ANALYTICAL PRICING FORMULA UNDER THREE-STATE REGIME-SWITCHING MODEL

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ABSTRACT

ANALYTICAL PRICING FORMULA UNDER THREE-STATE
REGIME-SWITCHING MODEL

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Economic and financial data display diverse behavior at different time intervals due to their dynamics and stochastic nature. To build explanatory models, different time periods with similar characteristics can be grouped together under a single regime. In this study, it is assumed that the states of the economy follow a homogeneous first-order continuous-time finite-state hidden Markov chain process.

We consider the valuation of European options in a three-state Markov switching extension of the Black-Scholes-Merton framework. In this context, the interest rate, drift, and volatility parameters of the underlying asset depend on the underlying market regime that switches among a finite number of states. Due to the additional source of randomness caused by the underlying Markov chain, the market is incomplete. The regime switching Esscher transform is applied to determine the equivalent martingale measure. Under this measure, the analytical formula for the regime-switching European options is derived. The option pricing procedure under this model has been studied in the literature for the two-state regime-switching framework. In this thesis, we utilize the joint density function of occupation times of the Markov chain proposed by Falzon to obtain the analytical solution for the three-state model. The calculations of the Greeks for the regime-switching European option by using the proposed method are presented.

Some of the exotic options can be represented in terms of the European options. We also consider this relationship for the barrier options and show how our method can be extended for both the valuation of regime-switching barrier options and their Greeks. The validity of the method is illustrated by presenting several examples and comparing them with the results existing in the literature.

Lastly, we consider the regime-switching guaranteed minimum maturity benefit valuation. Considering the long life of variable annuity contracts, insurance providers need to take into account both interest rate and mortality fluctuations in addition to stock market fluctuations. We consider interest rate, mortality, and underlying fund dynamics to be switching among the states, and we propose the formulae for two different models by assuming independent filtration for the mortality component. The first model assumes that both financial and mortality parameters are regulated by the same underlying Markov chain. On the other hand, the second model assumes that the parameters of the mortality model are based on a separate second Markov chain. This study is complemented with some numerical examples to highlight the implication of our approach on pricing these contracts under a regime-switching framework.

Keywords: Regime-switching models, Markov chain, occupation time, European options, barrier options, guaranteed minimum maturity benefit

ÖZ

ÜÇ-DURUMLU REJİM DEĞİŞİM MODELİ ALTINDA ANALİTİK FİYATLAMA FORMÜLÜ

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Ekonomik ve finansal veriler, dinamikleri ve stokastik yapıları nedeniyle farklı zaman aralıklarında farklı davranışlar sergilemektedir. Açıklayıcı modeller oluşturmak için benzer özelliklere sahip farklı zaman periyotları tek bir rejim altında gruplandırılabilir. Bu çalışmada, ekonominin durumlarının homojen bir birinci mertebeden, sürekli zamanlı, sonlu durumlu saklı Markov zinciri süreci izlediği varsayılmaktadır.

Black-Scholes-Merton çerçevesinin üç durumlu Markov rejim değişimi modeline genişletilmesi durumunda Avrupa tipi opsiyonların değerlendirilmesi problemini ele alınmıştır. Bu bağlamda, faiz oranını, dayanak varlığın sapma ve oynaklık parametreleri altta yatan Markov zincirine bağlıdır ve parametre değerleri sonlu sayıda durumlar arasında geçiş yapmaktadır. Altta yatan Markov zincirinin sebep olduğu belirsizlik nedeniyle, piyasa eksiktir (incomplete). Eşdeğer martingale ölçüsünü belirlemek amacıyla Markov rejim değiştirme modeli için Esscher dönüşümü uygulanarak, bu ölçü altında, parametreleri altta yatan Markov zincirine bağlı olan Avrupa tipi opsiyonlar için analitik formül türetilmiştir. Bu model altındaki fiyatlama prosedürü, literatürde altta yatan Markov zincirinin iki durumlu olduğu model altında incelenmiştir. Bu tezde altta yatan Markov zincirinin üç durumlu olduğu model için analitik çözümü elde etmek amacıyla Falzon tarafından önerilen ortak olasılık yoğunluk fonksiyonunun kullanılması önerilmiştir. Önerilen yöntem kullanarak altta yatan Markov zincirine göre parametre değerleri değişen Avrupa tipi opsiyonlar için parametre hassasiyetlerinin hesaplamaları sunulmuştur.

Egzotik opsiyonlardan bazıları Avrupa tipi opsiyonlar cinsinden ifade edilebilir. Bu ilişki bariyer opsiyonları açısından ele alımıştır ve hem altta yatan Markov zincirine göre parametre değerleri değişen bariyer opsiyonları için hem de bu opsiyonların parametre hassasiyeti hesaplamları için önerdiğimiz yöntemin nasıl genişletilebileceğini gösterilmiştir. Yöntemin geçerliliği, çeşitli örnekler ile ve bu yöntem ile elde edilen sonuçların literatürde var olan sonuçlarla karşılaştırılması ile gösterilmiştir.

Son olarak, vade sonunda minimum garanti ödemeli fayda opsiyonu değerlemesi Markov rejim değişimi modeli altında ele alınmıştır. Değişken annüite sözleşmelerinin uzun ömürlü oldukları düşünüldüğünde sigorta sağlayıcılarının borsadaki dalgalanmalara ek olarak hem faiz oranı hem de ölümlülük oranlarındaki dalgalanmaları dikkate almaları gerekmektedir. Faiz oranı, ölümlülük oranı ve temel fon parametrelerinin altta yatan Markov zincirine bağlı olduğu modeli ele alınmaktadır ve ölümlülük bileşeni için bağımsız filtreleme varsayımı altında iki farklı model önerilmektedir. İlk model, hem finansal hem de ölümlülük parametrelerinin aynı temel Markov zinciri tarafından düzenlendiğini varsayarken, ikinci model, ölüm modelinin parametrelerinin ayrı bir ikinci Markov zincirine dayandığını varsaymaktadır. Bu çalışma, Markov rejim değiştirme çerçevesi altında bu sözleşmelerin fiyatlandırılması konusundaki yaklaşımımızın etkisini göstermek için sayısal örneklerle tamamlanmıştır.

Anahtar Kelimeler: Rejim değişimi modelleri, Markov zinciri, işgal süresi, Avrupa tipi opsiyonlar, bariyer opsiyonları, vade sonunda mimimum garanti ödemeli fayda opsiyonları

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RS	Regime-Switching
GBM	Geometric Brownian Motion
CTMP	Continuous-time Markov process
FFT	Fast Fourier transform
i.i.d.	Independent and identically distributed
MC	Monte Carlo
SMC	Semi-Monte Carlo
CI	Confidence interval
LLN	Law of large numbers
CLT	Central limit theorem
$\mathbb{1}_E$	Indicator function of E
GMMB	Guaranteed minimum maturity benefit
OTC	Over the counter
FX	Foreign exchange
DO	Down-and-out
VA	Variable annuity

CHAPTER 1

INTRODUCTION

Regime-switching models are one of the powerful tools that explain the stochastic processes which have structural changes between different states. Because of their dynamic and stochastic nature, economic and financial data may exhibit different behavior for different time intervals. There are times when dramatic events disrupt the normal behavior of economies. A prominent example is the business cycle, and other examples are currency crises and stock market bubbles. Different time periods that share similar characteristics might be grouped together under a single regime in order to obtain explanatory models.

In conjunction with this, regime-switching models have been used to model different problems in various fields of finance and actuarial sciences including valuation of equity options [8, 9, 32, 43], interest rate instruments (bonds and interest rate derivatives) [14, 16, 15, 55], and energy and commodity derivatives [56, 27], portfolio selection [19], pricing and hedging variable annuities [34, 60, 28]

The remainder of this chapter is outlined as follows. Section 1.1 provides an literature review. In this section, the literature on the pricing of financial and actuarial products covered in this thesis is reviewed under the subsections of European option pricing, barrier option pricing, and guaranteed minimum maturity benefit valuation. In addition, the literature on occupation time distribution, which is one of the main building blocks of the thesis, is given. Motivation and the contribution of the thesis is presented in Section 1.2. The structure of the remaining chapters of the thesis is given in Section 1.3.

1.1 Literature Review

In the field of economics, the origin of the regime-switching models may be traced all the way back to the work of Goldfield and Quandt [29]. They demonstrated that the non-linearity in economic data can be explained by regime-switching regression models. Hamilton [33] utilized regime-switching model to identify business cycles and credited for popularizing the regime-switching models in economics and econometrics literature. Since then, regime-switching models have received a great deal of attention and have been applied in a variety of fields, including option valuation, risk management, actuarial valuation, and others.

In this study, since the pricing problem for European options, barrier options, and guaranteed minimum maturity benefit contracts is the primary focus of our investigation, we present the literature reviews for these sub-areas separately. Mind map of the studies used as main references for this study is illustrated in figure Figure 1.2.

1.1.1 European Option Pricing

The Black-Scholes [2] and Merton [49] model assumes that the underlying asset price dynamics are modeled by a geometric Brownian motion with constant appreciation rate and volatility. Despite the success of the Black-Scholes formula, numerous empirical studies have shown that the model has some drawbacks in terms of the features of the underlying assets, namely the leptokurtic feature, volatility smile, and volatility clustering phenomena of the asset return distribution [23, 47, 59, 58, 26]. In order to overcome the aforementioned shortcomings, various extensions of the standard Black-Scholes model have been proposed in the literature, including the jump-diffusion models, stochastic volatility models, and regime-switching models.

Numerous researchers have investigated the valuation of financial instruments under the regime-switching framework. For the European option, Naik [50], Guo [32], Elliott et al. [12], and McKinlay [48] provide an closed-form pricing formula. Closed-form formula which assume a model in which the volatility of the returns of the underlying risky assets is subject to random shifts first proposed by Naik in [50]. Naik [50] assumed two-state model and used occupation time distribution formula presented by Pedler [53]. Guo [32] also presented an explicit formula by deriving the formula for the probability density of occupation time for a two-state model.

One of the biggest problems encountered in option pricing problems under regime-switching framework is that the market is incomplete because of the presence of the additional source of randomness caused by the underlying Markov chain. Consequently, there is more than one equivalent martingale measure (EMM), and hence, more than one no-arbitrage price for an option.

Different approaches have been proposed for pricing and hedging derivative securities in incomplete financial markets. Guo [32] used the so-called change-of-state contract that pays one dollar when the current state switches to another state in order to complete the market. This change-of-contracts behaves like an insurance contract in a sense that it protects the holder from any losses resulting from the regime-switching. Elliot, Chan, and Siu [12] considered the option pricing problem when the underlying asset is driven by a Markov regime-switching geometric Brownian motion (GBM). They assumed that the market parameters, namely the market interest rate, the appreciation rate (drift), and the volatility of the underlying asset, were assumed to depend on a continuous-time hidden Markov model, which models the market modes (regimes). They utilized the regime-switching Esscher transform to determine the equivalent martingale measure. They also demonstrated that the results driven from Esscher transform when pricing a contingent claim, are equivalent to those driven from the minimum entropy martingale measure. The general form of the semi-closed formulation for European call options under regime switching framework is given by Elliot et al. [12] without making any specification about the joint density function of the occupation times. They also presented the general form of the characteristic function of the occupation times.

McKinlay [48] considered the model where a continuous-time finite Markov chain drived the drift and the volatility but not the interest rate. McKinlay [48] presented a proof for the two-state explicit formula and stated the formula for the three-state case. He also employed the Monte Carlo simulation and the Fourier transform method to obtain the European option prices and compared these results with the solutions obtained by numerical integration by considering Pedler's occupation time density function for a two-state model.

As stated by Zeng et al. [62], Zhang et al. [63], and Boyle et al. [4], and to the best of our knowledge, the pricing problem for regime-switching European options in a two-state regime-switching model can be solved analytically. However, closed-form solutions for models with more than two states are not yet available and N-state results are often obtained by simulation or numerical techniques.

Mamon and Rodrigo [46] obtained an explicit solution for European options in a regime-switching economy by considering the solution of a system of PDEs. Furthermore in a similar setting, Buffington and Elliot [8, 9] presented a method for pricing European and American options using partial differential equations.

Liu, Zhang, and Yin [43] employed a fast Fourier transform (FFT) approach in order to price the European option in a Markov regime-switching model. They also assumed a two-state economy. Zeng et al. [62] presented a numerical comparison of the Monte Carlo simulation and the finite-difference method for European option under a regime-switching framework. Bollen [3] presented a lattice method and Monte Carlo simulation for valuing both European and American options in regime-switching models.

1.1.2 Barrier Option Pricing

Numerous academics and practitioners have examined the valuation of barrier options, which is a significant subject in the theory and practice of finance. Merton [49] calculated the analytical value of our down and call option. Rubinstein and Reiner [57] derived closed-form pricing formulae for several single-barrier option types. Buhchen [7] investigated the risk-neutral pricing of double knock-out barrier options using the method of images. Hieber and Scherer [38] used Brownian bridge concept to present an efficient Monte-Carlo method for barrier option pricing in regime-switching framework. Closed-form estimate for the price of a European barrier option with time dependent parameters are provided by Lo, Lee and Hui [44]. Elliot, Siu and Chan [18] derived semi-analytical approximation formula for barrier options under the regime-switching model by considering the results given by Lo et al. [44] and they obtained a formula which can be calculated by using the joint conditional density of the occupation time of the Markov chain.

1.1.3 Guaranteed Mimimum Maturity Benefit Pricing

In Europe and Canada, variable annuities are known as unit-linked contracts and segregated funds, respectively. Equity linked contracts include embedded guarantees to protect the policyholder against the downside risk.

Variable annuity providers offer a variety of guarantee riders. Besides guaranteed minimum death benefit (GMDB) riders, three main types of guaranteed living benefit (GLB) riders exist: guaranteed minimum accumulation benefit (GMAB) riders, guaranteed mimimum maturity benefit (GMMB) riders, guaranteed minimum income benefit (GMIB) riders and guaranteed minimum withdrawal benefit (GMWB) riders. These guarantees are relevant to long-dated option pricing since they depend on the survival of the policyholder. The guarantees are subject to interest rate risk, mortality risk, and equity risk.

There are many studies in the literature that comprehensively address these risks [5, 34, 61, 54, 41, 42, 40, 24, 45]. Variable annuities provide long-term protection against the effect of inflation on fixed income in comparison with fixed annuities. Sales of variable annuities tend to increase when the stock market grows, but sales of fixed annuities tend to decrease as the stock market grows. Table 1.1 shows individual annuity sales in the last few years in the US. [1]

Table 1.1: Annuity Sales in the U.S. between 2017-2021

	Variable Annuity		**Fixed Annuity**		**Total**	
Year	**Amount ($ billions)**	**Percent Change**	**Amount ($ billions)**	**Percent Change**	**Amount ($ billions)**	**Percent Change**
2017	98.2	-	105.3	-	203.5	-
2018	100.2	0.02%	133.6	0.27%	233.8	14.9%
2019	101.9	0.017%	139.8	0.05%	241.7	3.4%
2020	98.6	-0.032%	120.5	-0.14%	219.1	-9.4%
2021	125.3	0.271%	129.2	0.07%	254.3	16.1%

[1] Source: http://www.iii.org/media/facts/statsbyissue/annuities/.

Considering these long-dated contracts, the regime-switching method is an effective method to explain the behavior of risks that contracts are based on.

Ignatieva [40] investigated pricing and hedging of the Guaranteed Minimum Benefits (GMBs) under regime switching framework via Fourier Space-Time-stepping algorithm. Pricing of equity-indexed annuities and GMDBs under a double regime-switching model by using the fast Fourier is presented by Fan et al. [24]. Mamon et al. [45] investigated the valuation of GMMBs by assuming the dynamics of interest rate, mortality rate, and stock index are modulated by a hidden Markov model. They employed the Fourier transform method and compared the results with Monte Carlo simulation method and also presented the evaluation of the Greeks. They also presented parameter estimation of the model by recursive HMM based filtering technique.

1.1.4 Occupation Time Distribution of Markov Chains

In order to calculate the option prices with this approach, the joint density function of the occupation time needs to be specified.

Several researchers studied the theoretical joint density function for a two-state Markov chain in literature. Darroch and Morris [11] stated the conditional moment generating function of occupation time. Pedler [53] explicitly derived the density function of the occupation time in a certain state for two state continuous time Markov chain process. Hsia [39] obtained the joint probability density function of the occupation time of a special three-state Markov process in which the $q_{ij} \neq 0$ for $i \neq j$ but $q_{ii} = 0$, $i = 1, 2, 3$. Good [30] derived and expression for the interior solution for the joint density function of occupation time of Markov chain when all the states have been visited at least once. Falzon [22] extended the Good [30]'s solution by also considering the boundary solutions. For the general case, Falzon [22, 21] and Pearce et al. [52] obtained the conditional joint probability density of the occupation time (accumulated sojourn time) of the three-state Markov chain in which the initial state of the process is known. Falzon [22] solved the Kolmogorov equations of the solution of the probability of the accumulated sojourn time in a three-state Markov chain. The solution was obtained in infinite sums of convolutions of modified Bessel functions.

Then Falzon [22] generalized the formula, first to four-states and finally to N states by using the relations between spanning trees and the probability densities.

1.2 Motivation and Contribution of the Thesis

The main motivation for this thesis is to obtain analytical pricing and hedging formulae for financial and actuarial contracts under the three-state regime-switching framework. The intuition behind the regime switching framework is that it allows the model parameters to switch according to the regimes of the market and/or economy. In pricing problems, it provides a more general environment in which the model becomes a weighted standard Black-Scholes formula in which the weights are determined according to the time spent on each state during the maturity of the contract. It also recovers the standard Black-Scholes formula when the number of states is one.

For the two-state model, this behavior of the regime-switching model is illustrated in Figure 1.1 [2]. It is observed that the regime-switching option prices converge for larger Falzon's formula is given for the case in which the Markov process starts from state 1. It is also possible to use the formula proposed by Falzon for cases in which the Markov chain starts from state 2, or state 3 based on the transition rate matrix provided in the remark below.

Remark. Let the initial state be $l \in \{1, 2, 3\}$. In the following without loss of generality we define a permutation of the states and corresponding transition rate matrix M so that

$$Q = PMP^T$$

is the transition rate matrix $Q = (q_{ij})$ with $i, j \in S$.

T values and are all between those given by the two classical Black-Scholes models. For the market which has a regime-switching behavior, using the standard Black-Scholes formula might be too restrictive.

Naik [50] , Guo [32], Elliott et al. [12], and McKinlay [48] provided a closed-form

[2] The parameters of this example are as follows: initial stock price $S_0 = 36$, strike price $K = 40$, the interest rates $(0.1, 0.3)$, and the volatilities are $(0.25, 0.35)$ for the two-state. The transition rate probabilities are $q_{ji} = 1$ for $i \neq j$.

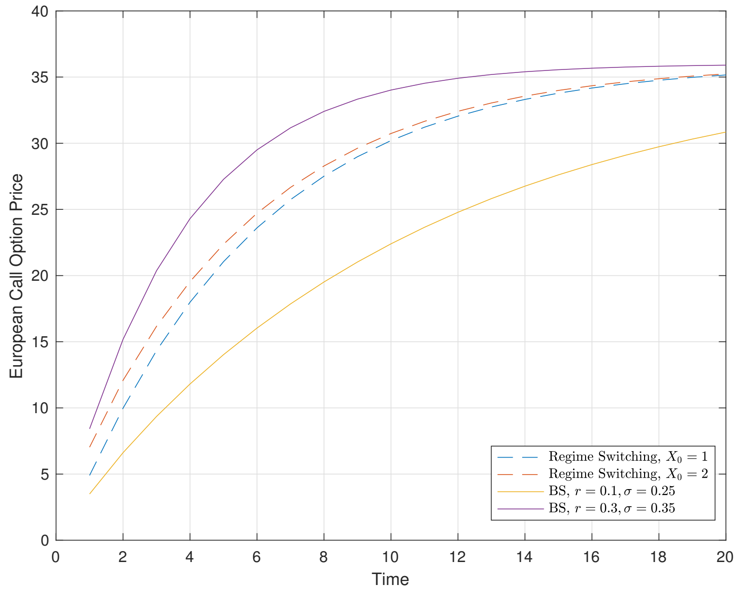


Figure 1.1: Black-Scholes vs Regime Switching

pricing formula for European options. Naik [50], Guo [32], and McKinlay [48] also specified the formula under different assumptions for the two-state model. Elliott et al. [12] presented a general formula for N-state case and also characteristic function of occupation times. The joint density of occupation times is needed to apply the presented formula.

Our aim is to extend this framework to the three-state economy. In order to put this method into practice, information about the joint probability distribution of occupation time of Markov chains is needed. For this reason, the distributions of occupation times of Markov chains are investigated in the existing literature.

We investigate the joint density function of the occupation time for the three-state Markov chain proposed by Falzon [22] and show that it can be used in the calculation of the Markov-modulated option pricing problem. To the best of our knowledge, the Markov-modulated option pricing formula by considering this approach has not yet been presented in the literature under the three-state regime-switching framework. We observe that the Falzon's formula for three states is also reduced to a two-state formula proposed by Pedler [53].

We also carry out numerical analysis, to show the impact of the proposed approach. In order to verify the validity of the results, we compare our result with the results existing in the literature. We observe that our approach needs less computation time in Monte Carlo approach for the two-state and three-state models.

Another advantage of the formulation is that it facilitates further analysis. By using this formulation we also calculate the option Greeks and compare the results with the finite-difference method results.

We extend this approach further for the valuation of barrier options and guaranteed minimum maturity benefit contracts, since these contracts can be written in terms of European options.

1.3 Structure of the Thesis

In Chapter 2, we provide a brief mathematical background, including related theories regarding Markov processes. The model presented in this thesis and in the Elliot et al. [12] is explained in detail.

The occupation time distributions are one of the key building blocks of the thesis. In Chapter 3, therefore, the details of Pedler [53] and Falzon [22]'s respective two-state and three-state occupation time distributions are introduced.

In Chapter 4, we describe the principal pricing formula for European options under the assumption that the model parameters (drift, volatility, and interest rate) are governed by a Markov chain. In this chapter, we also provide the proof of the formula. In the proposed formula, another expectation needs to be taken into account with respect to occupation time distributions. So, we describe how the Falzon's formula may be broken down into four separate sections and evaluated. Using this formula and the Leibniz rule, it is possible to calculate the Greeks: delta, gamma, rho, and vega calculations are presented.

We demonstrate that the comparison of the numerical integration results with the results obtained by using the FFT and Monte Carlo method. For the three-state model, we confirm that our results with the results presented in Zeng et al. [62]. Furthermore we calculate the option Greeks with the proposed approach and observe that they are consistent with the finite-difference results.

Proposed approach can be extent for some of the exotic options. Chapter 5 describes the application of the proposed approach to barrier options. The pricing formula for the regime-switching barrier options and corresponding Greeks are given. Numerical examples in order to show the accuracy of the proposed method are presented.

In Chapter 6 we demonstrate that the proposed approach presented in Chapter 4 can also be used in the valuation of Guaranteed Minimum Maturity Benefit (GMMB) contract, since it can be written in terms of the European options. We propose formulae for two different models by assuming independent filtration for the mortality component. The first model assumes that both financial and mortality parameters are regulated by the same underlying Markov chain. On the other hand, the second model assumes that the parameters of the mortality model are based on a separate second Markov chain. Numerical results for GMMB contract pricing models and corresponding Greeks are reported in Chapter 6.

Finally, Chapter 7 concludes the study with the main findings of the thesis and a discussion on potential future studies.

Figure 1.2: Literature Review

CHAPTER 2

PRELIMINARIES

In this study, we assume that the model parameters are modulated by a finite-state continuous-time Markov chain. The purpose of this chapter is to present the general structure of continuous-time Markov chains and some of their useful properties. In addition, we present the framework for the main model used in the thesis, based on Elliot et al. [12].

2.1 Continuous Time Markov Chains

A Markov chain $X_t, t \geq 0$ where X_t are random variables with values in finite or countable state-space $\mathcal{X}$ is a stochastic process which satisfy the Markov property.

Definition 2.1.1 (Norris [51])**.** The Markov assumption states that the future values of the process is independent of its past given its present. More explicitly, a continuous-time stochastic process taking values in the finite state space $\mathcal{X}$ has the Markov property if for any $j \in \mathcal{X}$

$$\mathbb{P}(X_t = j | X_r, 0 \leq r \leq s) = \mathbb{P}(X_t = j | X_s),$$

for all $r < s < t < \infty$ and $r, s, t \in \mathcal{T}$, with $\mathcal{T} \subseteq \mathbb{R}^+ := [0, \infty)$. If moreover it satisfies the condition

$$\mathbb{P}(X_{t+s} = j | X_s = i) = \mathbb{P}(X_t = j | X_0 = i) = P_{ij}(t)$$

it is called time-homogeneous.

Theorem 2.1.1. *The family $P(t)_{t \geq 0}$ is a stochastic semigroup which satisfies the following properties:*

i) $P(0) = \mathbf{1}$*,*

ii) $P(t)$ *is stochastic, that is*

a) P *has non-negative entries, i.e.,* $P_{ij} \geq 0$*,* $\forall i, j \in \mathcal{X}$*,*

b) rows of P *sum to* 1*, i.e.,* $\sum_{j \in \mathcal{X}} P_{ij} = 1$*,* $\forall i \in \mathcal{X}$*.*

iii) (Chapman-Kolmogorov equations) For all times s *and* t *the transition probability functions are obtained from* P_{ik} *and* P_{kj} *as*

$$P_{ij}(t+s) \sum_{k \in \mathcal{X}} P_{ik}(t) P_{kj}(s) \iff P(t+s) = P(t)P(s).$$

Definition 2.1.2. Let $\{X_t\}_{t \geq 0}$ be a continuous time Markov chain with transition probabilities $P(t) = (P_{ij}(t))$. The generator or infinitesimal generator of the Markov chain is the matrix

$$Q = \lim_{h \to 0} \frac{P(h) - \mathbf{1}}{h} \tag{2.1}$$

with entries as $Q = (q_{ij})_{i,j \in \mathcal{X}}$. It has the following properties:

(i) $0 \leq -q_{ii} =< \infty$ for all i,

(ii) $q_{ij} \geq 0$ for all $i \neq j$,

(iii) $\sum_{j \in \mathcal{X}} q_{ij} = 0$ for all i.

The quantities q_{ij} referred as instantaneous transition rates and indicates the rate at which the process transition from state i to state j. This rate measures the average number of transition from state i to state j. The diagonal entries $-q_{ii}$ are the overall rates of leaving state i.

Rearranging (2.1) gives the infinitesimal transition matrix yields

$$P(h) = Qh + \mathbf{1} + o(h), \tag{2.2}$$

which states that

$$P_{ij} = q_{ij}h + o(h) \tag{2.3}$$

$$P_{ii} = 1 + q_{ii}h + o(h). \tag{2.4}$$

In order to compute $P_{ij}(t)$ for all $t > 0$ and all states $i, j \in \mathcal{X}$ one need to determine a differential equation that $P_{ij}(t)$ must satisfy, and then solve the equation.

Theorem 2.1.2. *The transition probability function P_{ij} of a continuous-time Markov chain satisfy the system of differential equations for all pairs $i, j \in \mathcal{X}$, and $t \geq 0$*

$$P'_{ij}(t) = \sum_{k \in \mathcal{X} \backslash j} q_{kj} P_{ik}(t) + q_{jj} P_{ij}(t), \tag{2.5}$$

$$P'_{ij}(t) = \sum_{k \in \mathcal{X} \backslash i} q_{ik} P_{kj}(t) + q_{ii} P_{ij}(t). \tag{2.6}$$

The (2.5) *and* (2.6) *are called Kolmogorov's forward equation and backward equation, respectively.*

Theorem 2.1.3. *Let T_i be the time that a homogeneous continuous time Markov chain process spends in state i. Then we have*

$$\mathbb{P}(T_i > s + t | T_i > s) = \mathbb{P}(T_i > t)$$

for $s, t \geq 0$. Hence T_i is exponential distributed and memoryless.

The time spent in state i before transitioning to state j for $i \neq j$ is distributed exponentially with $t \sim \text{Exp}\,(-q_{ii})$. The pseudocode to generate a continuous time Markov chain is given in Algorithm 1.

Algorithm 1 Algorithm for Simulating a Time-Homogeneous Continuous-Time Markov Chain on a Discrete State Space

```
Choose an initial value for X_0 = Y_0 = e_i for some i ∈ I. Set n = 0, T_0 = 0
while T_n < T do
    Draw a random variable from exponential distribution with corresponding parameter depending on the current state of the chain, A_n ~ Exp(q_{Y_n, Y_n})
    Set n = n + 1
    Set T_{n+1} = T_n + A_n
    Simulate the new state Y_{n+1}
end while
```

Example 2.1.1. An example with the transition rate matrix

$$\begin{bmatrix} -5 & 5 & 0 \\ 1 & -2 & 1 \\ 3 & 1 & -4 \end{bmatrix}$$

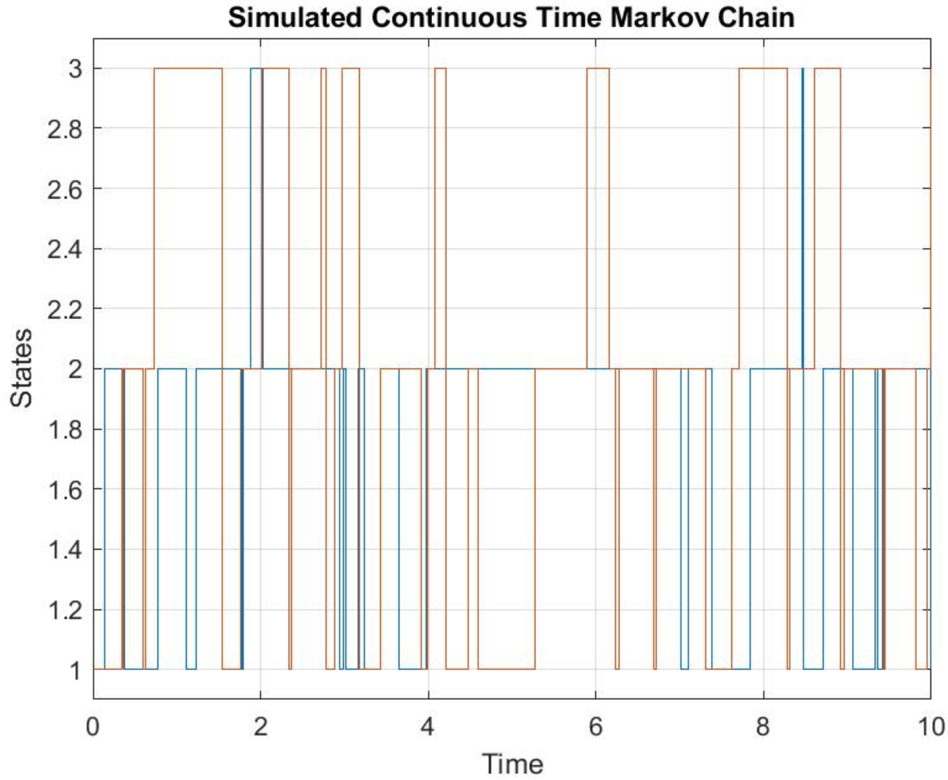


Figure 2.1: The two realizations of a continuous-time Markov chain

is considered. For $T = 10$ and the initial state is $X_0 = 1$, the two realizations (paths) of the Markov chain are illustrated in Figure 2.1.

