L^2 EXTENSION THEOREMS AND THEIR APPLICATIONS IN SEVERAL COMPLEX VARIABLES

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submitted by **AYLİN ÇOĞALMIŞ** in partial fulfillment of the requirements for the degree of **Master of Science in Mathematics Department, Middle East Technical University** by,

Prof. Dr. Halil Kalıpçılar
Dean, Graduate School of **Natural and Applied Sciences** _______________

Prof. Dr. Yıldıray Ozan
Head of Department, **Mathematics** _______________

Assist. Prof. Dr. Özcan Yazıcı
Supervisor, **Mathematics, METU** _______________

Examining Committee Members:

Assoc. Prof. Dr. Ali Ulaş Özgür Kişisel
Mathematics, METU _______________

Assist. Prof. Dr. Özcan Yazıcı
Mathematics, METU _______________

Prof. Dr. Uğur Gül
Mathematics, Hacettepe University _______________

____

____

Date:

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Surname: Aylin Çoğalmış

Signature :

ABSTRACT

L^2 EXTENSION THEOREMS AND THEIR APPLICATIONS IN SEVERAL COMPLEX VARIABLES

Çoğalmış, Aylin
M.S., Department of Mathematics
Supervisor: Assist. Prof. Dr. Özcan Yazıcı

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In this thesis we will survey existence of the solutions of $\overline{\partial}$- equation with L^2 estimates. The main part of the thesis involves the Ohsawa-Takegoshi Extension Theorem and its applications to the well-known problems so called Openness Conjecture and Suita Conjecture. In the last part, we consider some questions posed by Ohsawa and related answers to these questions.

Keywords: $\overline{\partial}$- Solution with L^2 Estimates, Ohsawa-Takegoshi Extension, Openness Conjecture, Suita Conjecture

ÖZ

L^2 GENİŞLETME TEOREMLERİ VE ÇOK DEĞİŞKENLİ KOMPLEKS ANALİZE UYGULAMALARI

Çoğalmış, Aylin
Yüksek Lisans, Matematik Bölümü
Tez Yöneticisi: Dr. Öğr. Üyesi. Özcan Yazıcı

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Bu tezde $\overline{\partial}$- denkleminin L^2 yaklaşımını sağlayan çözümleri üzerine çalışılmıştır. Bu tezin ana kısmında Ohsawa-Takegoshi Genişletme Teoremi ve bu teoremin Açıklık Sanısı ve Suita Sanısı olarak bilinen problemlere uygulamaları ele alınmıştır. Tezin son kısmında, Ohsawa tarafından önerilen bazı sorular ve bunlarla ilgili cevaplar üzerinde durulmuştur.

Anahtar Kelimeler: $\overline{\partial}$- denklemi, Ohsawa-Takegoshi Genişletme Teoremi, Açıklık Sanısı, Suita Sanısı

To my family and Onur.

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CHAPTER 1

INTRODUCTION

For a domain $\Omega \subseteq \mathbb{C}^n$ and a form $\alpha = \sum_{j=1}^{n} \alpha_j d\bar{z}_j$ on the domain, the equation $\bar{\partial} u = \alpha$ has many more alterations once $n > 1$. A necessary condition to solve this equation is that $\overline{\partial}\alpha = 0$. The equation for $n > 1$, also indicates how cumbersome it is to prove existence theorems and form the applications. In fact, the solution for the $\overline{\partial}$-problem is affected by the boundary of the domain. After great numbers of studies on the question, it turned out that the domain's being pseudo-convex plays a fundamental role for solving it. At the beginning, mathematicians studied $\overline{\partial}$-problem only on the products of polydiscs. Later, these results were extended for general domains. However, this time, due to the restrictions on the type of the differential equation systems, only for Cauchy-Riemann equations for an analytic function of several complex variables, one started to study more effective ways that can be applied to solve the general over determined systems for extension purposes.

D.C. Spencer and Garabedian [9] introduced such a way, on the other hand, this new treatment of the $\overline{\partial}$-problem for overdetermined systems brought some problematic results. Later, in 1958, Morrey [20] came up with a crucial idea that L^2 estimates are required for solving the problem. The method submitted by Morrey, then, improved and simplified by Kohn [17] and Ash [1]. In 1965, Lars Hörmander [16] simplified further what Kohn and Ash were working on by using basic consequences of "extensions of differential operators" and characterized the open sets for which estimates of Morrey-Kohn type are consistent. Then, to provide a valid proof for existence theorems and approximation theorems, L^2 estimates with density factor was introduced in the paper. Hörmander presented one of most vital results for solving $\overline{\partial}$-problem, regarding extension as well, which now we know as Hörmander's Estimate. Then, in 1987, Ohsawa presented the L^2-extension of holomorphic functions and following this, many other versions and proofs of the Ohsawa-Takegoshi Extension Theorem were obtained. In the light of these developments in the L^2 extension theory, in 1972, Suita [25] conjectured that for any bounded domain D in $\mathbb{C}$ one has $c_D^2 \leq \pi K_D$, where c_D is the logarithmic capacity of $\mathbb{C} \setminus D$ with respect to $z \in D$ and K_D is the Bergman kernel on the diagonal. Using elliptic functions and different domains, Suita obtained important results. Similarly, in 2001, Demailly and Kollár [8] proposed so called Openness Conjecture, which was proved by Guan and Zhou [12], using Ohsawa-Takegoshi Extension Theorem. Later, Berndtsson [3] gave a different proof in a more elementary way.

In this Masters thesis, in Chapter 2, in the first part, we provide some useful results from functional analysis revolving around the first order differential operators which are required for proving existence theorems, and the estimates we will use for the next section. In the second part, we will recall some important definitions, theorems on pseudo-convex domains in $\mathbb{C}^n$ and finish this chapter with Hörmander's Estimate. Chapter 3 is focused on the Ohsawa-Takegoshi Extension Theorem for domains in $\mathbb{C}^n$, which is of interest to solve extension problem with L^2 estimates of holomorphic functions. Being one of the most efficient results in several complex variables and complex geometry, the theorem has been proven in many different ways. Upon proving, we consider the methods used by Berndtsson mostly. Then, Chapter 4 is devoted to some applications of Ohsawa-Takegoshi Extension Theorem. Indeed, we will be working on Openness Conjecture and Suita Conjecture, which can be viewed as consequences of the theorem. Finally, in Chapter 5, we will consider a question posed by Ohsawa and see some answers to this question.

CHAPTER 2

PRELIMINARIES

2.1 Some Tools in Hilbert Space Theory

Let $D \subset \mathbb{C}^n$ be domain, u be a function defined in D and α be a L^2- form on D. The equation $\overline{\partial} u = \alpha$ can be solvable provided that

$$\overline{\partial}\alpha = 0.$$

Therefore, the so called $\overline{\partial}-$problem is only about giving a solution to the equation $\overline{\partial} u = \alpha$ where $\overline{\partial}\alpha = 0$.
The $\overline{\partial}-$ problem becomes more complicated in higher dimensions because unlike one-dimensional case, we lack of the compatibility condition $\overline{\partial}\alpha = 0$ in higher dimensions. At this point, by considering the weighted Hilbert Spaces $L^2(D, \varphi)$ where φ will be a special type of function we will define in the next section, we can also solve the problem in higher dimensions. However, to do so, we need some basic tools in Hilbert Space Theory.

Given Hilbert spaces H_1 and H_2, and $T : H_1 \to H_2$ a linear, closed, densely defined operator which satisfies the property $\langle Tf, g\rangle_2 = \langle f, T^*g\rangle_1$ for all $f \in H_1$ and $g \in H_2$. Define T^*: $H_2 \to H_1$ satisfying the same properties such that $T^{**} = T$ holds. By the definition of the adjoint operator, the orthogonal complement of the range $RanT$ of T is the null space $NullT^*$ of T^* .

Theorem 2.1.1. *Let T be a linear, closed, densely defined operator with domain DomT. Then the following are equivalent:*
(i) RanT is closed.

(ii)There exists some constant C such that $\|f\|_1 \leq C\|Tf\|_2$ for all $f \in DomT \cap \overline{RanT^*}$.

(iii)$RanT^*$ is closed.

(iv)There exists some constant C such that $\|g\|_2 \leq C\|T^*g\|_1$ for all $g \in DomT^* \cap \overline{RanT}$.

Proof. $(i) \Rightarrow (ii)$ Note that T is closed, in other words, T has a closed graph, so T is continuous by Closed Graph Theorem. Suppose that *RanT* is closed. Since the orthogonal complement of $\overline{RanT^*}$ is equal to *NullT*, we have $T|_{DomT\cap\overline{RanT^*}}$ is a closed linear map onto the subspace *RanT* of H_2. Also on $\overline{RanT^*}$, T is one to one, so the inverse of T exists.

Observe that the graph of T^{-1} is closed. To see this, assume that $(g_n, T^{-1}g_n)$ converges to (g, f) in norm. Then $g_n \to g$ and $T^{-1}g_n \to f$ in norm. Then, $TT^{-1}g_n \to Tf$, or equivalently, $g_n \to Tf$, and since the limit is unique, we have $g = Tf$. So $f = T^{-1}g$ and f is an element of the graph of T^{-1}. Thus, the graph of T^{-1} is closed. By Closed Graph Theorem, T^{-1} is continuous. Hence, $\|T^{-1}f\|_1 \leq C\|f\|_2$, applying the linear operator T to inside of each side of the inequality, we prove (ii).

$(ii) \Rightarrow (i)$ Suppose (ii) holds. Considering f as a vector, since T is a linear operator and $Tf = 0$ implies $f = 0$, *NullT* = $\{0\}$. The orthogonal complement of *RanT*, denoted by $RanT^{\perp}$, is equal to $NullT^*$. Then by Hilbert Space Theory, $NullT^{*\perp} = (RanT^{\perp})^{\perp} = RanT$.

Thus, we must show *RanT* =$\overline{RanT}$. By assumption of (ii),

$$T : DomT \cap \overline{RanT^*} \to RanT$$

is one-to-one (since $\overline{RanT^*} = NullT^{\perp}$, the functions on $\overline{RanT^*}$ are one-to-one), onto and closed, so by Closed Graph Theorem T is continuous. Then, $T|_{DomT\cap\overline{RanT^*}}$ is an isomorphism. Thus, $DomT \cap \overline{RanT^*} \cong RanT$ and $DomT \cap \overline{RanT^*}$ is closed subset of $DomT$. Since T is closed, *RanT* is closed.

$(iii) \Rightarrow (iv)$ Assume that $RanT^*$ is closed. The orthogonal complement of $\overline{RanT}$

is *NullT**, then, $T^*|_{DomT^* \cap \overline{RanT}}$ is closed, one-to-one linear map onto the subspace *RanT** of H_1. Hence, the inverse of T^* is a continuous map and this proves (iv).

$(iv) \Rightarrow (iii)$ Assume that there is a constant C such that $\|g\|_2 \leq C\|T^*g\|_1$ for all $g \in DomT^* \cap \overline{RanT}$. Since T^* is a linear operator and $T^*g = 0$ implies $g = 0$, *NullT**=$\{0\}$. The orthogonal complement of $RanT^{*\perp}$ equals to *NullT*, by Hilbert Space Theory,

$$NullT^{\perp} = (RanT^{*\perp})^{\perp} = \overline{RanT^*}.$$

Thus, we must show $\overline{RanT^*} = RanT^*$.
By assumption of (iv), $T^* : DomT^* \cap \overline{RanT} \to RanT^*$ is one-to-one, onto and since the graph of T^* is closed, T^* is closed. Then by Closed Graph Theorem, T^* is continuous and thus $T^*|_{DomT^* \cap \overline{RanT}}$ is an isomorphism. Thus, $DomT^* \cap \overline{RanT} \cong RanT^*$ and $DomT^* \cap \overline{RanT}$ is closed subset of $DomT^*$. Since T^* is closed, $RanT^*$ is closed.

$(ii) \Rightarrow (iv)$ Assume that (ii) holds. Consider for $g \in DomT^* \cap \overline{RanT}$,
$|(g, Tf)_2| = |(T^*g, f)_1| \leq ||T^*g||_1||f||_1 < C||T^*g||_1||Tf||_2$.
Hence, $|(g, h)_2| \leq ||T^*g||_1||h||_2$ for $g \in DomT^*$ and $h \in RanT$ which implies (iv).

□

Given a Hilbert Space H_3 and a closed, densely defined linear operator $S : H_2 \to H_3$, from now on we will assume that $ST = 0$. The space $RanT$ is contained in space $NullS$.

Theorem 2.1.2. *A necessary and sufficient condition for $RanT$ and $RanS$ to be closed is that $||g||_2^2 \leq C^2(||T^*g||_1^2 + ||Sg||_3^2)$ where $g \in DomT^* \cap DomS$ and $g \perp N = NullT^* \cap NullS$.*

Proof. Note that $ST = 0$ implies that $RanT \perp RanS^*$. To see this, let $Tf \in RanT$ and $S^*g \in RanS^*$, then $\langle Tf, S^*g\rangle = \langle STf, g\rangle = 0$ for all such f and g, so $RanT \perp RanS^*$.
By the fact that $RanT \perp RanS^*$ and N is perpendicular to this set, we have that $H_2 = \overline{RanT} \oplus N \oplus \overline{RanS^*}$. Now by Theorem 2.1.1$(ii)$, $RanT$ is closed if and only

if $||g||_2 \leq C||T^*g||_1$ where $g \in DomT^* \cap \overline{RanT}$ and by Theorem 2.1.1(iv), $RanS$ is closed if and only if $||g||_1 \leq C||Sg||_2$ where $g \in DomS \cap \overline{RanS^*}$.
Observe that since $g \in RanT$ and $g \in RanS^*$ and that $ST = 0$ implies that S vanishes on $RanT$, T^* vanishes on $RanS^*$, we have that

$$||g||_2 \leq C||T^*g||_1, \qquad ||g||_2 \leq C||Sg||_3.$$

Then, this shows that there exists $g \in H_2$ such that $g = g_1 \oplus g_2$, where $g_1 \in \overline{RanT}$, $g_2 \in \overline{RanS^*}$. Therefore, $RanT$ and $RanS$ are closed if and only if

$$||g||_2^2 \leq ||g_1||_2^2 + ||g_2||_2^2 \leq C^2\bigg(||T^*g||_1^2 + C||Sg||_3^2\bigg)$$

where $g \in DomT^* \cap DomS$ and $g \perp N$. This finishes the proof. □

Note that, $dimN = dim(NullS) - dim(\overline{RanT})$. This gives a strong possibility that in the applications, N shall be of finite dimension.

The inequality in 2.1.2 is a result which makes it easier to study than Theorem 2.1.1(ii) or (iv). A sufficient condition for 2.1.2 follows from compactness argument:

Theorem 2.1.3. *Suppose from any sequence $g_k \in DomT^* \cap DomS$ of bounded linear operators with $||g_k||_2$ bounded and $T^*g_k \to 0$ in H_1, $Sg_k \to 0$ in H_3, a strongly convergent subsequence can be chosen. Then,*

$$||g||_2^2 \leq C^2(||T^*g||_1^2 + ||Sg||_3^2) \tag{2.1.1}$$

holds and $N = NullT^* \cap NullS$ is finite dimensional.

Proof. Note that the null space is always closed. So $S^{-1}(\{0\}) = NullS$ and $(T^*)^{-1}(\{0\}) = NullT^*$ are closed. Then, $N = NullT^* \cap NullS$ is a closed subset of the H_2. Hence, N is a Hilbert Space. Since the unit sphere in N, say $\overline{B_1(0)}$, is closed and bounded, N is of finite dimension.
Now assume that the inequality (2.1.3) does not hold. Then, we could find a sequence $g_k \perp N \subseteq DomT^* \cap DomS$ with $||g_k||_2 = 1$ in H_1 and $T^*g_k = 0$, and $Sg_k = 0$ in H_3.
Hence, there is a strongly convergent subsequence of g_k. Let g be the strong limit

of g_k. Then, $||g||_2 = 1$ and $g \perp N$, but $T^*g = Sg = 0$ so that $g \in N$. Thus, $g \in N$ and $g = 0$, but this is a contradiction to $||g||_2 = 1$. Therefore, the inequality $||g||_2^2 \leq C^2(||T^*g||_1^2 + ||Sg||_3^2)$ holds. $\square$

Theorem 2.1.4. *Assume that A is a densely defined, linear operator in H_2, which is closed and F is a closed subspace of H_2 containing $RanT$ as a subset. Suppose*

$$||Af||_2^2 \leq ||T^*f||_1^2 + ||Sf||_3^2 \tag{2.1.2}$$

where $f \in DomT^* \cap DomS \cap F$. Then, we have $RanA^* \cap NullS \cap F \subset RanT$; if $g = A^*h, h \in DomA^*$ and $g \in NullS \cap F$, we can find $u \in DomT$ so that

$$Tu = g \textit{ and } ||u||_1 \leq ||h||_2.$$

Furthermore, if $v \in RanT$, we can choose $f \in DomA \cap DomT^*$ so that

$$T^*f = v \textit{ and } ||Af||_2 \leq ||v||_1.$$

Proof. Let $g = A^*h$ where $h \in DomA^*$ and $g \in NullS \cap F$. We must find $u \in H_1$ so that $||u||_1 \leq ||h||_2$ and $Tu = g$, or equivalently, $\langle u, T^*f\rangle_1 = \langle g, f\rangle_2$ for all $f \in DomT^*$. On the other hand, by Hahn- Banach Theorem, this is equivalent to proving $|\langle g, f\rangle_2| \leq \|h\|_2\|T^*f\|_1$ where $f \in DomT^*$. To clarify, observe that $T^*f \in RanT^* \subseteq H_1$, and let φ be defined so that $\varphi(T^*f) = \langle g, f\rangle_2$ for all $f \in DomT^*$. Then by $|\langle g, f\rangle_2| \leq \|h\|_2\|T^*f\|_1$, we have

$$||\varphi|| = ||h||_2.$$

Then by Hahn- Banach Theorem, φ extends to a linear functional on H_1, say $\tilde{\varphi}$ such that

$$||\tilde{\varphi}|| = ||h||_2.$$

Then, bu Riesz's Representation, $\tilde{\varphi}(T^*f) = \langle g, f\rangle_2 = \langle u, T^*f\rangle$ for some $u \in H_1$ with $||u||_1 = ||\tilde{\varphi}|| = ||h||_2$. Therefore, using Hahn-Banach Theorem, it is sufficient to prove that

$$|\langle g, f\rangle_2| \leq \|h\|_2\|T^*f\|_1$$

where $f \in DomT^*$.

If $f \perp NullS \cap F$, then $T^*f = 0$ because $RanT \subset NullS \cap F$ and so $f \in (RanT)^\perp = NullT^*$. Since $g \in Null \cap F$, it is enough to show that $|\langle g, f\rangle_2| \leq$

$\|h\|_2\|T^*f\|_1$ where $f \in DomT^*$ holds when $f \in NullS \cap F$. Then the inequality given in the theorem becomes $||Af||_2^2 \leq ||T^*f||_1^2$ and this gives,

$$|\langle g, f\rangle_2| = |\langle A^*h, f\rangle_2| = |\langle h, Af\rangle_2| \leq \|h\|_2\|Af\|_2 \leq \|h\|_2\|T^*f\|_1.$$

This finishes the first part of the proof.
For the second part, by the definition of $RanT^*$, $f \in NullS \cap F \cap DomT^*$ so that $T^*f = v$. But then, by $|\langle g, f\rangle_2| \leq \|h\|_2\|T^*f\|_1$, $f \in DomA$ and that
$\|Af\|_2 \leq \|v\|_1$ for $v \in RanT^*$.

