2F_1^{-1})F_2(F_2F_1^{-1})^{-1} = u_{g-5}C_2A_1^{-1}.$$

Since we have similar cases in the remaining parts of the paper, let us give some details before we proceed. It can be verified that the diffeomorphism $F_2F_1^{-1}$ send the curves $(\alpha_{g-5}, \gamma_6, a_1)$ to the curves (α_{g-5}, c_2, a_1). Then, we get

$$\begin{aligned} F_3 &= (F_2F_1^{-1})F_2(F_2F_1^{-1})^{-1} \\ &= (F_2F_1^{-1})u_{g-5}\Gamma_6A_1^{-1}(F_2F_1^{-1})^{-1} \\ &= u_{g-5}C_2A_1^{-1}. \end{aligned}$$

Thus, we have the elements $F_2F_3^{-1} = \Gamma_6C_2^{-1}$ and $T^4(\Gamma_6C_2^{-1})T^{-4} = \Gamma_{10}C_4^{-1}$, which are both contained in G.

Moreover, we have the following elements

$$\begin{aligned} F_4 &= (C_4\Gamma_{10}^{-1})F_1 = u_{g-1}C_4C_2^{-1}, \\ F_5 &= T^{-1}F_4T = u_{g-2}B_4B_2^{-1} \text{ and} \\ F_6 &= (F_4F_5)F_3(F_4F_5)^{-1} = u_{g-5}B_2A_1^{-1}, \end{aligned}$$

all of which are contained in the subgroup G. From this, we get the element $F_6F_3^{-1} = B_2C_2^{-1} \in G$. Also, we have $T(B_2C_2^{-1})T^{-1} = C_2B_3^{-1} \in G$, which gives rise to

$$B_2B_3^{-1} = (B_2C_2^{-1})(C_2B_3^{-1}) \in G.$$

This implies that $T^{-2}(B_2B_3^{-1})T^2 = B_1B_2^{-1}$ is in G. We also have the elements

$$\begin{aligned} T^2(C_2B_3^{-1})T^{-2} &= C_3B_4^{-1} \in G \text{ and} \\ T^{-2}(\Gamma_{10}C_4^{-1})T^2 &= \Gamma_8C_3^{-1} \in G, \end{aligned}$$

implying that $\Gamma_8B_4^{-1} = (\Gamma_8C_3^{-1})(C_3B_4^{-1}) \in G$. The conjugation of the element $\Gamma_8B_4^{-1}$ by T^{-7} is the element $\Gamma_1A_1^{-1} = A_2A_1^{-1}$ which is contained in G. By the proof of Theorem 2.1, the subgroup G contains the elements A_1, A_2, B_i and C_i for $i = 1, \ldots, r$. Then, in particular we have the elements $T^9A_2T^{-9} = \Gamma_{10} \in G$ and $C_2 \in G$. We conclude that $u_{g-1} = F_1(C_2\Gamma_{10}^{-1}) \in G$, which completes the proof. $\square$

4. Involution generators for $\mathrm{Mod}(N_g)$

In the first part of this section, where the genus of the surface N_g is even, we refer to Figure 4 for the involution generators ρ_1 and ρ_2 of N_g. The elements ρ_1 and ρ_2 are reflections about the indicated planes in Figure 4 in such a way that the rotation T, depicted in Figure 3, is given by $T = \rho_2\rho_1$.

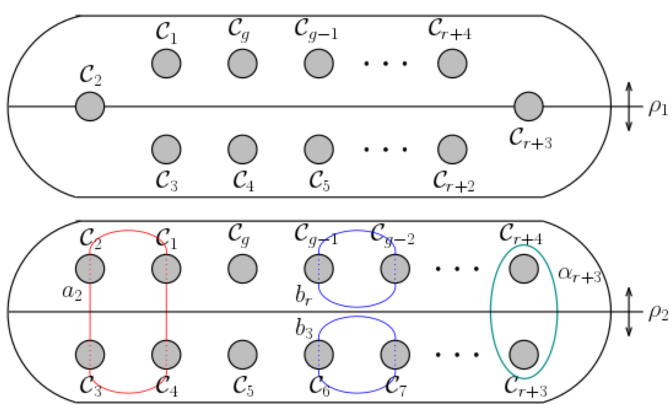

FIGURE 4. The reflections ρ_1 and ρ_2 for $g = 2r + 2$.