Henceforth, when we reference to the Markov chain, we shall assume that it is a homogeneous first-order continuous-time Markov chain.

2.2 Model

Consider a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\{W_t\}_{t\in\mathcal{T}}$ denote a standard Brownian motion on this complete probability space with respect to $\mathbb{P}$. Let $\mathcal{T}$ denote the time index set $[0, T]$ of the model. The states of the economy follows a continuous-time hidden Markov chain process $\{X_t\}_{t\in\mathcal{T}}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with a finite state-space $\mathcal{X} := (x_1, x_2, \ldots, x_N)$ and it takes values from the set of canonical unit vectors $\{e_1, e_2, \ldots, e_N\}$, where $e_i = (0, \ldots, 1, \ldots, 0)^\top \in \mathbb{R}^N$, where $^\top$ is a transpose of a vector. $\{X_t\}_{t\in\mathcal{T}}$ and $\{W_t\}_{t\in\mathcal{T}}$ are assumed to be independent.

Then by following Elliot, Aggoun and Moore [13] the semi-martingale representation of $\{X_t\}_{t\in\mathcal{T}}$ is written as

$$X_t = X_0 + \int_0^t QX_s ds + M_t \tag{2.7}$$

where $Q = (q_{ij})$ is the generator matrix, such that $\sum_{j=1}^{N} q_{ji} = 0$ and $q_{ij} \geq 0$ if $i \neq j$. Here, $\{M_t\}_{t\in\mathcal{T}}$ is an $\mathbb{R}^N$-valued martingale increment with respect to the filtration generated by $\{X_t\}_{t\in\mathcal{T}}$.

The instantaneous market interest rate $\{r_t\}_{t\in\mathcal{T}}$ of the bank account is given by

$$r_t := r(t, X_t) = \langle r, X_t \rangle, \tag{2.8}$$

where $r := (r_1, r_2, \ldots, r_N)^\top$ with $r_i > 0$ for each $i = 1, 2, \ldots, N$ and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^N$. Bank account dynamics are described by

$$dB_t = r_t B_t dt, \quad B_0 = 1.$$

It is assumed that the stock appreciation rate $\{\mu_t\}_{t\in\mathcal{T}}$ and the volatility $\{\sigma_t\}_{t\in\mathcal{T}}$ of risky underlying asset S also depend on $\{X_t\}_{t\in\mathcal{T}}$ and are described by

$$\mu_t := \mu(t, X_t) = \langle \mu, X_t \rangle, \quad \sigma_t := \sigma(t, X_t) = \langle \sigma, X_t \rangle, \tag{2.9}$$

where $\mu := (\mu_1, \mu_2, \ldots, \mu_N)^\top$ and $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_N)^\top$. The parameters are depend on the Markov chain X with $\sigma_i > 0$ for each $i = 1, 2, \ldots, N$. The dynamics of the stock price process $\{S_t\}_{t\in\mathcal{T}}$ is given by the following Markov-modulated geometric Brownian motion (GBM)

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t, \quad S_0 = s > 0, \tag{2.10}$$

for $t \in [0, T]$. The solution of (2.10) is

$$S_t = S_0 \exp\left(\int_0^t \left(\mu_u - \frac{1}{2}\sigma_u^2 \right) du + \int_0^t \sigma_u dW_u \right). \tag{2.11}$$

Let Z_t denote the logarithmic return $\ln\left(S_t/S_0\right)$ over the interval $[0, t]$. Hence, the stock price dynamics can be written as

$$S_t = S_u \exp(Z_t - Z_u), \tag{2.12}$$

where

$$Z_t = \int_0^t \left(\mu_t - \frac{1}{2}\sigma_s^2 \right) ds + \int_0^t \sigma_s dW_s. \tag{2.13}$$

The market is complete if and only if every contingent claim is attainable. Due to the presence of an additional source of uncertainty related to regime-switching, the market in Markov regime-switching model is in general incomplete. The standard Black-Scholes perfect replication approach cannot be used in this situation. For the fair valuation we need to determine an equivalent risk-neutral martingale measure to ensure that there are no arbitrage opportunities in the market described by the model [35, 36, 37]. There are two commonly used approaches in the literature. Guo [32] proposed to use Arrow-Debreu securities related to cost of switching to complete the market. Elliot et al. [12] suggested to use of a regime-switching version of the Esscher transform to obtain a price kernel for option pricing in an incomplete market. We will follow the latter.

First, we need to describe the related filtrations under $\mathbb{P}$:

i) $\mathcal{F}_t^X$ for filtration generated by $\{X_t\}_{t\in\mathcal{T}}$,

ii) $\mathcal{F}_t^W$ for filtration generated by $\{Z_t\}_{t\in\mathcal{T}}$,

iii) $\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^W$.

Let $\theta_t := \theta(t, X_t)$ be the regime-switching Esscher parameter, which depends on X_t, and be defined by

$$\theta(t, X_t) = \langle \theta, X_t \rangle = \sum_{i=1}^{N} \theta_i \langle X_t, e_i \rangle \tag{2.14}$$

where $\theta := (\theta_1, \theta_2, \ldots, \theta_N)^\top \in \mathbb{R}^N$.

Then, as stated by Elliot et al. [12], the regime-switching Esscher transform $\mathbb{Q}_\theta$ equivalent to $\mathbb{P}$ on $\mathcal{F}_t$ is defined by

$$\left.\frac{d\mathbb{Q}_\theta}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \frac{\exp\left(\int_0^t \theta_s dZ_s\right)}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(\int_0^t \theta_s dZ_s\right)\middle|\mathcal{F}_t^X\right]} \tag{2.15}$$

$$= \exp\left(\int_0^t \theta_s \sigma_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 \sigma_s^2 ds\right), \quad t \in \mathcal{T}. \tag{2.16}$$

Let $\{\theta_t\}_{t\in\mathcal{T}}$ be the family of risk-neutral regime-switching Esscher parameters. The fundamental theorem of asset pricing[35, 36, 37] states that the no arbitrage principle is equivalent to the existence of an equivalent martingale measure under which the discounted asset prices are martingale. The martingale condition is given by considering enlarged filtration

$$S_0 = \mathbb{E}_{\mathbb{Q}_{\widetilde{\theta}}}\left[exp(-\int_0^t r_s ds)S_t|F_t\right], \tag{2.17}$$

for any $t \in \mathcal{T}$. By the martingale condition, an equivalent martingale measure $\mathbb{Q}_{\widetilde{\theta}}$ is determined as

$$\widetilde{\theta}_t = \frac{r_t - \mu_t}{\sigma_t^2}, \quad t \in \mathcal{T}. \tag{2.18}$$

Then, $\widetilde{\theta}_t = \langle \widetilde{\theta}, X_t \rangle$, where

$$\widetilde{\theta} = \left(\frac{r_1 - \mu_1}{\sigma_1^2}, \frac{r_2 - \mu_2}{\sigma_2^2}, \ldots, \frac{r_N - \mu_N}{\sigma_N^2}\right). \tag{2.19}$$

The Radon-Nikodym derivative $\mathbb{Q}_{\widetilde{\theta}}$ with respect to $\mathbb{P}$ on $\mathcal{F}_t$ is given by

$$\left.\frac{d\mathbb{Q}_{\widetilde{\theta}}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left[\int_0^t \left(\frac{r_s - \mu_s}{\sigma_s}\right) dW(s) - \frac{1}{2}\int_0^t \left(\frac{r_s - \mu_s}{\sigma_s}\right)^2 ds\right]. \tag{2.20}$$

Then, the dynamics of the stock price and the logarithmic returns under $\mathbb{Q}_{\widetilde{\theta}}$ are as follows:

$$dS_t = r_t S_t dt + \sigma_t S_t d\widetilde{W}_t, \tag{2.21}$$

$$dZ_t = \left(r_t - \frac{1}{2}\sigma_t^2\right)dt + \sigma_t d\widetilde{W}_t, \tag{2.22}$$

where $\widetilde{W}_t = W_t + \int_0^t \left(\frac{r_s - \mu_s}{\sigma_s}\right) ds$ is a standard Brownian motion with respect to the enlarged filtration $\mathcal{F}_t$ under $\mathbb{Q}_{\widetilde{\theta}}$.

We define the path integral of r and σ^2 on the interval $[0, T]$ by

$$P_{t,T} = \int_t^T r(s)ds = \int_t^T \langle r, X_s \rangle ds, \tag{2.23}$$

and

$$V_{t,T} = \int_t^T \sigma^2(s)ds = \int_t^T \langle \sigma, X_s \rangle^2 ds, \tag{2.24}$$

respectively.

In order to calculate these equations we need to define the occupation time of the Markov chain. In the next chapter, definition of the occupation time of the Markov chain and the conditional joint density functions of the Markov chain are presented.

CHAPTER 3

OCCUPATION TIME OF A MARKOV CHAIN

The occupation time (accumulated sojourn time) of the Markov chain plays an important role in option pricing problems when the parameters of the underlying asset are modulated by the Markov chain. This is because we need to use the joint density function of the occupation time of the Markov chain for the valuation of the regime switching derivatives. The time that the Markov chain spends in a particular state is called the occupation time. In this chapter, definition of the occupation function is given first. In Section 3.2 Pedler's conditional joint probability density function of occupation times for two state Markov chain is given. Section 3.3 presents the conditional joint probability density function of occupation times for three state Markov chain stated by Falzon.

3.1 Occupation Time

Let $J_i(t,T)$ denote the occupation time of a homogeneous continuous-time Markov chain during a finite time interval.

Definition 3.1.1. The occupation time of the Markov chain process X for each subset of the state-space during a finite time interval $[t,T]$ is given by

$$J_i(t,T) := \int_t^T \mathbb{1}_{\{X(s)=i\}} ds \quad \text{for } i \in \{1,2,\dots,N\}.$$

Since J_i is the time spent by the process in state i during a finite interval of time $[t,T]$, length of time duration can be written as the summation of the occupation times

$$T - t = \sum_{i=1}^{N} J_i, \tag{3.1}$$

for $i \in \{1, 2, \ldots, N\}$.

3.2 Joint Density Function of Occupation Times: Two-State

Darroch and Morris [11] presented the Laplace transform of occupation times, and Pedler [53] stated the distribution of the occupation times in a two-state continuous-time Markov chain via inverse Laplace transform. Pedler's result is given in the following theorem.

Theorem 3.2.1 (Probability density function of the two-state Markov process [53, 6])**.** *Let Q denote transition rate matrix of a Markov chain X, with*

$$Q = \begin{bmatrix} -q_{12} & q_{12} \\ q_{21} & -q_{21} \end{bmatrix}$$

for $q_{12}, q_{21} \geq 0$ and the initial probability matrix be given as

$$\pi = \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix}.$$

For a two-state Markov-modulated model, the probability density function of the occupation time in state 1 is

$$\begin{aligned} f(J_1, t) = e^{-q_{12}J_1 - q_{21}(t-J_1)} \big\{ &\pi_1 \delta(t - J_1) + \pi_2 \delta(J_1) \\ &+ [\pi_1 (q_{12}q_{21}J_1/(t - J_1))^{1/2} + \pi_2 (q_{12}q_{21}(t - J_1)/J_1)^{1/2}] \\ &\times I_1[2(q_{12}q_{21}J_1(t - J_1))^{1/2}] \\ &+ (\pi_1 q_{12} + \pi_2 q_{21}) I_0[2(q_{12}q_{21}J_1(t - J_1))^{1/2}] \big\}, \end{aligned}$$

where δ is the Dirac delta function.

If we know the initial state of the chain, the probability density function of the occupation time can be written in more specific form. The probability density function of the occupation time of state 1 given the process starts in state 1 is

$$\begin{aligned} f_{X_0=(1,0)}(J_1, T) = e^{-q_{12}J_1 - q_{21}(T-J_1)} \Big\{ & \delta(T - J_1) + (q_{12}q_{21}J_1/(T - J_1))^{1/2} \\ & \times I_1\big[2(q_{12}q_{21}J_1(T - J_1))^{1/2}\big] \\ & + q_{12}I_0\big[2(q_{12}q_{21}J_1(T - J_1))^{1/2}\big]\Big\}, \end{aligned}$$

and

$$I_r(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k+r}}{k!(k+1)!},$$

which is the modified Bessel function of order r.

Similarly, the probability density function of the occupation time of state 1, given the process starts in state 2, is

$$\begin{aligned} f_{X_0=(0,1)}(J_1, T) = e^{-q_{12}J_1 - q_{21}(T-J_1)} \Big\{ & \delta(J_1) + (q_{12}q_{21}(T - J_1)/(J_1))^{1/2} \\ & \times I_1\big[2(q_{12}q_{21}J_1(T - J_1))^{1/2}\big] \\ & + \nu I_0\big[2(q_{12}q_{21}J_1(T - J_1))^{1/2}\big]\Big\}. \end{aligned}$$

Proof is given by Pedler [53].

3.3 Joint Density Function of Occupation Times: Three-State

In this section we present the conditional joint density of occupation time for three-state Markov chain. Falzon [22] set up the Kolmogorov equations for the evolution of the probability density of the states by considering the final state of the process. The set up of the problem and consequences are presented in the sequel.

Consider a three-state continuous-time finite state Markov process with state-space $S = \{1, 2, 3\}$. The transition rate matrix is given as $Q = (q_{ij})$ with $i, j \in S$. Let $\boldsymbol{J}(t) = (J_2(t), J_3(t)) \in \mathbb{R}^2_+$, where $J_k(t)$ is a random variable representing the time spent in state k (i.e., occupation time, accumulated sojourn time) up to time $t \in \mathbb{R}_+$. In order to reduce the dimension let $J_1(t) = t - J_2(t) - J_3(t)$.

Let f_i denote the probability density of $\boldsymbol{J}(t)$ at time t and final state i. Evolution of the probability density is given by the following system of Kolmogorov equations:

$$\frac{\partial f_1}{\partial t}(\boldsymbol{J}, t) = \sum_{i=1}^{3} q_{i1} f_i(\boldsymbol{J}, t), \tag{3.2}$$

$$\frac{\partial f_k}{\partial t}(\boldsymbol{J}, t) = \sum_{i=1}^{3} q_{ik} f_i(\boldsymbol{J}, t) - \frac{\partial f_k}{\partial J_k}(\boldsymbol{J}, t), \qquad k \in \{2, 3\}. \tag{3.3}$$

It is assumed that the process starts in state 1 so that $J_2 = J_3 = 0$ at $t = 0$. The initial probability density is given by $f_k(\boldsymbol{J}, 0) = \delta_{k0}\delta(\boldsymbol{J})$, where δ_{ij} is the Kronecker delta defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

and δ is the Dirac distribution at $\boldsymbol{J} = (0, 0)$.

Let H denotes the Heaviside unit step function. Then, let $H(\boldsymbol{J}) = H(J_2, J_3)$ defined by

$$H(\boldsymbol{J}) = \prod_{i=2}^{N=3} H(J_i) = \begin{cases} 0 & \text{if } \exists\, J_i < 0 \in \boldsymbol{J} \text{ for } i = 2, 3 \\ 1 & \text{otherwise.} \end{cases} \tag{3.4}$$

Falzon [20] solved the given Kolmogorov equations for the system via transform methods which yields the following theorem.

Theorem 3.3.1 ([22])**.** *The joint probability density of the accumulated sojourn time in a three-state Markov chain is given by*

$$\begin{aligned} f(\boldsymbol{J}, t) &= f_1(\boldsymbol{J}, t) + f_2(\boldsymbol{J}, t) + f_3(\boldsymbol{J}, t) \\ &= e^{-\epsilon(\boldsymbol{J}, t)}[\alpha_1(\boldsymbol{J}, t) + \alpha_2(\boldsymbol{J}, t) + \alpha_3(\boldsymbol{J}, t)] \end{aligned} \tag{3.5}$$

where J_i is the time spent in state $i \in \{1, 2, 3\}$, J_1 is defined to be $t - J_2 - J_3$ and $\epsilon(\boldsymbol{J}, t) = -\sum_{k=1}^{3} q_{kk} J_k$. Here, α_1, α_2 and α_3 can be expressed as

$$\begin{aligned} \alpha_1(\boldsymbol{J}, t) = \delta(\boldsymbol{J}) &+ q_{12}q_{21}H(J_2)\delta(J_3)F_{0,2}(J_2, 0, t) \\ &+ q_{13}q_{31}\delta(J_2)H(J_3)F_{0,2}(0, J_3, t) \\ &+ H(\boldsymbol{J})\gamma(\boldsymbol{J})[\mu_1 F_{1,4}(\boldsymbol{J}, t) + \mu_2 F_{1,3}(\boldsymbol{J}, t) + \mu_3 F_{1,2}(\boldsymbol{J}, t)] \\ &+ H(\boldsymbol{J})[2\mu_1 F_{0,3}(\boldsymbol{J}, t) + \mu_2 F_{0,2}(\boldsymbol{J}, t)], \end{aligned}$$

$$\begin{aligned}\alpha_2(\boldsymbol{J},t) = {} & q_{12}H(J_2)\delta(J_3)F_{0,1}(J_2,0,t) \\ & + H(\boldsymbol{J})q_{12}J_2[\mu_1 F_{1,3}(\boldsymbol{J},t) + \mu_2 F_{1,2}(\boldsymbol{J},t) + \mu_3 F_{1,1}(\boldsymbol{J},t)] \\ & + H(\boldsymbol{J})[q_{13}q_{31}q_{21}F_{0,2}(\boldsymbol{J},t) + q_{13}q_{32}F_{0,1}(\boldsymbol{J},t)],\end{aligned}$$

$$\begin{aligned}\alpha_3(\boldsymbol{J},t) = {} & q_{13}\delta(J_2)H(J_3)F_{0,1}(0,J_3,t) \\ & + H(\boldsymbol{J})q_{13}J_3[\mu_1 F_{1,3}(\boldsymbol{J},t) + \mu_2 F_{1,2}(\boldsymbol{J},t) + \mu_3 F_{1,1}(\boldsymbol{J},t)] \\ & + H(\boldsymbol{J})[q_{12}q_{21}q_{13}F_{0,2}(\boldsymbol{J},t) + q_{12}q_{23}F_{0,1}(\boldsymbol{J},t)],\end{aligned}$$

where the function $F_{m,n}$ is given by

$$F_{m,n}(\boldsymbol{J},t) = H(J_1)\sum_{p=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\gamma(\boldsymbol{J})^p J_1^{p+2i+j+n-1}(\mu_1 J_2 J_3)^i(\mu_2 J_2 J_3)^j(\mu_3 J_2 J_3)^k}{p!(p+2i+j+n-1)!(i+j+k+m)!i!j!k!}, \tag{3.6}$$

The proof can be found in Falzon [22]. The auxiliary functions used in the formulae are as follows:

$$\mu_1 = q_{12}q_{21}q_{13}q_{31}, \tag{3.7}$$

$$\mu_2 = q_{13}q_{32}q_{21} + q_{12}q_{23}q_{31}, \tag{3.8}$$

$$\mu_3 = q_{23}q_{32}, \tag{3.9}$$

$$\gamma(\boldsymbol{J}) = q_{12}q_{21}J_2 + q_{13}q_{31}J_3. \tag{3.10}$$

Remark. The functions $F_{m,n}$ can be written as

$$\begin{aligned}F_{m,n}(\boldsymbol{J},t) &= H(J_1)\sum_{p=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\gamma(\boldsymbol{J})^p J_1^{p+2i+j+n-1}(\mu_1 J_2 J_3)^i(\mu_2 J_2 J_3)^j(\mu_3 J_2 J_3)^k}{p!(p+2i+j+n-1)!(i+j+k+m)!i!j!k!} \\ &= H(J_1)J_1^{n-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\mu_1 J_1^2 J_2 J_3)^i(\mu_2 J_1 J_2 J_3)^j}{i!j!} \\ &\quad \times \sum_{k=0}^{\infty}\frac{(\mu_3 J_2 J_3)}{k!(i+j+m+k)!}\sum_{p=0}^{\infty}\frac{(\gamma(\boldsymbol{J})J_1)^p}{p!(p+2i+j+n-1)!} \\ &= H(J_1)J_1^{n-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\mu_1 J_1^2 J_2 J_3)^i(\mu_2 J_1 J_2 J_3)^j}{i!j!} \\ &\quad \times G_{i+j+m}(\mu_3 J_2 J_3)G_{2i+j+n-1}(\gamma(\boldsymbol{J})J_1).\end{aligned} \tag{3.11}$$

where

$$G_n(y) = \sum_{k=0}^{\infty}\frac{y^k}{(n+k)!k!}. \tag{3.12}$$

Noting that

$$\sqrt{y}I_1(2\sqrt{y}) = y\sum_{k=0}^{\infty}\frac{y^k}{k!(k+1)!}, \tag{3.13}$$

it is deduced that

$$G_n(y) = y^{-n/2}I_n(2\sqrt{y}). \tag{3.14}$$

Hence, $F_{m,n}$ can be calculated by using Bessel functions.

Remark. Note that when $J_2 = 0$ we may rewrite (3.11) as

$$\begin{aligned} F_{m,n}(0, J_3, t) &= H(J_1)\sum_d \frac{J_1^{n-1}(q_{31}q_{13}J_1J_3)^d}{m!d!(d+n-1)!} \\ &= H(J_1)\frac{J_1^{n-1}}{m!}G_{n-1}(q_{31}q_{13}J_1J_3). \end{aligned}$$

Similarly, when $J_3 = 0$ we have

$$F_{m,n}(0, J_3, t) = H(J_1)\frac{J_1^{n-1}}{m!}G_{n-1}(q_{21}q_{12}J_1J_2).$$

The joint density function of the occupation time for a two-state Markov chain is a special case of the three-state Markov chain and it is given in the following lemma.

Lemma 3.3.1. *The accumulated sojourn time for a two-state Markov process is*

$$\begin{aligned} w(J_2, t) = e^{-q_{12}t}\delta(J_2) &+ q_{12}e^{-q_{12}(t-J_2)-q_{21}J_2}F_{0,1}(J_2, 0, t) && (3.15) \\ &+ q_{12}q_{21}e^{-q_{12}(t-J_2)-q_{12}J_2}F_{0,2}(J_2, 0, t), && (3.16) \end{aligned}$$

where

$$F_{0,n}(J_2, 0, t) = \left(\sqrt{\frac{(t-J_2)}{(q_{12}q_{21}J_2)}}\right)^{n-1} I_{n-1}(2\sqrt{(q_{12}q_{21}J_2(t-J_2))}), \tag{3.17}$$

with I_r being the modified Bessel function of order r.

Proof. Let the transition rates to and from state 3 equal zero. Since $q_{13} = q_{31} = q_{23} = q_{32} = 0$ we get $q_{33} = 0$, $\mu_1 = \mu_2 = \mu_3 = 0$ and $\gamma(\boldsymbol{J}) = q_{12}q_{21}y_2$. Then

$$Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} & 0 \\ q_{21} & q_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.18}$$

Plugging in these values into the expression for $f(\boldsymbol{J}, t)$ in (3.5) yields

$$\begin{aligned} f(\boldsymbol{J}, t) = e^{(q_{11}J_1 + q_{22}J_2 + q_{33}J_3)} \Big[& \delta(\boldsymbol{J}) + q_{12}q_{21}H(J_2)\delta(J_3)F_{0,2}(J_2, 0, t) \\ & + q_{13}q_{31}\delta(J_2)H(J_3)F_{0,2}(0, J_3, t) \\ & + H(\boldsymbol{J})\gamma(\boldsymbol{J})[\mu_1 F_{1,4}(\boldsymbol{J}, t) + \mu_2 F_{1,3}(\boldsymbol{J}, t) + \mu_3 F_{1,2}(\boldsymbol{J}, t)] \\ & + H(\boldsymbol{J})[2\mu_1 F_{0,3}(\boldsymbol{J}, t) + \mu_2 F_{0,2}(\boldsymbol{J}, t)] \\ & + q_{12}H(J_2)\delta(J_3)F_{0,1}(J_2, 0, t) \\ & + H(\boldsymbol{J})q_{12}J_2[\mu_1 F_{1,3}(\boldsymbol{J}, t) + \mu_2 F_{1,2}(\boldsymbol{J}, t) + \mu_3 F_{1,1}(\boldsymbol{J}, t)] \\ & + H(\boldsymbol{J})[q_{13}q_{31}q_{21}F_{0,2}(\boldsymbol{J}, t) + q_{13}q_{32}F_{0,1}(\boldsymbol{J}, t)] \\ & + q_{13}\delta(J_2)H(J_3)F_{0,1}(0, J_3, t) \\ & + H(\boldsymbol{J})q_{13}J_2[\mu_1 F_{1,3}(\boldsymbol{J}, t) + \mu_2 F_{1,2}(\boldsymbol{J}, t) + \mu_3 F_{1,1}(\boldsymbol{J}, t)] \\ & + H(\boldsymbol{J})[q_{12}q_{21}q_{13}F_{0,2}(\boldsymbol{J}, t) + q_{12}q_{23}F_{0,1}(\boldsymbol{J}, t)]\Big] \\ = e^{(q_{11}J_1 + q_{22}J_2)} & \big[\delta(\boldsymbol{J}) + q_{12}F_{0,1}(J_2, 0, t) + q_{12}q_{21}F_{0,2}(J_2, 0, t)\big]. \end{aligned}$$

Since $q_{11} = -q_{12}$ and $q_{22} = -q_{21}$, it can be written as

$$\begin{aligned} w(J_2, t) = f(J_2, 0, t) = e^{-q_{12}t}\delta(J_2) &+ q_{12}e^{-q_{12}(t-J_2) - q_{21}J_2}F_{0,1}(J_2, 0, t) \\ &+ q_{12}q_{21}e^{-q_{12}(t-J_2) - q_{21}J_2}F_{0,2}(J_2, 0, t), \end{aligned}$$

and hence, the proof is completed. □

It has been observed that the joint density function of the occupation time of the three-state Markov process proposed by Falzon [22] can be reduced to the two-state formula. The reduced formula is consistent with the joint density function of the occupation time of the two-state Markov process formula proposed by Pedler [53]. This indicates that the Falzon's three-state formula can be considered as the general formula for the number of states $N \leq 3$.

3.3.1 Probabilistic Interpretation of the Formula

In order to make the formula more straightforward and provide a probabilistic interpretation, Falzon [22] reformulated the solution for the three-state accumulated sojourn time distribution using algebraic manipulation. A new expression for the formula in Theorem 3.3.1 may be interpreted as follows.

Corollary 3.3.1 ([22]). *Recall that the joint probability density of the accumulated sojourn time in a three-state Markov chain is given by*

$$\begin{aligned} f(\boldsymbol{J},t) &= f_1(\boldsymbol{J},t) + f_2(\boldsymbol{J},t) + f_3(\boldsymbol{J},t) \\ &= e^{-\epsilon(\boldsymbol{J},t)}[\alpha_1(\boldsymbol{J},t) + \alpha_2(\boldsymbol{J},t) + \alpha_3(\boldsymbol{J},t)], \end{aligned}$$

where J_i *is the time spent in state* $i = \{1,2,3\}$*,* J_1 *is defined to be* $t - J_2 - J_3$ *and* $\epsilon(\boldsymbol{J},t) = -\sum_{k=1}^{3} q_{kk}J_k$*. Here,* α_1, α_2 *and* α_3 *can be expressed in terms of the component functions* $F_{0,n}$ *and new defined function* $L_{m,n}$*:*

$$\begin{aligned} \alpha_1(\boldsymbol{J},t) = \{&\delta(\boldsymbol{J}) + q_{12}q_{21}\delta(J_3)F_{0,2}(J_2,0,t) + q_{13}q_{31}\delta(J_2)F_{0,2}(0,J_3,t) \\ &+ H(\boldsymbol{J})[q_{12}q_{13}L_{1,1}(\boldsymbol{J},t) + q_{12}q_{23}L_{0,1}(\boldsymbol{J},t) + q_{13}q_{32}L_{1,0}(\boldsymbol{J},t)]\} \end{aligned} \tag{3.19}$$

$$\begin{aligned} \alpha_2(\boldsymbol{J},t) = \{&q_{12}\delta(J_3)F_{0,1}(J_2,0,t) \\ &+ H(\boldsymbol{J})[q_{12}q_{13}L_{0,1}(\boldsymbol{J},t) + q_{12}q_{23}L_{-1,1}(\boldsymbol{J},t) + q_{13}q_{32}L_{0,0}(\boldsymbol{J},t)]\} \end{aligned} \tag{3.20}$$

$$\begin{aligned} \alpha_3(\boldsymbol{J},t) = \{&q_{13}\delta(J_2)F_{0,1}(0,J_3,t) \\ &+ H(\boldsymbol{J})[q_{12}q_{13}L_{1,0}(\boldsymbol{J},t) + q_{12}q_{23}L_{0,0}(\boldsymbol{J},t) + q_{13}q_{32}L_{1,-1}(\boldsymbol{J},t)]\} \end{aligned} \tag{3.21}$$

where

$$L_{m,n}(\boldsymbol{J},t) = H(J_1)\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty} \frac{y_{10}^{a+b-c+m}y_{20}^{d+c-b+n}y_{01}^{a}y_{21}^{b}y_{12}^{c}y_{02}^{d}}{a!b!c!d!(a+b-c+m)!(d+c-b+n)!} \tag{3.22}$$

$$\begin{aligned} &= H(J_1)\sum_{b=0}^{\infty}\sum_{c=0}^{\infty} \frac{y_{10}^{b-c+m}y_{20}^{c-b+n}y_{21}^{b}y_{12}^{c}}{b!c!} \\ &\quad\times \sum_{a=0}^{\infty}\sum_{d=0}^{\infty} \frac{(y_{10}y_{01})^{a}(y_{20}y_{02})^{d}}{a!d!(a+b-c+m)!(d+c-b+n)!} \\ &= H(J_1)\sum_{b=0}^{\infty}\sum_{c=0}^{\infty} \frac{y_{10}^{b-c+m}y_{20}^{c-b+n}y_{21}^{b}y_{12}^{c}}{b!c!} G_{m+b-c}(y_{10}y_{01})G_{n-b+c}(y_{20}y_{02}) \end{aligned} \tag{3.23}$$

and the terms in the numerator are:

$$y_{10} = q_{21}J_1, \quad y_{20} = q_{31}J_1, \quad y_{01} = q_{12}J_2,$$

$$y_{21} = q_{32}J_2, \quad y_{12} = q_{23}J_3, \quad y_{02} = q_{13}J_3.$$

In compact form the formula can be written as

$$\begin{aligned}f(\boldsymbol{J},t)=e^{-\epsilon(\boldsymbol{J},t)}\{&\delta(\boldsymbol{J})+q_{12}\delta(J_3)[q_{21}F_{0,2}(J_2,0,t)+F_{0,1}(J_2,0,t)]\\&+q_{13}\delta(J_2)[q_{31}F_{0,2}(0,J_3,t)+F_{0,1}(0,J_3,t)]\\&+H(\boldsymbol{J})\,(q_{12}q_{13}[L_{1,1}(\boldsymbol{J},t)+L_{0,1}(\boldsymbol{J},t)+L_{1,0}(\boldsymbol{J},t)]\\&+q_{12}q_{23}[L_{0,1}(\boldsymbol{J},t)+L_{-1,1}(\boldsymbol{J},t)+L_{0,0}(\boldsymbol{J},t)]\\&+q_{13}q_{32}[L_{1,0}(\boldsymbol{J},t)+L_{0,0}(\boldsymbol{J},t)+L_{1,-1}(\boldsymbol{J},t)])\}.\end{aligned}\tag{3.24}$$

We may, further define $L_m(J_i,t)$ in which one of the J_i is equal to zero, as follows

$$L_m(J_i,t)=H(J_1)\sum_a\frac{y_{(i-1)0}^{a+m}y_{0(i-1)}^{a}}{(a+m)!a!}\quad\text{for } i\in\{2,3\}.$$

Hence,

$$F_{0,m+1}(J_2,0,t)=q_{21}^{-m}L_m(J_2,t),$$

and

$$F_{0,m+1}(0,J_3,t)=q_{31}^{-m}L_m(J_3,t).$$

Therefore, the two-state formula can be rewritten by using these new expressions as

$$f(\boldsymbol{J},t)=e^{\epsilon(\boldsymbol{J},t)}[\underbrace{\delta(J_2)}_{A}+H(\boldsymbol{J})\{\underbrace{q_{12}L_1(J_2,t)}_{B}+\underbrace{q_{12}L_0(J_2,t)}_{C}\}].\tag{3.25}$$

The first term, A, indicates the boundary solution and arises from the cases when the Markov chain never leaves the initial state. The second term, B reflects events that have the same number of transitions from state 1 to state 2 as there are from state 2 to state 1, but with one less sojourn in state 2. Similarly, the third term comes from situations in which each state has an equal number of total sojourns and an extra transition into state 2, the final state.