$\square$

Recall that a differential form is said to be of type (p, q) if it can be represented in the form

$$f = \sum_{|I|=p, |J|=q,} f_{I,J}.dz^I \wedge d\overline{z}^J$$

where $I = (i_1, i_2, .., i_p)$ and $J = (j_1, j_2, .., j_q)$ are sequences of indices between 1 and n, and the summation is only over strictly increasing multi-indices and we write,

$$dz^I \wedge d\overline{z}^J = dz_{i_1} \wedge dz_{i_2} \wedge ... \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge d\overline{z}_{j_2} \wedge .. \wedge d\overline{z}_{j_q}$$

The coefficients $f_{I,J}$ might be linear functionals defined on $C^\infty(\Omega), \Omega \subset \mathbb{C}^n$, which are also called distributions and should be defined for arbitrary I and J so that they are antisymmetric both in the indices of I and of J. We also have the following operation:

$$\overline{\partial} f = \sum_{I,J} \frac{\partial f_{I,J}}{\partial \overline{z_k}} d\overline{z_k} \wedge dz^I \wedge d\overline{z}^J$$

which leads the form $\overline{\partial} f$ to be of type $(p, q+1)$ and that $\overline{\partial\partial} f = 0$.
For instance, let $f \in L^2_{(0,1)}(\Omega)$ where $\Omega \subset \mathcal{C}^2$. Then, the $\overline{\partial}$ operator applied to f gives the following

$$\overline{\partial} f = \frac{\partial f}{\partial \overline{z}_1} d\overline{z}_1 + \frac{\partial f}{\partial \overline{z}_2} d\overline{z}_2,$$

Hence, we have

$$\begin{aligned}
\overline{\partial\partial} f &= \frac{\partial}{\partial \overline{z}_1}\left(\frac{\partial f}{\partial \overline{z}_1} d\overline{z}_1 + \frac{\partial f}{\partial \overline{z}_2} d\overline{z}_2\right) \wedge d\overline{z}_1 + \frac{\partial}{\partial \overline{z}_2}\left(\frac{\partial f}{\partial \overline{z}_1} d\overline{z}_1 + \frac{\partial f}{\partial \overline{z}_2} d\overline{z_2}\right) \wedge d\overline{z}_2 \\
&= \frac{\partial^2 f}{\partial \overline{z}_1^2} d\overline{z}_1 \wedge d\overline{z}_1 + \frac{\partial^2 f}{\partial \overline{z}_1 \partial \overline{z}_2} d\overline{z}_2 \wedge d\overline{z}_1 + \frac{\partial^2 f}{\partial \overline{z}_2 \partial \overline{z}_1} d\overline{z}_1 \wedge d\overline{z}_2 + \frac{\partial^2 f}{\partial \overline{z}_2^2} d\overline{z}_2 \wedge d\overline{z}_2 \\
&= 0
\end{aligned}$$

where the last equality follows from the property that $d\bar{z}_1 \wedge d\bar{z}_2 = -d\bar{z}_2 \wedge d\bar{z}_1$.

The value of a linear functional g on a function $\varphi \in C^\infty$ is denoted by $\langle g, \varphi \rangle$. Two distributions g_1 and g_2 are said to be equal in their domain if $\langle g_1, \varphi \rangle = \langle g_2, \varphi \rangle$ for all $\varphi \in C^\infty$. A distribution is the sum of distributions g_1 and g_2 if $\langle g, \varphi \rangle = \langle (g_1 + g_2), \varphi \rangle$ for all $\varphi \in C^\infty$.

Let $\mathcal{F}$ be the space of distributions, consisting of (p, q) forms with coefficients in $\mathcal{F}$ is denoted by $\mathcal{F}_{(p,q)}$. In fact, we will use this notation with $\mathcal{F} = C^k(\overline{\Omega})$ and denote with $C_0^k(\Omega)$ the elements of $\mathcal{F}$ which vanish outside a big sphere. For φ which is measurable on Ω and locally bounded from above, we write the space of square integrable functions with respect to the factor $e^{-\varphi}$ in Ω by $L^2_{(p,q)}(\Omega, \varphi)$ and the norm in $L^2_{(p,q)}(\Omega, \varphi)$ is defined by

$$||f||^2_\varphi = \int |f(z)|^2 e^{-\varphi} dV \text{ where } f \in L^2_{(p,q)}(\Omega, \varphi)$$

and dV is the Lebesgue measure and

$$|f(z)|^2 = \langle f(z), f(z) \rangle = \sum |f_{I,J}(z)|^2.$$

Finally, let $L^2(\Omega, loc)$ denote the space of square integrable functions on all compact subsets of Ω, in other words, for any compact set K,

$$L^2(\Omega, loc) = \{f : f : K \to \mathbb{C}, \int_K |f(z)|^2 dz < \infty\}.$$

Note that, $L^2(\Omega, loc)$ is a complete metric space with the inner product

$$\langle f, g \rangle = \int f\overline{g} dz.$$

Thus, $L^2(\Omega, loc)$ is a Hilbert space. If p, q are fixed where $q > 0$, we denote the maximal (weak) differential operator $\overline{\partial}$ from $L^2_{(p,q-1)}(\Omega, \varphi)$ to $L^2_{(p,q)}(\Omega, \varphi)$ by T, so a form $u \in L^2_{(p,q-1)}(\Omega, \varphi)$ is in $DomT$ if and only if $\overline{\partial} u$ belongs to $L^2_{(p,q)}(\Omega, \varphi)$. Observe that if φ is continuous, then T is closed and densely defined. Likewise, the operator $\overline{\partial}$ defines a closed and densely defined operator S from $L^2_{(p,q)}(\Omega, \varphi)$ to $L^2_{(p,q+1)}(\Omega, \varphi)$. Then, by $\overline{\partial\partial} f = 0$, we have $ST = 0$.

Then, all the theorems provided above are also available to be applied here, which will help us to prove the following series of estimates. Before continuing with the estimates, consider the following proposition.

Proposition 2.1.5. *Let A be a linear operator as in Theorem 2.1.4. Let u be the solution of $Au = f$ with compact support. Then, there exists $u^v \in C_0^\infty$ with $u^v \to u$ and $||Au^v - Au|| \to 0$ as $v \to \infty$.*

For the proof, see the first section of [16].

Proposition 2.1.6. *$C^1_{(p,q)}(\overline{\Omega}) \cap DomT^*$ is dense in $DomT^* \cap DomS$ in the graph norm $f \mapsto (||f||^2 + ||T^*f||^2 + ||Sf||^2)^{1/2}$ if the boundary $\partial\Omega$ of Ω is of class C^2 and $\varphi \in C^1(\overline{\Omega})$.*
Further, $C^1_{(p,q-1)}(\overline{\Omega})$ is dense in $DomT$ in the graph norm $f \mapsto (||f||^2 + ||Tf||^2)^{1/2}$.

Proof. Let $\chi \in C^\infty(\overline{\Omega})$ and $f \in DomS$, then $\chi f \in DomS$ and

$$\begin{aligned} ||S(\chi f) - \chi(Sf)||_\varphi = ||\overline{\partial}(\chi f) - \chi\overline{\partial} f||_\varphi &= ||\chi\overline{\partial} f + (\overline{\partial}\chi)f - \chi\overline{\partial} f||_\varphi \\ &= [\int |(\overline{\partial}\chi)f|^2 e^{-\varphi}]^{1/2} \\ &\leq C.sup|grad\chi|.||f||_\varphi. \end{aligned}$$

Similar result is valid for T,

$$||T(\chi f) - \chi(Tf)||_\varphi \leq C.sup|grad\chi|.||f||_\varphi.$$

By the fact that

$$\begin{aligned} |\langle \chi f, Tu\rangle_\varphi - \langle f, T(\overline{\chi}u)\rangle_\varphi| &= \left| \int (\chi f\overline{\overline{\partial} u} - f\overline{\overline{\partial}(\overline{\chi}u)})e^{-\varphi} \right| \\ &= \left| \int (\chi f\overline{\overline{\partial} u} - f(\overline{\partial}\overline{\chi})u - f(\overline{\partial}u)\overline{\chi})e^{-\varphi} \right| \\ &\leq C.sup|grad\chi|.||u||_\varphi.||f||_\varphi \end{aligned} \tag{2.1.3}$$

we have,

$$|\langle T^*(\chi f), u\rangle_\varphi - \langle \chi T^* f, u\rangle_\varphi| \leq C.sup|grad\chi|.||u||_\varphi.||f||_\varphi.$$

From (2.1.3), we also see that if $f \in DomT^*$, then $\chi f \in DomT^*$ and

$$||T^*(\chi f) - \chi(T^* f)||_\varphi \leq C \sup |grad\chi|.||f||_\varphi.$$

Let $\chi \in C_0{}^\infty(\mathbb{C}^n)$, so $\chi = 0$ outside of compact set. Let χ satisfy the condition that $\chi(0) = 1$ and set $\chi^\epsilon(z) = \chi(\epsilon z)$. If $f \in DomT^* \cap DomS$, then $\chi^\epsilon f \in DomT^* \cap DomS$ and the following holds

$$\chi^\epsilon f \mapsto f,$$

$$S(\chi^\epsilon f) \mapsto Sf,$$

$$T^*(\chi^\epsilon f) \mapsto T^* f$$

in the appropriate L^2 spaces as $\epsilon \to 0$. Thus, this is enough to estimate for $\chi^\epsilon f$ with compact support. Note that the second item holds because $||S(\chi^\epsilon f) - \chi^\epsilon(Sf)||_\varphi \leq C.\epsilon.sup|grad\chi|.||f||_\varphi \to 0$.

To clarify, the reason we take $\chi^\epsilon f$ is that we want to change f and bring it into a smooth function as $\epsilon \to 0$ because as $\epsilon \to 0$, $\chi^\epsilon f \to f$. Moreover, $\chi^\epsilon f \equiv 0$ outside compact sets.
Then, $\chi^\epsilon f \to f$ means that approximating $\chi^\epsilon f$ is the same as approximating f. By the smoothness of $\chi^\epsilon f$, we will be able to eliminate the $\partial\Omega - integral$ in the Green's Formula and use it for later.

Starting the proof, approximate the elements $f \in DomT^* \cap DomS$ which are 0 outside a sphere large enough. By definition of T^* and since the elements of $DomT^*$ satisfy the Cauchy-Boundary Condition (in the weak sense), the proof will follow from the Proposition 2.1.5. □

Assume that the boundary $\partial\Omega$ of Ω satisfies C^2 and let ϱ be a real valued function in $C^2(\overline{\Omega})$, that is 0 on the boundary of Ω, is negative on Ω and be such that $|grad\varrho| = 1$ on $\partial\Omega$. Then, gradϱ is the exterior unit normal on $\partial\Omega$, so the Green's Formula may be written as, if $v, w \in Int(C^1(\overline{\Omega}))$,

$$\int\limits_\Omega \frac{\partial v}{\partial \overline{x}_j} \overline{w} e^{-\varphi} dV = -\int\limits_\Omega v\overline{(\frac{\partial w}{\partial x_j} - w\frac{\partial \varphi}{\partial x_j})} e^{-\varphi} dV + \int\limits_{\partial\Omega} \frac{\partial \varrho}{\partial x_j} v\overline{w} e^{-\varphi} ds$$

where dS is the Euclidean surface element on $\partial\Omega$. To see this,

Let $\Omega = \{\varrho < 0\}$ and on $\partial\Omega$, $|grad\varrho| = 1$, so grad ϱ is the unit normal and let $v, w \in C^1(\overline{\Omega})$, that is, the first partial derivatives of v and w exist. Recall the Divergence Formula:

$$\int\limits_{\Omega} \nabla.F.dV = \int\limits_{\partial\Omega} \langle F, \vec{n}\rangle.ds$$

where ∇ is given by $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, .., \frac{\partial}{\partial x_n})$, F is a vector field and $\vec{n}$ is the gradient of ϱ. Take $F = (0, 0, .., v\overline{w}e^{-\varphi}, 0, .., 0)$ where $v\overline{w}e^{-\varphi}$ is the j^{th}- term.

Consider $gradF = (0, 0, .., \frac{\partial}{\partial x_j}(v\overline{w}e^{-\varphi}), 0, 0, .., 0)$ and $grad\varrho = \frac{\partial\varrho}{\partial x_j}$

$$gradF = (0, 0, .., \frac{\partial v}{\partial x_j}\overline{w}e^{-\varphi} + e^{-\varphi}v(\frac{\partial\overline{w}}{\partial x_j} - \overline{w}\overline{\frac{\partial\varphi}{\partial x_j}}), 0, 0, .., 0)$$

Then, we have,

$$\begin{aligned}
\int\limits_{\Omega} \nabla.F.dV = \int\limits_{\Omega} \frac{\partial}{\partial x_j}(v.\overline{w}.e^{-\varphi})dV \quad &= \quad \int\limits_{\Omega} \left[\frac{\partial v}{\partial x_j}\overline{w}e^{-\varphi} + ve^{-\varphi}(\frac{\partial\overline{w}}{\partial x_j} - \overline{w}\overline{\frac{\partial\varphi}{\partial x_j}})\right] dV \\
&\Longleftrightarrow \int\limits_{\Omega} \nabla.F.dV = \int\limits_{\partial\Omega} \langle F, \vec{n}\rangle.ds \\
&\Longleftrightarrow \int\limits_{\Omega} \frac{\partial v}{\partial x_j}\overline{w}e^{-\varphi}dV + \int\limits_{\Omega} ve^{-\varphi}(\frac{\partial\overline{w}}{\partial x_j} - \overline{w}\overline{\frac{\partial\varphi}{\partial x_j}})dV \\
&= \int\limits_{\partial\Omega} v\overline{w}e^{-\varphi}\frac{\partial\varrho}{\partial x_j}ds
\end{aligned}$$

$$\Longleftrightarrow \int\limits_{\Omega} \frac{\partial v}{\partial x_j}\overline{w}e^{-\varphi}dV = \int\limits_{\partial\Omega} v\overline{w}e^{-\varphi}\frac{\partial\varrho}{\partial x_j}ds - \int\limits_{\Omega} ve^{-\varphi}\overline{(\frac{\partial w}{\partial x_j} - w\frac{\partial\varphi}{\partial x_j})}dV$$

Repeating the same procedure for y_j-terms, we obtain the z_j-terms:

$$\delta_j = \frac{\partial w}{\partial z_j} - w\frac{\partial\varphi}{\partial z_j} = e^{\varphi}\frac{\partial(we^{-\varphi})}{\partial z_j} \tag{2.1.4}$$

Then we obtain,

$$\int\limits_{\Omega} \frac{\partial v}{\partial \overline{z}_j}\overline{w}e^{-\varphi}dV = -\int\limits_{\Omega} v\overline{\delta_j w}e^{-\varphi}dV + \int\limits_{\partial\Omega} \frac{\partial \varrho}{\partial z_j}v\overline{w}e^{-\varphi}ds \tag{2.1.5}$$

Note that, for $\varphi \in C^2$, we have commutation relations, in other words, consider the following,

$$\begin{aligned}
\Big(\delta_k \frac{\partial}{\partial \overline{z}_j} - \frac{\partial}{\partial\, \overline{z}_j}\delta_k\Big)w &= \frac{\partial^2 w}{\partial z_k \partial \overline{z}_j} - \frac{\partial w \partial \varphi}{\partial \overline{z}_j \partial z_k} - \frac{\partial^2 w}{\partial \overline{z}_j \partial z_k} + \frac{\partial}{\partial \overline{z}_j}\Big(w\frac{\partial \varphi}{\partial z_k}\Big) \\
&= \frac{\partial^2 w}{\partial z_k \partial \overline{z}_j} - \frac{\partial w \partial \varphi}{\partial \overline{z}_j \partial z_k} - \frac{\partial^2 w}{\partial \overline{z}_j \partial z_k} + \Big(\frac{\partial w}{\partial \overline{z}_j}\frac{\partial \varphi}{\partial \overline{z}_k} + w\frac{\partial^2 \varphi}{\partial \overline{z}_j \partial z_k}\Big) \\
&= w\frac{\partial^2 \varphi}{\partial \overline{z}_j \partial z_k}
\end{aligned}$$

Thus we have,

$$\Big(\delta_k \frac{\partial}{\partial \overline{z}_j} - \frac{\partial}{\partial \overline{z}_j}\delta_k\Big)w = w\frac{\partial^2 w}{\partial \overline{z}_j \partial z_k} \tag{2.1.6}$$

with $w \in C^2$, which implies that

$$\begin{aligned}
\int\limits_{\Omega} \overline{\overline{\delta_j(v)}\delta_k(w)e^{-\varphi}}dV &- \int\limits_{\Omega} \frac{\partial v}{\partial \overline{z}_k}\frac{\partial w}{\partial \overline{z}_j}e^{-\varphi}dV \qquad (2.1.7)\\
&= -\int\limits_{\Omega} \frac{\partial}{\partial z_j}\overline{\delta_k(w)}ve^{-\varphi}dV + \int\limits_{\partial\Omega} \frac{\partial \varrho}{\partial z_j}\overline{\delta_k(w)}ve^{-\varphi}ds \\
&+ \int\limits_{\Omega} v\overline{\delta_k(\frac{\partial \overline{w}}{\partial z_k})}e^{-\varphi}dV - \int\limits_{\partial\Omega} \frac{\partial \varrho}{\partial \overline{z}_k}v\frac{\partial w}{\partial \overline{z}_k}e^{-\varphi}ds \\
&= \int\limits_{\Omega} v\overline{w}\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k}e^{-\varphi}dV + \int\limits_{\partial\Omega} \frac{\partial \varrho}{\partial z_j}\overline{\delta_k(w)}ve^{-\varphi}ds \\
&- \int\limits_{\partial\Omega} \frac{\partial \varrho}{\partial \overline{z}_k}v\frac{\overline{\partial} w}{\partial \overline{z}_j}e^{-\varphi}ds. \qquad (2.1.8)
\end{aligned}$$

Indeed, the last equality is a direct result of (2.1.5) and (2.1.6) if $w \in Int(C^2(\overline{\Omega}))$ and it follows when $w \in Int(C^1(\overline{\Omega}))$ because $Int(C^2(\overline{\Omega}))$ is a dense subset. In other words, working with C^1 is enough.

For convenience, we shall describe the space of $Int(C^1_{(p,q)}(\overline{\Omega}))\cap DomT^*$ in the Proposition 2.1.6. Consider the inner product defined below,

$$\langle\overline{\partial}u, f\rangle_\varphi = \int\limits_\Omega \langle\overline{\partial}u, f\rangle e^{-\varphi}dV$$

where $f \in Int(C^1_{(p,q)}(\overline{\Omega}))$ and $u \in Int(C^1_{(p,q-1)}(\overline{\Omega}))$. Note that, the reason why we take $\langle\overline{\partial}u, f\rangle$ with $f \in C^1_{(p,q)}(\overline{\Omega})$ is to make sure the agreement in the number of wedge products. Now, let us pass from the differentiations of u to the ones of f. Observe that $\overline{\partial} = T$ so, $\langle Tu, f\rangle = \langle u, T^*f\rangle$.