Theorem 4.1. *For $g = 2r+2 \geq 26$, the mapping class group $\mathrm{Mod}(N_g)$ is generated by the involutions ρ_1, ρ_2 and $\rho_2A_2B_rB_3u_{r+3}$.*

Proof. Consider the surface N_g as in Figure 4. It follows from

$$\rho_2(a_2) = a_2 \text{ and } \rho_2(b_r) = b_3$$

and also ρ_2 reverses the given orientation of a neighbourhood of a two-sided simple closed curve that

$$\rho_2 A_2 \rho_2 = A_2^{-1} \text{ and } \rho_2 B_r \rho_2 = B_3^{-1}.$$

Since $\rho_2 u_{r+3} \rho_2 = u_{r+3}^{-1}$, one can verify that the element $\rho_2 A_2 B_r B_3 u_{r+3}$ is an involution. Let $H_1 = A_2 B_r B_3 u_{r+3}$ and let H be the subgroup of $\mathrm{Mod}(N_g)$ generated by the set

$$\{\rho_1, \rho_2, \rho_2 H_1\}.$$

It is clear that H_1 and $T = \rho_2\rho_1$ are contained in the subgroup H. By Theorem 2.1, we need to prove that the subgroup H contains the elements $A_1A_2^{-1}$, $B_1B_2^{-1}$ and u_{r+3}. Let H_2 be the conjugation of H_1 by T^7. Thus

$$H_2 = T^7 H_1 T^{-7} = \Gamma_8 C_2 C_6 u_{r+10} \in H.$$

Let

$$H_3 = (H_2H_1)H_2(H_2H_1)^{-1} = \Gamma_8 B_3 C_6 u_{r+10},$$

which is also in H. From this, we get the element $H_2H_3^{-1} = C_2B_3^{-1} \in H$ implying that $T(C_2B_3^{-1})T^{-1} = B_3C_3^{-1} \in H$. One can easily see that $B_iC_i^{-1} \in H$ by conjugating $B_3C_3^{-1}$ with powers of T. Also, since $T(B_3C_3^{-1})T^{-1} = C_3B_4^{-1} \in H$, similarly $C_iB_{i+1}^{-1} \in H$ by conjugating $C_3B_4^{-1}$ with powers of T. Hence, we have the elements

$$B_iB_{i+1}^{-1} = (B_iC_i^{-1})(C_iB_{i+1}^{-1})$$

which are in H for all $i = 1, \ldots, r-1$. Moreover, $B_iB_j^{-1} \in H$ by the transitivity. In particular $B_1B_2^{-1} \in H$. Now, we have the following elements

$$\begin{array}{rcl}
H_4 & = & (B_7B_3^{-1})H_1 = A_2B_7B_ru_{r+3} \text{ if } r \neq 16, 17, 18, \\
(H_4 & = & (B_9B_3^{-1})H_1 = A_2B_9B_ru_{r+3} \text{ if } r = 16, 17, 18,) \\
H_5 & = & T^6H_4T^{-6} = \Gamma_7B_{10}B_2u_{r+9} \text{ if } r \neq 16, 17, 18, \\
(H_5 & = & T^6H_4T^{-6} = \Gamma_7B_{12}B_2u_{r+9} \text{ if } r = 16, 17, 18,) \\
H_6 & = & (H_5H_4)H_5(H_5H_4)^{-1} = \Gamma_7B_{10}A_2u_{r+9} \text{ if } r \neq 16, 17, 18, \\
(H_6 & = & (H_5H_4)H_5(H_5H_4)^{-1} = \Gamma_7B_{12}A_2u_{r+9} \text{ if } r = 16, 17, 18,)
\end{array}$$

which are all contained in H. Thus, we get the element $H_6H_5^{-1} = A_2B_2^{-1} \in H$. On the other hand, since $C_1B_2^{-1}$ is contained in H, the subgroup H contains the following elements

$$\begin{array}{c}
T^{-2}(C_1B_2^{-1})T^2 = A_1B_1^{-1}, \\
(A_1B_1^{-1})(B_1B_2^{-1}) = A_1B_2^{-1}, \\
(A_2B_2^{-1})(B_2A_1^{-1}) = A_2A_1^{-1}.
\end{array}$$

It follows from T, $A_1A_2^{-1}$ and $B_1B_2^{-1}$ are in H that the Dehn twists A_1, A_2, B_i and C_i are also in H for $i = 1, \ldots, r$. This implies that

$$u_{r+3} = (B_3^{-1}B_r^{-1}A_2^{-1})H_1 \in H,$$

which completes the proof. □

In the second part of this section, where the genus of the surface N_g is odd, we refer to Figure 5 for the involution generators ρ_1 and ρ_2 of N_g. Similarly, the elements ρ_1 and ρ_2 are reflections about the indicated planes in Figure 5 such that the rotation T in Figure 3 is given by $T = \rho_2\rho_1$. In the proof of the following theorem, we use the crosscap transposition supported on the one holed Klein bottle

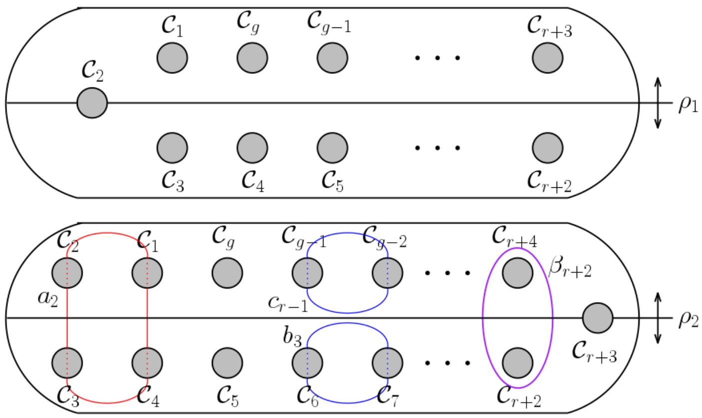

FIGURE 5. The reflections ρ_1 and ρ_2 for $g = 2r + 1$.

whose boundary is the curve β_i shown in Figure 1. Let us denote this crosscap transposition by v_i. Note that the rotation T sends β_i to β_{i+1} and the crosscap $\mathcal{C}_i$ to $\mathcal{C}_{i+1}$, which implies that $Tv_iT^{-1} = v_{i+1}$.