Let $f_{(i)}(\boldsymbol{J},t)$ denotes the probability density when the final state of the process is i. Then the joint probability density can be written in terms of the component densities $f_{(1)}(\boldsymbol{J},t)$, $f_{(2)}(\boldsymbol{J},t)$, and $f_{(3)}(\boldsymbol{J},t)$ as follows:

$$\begin{aligned}f_{(1)}(\boldsymbol{J},t)=e^{-\epsilon(\boldsymbol{J},t)}\{&\delta(\boldsymbol{J})+q_{12}\delta(J_3)L_1(J_2,t)+q_{13}\delta(J_2)L_1(J_2,t)\\&+H(\boldsymbol{J})[q_{12}q_{13}L_{1,1}(\boldsymbol{J},t)+q_{12}q_{23}L_{0,1}(\boldsymbol{J},t)+q_{13}q_{32}L_{1,0}(\boldsymbol{J},t)]\},\end{aligned}$$

$$\begin{aligned}f_{(2)}(\boldsymbol{J},t)=e^{-\epsilon(\boldsymbol{J},t)}\{&q_{12}\delta(J_3)L_0(J_2,t)\\&+H(\boldsymbol{J})[q_{12}q_{13}L_{0,1}(\boldsymbol{J},t)+q_{12}q_{23}L_{-1,1}(\boldsymbol{J},t)+q_{13}q_{32}L_{0,0}(\boldsymbol{J},t)]\},\end{aligned}$$

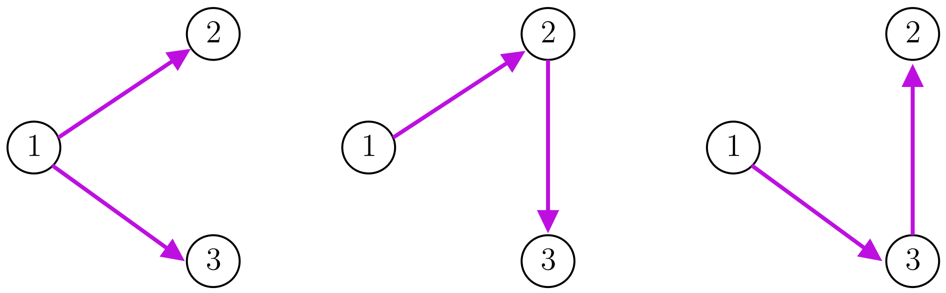


Figure 3.1: Spanning trees of a 3-node digraph rooted from node 1 [22]

$$
\begin{aligned}
f_{(3)}(\boldsymbol{J},t) = e^{-\epsilon(\boldsymbol{J},t)}\{& q_{13}\delta(J_2)L_0(J_3,t) \\
&+ H(\boldsymbol{J})[q_{12}q_{13}L_{1,0}(\boldsymbol{J},t) + q_{12}q_{23}L_{0,0}(\boldsymbol{J},t) + q_{13}q_{32}L_{1,-1}(\boldsymbol{J},t)]\}.
\end{aligned}
$$

In these probability density functions, the terms $L_{m,n}$ can be rewritten as

$$
\begin{aligned}
L_{m,n}(\boldsymbol{J},t) &= H(J_1)\sum_a\sum_b\sum_c\sum_d \frac{y_{10}^{a+b-c+m}y_{20}^{d+c-b+n}y_{01}^a y_{21}^b y_{12}^c y_{02}^d}{a!b!c!d!(a+b-c+m)!(d+c-b+n)!} \qquad (3.26)\\
&= y_{10}^m y_{20}^n H(J_1)\sum_a\sum_b\sum_c\sum_d \frac{(y_{01}y_{10})^a(y_{02}y_{20}^d)\left(\frac{y_{12}y_{20}}{y_{10}}\right)^c\left(\frac{y_{21}y_{10}}{y_{20}}\right)^b}{a!b!c!d!(a+b-c+m)!(d+c-b+n)!}
\end{aligned}
$$

Each of the components in the summands' numerators may be interpreted as a two-step transition event. The first two elements represent a route that starts in state 1 and ends in state 1. The third component relates to transactions between state 2 and state 1 with a stopover in state 3 in between. These occurrences replace an equal number of state 2 to state 1 transitions, according to the third factor's denominator. Similar interpretations apply to the fourth component. For example, $L_{0,0}$ counts all of the potential transition cycles with state 1 serving as both the starting and ending states.

The subscripts in $L_{m,n}$ gives information about the transitions; m additional transition from state 2 to state 1 and n additional transition from state 3 to state 1, as well as $m+n$ full sojourns in state 1. Falzon [22] also gave an explanation about the relation with each q_{kj} a directed line in a three-node spanning tree. The coefficients of the $L_{m,n}$ in each density f_i show the sequence of the edges in each directed tree that starts at node 1. In Figure 3.1, three such directed trees are illustrated.

Also, subscripts in L is determined by the coefficients to assure the final state is i, for example, the first interior term in f_1 is $q_{12}q_{13}L_{1,1}$, which is related to the first tree in the Figure 3.1. In order to make state 1 the final state further transitions are required as follows:

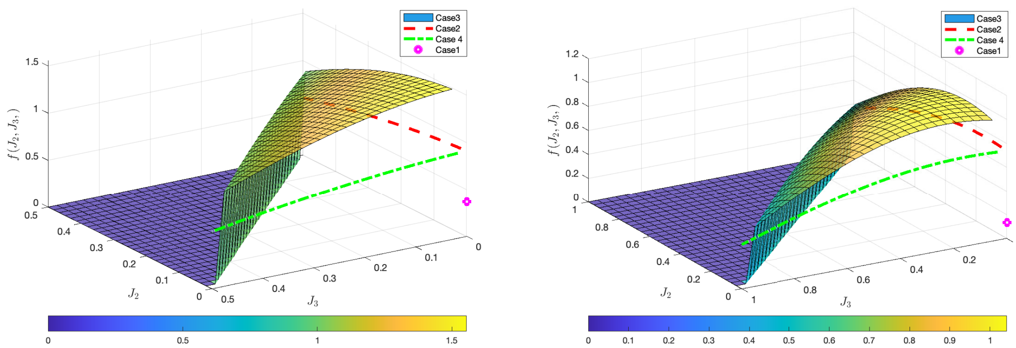
$$2 \longrightarrow 1 \qquad 3 \longrightarrow 1$$

Therefore we require $m = 1$ and $n = 1$ for this to occur. Similarly, it is required to have $m = 0$, $n = 1$ for the second term and $m = 1$, $n = 0$ for the third term, in order to make state 1 the final state. Falzon [22] using the relations between spanning trees and the probability densities first generalize formula for four-state case and then N-state case. Further information about the generalization to N-state case can be found in Falzon [22].

To demonstrate the behavior of the distribution, we consider a three-state Markov chain with the transition rate matrix

$$Q = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}.$$

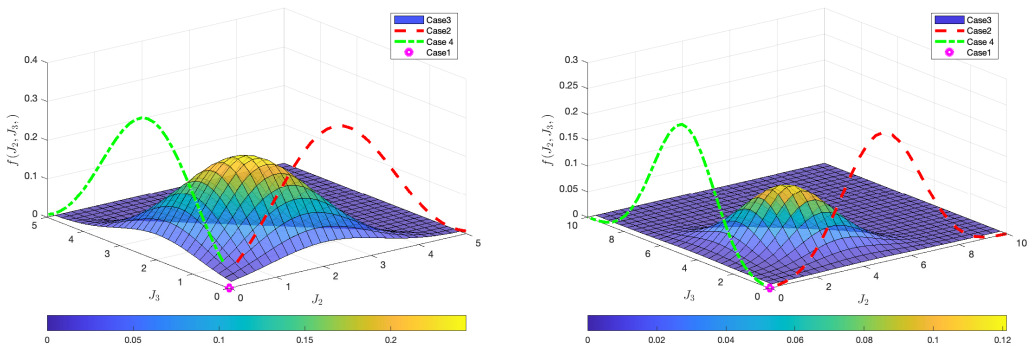
The behavior of the conditional joint distribution of the three-state Markov chain for different values of T are illustrated in the Figure 3.2, Figure 3.3, and Figure 3.4.



(a) $T = 0.5$ (b) $T = 1$

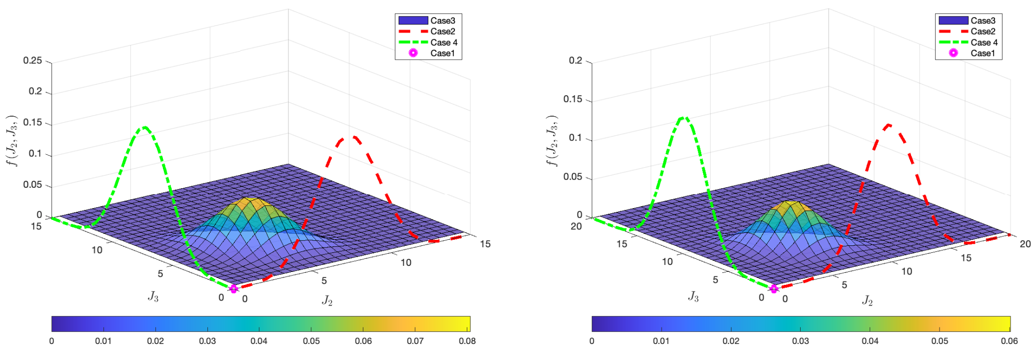
Figure 3.2: Conditional joint distribution of occupation times

Remark. Falzon [22] stated that they used appropriate finite sums to approximate the infinite summing in order to put the formulas $F_{m,n}$ and $L_{m,n}$ into practice. The



(a) $T = 5$ (b) $T = 10$

Figure 3.3: Conditional joint distribution of occupation times



(a) $T = 15$ (b) $T = 20$

Figure 3.4: Conditional joint distribution of occupation times

stopping condition used was that the difference between two successive partial sums must be smaller than the selected precision. In the majority of instances, they found that the stopping condition was satisfied before less than term terms of each sum were evaluated with 15 decimal digit precision. In Example 4.1.4, we have included an application result that supports this claim.

Falzon's formula is given for the case in which the Markov process starts from state 1. It is also possible to use the formula proposed by Falzon for cases in which the Markov chain starts from state 2, or state 3 based on the transition rate matrix provided in the remark below.

Remark. Let the initial state be $l \in \{1, 2, 3\}$. In the following without loss of generality we define a permutation of the states and corresponding transition rate matrix

M so that

$$Q = PMP^T$$

is the transition rate matrix $Q = (q_{ij})$ with $i, j \in S$.

CHAPTER 4

EUROPEAN OPTION PRICING UNDER REGIME-SWITCHING FRAMEWORK

In this chapter we formulate the option pricing formula under regime-switching model when the underlying Markov chain has three states. The model dynamics are introduced in Section 2.2. We first state the general N-state from introduced by Elliot et al. [12], then in Section 4.1.1 formula for the three-state Markov chain is presented with its proof. Since the Monte Carlo method is widely used as a true-value benchmark for assessing other numerical algorithms we present the Monte Carlo algorithm for the regime-switching European option pricing problem. Fast Fourier transform (FFT) is an another conventional method for the option pricing problem under regime-switching context. We briefly summarize these methods in Section 4.1.2 and Section 4.1.3, respectively. Then we outline numerical examples that are performed to investigate the assessment of the proposed method with other methods in Section 4.1.4. We also present an example in Section 4.1.5 demonstrating that the infinite sum given in the main pricing formula can be stated by the finite sum of acceptable value.

The Greeks are important tools in financial risk management. Each Greek measures the sensitivity of the value of derivatives or a portfolio to a small change in a given underlying parameter. Hence, we also investigate the computation of the Greeks under the regime-switching framework by using proposed formula given in Section 4.1.1. European call option Greeks are obtained by taking derivatives with respect to the corresponding parameter and utilizing double Leibniz rule. We compare the results which are obtained by the finite difference method with the results of the analytical solution.

4.1 Implementation of Regime-Switching Option Pricing

The general form of the N-state regime switching formula is stated without proof by Elliot et al. [9, 8, 12] as follows:

$$C(t,T,S_t) := \int_t^T \cdots \int_t^T C(t,T,S_t,P_{t,T},V_{t,T}) f_{X_t}(J_1,J_2,\ldots,J_N) dJ_1 \ldots dJ_N, \tag{4.1}$$

where

$$C(t,T,S_t,P_{t,T},V_{t,T}) = S_t N(d_{1,t,T}) - K\exp(-P_{t,T}) N(d_{2,t,T}), \tag{4.2}$$

and

$$d_{1,t,T} = (V_{t,T})^{-1/2}\left(\ln\frac{S_t}{K} + P_{t,T} + \frac{1}{2}V_{t,T}\right),$$
$$d_{2,t,T} = d_{1,t,T} - (V_{t,T})^{1/2}.$$

In the following section, by using the relation given in (3.1), we present the formula under three-state can be obtained as a 2-fold integral. When the market interest rate of the bank account is a constant McKinlay [48] presented the formula for two-state and three-state cases. In the following subsection, we extend the proof presented by McKinlay [48] for the model in which all parameters are governed by the Markov chain and the number of state is three.

4.1.1 Formula for the Three-State Model

In the sequel, we use $f_{X(t)}(J_2,J_3)$ instead of $f(J_2,J_3,T)$ for the conditional joint density function of the occupation times when the state of the Markov process at time t is given.

Theorem 4.1.1. *Let the parameters of the underlying asset and the market interest rate depend on the underlying three-state continuous time Markov chain. Then, the European call option price at time* t *with a strike price* K *with maturity* T *is given by*

$$C(S(t),t,T) = \int_0^{T-t}\int_0^{T-t-J_3} C^{BS}\big(S(t),K,\bar{p},T-t,\bar{v}\big) f_{X(t)}(J_2,J_3) dJ_2 dJ_3 \tag{4.3}$$

where $0 \leq J_2 < T-t$, $0 \leq J_3 < T-t$, $\bar{p} = \frac{p_{X(t)}(J_2,J_3)}{T-t}$, $\bar{v} = \sqrt{\frac{v_{X(t)}(J_2,J_3)}{T-t}}$, *and* $C^{BS}(S(t), K, r, T-t, \sigma)$ *is the Black-Scholes formula for the call option and* $f_{X(t)}(J_2, J_3)$ *is the joint density of the occupation times of state two and state three during the time interval* $[t, T]$ *conditional on* $X(t)$.

Proof. The price of a European type option with payoff $g(S(T))$ at maturity is given by

$$C(t) = \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^T r_s ds\Big) g(S(T))\Big|\mathcal{F}_t\Big] \tag{4.4}$$

Using double expectation law, we can write

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^T r_s ds\Big) g(S(T))\Big|\mathcal{F}_t\Big] \\
&\qquad = \mathbb{E}^{\mathbb{Q}}\Big[\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^T r_s ds\Big) g(S(T))\Big|\mathcal{F}_t^X \vee \mathcal{F}_t^W\Big]\Big|\mathcal{F}_t\Big].
\end{aligned} \tag{4.5}$$

Then, the inner integral can be written as

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^T r_s ds\Big) g(S(T))\Big|\mathcal{F}_t^X \vee \mathcal{F}_t^W\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^T r_s ds\Big) \\
&\quad \times g\Big(S_0 \exp\Big\{\int_0^T r_s ds - \frac{1}{2}\int_0^T \sigma_s^2 ds + \int_0^T \sigma_s d\widetilde{W}(s)\Big\}\Big)\Big|\mathcal{F}_t^X \vee \mathcal{F}_t^W\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^T r_s ds\Big) \\
&\quad \times g\Big(S(t) \exp\Big\{\int_t^T r_s ds - \frac{1}{2}\int_t^T \sigma_s^2 ds + \int_t^T \sigma_s d\widetilde{W}(s)\Big\}\Big)\Big|\mathcal{F}_t^X \vee \mathcal{F}_t^W\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[\exp(P_{t,T}) g\Big(S(t) \exp\Big\{P_{t,T} - \frac{1}{2}V_{t,T} + \sqrt{V_{t,T}}x\Big\}\Big)\Big|\mathcal{F}_t^X \vee \mathcal{F}_t^W\Big] \\
&= \int_{\mathbb{R}} \exp(P_{t,T}) g\Big(S(t) \exp\Big\{P_{t,T} - \frac{1}{2}V_{t,T} + \sqrt{V_{t,T}}x\Big\}\Big)\phi(x)dx,
\end{aligned}$$

where $\phi(x)$ stands for the probability density of the standard normal distribution (with zero mean and unit variance). Hence, (4.5) can be written as

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^T r_s ds\Big) g(S(T))\Big|\mathcal{F}_t\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[\int_{\mathbb{R}} \exp(P_{t,T}) g\Big(S(t) \exp\Big\{P_{t,T} - \frac{1}{2}V_{t,T} + \sqrt{V_{t,T}}x\Big\}\Big)\phi(x)dx\Big|\mathcal{F}_t\Big].
\end{aligned} \tag{4.6}$$

Since we consider the underlying Markov chain has three state, let $f_{X(t)}(J_2, J_3)$ be the conditional joint density of the occupation time of the states during the time interval $[t, T]$, given $X(t) = 1$, and for $0 \le J_2 < T - t$, $0 \le J_3 < T - t$ define

$$v_{X(t)}(J_2, J_3) = \sigma_1^2(T - t - J_2 - J_3) + \sigma_2^2(J_2) + \sigma_3^2(J_3), \tag{4.7}$$

and

$$p_{X(t)}(J_2, J_3) = r_1(T - t - J_2 - J_3) + r_2(J_2) + r_3(J_3). \tag{4.8}$$

Note that $V_{t,T} = v_{X(t)}(J_2, J_3)$ and $P_{t,T} = p_{X(t)}(J_2, J_3)$, where J_2 and J_3 are the total time spent by the Markov chain process X at state 2 and state 3, respectively, during the time interval $[t, T]$ where $X(t) = 1$. We may therefore rewrite (4.6) as

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right) g(S(T))\Big|\mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\Bigg[\int_{\mathbb{R}} \exp(p_{X(t)}(J_2, J_3)) g\Bigg(S(t)\exp\left\{p_{X(t)}(J_2, J_3) - \frac{1}{2}v_{X(t)}(J_2, J_3)\right. \\
&\quad \left.+\sqrt{v_{X(t)}(J_2, J_3)}x\right\}\Bigg)\times\phi(x)dx\Big|\mathcal{F}_t\Bigg] \\
&= \int_0^{T-t}\int_0^{T-t-J_3}\int_{\mathbb{R}} \exp(p_{X(t)}(J_2, J_3)) g\Bigg(S(t)\exp\left\{p_{X(t)}(J_2, J_3) - \frac{1}{2}v_{X(t)}(J_2, J_3)\right. \\
&\quad \left.+\sqrt{v_{X(t)}(J_2, J_3)}x\right\}\Bigg)\phi(x) f_{X(t)}(J_2, J_3) dx dJ_2 dJ_3 \\
&= \int_0^{T-t}\int_0^{T-t-J_3}\Bigg[\exp(p_{X(t)}(J_2, J_3)) \\
&\quad \times\int_{\mathbb{R}}\left(S(t)\exp\left\{p_{X(t)}(J_2, J_3) - \frac{1}{2}v_{X(t)}(J_2, J_3) + \sqrt{v_{X(t)}(J_2, J_3)}x\right\} - K\right)^+ \phi(x)dx\Bigg] \\
&\quad \times f_{X(t)}(J_2, J_3) dJ_2 . dJ_3
\end{aligned}$$

We now consider the inner integral

$$\begin{aligned}
&\int_{\mathbb{R}}\left(S(t)\exp\left\{p_{X(t)}(J_2, J_3) - \frac{1}{2}v_{X(t)}(J_2, J_3) + \sqrt{v_{X(t)}(J_2, J_3)}x\right\} - K\right)^+ \phi(x)dx \\
&= \int_{\mathbb{R}} \mathbb{1}_{\{S_t \exp[p_{X(t)}(J_2,J_3) - \frac{1}{2}v_{X(t)}(J_2,J_3) + \sqrt{v_{X(t)}(J_2,J_3)}x] > K\}} \\
&\quad \times S(t)\exp\left\{p_{X(t)}(J_2, J_3) - \frac{1}{2}v_{X(t)}(J_2, J_3) + \sqrt{v_{X(t)}(J_2, J_3)}x\right\}\phi(x)dx \\
&\quad - \int_{\mathbb{R}} \mathbb{1}_{\{S(t) \exp[p_{X(t)}(J_2,J_3) - \frac{1}{2}v_{X(t)}(J_2,J_3) + \sqrt{v_{X(t)}(J_2,J_3)}x] > K\}} K\phi(x)dx.
\end{aligned} \tag{4.9}$$

We have:

$$
\begin{aligned}
&S(t)\exp\left\{p_{X(t)}(J_2,J_3)-\frac{1}{2}v_{X(t)}(J_2,J_3)+\sqrt{v_{X(t)}(J_2,J_3)}x\right\}>K\\
\iff\ &p_{X(t)}(J_2,J_3)-\frac{1}{2}v_{X(t)}(J_2,J_3)+\sqrt{v_{X(t)}(J_2,J_3)}x>\ln\left(\frac{K}{S(t)}\right)\\
\iff\ &x>\frac{\ln(K/S(t))-p_{X(t)}(J_2,J_3)+v_{X(t)}(J_2,J_3)/2}{\sqrt{v_{X(t)}(J_2,J_3)}}.
\end{aligned}
$$

Define $h:=(\ln(K/S(t))-p_{X(t)}(J_2,J_3)+v_{X(t)}(J_2,J_3)/2)/\sqrt{v_{X(t)}(J_2,J_3)}$, then (4.9) becomes

$$
\begin{aligned}
&\int_{\mathbb{R}}\left(S(t)\exp\left\{p_{X(t)}(J_2,J_3)-\frac{1}{2}v_{X(t)}(J_2,J_3)+\sqrt{v_{X(t)}(J_2,J_3)}x\right\}-K\right)^+\phi(x)dx\\
=&\int_{\mathbb{R}}\mathbb{1}_{\{x>-h\}}S(t)\exp\left\{p_{X(t)}(J_2,J_3)-\frac{1}{2}v_{X(t)}(J_2,J_3)+\sqrt{v_{X(t)}(J_2,J_3)}x\right\}\phi(x)dx\\
&-\int_{\mathbb{R}}\mathbb{1}_{\{x>-h\}}K\phi(x)dx. \qquad (4.10)
\end{aligned}
$$

The first integral of (4.10) can be calculated as

$$
\int_{\mathbb{R}}\mathbb{1}_{\{x>-h\}}S(t)\exp\left\{p_{X(t)}(J_2,J_3)-\frac{1}{2}v_{X(t)}(J_2,J_3)+\sqrt{v_{X(t)}(J_2,J_3)}x\right\}\phi(x)dx \qquad (4.11)
$$

$$
\begin{aligned}
=&S(t)e^{p_{X(t)}(J_2,J_3)-v_{X(t)}(J_2,J_3)/2}\int_{\mathbb{R}}\mathbb{1}_{\{x>-h\}}e^{\sqrt{v_{X(t)}(J_2,J_3)}x}\phi(x)dx\\
=&S(t)e^{p_{X(t)}(J_2,J_3)-v_{X(t)}(J_2,J_3)/2}\frac{1}{\sqrt{2\pi}}\int_{-h}^{\infty}e^{\sqrt{v_{X(t)}(J_2,J_3)}x-x^2/2}dx\\
=&S(t)e^{p_{X(t)}(J_2,J_3)-v_{X(t)}(J_2,J_3)/2}\frac{1}{\sqrt{2\pi}}\int_{-h}^{\infty}e^{-\frac{(x-\sqrt{v_{X(t)}(J_2,J_3)})^2}{2}+\frac{v_{X(t)}(J_2,J_3)}{2}}dx\\
=&S(t)e^{p_{X(t)}(J_2,J_3)}\frac{1}{\sqrt{2\pi}}\int_{-h}^{\infty}e^{-\frac{(x-\sqrt{v_{X(t)}(J_2,J_3)})^2}{2}}dx\\
=&S(t)e^{p_{X(t)}(J_2,J_3)}\frac{1}{\sqrt{2\pi}}\int_{-h-\sqrt{v_{X(t)}(J_2,J_3)}}^{\infty}e^{-y^2/2}dy \quad \left(\text{where } y=x-\sqrt{v_{X(t)}(J_2,J_3)}\right)\\
=&S(t)e^{p_{X(t)}(J_2,J_3)}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{h+\sqrt{v_{X(t)}(J_2,J_3)}}e^{-y^2/2}dy\\
=&S(t)e^{p_{X(t)}(J_2,J_3)}\Phi\big(h+\sqrt{v_{X(t)}(J_2,J_3)}\big)\\
=&S(t)e^{p_{X(t)}(J_2,J_3)}\Phi(d_1),
\end{aligned}
$$

where Φ is the standard normal cumulative distribution function and $d_1=h+\sqrt{v_{X(t)}(J_2,J_3)}$.

Now, consider the second integral on the right-hand-side of (4.10), which can be calculated as

$$
\begin{aligned}
\int_{\mathbb{R}} \mathbb{1}_{x>-h} K\phi(x)dx &= \frac{1}{\sqrt{2\pi}} \int_{-h}^{\infty} K e^{-x^2/2} dx \\
&= \frac{K}{\sqrt{2\pi}} \int_{-\infty}^{h} e^{-x^2/2} dx \\
&= K\Phi(d_1 - \sqrt{v_{X(t)}(J_2, J_3)}).
\end{aligned}
$$

Therefore, inserting these into (4.4) we obtain the European call option price

$$
\begin{aligned}
C(t) = &\int_0^{T-t} \int_0^{T-t-J_3} \exp(-p_{X(t)}(J_2, J_3)) \big[S(t) \exp(p_{X(t)}(J_2, J_3)) \Phi(d_1) \\
&- K\Phi(d_1 - \sqrt{v_{X(t)}(J_2, J_3)})\big] f_{X(t)}(J_2, J_3) dJ_2 dJ_3 \\
= &\int_0^{T-t} \int_0^{T-t-J_3} \big[S(t)\Phi(d_1) - K \exp(-p_{X(t)}(J_2, J_3)) \Phi(d_1 - \sqrt{v_{X(t)}(J_2, J_3)})\big] \\
&\times f_{X(t)}(J_2, J_3) dJ_2 dJ_3 \\
= &\int_0^{T-t} \int_0^{T-t-J_3} C^{BS}\left(S(t), K, \frac{p_{X(t)}(J_2, J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2, J_3)}{T-t}}\right) \\
&\times f_{X(t)}(J_2, J_3) dJ_2 dJ_3, \qquad (4.12)
\end{aligned}
$$

where C^{BS} is the standard Black-Scholes formula with the corresponding parameters. This completes the proof. □

Remark. Similarly, it can be showed that the put option prices can be expressed as

$$
P(S(t), t, T) = \int_0^{T-t} \int_0^{T-t-J_3} P^{\mathrm{BS}}\big(S(t), K, \bar{p}, T-t, \bar{v}\big) f_{X(t)}(J_2, J_3) dJ_2 dJ_3,
$$

where P^{BS} is the standard Black-Scholes put option formula. Regime-switching put option formula can also be obtained by regime-switching version of the put-call parity as follows

$$
\begin{aligned}
P(S(t), t, T) + S(t) &= C(S(t), t, T) \\
&\quad + \int_0^{T-t} \int_0^{T-t-J_3} K \exp(-p_{X(t)}(J_2, J_3)) f_{X(t)}(J_2, J_3) dJ_2 dJ_3 \\
P(S(t), t, T) &= C(S(t), t, T) - S(t) \\
&\quad + \int_0^{T-t} \int_0^{T-t-J_3} K \exp(-p_{X(t)}(J_2, J_3)) f_{X(t)}(J_2, J_3) dJ_2 dJ_3.
\end{aligned}
$$

We examine the Falzon's formula (for the conditional joint distribution of the accumulated sojourn time for the Markov chain) in four parts for better understanding. It is assumed that the initial state is known to be state 1. Recall the joint probability density of the accumulated sojourn time in a three-state Markov chain is given by

$$
\begin{aligned}
f(\boldsymbol{J}, t) = e^{(q_{11}J_1 + q_{22}J_2 + q_{33}J_3)} \big[& \delta(\boldsymbol{J}) + q_{12}q_{21}H(J_2)\delta(J_3)F_{0,2}(J_2, 0, t) \\
& + q_{13}q_{31}\delta(J_2)H(J_3)F_{0,2}(0, J_3, t) \\
& + H(\boldsymbol{J})\gamma(\boldsymbol{J})[\mu_1 F_{1,4}(\boldsymbol{J}, t) + \mu_2 F_{1,3}(\boldsymbol{J}, t) + \mu_3 F_{1,2}(\boldsymbol{J}, t)] \\
& + H(\boldsymbol{J})[2\mu_1 F_{0,3}(\boldsymbol{J}, t) + \mu_2 F_{0,2}(\boldsymbol{J}, t)] \\
& + q_{12}H(J_2)\delta(J_3)F_{0,1}(J_2, 0, t) \\
& + H(\boldsymbol{J})q_{12}J_2[\mu_1 F_{1,3}(\boldsymbol{J}, t) + \mu_2 F_{1,2}(\boldsymbol{J}, t) + \mu_3 F_{1,1}(\boldsymbol{J}, t)] \\
& + H(\boldsymbol{J})[q_{13}q_{31}q_{21}F_{0,2}(\boldsymbol{J}, t) + q_{13}q_{32}F_{0,1}(\boldsymbol{J}, t)] \\
& + q_{13}\delta(J_2)H(J_3)F_{0,1}(0, J_3, t) \\
& + H(\boldsymbol{J})q_{13}J_2[\mu_1 F_{1,3}(\boldsymbol{J}, t) + \mu_2 F_{1,2}(\boldsymbol{J}, t) + \mu_3 F_{1,1}(\boldsymbol{J}, t)] \\
& + H(\boldsymbol{J})[q_{12}q_{21}q_{13}F_{0,2}(\boldsymbol{J}, t) + q_{12}q_{23}F_{0,1}(\boldsymbol{J}, t)]\big],
\end{aligned}
$$

where $\boldsymbol{J}(t) = (J_2(t), J_3(t)) \in \mathbb{R}^2_+$ and $H(J)$ denote the Heaviside unit step function. $H(\boldsymbol{J})$ given in (3.4) denotes $H(J_2)H(J_3)$ which is equal to 0 for either $J_2 \leq 0$ or $J_3 \leq 0$ and 1 when $J_2 > 0$ and $J_3 > 0$.