Let $u = \sum_{|I|=p,|K|=q-1} u_{I,K}dz^I \wedge d\overline{z}^K$, then,

$$\overline{\partial}u = (-1)^p \sum_{I,K}\sum_j \frac{\partial u_{I,K}}{\partial\overline{z}_j}dz^I \wedge d\overline{z}_j \wedge d\overline{z}^K.$$

Plus, by Green's Formula we have,

$$\begin{aligned}
\langle\overline{\partial}u, f\rangle_\varphi &= (-1)^p \int\limits_\Omega \sum_{I,K}\sum_j \frac{\partial u_{I,K}}{\partial\overline{z}_j}\overline{f_{I,jK}}e^{-\varphi}dV \\
&= (-1)^{p-1}\int\limits_\Omega \sum_{I,K}\sum_j u_{I,K}\overline{\delta_j(f_{I,jK})}e^{-\varphi}dV + (-1)^p\int\limits_{\partial\Omega}\sum_{I,K}\sum_{j=1^n}\frac{\partial\varrho}{\partial\overline{z}_j}u_{I,K}\overline{f}_{I,jK}e^{-\varphi}ds
\end{aligned}$$

Also note that, since $\langle\overline{\partial}u, f\rangle_\varphi = \langle u, T^*f\rangle_\varphi$ and $T^*f = \overline{\delta_j(f_{I,jK})}$, we have that

$$\sum_{I,K}\sum_{j=1}^n \frac{\partial\varrho}{\partial\overline{z}_j}u_{I,K}\overline{f}_{I,jK}e^{-\varphi}ds = \sum_{I,K}u_{I,K}\sum_{j=1}^n \overline{f_{I,jK}\frac{\partial\varrho}{\partial z_j}}e^{-\varphi}ds.$$

Since $Int(C^1_{(p,q-1)})$ is dense in $DomT$ for the graph norm given in Proposition 2.1.6, we conclude that for any $u \in DomT$,

$$\int_{\partial\Omega}\sum_{I,K}u_{I,K}\sum_{j=1}^n \overline{f_{I,jK}\frac{\partial\varrho}{\partial z_j}}e^{-\varphi}ds = 0.$$

Therefore, for all I, K and $T^*f = (-1)^{p-1}\sum\limits_{I,K}\sum\limits_{j=1}^n \delta_j f_{I,jK}dz^I \wedge d\overline{z}^K$ with $|I| =$

$p, |K| = q - 1$, it holds that

$$f \in DomT^* \text{ if } \sum_{j=1}^{n} f_{I,jK} \frac{\partial \varrho}{\partial z_j} = 0 \text{ on } \partial\Omega. \tag{2.1.9}$$

If $f \in C^1_{(p,q)}(\overline{\Omega}) \cap DomT^*$, then

$$\begin{aligned} ||T^* f||^2_\varphi + ||Sf||^2_\varphi &= \sum_{I,K} \sum_{j,k=1}^{n} \int_\Omega \delta_j f_{I,jK} \overline{\delta_k f_{I,kK}} e^{-\varphi} dV \\ &+ \sum_{I,J,L} \sum_{i,l=1} \int_\Omega \frac{\partial f_{I,J}}{\partial \overline{z}_j} \overline{\frac{\partial f_{I,L}}{\partial \overline{z}_l}} \epsilon^{iJ}_{lL} e^{-\varphi} dV \end{aligned} \tag{2.1.10}$$

where $\epsilon^{iJ}_{lL} = 0$ if $j \in J$, in other words $d\overline{z}^J \wedge d\overline{z}_j$ with $d\overline{z}_j = 0$, $l \notin L$ and $\{j\} \cup J = \{l\} \cup L$, in which case ϵ^{iJ}_{lL} is the sign of the permutation $\binom{jJ}{lL}$.
To clarify, consider the following illustration, if $(1, 2, 3) \neq (2, 3, 4)$, then $f_{I,jJ} dz_I \wedge d\overline{z}_1 \wedge d\overline{z}_2 \wedge d\overline{z}_3 \iff 0.dz_I \wedge d\overline{z}_1 \wedge d\overline{z}_2 \wedge d\overline{z}_3$ in the inner product and otherwise, $\epsilon^{iJ}_{lL} = 1$.

We shall rearrange the terms in the last sum. We will consider two cases:
Assume that $j = l$, which implies that $J = L$ and $j \notin J$ if $\epsilon^{iJ}_{lL} \neq 0$. Then the last sum becomes:

$$\sum_{I,J} \sum_{j \notin J} \int_\Omega \left| \frac{\partial f_I J}{\partial \overline{z}_j} \right|^2 e^{-\varphi} dV$$

Next, assume that $j \neq l$. In this case, since $\{j\} \cup J = \{l\} \cup L$, this implies that $j \in L$, $l \in J$ and if $\epsilon^{iJ}_{lL} \neq 0$, we have $L/\{j\} = J/\{l\} = K$, or that, $lK = J$ and $jK = L$. Then, by $\epsilon^{iJ}_{lL} = \epsilon^{jJ}_{jlK} \epsilon^{jlK}_{ljK} \epsilon^{ljk}_{lL} = -\epsilon^{J}_{lK} \epsilon^{jK}_{L}$, the last sum becomes:

$$-\sum_{I,K} \sum_{j \neq l} \int_\Omega \frac{\partial f_{I,lK}}{\partial \overline{z}_j} \overline{\frac{\partial f_{I,jk}}{\partial \overline{z}_l}} e^{-\varphi} dV.$$

Therefore, the equation (2.1.10) becomes the following:

$$||T^*f||^2_\varphi + ||Sf||^2_\varphi = \sum_{I,K}\sum_{j,K}\int_\Omega \delta_j f_{I,jK}\overline{\delta_k f_{I,kK}}e^{-\varphi}dV - \sum_{I,K}\sum_{j,k}\int_\Omega \frac{\partial f_{I,jK}}{\overline{z}_k}\overline{\frac{\partial f_{I,kK}}{\partial \overline{z}_l}}e^{-\varphi}dV + \sum_{I,J}\sum_{j}\int_\Omega \left|\frac{\partial f_{I,J}}{\partial \overline{z}_j}\right|^2 e^{-\varphi}dV.$$

After rearranging the terms in the last sum and hence in the whole equation, we now turn our attention to integration by parts, and to moving all differentiations to the right.

By using previous results, we obtain that,

$$||T^*f||^2_\varphi + ||Sf||^2_\varphi = \sum_{I,K}\sum_{j,k}\int_\Omega f_{I,jK}\overline{f_{I,kK}}\frac{\partial^2\varphi}{\partial z_j \partial \overline{z}_k}e^{-\varphi}dV + \sum_{I,J}\sum_{j}\int_\Omega \left|\frac{\partial f_{I,J}}{\partial \overline{z}_j}\right|^2 e^{-\varphi}dV - \sum_{I,K}\sum_{j,k}\int_{\partial\Omega} f_{I,jK}\frac{\partial\varrho}{\partial \overline{z}_k}\overline{\frac{\partial f_{I,kK}}{\partial \overline{z}_j}}e^{-\varphi}ds.$$

Let $g = \sum_k f_{I,kK}\frac{\partial\varrho}{\partial z_k}$, and $g \equiv 0$ on $\partial\Omega$, so, $\nabla g \parallel \nabla\varrho$, which also implies that $\overline{\nabla g} \parallel \overline{\nabla \varrho}$. Then, the j^{th} term of ∇g is equal to

$$\sum_k \left(\frac{\partial f_{I,kK}}{\partial \overline{z}_j}\frac{\partial\varrho}{\partial z_k} + f_{I,kK}\frac{\partial^2\varrho}{\partial \overline{z}_j \partial z_k}\right) = C\frac{\partial\varrho}{\partial \overline{z}_j} \text{ for some constant } C.$$

Thus, for any point on the boundary, it holds that

$$\sum_k \left(\frac{\partial f_{I,kK}}{\partial \overline{z}_j}\frac{\partial\varrho}{\partial z_k} + f_{I,kK}\frac{\partial^2\varrho}{\partial \overline{z}_j \partial z_k}\right) = C\frac{\partial\varrho}{\partial \overline{z}_j} \text{ for } j = 1, 2, .., n.$$

Multiplying this equation with $\overline{f_{I,jK}}$ we get,

$$\sum_{j,k} \left(\overline{f_{I,jK}}\frac{\partial f_{I,kK}}{\partial \overline{z}_j}\frac{\partial\varrho}{\partial z_k} + \overline{f_{I,jK}}f_{I,kK}\frac{\partial^2\varrho}{\partial \overline{z}_j \partial z_k}\right) = 0 \text{ on } \partial\Omega,$$

by 2.1.9. Thus, we have proved the following:

Proposition 2.1.7. *The following identity is valid when* $f \in Int(C^1_{(p,q)}(\overline{\Omega})) \cap Dom T^*$:

$$||T^*f||^2_\varphi + ||Sf||^2_\varphi = \sum_{I,K}\sum_{j,k}\int_\Omega f_{I,jK}\overline{f_{I,kK}}\frac{\partial^2\varphi}{\partial z_j \partial \overline{z}_k}e^{-\varphi}dV + \sum_{I,J}\sum_{j}\int_\Omega \left|\frac{\partial f_{I,J}}{\partial \overline{z}_j}\right|^2 e^{-\varphi}dV + \sum_{I,K}\sum_{j,k}\int_{\partial\Omega} f_{I,jK}\overline{f_{I,kK}}\frac{\partial^2\varrho}{\partial z_j \partial \overline{z}_k}e^{-\varphi}ds.$$

Now consider the first sum of the last proposition,

$$\sum_{I,K}\sum_{j,k}\int_{\Omega} f_{I,jK}\overline{f_{I,kK}}\frac{\partial^2\varphi}{\partial z_j\partial\overline{z}_k}e^{-\varphi}dV.$$

Being ascribed to the weight function $e^{-\varphi}$, this term will be practical for the estimates that we will do later, and will allow us to extend the work quoted that is defined for the surface integral in the last proposition. However, to be able to continue, we need to recall few definitions in pluripotential theory.

2.2 Hörmander's Solution to the $\overline{\partial}$ Problem

Definition 2.2.1. *A function $\varphi : \Omega \to \mathbb{C}$ is said to be $plurisubharmonic$ provided that it is upper semi-continuous and locally integrable and*

$$\sum_{j,k=1}^{n} t_j\bar{t}_k\frac{\partial^2\varphi}{\partial z_j\partial\overline{z}_k} \geq 0$$

for any $t = (t_1, t_2, \ldots, t_n) \in \mathbb{C}^n$, in the sense of distributions. That is,

$$\int_{\Omega}\varphi(z)\sum_{j,k} t_j\overline{t_k}\frac{\partial^2 u}{\partial z_j\partial\overline{z}_k}dV \geq 0$$

for all $\varphi \in C_0^\infty(\Omega)$.

For $\chi \in C(\Omega)$, e^χ is a lower bound for plurisubharmonicity of φ if the difference

$$\sum_{j,k=1}^{n} t_j\bar{t}_k\frac{\partial^2\varphi}{\partial z_j\partial\overline{z}_k} - e^\chi\sum_{1}^{n}|t_j|^2$$

is a positive measure for arbitrary complex numbers t_j.

Definition 2.2.2. *A set $M \subset \mathbb{R}^n$ is a C^k-smooth hypersurface if at each point $p \in M$, there exists a k-times continuously differentiable function $\varrho : V \to \mathbb{R}$ with non-vanishing derivative, defined in a neighborhood V of p such that*

$$M \cap V = \{x \in V : \varrho(x) = 0\}.$$

The function ϱ is called the $defining\ function$ of M at p.

Adapting these definitions to what we have been working with, we have the following definition.

Definition 2.2.3. *Let Ω be open and $\partial\Omega$ is smooth. The boundary $\partial\Omega$ is said to be pseudo-convex if at every point of $\partial\Omega$,*

$$\sum_{j,k=1}^{n} t_j \bar{t}_k \frac{\partial^2 \varrho}{\partial z_j \partial \overline{z}_k} \geq 0 \text{ whenever } \sum_{j=1}^{n} t_j \frac{\partial \varrho}{\partial z_j} = 0,$$

where $(t_1, t_2, ..., t_n) \in \mathbb{C}^n$.

Remark 2.2.4. *Recall that $\sum_{j,k=1}^{n} t_j \bar{t}_k \frac{\partial^2 \varrho}{\partial z_j \partial \overline{z}_k}$ is called the Hermitian form and if, in this definition, the Hermitian form is strictly positive for all such $t \neq 0$, then the boundary $\partial\Omega$ is said to be strictly pseudo-convex.*

Remark 2.2.5. *Observe that the components t_j, t_k appearing in the definition correspond to $f_{I,jK}, f_{I,kK}$ of the Proposition 2.1.7, respectively.*

If $\partial\Omega$ is pseudo-convex, it follows from

$$\sum_{j=1}^{n} f_{I,jK} \frac{\partial \varrho}{\partial z_j} = 0 \text{ on } \partial\Omega, \text{ for all } I, K$$

that the last sum of the Proposition 2.1.7,

$$\sum_{I,K} \sum_{j,k} \int_{\partial\Omega} f_{I,jK} \overline{f_{I,kK}} \frac{\partial^2 \varrho}{\partial z_j \partial \overline{z}_k} e^{-\varphi} ds$$

is non-negative. This gives us the following theorem:

Theorem 2.2.6. *If $\partial\Omega$ is pseudo-convex, and $f \in Int(C^1_{(p,q)}(\overline{\Omega})) \cap DomT^*$, then*

$$\int_{\Omega} \sum_{I,K} \sum_{j,k} f_{I,jK} \overline{f_{I,kK}} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} e^{-\varphi} dV \leq ||T^* f||^2_{\varphi} + ||Sf||^2_{\varphi}$$

Note that, for future purposes, we shall choose φ so that the Hermitian form

$$\sum_{j} \sum_{k} t_j \bar{t}_k \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k}$$

is strictly positive at every point of Ω. In other words, we shall choose φ strictly plurisubharmonic.

Combination of all of the definitions given above, we have an extended definition.

Definition 2.2.7. *An open set Ω is pseudo-convex if there exists a plurisubharmonic function $\varphi \in \Omega$ such that*

$$\Omega_\varphi = \{z : z \in \Omega, \varphi(z) < c\}$$

is relatively compact in Ω, for any real number c.

So far, we have just worked on the tools we need for dealing with extension problems. Now, we will start considering existence theorems and provide their proofs.

Using Proposition 2.1.6, and Theorem 2.2.6 with a modified version of Theorem 2.1.3 by taking $F = H_2$ provides the following.

Theorem 2.2.8. *[16] Assume that $\Omega \subseteq \mathbb{C}^n$ is domain with a C^2-pseudo-convex boundary. Let $\varphi \in C^2(\Omega)$ be plurisubharmonic in Ω and let e^χ with $\chi \in C(\Omega)$ be the lowest eigenvalue of the matrix $\left(\frac{\partial^2\varphi}{\partial z_j \partial \bar{z}_k}\right)$. For every $f \in L^2_{(p,q)}(\Omega, \varphi), q > 0$, such that $\overline{\partial} f = 0$ and*

$$\int_\Omega |f|^2 e^{-(\varphi+\chi)} dV < \infty,$$

we can then find a form $u \in L^2_{(p,q-1)}(\Omega, \varphi)$ such that $\overline{\partial} u = f$ and

$$q \int_\Omega |u|^2 e^{-\varphi} dV \le \int_\Omega |f|^2 e^{-(\varphi+\chi)} dV.$$

Before continuing with the proof, we first try to understand the theorem. The theorem simply states that for every $f \in L^2_{(p,q)}(\Omega, \varphi)$ with the given properties, we can find a $(p, q-1)$ form u such that $\overline{\partial} u = f$, and the weighted norm of u is preserved with respect to the weighted norm of f. Now to prove this theorem, observe that, in the view of Theorem 2.2.6, Proposition 2.1.7 stated in last section infers as follows,

$$\sum f_{I,jK} \overline{f}_{I,kK} \text{ can be viewed as } (f_j)\left(\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k}\right)(\overline{f}_k) \ge e^\chi f_j \overline{f}_k$$

where f_j and $\overline{f}_k$ are vectors. Then, the inequality in Theorem 2.2.6 becomes

$$||T^* f||^2_\varphi + ||Sf||^2_\varphi \ge \int_\Omega \sum_{I,K} \sum_{j,k} e^\chi f_{I,jK} \overline{f_{I,kK}} e^{-\varphi} dV.$$

But why do we need "$e^\chi f_{I,jK} \overline{f_{I,kK}}$" part in the first place? It is because we want to be able to apply Theorem 2.1.3 in which this term $e^{\chi} f_{I,jK} \overline{f_{I,kK}}$ is equal to $||Af||_2^2$ where $A : H_2 \to H_2$. Let us now move to the proof.

Proof. Putting what we are given in the theorem, we have,

$$L^2_{(p,q-1)} \quad \xrightarrow{(T=\overline{\partial})} \quad L^2_{(p,q)} \quad \xrightarrow{(S=\overline{\partial})} \quad L^2_{(p,q+1)}.$$

Let $\chi/2$ be a real number and let $A : L^2_{(p,q)} \to L^2_{(p,q)}$ be defined by

$$A(f) = e^{\chi/2} f$$

so that the desired $||Af||_2^2$ will be obtained as described above. Then, the adjoint operator $A^* : L^2_{(p,q)} \to L^2_{(p,q)}$ becomes

$$A^*(e^{\chi/2} f) = f.$$

Now consider the following,

$$\langle f, A^* g\rangle = \langle Af, g\rangle = \langle e^{\chi/2} f, g\rangle = \langle f, e^{\chi/2} g\rangle \quad \Rightarrow \langle f, A^* g\rangle = \langle f, g\, e^{\chi/2}\rangle$$

$$\iff \quad A^* g = e^{\chi/2} g \Rightarrow A^* = A$$

Thus, A is a $self\ adjoint$ operator.

Hence, we have that A is self-adjoint , closed and densely defined operator. Moreover, by construction,

$$||Af||_2^2 = ||e^{\chi/2} f||_2^2 \leq ||T^* f||_1^2 + ||Sf||_3^2.$$

Thus, if $f = A^* g$ where $g \in Dom A^*$, then by Theorem 2.1.3, there exists a form $u \in Dom T = L^2_{(p,q-1)}$ such that $Tu = f$, or $\overline{\partial} u = f$, and

$$||u||_1 = ||g||_2\,,$$

or equivalently,

$$\int_\Omega |u|^2 e^{-\varphi} dV \leq \int_\Omega \left|e^{-\chi/2} f\right|^2 e^{-\varphi} dV.$$

□

Since restricting the boundary is a little cumbersome for extension problem, our next purpose is to remove the condition of smoothness on the boundary $\partial\Omega$ of Ω in the previous theorem. That is why we modified the definitions and extended them to the Definition 2.2.3, because, then, for a domain Ω with boundary $\partial\Omega \subset C^2$, the boundary is already pseudo-convex by Definition 2.2.3 if and only if Ω is pseudo-convex as in Definition 2.2.7. With restrictions being removed, we state the previous theorem below.

Theorem 2.2.9. *[16]Let Ω be a pseudo-convex open set in $\mathbb{C}^n$, φ be a plurisubharmonic function in Ω and e^χ with $\chi \in C^2(\Omega)$ be a lower bound for plurisubharmonicity of φ. Then, for every $f \in L^2_{(p,q)}(\Omega, loc), q > 0$ such that $\overline{\partial} f = 0$ and*

$$\int_\Omega |f|^2 e^{-(\varphi+\chi)} dV < \infty,$$

one can find a form $u \in L^2_{(p,q-1)}(\Omega, \varphi)$ such that $\overline{\partial} u = f$ and

$$q \int_\Omega |u|^2 e^{-\varphi} dV \leq \int_\Omega |f|^2 e^{-(\varphi+\chi)} dV.$$

Note that, a proof for above theorem is given in [16]. We now give some consequences of Theorem 2.2.8.