Theorem 4.2. *For $g = 2r+1 \geq 27$, the mapping class group $\mathrm{Mod}(N_g)$ is generated by the involutions ρ_1, ρ_2 and $\rho_2 A_2 C_{r-1} B_3 v_{r+2}$.*

Proof. We will follow the proof of Theorem 4.1, closely. Let us consider the surface N_g as in Figure 5. Since

$$\rho_2(a_2) = a_2 \text{ and } \rho_2(c_{r-1}) = b_3$$

and also since ρ_2 reverses the given orientation of a neighbourhood of a two-sided simple closed curve, we get

$$\rho_2 A_2 \rho_2 = A_2^{-1} \text{ and } \rho_2 C_{r-1} \rho_2 = B_3^{-1}.$$

By the fact that $\rho_2 v_{r+2} \rho_2 = v_{r+2}^{-1}$, it can be easy to verify that the element $\rho_2 A_2 C_{r-1} B_3 \phi_{r+2,r+4}$ is an involution. Let $E_1 = A_2 C_{r-1} B_3 v_{r+2}$ and let K denote the subgroup of $\mathrm{Mod}(N_g)$ generated by the set

$$\{\rho_1, \rho_2, \rho_2 E_1\}.$$

It is easy to see that E_1 and $T = \rho_2\rho_1$ are in K. By Theorem 2.1, we need to show that K contains the elements $A_1A_2^{-1}$, $B_1B_2^{-1}$ and v_{r+2}. Let E_2 be the following:

$$E_2 = T^7 E_1 T^{-7} = \Gamma_8 C_2 C_6 v_{r+9} \in K.$$

Consider the element

$$E_3 = (E_2E_1)E_2(E_2E_1)^{-1} = \Gamma_8 B_3 C_6 v_{r+9},$$

which belongs to K. One can conclude that the element $E_2E_3^{-1} = C_2B_3^{-1} \in K$, which implies that $T(C_2B_3^{-1})T^{-1} = B_3C_3^{-1} \in K$. From this, we get the elements $B_iC_i^{-1} \in H$ by conjugating $B_3C_3^{-1}$ with powers of T. Also, since $T(B_3C_3^{-1})T^{-1} = C_3B_4^{-1} \in K$, $C_iB_{i+1}^{-1} \in K$ by again conjugating $C_3B_4^{-1}$ with powers of T. Thus, we get the elements

$$B_iB_{i+1}^{-1} = (B_iC_i^{-1})(C_iB_{i+1}^{-1}),$$

which belong to K for all $i = 1, \ldots, r-1$. Also, using the transitivity $B_iB_j^{-1} \in K$. In particular $B_1B_2^{-1} \in K$. Moreover, we have the elements

$$\begin{array}{rcl} E_4 &=& (B_7B_3^{-1})E_1 = A_2B_7C_{r-1}v_{r+2} \text{ if } r \neq 16, 17, 18, 19, \\ (E_4 &=& (B_9B_3^{-1})E_1 = A_2B_9C_{r-1}v_{r+2} \text{ if } r = 16, 17, 18, 19,) \\ E_5 &=& T^6E_4T^{-6} = \Gamma_7B_{10}B_2v_{r+8} \text{ if } r \neq 16, 17, 18, 19, \\ (E_5 &=& T^6E_4T^{-6} = \Gamma_7B_{12}B_2v_{r+8} \text{ if } r = 16, 17, 18, 19,) \\ E_6 &=& (E_5E_4)E_5(E_5E_4)^{-1} = \Gamma_7B_{10}A_2v_{r+8} \text{ if } r \neq 16, 17, 18, 19, \\ (E_6 &=& (E_5E_4)E_5(E_5E_4)^{-1} = \Gamma_7B_{12}A_2v_{r+8} \text{ if } r = 16, 17, 18, 19,) \end{array}$$

which are all contained in the subgroup K. Thus, we conclude that the element $E_6E_5^{-1} = A_2B_2^{-1} \in K$.

Since the element $C_1B_2^{-1} \in K$, as in the proof of Theorem 4.1, one can conclude that the Dehn twists A_1, A_2, B_i and C_j are in K for $i = 1, \ldots, r$ and $j = 1, \ldots, r-1$. This implies that $v_{r+2} = (B_3^{-1}C_{r-1}^{-1}A_2^{-1})E_1 \in K$, which finishes the proof. □

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