The formula includes four cases. The breakdown of the formula according to all possible cases is as follows. Note that since it is assumed that the initial state is 1, i.e., the process starts from state 1, this formula includes all cases in which $J_1 > 0$. In the sequel, we use the notation $f^k_{X(0)=1}(J_2, J_3)$ for $k \in \{1, 2, 3, 4\}$ where k represents the case number.

case 1) $J_1 > 0, J_2 = 0, J_3 = 0$

$$
f^1_{X(0)=1}(J_2, J_3) = f(\boldsymbol{J}, t) = e^{(q_{11}J_1 + q_{22}J_2 + q_{33}J_3)}(\delta(\boldsymbol{J})) \tag{4.13}
$$

This is the scenario in which the Markov chain does not change its state; the Markov chain remains in its initial state. We can utilize the conventional Black-Scholes formula with the initial state parameters in this scenario. Figure 4.1 depicts a realization of a Markov chain for this scenario.

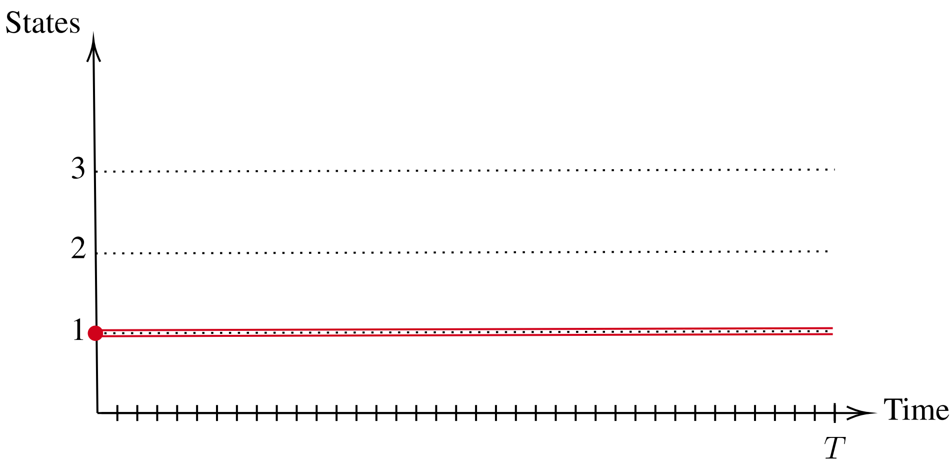


Figure 4.1: Realization of Case 1

case 2) $J_1 > 0, J_2 > 0, J_3 = 0$

$$\begin{aligned} f^2_{X(0)=1}(J_2, J_3) = f(\boldsymbol{J}, t) = e^{(q_{11}J_1 + q_{22}J_2 + qq_{33}J_3)}[q_{12}q_{21}H(J_2)\delta(J_3)F_{0,2}(J_2, 0, t) \\ + q_{12}H(J_2)\delta(J_3)F_{0,1}(J_2, 0, t)] \end{aligned} \tag{4.14}$$

This part of the formula considers cases where the Markov chain spends time in state 1 and state 2 but never goes into state 3. Figure 4.2 demonstrates how a Markov chain behave in this case.

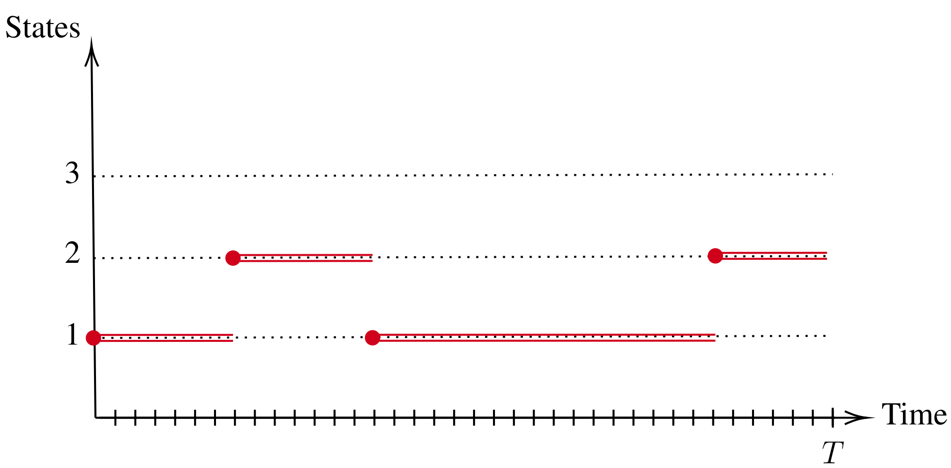


Figure 4.2: Realization of Case 2

case 3) $J_1 > 0, J_2 > 0, J_3 > 0$

$$
\begin{aligned}
f^3_{X(0)=1}(J_2, J_3) = f(\boldsymbol{J}, t) \\
&= e^{(q_{11}J_1+q_{22}J_2+q_{33}J_3)}\big[H(\boldsymbol{J})\gamma(\boldsymbol{J})[\mu_1 F_{1,4}(\boldsymbol{J}, t) + \mu_2 F_{1,3}(\boldsymbol{J}, t) \\
&\quad + \mu_3 F_{1,2}(\boldsymbol{J}, t)]H(\boldsymbol{J})[2\mu_1 F_{0,3}(\boldsymbol{J}, t) + \mu_2 F_{0,2}(\boldsymbol{J}, t) \\
&\quad + H(\boldsymbol{J})q_{12}J_2[\mu_1 F_{1,3}(\boldsymbol{J}, t) + \mu_2 F_{1,2}(\boldsymbol{J}, t) + \mu_3 F_{1,1}(\boldsymbol{J}, t)] \\
&\quad + H(\boldsymbol{J})[q_{13}q_{31}q_{21}F_{0,2}(\boldsymbol{J}, t) + q_{13}q_{32}F_{0,1}(\boldsymbol{J}, t)] \\
&\quad + H(\boldsymbol{J})q_{13}y_3[\mu_1 F_{1,3}(\boldsymbol{J}, t) + \mu_2 F_{1,2}(\boldsymbol{J}, t) + \mu_3 F_{1,1}(\boldsymbol{J}, t)] \\
&\quad + H(\boldsymbol{J})[q_{12}q_{21}q_{13}F_{0,2}(\boldsymbol{J}, t) + q_{12}q_{23}F_{0,1}(\boldsymbol{J}, t)]\big] \qquad (4.15)
\end{aligned}
$$

This part of the formula deals with the case that all states visited at least once, and such a chain is illustrated in Figure 4.3.

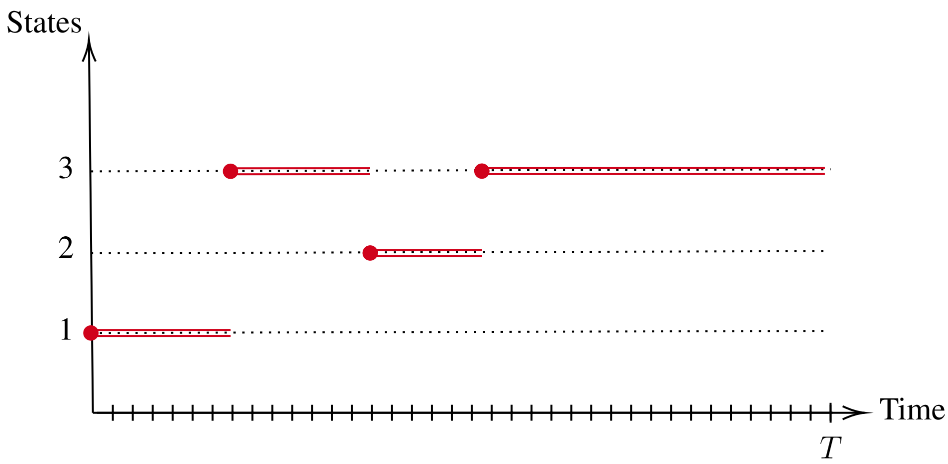


Figure 4.3: Realization of Case 3

case 4) $J_1 > 0, J_2 = 0, J_3 > 0$

$$
\begin{aligned}
f^4_{X(0)=1}(J_2, J_3) = f(\boldsymbol{J}, t) = e^{(q_{11}J_1+q_{22}J_2+q_{33}J_3)}[q_{13}q_{31}\delta(J_2)H(J_3)F_{0,2}(0, J_3, t) \\
+ q_{13}\delta(J_2)H(J_3)F_{0,1}(0, J_3, t)] \qquad (4.16)
\end{aligned}
$$

Similar to case 2, this part of the formula considers the situations where the Markov chain spends time in state 1 and state 3 but never goes into state 2. Figure 4.4 displays the Markov chain's behavior in this scenario.

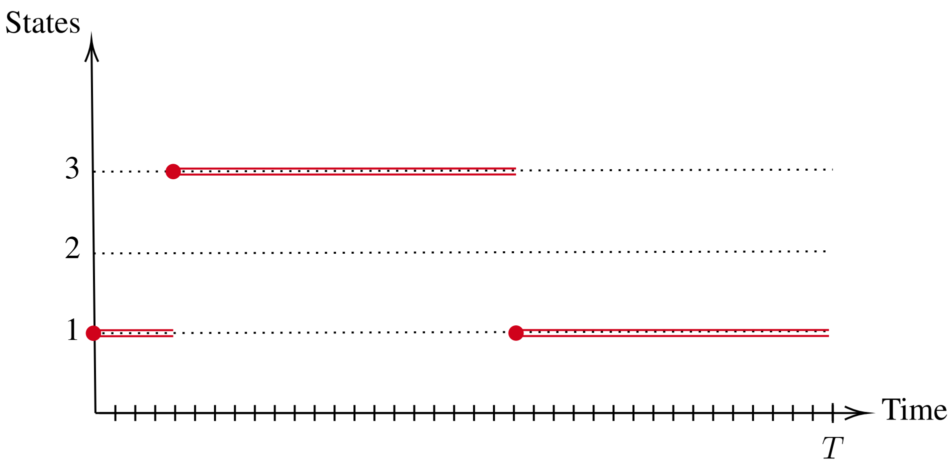


Figure 4.4: Realization of Case 4

4.1.2 Monte Carlo Simulation

The basic idea of Monte Carlo simulations is the producing the large amounts of repeated random samplings in order to approximate results. The Law of Large Numbers (LLN) and the Central Limit Theorem (CLT) provide the theoretical foundation upon which these broad class of computational algorithms are built.

In addition to their many other benefits, simulation techniques are renowned for their applicability to complicated systems and their straightforward implementation, particularly in light of improvement of the computational power.

It is a well known fact that order of converge speed for Monte Carlo simulation is $\mathcal{O}(1/\sqrt{N})$, where N is the number of sample paths. To put it another way, increasing the sample size one hundredfold is required to improve the estimate by one significant digit.

In the field of finance, having the ability to quickly collect relevant data and make pricing and risk management decisions is essential. When we compare Monte Carlo simulation method with the other numerical methods, namely, tree methods and finite-difference algorithms, we can observe that these methods offer substantially faster convergence rates of order $\mathcal{O}(1/N)$ and $\mathcal{O}(1/N^2)$, respectively. However using variance reduction methods (control variates, antithetic variates) or a quasi-Monte Carlo methodology may increase efficiency of the Monte Carlo method even more.

In this study, Monte Carlo (MC) algorithm is used as benchmark values in numerical experiments. In order to price path independent options it is adequate to obtain the final stock price S_T at the maturity, not the whole stock price trajectory $\{S_t\}_{t\in[0,T]}$. Hence, first we need to simulate the Markov chain for each path and obtain the occupation times J_1, J_2, and J_3 for the three-state Markov chain. By using the independent increments property of Brownian motion and properties of the sum of two independent normally distributed random variables we can write

$$
\begin{aligned}
(\ln S_t|\mathcal{F}_t) = \ln S_0 & \\
&+ \int_0^{J_1} \left(r_1 - \frac{1}{2}\sigma_1^2 \right) ds + \int_0^{J_1} \sigma_1 d\widetilde{W}(s) \\
&+ \int_{J_1}^{J_1+J_2} \left(r_2 - \frac{1}{2}\sigma_2^2 \right) ds + \int_{J_1}^{J_1+J_2} \sigma_2 d\widetilde{W}(s) \\
&+ \int_{J_1+J_2}^{T} \left(r_3 - \frac{1}{2}\sigma_3^2 \right) ds + \int_{J_1+J_2}^{T} \sigma_3 d\widetilde{W}(s) \\
&= \ln(S_0) + r_1 J_1 + r_2 J_2 + r_3 J_3 - \frac{1}{2}\left(\sigma_1^2 J_1 + \sigma_2^2 J_2 + \sigma_3^2 J_3 \right) \\
&\quad + \sigma_1 \widetilde{W}(J_1) + \sigma_2(\widetilde{W}(J_1+J_2) - \widetilde{W}(J_1)) + \sigma_3(\widetilde{W}(T) - \widetilde{W}(J_1+J_2)) \\
&\overset{d}{=} \mathcal{N}\left(\mu^*, \sigma^* \right),
\end{aligned}
$$

where

$$
\mu^* = \ln(S_0) + r_1 J_1 + r_2 J_2 + r_3 J_3 - \frac{1}{2}\left(\sigma_1^2 J_1 + \sigma_2^2 J_2 + \sigma_3^2 J_3 \right)
$$

and

$$
\sigma^* = \sigma_1^2 J_1 + \sigma 2^2 J_2 + \sigma_3^2 J_3.
$$

Hence, to simulate the conditional random variable $\ln S_t|\mathcal{F}_t$ we only need one pseudo-random sample. Pseudo-code for option pricing via Monte Carlo approach under regime-switching framework is given in Algorithm 2.

4.1.3 Fast Fourier Transform (FFT)

There has been a great deal of use of fast Fourier transforms in the valuation of financial derivatives. The approach is applicable to problems where the characteristic functions of the underlying price process can be derived analytically. Carr and Madan [10]

Algorithm 2 Monte Carlo Simulation for Markov Modulated Option Pricing

Require:

1: Initial stock price S_0, Strike price K, Maturity time T

2: Regime-switching interest rate vector r, Regime-switching volatility vector σ

3: Initial state of the Markov chain $X_0 = j$

4: Transition rate matrix of the Markov chain Q

5: Number of simulation paths n_r,

Ensure: European call option price C

6: **for** $k = 1, 2, \dots, n_r$ **do**

7: Simulate a Markov chain on which the parameters of the simulated path will depend on.

8: Find the states of the simulated chain.

9: Calculate the $P^k_{(t,T)}$, and $V^k_{(t,T)}$ given in (2.23), (2.24), respectively.

10: Simulate the Markov-modulated Stock prices $S^k(T)$

$$S^k(T) = S_0 \exp\left\{(P^k_{(0,T)} - V^k_{(0,T)}/2) + \sqrt{V^k_{(0,T)}}z\right\} \tag{4.17}$$

where z is a random number from the standard normal distribution.

11: Calculate the discounted final payoff for each simulation

$DP[k] \leftarrow \exp\{-P^k_{(0,T)}\} \max(S^k(T) - K, 0)$

12: **end for**

13: Calculate the Monte Carlo option price

$$C = \frac{1}{n_r} \sum_{k=1}^{n_r} DP[k]$$

presented the fundamental idea of using FFT method to price European-style options under the variance gamma model.

For the sake of completeness, we present a brief introduction of the application of fast Fourier transform in the sequel.

The Fourier transform of a piecewise continuous real function f, which satisfies the integrability condition

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty, \tag{4.18}$$

is defined as

$$\hat{f}(u) = \int_{-\infty}^{\infty} e^{iux} f(x)dx, \tag{4.19}$$

for $u \in \mathbb{R}$, where $i = \sqrt{-1}$. The inverse transform of f is defined by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \hat{f}(u)du. \tag{4.20}$$

Fourier transform of $C(k)$ can be obtain by assuming $f(x)$ is the density function of log-price ($S_T = S_0 e^{Z_T}$). The details of the model given in Section 2.2. Under the risk-neutral probability measure Q, European call option price at time $t = 0$ with maturity $T > 0$ and strike price $K > 0$ is given by

$$C(K) = \mathbb{E}\Big[\exp\Big(-\int_0^T r_t dt \Big)(S_T - K)^+ \Big], \tag{4.21}$$

where $r_t = r(t, X_t) = \langle r, X_t \rangle$ is the Markov-modulated interest rate. Define the modified log-strike as $k = \ln\left(\frac{K}{S_0}\right)$ to guarantee $k = 0$ will correspond to the at-the-money case for all possible conditions. Then (4.21) can be written as

$$C(k) = S_0 \mathbb{E}\Big[\exp\Big(-\int_0^T r_t dt \Big)(e^{Z_T} - e^k)^+ \Big]. \tag{4.22}$$

Fourier transform of the European call option price is

$$\begin{aligned}
\hat{C}(u) &= \int_{-\infty}^{\infty} e^{iuk} C(k)dk \\
&= \int_{-\infty}^{\infty} e^{iuk} S_0 \mathbb{E}\Big[\exp\Big(-\int_0^T r_t dt \Big)(e^{Z_T} - e^k)^+ \Big] dk \\
&= S_0 \int_{-\infty}^{\infty} e^{iuk} \int_{-\infty}^{\infty} \exp\Big(-\int_0^T r_t dt \Big)(e^y - e^k)^+ f(y)dydk \\
&= S_0 \int_{-\infty}^{\infty} \exp\Big(-\int_0^T r_t dt \Big) e^{iuk} \int_{-\infty}^{\infty} (e^y - e^k)^+ f(y)dydk \\
&= S_0 \int_{-\infty}^{\infty} \exp\Big(-\int_0^T r_t dt \Big) f(y) \int_{-\infty}^{\infty} e^{iuk}(e^y - e^k)^+ dkdy
\end{aligned}$$

since the order of integration does not matter due to the Fubini's theorem. Last equation can be written in the form

$$\hat{C}(u) = S_0 \int_{-\infty}^{\infty} \exp\Big(-\int_0^T r_t dt \Big) f(y) \int_{-\infty}^{y} e^{iuk}(e^y - e^k)^+ dkdy \tag{4.23}$$

since $\int_y^{\infty} e^{iuk}(e^y - e^k)dk = 0$. Since the inner integral of (4.23) does not decay to 0 as k goes to $-\infty$, Carr and Madan [10] suggested to use a damping factor parameter

ρ to define the modified call price[1] as

$$c(k) = e^{\rho k}\frac{C(k)}{S_0}, \quad -\infty < k < \infty. \tag{4.24}$$

The Fourier transform of the dampened call option price can be written as

$$\begin{aligned}
\hat{c}(u) &= \int_{-\infty}^{\infty} e^{iuk} c(k) dk \\
&= \int_{-\infty}^{\infty} e^{iuk} e^{\rho k} \frac{C(k)}{S_0} dk \\
&= \int_{-\infty}^{\infty} e^{iuk} e^{\rho k} \mathbb{E}\Big[\exp\Big(-\int_0^T r_t dt\Big)(e^{Z_T} - e^k)^+\Big] dk \\
&= \mathbb{E}\Big[\int_{-\infty}^{\infty} e^{iuk} e^{\rho k} \mathbb{E}\Big[\exp\Big(-\int_0^T r_t dt\Big)(e^{Z_T} - e^k)^+ \Big| \mathcal{F}_T^X\Big] dk\Big] \\
&= \mathbb{E}\Big[\int_{-\infty}^{\infty} e^{iuk} e^{\rho k} \int_{-\infty}^{\infty} \exp\Big(-\int_0^T r_t dt\Big)(e^y - e^k)^+ f_{\mathcal{F}_T^X}(y) dy dk\Big]
\end{aligned}$$

By using the tower property of conditional expectation [2]

$$\begin{aligned}
\hat{c}(u) &= \mathbb{E}\Big[\int_{-\infty}^{\infty} e^{iuk} e^{\rho k} \mathbb{E}\Big[\exp\Big(-\int_0^T r_t dt\Big)(e^{Z_T} - e^k)^+ \Big| \mathcal{F}_T^X\Big] dk\Big] \\
&= \mathbb{E}\Big[\int_{-\infty}^{\infty} e^{iuk} e^{\rho k} \int_{-\infty}^{\infty} \exp\Big(-\int_0^T r_t dt\Big)(e^y - e^k)^+ f_{\mathcal{F}_T^X}(y) dy dk\Big] \\
&= \mathbb{E}\Big[\int_{-\infty}^{\infty} e^{iuk} e^{\rho k} \exp\Big(-\int_0^T r_t dt\Big) \int_{k}^{\infty} (e^y - e^k) f_{\mathcal{F}_T^X}(y) dy dk\Big] \\
&= \mathbb{E}\Big[\int_{-\infty}^{\infty} \exp\Big(-\int_0^T r_t dt\Big) f_{\mathcal{F}_T^X}(y) \int_{k}^{\infty} e^{iuk} e^{\rho k}(e^y - e^k) dk dy\Big] \\
&= \mathbb{E}\Big[\int_{-\infty}^{\infty} \exp\Big(-\int_0^T r_t dt\Big) f_{\mathcal{F}_T^X}(y) \int_{k}^{\infty} e^{(iu+\rho)k} e^y - e^{(iuk+\rho+1)k} dk dy\Big] \\
&= \mathbb{E}\Big[\int_{-\infty}^{\infty} \exp\Big(-\int_0^T r_t dt\Big) f_{\mathcal{F}_T^X}(y) \Big(\frac{e^{(iu+\rho+1)y}}{iu+\rho} - \frac{e^{(iu+\rho+1)y}}{iu+\rho+1}\Big) dy\Big] \\
&= \mathbb{E}\Big[\exp\Big(-\int_0^T r_t dt\Big)\Big(\frac{\phi_{\mathcal{F}_T}(u - i(1+\rho))}{iu+\rho} - \frac{\phi_{\mathcal{F}_T}(u - i(1+\rho))}{iu+\rho+1}\Big)\Big] \\
&= \frac{\mathbb{E}\big[\exp\big(-\int_0^T r_t dt\big)\phi_{\mathcal{F}_T}(u - i(1+\rho))\big]}{(iu+\rho)(iu+\rho+1)}
\end{aligned} \tag{4.25}$$

where $f_{\mathcal{F}_T^X}(y)$ is the conditional density function of y given $\mathcal{F}_T$ and

$$\phi_{\mathcal{F}_T}(u) = \mathbb{E}[e^{iuY} | \mathcal{F}_T] = \int_{-\infty}^{\infty} e^{iuy} f_{\mathcal{F}_T^X}(y) dy. \tag{4.26}$$

[1] Positive damping parameter for call options and negative damping parameter for put options are considered.
[2] If $\mathcal{H} \subseteq \mathcal{G}$, then $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{G}]$, a.s.

The option pricing formula can be written by using the inverse Fourier transform formula as follows:

$$C(k) = \frac{e^{-\rho k} S_0}{2\pi} \int_{-\infty}^{\infty} e^{-iuk} \hat{c}(u) du \tag{4.27}$$

$$= \frac{e^{-\rho k} S_0}{\pi} \int_{0}^{\infty} e^{-iuk} \hat{c}(u) du. \tag{4.28}$$

Thus, in order to calculate the option pricing formula the distribution of the log-price is required to calculate the expectation in (4.25). Under probability measure $\mathbb{Q}$,

$$S_T = S_0 \exp\left\{ \int_0^T \left(\mu_s - \tfrac{1}{2}\sigma_s^2\right) ds + \int_0^T \sigma_s dW_s \right\}. \tag{4.29}$$

Define the logarithmic returns as $Z(T) = \log(S_T/S_0)$ during the time horizon $[0, T]$. $Z(T)$ is normally distributed conditional on $\mathcal{F}_T$ with the following mean and variance

$$\mathbb{E}[Z(T)|\mathcal{F}_T] = \int_0^T \left(\mu_s - \tfrac{1}{2}\sigma_s^2\right) ds \tag{4.30}$$

and

$$\text{Var}[Z(T)|\mathcal{F}_T] = \int_0^T \sigma_s^2 dt. \tag{4.31}$$

Then the characteristic function of Z_T can be written in terms of the following predefined notations:

$$L_{t,T} = \int_t^T \langle \mu, X_s \rangle ds = \sum_{i=1}^{N} \mu_i J_i(t, T), \tag{4.32}$$

$$P_{t,T} = \int_t^T \langle r, X_s \rangle ds = \sum_{i=1}^{N} r_i J_i(t, T), \tag{4.33}$$

$$V_{t,T} = \int_t^T \langle \sigma, X_s \rangle^2 ds = \sum_{i=1}^{N} \sigma_i^2 J_i(t, T). \tag{4.34}$$

The characteristic function is therefore obtained as

$$\phi_{\mathcal{F}_T}(u) = \exp\left(iu(L_{0,T} - \frac{1}{2}V_{0,T}) - \frac{1}{2}u^2 V_{0,T} \right), \tag{4.35}$$

which yields to

$$
\begin{aligned}
\hat{c}(u) &= \frac{\mathbb{E}\big[\exp\big(-\int_0^T r_t dt\big)\phi_{\mathcal{F}_T}(u-i(1+\rho))\big]}{(iu+\rho)(iu+\rho+1)} \\
&= \frac{\mathbb{E}\bigg[\exp\big(-P_{0,T}\big)\exp\bigg(iu+(1+\rho)(LV_{0,T}(-\frac{1}{2})-\frac{1}{2}(u-i(1+\rho))^2V_{0,T}\bigg)\bigg]}{(iu+\rho)(iu+\rho+1)} \\
&= \frac{\mathbb{E}\bigg[\exp\bigg((1+\rho)\bigg(LV_{0,T}(\frac{1}{2})\bigg)-P_{0,T}-\frac{1}{2}u^2V_{0,T}+iu\bigg(LV_{0,T}(\frac{1}{2}+\rho\bigg)\bigg)\bigg)\bigg]}{(iu+\rho)(iu+\rho+1)},
\end{aligned}
$$

where

$$
LV_{0,T}(\alpha) = L_{0,T} + \alpha V_{0,T}.
$$

Still, we could not calculate this expression without knowing the values of $L_{0,T}$, $V_{0,T}$, and $P_{0,T}$ which depend on the occupation time of the Markov chain.

Recall that we define the occupation time of the Markov chain as follows:

$$
J_i(0,T) = \int_0^T \langle X_u, e_i\rangle du = \int_0^T \mathbb{1}_{\{X(u)=i\}} du. \tag{4.36}
$$

Hence $\sum_{i=1}^N J_i(0,T) = T$, and

$$
L_{0,T} = \sum_{i=1}^{N-1}(\mu_i - \mu_N)J_i(0,T) + \mu_N T, \tag{4.37}
$$

$$
V_{0,T} = \sum_{i=1}^{N-1}(\sigma_i^2 - \sigma_N^2)J_i(0,T) + \sigma_N^2 T, \tag{4.38}
$$

$$
P_{0,T} = \sum_{i=1}^{N-1}(r_i - r_N)J_i(0,T) + r_N T. \tag{4.39}
$$

It follows that

$$
\begin{aligned}
\hat{c}(u) &= \frac{\mathbb{E}\bigg[\exp\bigg((1+\rho)\bigg(LV_{0,T}(\frac{1}{2}\rho)\bigg)-P_{0,T}-\frac{1}{2}u^2V_{0,T}+iu\bigg(LV_{0,T}(\frac{1}{2}+\rho\bigg)\bigg)\bigg)\bigg]}{(iu+\rho)(iu+\rho+1)} \\
&= \frac{\exp(B(u)T)\mathbb{E}\bigg[\exp\bigg(i\sum_{j=1}^{N-1}A(u,j)J_j(0,T)\bigg)\bigg]}{(iu+\rho)(iu+\rho+1)},
\end{aligned} \tag{4.40}
$$

where, for $j = 1, \ldots, N-1$,

$$\begin{aligned}
A(u,j) =& \left[(\mu_j - \mu_N) + \left(\frac{1}{2} + \rho\right)(\sigma_j^2 - \sigma_N^2)\right]u + \frac{1}{2}u^2(\sigma_j^2 - \sigma_N^2)i \\
&+ \left[(r_j - r_N) - (1+\rho)(\mu_j - \mu_N) - \frac{1}{2}\rho(1+\rho)(\sigma_j^2 - \sigma_N^2)\right]i, \\
B(u) =& iu\left[\mu_N + \left(\frac{1}{2} + \rho\right)\sigma_N^2\right] - \frac{1}{2}u^2\sigma_N^2 + (1+\rho)\mu_N - r_N + \frac{1}{2}\rho(1+\rho)\sigma_N^2.
\end{aligned}$$

Consequently, we can obtain the solution by using the characteristic function of the occupation times of the Markov chain. Recall the following lemma suggested by Elliot et al. [12].

Lemma 4.1.1. *Consider the N state Markov switching model. Let*

$$J(t,T) := (J_1(t,T), J_2(t,T), \ldots, J_N(t,T))$$

denote the vector of occupation times and D denote a diagonal matrix consisting of the elements in the vector $\nu := (\nu_1, \nu_2, \ldots, \nu_N)^\top \in \mathbb{R}^N$ as its diagonal, i.e. $D = \operatorname{diag}(\nu)$. Then, the conditional characteristic function of $J(t,T)$ is given by

$$\begin{aligned}
\mathbb{E}\left[\exp(i\langle \nu, J(t,T)\rangle)\middle|\mathcal{F}_t^Z\right] &= \mathbb{E}[\exp\left(i\sum_{j=1}^{N}\nu_k, J_j(t,T)\right)|\mathcal{F}_t^Z] \\
&= \langle \exp[(Q+iD)(T-t)]X_t, \mathbf{1}\rangle \qquad (4.41)
\end{aligned}$$

where $i = \sqrt{-1}$ and $\mathbf{1} = (1,1,\ldots,1) \in \mathbb{R}^N$.