Theorem 2.2.10. *[16] Assume that Ω is a pseudo-convex, bounded domain in $\mathbb{C}^n$ and let $\delta = sup_{z,w\in\Omega}|z - w|$ be the diameter of Ω and φ be plurisubharmonic on Ω. Then, for any $f \in L^2_{(p,q)}(\Omega, \varphi), q > 0$ with $\overline{\partial} f = 0$, we can find a form $u \in L^2_{(p,q-1)}(\Omega, \varphi)$ such that $\overline{\partial} u = f$ and*

$$q \int_\Omega |u|^2 e^{-\varphi} dV \leq e\, \delta^2 \int_\Omega |f|^2 e^{-\varphi} dV.$$

Proof. Let a be a positive real number. Then, $\varphi(z) + a|z|^2$ is strictly plurisubharmonic. We can apply Theorem 2.2.9 because there is no restriction on the boundary. Assume that $0 \in \Omega$, then $|z| \leq \delta$ when $z \in \Omega$. Choose $a = e^\chi$, applying the above theorem, for every $f \in L^2_{(p,q)}(\Omega, \varphi), q > 0$ with $\overline{\partial} f = 0$, we have

$$q \int_\Omega |u|^2 e^{-\varphi} dV \leq \int_\Omega |f|^2 e^{-(\varphi+\chi)} dV.$$

Now replacing φ by $\varphi + a|z|^2$, we get,

$$\begin{aligned} q\int_\Omega |u|^2 e^{-(\varphi(z)+a|z|^2)}dV &= q\int_\Omega |u|^2 e^{-\varphi(z)}e^{-a|z|^2}dV \\ &\leq q\int_\Omega |u|^2 e^{-(\varphi(z)+a\delta^2)}dV \\ &\leq \int_\Omega |f|^2 e^{-(\varphi+\chi)}e^{-a\delta^2}dV \\ &= \int_\Omega |f|^2 e^{-\varphi}e^{-\chi}e^{-a\delta^2} \\ &= \int_\Omega |f|^2 e^{-\varphi}a^{-1}e^{-a\delta^2}dV \\ &\leq a^{-1}e^{-a\delta^2}\int_\Omega |f|^2 e^{-\varphi}dV. \end{aligned}$$

Letting $a = \delta^{-2}$, the right hand side reaches its minimum with respect to a and the proof is finished. □

Now for future purposes, we will give a modified version of the Hörmander's estimates. This version will be used to prove the Ohsawa-Takegoshi Extension Theorem.

Theorem 2.2.11. *(Hörmander's Estimate) Let Ω be a pseudo-convex domain in $\mathbb{C}^n$ and φ be a C^2-strongly plurisubharmonic function in Ω. Then, for every $f \in L^2_{loc,(0,1)}(\Omega)$ with $\overline{\partial} f = 0$, there exists a form $u \in L^2_{loc}(\Omega)$ solving $\overline{\partial} u = f$ and such that*

$$\int_\Omega |u|^2 e^{-\varphi}d\lambda \leq \int_\Omega |f|^2_{i\partial\overline{\partial}\varphi}e^{-\varphi}d\lambda$$

where $|f|^2_{i\partial\overline{\partial}\varphi} = \sum_{j,k}\varphi^{j\overline{k}}\overline{f}_j f_k$ and $\varphi^{j\overline{k}}$ is the inverse of the Hessian matrix.

Proof. Let A be a linear operator given such that

$$A^2 = \left[\frac{\partial^2\varphi}{\partial z_j \partial \overline{z}_j}\right]$$

which is the Hessian matrix. Note that, since $A = A^*$, A is unitary and Hermitian, it is likely to get that A^2 is the Hessian. To see that A is Hermitian, consider

$$\overline{A} = \overline{\frac{\partial^2\varphi}{\partial z_j \partial \overline{z_k}}} = \frac{\partial}{\partial \overline{z_j}}\frac{\overline{\partial\varphi}}{\partial \overline{z_k}} = \frac{\partial}{\partial \overline{z_j}}\frac{\partial\varphi}{\partial z_k} = A^T,$$

which implies that $A = \overline{A^T}$, so A is Hermitian.

Let f be a form in $C^1_{(0,1)}(\overline{\Omega})$, then since Ω is pseudo-convex, by Theorem 2.2.6 we must have,

$$\int_\Omega \sum_{I,k} \sum_{j,k} f_{I,jk} \overline{f}_{I,kK} A^2 e^{-\varphi} dV \leq ||T^* f||_\varphi^2 + ||Sf||_\varphi^2.$$

Indeed, considering $||Af||_\varphi^2 = \langle Af, Af \rangle_\varphi$, we see

$$||Af||_\varphi^2 = \langle f, A^2 f \rangle_\varphi = \langle f, \partial\overline{\partial}\varphi f \rangle_\varphi = \int_\Omega \sum_{I,k} \sum_{j,k} f_{I,jk} \overline{f}_{I,kK} A^2 e^{-\varphi} dV.$$

Hence,we have the following inequality

$$||Af||_\varphi^2 \leq ||T^* f||_\varphi^2 + ||Sf||_\varphi^2.$$

We define h by $f = A^* h = Ah$, then applying Theorem 2.1.4, we see that,
A is a closed, densely defined linear operator in $L^2_{(p,q+1)}(\Omega)$ and we take the closed subspace F of $L^2_{(p,q+1)}(\Omega)$ as the big space itself in the theorem. Since it holds that for $f \in DomA^*$,

$$||Af||_2^2 \leq ||T^* f||_1^2 + ||Sf||_3^2.$$

Then, for $g = A^* h$ where $h \in DomA^*$, there exists a form $u \in L^2_{(p,q)}(\Omega)$ such that

$$\overline{\partial} u = f \text{ satisfying that } ||u||_1 \leq ||h||_2.$$

Now, $\langle f, f \rangle = \langle Ah, Ah \rangle = \langle h, A^2 h \rangle = \langle h, \partial\overline{\partial}\varphi h \rangle = \sum h_i \overline{h}_j \frac{\partial^2 \varphi}{\partial z_i \partial \overline{z}_j}$. This implies that

$$||h||_2 = \sum f_i \overline{f}_j \varphi^{i\overline{j}} := |f|^2_{\partial\overline{\partial}\varphi}$$

where $\varphi^{i\overline{j}}$ is the inverse of $\frac{\partial^2 \varphi}{\partial z_i \partial \overline{z}_j}$. This finishes the proof. $\square$

CHAPTER 3

OHSAWA-TAKEGOSHI EXTENSION THEOREM

For U is a complex submanifold of $\mathbb{C}^n$, and W is its regular, closed subspace, it has been studied for a long time that any holomorphic function f on W can be holomorphically extended to a function F on U. However, it is important to note that, the holomorphic extension of bounded functions is remarkably more difficult than other types, that is why instead of working with bounded functions, it was preferred to work with L^2 holomorphic functions. Later, the studies started to evolve around holomorphic extensions with growth conditions in $U = \mathbb{C}^n$. Then, in 1987, Ohsawa and Takegoshi [23] presented their study considering the extension problem of L^2 holomorphic functions with that U is a bounded domain in $\mathbb{C}^n$. We will survey the Ohsawa-Takegoshi Extension Theorem and give a proof of the theorem with the similar method provided in [4].

Theorem 3.0.1. *Let Ω be a bounded pseudo-convex domain in $\mathbb{C}^n$ and φ an arbitrary plurisubharmonic function in Ω. Assume that H is a complex linear subspace of $\mathbb{C}^n$ and denote $\Omega^* := \Omega \cap H$.*
Then, for every holomorphic function f on Ω^*, there exists a holomorphic function F in Ω such that $F = f$ on Ω^* and

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \leq C_\Omega \int_{\Omega^*} |f|^2 e^{-\varphi} d\lambda^*,$$

where C_Ω is a constant depending only on n and on an upper bound for the diameter of Ω.

Proof. Without loss of generality, assume that $H = \{z_1 = 0\}$ and $\Omega \subset \{|z_1| < 1\}$. Let us first explain why it is enough to take H as a hyperplane. For simplicity, let $n = 4$ and $\Omega \subset \mathbb{C}^4$, and let $H = \{z_1 = z_2 = z_3 = 0\}$, which is a complex line, then

$H \subseteq \mathbb{C}$. Now consider,

$$(i)\ \ \Omega^* = \Omega \cap H \subset \tilde{\Omega} := \Omega \cap \{z_2 = z_3 = 0\} \subset \mathbb{C}^2$$

and $H \subset \tilde{\Omega}$. Then, $f \in \mathcal{O}(\Omega^*)$ which implies that there is a function $F \in \mathcal{O}(\tilde{\Omega})$ such that $F = f$ on $\tilde{\Omega}$ and

$$\int_{\tilde{\Omega}} |F|^2 e^{-\varphi} d\lambda \leq C_{\tilde{\Omega}} \int_{\Omega^*} |f|^2 e^{-\varphi} d\lambda^*.$$

Thus, we levelled up one dimension.

Continuing for the next level-up, consider,

$$(ii)\ \ \tilde{\tilde{\Omega}} = \Omega \cap \{z_3 = 0\} \subset \mathbb{C}^3$$

and $\tilde{\Omega} \subset \tilde{\tilde{\Omega}}$. Then, $F \in \mathcal{O}(\tilde{\Omega})$, which implies that there is a function $\tilde{F} \in \mathcal{O}(\tilde{\tilde{\Omega}})$ such that $\tilde{F} = F$ on $\tilde{\Omega}$ and the norm condition holds. Moreover, $\tilde{\tilde{\Omega}} \subset \mathbb{C}^3$, and $\tilde{\tilde{\Omega}} \subset \Omega \subset \mathbb{C}^4$.

By this process, we obtain an extension on Ω with the required norm. Also note that, we may assume that $\Omega \subset \{|z_1| < 1\}$ because Ω is a bounded domain, so it is contained in an $M - ball$, that is

$$\Omega \subset \{|z_1| < M\} \Rightarrow \Omega \subset \{|\frac{z_1}{M}| < 1\},$$

with change of coordinates $\Omega \subset \{|z_1| < 1\}$.

Now by approximating Ω from inside (taking an increasing sequence of strongly pseudo-convex domains Ω_n, and letting $\Omega_n \to \cup_{n\in\mathbb{N}}\Omega_n$ gives Ω as $n \to \infty$), and approximating φ from above, (taking a decreasing sequence of smooth plurisubharmonic functions φ_n and letting $\varphi_n \to \varphi$ as $n \to \infty$), we may assume that Ω is strongly pseudo-convex domain with a smooth boundary, φ is plurisubharmonic function which is smooth up to boundary, and f is defined in a neighborhood of $\overline{\Omega}^*$ in H. Let $\alpha = \overline{\partial}(\chi(z_1)f(z^*))$, where z^* represents the remaining $(n-1)$-variables, $\chi = 1$ near 0, $\{|z_1| < \epsilon\}$ is the support of α, and $\overline{\partial}\alpha = 0$. Observe that, since f is defined on $\Omega \cap H$, f is a function of $(n-1)$-variables. Let $\varphi = 2\log|z_1|$.

Then, by Hörmander's Estimate, we already have that, there exists $u \in L^2_{loc}(\Omega)$ such that $\overline{\partial}u = \alpha$, and the following holds

$$\int_{\Omega} |u|^2 e^{-\varphi} d\lambda \leq \int_{\Omega} |\alpha|^2_{i\partial\overline{\partial}\varphi} e^{-\varphi} d\lambda.$$

Now consider the following function,

$$\tilde{f}(z_1, z^*) = \chi(z_1)f(z^*) - u(z_1, z^*).$$

We will examine if this function $\tilde{f}$ might be an analytic extension for f. Let us start with checking the analyticity first and follow this with checking if it is an extension.

Observe that $\overline{\partial}\tilde{f} = \overline{\partial}(\chi(z_1)f(z^*) - u(z_1, z^*)) = \alpha - \alpha = 0$. Thus, $\tilde{f}$ is analytic. For checking the extension property, consider the following,

$$\text{on } H, z_1 = 0 \text{ which implies that } \chi(z_1) = 1 \text{ because } \chi = 1 \text{ near } 0.$$

Also, observe that, on $z_1 = 0$, $\tilde{f}(0, z^*) = f(z^*) - u(0, z^*)$, indicating that $\tilde{f}$ does not seem like an extension near $z_1 = 0$. Hence, for $\tilde{f}$ to be an extension-like, $u \equiv 0$ near $z_1 = 0$. Then, we must check if $u(0, z^*) \equiv 0$. Observe that,

$$\int_\Omega |u|^2 \frac{1}{|z_1|^2} d\lambda \leq \int_{\Omega \cap \{|z_1| < \epsilon\}} |\alpha|^2 \frac{1}{|z_1|^2} d\lambda < \infty.$$

Assume that $u \neq 0$ on H, so $|u| > M$ for some M. Then, there is a neighborhood V around $c \in H$ so that $u \neq 0$ in V, but then, we would have,

$$\int_\Omega |u|^2 \frac{1}{|z_1|^2} d\lambda > \int_V M^2 \frac{1}{|z_1|^2} d\lambda \;\Rightarrow\; \int_{\Omega \cap H} |u|^2 \frac{1}{|z_1|^2} d\lambda > \int_{V \cap H} M^2 \frac{1}{|z_1|^2} d\lambda.$$

Since,

$$\int_{V \cap H} M^2 \frac{1}{|z_1|^2} d\lambda \approx \int_{\mathbb{D}} \frac{dz}{|z_1|^2} = \infty,$$

we have that $u \equiv 0$ on H, which shows that $\tilde{f}$ is an analytic extension of f.

Let $F \in A^2(\Omega, e^{-\varphi}) := \mathcal{O}(\Omega) \cap L^2(\Omega, e^{-\varphi})$ be the function satisfying $F = f$ on H with the minimal norm in $L^2(\Omega, e^{-\varphi})$, in other words, suppose that

$$||F||^2_\varphi = \int_\Omega |F|^2 e^{-\varphi} dV \text{ is minimal in } L^2(\Omega, e^{-\varphi}).$$

$$A^2_H(\Omega, e^{-\varphi}) = \{f \in A^2(\Omega, e^{-\varphi}) : f = 0 \text{ on } H\} \text{ is a closed subset of } A^2(\Omega, e^{-\varphi}),$$

where $H = \{z_1 = 0\} \cong \mathbb{C}^{n-1}$. Then,

$$A^2(\Omega, e^{-\varphi}) = A^2_H(\Omega, e^{-\varphi}) \oplus \left[A^2_H(\Omega, e^{-\varphi})\right]^\perp.$$

Since $F \in A^2(\Omega, e^{-\varphi})$, we can express F as follows,

$$F = F' + F'' \text{ where } F' \in A^2_H, F'' \in \left[A^2_H\right]^\perp.$$

Let $F' = 0$ so that F has the minimal norm. Then, $F = F''$, $F \in [A^2_H]^\perp$ and so F is perpendicular to the space $z_1 A^2(\Omega, e^{-\varphi})$.

To see this, let $g \in A^2_H(\Omega, e^{-\varphi})$. For simplicity, choose g to be a functional in two variables, z_1, z_2 only. Then, the power series of g will be of the following form

$$g(z) = a_0 + a_1 z_1 + a_2 z_2 + a_3 z_1^2 + a_4 z_1 z_2 + a_5 z_2^2 + \dots.$$

Since $g \in A^2_H(\Omega, e^{-\varphi})$, $g = 0$ on H. Thus,

$$g(z) = a_0 + a_2 z_2 + a_5 z_2^2 + \dots$$

which implies that for g to be 0 on H, we must have $a_0 = a_2 = a_5 = \dots = 0$, and so g consists of terms containing z_1. That is,

$$g(z) = a_1 z_1 + a_3 z_1^2 + a_4 z_1 z_2 + \dots.$$

Or equivalently,

$$g(z) = z_1 \tilde{g} \text{ where } \tilde{g} \in A^2(\Omega, e^{-\varphi}).$$

Hence, for $g \in A^2_H(\Omega, e^{-\varphi})$,

$$\langle F, g\rangle = \langle F, z_1 \tilde{g}\rangle = 0.$$

Therefore, F is perpendicular to the space $z_1 A^2(\Omega, e^{-\varphi})$. Note that, this also means that $\overline{z}_1 F$ is perpendicular to $A^2(\Omega, e^{-\varphi})$.

Now choose such α with the minimal norm. Then, $\alpha \perp Ker\overline{\partial}^*$. To clarify, $Ker\overline{\partial}^* = \left[Ran\overline{\partial}\right]^\perp$ and $L^2(\Omega, e^{-\varphi}) = Ker\overline{\partial}^* \oplus \overline{Ran\overline{\partial}}$. Let $\alpha = \alpha_1 + \alpha_2$ where $\alpha_1 \in \overline{Ran\overline{\partial}}$ and $\alpha_2 \in Ker\overline{\partial}^*$. Then,

$$\overline{\partial}^* \alpha = \overline{\partial}^* \alpha_1 + \overline{\partial}^* \alpha_2 = \overline{\partial}^* \alpha_1 = \overline{z}_1 F.$$

Note that, since α has the minimal norm, $\alpha = \alpha_1 \in \overline{Ran\overline{\partial}} = \left[Ker\overline{\partial}^*\right]^{\perp}$. This implies that there exists a sequence α_n such that $\lim_{n\to\infty} \overline{\partial}\alpha_n = \alpha$, and $\overline{\partial}\alpha = \lim_n \overline{\partial}\overline{\partial}\alpha_n = 0$. Thus, $\overline{\partial}\alpha = 0$.

Now we will check if α satisfies the $\overline{\partial}$-Neumann Boundary Condition, that is,

$$\sum_j \alpha_j \rho_j = 0 \text{ on } \partial\Omega, \text{ where } \rho \text{ is a defining function for } \Omega.$$

Observe that, by [16], an element $\alpha \in Int(C^1_{(0,1)}(\overline{\Omega}))$ belongs to $DomT^*$ if and only if

$$\sum_{j=1}^{n} \alpha_{I,jK} \frac{\partial \varrho}{\partial z_j} = 0 \text{ on } \partial\Omega \text{ for all } I, K, \text{ and } \varrho \text{ is a defining function.}$$

Now, in our case, $\alpha \in L^2(\Omega, e^{-\varphi})$ belongs to $Dom\overline{\partial}^*$ because $\overline{\partial}^*\alpha = \overline{z}_1 F$, but again by [16],

$$\sum_{j=1}^{n} \alpha_j \rho_j = 0 \text{ on } \partial\Omega,$$

where α_j are the coefficient functions and ρ_j are the partial derivatives of ρ.

Consider then,

$$\begin{aligned}
\int_\Omega |F|^2 e^{-\varphi} d\lambda &= \int_\Omega \frac{F(z)}{z_1} \overline{(\overline{\partial}^*\alpha)} e^{-\varphi} d\lambda \qquad (\overline{\partial}^*\alpha = \overline{z}_1 F) \\
&= \left\langle \frac{F}{z_1}, \overline{\partial}^*\alpha \right\rangle_{e^{-\varphi}} \\
&= \left\langle \overline{\partial}\left(\frac{F}{z_1}\right), \alpha \right\rangle_{e^{-\varphi}} \\
&= \left\langle \left(\overline{\partial}F\right)\frac{1}{z_1} + F\overline{\partial}\left(\frac{1}{z_1}\right), \alpha \right\rangle_{e^{-\varphi}} \\
&= \left\langle F\, \overline{\partial}\left(\frac{1}{z_1}\right), \alpha \right\rangle_{e^{-\varphi}} \qquad (3.0.1)
\end{aligned}$$

where the last equality follows from the fact that F is an analytic extension of f and $\overline{\partial}F = 0$.

Consider the function $h : z \mapsto z_1$ and that $\overline{\partial}(\frac{1}{h}) = \pi(\overline{\partial h}/|\partial h|^2)dV$ where V is the space on which h vanishes and dV is the surface measure. Thus,

$$\overline{\partial}(\frac{1}{z_1}) = \pi\frac{\overline{\partial z_1}}{|\partial z_1|^2}d\lambda^* = \pi\partial\overline{z}_1 d\lambda^*.$$

Then, continuing with equation (3.0.1),

$$\begin{aligned}
\Big\langle F\,\overline{\partial}\Big(\frac{1}{z_1}\Big), \alpha\Big\rangle_{e^{-\varphi}} &= \Big\langle F\pi d\overline{z}_1, \alpha\Big\rangle_{e^{-\varphi}}, \qquad \text{since } \overline{\partial}\Big(\frac{1}{z_1}\Big) \text{ has support on } \Omega^* \text{ and 0 outside,}\\
&= \int_{\Omega^*} f\pi\overline{\alpha}d\overline{z}_1 e^{-\varphi}d\lambda^*\\
&= \pi\int_{\Omega^*} f\,\overline{\alpha}\,d\overline{z}_1 e^{-\varphi}d\lambda^*\\
&\leq \pi\Big(\int_{\Omega^*}|f|^2e^{-\varphi}d\lambda^*\Big)^{\frac{1}{2}}.\Big(\int_{\Omega^*}|\alpha_1|^2e^{-\varphi}d\lambda^*\Big)^{\frac{1}{2}}
\end{aligned}$$

where the last inequality follows from Hölder's Inequality.