The proof of the lemma is given in Section A.1.

The result in the lemma is used in the (4.40) by setting $\nu_j = A(u,j)$. This yields to

$$\begin{aligned}
\hat{c}(u) &= \frac{\exp(B(u)T)\mathbb{E}\left[\exp\left(i\sum_{j=1}^{N-1}A(u,j)J_j(0,T)\right)\right]}{(iu+\rho)(iu+\rho+1)} \\
&= \frac{\exp(B(u)T)\langle \exp[(Q+iD^*)(T)]X_0, \mathbf{1}\rangle}{(iu+\rho)(iu+\rho+1)},
\end{aligned}$$

where $D^* = \operatorname{diag}(A(u))$ with $A(u) = (A(u,1), A(u,2), \ldots, A(u,N))^\top \in \mathbb{R}^N$. Consequently by using the inverse Fourier transform option price is obtained:

$$\begin{aligned}
C(k) &= \frac{e^{-\rho k}S_0}{2\pi}\int_{-\infty}^{\infty} e^{-iuk}\hat{c}(u)du \\
&= \frac{e^{-\rho k}S_0}{\pi}\int_{0}^{\infty} e^{-iuk}\hat{c}(u)du \\
&= \frac{e^{-\rho k}S_0}{\pi}\int_{0}^{\infty} e^{-iuk}\frac{\exp(B(u)T)\langle \exp[(Q+iD^*)(T)]X_0, \mathbf{1}\rangle}{(iu+\rho)(iu+\rho+1)}du.
\end{aligned}$$

An approximation for $C(k)$ can be found by numerical integration: for simplicity take $u_j = j\Delta_u$, $j = 0, 1, \ldots, N-1$, where Δ_u is the grid size in the variable u. Then, the approximation can be written by the following summation

$$C(k) \approx \frac{e^{-\rho k} S_0}{\pi} \sum_{j=0}^{N-1} e^{-iu_j k} \hat{c}(u_j)\Delta_u. \tag{4.42}$$

Let Δ_k be the grid size in modified log-strike k as follows

$$k_l = \left(l - \frac{N}{2}\right)\Delta_k, \tag{4.43}$$

for $l = 0, 1, \ldots, N-1$. Then

$$c(k_l) \approx \frac{e^{-\rho l \Delta_k} e^{\rho(N/2)\Delta_k} S_0}{\pi} \sum_{j=0}^{N-1} e^{-ijl\Delta_u\Delta_k} e^{ij\Delta_u(N/2)\Delta_k} \hat{c}(j\Delta_u)\Delta_u, \tag{4.44}$$

for $l = 0, 1, \ldots, N-1$. We note from the study of Carr and Madan [10] that

$$\Delta_u \Delta_k = \frac{2\pi}{N}. \tag{4.45}$$

Hence,

$$c(k_l) \approx \frac{e^{-\rho l \Delta_k} e^{\rho(N/2)\Delta_k} S_0 \Delta_u}{\pi} \sum_{j=0}^{N-1} e^{-ijl(2\pi/N)} e^{ij\pi} \hat{c}(j\Delta_u), \tag{4.46}$$

for $l = 0, 1, \ldots, N-1$. Among the parameters Δ_u, Δ_k and N only two can be chosen and the third will depend on that parameters by the restriction (4.45). As stated by Carr and Madan [10] choosing small Δ_k to obtain a fine grid for the integration requires to have relatively large Δ_u with few strikes lying in the desired region near the stock price. In order to increase the accuracy of the integration, we utilize Simpson's rule weightings into the summation to obtain

$$c(k_l) \approx \frac{e^{-\rho l \Delta_k} e^{\rho(N/2)\Delta_k} S_0 \Delta_u}{\pi} \sum_{j=0}^{N-1} e^{-ijl(2\pi/N)} e^{ij\pi} \hat{c}(j\Delta_u) w(j), \tag{4.47}$$

for $l = 0, 1, \ldots, N-1$ where the weights $w(j)$ are

$$w(j) = \begin{cases} \frac{1}{3}, & \text{if } j = 0, \\ \frac{4}{3}, & \text{if } j \text{ is odd}, \\ \frac{2}{3}, & \text{if } j \text{ is even}. \end{cases} \tag{4.48}$$

4.1.4 Numerical Implementation

To assess the performance of the proposed approach, we compare our results with the ones existing in the literature. First we consider the example given by Liu et al. [43]and replicate their study. We also compare computation times of Monte Carlo, FFT and proposed numerical integration method results. Second, we consider the Markov-modulated European put option pricing example given by Zeng et al. [62] for the two-state and three-state cases. We also show in Section 4.1.5 how the infinite sum given in the main pricing formula may be represented by the finite sum of acceptable value.

4.1.4.1 Two-State

Example 4.1.1. We investigate the same problem given in Liu, Zhang and Yin [43]. The considered option has the following maturity $T = 1$, and initial asset price $S(0) = 100$. The two-state Markov chain model is considered with the parameters $q_{12} = 20, q_{21} = 30, \mu_1 = r_1 = 0.05, \mu_2 = r_2 = 0.1, \sigma_1 = 0.5, \sigma_2 = 0.3$. Hence, the generator matrix is

$$Q = \begin{bmatrix} -20 & 20 \\ 30 & -30 \end{bmatrix}.$$

We calculated the option price by using the formula given in (4.12). Option prices are given in Table 4.1 and Table 4.2.

Table 4.1: Option prices, $X(0) = 1$

$\ln(K/S_0)$ (K)	Monte Carlo	FFT	Numerical Integration (Pedler)
-0.3 **(74.082)**	34.9863 ± 0.2599	34.7736	34.7735
-0.2 **(81.873)**	29.5938 ± 0.2463	29.6558	29.6958
-0.1 **(90.484)**	24.7160 ± 0.2316	24.7635	24.7634
0 (100)	20.1046 ± 0.2159	20.1160	20.1160
0.1 (110.517)	15.9563 ± 0.1990	15.8806	15.8806
0.2 (122.140)	12.2101 ± 0.1795	12.1569	12.1570
0.3 (134.986)	9.1409 ± 0.1577	9.0059	9.0059

Numerical integral results obtained with the distribution function proposed by Pedler [53] are consistent with the results obtained by Monte Carlo and FFT methods.

Table 4.2: Option prices, $X(0) = 2$

$\ln(K/S_0)$ (K)	**Monte Carlo**	**FFT**	**Numerical Integration (Pedler)**
$\mathbf{-0.3\ (74.082)}$	34.6853 ± 0.2539	34.747	34.7416
$\mathbf{-0.2\ (81.873)}$	29.8875 ± 0.2451	29.6423	29.6423
$\mathbf{-0.1\ (90.484)}$	24.6601 ± 0.2304	24.6886	24.6884
$\mathbf{0\ (100)}$	20.004 ± 0.2189	20.0224	20.0224
$\mathbf{0.1\ (110.517)}$	15.8254 ± 0.1970	15.7735	15.7735
$\mathbf{0.2\ (122.140)}$	12.1263 ± 0.1772	12.0433	12.0434
$\mathbf{0.3\ (134.986)}$	8.7514 ± 0.1518	8.8932	8.8932

We also compare the computation times of Monte Carlo, FFT and numerical integration approaches and the results are given in the Table 4.3. It can be observed that the proposed method requires less computation time according to the the Monte Carlo and FFT methods.

Table 4.3: Computation Times, $X(0) = 1$

$\ln(K/S_0)$ (K)	**Monte Carlo**	**FFT**	**Numerical Integration (Pedler)**
$\mathbf{-0.3\ (74.082)}$	270.222515	0.390784	0.002510

4.1.4.2 Three-State

Example 4.1.2 (Validation of the Formula)**.** In order to assess the performances of the proposed pricing method, we consider a regime-switching dynamic driven by a three-state Markov chain, i.e., $N = 3$. We evaluate the pricing formula given in (4.3) by using standard integration routines. We implemented our valuation model in the MATLAB® environment, using the build-in integration functions.

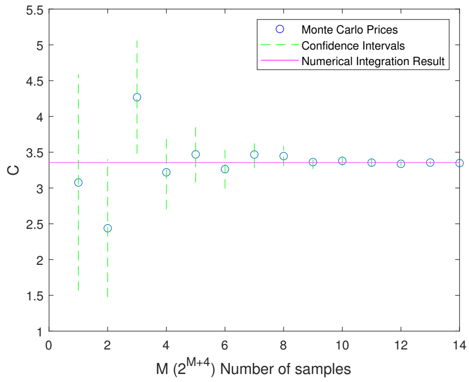
We consider the Markov-modulated European put option pricing example given by Zeng et al. [62]. They compared the Monte Carlo and finite difference method results. The parameters are $S_0 = 36$, $K = 40$, $T = 1$, (they considered constant interest rate parameters $r = 0.1$), the interest rates are $(0.1, 0.1)$, $(0.1, 0.1, 0.1)$ and the volatilities are $(0.15, 0.25)$, $(0.15, 0.25, 0.35)$ for the two-states, three-states respectively. The transition rate probabilities are $q_{ji} = 1, j \neq i$. The result are given in Table 4.4.

We can conclude that our results are consistent with the results given Zeng et al. [62].

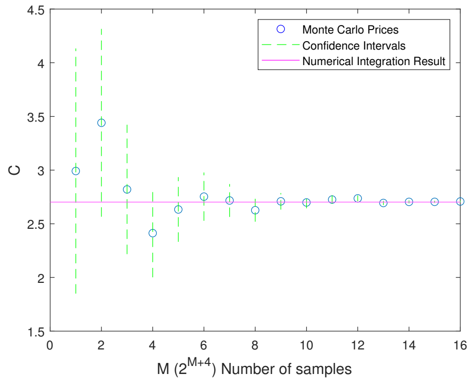
Table 4.4: Comparison of the results

	MC (Zeng et al.) (500,000 simulation)	MC (Our Results) (500,000 simulation)	Proposed Method
$P_{X_0=1}$ (2-state)	2.7022±0.0015	2.7010±0.096	2.7023
$P_{X_0=2}$ (2-state)	3.3203±0.0017	3.3259±0.0118	3.3203
$P_{X_0=1}$ (3-state)	3.3562±0.0016	3.3524±0.0118	3.3566
$P_{X_0=2}$ (3-state)	3.7653±0.0018	3.7596±0.0132	3.7643
$P_{X_0=3}$ (3-state)	4.2508±0.0020	4.2548±0.0147	4.2511

Example 4.1.3. The convergence graphs of Monte Carlo approach for two-sate and three state are given in Figure 4.5b and Figure 4.5a, respectively.



(a) Convergence Graph (Three-state)



(b) Convergence Graph (Two-state)

Figure 4.5b and Figure 4.5a show the 96% confidence intervals for the number of replications used by the Monte Carlo method. The dots and the pink line represent the Monte Carlo prices and numerical integration results, respectively. It can be observed from figures that roughly after 2^{14} samples, the Monte Carlo results converge to the numerical integration results. Table 4.5 presents the Monte Carlo results with 96% confidence interval and proposed numerical integral results for two and three-state cases by considering different numbers of states. The calculation times for the Monte Carlo and proposed methods are listed in columns four and six, respectively. The proposed method requires less computation time than the Monte Carlo method.

Table 4.5: Comparison of the results

States	**Number of samples**	**Monte Carlo (MC) Price with CI**	**Time (MC)**	**Numerical Integration (NI)**	**Time (NI)**
2-State	100,000	2.7001 CI = [2.6780 2.722]	52.061607	2.7023	0.282781
	200,000	2.6978 CI = [2.6822 2.7134]	238.337820		
	500,000	2.7010 CI = [2.6912 2.7109]	1601.545945		
3-State	100,000	3.3379 CI = [3.3109 3.3650]	148.334582	3.3566	3.368099
	200,000	3.3422 CI = [3.3231 3.3613]	485.029417		
	500,000	3.3668 CI = [3.3547 3.3790]	3593.297081		

4.1.5 Numerical Calculation Details

Example 4.1.4. We consider another numerical example is to illustrate the analysis for the number terms in the calculation of component function F given in (3.11). We calculate Euoropean call option prices with the initial stock price $S_0 = 26$, strike price $K = 40$, initial state of the Markov chain $X_0 = 1$, and the remaining parameters are given in Table 4.6.

Table 4.6: Parameter set for the calculation of Eureopen options under RS

	Transition Rate Matrix			**Model Parameters**	
States	**State 1**	**State 2**	**State 3**	**Interest Rate**	**Volatility**
State 1	-2	1	1	0.1	0.15
State 2	1	-2	1	0.2	0.25
State 3	1	1	-2	0.3	0.35

Let p denotes the number terms in the calculation of component function F. For different values of p and T the corresponding European call option prices are obtained. To show the convergence of the pricing formula we take the difference of prices for subsequent values of p for each T. The results are illustrated in Figure 4.6. Figure 4.7 illustrates the corresponding CDF values with respect to p values. From both figures we can observe that after the ninth term the calculations converges.

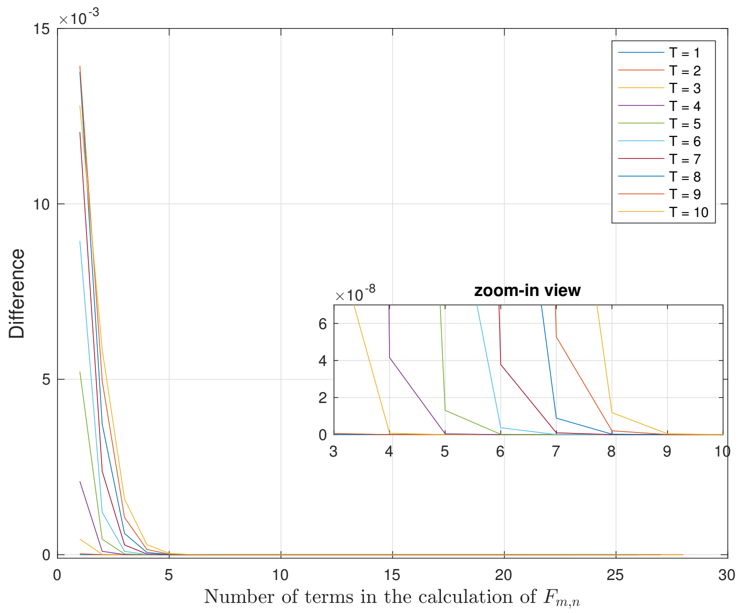


Figure 4.6: Number of terms vs difference between two subsequent term

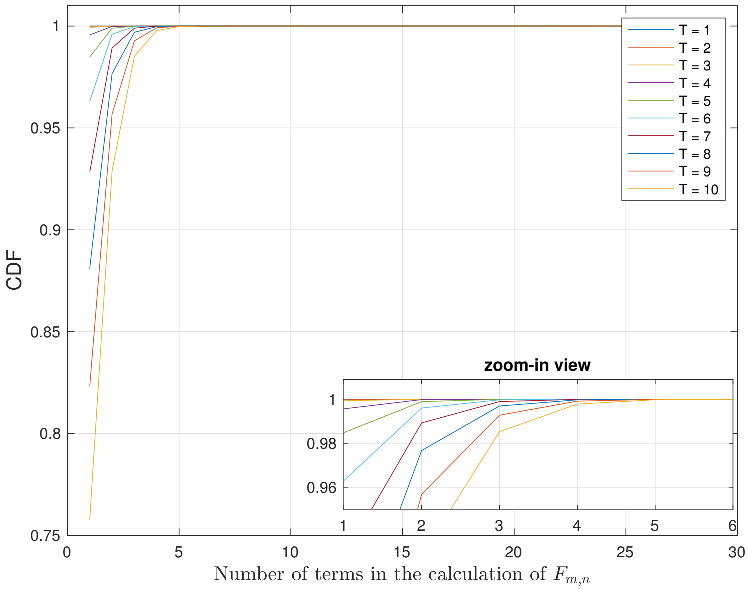


Figure 4.7: Number of terms vs CDF

4.1.6 Implied Volatility

Stochastic variability in the market parameters is not reflected in the Black-Scholes model. Hence, one of the drawback of the Black-Scholes model is that its failure to capture the volatility smile. The implied volatility of the underlying asset, rather than being constant, should change with respect to maturity and exercise price of option.

We take the Markov-modulated European (call or put) prices as observed prices and back out the implied volatility from Black-Scholes formula. We considered the parameters given in the Zeng et al. [62] as follows: $S_0 = 36$, $K = 40$, $T = 1$, (Zeng et al. considered constant interest rate parameters $r = 0.1$), the interest rates $(0.1, 0.1, 0.1)$ and the volatilities are $(0.15, 0.25, 0.35)$ for the three-states. The transition rate matrix is

$$Q = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

We plot the implied volatility against the strike price K and time the maturity as given in Figure 4.8

The unique form of the curve illustrates that the implied volatilities for out-of-the-money options are often higher than those of at-the-money options. The implied

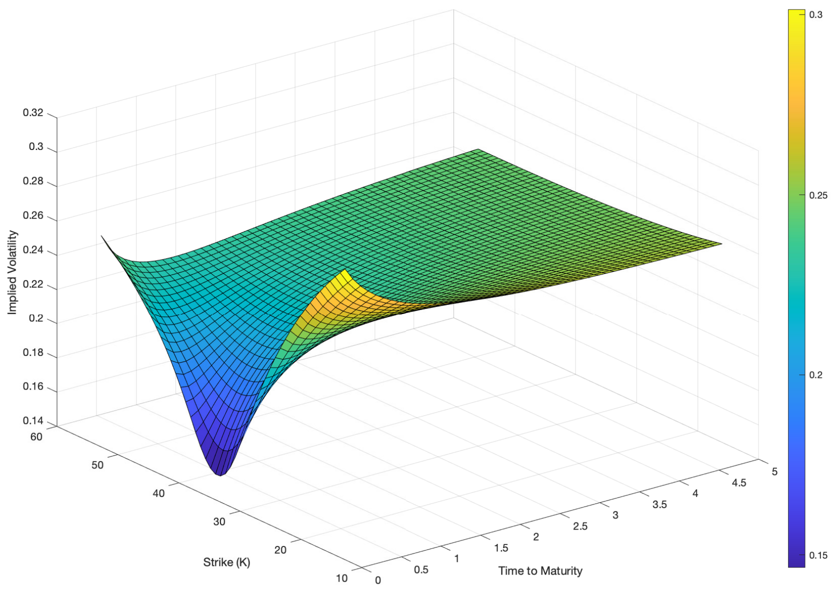


Figure 4.8: Implied Volatility Curves

volatility reaches its minimum at $K = 36$ (at money) and increases as K moves away from $K = 36$. This is the well-known volatility smile phenomenon in stock options, and it can be observed that the regime-switching models can exhibit this behavior.

4.2 Calculation of the Greeks

In this thesis we focus only on the first order Greeks that are represented by the first derivatives with respect to the corresponding parameters. The second or higher order Greeks can be obtained by following the same methodology.

The risk can be controlled and managed by the use of the sensitivity parameter. Option market makers and traders at financial institutions often make use of Greeks as a risk management tool for option positions. Greeks can be used to quantify the risk associated with individual stock options.

In the sequel, the formulae of the Greeks, namely, Delta, Gamma, Rho and Vega are presented for regime-switching European option. To illustrate the behavior of the Greeks numerical example is given.

Corollary 4.2.1. *Let C_Δ, C_Γ denote the Delta and Gamma of an attainable Euro-*

pean call option with strike price K and payoff $(S_T - K)^+$ at the maturity date T. Similarly let C_{ρ_i} and C_{ν_i} denote the state dependent Rho and Vega of the corresponding option for state $i \in \{1,2,3\}$. Formulae for the regime-switching European call option Greeks are given as follows:

(i) Delta and Gamma:

$$C_G(t) = \int_0^{T-t}\int_0^{T-t-J_3} C_G^{BS}\left(S, K, \frac{p_{X(t)}(J_2,J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2,J_3)}{T-t}}\right) \times f_{X(t)}(J_2,J_3)dJ_2dJ_3$$

for $G \in \{\Delta, \Gamma\}$.

(ii) Rho for $i \in \{1,2,3\}$:

$$C_{\rho_i}(t) = \int_0^{T-t}\int_0^{T-t-J_3} J_i K e^{-p_{X(t)}(J_2,J_3)}\Phi(d_2) f_{X(t)}(J_2,J_3)dJ_2dJ_3,$$

(iii) Vega for $i \in \{1,2,3\}$:

$$C_{\nu_i}(t) = \int_0^{T-t}\int_0^{T-t-J_3} K e^{-p_{X(t)}(J_2,J_3)}\phi(d_2)\frac{J_i\sigma_i}{\sqrt{\bar{\sigma}}} f_{X(t)}(J_2,J_3)dJ_2dJ_3. \tag{4.49}$$

where $\bar{\sigma} = \sigma_1^2(T - J_2 - J_3) + \sigma_2^2 J_2 + \sigma_3^2 J_3$.

Proof. The price at time t of a European call option with payoff $(S_T - K)^+$ at maturity is given by

$$C(S,K,r,T-t,\sigma) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \tag{4.50}$$

where

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{(T-t)}}, \tag{4.51}$$

$$d_2 = d_1 - \sigma\sqrt{(T-t)} = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{(T-t)}} \tag{4.52}$$

By substituting the corresponding regime-switching parameters the formula given by (4.50) becomes

$$C_G^{\mathrm{BS}}\left(S, K, \frac{p_{X(t)}(J_2,J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2,J_3)}{T-t}}\right) = S\Phi(d_1) - Ke^{-p_{X(t)}(J_2,J_3)}\Phi(d_2), \tag{4.53}$$

where

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + r_1(T - J_2 - J_3) + r_2 J_2 + r_3 J_3 + \frac{\sigma_1^2(T-J_2-J_3)+\sigma_2^2 J_2+\sigma_3^2 J_3}{2}}{\sqrt{\sigma_1^2(T - J_2 - J_3) + \sigma_2^2 J_2 + \sigma_3^2 J_3}}, \quad (4.54)$$

$$\begin{aligned} d_2 &= d_1 - \sqrt{\frac{\upsilon_{X(t)}(J_2, J_3)}{T-t}}\sqrt{(T-t)} && (4.55)\\ &= d_1 - \sqrt{\sigma_1^2(T - J_2 - J_3) + \sigma_2^2 J_2 + \sigma_3^2 J_3} && (4.56)\\ &= \frac{\ln\left(\frac{S}{K}\right) + r_1(T - J_2 - J_3) + r_2 J_2 + r_3 J_3 - \frac{\sigma_1^2(T-J_2-J_3)+\sigma_2^2 J_2+\sigma_3^2 J_3}{2}}{\sqrt{\sigma_1^2(T - J_2 - J_3) + \sigma_2^2 J_2 + \sigma_3^2 J_3}} && (4.57) \end{aligned}$$

Now, we state the following remark, which is useful in the proof of the main results:

Remark. (4.54) can be written as

$$\begin{aligned} d_1\sqrt{\sigma_1^2(T - J_2 - J_3) + \sigma_2^2 J_2 + \sigma_3^2 J_3} = \ln\left(\frac{S}{K}\right) &+ r_1(T - J_2 - J_3) + r_2 J_2 + r_3 J_3 \\ &+ \frac{\sigma_1^2(T - J_2 - J_3) + \sigma_2^2 J_2 + \sigma_3^2 J_3}{2} \end{aligned}$$

$$\begin{aligned} \Rightarrow \ln(S) - \ln(K) + r_1(T - J_2 - J_3) + r_2 J_2 + r_3 J_3 &= d_1\sqrt{\bar{\sigma}} - \frac{\bar{\sigma}}{2} \\ &= \frac{1}{2}(d_1^2 - (d_1 - \sqrt{\bar{\sigma}})^2) \end{aligned}$$

$$\begin{aligned} &\Rightarrow \ln(S) + \ln\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{d_1^2}{2} = \ln(K) - (\bar{r}) + \ln\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{d_2^2}{2} \\ &\Rightarrow S\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} = Ke^{-\bar{r}}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_2^2}{2}} \\ &\Rightarrow S\phi(d_1) = Ke^{-\bar{r}}\phi(d_2) \end{aligned}$$

where $\bar{\sigma} = \sigma_1^2(T - J_2 - J_3) + \sigma_2^2 J_2 + \sigma_3^2 J_3$ and $\bar{r} = r_1(T - J_2 - J_3) + r_2 J_2 + r_3 J_3$

First first we prove (i)

(i) Delta and Gamma:

$$\begin{aligned}
C_\Delta(t) &= \frac{\partial C(t)}{\partial S} \\
&= \frac{\partial}{\partial S}\int_0^{T-t}\int_0^{T-t-J_3} C^{\mathrm{BS}}\left(S, K, \frac{p_{X(t)}(J_2,J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2,J_3)}{T-t}}\right) \\
&\quad \times f_{X(t)}(J_2,J_3)dJ_2dJ_3 \\
&= \int_0^{T-t}\int_0^{T-t-J_3} \frac{\partial}{\partial S}C^{\mathrm{BS}}\left(S, K, \frac{p_{X(t)}(J_2,J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2,J_3)}{T-t}}\right) \\
&\quad \times f_{X(t)}(J_2,J_3)dJ_2dJ_3 \\
&= \int_0^{T-t}\int_0^{T-t-J_3} C_\Delta^{BS}\left(S, K, \frac{p_{X(t)}(J_2,J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2,J_3)}{T-t}}\right) \\
&\quad \times f_{X(t)}(J_2,J_3)dJ_2dJ_3,
\end{aligned}$$

where the result follows from the Leibniz integral rule since C^{BS} is continuous in T and S, and $f_{X(t)}(J_2, J_3)$ is continuous in J_2 and J_3 on $(0, T-t)$. The result for the C_Γ can be proved in a similar way.

(ii) Rho:

$$\begin{aligned}
C_{\rho_i}(t) &= \frac{\partial C(t)}{\partial r_1} \\
&= \frac{\partial}{\partial r_i}\int_0^{T-t}\int_0^{T-t-J_3} C^{\mathrm{BS}}\left(S, K, \frac{p_{X(t)}(J_2,J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2,J_3)}{T-t}}\right) \\
&\quad \times f_{X(t)}(J_2,J_3)dJ_2dJ_3 \\
&= \int_0^{T-t}\int_0^{T-t-J_3} \frac{\partial}{\partial r_i}C^{\mathrm{BS}}\left(S, K, \frac{p_{X(t)}(J_2,J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2,J_3)}{T-t}}\right) \\
&\quad \times f_{X(t)}(J_2,J_3)dJ_2dJ_3 \\
&= \int_0^{T-t}\int_0^{T-t-J_3} \left(S\phi(d_1)\frac{\partial d_1}{\partial r_i} + J_iKe^{-p_{X(t)}(J_2,J_3)}\Phi(d_2)\right. \\
&\quad \left. - Ke^{-p_{X(t)}(J_2,J_3)}\phi(d_2)\frac{\partial d_2}{\partial r_i}\right) f_{X(t)}(J_2,J_3)dJ_2dJ_3 \qquad (4.58) \\
&= \int_0^{T-t}\int_0^{T-t-J_3} J_iKe^{-p_{X(t)}(J_2,J_3)}\Phi(d_2)f_{X(t)}(J_2,J_3)dJ_2dJ_3 \qquad (4.59)
\end{aligned}$$

(iii) Vega:

$$
\begin{aligned}
C_{\nu_1 i}(t) &= \frac{\partial C(t)}{\partial \sigma_i} \\
&= \frac{\partial}{\partial \sigma_i} \int_0^{T-t} \int_0^{T-t-J_3} C^{\text{BS}} \left(S, K, \frac{p_{X(t)}(J_2, J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2, J_3)}{T-t}} \right) \\
&\quad \times f_{X(t)}(J_2, J_3) dJ_2 dJ_3 \\
&= \int_0^{T-t} \int_0^{T-t-J_3} \frac{\partial}{\partial \sigma_i} C^{\text{BS}} \left(S, K, \frac{p_{X(t)}(J_2, J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2, J_3)}{T-t}} \right) \\
&\quad \times f_{X(t)}(J_2, J_3) dJ_2 dJ_3 \qquad (4.60) \\
&= \int_0^{T-t} \int_0^{T-t-J_3} K e^{-p_{X(t)}(J_2,J_3)} \phi(d_2) \frac{J_i \sigma_i}{\sqrt{\bar{\sigma}}} f_{X(t)}(J_2, J_3) dJ_2 dJ_3. \qquad (4.61)
\end{aligned}
$$

The proof is completed. □

Remark. We were unable to write down explicitly the theta Greek due to complicated calculation, so the finite difference method is used to calculate the value of theta. In addition, we also used the finite difference method to validate other Greek calculations, namely, delta, gamma, rho, and vega, to check the accuracy of the proposed method. Therefore, the corresponding formulae for using the finite difference method are presented below.

We employ the finite difference approximation result to obtain numerical results for the calculation of the sensitivities. For simplicity we denote the European Call option pricing formula as follows

$$
C(S, d) := C \left(S, K, \frac{p_{X(t)}(J_2, J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2, J_3)}{T-t}} \right), \qquad (4.62)
$$

where d represents the vector of the other parameters as $d = \left[K, \frac{p_{X(t)}(J_2,J_3)}{T-t}, T-t \right]$.

Delta of the corresponding option can be calculated with the following central difference second-order approximation formula

$$
\begin{aligned}
C_\Delta(S, d) &= \frac{C(S+\delta, d) - C(S-\delta, d)}{2\delta} + \mathcal{O}(\delta^2) \\
&\approx \frac{C(S+\delta, d) - C(S-\delta, d)}{2\delta},
\end{aligned}
$$

where $\delta \neq 0$ is a small change in the stock price S.

The other first order sensitivities, namely, rho, vega, and theta can be obtained similarly by considering the small perturbations on the corresponding parameter, i.e., $r_i \pm \delta_{r_i}$, $\nu_i \pm \delta_{\nu_i}$ and $\tau \pm \delta_\tau$, respectively. Since gamma is a second order sensitivity it can be calculated by the following formula

$$
\begin{aligned}
C_\Gamma(S,d) &= \frac{C(S+\delta,d) - 2C(S,d) + C(S-\delta,d)}{\delta^2} + \mathcal{O}(\delta^2) \\
&\approx \frac{C(S+\delta,d) - 2C(S,d) + C(S-\delta,d)}{2\delta^2}.
\end{aligned}
$$

For theta, denote the European Call option pricing formula by $C(T-t, d^\Theta)$ where $d^\Theta = \left[S, K, \frac{p_{X(t)}(J_2,J_3)}{T-t}\right]$. Theta of the option formula is given as

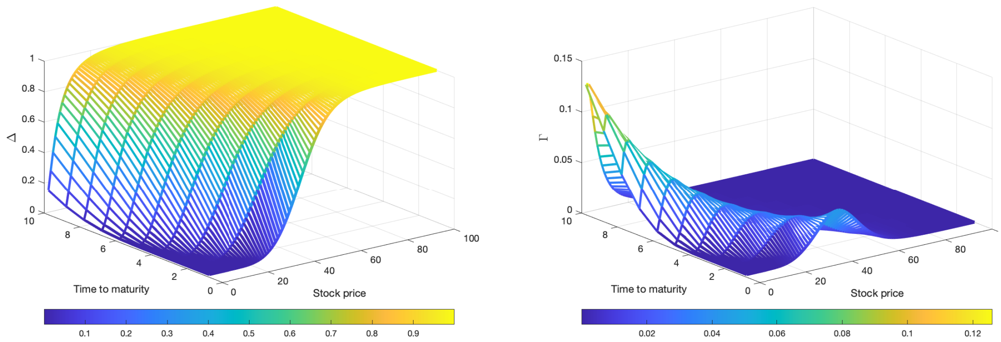
$$
\begin{aligned}
C_\Theta(T-t, d^\Theta) &= \frac{C(T-t+\delta t, d^\Theta) - C(T-t-\delta t, d^\Theta)}{2\delta t} + \mathcal{O}(\delta t^2) \\
&\approx \frac{C(T-t+\delta t, d^\Theta) - C(T-t-\delta t, d^\Theta)}{2\delta t}.
\end{aligned}
$$

Rho and Vega of the corresponding option can be obtained similarly by using the central difference formula.

4.2.1 Numerical Implementation

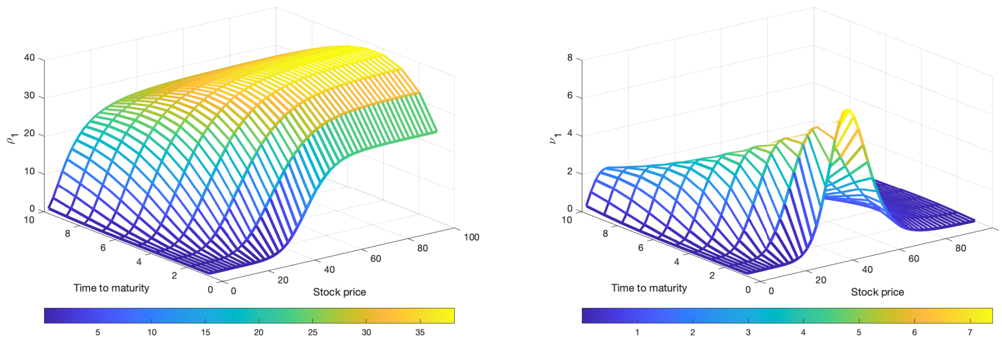
We consider a European call option with parameter set in which strike price $K = 50$, interest rates are $(0.1, 0.2, 0.3)$ and the volatility rates are $(0.15, 0.25, 0.35)$ for three-states. The transition rate probabilities are $q_{ji} = 1, j \neq i$. By using the formulae presented in Corollary 4.2.1 delta, gamma, rho and vega sensitivities are illustrated for different initial stock prices and maturities in Figure 4.9, Figure 4.10, Figure 4.11, and Figure 4.12 respectively. Theta is calculated via finite-difference method and illustrated in Figure 4.13.

It can be observed from the figures that Delta, Gamma, and Theta of the regime-switching European options are quite similar to the standard European option's Greeks since these Greeks are calculated as linear combinations of the standard European options. The Delta of a standard European call option, for example, is equal to $\Phi(d_1)$, and when the call option is deep in the money, $\Phi(d_1)$ approaches 1, but never exceeds



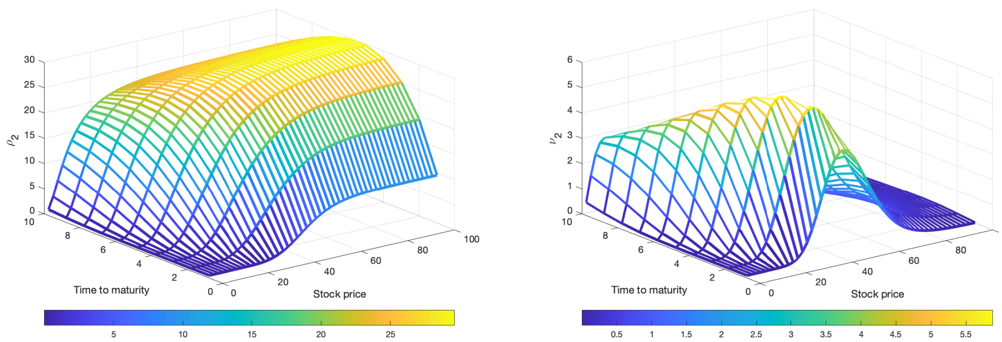
(a) Delta (b) Gamma

Figure 4.9: Delta and Gamma surfaces



(a) ρ_1 (b) ν_1

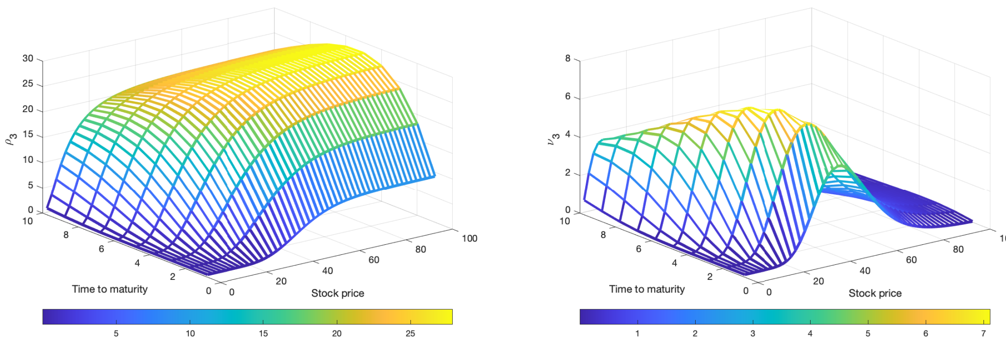
Figure 4.10: Rho and Vega surfaces: ρ_1, ν_1



(a) ρ_2 (b) ν_2

Figure 4.11: Rho and Vega surfaces: ρ_2, ν_2

1, because it is a cumulative distribution function. In the same way, the Gamma of the European option with a regime switch acts similar to the Gamma of the standard European option.



(a) ρ_3 (b) ν_3

Figure 4.12: Rho and Vega surfaces: ρ_3, ν_3

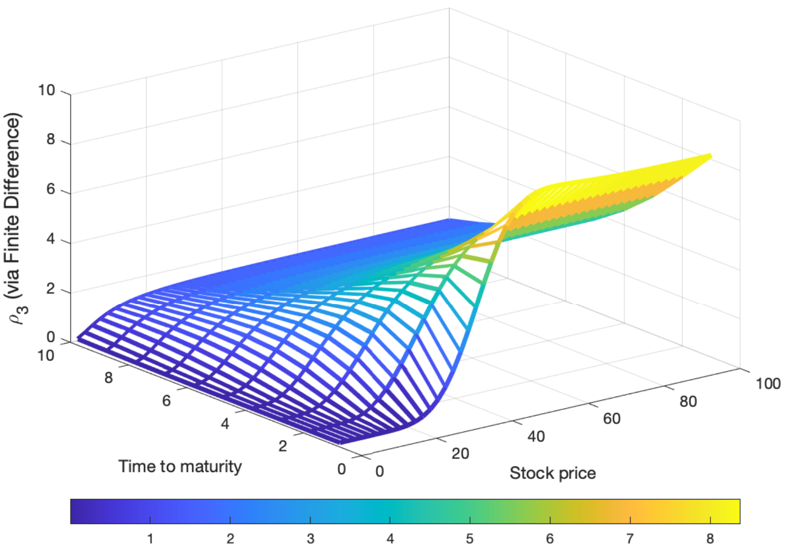


Figure 4.13: Theta surface

The Rho and Vega surfaces for the regime-switching options are given in Figure 4.10, Figure 4.11, and Figure 4.12 for states one, two, and three, respectively. The general behavior of the Rho and Vega Greeks is similar to that of the corresponding Greeks of the European option. However, it is observed that the Rho and Vega values for state one are higher than the values of the other states. Since for this example, we assume that the process starts from state one, the the influence of the associated Greeks for state one is greater than the others.

We compare the results obtained by the analytical solution with the results obtained by the finite difference method. The results of the finite difference method as well as the difference between the analytical solution and the finite difference results are plotted in Section A.4. Right-hand panels show the differences between the finite difference results and the left-hand panels show the results of the finite difference method. The difference between the two methods is in the order of 10^{-3}.

CHAPTER 5

EXTENSION TO BARRIER OPTIONS

In this chapter, we demonstrate how the proposed method can be applied to barrier option pricing problems. It is possible to express barrier options in terms of European options. As a consequence of this, the pricing formula for the regime-switching barrier options and corresponding Greeks can be derived by using this connection. Numerical examples are presented in order to show the accuracy of the proposed method. Lastly, we present the specific form of the regime-switching barrier option formula proposed by Elliot, Siu and Chan [18] for the three-state regime-switching model.

5.1 Barrier Option Pricing

Barrier options are one of the commonly traded exotic options in the OTC and FX markets. Barrier options are important tools in the sense that many exotic options can be decomposed into barrier options. The payoff of a barrier option depends on whether or not the stock price S reaches a pre-specified barrier level B during the life of the option. A barrier option can be an "out" or "in" option. An out option becomes worthless when the underlying asset price crosses a predefined barrier and the opposite is true for an in option. Merton [49] derived a closed-form solution for down-and-out call option. Rubinstein and Reiner [57] presented pricing formulas for all eight types of barrier options. The following formulae used in the valuation of the barrier options under Black-Scholes setting with no rebate [1]

$$A_1 = \Xi S \Phi(\Xi x_1) - \Xi K e^{-rT} \Phi(\Xi x_1 - \Xi\sigma\sqrt{T}) \tag{5.1}$$

$$A_2 = \Xi S \Phi(\Xi x_2) - \Xi K e^{-rT} \Phi(\Xi x_2 - \Xi\sigma\sqrt{T}) \tag{5.2}$$

$$A_3 = \Xi S \left(\frac{B}{S}\right)^{2(\mu+1)} \Phi(\Psi y_1) - \Xi K e^{-rT} \left(\frac{B}{S}\right)^{2\mu} \Phi(\Psi y_1 - \Psi\sigma\sqrt{T}) \tag{5.3}$$

$$A_4 = \Xi S \left(\frac{B}{S}\right)^{2(\mu+1)} \Phi(\Psi y_2) - \Xi K e^{-rT} \left(\frac{B}{S}\right)^{2\mu} \Phi(\Psi y_2 - \Psi\sigma\sqrt{T}) \tag{5.4}$$

where

$$x_1 = \frac{\ln(S/K)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T}, \quad x_2 = \frac{\ln(S/B)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T},$$

$$y_1 = \frac{\ln(B^2/SK)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T}, \quad y_2 = \frac{\ln(B/S)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T},$$

$$\mu = \frac{r - \sigma^2/2}{\sigma^2}.$$

Corresponding in and out barrier option formulae are given in Table 5.1.

5.1.1 Regime-Switching Barrier Option Pricing

In order to examine the formulae given in the literature for fixed parameters under the regime-switching model, another expectation with occupation time distributions should be taken over these formulae.

For example, we consider down-and-out (DO) call option, in which it pays $(S_T - K)^+$ at maturity if the stock price process does not go below a pre-specified barrier B up to maturity. Pricing formula for this option at time t is

$$C^B_{DO}(S(t), K, r, T-t, \sigma) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[(S_T - K)^+ \mathbb{1}_{\{\tau_B > T\}} | \mathcal{F}_t]$$

where τ_B is the first passage time defined as $\tau_B = \inf\{t > 0 : S_t \leq B\}$. Closed-form solution of European down-and-out call option for $B < K$ is presented by Merton [49], Rubinstein and Reiner [57] and Buchen [7] as follows under constant

[1] Rebate is the payment to the option holder if the barrier level is reached.

Table 5.1: Barrier option formulae

Barrier Type	**Barrrier Setting**	**Price**	Ξ, Ψ
Down-and-in call	$S_0 > B$	$C_{\text{DI}(K>B)} = A_3$ $C_{\text{DI}(K<B)} = A_1 - A_2 + A_4$	$\Xi = 1, \Psi = 1$ $\Xi = 1, \Psi = 1$
Up-and-in call	$S_0 < B$	$C_{\text{UI}(K>B)} = A_1$ $C_{\text{UI}(K<B)} = A_2 - A_3 + A_4$	$\Xi = 1, \Psi = -1$ $\Xi = 1, \Psi = -1$
Down-and-in put	$S_0 > B$	$P_{\text{DI}(K>B)} = A_2 - A_3 + A_4$ $P_{\text{DI}(K<B)} = A_1$	$\Xi = -1, \Psi = 1$ $\Xi = -1, \Psi = 1$
Up-and-in put	$S_0 < B$	$P_{\text{UI}(K>B)} = A_1 - A_2 + A_4$ $P_{\text{UI}(K<B)} = A_3$	$\Xi = -1, \Psi = -1$ $\Xi = -1, \Psi = -1$
Down-and-out call	$S_0 > B$	$C_{\text{DO}(K>B)} = A_1 - A_3$ $C_{\text{DO}(K<B)} = A_2 - A_4$	$\Xi = 1, \Psi = 1$ $\Xi = 1, \Psi = 1$
Up-and-out call	$S_0 < B$	$C_{\text{UO}(K>B)} = 0$ $C_{\text{UO}(K<B)} = A_1 - A_2 + A_3 - A_4$	$\Xi = 1, \Psi = -1$ $\Xi = 1, \Psi = -1$
Down-and-out put	$S_0 > B$	$P_{\text{DO}(K>B)} = A_1 - A_2 + A_3 - A_4$ $P_{\text{DO}(K<B)} = 0$	$\Xi = -1, \Psi = 1$ $\Xi = -1, \Psi = 1$
Up-and-out put	$S_0 < B$	$P_{\text{UO}(K>B)} = A_2 - A_3$ $P_{\text{UO}(K<B)} = A_1 - A_3$	$\Xi = 1, \Psi = -1$ $\Xi = 1, \Psi = -1$

parameter assumption

$$\begin{aligned} C^B_{\text{DO}}(S(t), K, r, T-t, \sigma) = {} & C^{\text{BS}}(S(t), K, r, T-t, \sigma) \\ & - \left(\frac{S(t)}{B}\right)^{(1-2r/\sigma^2)} C^{\text{BS}}\left(\frac{B^2}{S(t)}, K, r, T-t, \sigma\right) \end{aligned}$$

where $C^{\text{BS}}(S(t), K, r, T-t, \sigma)$ is the plain (vanilla) European call. Since we assume that the model's parameters depend on the underlying Markov chain, by taking another expectation with respect to the joint occupation time density function, we obtained the representation for the regime-switching down-and-out call option formula:

$$
\begin{aligned}
C_{\mathrm{DO}}(t) = & \int_0^{T-t}\int_0^{T-t-J_3} C_{DO}^{B}\Bigg(S(t), K, \frac{p_{X(t)}(J_2,J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2,J_3)}{T-t}}\Bigg) \\
& \times f_{X(t)}(J_2,J_3)dJ_2dJ_3 \\
= & \int_0^{T-t}\int_0^{T-t-J_3} \Bigg(C^{BS}\Bigg(S(t), K, \frac{p_{X(t)}(J_2,J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2,J_3)}{T-t}}\Bigg) \\
& - \left(\frac{S(t)}{B}\right)^{(1-2p_{X(t)}(J_2,J_3))/v_{X(t)}(J_2,J_3)} \\
& \times C^{BS}\Bigg(\frac{B^2}{S(t)}, K, \frac{p_{X(t)}(J_2,J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2,J_3)}{T-t}}\Bigg)\Bigg) \\
& \times f_{X(t)}(J_2,J_3)dJ_2dJ_3.
\end{aligned}
$$

where B is the barrier level and $f_{X(t)}(J_2, J_3)$ is the joint probability density of the occupation time of the three-state Markov chain given by Theorem 3.3.1 in Chapter 3.

5.1.2 Numerical Implementation

In this subsection in order validate the our result, we use the results presented by Hieber and Scherer [38] in the first example. In the second example a Monte Carlo based algorithm is used as a benchmark values in numerical experiment.

Example 5.1.1. Hieber and Scherer [38] used Brownian bridge concept to present an efficient Monte-Carlo method for barrier option pricing in regime-switching framework. They used the Brownian bridge to calculate probability of barrier crossing between regime switching times. In this method, they only needed to control the barrier hits at regime change times, so there was no need to simulate the underlying presence between regime change.

In this section, we compare the result of our proposed method with the results presented by Hieber and Scherer [38]. We considered the six different scenarios given in Hieber and Scherer, $\theta_j = (-q_{11}, -q_{22}, \sigma_1, \sigma_2, B, K)$ with $S_0 = 1, T = 1$ and interest rate is assumed to be constant and equal to 3%. The scenarios with corresponding parameters are shown in Table 5.2.

Table 5.2: The scenarios considered for calculating barrier option prices

	Parameters					
Scenarios	$-q_{11}$	$-q_{22}$	σ_1	σ_2	B	K
θ_1	0.8	0.6	0.15	0.25	0.6	0.6
θ_2	0.8	0.6	0.15	0.25	0.8	0.8
θ_3	0.8	0.6	0.15	0.25	0.9	0.9
θ_4	0.2	0.1	0.1	0.25	0.8	0.8
θ_5	1.0	0.6	0.1	0.25	0.8	0.8
θ_6	3.0	2.0	0.1	0.25	0.8	0.8

Table 5.3 gives the result of down-and-out call option prices under two-state regime framework. It can be observed that our findings are consistent with the results obtained by Hieber and Scherer [38].

Table 5.3: Comparison of the results

	Results					
Scenarios	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6
Hieber and Scherer	0.4177	0.2217	0.1186	0.2232	0.2233	0.2225
The proposed method	0.4177	0.2217	0.1179	0.2232	0.2222	0.2215

In the following example, we consider a three-state regime-switching framework with the assumption that all model parameters are governed by the same underlying Markov chain.

Example 5.1.2. In order to show the comparison of the method with the Monte Carlo simulation approach we consider an example with the initial stock price $S_0 = 100$, strike price $K = 100$, barrier level $B = 80$, maturity time $T = 1$, initial state $X_0 = 1$ and the remaining parameters given as:

Table 5.4: Parameter set for the calculation of barrier options under RS

	Transition Rate Matrix			**Model Parameters**	
States	**State 1**	**State 2**	**State 3**	**Interest Rate**	**Volatility**
State 1	-1	0.5	0.5	0.04	0.25
State 2	0.5	-1	0.5	0.06	0.35
State 3	0.5	0.5	-1	0.08	0.45

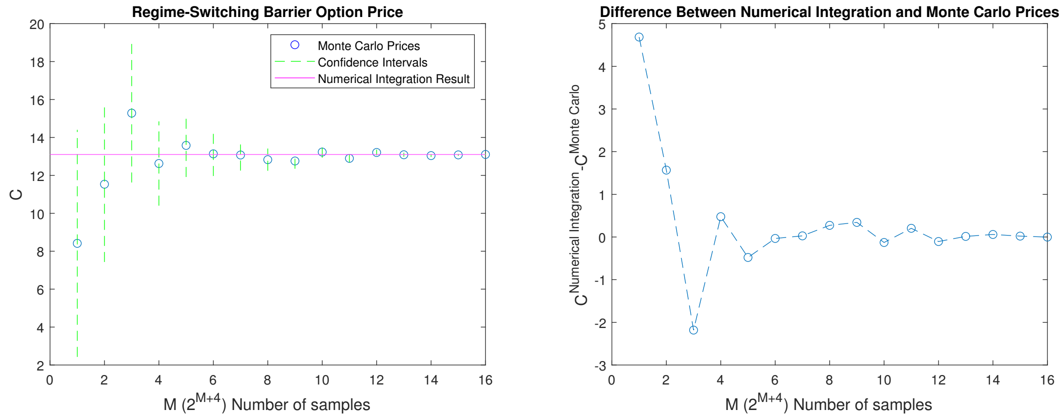


Figure 5.1: Convergence Graph

It can be observed in Figure 5.1 that the Monte Carlo results converge to the proposed method's result after the 2^{10} number of samples. The corresponding barrier option for different time to maturity and initial stock price values are presented in Figure 5.2.

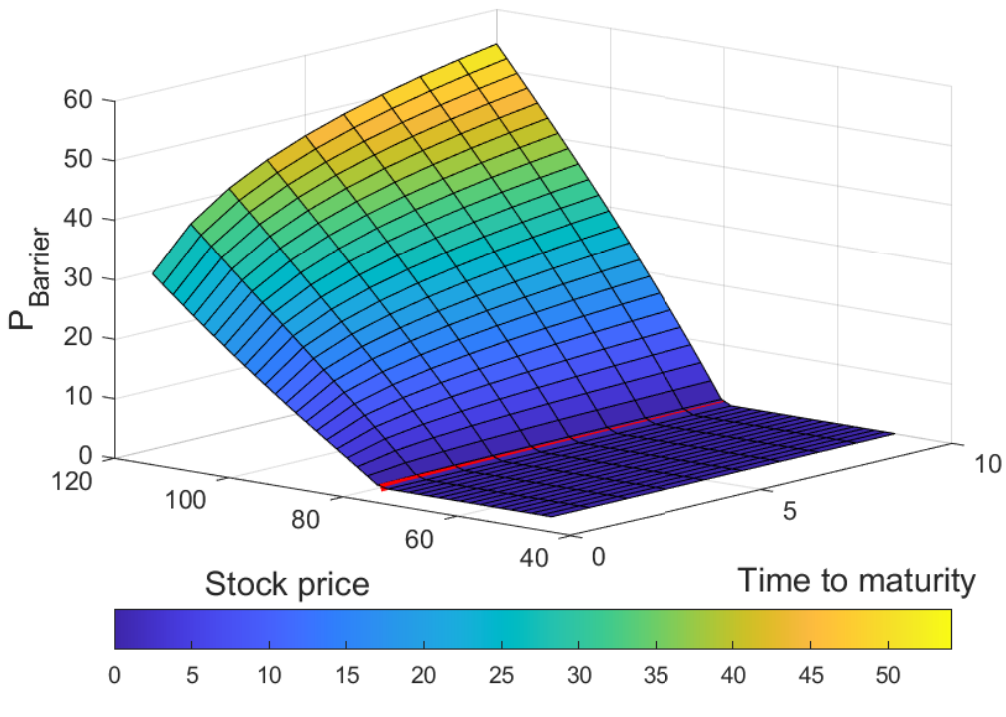


Figure 5.2: Price surface of the regime-switching barrier option

The red line indicates the barrier level, and the value of the down-and-out call option decreases as the stock price declines, and it is worthless below the barrier by definition.

5.2 Sensitivity Analysis for Barrier Options

In this section derivation of the Greeks for regime-switching down-and-out barrier options are investigated. Greeks for other type of barrier options can be obtained similarly.

We first consider the calculation of Delta of the associated regime-switching barrier option. It can be obtained by the same approach that we presented in Corollary 4.2.1 as

$$
\begin{aligned}
C^{\Delta}_{\text{DO}} =& \frac{\partial C_{\text{DO}}}{\partial S} \\
=& \frac{\partial}{\partial S} \int_0^{T-t} \int_0^{T-t-J_3} C^{B}_{DO}\left(S(t), K, \frac{p_{X(t)}(J_2, J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2, J_3)}{T-t}} \right) \\
& \times f_{X(t)}(J_2, J_3) dJ_2 dJ_3 \\
C^{\Delta}_{\text{DO}} =& \frac{\partial}{\partial S} \int_0^{T-t} \int_0^{T-t-J_3} \left(C^{BS}\left(S(t), K, \frac{p_{X(t)}(J_2, J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2, J_3)}{T-t}} \right) \right. \\
& - \left(\frac{S(t)}{B} \right)^{(1-2p_{X(t)}(J_2,J_3))/v_{X(t)}(J_2,J_3)} \\
& \left. \times C^{BS}\left(\frac{B^2}{S(t)}, K, \frac{p_{X(t)}(J_2, J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2, J_3)}{T-t}} \right) \right) \\
& \times f_{X(t)}(J_2, J_3) dJ_2 dJ_3.
\end{aligned}
$$

By using Leibniz rule we can write

$$
\begin{aligned}
C_{\text{DO}}^{\Delta} = & \int_0^{T-t}\int_0^{T-t-J_3} \frac{\partial}{\partial S}\left(C^{BS}\left(S(t), K, \frac{p_{X(t)}(J_2,J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2,J_3)}{T-t}}\right)\right. \\
& -\left(\frac{S(t)}{B}\right)^{(1-2p_{X(t)}(J_2,J_3))/v_{X(t)}(J_2,J_3)} \\
& \left.\times C^{BS}\left(\frac{B^2}{S(t)}, K, \frac{p_{X(t)}(J_2,J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2,J_3)}{T-t}}\right)\right) \\
& \times f_{X(t)}(J_2,J_3)dJ_2dJ_3 \\
= & \int_0^{T-t}\int_0^{T-t-J_3} C_{\Delta}^{BS}\left(S(t), K, \frac{p_{X(t)}(J_2,J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2,J_3)}{T-t}}\right) \\
& -\left\{\left(1-\frac{2\bar{r}}{\bar{\sigma}}\right)\left(\frac{S(t)}{B}\right)^{(-2\bar{r}/\bar{\sigma})}\frac{1}{B}\right. \\
& \times C^{BS}\left(\frac{B^2}{S(t)}, K, \frac{p_{X(t)}(J_2,J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2,J_3)}{T-t}}\right) \\
& \left.-\frac{B^2}{S(t)^2}\Phi(d_1^B)\left(\frac{S(t)}{B}\right)^{(1-2\bar{r}/\bar{\sigma})}\right\} \times f_{X(t)}(J_2,J_3)dJ_2dJ_3
\end{aligned}
$$

where

$$
d_1^B = \frac{\ln\left(\frac{B^2}{S(t)K}\right) + (\bar{r}+0.5\bar{\sigma})(T-t)}{\sqrt{\bar{\sigma}(T-t)}},
$$

$$
\bar{\sigma} = \frac{\sigma_1^2(T-t-J_2-J_3)+\sigma_2^2J_2+\sigma_3^2J_3}{T-t},
$$

and

$$
\bar{r} = \frac{r_1(T-t-J_2-J_3)+r_2J_2+r_3J_3}{T-t}.
$$

Similarly the gamma of the down-and-out barrier option is stated as

$$
C_{\text{DO}}^{\Gamma} = \frac{\partial C_{\text{DO}}}{\partial S^2} = \frac{\partial C_{D0}^{\Delta}}{\partial S}.
$$

Hence,

$$C_{\text{DO}}^{\Gamma} = \int_0^{T-t}\int_0^{T-t-J_3} C_{\Gamma}^{BS}\left(S(t), K, \frac{p_{X(t)}(J_2,J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2,J_3)}{T-t}}\right)$$

$$-\left\{\left(1-\frac{2\bar{r}}{\bar{\sigma}}\right)\frac{1}{B}\left[\left(-\frac{2\bar{r}}{\bar{\sigma}}\right)\left(\frac{S(t)}{B}\right)^{\left(-\frac{2\bar{r}}{\bar{\sigma}}-1\right)}\right.\right.$$

$$\times C^{BS}\left(\frac{B^2}{S(t)}, K, \frac{p_{X(t)}(J_2,J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2,J_3)}{T-t}}\right)$$

$$\left.-\frac{B^2}{S(t)}\Phi(d_1^B)\left(\frac{S(t)}{B}\right)^{\left(-\frac{2\bar{r}}{\bar{\sigma}}\right)}\right]$$

$$+\frac{B^2}{S(t)}\Phi(d_1^B)\left(\frac{S(t)}{B}\right)^{\left(1-\frac{2\bar{r}}{\bar{\sigma}}\right)} - \frac{B^2}{S(t)}\phi(d_1^B)\frac{\partial d_1^B}{\partial S(t)}\left(\frac{S(t)}{B}\right)^{\left(1-\frac{2\bar{r}}{\bar{\sigma}}\right)}$$

$$\left.-\frac{B^2}{S(t)}\Phi(d_1^B)\left(1-\frac{2\bar{r}}{\bar{\sigma}}\right)\left(\frac{S(t)}{B}\right)^{\left(-\frac{2\bar{r}}{\bar{\sigma}}\right)}\frac{1}{B}\right\}.$$

Corresponding interest rate sensitivities can be written as

$$C_{\text{DO}}^{\rho_i} = \int_0^{T-t}\int_0^{T-t-J_3}\frac{\partial}{\partial r_i}\left(C^{BS}\left(S(t), K, \frac{p_{X(t)}(J_2,J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2,J_3)}{T-t}}\right)\right.$$

$$-\left(\frac{S(t)}{B}\right)^{(1-2p_{X(t)}(J_2,J_3))/v_{X(t)}(J_2,J_3)}$$

$$\left.\times C^{BS}\left(\frac{B^2}{S(t)}, K, \frac{p_{X(t)}(J_2,J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2,J_3)}{T-t}}\right)\right)$$

$$\times f_{X(t)}(J_2,J_3)dJ_2dJ_3$$

$$= \int_0^{T-t}\int_0^{T-t-J_3}\left(\frac{\partial\left(C^{BS}\left(S(t), K, \bar{r}, T-t, \sqrt{\bar{\sigma}}\right)\right)}{r_i}\right.$$

$$-\left\{\log\left(\frac{S(t)}{B}\right)\left(\frac{S(t)}{B}\right)^{\left(1-\frac{2\bar{r}}{\bar{\sigma}}\right)} - \frac{2J_i}{\bar{\sigma}}\left(C^{BS}\left(\frac{B^2}{S(t)}, K, \bar{r}, T-t, \sqrt{\bar{\sigma}}\right)\right)\right.$$

$$\left.\left.+\frac{\partial\left(C^{BS}\left(B^2/S(t), K, \bar{r}, T-t, \sqrt{\bar{\sigma}}\right)\right)}{r_i}\left(\frac{S(t)}{B}\right)^{\left(1-\frac{2\bar{r}}{\bar{\sigma}}\right)}\right\}\right)$$

$$\times f_{X(t)}(J_2,J_3)dJ_2dJ_3$$

Hence,

$$
\begin{aligned}
C_{\text{DO}}^{\rho_i} = \int_0^{T-t}\int_0^{T-t-J_3} & \Big(J_i K e^{-p_{X_t}(J_2,J_3)}\Phi(d_2) \\
& - \left\{\log\left(\frac{S(t)}{B}\right)\left(\frac{S(t)}{B}\right)^{\left(1-\frac{2\bar{r}}{\bar{\sigma}}\right)} - \frac{2J_i}{\bar{\sigma}}\left(C^{BS}\left(\frac{B^2}{S(t)}, K, \bar{r}, T-t, \sqrt{\bar{\sigma}}\right)\right)\right. \\
& \left. + J_i K e^{-p_{X_t}(J_2,J_3)}\Phi(d_2^B)\left(\frac{S(t)}{B}\right)^{\left(1-\frac{2\bar{r}}{\bar{\sigma}}\right)}\right\}\Big) \\
& \times f_{X(t)}(J_2,J_3)dJ_2dJ_3.
\end{aligned}
$$