Therefore, for the demanded inequality, we only need to estimate

$$\int_{\Omega^*}|\alpha_1|^2e^{-\varphi}d\lambda^*.$$

Consider the following equality,

$$\begin{aligned}
\sum(\alpha_j\overline{\alpha}_k e^{-\varphi})_{j\overline{k}} &= \sum\frac{\partial^2}{\partial z_j\partial\overline{z}_k}(\alpha_j\overline{\alpha}_k e^{-\varphi})\\
&= (-2\,\mathsf{Re}\,(\overline{\partial}\overline{\partial}^*\alpha.\,\alpha) + |\overline{\partial}^*\alpha|^2 + \sum|\alpha_{j,\overline{k}}|^2 - |\overline{\partial}\alpha|^2 + \sum\varphi_{j,\overline{k}}\alpha_j\overline{\alpha}_k)e^{-\varphi}.
\end{aligned}$$

From integrating by parts and continuing calculation for any function ω, and any α with $\sum_j \alpha_j\rho_j = 0$ on $\partial\Omega$, we obtain,

$$\begin{aligned}
\int_\Omega\sum\omega_{j,\overline{k}}\alpha_j\overline{\alpha}_k e^{-\varphi}d\lambda &- \int_{\partial\Omega}\sum\rho_{j\overline{k}}\alpha_j\overline{\alpha}_k e^{-\varphi}\omega\frac{ds}{|\partial\rho|}\\
&= \int_\Omega\Big(-2\mathsf{Re}\,(\overline{\partial}\overline{\partial}^*\alpha.\,\alpha) + |\overline{\partial}^*\alpha|^2\\
&+ \sum|\alpha_{j,\overline{k}}|^2 - |\overline{\partial}\alpha|^2 + \sum\varphi_{j,\overline{k}}\alpha_j\overline{\alpha}_k\Big)e^{-\varphi}\omega d\lambda.
\end{aligned}$$

Since we have $\overline{\partial}\alpha = 0$ and $\overline{\partial}^*\alpha = \overline{z}_1F$, for ω is a negative function of variable z_1, we would have

$$
\begin{aligned}
\int_\Omega \omega_{1\bar{1}}|\alpha_1|^2 e^{-\varphi} d\lambda \quad &- \quad \int_{\partial\Omega} \sum \rho_{j\bar{k}} \alpha_j \overline{\alpha}_k e^{-\varphi} \omega \frac{ds}{|\partial\rho|} \\
&= \quad \int_\Omega -2 \text{ Re}\,(F\overline{\alpha}_1)\omega e^{-\varphi} d\lambda + \int_\Omega \omega e^{-\varphi} d\lambda \\
&+ \quad \int_\Omega |\alpha_{1,\bar{1}}|^2 e^{-\varphi} d\lambda - \int_\Omega \omega e^{-\varphi} d\lambda + \int_\Omega \varphi_{1\bar{1}} |\alpha_1|^2 \omega e^{-\varphi} d\lambda \\
&= \quad -2 \text{ Re} \left[\int_\Omega F\overline{\alpha}_1 \omega e^{-\varphi} d\lambda \right] + \int_\Omega |\alpha_{1,\bar{1}}|^2 \omega e^{-\varphi} d\lambda \\
&+ \quad \int_\Omega \varphi_{1\bar{1}} |\alpha_1|^2 \omega e^{-\varphi} d\lambda,
\end{aligned}
$$

which then gives that,

$$
\begin{aligned}
\int_\Omega \omega_{1\bar{1}}|\alpha_1|^2 e^{-\varphi} d\lambda \quad &= \quad -2 \text{ Re} \left[\int_\Omega F\ \overline{\alpha}_1 \omega e^{-\varphi} d\lambda \right] + \int_{\partial\Omega} \sum \rho_{j\bar{k}} \alpha_j \overline{\alpha}_k e^{-\varphi} \omega \frac{ds}{|\partial\rho|} \\
&+ \quad \int_\Omega |\alpha_{1,\bar{1}}|^2 \omega e^{-\varphi} d\lambda + \int_\Omega \varphi_{1\bar{1}} |\alpha_1|^2 \omega e^{-\varphi} d\lambda.
\end{aligned}
$$

Hence we get,

$$
\int_\Omega \omega_{1\bar{1}}|\alpha_1|^2 e^{-\varphi} d\lambda \le -2 \text{ Re} \left[\int_\Omega F\overline{\alpha}_1 \omega e^{-\varphi} d\lambda \right].
$$

Now let $\omega := 2\log|z_1| + |z_1|^{2\delta} - 1$, where $0 < \delta < 1$. Then,

$$
\omega_{1\bar{1}} = \pi\delta_0' + \delta^2 |z_1|^{2\delta-2},
$$

and for some $t > 0$,

$$
\begin{aligned}
& \int_\Omega (\pi\delta_0' + \delta^2|z_1|^{2\delta-2})|\alpha_1|^2 e^{-\varphi} d\lambda \\
\le \quad & -2\text{Re} \left[\int_\Omega F\overline{\alpha}_1 \log|z_1|^2 e^{-\varphi} d\lambda + \int_\Omega F\overline{\alpha}_1 |z_1|^{2\delta} e^{-\varphi} d\lambda - \int_\Omega F\overline{\alpha}_1 e^{-\varphi} d\lambda \right] \\
\iff \quad & \pi \int_{\Omega^*} |\alpha_1|^2 e^{-\varphi} d\lambda^* + \delta^2 \int_\Omega |\alpha_1|^2 |z_1|^{2\delta-2} e^{-\varphi} d\lambda \\
\le \quad & t \int_\Omega |F|^2 e^{-\varphi} d\lambda + \frac{1}{t} \int_\Omega |\alpha_1|^2 \omega^2 e^{-\varphi} d\lambda.
\end{aligned}
$$

Now observe that $|a+b|^2 = |a|^2 + |b|^2 + \langle a,b\rangle + \overline{\langle a,b\rangle}$ and $\langle a,b\rangle + \overline{\langle a,b\rangle} = 2 \text{ Re}\,\langle a,b\rangle$. Thus, $|a+b|^2 = |a|^2 + |b|^2 + 2 \text{ Re}\,\langle a,b\rangle$ and $-2 \text{ Re}\,\langle a,b\rangle = |a|^2 + |b|^2 - |a+b|^2$.

Then, $-2\ \mathsf{Re}\,\langle a, b\rangle \leq |a|^2 + |b|^2$. Regarding this, consider,

$$\int_\Omega F\overline{\alpha}_1 \omega e^{-\varphi} d\lambda \leq \int_\Omega \langle F, \alpha_1\rangle_{\omega e^{-\varphi}} d\lambda,$$

$$\begin{aligned} -2\ \mathsf{Re}\left[\int_\Omega F\overline{\alpha}_1\omega e^{-\varphi}d\lambda\right] &= -2\ \mathsf{Re}\left(\langle F, \alpha_1\rangle_{\omega e^{-\varphi}}\right) \\ &= -2\ \mathsf{Re}\left(\langle tF, 1/t\ \alpha_1\overline{\omega}\rangle_{e^{-\varphi}}\right) \\ &\leq t\int_\Omega |F|^2 e^{-\varphi}d\lambda + \frac{1}{t}\int_\Omega |\alpha_1|^2\omega^2 e^{-\varphi}d\lambda. \end{aligned}$$

Take t such that $\omega^2 \leq \delta^2 t|z_1|^{2\delta-2}$ in $\{|z_1| \leq 1\}$, and with regard to

$$\int_\Omega \omega_{1\bar{1}}|\alpha_1|^2 e^{-\varphi}d\lambda \leq -2\ \mathsf{Re}\left[\int_\Omega F\overline{\alpha}_1\omega e^{-\varphi}d\lambda\right],$$

we get,

$$\begin{aligned} t\int_\Omega |F|^2 e^{-\varphi}d\lambda + \frac{1}{t}\int_\Omega |\alpha_1|^2\omega^2 e^{-\varphi}d\lambda &\leq t\int_\Omega |F|^2 e^{-\varphi}d\lambda + \frac{1}{t}\int_\Omega |\alpha_1|^2\delta^2 t|z_1|^{2\delta-2}e^{-\varphi}d\lambda \\ &\leq t\int_\Omega |F|^2 e^{-\varphi}d\lambda + \delta^2\int_\Omega |\alpha_1|^2|z_1|^{2\delta-2}e^{-\varphi}d\lambda. \end{aligned}$$

Thus,

$$\frac{1}{t}\int_\Omega |\alpha_1|^2\omega^2 e^{-\varphi}d\lambda \leq \delta^2\int_\Omega |\alpha_1|^2|z_1|^{2\delta-2}e^{-\varphi}d\lambda.$$

However, we also have that

$$\pi\int_{\Omega^*}|\alpha_1|^2 e^{-\varphi}d\lambda^* + \delta^2\int_\Omega |\alpha_1|^2|z_1|^{2\delta-2}e^{-\varphi}d\lambda \leq t\int_\Omega |F|^2 e^{-\varphi}d\lambda + \frac{1}{t}\int_\Omega |\alpha_1|^2\omega^2 e^{-\varphi}d\lambda.$$

Hence,

$$\begin{aligned} \pi\int_{\Omega^*}|\alpha_1|^2 e^{-\varphi}d\lambda^* + \frac{1}{t}\int_\Omega |\alpha_1|^2\omega^2 e^{-\varphi}d\lambda &\leq t\int_\Omega |F|^2 e^{-\varphi}d\lambda + \frac{1}{t}\int_\Omega |\alpha_1|^2\omega^2 e^{-\varphi}d\lambda \\ &\Rightarrow \pi\int_{\Omega^*}|\alpha_1|^2 e^{-\varphi}d\lambda^* \leq t\int_\Omega |F|^2 e^{-\varphi}d\lambda. \end{aligned}$$

which gives us,

$$\left[\int_\Omega |F|^2 e^{-\varphi}d\lambda\right]^2 \leq \pi\left[\int_{\Omega^*}|f|^2 e^{-\varphi}d\lambda^*\right].\left[\frac{t^2}{\pi^2}\int_\Omega |F|^2 e^{-\varphi}d\lambda\right].$$

Therefore, we obtain the norm condition

$$\int_{\Omega} |F|^2 e^{-\varphi} d\lambda \leq \frac{t^2}{\pi} \int_{\Omega^*} |f|^2 e^{-\varphi} d\lambda^*$$

$\square$

CHAPTER 4

APPLICATIONS OF OHSAWA-TAKEGOSHI EXTENSION THEOREM

4.1 Openness Conjecture

Choose an open set $U \subseteq \mathbb{C}$ and $x \in U$. If we take a holomorphic function f on a neighbourhood of x, say V, denote the germ of f at x by f_x (equivalently, f_x denotes the equivalence class of function elements (f, V), where two function elements are equivalent at x if they agree on an open neighbourhood of x).

Let u be a plurisubharmonic function defined in a neighborhood of $0 \in \mathbb{C}^n$ satisfying that e^{-u} is an element of L^1.With this assumption, Demailly and Kollár [8] presented the following result called Openness Conjecture,

There is a number $p > 1$ such that e^{-u} is an element of L^p, provided that the neighborhood is shrunk. The conjecture simply states that if $|f|^2 e^{-u}$ is locally integrable, then there exists $\epsilon_0 > 0$ satisfying that $|f|^2 e^{-(1+\epsilon_0)u}$ is also locally integrable, or equivalently, the set

$$\{p \in \mathbb{R} : |f|^2 e^{-pu} \text{ is locally integrable}\}$$

is an open set. Now, define the multiplier ideal sheaf as follows (see [21]),

$$I(u) = \left\{ f_x : f \in \mathcal{O}(V), \int\limits_V |f|^2 e^{-u} < \infty, V \subset U \text{ open}, x \in V \right\}.$$

We also define

$$I_+(u) := \cup_{\epsilon > 0} I((1+\epsilon)u).$$

With these sets being defined, the Openness Conjecture can also be written as follows, For a subharmonic function u on X and $I(u)$ is the germ of holomorphic functions $f \in \mathcal{O}_X$. It holds that

$$I_+(u) = I(u).$$

In 2013, Berndtsson [3] gave a proof for above statement of the conjecture, which has lead to a great progress in the view of proving the conjecture.

Theorem 4.1.1. *[19] If $u_1 \leq u_2 \leq \ldots$ are plurisubharmonic functions on U and $u = \lim_j u_j$ is locally bounded above on U, then $I(u) = I(u_j)$ for some j.*

Observe that the theorem above implies that the Openness Conjecture is true, because if we take u as follows

$$u \leq 0,$$

then by taking u_j as $(1 + \frac{1}{j})u$ in the Theorem 4.1.1, we see that

$$I_+(u) = I((1 + \frac{1}{j})u), \text{ and by limit condition, we get } I_+(u) = I(u).$$

Now, to be able to prove this theorem, we need some additional results. First we consider these results.

Proposition 4.1.2. *Let $\Delta \subset \mathbb{C}$ denote the unit disc. Given $k \in \mathbb{N}$ and a holomorphic function F defined in a neighbourhood of $\bar{\Delta}$ and such that $F \neq 0$ on $\Delta \setminus \{0\}$. Assume that $G \in \mathcal{O}(\Delta)$ satisfies $\lim_{z\to 0} \frac{G(z)}{F(z)} = 0$, and $t \in \Delta \setminus \{0\}$, for any k-th roots of unity w, we have $F(wt) = G(wt)$. Then,*

$$\sup_{\Delta} |G| \geq c_1 |t|^{-k}, \text{ where } c_1 = \min_{|s|=1} |F(s)| > 0.$$

Proof. Let $F(s) = s^p F_1(s)$ with p is an element of $\mathbb{N} \cup \{0\}$ and $F_1 \neq 0$ on $\bar{\Delta}$. By eliminating F_1, we may assume that $F(s) = s^p$. We define an auxiliary function,

$$G_1(s) = \frac{1}{k} \sum_w w^{-p} G(ws).$$

Then, for w_0 is a k-th root of unity and by $F(ww_0t) = (ww_0t)^p$, we have the following,

$$G_1(w_0t) = \frac{1}{k} \sum_w w^{-p} G(ww_0t) = \frac{1}{k} \sum_w w_0^p t^p = \frac{w_0^p}{k} \sum_w t^p = \frac{w_0^p t^p}{k}.k = w_0^p t^p$$

Hence, the function G_1 is holomorphic on Δ and $\lim_{z\to 0} \frac{G_1(z)}{F(z)} = 0$ because as $s \to 0$, and $z \to 0$, $G_1(z) = \frac{1}{k} \sum_w w^{-p} G(wz)$ and $G = o(F)$ at 0. Moreover, for any

$t \in \Delta \setminus \{0\}$ and $F(wt) = G_1(wt)$.

Then, this function G_1 satisfies what the function G satisfies in the proposition. Also, its Taylor Series consists of monomials of the form s^q where k divides $q - p > 0$. To make it clear, we can give a simple example for this part with taking $k = 2$ and $p = 3$. First consider the explicit formula of the Taylor Series for G_1,

$$\begin{aligned} G_1(s) = \frac{1}{k}\sum_w w^{-p}G(ws) &= \frac{1}{k}\sum_w w^{-p}\Big[\sum_{n=1}^{\infty} a_n(ws)^n\Big] \\ &= \frac{1}{2}\Big[\sum_{n=1}^{\infty} a_n s^n\Big] + \frac{1}{2}(-1)^{-p}\Big[\sum_{n=1}^{\infty} a_n s^n(-1)^n\Big], \end{aligned}$$

then for $k = 2$ and $p = 3$, the last two summations become,

$$\begin{aligned} \frac{1}{2}\Big[\sum_{n=1}^{\infty} a_n s^n\Big] + \frac{1}{2}(-1)\Big[\sum_{n=1}^{\infty} a_n s^n(-1)^n\Big] &= \frac{1}{2}\Big[\sum_{n=1}^{\infty} a_n s^n(1-(-1)^n)\Big] \\ &= \begin{cases} 0 & n = 2k \\ \sum_{n=1}^{\infty} a_n s^n & n = 2k+1. \end{cases} \end{aligned}$$

Then, the Taylor Series for G_1 in the case of $k = 2$ and $p = 3$ consists only of monomials s^q for which $q - 3$ is divisible by $k = 2$. Coming back to the proof of the proposition, in particular, $q \geq p + k$ and so $\frac{G_1(s)}{s^{p+k}}$ is holomorphic on Δ. Hence, for any $t \in \Delta \setminus \{0\}$,

$$\sup_{s\in\Delta}|G_1(s)| = \sup_{s\in\Delta}\left|\frac{G_1(s)}{s^{p+k}}\right| \geq \left|\frac{G_1(t)}{t^{p+k}}\right| = \left|\frac{t^p}{t^{p+k}}\right| = |t|^{-k}$$

where the first equality follows from the fact that a holomorphic function takes its maximum value on the boundary and since $s \in \Delta$, on the boundary $|s| = 1$ and the last two equations follows from the fact that $G_1(t) = F(t) = t^p$ because 1 is a root of unity, so $F(wt) = G_1(wt)$. Therefore, we get,

$$\sup_{s\in\Delta}|G_1(s)| \geq |s|^{-k},$$

$$\sup_{\Delta}|G| \geq c_1|t|^{-k}, c_1 = \min_{|s|=1}|F(s)| > 0.$$

□

For the final required result to prove the theorem, consider the following construction. Given a measure $g : W \to \mathbb{C}$, let $||g|| \in [0, \infty]$ be defined such that

$$||g||^2 = \inf_j \int_W |g|^2 e^{-u_j}.$$

Lemma 4.1.3. *Let $f \in \mathcal{O}(U)$. The germ of f at x, is an element of $J = \bigcup_j I(u_j)$ if and only if for arbitrary small neighbourhood $V \subset U$ around x and for any hyperplane $P_0 \subseteq \mathbb{C}^m$,*

$$\liminf dist(x, P)||f|_{V\cap P}|| = 0$$

as P is parallel to P_0 and $dist(x, P)$ converges to 0.

Before continuing with the proof, assume that V is polydisc around 0 and P_0 denote the set of all points whose first coordinate is 0. Then, the lemma states the following:

$$\int_V |f|^2 e^{-u} < \infty \text{ if and only if } \liminf_{s\to 0} |s|^2.\left[\int_{V\cap\{z_1=s\}} |f|^2 e^{-u}\right] = 0.$$

Since $\int_V |f|^2 e^{-u} = \int_{\mathbb{C}} \left(\int_{V\cap\{z_1=s\}} |f|^2 e^{-u}\right) d\lambda_2(s)$, the lemma provides the convergence of the following integral

$$\int_{\{|s|<r\}} \varphi(s) d\lambda_2(s)$$

if and only if $\liminf_{s\to 0} |s|^2\varphi(s) = 0$, where $\varphi(s) = \int_{V\cap\{z_1=s\}} |f|^2 e^{-u}$.

Proof. Suppose that $x = 0$.