Vega can be obtained as follows

$$
\begin{aligned}
C_{\text{DO}}^{\nu_i} = & \int_0^{T-t}\int_0^{T-t-J_3} \frac{\partial}{\partial\sigma_i}\left(C^{BS}\left(S(t), K, \frac{p_{X(t)}(J_2,J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2,J_3)}{T-t}}\right)\right. \\
& - \left(\frac{S(t)}{B}\right)^{(1-2p_{X(t)}(J_2,J_3))/v_{X(t)}(J_2,J_3)} \\
& \left.\times C^{BS}\left(\frac{B^2}{S(t)}, K, \frac{p_{X(t)}(J_2,J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2,J_3)}{T-t}}\right)\right) \\
& \times f_{X(t)}(J_2,J_3)dJ_2dJ_3 \\
= & \int_0^{T-t}\int_0^{T-t-J_3}\Bigg(\frac{\partial\left(C^{BS}\left(S(t), K, \bar{r}, T-t, \sqrt{\bar{\sigma}}\right)\right)}{\sigma_i} \\
& - \left\{\log\left(\frac{S(t)}{B}\right)\left(\frac{S(t)}{B}\right)^{\left(1-\frac{2\bar{r}}{\bar{\sigma}}\right)} + \frac{4\bar{r}\sigma_i J_i}{\bar{\sigma}}\left(C^{BS}\left(\frac{B^2}{S(t)}, K, \bar{r}, T-t, \sqrt{\bar{\sigma}}\right)\right)\right. \\
& \left. + \frac{\partial\left(C^{BS}\left(B^2/S(t), K, \bar{r}, T-t, \sqrt{\bar{\sigma}}\right)\right)}{\sigma_i}\left(\frac{S(t)}{B}\right)^{\left(1-\frac{2\bar{r}}{\bar{\sigma}}\right)}\right\}\Bigg) \\
& \times f_{X(t)}(J_2,J_3)dJ_2dJ_3.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
C_{\text{DO}}^{\nu_i} = \int_0^{T-t}\int_0^{T-t-J_3} & \Bigg(K e^{-p_{X_t}(J_2,J_3)}\phi(d_2)\frac{J_i\sigma_i}{\sqrt{\bar{\sigma}}} \\
& - \left\{\log\left(\frac{S(t)}{B}\right)\left(\frac{S(t)}{B}\right)^{\left(1-\frac{2\bar{r}}{\bar{\sigma}}\right)} + \frac{4\bar{r}\sigma_i J_i}{\bar{\sigma}}\left(C^{BS}\left(\frac{B^2}{S(t)}, K, \bar{r}, T-t, \sqrt{\bar{\sigma}}\right)\right)\right. \\
& \left. + K e^{-p_{X_t}(J_2,J_3)}\phi(d_2^B)\frac{J_i\sigma_i}{\sqrt{\bar{\sigma}}}\left(\frac{S(t)}{B}\right)^{\left(1-\frac{2\bar{r}}{\bar{\sigma}}\right)}\right\}\Bigg) \\
& \times f_{X(t)}(J_2,J_3)dJ_2dJ_3.
\end{aligned}
$$

5.2.1 Numerical Implementation

In order to present the Delta and Gamma Greeks of the regime-switching barrier option we consider an example with the istrike price $K = 50$, barrier level $B = 5$, initial state $X_0 = 1$ and the remaining parameters given as:

Table 5.5: Parameter set for the calculation of barrier options under RS

	Transition Rate Matrix			Model Parameters	
States	**State 1**	**State 2**	**State 3**	**Interest Rate**	**Volatility**
State 1	-2	1	1	0.1	0.15
State 2	1	-2	1	0.2	0.25
State 3	1	1	-2	0.3	0.35

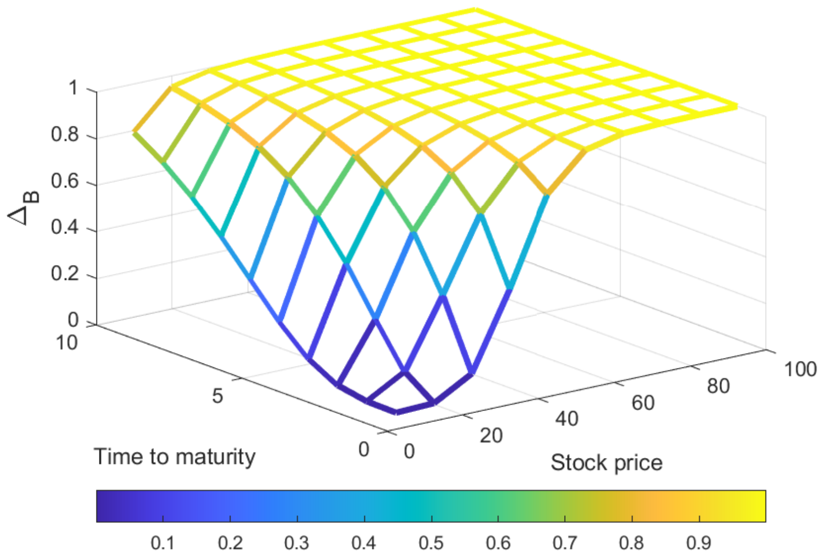


Figure 5.3: Delta of the regime-switching Barrier option

It can be observed from Figure 5.3 and Figure 5.5 that even though their general behavior is similar to the Greeks of the European option, the behavior of the corresponding Greeks near the barrier level demonstrates fluctuations. Delta and Gamma of the regime-switching barrier option grows faster near the barrier and maturity gets closer according to the regime-switching European options.

The next subsection describes another approach proposed by Elliot, Siu, and Chan [18]. Lo, Lee, and Hui [44] presented a closed-form approximation to price of the up-and-

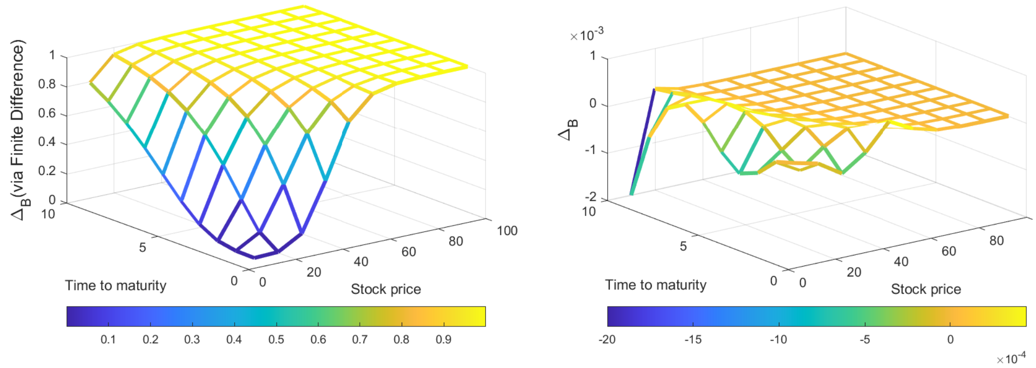


Figure 5.4: Comparison with Finite Difference: B_Δ

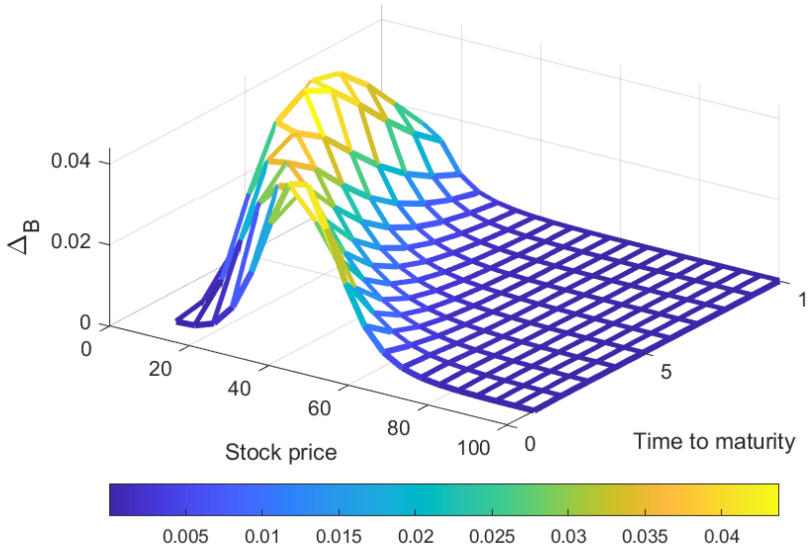


Figure 5.5: Gamma of the regime-switching Barrier option

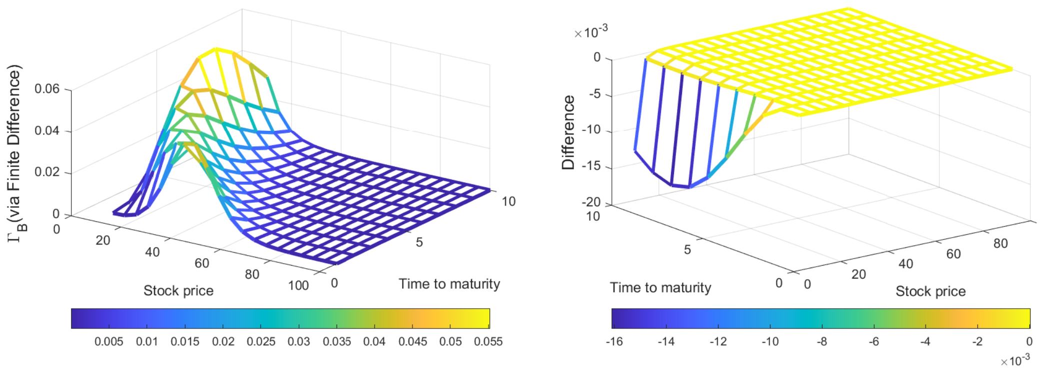


Figure 5.6: Comparison with Finite Difference: B_Γ

out call option with time-dependent parameters. Elliot, Siu, and Chan [18] modified the formula presented by Lo et al. [44] to price up-and-out put option under regime-switching framework. Elliot et al. [18] reconsidered the formula in terms of the joint density function of the occupancy time and converted relevant parts of the formula to fit the regime-switching framework. We give the details of the formula based on the idea that the parameters of the model are controlled by the three-state Markov chain.

5.2.2 Regime-Switching Barrier Option Pricing by Elliot et al. [18]

In this section we present the (semi)-analytical approximation to price the up-and-out put option under regime-switching framework suggested by Elliot et al. [18] based on the study of Lo, Lee and Hui [44] .

Lemma 5.2.1. *Let $\tau := T - t$ denotes the time to maturity for each $\tau \in \mathcal{T} = [0, T]$. Then the price of the up-and-out put option $P_{UO}(t, S(t))$ can be approximated as follows:*

$$
\begin{aligned}
P_{UO}^{X}(t, S(t)) \approx & - \exp\left(c_3(\tau) + c_2(\tau) + c_1(\tau) + S(t)\right) B \left[\Phi\left(\frac{S(t) + c_1(\tau) + 2c_2(\tau)}{\sqrt{2c_2(\tau)}} \right) \right. \\
& \left. - \Phi\left(\frac{S(t) + c_1(\tau) + 2c_2(\tau) - \ln(K/B)}{\sqrt{2c_2(\tau)}} \right) \right] + K \exp(c_3(\tau)) \\
& \times \left[\Phi\left(\frac{S(t) + c_1(\tau)}{\sqrt{2c_2(\tau)}} \right) - \Phi\left(\frac{S(t) + c_2(\tau) - \ln(K/B)}{\sqrt{2c_2(\tau)}} \right) \right] \\
& + \exp(c_3(\tau) + (\beta - 1)(S(t) + c_1(\tau)) + (\beta - 1)^2 c_2(\tau)) B \\
& \times \left[\Phi\left(- \frac{S(t) + c_1(\tau) + 2(\beta - 1)c_2(\tau)}{\sqrt{2c_2(\tau)}} \right) \right. \\
& \left. - \Phi\left(- \frac{S(t) + c_1(\tau) 2(\beta - 1)c_2(\tau) + \ln(K/B)}{\sqrt{2c_2(\tau)}} \right) \right] \\
& - K \exp(c_3(\tau) + \beta(S(t) + c_1(\tau)) + \beta^2 c_2(\tau)) \\
& \times \left[\Phi\left(- \frac{S(t) + c_1(\tau) + 2\beta c_2(\tau)}{\sqrt{2c_2(\tau)}} \right) \right. \\
& \left. - \Phi\left(- \frac{S(t) + c_1(\tau) + 2\beta c_2(\tau) + \ln(K/B)}{\sqrt{2c_2(\tau)}} \right) \right]
\end{aligned}
\tag{5.5}
$$

where Φ is the cumulative normal distribution and

$$c_1(\tau) := \int_0^\tau \left(r(u) - \frac{1}{2}\sigma^2(u) \right) du, \tag{5.6}$$

$$c_2(\tau) := \frac{1}{2}\int_0^\tau \sigma^2(u)du, \tag{5.7}$$

$$c_3(\tau) := -\int_0^\tau r(u)du, \tag{5.8}$$

$$\beta(t) := -\frac{c_1(t)}{c_2(t)}.$$

The adjustable parameter, which controls the movement of the barrier is chosen by Lo, Lee, and Hui [44] to be an optimal value that minimizes the given path integral

$$\int_0^t [Z^*(u)]^2 du,$$

where $Z^*(t) := \ln(S(t)/B) = -c_1(\tau) - \beta c_2(t)$, which is an absorbing moving barrier along the axis given by the value $\ln(S(t)/B)$. Optimal value β_{opt} for β is

$$\beta opt = \frac{\int_0^t c_1(\tau)c_2(\tau)d\tau}{\int_0^t [c_2(\tau)]^2 d\tau}. \tag{5.9}$$

For constant model parameters βopt is equal to

$$\beta opt = 1 - 2r\sigma^{-2}. \tag{5.10}$$

For regime switching parameters the value of β is given as:

$$\beta(t) := -\frac{c_1(t)}{c_2(t)}. \tag{5.11}$$

Note that, when the model parameters are constant (5.11) is reduced to (5.10). Since the model parameters depend on the underlying Markov chain, (5.6), (5.7), (5.8) and (5.11) can be written with the help of occupation time of the Markov chain as follows:

$$c_1(\tau) = \sum_{i=1}^{N} \left(r_i - \frac{1}{2}\sigma_i^2 \right) J_i(t,T), \tag{5.12}$$

$$c_2(\tau) = \frac{1}{2}\sum_{i=1}^{N} \sigma_i^2 J_i(t,T), \tag{5.13}$$

$$c_3(\tau) = -\sum_{i=1}^{N} r_i J_i(t,T), \tag{5.14}$$

$$\beta(t) = -\sum_{i=1}^{N} \left(\frac{r_i \frac{1}{2}\sigma_i^2}{\sigma_i^2} \right) J_i(0,t). \tag{5.15}$$

For each $t \in \mathcal{T}$, the minimal $\sigma-$field is defined as $\mathcal{F}(t) := \mathcal{F}^W(t) \vee \mathcal{F}^X(t)$ and $\mathbb{F} := \{\mathbb{F}(t) | t \in \mathcal{T}\}$. Now given the current information $\mathcal{F}(t)$, the conditional price of the up-and-out put option is given by:

$$P^*_{UO}(t, S(t), \beta(t)) := \mathbb{E}^Q[P(t, S(t), X) | \mathcal{F}(t)].$$

Note that

$$\beta(t) = -\sum_{i=1}^{N} \left(\frac{r_i - \frac{1}{2}\sigma_i^2}{\sigma_i^2} \right) J_i(0,t) \in \mathcal{F}(t) \subset \mathcal{F}.$$

Let $\boldsymbol{J}(t,T) = (J_1(t,T), J_2(t,T), \ldots, J_N(t,T))' \in [t,T]^N$. Write $f_{X(t),}(\boldsymbol{j})$ for the joint conditional density function of $\boldsymbol{J}(t,T)$ given $\mathcal{F}(t)$, where $\boldsymbol{j} \in [t,T]^N$. Then the conditional price of the up-and-out put option can be written as

$$P^*_{UO}(t, S(t), \beta(t)) = \int_{[t,T]^N} P^X_{UO}(t, S(t)) f_{X(t),}(\boldsymbol{j}) d\boldsymbol{j}. \tag{5.16}$$

By using this lemma stated by Elliot, Siu and Chan [18] and utilizing the joint conditional density function of occupation times given in Theorem 3.3.1 in Chapter 3, we obtained the following corollary.

Corollary 5.2.1. *Consider three-state case under which the parameters of the underlying asset and the market interest rate depend on the underlying three-state continuous time Markov chain. Then, the up-and out put option price with a strike price K and the barrier level B at time t is given as*

$$\begin{aligned} P^*_{UO}(t, S(t), \beta(t)) &= \int_t^T \int_t^{T-J_3} P^X_{UO}(t, S(t)) f_{X(t)}(J_2, J_3) dJ_2 dJ_3 & (5.17) \\ &= \psi(t,T) + \xi(t,T) + \eta(t,T) + \zeta(t,T) & (5.18) \end{aligned}$$

where

$$\psi(t,T) = \int_t^T \int_t^{T-J_3} P^X_{UO}(t, S(t)) f^1_{X(t)}(J_2, J_3) dJ_2 dJ_3,$$

$$\xi(t,T) = \int_t^T \int_t^{T-J_3} P^X_{UO}(t, S(t)) f^2_{X(t)}(J_2, J_3) dJ_2 dJ_3,$$

$$\eta(t,T) = \int_t^T \int_t^{T-J_3} P_{UO}^X(t,S(t)) f_{X(t)}^3(J_2,J_3)\, dJ_2 dJ_3,$$

$$\zeta(t,T) = \int_t^T \int_t^{T-J_3} P_{UO}^X(t,S(t)) f_{X(t)}^4(J_2,J_3)\, dJ_2 dJ_3,$$

where $f_{X(t)}^1(J_2,J_3)$*,* $f_{X(t)}^2(J_2,J_3)$*,* $f_{X(t)}^3(J_2,J_3)$*, and* $f_{X(t)}^4(J_2,J_3)$ *are stated in* (4.13)*,* (4.14)*,* (4.15)*, and* (4.16)*, respectively. Here* $0 \leq J_2 < T-t$*,* $0 \leq J_3 < T-t$*, and* $f_{X(t)}(J_2,J_3)$ *is the joint density of the occupation times of state two and state three during the time interval* $[t,T]$ *conditional on* $X(t)$*.*

CHAPTER 6

VALUATION OF GUARANTEED MIMIMUM MATURITY BENEFIT CONTRACTS UNDER REGIME-SWITCHING FRAMEWORK

We demonstrate that the proposed approach presented in Chapter 4 can also be used in the valuation of Guaranteed Minimum Maturity Benefit (GMMB) contract, since it can be written in terms of the European options. We propose formulae for two different models by assuming independent filtration for the mortality component. The first model assumes that both financial and mortality parameters are regulated by the same underlying Markov chain. On the other hand, the second model assumes that the parameters of the mortality model are based on a separate second Markov chain.

6.1 Guaranteed Minimum Maturity Benefit Pricing

Variable annuity (VA) contracts have recently become more popular as the need for products that can cover the longevity risk associated with an aging population has increased. The guarantees in VA contracts offers policyholders a range of investment options and protect their investment funds from losses. In this sense, these guarantees have characteristics similar to financial options.

There are two main types of guarantees: guaranteed minimum death benefits (GMDB) and guaranteed minimum living benefits. Guaranteed minimum living benefits can be divided into three groups: GMxB, where x stands for the type of guarantee rider, such as maturity (M), income (I), or withdrawal (W).

The assumption of constant parameters may not represent the real world dynamics accurately for the valuation of variable annuities due to the long-term nature of these contracts. For this reason, in this study, the pricing problem is investigated under the regime-switching parameter assumption.

In addition to the Markov-modulated drift and diffusion rates for the equity process and the Markov-modulated interest rate, we assume that the mortality rate also relies on the underlying Markov chain. Under this framework we consider two models: for the first model, one Markov chain governs all model parameters (equity process parameters, interest rate, and mortality rate), but for the second model, we assume that the mortality rate is modified by another Markov chain.

6.1.1 Model 1 : Common Markov Chain for the Model Parameters

We assume that all of the model parameters depend on a single Markov chain. The details of the Markov chain and the fund dynamics are given in Section 2.2 in Chapter 2. Markov modulated mortality rate is given by

$$\kappa_t := \kappa(t, X_t) = \langle \kappa, X_t \rangle$$

where $\{X_t\}_{t\in\mathcal{T}}$ is the Markov chain. We define the path integral of κ on the interval $[0, T]$ by

$$M_{t,T} = \int_t^T \kappa(s)ds = \int_t^T \langle \kappa, X_s \rangle ds. \tag{6.1}$$

Assuming that the underlying Markov chain has three states with the conditional joint density of the occupation times of the states during the time interval $[t, T]$, given $X(t) = 1$, and for $0 \leq J_2 \leq T - t, 0 \leq J_3 \leq T - t$, we can define

$$m_{X(t)}(J_2, J_3) = \kappa_1(T - t - J_2 - J_3) + \kappa_2 J_2 + \kappa_3 J_3.$$

Therefore the expected survival probability, say $L(t, T)$, can be represented as

$$\begin{aligned} L(t,T) &= \mathbb{E}^{\mathbb{Q}}\big[e^{-\int_t^T \kappa(s)ds}|\mathcal{F}_t\big] \\ &= \int_0^{T-t}\int_0^{T-t-J_3} \exp(-m_{X(t)}(J_2,J_3))f_{X(t)}(J_2,J_3)dJ_2dJ_3. \end{aligned}$$

A GMMB guarantees the minimum level of benefit at maturity conditional on the policyholder survival function and can be written as the combination of the survival function $L(t,T)$, the zero-coupon bond $B(t,T)$ and the European call $C(S,t,T)$. The insurer's liability is $(G - F_T)^+$, where F_T is the policyholder's fund level at maturity T and G is the minimum guarantee. If the guarantee exceeds the fund value at contract maturity T, the insurance company must pay the difference between G and F_T, i.e. $(G - F_T)$, with F_T is related to the performance of the stock index S_T by

$$F_T = F_0 \frac{S_T}{S_0} e^{-\psi T}$$

where ψ is the constant continuously compounded management charge rate. For simplicity we assume $F_0 = S_0$ and $\psi = 0$. The fair value of a GMMB at time t with no lapse[1] assumption is given as

$$P_{\text{GMMB}} = \mathbb{E}^{\mathbb{Q}}\big[e^{-\int_t^T \kappa(s)ds} e^{-\int_t^T r(s)ds} \max(G, F_T)\big|\mathcal{F}_t\big]$$

Assuming independence between the survival function and the financial components the price of GMMB contract can be written as

$$\begin{aligned}
P_{\text{GMMB}} &= \mathbb{E}^{\mathbb{Q}}\big[e^{-\int_t^T \kappa(s)ds} e^{-\int_t^T r(s)ds} \max(G, F_T)\big|\mathcal{F}_t\big] \\
&= \mathbb{E}^{\mathbb{Q}}\big[e^{-\int_t^T \kappa(s)ds}\big|\mathcal{F}_t\big]\mathbb{E}^{\mathbb{Q}}\big[e^{-\int_t^T r(s)ds} \max(G, F_T)\big|\mathcal{F}_t\big] \\
&= L(t,T)\,\mathbb{E}^{\mathbb{Q}}\big[e^{-\int_t^T r(s)ds} \max(G, F_T)\big|\mathcal{F}_t\big]
\end{aligned}$$

Furthermore, the financial component can be decomposed to a unit zero-coupon fund and a European call option written on the fund F with strike price G as follows:

$$\begin{aligned}
P_{\text{GMMB}} &= L(t,T)\mathbb{E}^{\mathbb{Q}}\big[e^{-\int_t^T r(s)ds} \max(G, F_T)\big|\mathcal{F}_t\big] \\
&= L(t,T)\big(G\mathbb{E}^{\mathbb{Q}}\big[e^{-\int_t^T r(s)ds}\big|\mathcal{F}_t\big] + \mathbb{E}^{\mathbb{Q}}\big[e^{-\int_t^T r(s)ds} \max(0, F_T - G)\big|\mathcal{F}_t\big]\big) \\
&= L(t,T)(GB(t,T) + C(F,t,T)) \qquad (6.2)
\end{aligned}$$

where

$$\begin{aligned}
B(t,T) &= \mathbb{E}^{\mathbb{Q}}\big[e^{-\int_t^T r(s)ds}\big|\mathcal{F}_t\big] \\
&= \int_0^{T-t}\int_0^{T-t-J_3} \exp\big\{-p_{X(t)}(J_2, J_3)\big\}\, f_{X(t)}(J_2, J_3)dJ_2dJ_3,
\end{aligned}$$

and $C(F,t,T)$ is given in (4.3).

[1] The probability that policyholders may terminate their policies early for a variety of reason is referred to as lapse risk.

6.1.2 Model 2 : Independent Markov Chains for Mortality and Financial Parameters

In this model, we suppose that the mortality rate is controlled by different homogeneous continuous-time Markov chain process $\{X_t^M\}_{t\in\mathcal{T}}$ with a finite state space. We assume that $\{X_t^M\}_{t\in\mathcal{T}}$, $\{X_t\}_{t\in\mathcal{T}}$, and $\{W_t\}_{t\in\mathcal{T}}$ are independent. Semi-martingale representation of $\{X_t^M\}_{t\in\mathcal{T}}$ is given as

$$X_t^M = X_0^M + \int_0^t Q^M X_s ds + M_t^M, \tag{6.3}$$

where Q^M is the generator matrix with $Q^M = \left(q_{ij}^M\right)$, $1 \le i,j \le N$, $\sum_{j=1}^N q_{ij}^M = 0$ and $q_{ij}^M \ge 0$ if $i \ne j$. The mortality rate is given by

$$\kappa_t := \kappa(t, X_t^M) = \langle \kappa, X_t^M \rangle.$$

The filtration generated by $\{X_t^M\}_{t\in\mathcal{T}}$ is denoted by $\mathcal{F}_t^{X^M}$.

It is assumed that the interest rate and volatility parameter of the underlying fund is modulated by the other Markov chain process $\{X_t\}_{t\in\mathcal{T}}$. The enlarged filtration is given by $\mathcal{G}_t = \mathcal{F}_t^{X^M} \vee \mathcal{F}_t^X \vee \mathcal{F}_t^W$. The pricing formula for the GMMB at time t is given as

$$\begin{aligned}
P_{\text{GMMB}} &= \mathbb{E}^{\mathbb{Q}}\big[e^{-\int_t^T \kappa(s)ds} e^{-\int_t^T r(s)ds} \max(G, F_T)\big|\mathcal{G}_t\big] \\
&= \mathbb{E}^{\mathbb{Q}}\big[e^{-\int_t^T \kappa(s)ds}\big|\mathcal{G}_t\big]\mathbb{E}^{\mathbb{Q}}\big[e^{-\int_t^T r(s)ds} \max(G, F_T)\big|\mathcal{G}_t\big] \\
&= \mathbb{E}^{\mathbb{Q}}\big[e^{-\int_t^T \kappa(s)ds}\big|\mathcal{F}_t^{X^M}\big]\mathbb{E}^{\mathbb{Q}}\big[e^{-\int_t^T r(s)ds} \max(G, F_T)\big|\mathcal{F}_t\big] \\
&= L^M(t,T)(GB(t,T) + C(F,t,T)).
\end{aligned} \tag{6.4}$$

In contrast to Model 1, the survival probability for this model must be computed using the parameters of the Markov chain process $\{X_t^M\}_{t\in\mathcal{T}}$. This model reduces to Model 1 if $\{X_t^M\}_{t\in\mathcal{T}}$ and $\{X_t\}_{t\in\mathcal{T}}$ have the same generator matrix.

6.1.3 Numerical Implementation

We consider an example to illustrate the application of proposed method with the parameters $S_0 = 36$, $T = 1$, $G = 50$, the interest rates $r = (0.1, 0.15, 0.2)$, the volatility rates $\sigma = (0.15, 0.25, 0.35)$, and the mortality rates $\kappa = (0.3, 0.4, 0.5)$

for three states. For Model 1, the transition rate probabilities are $q_{ji} = 1, j \neq i$. For Model 2, we assume that the transition rate probabilities for $\{X_t\}_{t\in\mathcal{T}}$ are $q_{ji} = 1, j \neq i$ and for $\{X_t^M\}_{t\in\mathcal{T}}$ are $q_{ji} = 0.5, j \neq i$.

The left panels of Figure 6.1 and Figure 6.2 demonstrate the comparison of the GMMB contract prices calculated with the proposed approach and those obtained through Monte Carlo simulation for Model 1 and Model 2, respectively. The right panels illustrate the difference between the proposed price and the Monte Carlo price. It is observed that for a number of samples greater than 2^{13}, Monte Carlo prices converges to the price obtained by the proposed method.

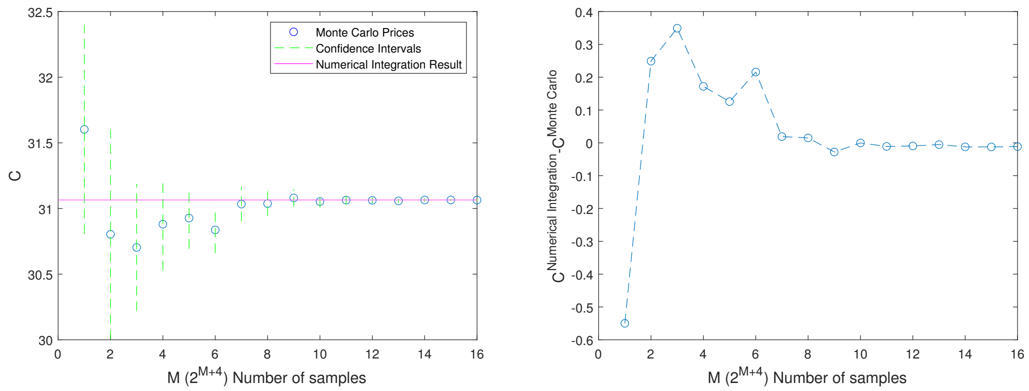


Figure 6.1: Convergence Graph for Model 1

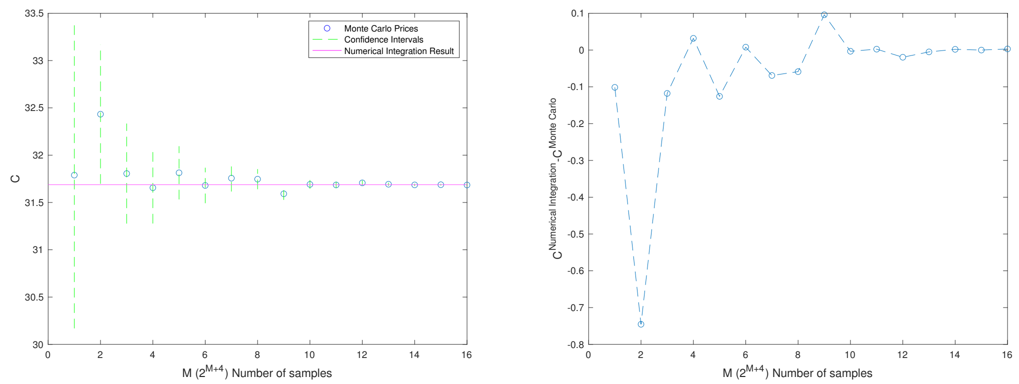


Figure 6.2: Convergence Graph for Model 2

Table 6.1: GMMB Results

	Number of samples	**Monte Carlo (MC) Price with CI**	**Time (MC)**	**Numerical Integration (NI)**	**Time (NI)**
GMMB 3-State Model 1	100,000	31.0674 CI = [31.0487 31.0861]	58.539949	31.0526	4.624355
	200,000	31.0705 CI = [31.0573 31.0837]	272.945588		
	500,000	31.0624 CI = [31.0540 31.0708]	2042.429812		
GMMB 3-State Model 2	100,000	31.7035 CI = [31.6842 31.7229]	113.687162	31.6877	5.095991
	200,000	31.6956 CI = [31.6830 31.7092]	542.694302		
	500,000	31.6935 CI = [31.6848 31.7021]	3543.745838		

Table 6.1.3 reports the results for GMMB contract prices for Model 1 and Model 2 obtained using Monte Carlo simulations and the proposed approach for different numbers of samples. It is observed that the prices obtained by the Monte Carlo approach are close to those obtained by the proposed approach. Columns four and six list the computation times for the Monte Carlo and proposed methods, respectively. The proposed method requires less computation time than the Monte Carlo method.