($\Rightarrow$) Let $f_0 \in J$, pick j and a neighbourhood V around 0 such that $\int_V |f|^2 e^{-u_j} < \infty$. For a hyperplane P_0, changing the necessary coordinates one can obtain that $P_0 || \{z \in \mathbb{C}^m : z_1 = 0\}$. Then, using Fubini's Theorem, one gets the following:

$$\infty > \int_V |f|^2 e^{-u_j} = \int_{\mathbb{C}} \left[\int_{V\cap\{z_1=r\}} |f|^2 e^{-u_j}\right] d\lambda_2(r).$$

Since $\int |r|^2 d\lambda_2(r)$ is divergent on an arbitrary neighbourhood of $0 \in \mathbb{C}$, the above inequality implies that

$$\liminf_{r\to 0} |r|^2. \left[\int_{V\cap\{z_1=r\}} |f|^2 e^{-u_j}\right] = 0.$$

Indeed, assume that $\varphi(r) = \int_{V\cap\{z_1=r\}} |f|^2 e^{-u_j}$, if we have

$$\liminf dist(x, P). \left[\int_{V\cap\{z_1=r\}} |f|^2 e^{-u_j}\right] \neq 0,$$

then there is some nonzero c such that as $r \to 0$,

$$\liminf dist(x, P). \left[\int\limits_{V \cap \{z_1 = r\}} |f|^2 e^{-u_j} \right] = c.$$

This implies that

$$\varphi(r) \geq \frac{c}{|r|^2}.$$

Thus, the integral

$$\int_{\mathbb{C}} \varphi(r)$$

diverges. This would be a contradiction to assumption of $\varphi(r) < \infty$. Thus,we obtain the demanded result as $r \to 0$,

$$\liminf dist(x, P). \left[\int\limits_{V \cap \{z_1 = r\}} |f|^2 e^{-u_j} \right] = 0.$$

($\Leftarrow$) Suppose on the contrary that $f_0 \notin J$. Let V be pseudo-convex. Given a holomorphic function $\alpha : \Delta \to U$, and $\alpha(0) = 0$, use $J \circ \alpha$ to denote the pullback $A = \{g \circ \alpha : g \in J\} \subset \mathcal{O}_{(\mathbb{C},0)}$.

Then, by the assumption there exists a non-zero α so that $f_0 \circ \alpha \notin \mathcal{O}_{(\mathbb{C},0)}(J \circ \alpha)$. We choose a hyperplane P_0 through $0 \in \mathbb{C}^m$ which does not include $\alpha(\Delta)$. Take $P_0 = \{z_1 = 0\}$ as the chosen hyperplane. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a holomorphic function in a neighbourhood of $\bar{\Delta}$ such that $F = f_0 \circ \alpha \neq 0$ on $\bar{\Delta} \setminus \{0\}$. Take $\alpha_1(s) = s^k$, where $s \in \Delta$ and $\alpha(\bar{\Delta}) \subset V$. Hence, there exists $c_2, c_2' \in \mathbb{C}$, for $g \in \mathcal{O}(V)$, and for any j,

$$\max_{\alpha(\bar{\Delta})} |g|^2 \leq c_2' \int\limits_V |g|^2 \leq c_2^2 \int\limits_V |g|^2 e^{-u} \leq c_2^2 \int\limits_V |g|^2 e^{-u_j} \tag{4.1.1}$$

where the first inequality is a result of subharmonicity and the middle inequality follows from $u < 0$. $P_r = \{z \in \mathbb{C}^m : z_1 = r\}$. We must find a bound for $||f|_{V \cap P_r}||$ from below. Let $r \in \Delta \setminus \{0\}$, and assume that $||f|_{V \cap P_r}|| < \infty$. Then, using Ohsawa-Takegoshi Extension Theorem, we can find a function $g \in \mathcal{O}(V)$ which is equal to f on $V \cap P_r$ and for sufficiently large j such that,

$$\int\limits_V |g|^2 e^{-u_j} \leq c_3^2 ||f|_{V \cap P_r}||^2 \tag{4.1.2}$$

where c_3 is independent of j and r. Note that, this inequality holds because if j is large enough, $||f|_{V\cap P_r}||^2 \leq ||f|_{V\cap P_r}|| + \epsilon$. In other words, for all $\epsilon > 0$, we can find some sufficiently large j such that taking e^{-u_j} as the weight factor in Ohsawa-Takegoshi Extension Theorem, we can extend the function f. With some j, the following holds

$$\int\limits_{V\cap P_r} |f|^2 e^{-u_j} \leq 2||f|_{V\cap P}||^2$$

and if we take u as u_j in the Ohsawa-Takegoshi Extension Theorem, we get the extension function g.

Let $G = g \circ \alpha$, with that its germ at 0 is an element of $J \circ \alpha$. Since $F = f_0 \notin \mathcal{O}_{(\mathbb{C},0)}(J \circ \alpha)$, the following must hold

$$\lim_{z\to 0} \frac{G(z)}{F(z)} = 0,$$

because if not, using Comparison Test, F and G have the same convergence and that $f_0 \circ \alpha \notin J \circ \alpha$ implies that $F \notin J$, which means that $\int |F|^2 e^{-u} = \infty$. However, $G \in J$, so $\int |G|^2 e^{-u} < \infty$. This gives a contradiction, and thus, we must have $G = o(F)$ at 0.

Further, for any choice of k-th root $\sqrt[k]{r}$, we have $F(\sqrt[k]{r}) = G(\sqrt[k]{r})$, because $\alpha_1(s) = s^k, s \in \Delta$ holds. Hence, by Proposition 4.1.2,with taking $k = 1$, we have

$$\max_{\alpha(\bar{\Delta})} |g| = \max_{\bar{\Delta}} |G| \geq \frac{c_1}{|r|}. \tag{4.1.3}$$

Putting the inequalities 4.1.1, 4.1.2 and 4.1.3 together, we have,

$$||f|_{V\cap P_r}|| \geq \frac{c_1}{c_2 c_3 |r|}, r \in \Delta \setminus \{0\}. \tag{4.1.4}$$

Then,

$$\liminf dist(0, P_r)||f|_{V\cap P_r}|| \geq \frac{c_1}{c_2 c_3},$$

which contradicts to the assumption. □

Now we prove Theorem 4.1.1.

Proof. Let $J = \bigcup_j I(u_j)$, which consists of germs of u_j at x. Suppose $P \subset \mathbb{C}^m$ is a complex hyperplane and let W be a relatively open subset of P.

Then, by Dominated Convergence Theorem, we have,

$$||g||^2 = \begin{cases} \infty & or \\ \lim\limits_j \int\limits_W |g|^2 e^{-u_j} = \int\limits_W |g|^2 e^{-u} \end{cases}$$

Let $dist(x, P)$ denote the Euclidean distance between x and P. Consider for $m > 1$, since J is a subset of $I(u)$ and $I(u)$ is finitely generated, it only remains to show that $f_x \in I(u) \Rightarrow f_x \in J$, which will be done by induction on the dimension m.

1. For dimension one,

$$\begin{aligned} f_x \in J \iff & \liminf dist(x, P).||f|_{V\cap P}|| = 0 \\ \iff & \liminf_{z_0 \to x} dist(x, z_0). \inf \int_{V\cap\{z_0\}} |f|^2 e^{-u_j} = 0 \\ \iff & \lim_{z_0 \to x} |z_0 - x|. \inf |f(z_0)|^2 e^{-u_j(z_0)} = 0 \\ \iff & \lim_{z_0 \to x} |z_0 - x|.|f(z_0)|^2 e^{-u(z_0)} = 0, \\ \iff & f_x \in I(u). \end{aligned}$$

   The last equivalence follows from Lemma 4.1.3 with $u_j = u$ for all j.

2. Assume that it holds for $m-1$. Let $f_x \in I(u)$, then $\int |f|^2 e^{-u} < \infty$, but then by induction hypothesis,

   $\displaystyle\int_{V\cap P_r} |f|^2 e^{-u} < \infty$ implies that there exists some j such that $\displaystyle\int_{V\cap P_r} |f|^2 e^{-u_j} < \infty$.

   Then, by taking u_j as u, and $I(u)$ as J in Lemma 4.1.3,

$$\liminf dist(x, P_r)^2 \int_{V\cap P_r} |f|^2 e^{-u} = 0,$$

   and thus,

$$\liminf dist(x, P_r) \int_{V\cap P_r} |f|^2 e^{-u} = 0.$$

   Now, note that $||f|_{V\cap P_r}|| = \inf \int_{V\cap P_r} |f|^2 e^{-u_j}$ and as u_j goes to u, when j goes to infinity, since $\int_{V\cap P_r} |f|^2 e^{-u_j} < \infty$,

$$\inf \int_{V\cap P_r} |f|^2 e^{-u_j} = \int_{V\cap P_0} |f|^2 e^{-u}$$

   by Dominated Convergence Theorem. Then, $f_x \in J$.

$\square$

4.2 Suita Conjecture

Suita [25] conjectured that for a bounded domain $D \subset \mathbb{C}$, one has

$$c_D^2 \leq \pi K_D \tag{4.2.1}$$

where $c_D(z) = e^{\lim\limits_{w \to z}(G_D(w,z) - \log|w-z|)}$ is the logarithmic capacity of the complement of D with respect to $z \in D$, $G_D(w,z)$ is the (negative) Green's function for D with pole at z and $K_D(z) = \sup\{|f(z)|^2 : f \in \mathcal{O}(D), ||f||_2 \leq 1\}$ is the Bergman kernel on the diagonal. Note that $c_D|dz|$ is an invariant metric, called the Suita metric, and its curvature is given by

$$Curv_{c_D|dz|} = -\frac{\partial^2 \log(c_D)}{\partial z \partial \overline{z}} \frac{1}{c_D^2}.$$

On the other hand, Suita also observed that

$$\frac{\partial^2 \log(c_D)}{\partial z \partial \overline{z}} = \pi K_D$$

which gives us the result that (4.2.1) is actually equivalent to that

$$Curv_{c_D|dz|} \leq -1.$$

It was also concluded that if D is simple connected, $Curv_{c_D|dz|} = -1$, so (4.2.1) becomes $c_D^2 = \pi K_D$ on ∂D if D is a domain with smooth boundary.
In 1995, Ohsawa [24] observed that Suita Conjecture can be viewed as an extension problem, for $z_0 \in D$, there is a function $f \in \mathcal{O}(D)$ such that $f(z_0) = 1$ and

$$\int_D |f|^2 d\lambda \leq \frac{\pi}{c_D^2(z_0)}.$$

He, then, proved that

$$c_D^2 \leq C_0 \pi K_D,$$

where $C_0 = 750$. Later, in 2007, Błocki [7] developed this constant to $C_0 = 2$,, and in 2011, (See [13]), it was improved further to $C_0 = 1.95388\ldots$. Finally, Błocki [5] concluded the result for the optimal constant $C_0 = 1$, mainly using Hörmander's solution to $\overline{\partial}$ equation using L^2 estimate.

Theorem 4.2.1. *[5] Assume that $\Omega \subset \mathbb{C}^{n-1} \times D$ is pseudo-convex with $0 \in D \subseteq \mathbb{C}$ and is bounded. Then, given a function f which is holomorphic in $\Omega' := \Omega \cap \{z_n = 0\}$*

and a plurisubharmonic function φ in Ω, there exists a holomorphic extension F of f on Ω with

$$||F||^2_\varphi \leq \frac{\pi}{(c_D(0))^2}||f||^2_\varphi.$$

Proof. Recall that Hörmander's Estimate [4] states that for a pseudo-convex domain Ω in $\mathbb{C}^n$ and $\alpha = \sum\limits_j \alpha_j d\overline{z}_j \in L^2_{loc(0,1)}(\Omega)$ such that $\overline{\partial}\alpha = 0$, for any plurisubharmonic φ in Ω, we can find $u \in L^2_{loc}(\Omega)$ such that $\overline{\partial}u = \alpha$ and

$$\int\limits_\Omega |u|^2 e^{-\varphi} d\lambda \leq \int\limits_\Omega |\alpha|^2_{i\partial\overline{\partial}\varphi} e^{-\varphi} d\lambda,$$

noting that $|\alpha|^2_{i\partial\overline{\partial}\varphi} = \sum\limits_{j,k} \varphi^{i\overline{k}}\overline{\alpha}_j\alpha_k$. With a little alteration, we can use the following theorem for the proof.

Theorem 4.2.2. *Let $\alpha \in L^2_{loc,(0,1)}(\Omega)$ be a $\overline{\partial}$-closed form in a pseudo-convex domain Ω in $\mathbb{C}^n$. Assume that φ is plurisubharmonic in Ω, $\psi \in W^{1,2}_{loc}(\Omega)$ is locally bounded from above and they satisfy $|\overline{\partial}\psi|^2_{i\partial\overline{\partial}\varphi} \leq H < 1$ in Ω for some $H \in L^\infty_{loc}(\Omega)$.*

Then, there is $u \in L^2_{loc}(\Omega)$ solving $\overline{\partial}u = \alpha$ and such that for every $b > 0$

$$\int\limits_\Omega |u|^2(1-H)e^{2\psi-\varphi}d\lambda \leq (1+\frac{1}{b})\int_\Omega |\alpha|^2_{i\partial\overline{\partial}\varphi}\frac{1+bH}{1-H}e^{2\psi-\varphi}d\lambda.$$

Here $W^{1,2}_{loc}(\Omega)$ is the Sobolev space.

Proof. Assume that φ is plurisubharmonic, smooth and ψ is bounded in Ω. Set u to be the minimal solution to $\overline{\partial}u = \alpha$ in $L^2(\Omega, e^{\psi-\varphi})$. This implies that $u \perp \ker\overline{\partial}$ in the set.

Then, $v := ue^\psi$ is perpendicular to $\ker\overline{\partial}$ in $L^2(\Omega, e^{-\varphi})$ because

$$\begin{aligned}
\overline{\partial}v = \overline{\partial}(ue^\psi) &= (\overline{\partial}u)e^\psi + u\overline{\partial}(e^\psi) \\
&= e^\psi\overline{\partial}u + u(\overline{\partial}\psi)e^\psi \\
&= \alpha e^\psi + ue^\psi\overline{\partial}\psi.
\end{aligned}$$

Hence, it is the minimal solution to $\overline{\partial}v = \beta$ where $\beta = e^\psi(\alpha + u\overline{\partial}\psi) \in L^2_{loc,(0,1)}(\Omega)$.

Then, using Hörmander's Estimate, we get, $\int_\Omega |v|^2 e^{-\varphi} d\lambda \leq \int_\Omega |\beta|^2_{i\partial\overline{\partial}\varphi} e^{-\varphi} d\lambda$, or equivalently,

$$\begin{aligned} \int_\Omega |u|^2 e^{2\psi-\varphi} d\lambda &\leq \int_\Omega |\alpha + u\overline{\partial}\psi|^2_{i\partial\overline{\partial}\varphi} e^{2\psi-\varphi} d\lambda \\ &\leq \int_\Omega (|\alpha|^2_{i\partial\overline{\partial}\varphi} + 2|u|\sqrt{H}|\alpha|_{i\partial\overline{\partial}\varphi} + |u|^2 H) e^{2\psi-\varphi} d\lambda. \end{aligned}$$

To see this,

$$\begin{aligned} \int_\Omega |v|^2 e^{-\varphi} d\lambda &= \int_\Omega |u|^2 e^{2\psi} e^{-\varphi} d\lambda = \int_\Omega |u|^2 e^{2\psi-\varphi} d\lambda \\ &\leq \int_\Omega |\beta|^2_{i\partial\overline{\partial}\varphi} e^{-\varphi} d\lambda = \int_\Omega e^{2\psi}(\alpha + u\overline{\partial}\psi)^2 e^{-\varphi} d\lambda \\ &= \int_\Omega |\alpha + u\overline{\partial}\psi|^2 e^{2\psi-\varphi} d\lambda \leq \int_\Omega (|\alpha|^2 + 2|\alpha||u\overline{\partial}\psi| + |u|^2|\overline{\partial}\psi|^2) e^{2\psi-\varphi} d\lambda \\ &\leq \int_\Omega (|\alpha|^2 + 2|\alpha||u|\sqrt{H} + |u|^2 H) e^{2\psi-\varphi} d\lambda, \end{aligned}$$

where the very last inequality is a result of $|\overline{\partial}\psi|^2_{i\partial\overline{\partial}\varphi} \leq H$.

Assume now that $t > 0$, then consider

$$0 \leq \left(\frac{|\alpha|\sqrt{H}}{\sqrt{t(1-H)}} - |u|\sqrt{t(1-H)} \right)^2,$$

$$2|\alpha||u|\sqrt{H} \leq \frac{|\alpha|^2 H}{t(1-H)} + |u|^2 t(1-H).$$

Thus, we have for $t > 0$,

$$\int_\Omega |u|^2(1-H) e^{2\psi-\varphi} d\lambda \leq \int_\Omega \left[|\alpha|^2_{i\partial\overline{\partial}\varphi} + |\alpha|^2_{i\partial\overline{\partial}\varphi} \frac{H}{t(1-H)} + t|u|^2 - t|u|^2 H \right] e^{2\psi-\varphi} d\lambda.$$

Then, upon taking $t := \frac{1}{b+1}$, we get the estimate given in the theorem. $\square$

Continuing the proof, for any $\epsilon > 0$, define

$$\alpha := \overline{\partial}(f(z')\chi(-2\log|z_n|)) = \frac{-f(z')\chi'(-2\log|z_n|)}{\overline{z}_n} d\overline{z}_n$$

where $z = (z', z_n)$ and $\chi \in C^{0,1}(\mathbb{R})$ is non-decreasing and satisfies the condition that $\chi = 0$ on $\{t \leq -2\log\epsilon\}$, $\chi(\infty) = 0$ (which we will explicitly define later).

Then, α is defined in Ω when ϵ is small enough, supp $\alpha \subset \{|z_n| \leq \epsilon\}$ and for any solution of $\overline{\partial} u = \alpha$, the following

$$F := f(z')\chi(-2\log|z_n|) - u \tag{4.2.2}$$

is a holomorphic extension of f given that $u = 0$ on Ω'. To make it clear, for $\Omega' = \{z_n = 0\} \cap \Omega$, and $|z_n| < \epsilon$, we see that $-2\log|z_n| \geq -2\log\epsilon$. Now if $\chi = 0$ on $\{t \leq -2\log\epsilon\}$, then $f\chi = 0$ there. So f has a support on $|z_n| < \epsilon$. That is why F can be the extension of f on Ω.

Since we want to apply the theorem above and we can find sufficient weights $\tilde{\varphi}, \psi$, we have to choose the following functions on $\mathbb{R}_+$.

$$\begin{aligned} h(t) :=& \ -\log(t + e^{-t} - 1) \\ g(t) :=& \ -\log(t + e^{-t} - 1) + \log(1 - e^{-t}) \end{aligned}$$

These functions solve the ODE:

$$\left(1 - \frac{(g')^2}{h''}\right) e^{2g-h+t} \geq 1,$$

where $g \in C^{0,1}(\mathbb{R}_+)$, $h \in C^{1,1}(\mathbb{R}_+)$ such that h has a positive second derivative and is nonincreasing and $g + \log t$, $h + \log t$ vanish at ∞.

We also see that,

$$\left(1 - \frac{(g')^2}{h''}\right) e^{2g-h+t} = 1 \tag{4.2.3}$$

$$\lim_{t\to\infty}(h(t) - 2g(t) + \log(-h'(t))) = 0. \tag{4.2.4}$$

Observe that, in fact we have $g = \log(-h')$, the ODE (4.2.3) becomes

$$\frac{d^2}{dt^2}(e^{-h}) = e^{-t}.$$

After this arrangement, solving the posed ODE is trivial.
Now denote $G = G_D(.,0)$, then for a harmonic function v which is bounded in D, we can also express G as $G = \log|r| + v$. Then for some constants c, c', we have that in D, by the definition of Green's Function,

$$|2G - 2\log|r|| \leq c, \text{ and } |2G_r - \frac{1}{r}| \leq c' \text{ near origin.}$$

Set $N := -2\log\epsilon - c$ and choose $\delta = N^{-\frac{1}{2}}$ such that

$$\lim_{\epsilon\to 0}\delta(\epsilon) = 0.$$

Then, we define the following functions,

$$\nu(t) := \begin{cases} h(t), & t < N \\ -\delta\log(t - N + a) + b, & t \geq N \end{cases}$$

$$\gamma(t) := \begin{cases} g(t), & t < N \\ -\delta\log(t - N + a) + \tilde{b}, & t \geq N \end{cases}$$

where $a, b, \tilde{b}$ are chosen so that $\nu \in C^{1,1}$ and $\gamma \in C^{0,1}$, so we determine a, b and $\tilde{b}$ as follows,

$$a = a(\epsilon) = -\frac{\delta}{h'(N)}$$
$$b = b(\epsilon) = h(N) + \delta\log a$$
$$\tilde{b} = \tilde{b}(\epsilon) = g(N) + \delta\log a.$$

Observe that, as $\epsilon \to 0$, $a(\epsilon) \to \infty$.

After this set up, we define $\tilde{\varphi}$ and ψ explicitly as given below,

$$\tilde{\varphi} := \varphi + 2G + \nu(-2G), \quad \psi := \gamma(-2G).$$

Note that, $\tilde{\varphi}$ is plurisubharmonic in Ω because Green's Function is harmonic and φ is plurisubharmonic on Ω. Also, note that, ψ is an element in the Sobolev space given in the theorem and is bounded from above. Then,

$$|\overline{\partial}\psi|^2_{i\partial\overline{\partial}\tilde{\varphi}} \leq |\overline{\partial}\psi|^2_{i\partial\overline{\partial}\nu(-2G)} = \frac{(\gamma'(-2G))}{\nu''(-2G)} = \begin{cases} < 1 & in\ \Omega \\ = \delta & on\ supp\alpha, \end{cases}$$

and $supp\alpha \subset \{-2G \geq N\}$.

Now to grasp a little better, recalling the functions γ and ν, γ is defined using g, then γ' has g', and ν is defined using h, then ν'' has h''. So the last equality we get will be in similar to the ODE we mentioned earlier. Then

$$\left(1 - \frac{(g')^2}{h''}\right) < 1 \Rightarrow \frac{(g')^2}{h''} < 1.$$

Continuing the proof, we also have that

$$\alpha = \overline{\partial}(f(z')\chi(-2\log|z_n|)) = \frac{-f(z')\chi'(-2\log|z_n|)}{\overline{z}_n}d\overline{z}_n.$$

Then,

$$|\alpha|^2_{i\partial\overline{\partial}\tilde{\varphi}} = \frac{|f(z')|^2(\chi'(-2\log|z_n|))^2}{|z_n|^2}|d\overline{z}_n|^2$$

where $|d\overline{z}_n|^2 = (1/(4|\partial z_n|^2\nu''(-2G))$ by above theorem (see [5]). However, the function

$$-\delta\log(t-N+a)+t$$

is non-decreasing function in the variable t. Then, it follows from 4.2.3,

$$\left(1-\frac{(\gamma')^2}{\nu''}\right)e^{2\gamma-\nu+t} = \begin{cases} = 1 & on\ \{t<N\}, \\ \geq (1-\delta)e^{2g(N)-h(N)+N} & on\ \{t\geq N\}. \end{cases}$$

Then, for ϵ is small enough, the above term is greater than or equal to 1 on $\mathbb{R}^+\cup\{0\}$.

Now, the above theorem provides a solution u_ϵ for $\overline{\partial}u=\alpha$ so that

$$\int_\Omega |u|^2 e^{-\varphi}d\lambda \leq \int_\Omega |u|^2(1-|\overline{\partial}\psi|^2_{i\partial\overline{\partial}\tilde{\varphi}})e^{2\psi-\varphi} \leq \frac{1+\sqrt{\delta}}{1-\sqrt{\delta}}A(\epsilon),$$

where

$$A(\epsilon) := \int_\Omega |\alpha|^2_{i\partial\overline{\partial}(\nu(-2G))}e^{2\psi-\varphi}d\lambda.$$

This gives us,

$$\overline{\lim_{\epsilon\to 0}}A(\epsilon) \leq \overline{\lim_{\epsilon\to 0}}B(\epsilon)\int\limits_{\Omega'}|f|^2e^{-\varphi}d\lambda',$$

$$B(\epsilon) := \int_{|r|\leq\epsilon}\frac{|\alpha|^2_{i\partial\overline{\partial}(\nu(-2G))}}{|f(z')|^2}e^{2\psi-\nu(-2G)-2G}d\lambda(r).$$

This holds because α depends on ϵ, supp $\alpha \subset \{|z_n|\leq\epsilon\}$ and u depends on ϵ.

By Fubini Theorem, we separate $A(\epsilon)$, noting that $|z_n|>\epsilon$, $\chi'=0$. Then, for $|z_n|\leq\epsilon$, consider the integral $A(\epsilon)$,

$$A(\epsilon) = \int\limits_{z'}\int\limits_{z_n}A(\epsilon)_{\text{inside}}dz_n dz' = \int\limits_{z'}|f(z')|^2e^{-\varphi}dz'\int\limits_{z_n}\frac{|\alpha|^2_{i\partial\overline{\partial}(\nu(-2G))}}{|f(z')|^2}e^{2\psi-\nu(-2G)-2G}dz_n$$

Since G depends on 1-coordinate z_n, ($G = \log|r| + v = G(0, .)$), we obtain,

$$A(\epsilon) = \int\limits_{z'} |f(z')|^2 e^{-\varphi} dz' \int\limits_{|r|\le\epsilon} \frac{|\alpha|^2_{i\partial\bar\partial(\nu(-2G))}}{|f(z')|^2} e^{2\psi-\nu(-2G)-2G+2} dz_n.$$

We now want to replace G by $\log|r|$ in the above integral of $B(\epsilon)$. Note that,

$$\lim_{\epsilon\to0} \sup_{\{|r|\le\epsilon\}} |r|^2 e^{-2G} = \frac{1}{(c_D(0))^2}$$

and near origin,

$$|2G_r - \frac{1}{r}| \le c', \text{ or equivalently,} \tag{4.2.5}$$

$$|2G_r - \frac{1}{r}|^2 = |4G_r^2 - 4G_r\frac{1}{r} + \frac{1}{r^2}| \le (c')^2,$$
$$4|G_r|^2|r|^2 \ge (1 - c'|r|)^2.$$

Plus, on $\{|r| \le \epsilon\}$ we have

$$\begin{aligned} 2\psi - \nu(-2G) &\le 2\gamma(-2\log|r| - c) - \nu(-2\log|r| - c) \\ &\le 2\gamma(-2\log|r|) - \nu(-2\log|r|) + \delta\log(1 + \frac{c}{a}) \end{aligned}$$

and

$$\nu''(-2G) \ge \nu''(-2\log|r| + c) \ge \left(\frac{a+c}{a+2c}\right)^2 \nu''(-2\log|r|)$$

Thus, by the conditions on δ and a, we get,

$$\overline{\lim_{\epsilon\to0}} B(\epsilon) \le \frac{1}{(c_D(0))^2} \overline{\lim_{\epsilon\to0}} \tilde{B}(\epsilon)$$

where $\tilde{B}(\epsilon) = \int\limits_{\{|r|\le\epsilon\}} \frac{(\chi'(-2\log|r|))^2}{|r|^2\nu''(-2\log|r|)} e^{2\gamma(-2\log|r|)-\nu(-2\log|r|)} d\lambda(r) = \pi \int\limits_{N+c}^{\infty} \frac{(\chi')^2 e^{2\gamma-\nu}}{\nu''} dt.$

By Schwarz inequality, choosing χ such that it is increasing and $\chi(\infty) = \int\limits_{N+c}^{\infty} \chi' dt = 1$ holds, we see that

$$\chi(t) := \begin{cases} 0 & t < N + c \\ \frac{1}{C} \int\limits_{N+c}^{t} w ds & t \ge N + c \end{cases}$$

where $w = \nu'' e^{\nu - 2\gamma}$ and $C = \int\limits_{N+c}^{\infty} w dt$. Then,

$$\tilde{B}(\epsilon) = \frac{\pi}{\int\limits_{N+c}^{\infty} \nu'' e^{\nu-2\gamma} dt}$$

and

$$\begin{aligned}
\int_{N+c}^{\infty} \nu'' e^{\nu-2\gamma} dt &= \delta e^{b-2\tilde{b}} \int_{c}^{\infty} (t+a)^{\delta-2} dt \\
&= \frac{1}{1-\delta} \left(1 + \frac{c}{a}\right)^{\delta-1} e^{h(N)} e^{-2g(N)} e^{\log(-h'(N))}.
\end{aligned}$$

By $\lim\limits_{t\to\infty} (h(t) - 2g(t) + \log(-h'(t))) = 0$, and $\lim\limits_{\epsilon\to 0} \delta(\epsilon) = 0$, and $\lim\limits_{\epsilon\to 0} a(\epsilon) = \infty$,we have

$$\lim_{\epsilon\to 0} \tilde{B}(\epsilon) = \pi.$$

Thus, we obtain,

$$\overline{\lim_{\epsilon\to 0}} \int_{\Omega} |u_\epsilon|^2 e^{-\varphi} d\lambda \leq \frac{\pi}{(c_D(0))^2} \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda.$$

Considering the case that $0 < \tilde{\epsilon} \leq \epsilon$,

$$\begin{aligned}
\int_{\{|r| \leq \tilde{\epsilon}\}} \left(1 - \frac{(\gamma'(-2G))^2}{\nu''(-2G)}\right) e^{2\psi - \nu(-2G) - 2G} d\lambda(r) &\geq \frac{1}{C(\epsilon)} \int_{\{|r|\leq\tilde{\epsilon}\}} \frac{e^{2\gamma(-2\log|r|) - \nu(-2\log|r|)}}{|r|^2} d\lambda(r) \\
&= \pi \frac{e^{2\tilde{b}-b}}{C(\epsilon)} \int_{-2\log\tilde{\epsilon}}^{\infty} (t - N + a)^{-\delta} dt = \infty
\end{aligned}$$

and by the previous theorem, for $u = u_\epsilon$ of $\overline{\partial} u = \alpha$ such that the inequality (4.2.5) holds, we have $u = 0$ almost everywhere on Ω'. Therefore, the weak limit of $F = F_\epsilon$ is the required extension satisfying the norm condition.

□

CHAPTER 5

OHSAWA'S QUESTION

The study of extending holomorphic functions with L^2 estimates presented in many different versions with various methods gave the most important results in several complex analysis concerning about this topic. These studies include important Ohsawa-Takegoshi Extension Theorem and other types of extension theorems provided by different mathematicians. Recently, Guan and Zhou found the way to establish an optimal form for the L^2 extension theorem (see [11], [14], and [15]). Morever, many other results have been obtained from these extension theorems related to Bergman Kernel (see [11]). This provided the unavoidable connection between the L^2 extension problem and Suita Conjecture. Related to this connection, many questions have been proposed some of which are still unanswered. We are going to survey some of these questions posed and some answers to them.

For example, Ohsawa proposed the following question in [22].
Given subharmonic functions φ and ψ on $\mathbb{C}$ satisfying

$$\int_{\mathbb{C}} e^{-\psi} < \infty,$$

does there exist a holomorphic function f with $f(0) = 1$ and

$$\int_{\mathbb{C}} |f|^2 e^{-(\psi+\varphi)} \leq e^{-\varphi(0)} \int_{\mathbb{C}} e^{-\psi} \ ?$$

Observe that when ψ and φ do not depend on $arg z$, the answer is positive. To see this,

$$\psi(z) = \psi(|z|) \text{ and } \varphi(z) = \varphi(|z|)$$

Then, if we take $f \equiv 1$, we have

$$\begin{aligned}\int\limits_{\mathbb{C}} e^{-\varphi(|z|)}e^{-\psi(|z|)} &= \int\limits_0^{2\pi}\int\limits_0^{\infty} e^{-\varphi(r)-\psi(r)} r dr d\theta \\ &\leq \int\limits_0^{2\pi}\int\limits_0^{\infty} e^{-\varphi(0)}e^{-\psi(r)} r dr d\theta \\ &\leq e^{-\varphi(0)}\int\limits_0^{2\pi}\int\limits_0^{\infty} e^{-\psi(r)} r dr d\theta \\ &= e^{-\varphi(0)}\int\limits_{\mathbb{C}} e^{-\psi(|z|)}.\end{aligned}$$

Note that, the fact that φ is a subharmonic function which is independent of $arg(z)$ and e^t is a convex increasing function implies that $e^{\varphi(t)}$ is a subharmonic function. Moreover, the function $e^{\varphi(t)}$ is a real-valued and increasing function. To clarify, observe that since φ is subharmonic, considering its Laplacian in polar coordinates we see that $r\varphi_r$ is an increasing function and so $\varphi_r > 0$ at any point in the domain because otherwise it would give a contradiction to $r\varphi_r \geq 0$. Then, φ is increasing and the inequality above follows.
On the other hand, if one of ψ or φ depends on $arg(z)$, the question is that what other conditions should be forced so that this inequality holds. In fact, in [11], Guan and Zhou proved an extension theorem implying the following result concerning this question.

Theorem 5.0.1. *[26] Let ψ be subharmonic on $\mathbb{C}$ with the property that $i\partial\bar{\partial}\psi$ is independent of $arg(z)$ and satisfies the following:*

$$\int\limits_{\mathbb{C}} e^{-\psi} < \infty.$$

Then for every φ subharmonic on $\mathbb{C}$ one can find a holomorphic function F on $\mathbb{C}$ with $F(0) = 1$ and

$$\int\limits_{\mathbb{C}} |F|^2 e^{-\varphi-\psi} \leq e^{-\varphi(0)}\int\limits_{\mathbb{C}} e^{-\psi}.$$

Let us now consider the extension theorem provided by Guan and Zhou, then using their result, we will give a proof for the theorem above.

Let f_X be a non-negative function in $C^\infty((-X,\infty))$ and $X \in (-\infty,\infty]$ such that

$$\int_{-X}^{\infty} f_X(t)e^{-t}dt < \infty \text{ and,}$$

$$\left[\int_{-X}^{t} f_X(t_1)e^{-t_1}dt_1\right]^2 > f_X(t)e^{-t}\int_{-X}^{t}\int_{-X}^{t_2} f_X(t_1)e^{-t_1}dt_1dt_2 \tag{5.0.1}$$

for any $t \in (-X,\infty)$, or equivalently,

$F(t) = \log\left(\int_{-X}^{t}\int_{-X}^{t_2} f_X(t_1)e^{-t_1}dt_1dt_2\right)$ is strictly concave for any $t \in (-X,\infty)$.

Remark 5.0.2. *An important class of functions $f_X(t)$ satisfying the inequality (5.0.1) contains the functions satisfying the following:*

1. $\frac{d}{dt}f_X(t)e^{-t} > 0,$ *for* $t \in (-X,a)$
2. $\frac{d}{dt}f_X(t)e^{-t} \leq 0,$ *for* $t \in [a,\infty)$
3. $\frac{d^2}{dt^2}\log(f_X(t)e^{-t}) < 0,$ *for* $t \in (-X,a)$

where $a \geq -X$ is constant.

Theorem 5.0.3. *Let $D \subseteq \mathbb{C}^n$ be pseudo-convex and l be an integer such that $1 \leq l \leq n$. Let φ be a function which is plurisubharmonic on D, and let $H = D \cap \{z_n = \cdots = z_{n-l+1} = 0\}$. Then, given an arbitrary holomorphic function f on H such that*

$$\int_H |f|^2 e^{-\varphi} dV_H < \infty$$

one can find a holomorphic extension F of f on D satisfying

$$\int_D f_X(-\log(|z_n|^2 + \cdots + |z_{n-l+1}|^2)|F|^2 e^{-\varphi}dV_D \leq \frac{\pi^l}{l!}\int_{-X}^{\infty} f_X(t)e^{-t}\int_H |f|^2e^{-\varphi}dV_H$$

where $f_X(t)$ is a positive function in $C^\infty((-X,\infty))$ such that

$$F(t) = \log\left(\int_{-X}^{t}\int_{-X}^{t_2} f_X(t_1)e^{-t_1}dt_1dt_2\right)$$

is strictly concave for any $t \in (-X,\infty)$.

Theorem 5.0.4. *Let $D \subseteq \mathbb{C}^n$ be pseudo-convex, and l be an integer such that $1 \leq l \leq n$. $H = D \cap \{z_n = \cdots = z_{n-l+1} = 0\}$ and,*

$$\psi(z) = \rho\left(\left(\sum_{j=n-l+1}^{n} |z_j|^2\right)^{1/2}\right)$$

where ρ does not depend on $arg(z)$, is subharmonic on $\mathbb{C}$. Then, for any φ which is subharmonic on D and any f holomorphic on H with

$$\int_H |f|^2 e^{-\varphi} dV_H < \infty,$$

one can find a holomorphic extension F of f on D, satisfying

$$\int_D |F|^2 e^{-\varphi-\psi} dV_D \leq 2\frac{\pi^l}{l!} \int_0^{e^{X/2}} e^{-\rho(t)} t dt \int_H |f|^2 e^{-\varphi} dV_H \tag{5.0.2}$$

where $X = \sup_{z \in D}\left(l \, \log\left(\sum_{j=n-l+1}^{n} |z_j|^2\right)\right)$.

Note that ρ is independent of the variable $arg(z)$, so we can regard ρ as a function defined on $\mathbb{R}^+ \cup \{0\}$ and thus,
$\rho\left(\left(\sum_{j=n-l+1}^{n} |z_j|^2\right)^{1/2}\right)$ and $\rho(t)$ in the equation (5.0.2) is significant. Our aim is to provide a proof of Theorem 5.0.4 and with the help of Theorem 5.0.3.

Proof. Assume that ρ is a smooth and strictly subharmonic function and

$$\Psi(z) = l.\, log\left(\sum_{j=n-l+1}^{n} |z_j|^2\right) \text{ and } f_X(t) = e^{-\rho(e^{-t/2})}.$$

Since ρ does not depend on $arg(z)$, take the function ρ on $\mathbb{R}^+ \cup \{0\}$. Observe that

$$\begin{aligned}
f_X(-\Psi(z)) &= e^{-\rho\left(e^{\left(\frac{\Psi(z)}{2}\right)}\right)} \\
&= e^{-\rho\left(e^{\frac{1}{2}l.\ log\left(\sum\limits_{j=n-l+1}^{n}|z_j|^2\right)}\right)} \\
&= e^{-\rho\left(e^{log\left(\sum\limits_{j=n-l+1}^{n}|z_j|^2\right)^{l/2}}\right)} \\
&= e^{-\rho\left(\left(\sum\limits_{j=n-l+1}^{n}|z_j|^2\right)^{l/2}\right)} \\
&= e^{-\psi(z)}.
\end{aligned}$$

Set $T = e^{-t/2}$, and we get, $dT = \frac{-1}{2}\, e^{-t/2}\, de^{-t/2}$,

$$\begin{aligned}
\int\limits_{-X}^{\infty} f_X(t)e^{-t}dt &= \int\limits_{-X}^{\infty} e^{-\rho(e^{-t/2})}e^{-t}dt = -2\int\limits_{t=-X}^{t=\infty} e^{-\rho(e^{-t/2})}e^{-t/2}de^{-t/2} \\
&= 2\int\limits_{t=\infty}^{t=-X} e^{-\rho(T)}TdT = 2\int\limits_{0}^{e^{(X/2)}} e^{-\rho(T)}TdT.
\end{aligned}$$

However, $f_X(t)$ satisfies Remark (1), (2), (3). To see this, consider,

$$\begin{aligned}
\frac{d}{dt}(f_X(t)e^{-t}) &= \frac{d}{dT}\left(e^{-\rho(e^{-t/2})}e^{-}t\right) = \frac{d}{dT}\left(e^{-\rho(T)}T^2\right)\frac{dT}{dt},\ \text{where}\ \frac{dT}{dt} = \frac{-1}{2}e^{-t/2} \\
&= (2Te^{-\rho(T)} - T^2\rho'(T)e^{-\rho(T)})(\frac{-1}{2}T) \\
&= \left(\frac{-T^2}{2}\right)e^{-\rho(T)}(2 - T\rho'(T)).
\end{aligned}$$

Considering next,

$$\log(f_X(t)e^{-t}) = \log(e^{-\rho(T)}T^2) = \log T^2 - \rho(T),$$

taking derivative with respect to dt, we get,

$$\frac{d}{dt}(\log(f_X(t)e^{-t})) = \frac{d}{dt}(\log T^2 - \rho(T)) = \frac{2}{T} - \rho'(T).$$

Finally, for the last property, consider,

$$\begin{aligned}
\frac{d^2}{dt^2}(\log(f_X(t)e^{-t})) &= \frac{d}{dt}(\frac{d}{dT}\log(e^{-\rho(T)}T^2))\frac{dT}{dt} \\
&= \frac{d}{dT}\left[\frac{d}{dT}\log(e^{-\rho(T)}T^2)\frac{dT}{dt}\right]\frac{dT}{dt} \\
&= \frac{d}{dT}\frac{d}{dT}\log(e^{-\rho(T)}T^2)\left(\frac{dT}{dt}\right)^2 + \frac{d}{dT}\log(e^{-\rho(T)}T^2)\frac{d^2T}{dt^2} \\
&= \left(\frac{-2}{T^2} - \rho''(T)\right)\frac{1}{4}T^2 + \left(\frac{2}{T} - \rho'(T)\right)\frac{1}{4}T \\
&= \frac{-1}{4}T(\rho'(T) + \rho''(T)T),
\end{aligned}$$

where the first equation is a result of $f_X(t)e^{-t} = e^{-\rho(T)}T^2$.
Now observe that the term

$$\rho'(T) + \rho''(T)T = \frac{d}{dT}(T\rho'(T))$$

is a positive number for $a = \{T : 2 - T\rho'(T) = 0\}$, then $T\rho'(T)$ is an increasing function there. Hence, $T\rho'(T)$ can be equal to 2 only once, then the derivative of $f_x(t)e^{-t}$ can change sign only once. Since the value of $T\rho'(T)$ is smaller at the points smaller than a, when $T \in (-\infty, a)$,

$$2 - T\rho'(T) > 0.$$

However, T and t are inversely related by construction. Thus, for $t \in (-A, a)$, we have

$$2 - T\rho'(T) < 0.$$

Hence, $f_x(t)$ satisfies Remark 5.0.2 (1) and (2). Moreover, since $\rho'(T) + \rho''(T)T$ is positive, by definition this shows that ρ is strictly plurisubharmonic on $\mathbb{C}$. Then, again by definition, it follows that

$$\frac{d^2}{dt^2}(\log(f_X(t)e^{-t})) < 0,$$

which shows that for $t \in (-X, a)$ with $a \geq -X$, the function $f_X(t)$ satisfies the third property of the remark above. Therefore, $f_X(t)$ satisfies the conditions given in the remark.
This also implies the sets of type $\frac{d}{dt}f_X(t)e^{-t} > 0$ or $\frac{d}{dt}f_X(t)e^{-t} \leq 0$ are intervals.
Then, by Theorem 5.0.3:

$$\int\limits_D f_X(-l\log(|z_n|^2+\cdots+|z_{n-l+1}|^2))|F|^2e^{-\varphi}dV_D$$

$$\begin{aligned}
&= \int\limits_D e^{-\rho(e^{\frac{-1}{2}(-l.\log(|z_n|^2+\cdots+|z_{n-l+1}|^2))})}|F|^2e^{-\varphi}dV_D \\
&= \int\limits_D e^{-\rho(e^{\log(|z_n|^2+\cdots+|z_{n-l+1}|^2)})^{\frac{l}{2}}}|F|^2e^{-\varphi}dV_D \\
&= \int\limits_D e^{-\rho\left(\left(\sum\limits_{j=n-l+1}^{n}|z_j|^2\right)^{l/2}\right)}|F|^2e^{-a}dV_D \\
&= \int\limits_D e^{-\psi}|F|^2e^{-\varphi}dV_D \\
&\leq \frac{\pi^l}{l!}\int\limits_{-X}^{0} e^{-\rho(e^{-t/2})}e^{-t}dt\int\limits_H |f|^2e^{-\varphi}dV_H \\
&= \frac{\pi^l}{l!}\int\limits_{0}^{e^{X/2}} e^{-\rho(t)}tdt\int\limits_H |f|^2e^{-\varphi}dV_H.
\end{aligned}$$

□

After proving Theorem 5.0.4, we now ready to prove Theorem 5.0.1. In the previous theorem, taking $n=1$, we obtain this less strict, or more general, condition for which Ohsawa's question still has a positive answer.

Proof of Theorem 5.0.1. Given $R>0$, define $D(0,R)$ to be the disc centered at 0 with radius R. Then using Riesz's Decomposition, we obtain

$$\psi(z)=\frac{1}{2\pi}\int\limits_{D(0,R)}\log|z-\xi|i\partial\bar{\partial}\psi(\xi)+h(z)$$

where h is a harmonic function on the given disc.

Let $\psi_1(z)=\frac{1}{2\pi}\int\limits_{D(0,R)}\log|z-\xi|i\partial\bar{\partial}\psi(\xi)$, then ψ_1 is a radial subharmonic function, in other words, ψ_1 does not depend on $arg(z)$. Then, by taking $D=\mathbb{C}$, $H=\{0\}$, $l=1$, φ as $\varphi+h$ and ψ_1 as the radial function in Theorem 5.0.4, we can find for any holomorphic function f on H with $\int\limits_H|f|^2e^{-\varphi}dV_H<\infty$, and a holomorphic function F on D such that $F=f$ on H and,

$$\int\limits_D|F|^2e^{-\varphi-h-\psi_1}dV_D\leq 2\frac{\pi^l}{l!}\int\limits_0^{e^{X/2}}e^{-\rho(t)}tdt\int\limits_H|f|^2e^{-(\varphi+h)}dV_H.$$

where $\rho = \psi_1$ as ψ_1 is radial in the theorem.

$$\begin{aligned}\int\limits_{D(0,R)} |F|^2 e^{-\varphi-h-\psi_1} dV_D &\leq 2\pi \int\limits_0^{e^{X/2}} e^{-\psi_1(t)} t dt \int\limits_{H=\{0\}} 1.e^{-\varphi-h} dV_H \\ &= 2\pi \left(\int\limits_0^R e^{-\psi_1(r)} r dr \right) e^{(-\varphi-h)(0)} \\ &= \left(\iint\limits_{D(0,R)} e^{-\psi_1} \right) e^{(-\varphi-h)(0)},\end{aligned}$$

here the first equality comes from that $X = \sup\{1. \log |R|^2\} = 2\log|R|$, and

$$\iint\limits_{D(0,R)} e^{-\psi_1} = \int\limits_0^{2\pi} \int\limits_0^R e^{-\psi_1(r)} r dr d\theta = \iint\limits_{D(0,R)} e^{-\psi_1(|z|)} dA$$

.

Since h is a harmonic function, e^x is a non-decreasing, convex function we have that e^{-h} is a subharmonic function. Thus, we see

$$e^{-h(0)} \iint\limits_{D(0,R)} e^{-\psi_1} \leq \iint\limits_{D(0,R)} e^{-\psi_1-h} = \iint\limits_{D(0,R)} e^{-\psi} < \infty.$$

Thus,

$$\iint\limits_{D(0,R)} |F|^2 e^{-\varphi-\psi} \leq e^{-\varphi(0)} \iint\limits_{D(0,R)} e^{-\psi} \leq e^{-\varphi(0)} \iint\limits_{\mathbb{C}} e^{-\psi} < \infty.$$

$\square$

Observe, the condition of $i\partial\bar{\partial}\psi$ is radial cannot be dropped. In [26], a counter example for the case of $i\partial\bar{\partial}\psi$ is not radial is provided. Assume that $i\partial\bar{\partial}\psi$ is a function depending on arg(z).

Consider the functions of the Corollary 5.0.1 as $\varphi = 2(1-\alpha)\log|z+1|$ and $\psi = 2\alpha\log|z+1| + h(|z+1|^2)$ where $0 < \alpha < 1$ is a constant. It is obvious that ψ and φ are subharmonic functions. Let

$$h(x) = \begin{cases} R_1 + \frac{1}{4} & x < R_1 \\ (x-R_1)^2 + R_1 + \frac{1}{4} & R_1 \leq x \leq R_1 + \frac{1}{2} \\ x & x > R_1 + \frac{1}{2} \end{cases}$$

where $R_1 > 0$ will be determined later. Observe that,

$$h'(x) = \begin{cases} 0, & x < R_1 \\ 2(x - R_1), & R_1 \leq x \leq R_1 + \frac{1}{2} \\ 1, & x > R_1 + \frac{1}{2} \end{cases}$$

So, $h' > 0$, which shows that h is increasing; plus, h is convex because the domain of h is $(-\infty, R_1] \cup [R_1, R_1 + \frac{1}{2}] \cup [R_1 + \frac{1}{2}, \infty)$ and this set is convex.
Now, on $\mathbb{R}$, h is convex and $|z+1|^2$ is subharmonic on $\mathbb{C}$ because on $\mathbb{C}$, for

$$g(z) \;=\; |z+1|^2$$

both g_{xx} and g_{yy} are positive. Consider for $K = \{|z+1|^2 < R_1 + \frac{1}{2}\}$ and $L = \{|z+1|^2 > R_1 + \frac{1}{2}\}$

$$\begin{aligned} \int\limits_{\mathbb{C}} e^{-\psi} &= \int\limits_K e^{-(2\alpha \log|z+1| + h(|z+1|^2))} + \int\limits_L e^{-(2\alpha \log|z+1| + h(|z+1|^2))} \\ &= \int\limits_K \frac{1}{|z+1|^{2\alpha}} e^{-h(|z+1|^2)} + \int\limits_L \frac{1}{|z+1|^{2\alpha}} e^{-h(|z+1|^2)} < \infty, \end{aligned}$$

where the first integral converges by p-test. With respect to $\varphi + \psi = 2\log|z+1| + h(|z+1|^2)$, square integrable holomorphic functions have an orthogonal basis $\{|z+1|^k\}, k \geq 1$. Therefore, minimal L^2-norm element with $f(0) = 1$ is given as $f(z) = z + 1$. Then,

$$\begin{aligned} \int |f|^2 e^{-\varphi-\psi} = I_1 \;&:=\; \int\limits_{\mathbb{C}} |z+1|^2 \frac{1}{|z+1|^2} e^{-h(|z+1|^2)} = \int\limits_{\mathbb{C}} e^{-h(|z+1|^2)} \\ &= \int\limits_K e^{-R_1 - \frac{1}{4}} dA + \int\limits_L e^{-h(|z+1|^2)} dA \\ &= \pi R_1 e^{-(R_1 + \frac{1}{4})} + \int\limits_L e^{-h(|z+1|^2)} dA. \end{aligned}$$

$$
\begin{aligned}
I_2 &:= e^{-\varphi(0)} \int\limits_{\mathbb{C}} e^{-\psi} = \int\limits_{\mathbb{C}} \frac{1}{|z+1|^{2\alpha}} e^{-h(|z+1|^2)} \\
&= \int\limits_{K} \frac{1}{|z+1|^{2\alpha}} e^{-(R_1+\frac{1}{4})} + \int\limits_{L} \frac{1}{|z+1|^{2\alpha}} e^{-h(|z+1|^2)} \\
&= e^{-(R_1+\frac{1}{4})} \int\limits_{K} \frac{1}{|z+1|^{2\alpha}} + \int\limits_{L} \frac{1}{|z+1|^{2\alpha}} e^{-h(|z+1|^2)} \\
&= e^{-R_1-\frac{1}{4}} \left(\frac{\pi}{1-\alpha} R_1^{1-\alpha} \right) + \int\limits_{L} \frac{1}{|z+1|^{2\alpha}} e^{-h(|z+1|^2)}.
\end{aligned}
$$

For R_1 large enough,

$$
\begin{aligned}
I_1 - I_2 &= e^{-R_1-\frac{1}{4}} \pi R_1 - e^{-R_1-\frac{1}{4}} \left(\frac{\pi}{1-\alpha} R_1^{1-\alpha} \right) + \int\limits_{L} e^{-h(|z+1|^2)} \left(1 - \frac{1}{|z+1|^{2\alpha}} \right) \\
&= e^{-(R_1+\frac{1}{4})} \pi \left(R_1 - \frac{R_1^{1-\alpha}}{1-\alpha} \right) + \int\limits_{L} e^{-h(|z+1|^2)} \left(1 - \frac{1}{|z+1|^{2\alpha}} \right) > 0
\end{aligned}
$$

So, $I_1 > I_2$. Then the minimal L^2-extension with respect to $\varphi + \psi$ such that $f(0) = 1$ exceeds $e^{-\varphi(0)} \int\limits_{\mathbb{C}} e^{-\psi}$. Then, the question posed by Ohsawa for this case fails.

We now consider another answer for Ohsawa's question provided in [10].

Theorem 5.0.5. *[10] There are subharmonic functions ψ and φ on $\mathbb{C}$ such that*

1. $\int\limits_{\mathbb{C}} e^{-\psi} < \infty$
2. $\varphi(0) \in (-\infty, \infty)$
3. *For any holomorphic function f on $\mathbb{C}$ such that $f(0) = 1$, $\int\limits_{\mathbb{C}} |f|^2 e^{-\varphi-\psi} = \infty$ holds.*

Proof. Let $\psi = 2max\{c_1 \log|z-1|, c_2 \log|z-1|\}$ and $\varphi = (1-c_1)(\log|z-1|)^2$, $c_1 \in (\frac{1}{2}, 1)$, $c_2 \in (1, \frac{3}{2})$. Assume not. There is a holomorphic function f on $\mathbb{C}$ such that $f(0) = 1$ and $\int\limits_{\mathbb{C}} |f|^2 e^{-\varphi-\psi} < \infty$. Also, observe that $(\psi + \varphi)|_{|z-1|<1} = 2\log|z-1|$, and, for $|z-1| < 1$,

$$
\psi = 2c_1 \log|z-1| \text{ and } \varphi = (1-c_1)\log|z-1|^2.
$$

Consider, $\psi + \varphi = 2\log|z-1|$, then $e^{-\varphi-\psi} = e^(\log|z-1|)^2 = \frac{1}{|z-1|^2}$. Therefore, it holds that

$$\int_{\mathbb{C}} |f|^2|\frac{1}{|z-1|^2} < \infty$$

because p-test holds if and only if $|f|^2 = 0$ at $z = 1$, as $z = 1$ is a singularity.

Then, for some number M,

$$\begin{aligned}
\psi + \varphi - 2(1-c_1+c_2)\log|z| &= 2c_2\log|z-1| + 2(1-c_1)\log|z-1| - 2(1-c_1+c_2)\log|z| \\
&= 2(1-c_1+c_2)(\log|z-1| - \log|z|) \\
&= 2(1-c_1+c_2)\log\frac{|z-1|}{|z|} \\
&< M.
\end{aligned}$$

$$\begin{aligned}
\infty > \int_{\mathbb{C}} |f|^2 e^{-(\varphi+\psi)} &> \int_{\mathbb{C}} |f|^2 e^{-M-2(\log|z|)(1-c_1+c_2)}, \text{ letting } e^{-M} = \tilde{c}, \\
&> \int_{\mathbb{C}} |f|^2\tilde{c}|z|^{-2(1-c_1+c_2)} \\
&= \tilde{c}\int_{\mathbb{C}} \frac{|f|^2}{|z|^{2(1-c_1+c_2)}}
\end{aligned}$$

Now, f is holomorphic then its Taylor Series will be in the form of $\sum a_n z^n$, but then, $|f|^2$ will be of the form $|z|^2$, so the inside of the last integral will be $\frac{1}{|z|^k}$ and for $k < 1$, the integral diverges, which is a contradiction. Thus, the degree of f must be 0, and so f is constant. However, now we have $f(0) = 1$ and $f(1) = 0$. This is a contradiction. $\square$

REFERENCES

[1] Ash, M. E. (1964). The basic estimate of the $\overline{\partial}$-Neumann problem in the non-Kählerian case. Amer. J. Math., 86, 247-254.

[2] Berndtsson, B. (2005). Integral formulas and the Ohsawa-Takegoshi extension theorem. Sci. China Ser. A-Math. 48, 61–73

[3] Berndtsson, B. (2013). The openness conjecture for plurisubharmonic functions, arXiv, https://arxiv.org/abs/1305.5781

[4] Błocki, Z. (2009). Some applications of the Ohsawa–Takegoshi extension theorem, Expositiones Mathematicae, Volume 27, Issue 2, Pages 125-135, ISSN 0723-0869,

[5] Błocki, Z. (2013). Suita conjecture and the Ohsawa-Takegoshi extension theorem, Inventiones mathematicae, 193, 149-158

[6] Błocki, Z. (2017). Suita Conjecture from the One-dimensional Viewpoint. In: Andersson, M., Boman, J., Kiselman, C., Kurasov, P., Sigurdsson, R. (eds) Analysis Meets Geometry. Trends in Mathematics. Birkhauser, Cham.

[7] Błocki, Z. (2007). Some estimates for the Bergman kernel and metric in terms of logarithmic capacity. Nagoya Mathematical Journal, 185, 143–150.

[8] Demailly, J-P., Kollar, J. (2001). Semicontinuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds, Ann Sci Ecole Norm Sup 34, pp 525-556.

[9] Garabedian, P. R., Spencer, D. C. (1952). Complex boundary problems. Trans. Amer. Math. Soc., 73, 223-242.

[10] Guan, Q. A. (2020). A remark on the extension of L^2 holomorphic functions , Int. J. Math. 31, 2050017, 3pp.

[11] Guan, Q. A., Zhou, X. Y. (2015). A solution of an L^2 extension problem with an optimal estimate and applications, Ann. Math. 181, 1139-1208.

[12] Guan, Q. A., Zhou, X. Y. (2015). A proof of Demailly's strong openness conjecture, Ann. Math. 182, 1-12.

[13] Guan, Q.A., Zhou, X., Zhu, L. (2012). On the Ohsawa-Takegoshi L^2 extension theorem and the Bochner-Kodaira identity with non-smooth twist factor. J. Math. Pures Appl. 97, 579–601.

[14] Guan, Q. A., Zhou, X. Y. (2015). Optimal constant in an L^2 extension problem and a proof of a conjecture of Ohsawa. Sci. China Math. 58, 35–59.

[15] Guan, Q. A., Zhou, X. Y. (2012). Optimal constant problem in the L^2 extension theorem. C. R. Math. Acad. Sci. Paris. 350, 753-756.

[16] Hörmander, L. (1965). L^2 estimates and existence theorems for the $\bar{\partial}$ operator. Acta Mathematica, 113(none) 89-152.

[17] Kohn, J. J. (1963). Harmonic integrals on strongly pseudo-convex manifolds I. Ann. Math. (2), 78, 112-148.

[18] Krantz, S. G. (2000). Function Theory of Several Complex Variables Second Edition. AMS Chelsea Publishing, American Mathematical Society.

[19] Lempert, L. (2014). Modules of square integrable holomorphic germs, arXiv, https://arxiv.org/pdf/1404.0407.

[20] Morrey, C. B. (1958). The analytic embedding of abstract real analytic manifolds. Ann. Math. (2), 68, 159-201.

[21] Nadel, A. M., (1990). Multiplier Ideal Sheaves and Kähler-Einstein Metrics of Positive Scalar Curvature, Ann. of Math. 132, 549–596.

[22] Ohsawa, T. (2017). On the Extension of L^2 Holomorphic Functions VIII—a remark on a theorem of Guan and Zhou , Int. J. Math. 28, 1740005, 12pp.

[23] Ohsawa, T. (2001). On the Extension of L^2 Holomorphic Functions. V. Effects of generalization, Nagoya Math. J. 161, 1-21.

[24] Ohsawa, T.(1995). Addendum to "On the Bergman kernel of Hyperconvex Domains". Nagoya Mathematical Journal, 137, 145–148.

[25] Suita, N. (1972). Capacities and Kernels on Riemann Surfaces. Arch. Ration. Mech. Anal. 46, 212–217.

[26] Yao, S., Li, Z., Zhou, X. (2017). On the Optimal L^2 Extension Theorem and a Question of Ohsawa. Nagoya Mathematical Journal, 245, 154-165.