Figure 6.3 and Figure 6.3 illustrate the GMMB contract prices for Model 1 and Model 2 respectively.

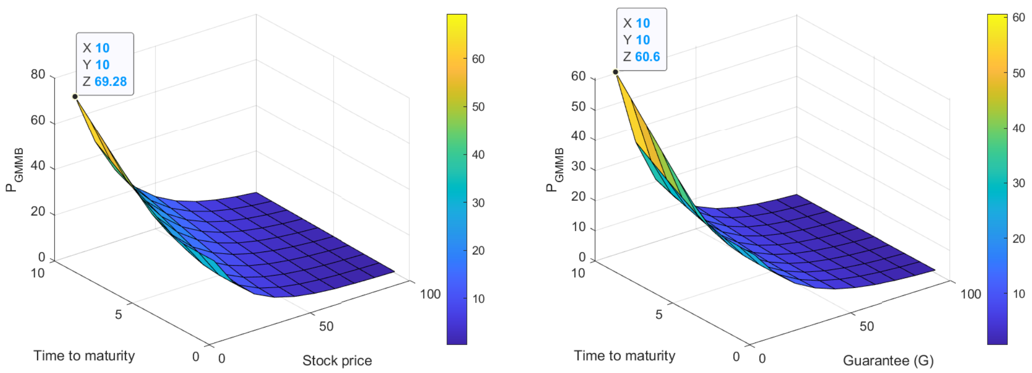


Figure 6.3: GMMB Surface Graphs (Model 1)

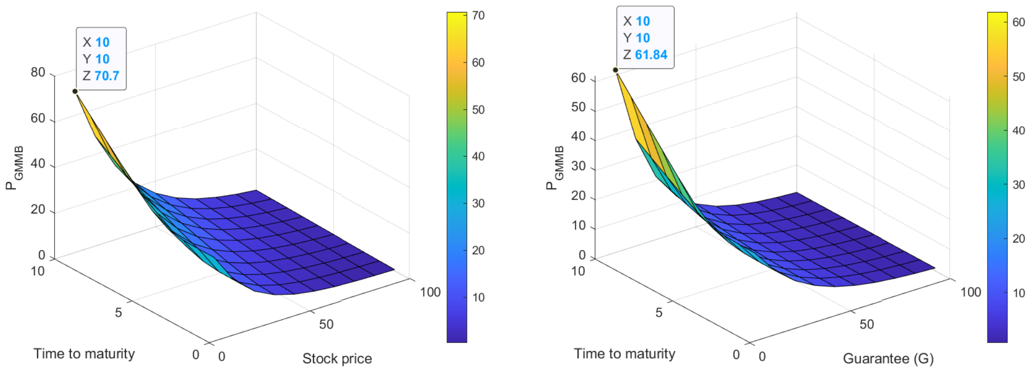


Figure 6.4: GMMB Surface Graphs (Model 2)

6.2 Sensitivity Analysis for GMMB

It is clear from (6.2) that the GMMB formula can be divided into two parts: the mortality component and the financial component. As a result, the Greek calculation can be performed separately on the mortality and financial components. For simplicity we first consider the Model 1. As performed by Ignatieva, Song and Ziveyi [40], and Mamon, Xiong and Zhao [45] we can write this approach as follows:

$$\begin{aligned} P_{\text{GMMB}} &= L(t,T)(GB(t,T) + C(F,t,T)) \\ &= \text{Mortality}(t,T) \times \text{Financial}(t,T,G) \end{aligned}$$

We first consider the financial Greeks. Calculation of delta and gamma Greeks of the associated European option can be obtained by the same approach that we presented in Corollary 4.2.1 as

$$\begin{aligned} C_G(t) = & \int_0^{T-t} \int_0^{T-t-J_3} C_G^{\text{BS}} \left(S, K, \frac{p_{X(t)}(J_2,J_3)}{T-t}, T-t, \sqrt{\frac{v_{X(t)}(J_2,J_3)}{T-t}} \right) \\ & \times f_{X(t)}(J_2,J_3) dJ_2 dJ_3 \end{aligned}$$

for $G \in \{\Delta, \Gamma\}$. Hence the delta and gamma Greeks of the GMMB is given as

$$\frac{\partial P_{\text{GMMB}}}{\partial S} = L(t,T)\, C_\Delta(t), \tag{6.5}$$

$$\frac{\partial P_{\text{GMMB}}}{\partial S^2} = L(t,T)\, C_\Gamma(t). \tag{6.6}$$

Similarly, the vega of the GMMB can be written as

$$\frac{\partial P_{\text{GMMB}}}{\partial \nu_i} = L(t,T)\, C_{\nu_i}(t) \tag{6.7}$$

where $C_{\nu_i}(t)$ is given in (4.49) for $i \in \{1,2,3\}$. In order to determine the GMMB's interest rate sensitivity, we must examine both the zero-coupon bond and the European option, as both are depend on the interest rate parameter. Differentiating corresponding zero-coupon bond with respect to r_i, we obtain:

$$\begin{aligned} \frac{\partial B(t,T)}{\partial r_i} &= \frac{\partial \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} \middle| \mathcal{F}_t \right]}{\partial r_i} \\ &= \frac{\partial}{\partial r_i} \left(\int_0^{T-t} \int_0^{T-t-J_3} \exp\{-p_{X(t)}(J_2,J_3)\} f_{X(t)}(J_2,J_3) dJ_2 dJ_3 \right) \end{aligned}$$

By Leibniz rule it can be written as

$$\begin{aligned} \frac{\partial B(t,T)}{\partial r_i} &= \int_0^{T-t} \int_0^{T-t-J_3} \frac{\partial}{\partial r_i} \left(\exp\{-p_{X(t)}(J_2,J_3)\} f_{X(t)}(J_2,J_3) dJ_2 dJ_3 \right) \\ &= \int_0^{T-t} \int_0^{T-t-J_3} (-J_i) \exp\{-p_{X(t)}(J_2,J_3)\} f_{X(t)}(J_2,J_3) dJ_2 dJ_3. \end{aligned} \tag{6.8}$$

Hence, the sensitivity of the GMMB contract with respect to Markov modulated interest rate parameter can be written with the aid of (6.8) and (4.59) as

$$\frac{\partial P_{\text{GMMB}}}{\partial r_i} = L(t,T)\left(G\frac{\partial B(t,T)}{\partial r_i} + \frac{\partial C(F,t,T)}{\partial r_i}\right). \tag{6.9}$$

In order to calculate the sensitivity with respect to mortality component we differentiate the GMMB with respect to Markov modulated mortality parameter κ_i, yields

$$\frac{\partial P_{\text{GMMB}}}{\partial \kappa_i} = \frac{\partial L(t,T)}{\partial \kappa_i}\left(GB(t,T) + C(F,t,T)\right), \tag{6.10}$$

where

$$\begin{aligned}
\frac{\partial L(t,T)}{\partial \kappa_i} &= \frac{\partial}{\partial \kappa_i}\int_0^{T-t}\int_0^{T-t-J_3} \exp\{-m_{X(t)}(J_2,J_3)\} f_{X(t)}(J_2,J_3)dJ_2dJ_3 \\
&= \int_0^{T-t}\int_0^{T-t-J_3} \frac{\partial}{\partial \kappa_i}\left(\exp\{-m_{X(t)}(J_2,J_3)\} f_{X(t)}(J_2,J_3)dJ_2dJ_3\right) \\
&= \int_0^{T-t}\int_0^{T-t-J_3} (-J_i)\exp\{-m_{X(t)}(J_2,J_3)\} f_{X(t)}(J_2,J_3)dJ_2dJ_3.
\end{aligned}$$

In the calculation of the theta sensitivity, all three distinct parts, namely the mortality component, zero-coupon bond, and European option, must be taken into account because they are all influenced by the parameter T. The calculation of theta is given below:

$$\begin{aligned}
\frac{\partial P_{\text{GMMB}}}{\partial (T-t)} &= \frac{\partial L(t,T)}{\partial (T-t)}(GB(t,T) + C(F,t,T)) \\
&\quad + L(t,T)\frac{\partial}{\partial (T-t)}\big(GB(t,T) + C(F,t,T)\big) \\
&= \frac{\partial L(t,T)}{\partial (T-t)}(GB(t,T) + C(F,t,T)) \\
&\quad + L(t,T)\left(G\frac{\partial B(t,T)}{\partial (T-t)} + \frac{\partial C(F,t,T)}{\partial (T-t)}\right),
\end{aligned}$$

where the terms $\frac{\partial L(t,T)}{\partial (T-t)}$, $\frac{\partial B(t,T)}{\partial (T-t)}$, and $\frac{\partial C(F,t,T)}{\partial (T-t)}$ can be obtained via finite difference method.

6.2.1 Numerical Implementation: GMMB Greeks

In this section, to illustrate the sensitivity analysis results for GMMB, we consider an example. Under Model 1, in which all model parameters, namely, mortality and

Table 6.2: Parameter set for the calculation of Greeks of GMMB

	Transition Rate Matrix			Model Parameters		
States	**State 1**	**State 2**	**State 3**	**Interest Rate**	**Volatility**	**Mortality Rate**
State 1	-2	1	1	0.1	0.15	0.3
State 2	1	-2	1	0.15	0.25	0.4
State 3	1	1	-2	0.2	0.35	0.5

finance, modulated by the same underlying Markov chain. The considered GMMB contract has the guarantee level $G = 50$, the initial state of the underlying Markov chain is given as state 1, $X_0 = 1$, and the remaining parameters are given in Table 6.2.

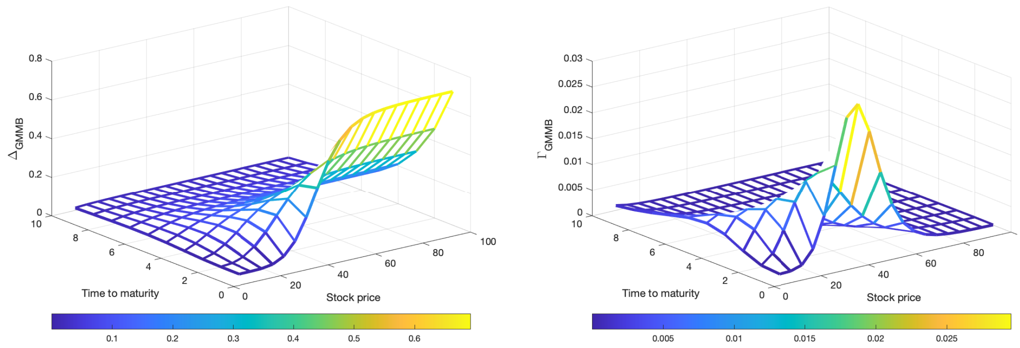
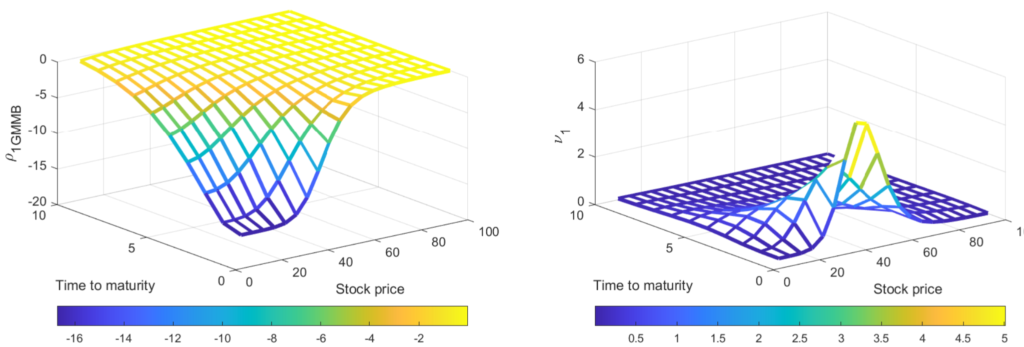


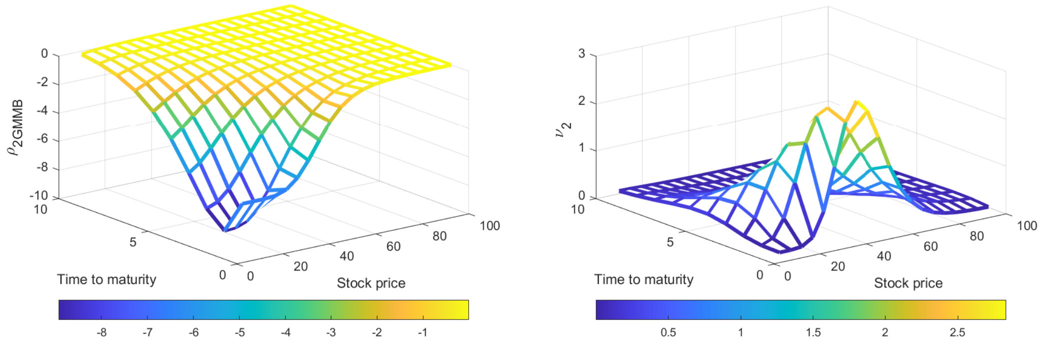
Figure 6.5: Delta and Gamma of the regime-switching GMMB



(a) ρ_1 (b) ν_1

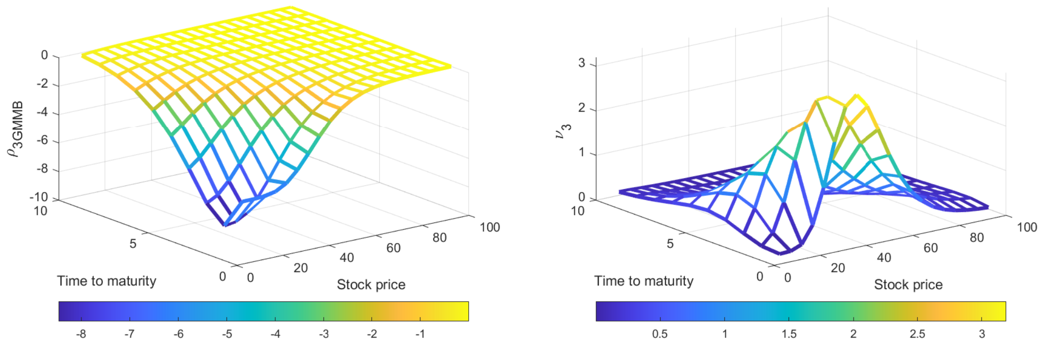
Figure 6.6: Rho and Vega surfaces: ρ_1, ν_1

The Greek calculation is performed separately on the mortality and financial components. Even though the general behavior of the financial Greeks for small time to maturity values is similar to the regime-switching European options, for larger values of the time to maturity, the values of the financial Greeks get closer to zero. The reason for this tendency is the multiplication of the financial Greek component with



(a) ρ_1 (b) ν_1

Figure 6.7: Rho and Vega surfaces: ρ_2, ν_2



(a) ρ_1 (b) ν_1

Figure 6.8: Rho and Vega surfaces: ρ_3, ν_3

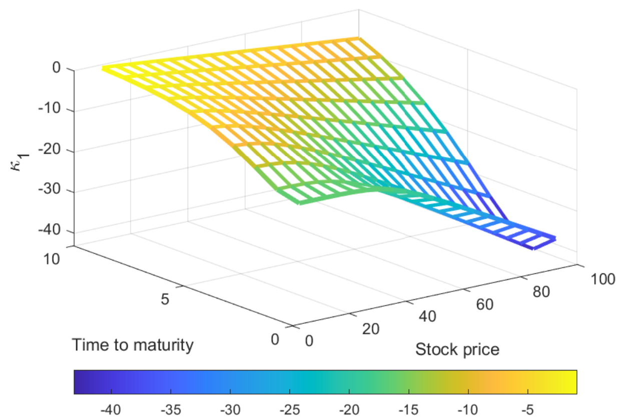


Figure 6.9: κ_1 of the regime-switching GMMB

the mortality component. Similar to the Rho and Vega sensitivities, the sensitivity of the mortality component, Kappa values for state 1 is higher than the other states. Because we assume in this example that the process begins at state 1, the associated

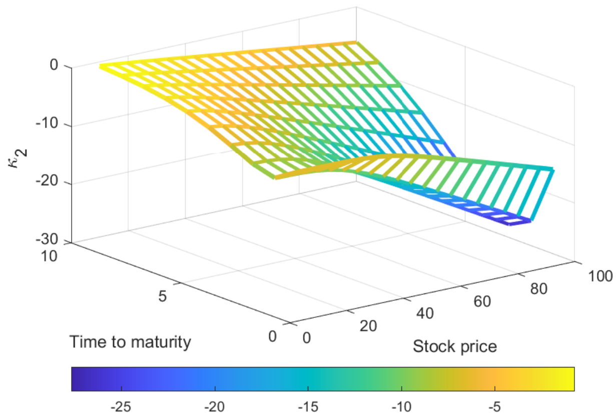


Figure 6.10: κ_2 of the regime-switching GMMB

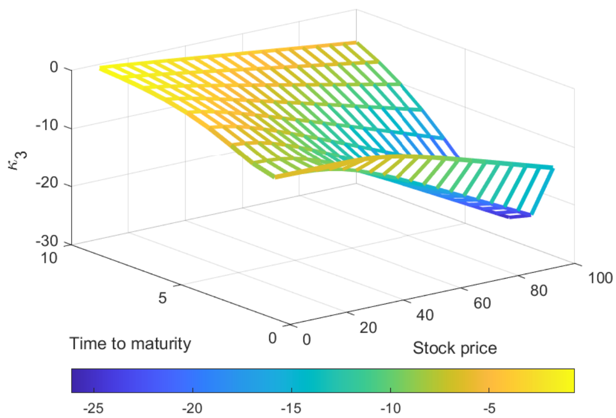


Figure 6.11: κ_3 of the regime-switching GMMB

Greeks have a greater influence than the others.

CHAPTER 7

CONCLUSION AND FUTURE WORK

In this thesis, we first consider the option pricing problem when the appreciation rate and the volatility of the underlying asset, as well as the market interest rate parameters, are driven by a continuous-time finite-state hidden Markov chain. We follow the Elliot et al. [12]'s approach in which the regime-switching Esscher transform is utilized to determine the equivalent martingale measure. We consider the proposed closed-form formulae for the European option pricing under regime-switching framework by several researchers, namely, Elliot et al. [12], McKinlay [48], Naik [50]. We elaborate the formula for the general three-state case in which all of the model parameters are depend on the underlying Markov chain.

In order to use the proposed regime-switching option pricing formula, one needs to obtain the joint density function of occupation times. Several researchers studied occupation time distribution function for the two-state case. Pedler [53], Guo [31], and Fuh et al. [25] obtained the joint probability density function of occupation times of the two-state Markov chain.

In two-state regime-switching models, the pricing problem for European options can be solved analytically, but in more complex models with more than two states, closed-form solutions have not been found, as stated by Zeng et al. [62], Zhang et al. [63], Boyle et al. [4]

In this study, in order to extend the analytical solution for more than two states we propose to utilize Falzon's formula. Falzon [22, 21, 52] presented the joint density function of the occupation time for three-state case. For simplicity, in our applications we break down the proposed conditional joint density function into four parts, according to possible visits of the process.

We observe that the Falzon's formula can be reduced to the two-state case by considering suitable parameters and it is consistent with the formula proposed by Pedler [53]. The numerical example given by Liu et al. [43] is considered in order to test the proposed method by comparing it with Monte Carlo and FFT methods. Consistent results are obtained and it is observed that in terms of computation time proposed method requires less time.

The conditional joint probability density function derived by Falzon has the condition that the Markov process starts from state one. In order to generalize this initial state condition we transform the corresponding transition rate by the help of permutation matrix. Zeng et al.[62]'s study is used as a benchmark to test our results when the initial state of the Markov chain starts from any of the three-states. We obtained consistent results for this scenarios as well.

Falzon [22] reconsidered the functions $F_{m,n}$ as sums of products of Bessel functions in order to obtain the feasible form for numerical computation. Falzon [22] stated that the infinite summation in the formula can be approximated by a suitable finite sum. We conducted a numerical example to check how many terms are needed to obtain a sufficient result. To demonstrate the convergence of the pricing formula, we take the difference between prices for successive values of the number of terms in the summation for each T. It is observed that even though the required number of terms rises as T grows, its rate is less than the increase of T.

Empirical studies show that the implied volatility of the underlying asset, rather than being constant, should change with respect to the maturity and exercise price of the option. Hence, another concern that needs to be addressed is whether the stylized facts of volatility smile can be better explained using a regime-switching model. We take the Markov-modulated European call prices as observed prices and back out the implied volatility from the Black-Scholes formula. The implied volatility surfaces with respect to the strike price and time to expiry is illustrated. The implied volatility reaches a minimum at the strike price corresponding to an at-the-money option, and it increases as it moves away from the strike price, which is known as the volatility smile phenomenon. For options with longer maturities, the volatility smile becomes flatter, which can be explained by the fact that B-S model pricing errors are higher than those for shorter maturities. A further observation is that the volatility smile is asymmetric in relation to the strike, which is in agreement with market observations and supports the idea that the regime-switching model has practical utility.

Sensitivity analysis, which examines how the value of a contingent claim changes as a result of changes in a particular parameter's value, is fundamental to risk management in the derivatives market. Using the proposed closed-form solution, we have shown that Greeks, namely Delta, Gamma, Rho and Vega can be calculated with the help of the double Leibnez rule. The finite difference method is the common methodology for sensitivity computations of the regime-switching models. As a consequence, we compare our findings with those obtained using the finite difference approach in order to verify the Greek formulas.

Our approach can further be extended for the valuation of the some of the exotic options with can be expressed in terms of the European options. As a consequence of this, the pricing formula for the regime-switching barrier options and corresponding Greeks can be derived by using this connection. In this thesis we examine down-and-out barrier options, however the other knock-out and knock-in barrier options can be considered similarly. Several numerical examples are presented to illustrate that the recommended methodology yields precise results.

Due to their long-term nature, variable annuities may include regime switching during their maturity time. It makes them suitable candidates to be modeled with regime-switching models. In this study we considered GMMB contracts which are insurance products with embedded option features. Using the assumption that the mortality component undergoes its own unique filtering, we derive the formulae for two distinct models. The first model assumes that the Markov chain that controls the financial parameters also controls the mortality parameters. The second model, on the other hand, assumes that the mortality model's parameters are obtained from a separate second Markov chain. The GMMB formula for two models can be divided into two parts: the mortality component and the financial component. As a result, the Greek calculations are performed separately on the mortality and financial components. Numerical experiments to check the accuracy of the proposed method are performed.

To sum up, in this thesis we generalize the Black-Scholes framework by considering three-state regime-switching framework. Consequently, the constant parameter disadvantage of the Black-Scholes model has been mitigated, and the theoretical elegance of the Black-Scholes model has been exploited. Moreover it is demonstrated that the proposed model can capture the asymmetric structure of the volatility smile phenomenon. The approach is flexible because it can also be used in the valuation of other financial instruments. Hence, the method can be useful for option valuation and hedging for practitioners as well in terms of accuracy and computation speed. Figure 7.1 summarizes the main contributions of the thesis to the existing literature.

Our results could be extended in a number of directions. The first is to extend current work to the N-state case. Falzon [22] and Pearce et al. [52] deduced the solution for case $N = 4$ by applying probabilistic interpretation to the three-state solution. Then, using graph theory and the matrix tree theorem, extend the probabilistic reasoning to get the general form for the case with n states. By using the formula for the N-state case, it is possible to extend the regime switching option pricing problem.

Another important direction is to calibrate the model parameters to real market data. The probability distribution observed in financial and actuarial data changes over time. Hence, in order to determine the appropriate number of states as well as the parameters of the models to be used, parameter estimation needs to be utilized as a first step. As a future extension, with real data parameter estimation can be considered first, and then the proposed pricing method can be applied.

The risk factors namely, the market risk and the mortality rate risk may exhibit more complex relation in real market data. Another relevant future direction is to consider the correlation between the risk factors via diffusion factor in addition to the underlying Markov chain.

The proposed method used in the pricing of regime-switching European, barrier options and GMMB contracts can be applicable to other financial contracts such as quando options, spread options, and power options.

Figure 7.1: Contribution to Literature

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APPENDIX A

PROOF OF SOME THEOREMS

A.1 Proof of Lemma 4.1.1

Proof. The proof is a modification of Lemma 5.1 in Elliot and Siu [17]. Consider an $\mathbb{R}^N$ valued process $Y(t) := \{Y(t,u)|u \in [t,T]\}$

$$\begin{aligned} Y(t,u) :=& \exp(i\langle \nu, J(t,u)\rangle)X(u) \\ =& \exp\left(i\int_t^u \langle \nu, X(s)\rangle ds\right)X(u). \end{aligned}$$

Then,

$$dY(t,u) = i\langle \nu, X(u)\rangle Y(t,u)du + \exp(i\langle \nu, J(t,u)\rangle)dX(u).$$

Semi-martingale dynamics of the Markov chain X is given in (2.7) can be written as

$$dX(u) = QX(u)du + dM(u).$$

Note that

$$\begin{aligned} \langle \nu, X(u)\rangle Y(t,u) &= \text{diag}\, \nu Y(t,u) \\ &= DY(t,u). \end{aligned}$$

Consequently,

$$\begin{aligned} dY(t,u) &= i\langle \nu, X(u)\rangle Y(t,u)du + \exp(i\langle \nu, J(t,u)\rangle)(QX(u)du + dM(u)) \\ &= (Q + i\,\text{diag}\,\nu)Y(t,u) + \exp(i\langle \nu, J(t,u)\rangle)dM(u). \end{aligned}$$

Hence,

$$\begin{aligned} Y(t,u) &= Y(t,t) + \int_t^u (Q + i\,\text{diag}\,\nu)Y(t,s)ds + \int_t^u \exp(\langle \nu, J(t,s)\rangle)dM(s) \\ &= X(t) + \int_t^u (Q + i\,\text{diag}\,\nu)Y(t,s)ds + \int_t^u \exp(\langle \nu, J(t,s)\rangle)dM(s). \end{aligned}$$

Since the final integral in this expression is a martingale, taking expectation by conditioning both sides on $\mathcal{F}_t^Z$ gives

$$\mathbb{E}[Y(t,u)|\mathcal{F}_t^Z] = X(t) + \int_t^u (Q + i\,\mathrm{diag}\,\nu)\mathbb{E}[Y(t,s)|\mathcal{F}_t^Z]ds.$$

Solving yields

$$\begin{aligned}\mathbb{E}[Y(t,T)|\mathcal{F}_t^Z] &= X(t)\exp((Q + i\,\mathrm{diag}\,\nu)(T-t)) \\ &= X(t)\exp((Q+iD)(T-t))\end{aligned}$$

Consequently the conditional characteristic function of $J(t,T)$ is

$$\Phi_{J(t,T)|\mathcal{F}_t^Z}(\nu) = \mathbb{E}[\exp(i\langle \nu, J(t,T)\rangle)|\mathcal{F}_t^Z] \tag{A.1}$$

Since $\langle X(T), \mathbf{1}\rangle = 1$ we can plug it (A.1). Hence

$$\begin{aligned}\Phi_{J(t,T)|\mathcal{F}_t^Z}(\nu) &= \mathbb{E}[\exp(i\langle \nu, J(t,T)\rangle)\langle X(T), \mathbf{1}\rangle|\mathcal{F}_t^Z] \\ &= \mathbb{E}[\langle \exp(i\langle \nu, J(t,T)\rangle)X(T), \mathbf{1}\rangle|\mathcal{F}_t^Z] \\ &= \langle \mathbb{E}[\exp(i\langle \nu, J(t,T)\rangle)X(T)|\mathcal{F}_t^Z], \mathbf{1}\rangle \\ &= \langle \mathbb{E}[Y(t,T)|\mathcal{F}_t^Z], \mathbf{1}\rangle \\ &= \langle X(t)\exp((Q + i\,\mathrm{diag}\,\nu)(T-t)), \mathbf{1}\rangle \\ &= \langle X(t)\exp((Q+iD)(T-t)), \mathbf{1}\rangle.\end{aligned}$$

□

A.2 Pseudo-Code for Barrier Option Pricing via MC Approach Under RS

Algorithm 3 Monte Carlo Simulation for RS Down-and-Out Barrier Option

Require:

Initial stock price S_0, Strike price K, Barrier level B, Maturity time T

Regime-switching interest rate vector r, Regime-switching volatility vector σ

Initial state of the Markov chain $X_0 = j$

Transition rate matrix of the Markov chain Q

Number of simulation paths n_r,

Ensure: Regime-switching down-and-out barrier option price C_{DO}

Create a $k \times 1$ random vector called RV

for $k = 1, 2, \ldots, n_r$ **do**

  Simulate a Markov chain on which the parameters of the simulated path will depend on.

  Find the states of the simulated chain.

  Calculate the $P^k_{(t,T)}$ and $V^k_{(t,T)}$ given in (2.23), (2.24).

  Simulate the Markov-modulated Stock prices $S^k(T)$ and $S^k_B(T)$

$$S^k(T) = S_0 \exp\left\{(P^k_{(0,T)} - V^k_{(0,T)}/2) + \sqrt{V^k_{(0,T)}}RV(k)\right\} \tag{A.2}$$

$$S^k_B(T) = \left(\frac{B^2}{S_0}\right) \exp\left\{(P^k_{(0,T)} - V^k_{(0,T)}/2) + \sqrt{V^k_{(0,T)}}RV(k)\right\} \tag{A.3}$$

  Calculate $\xi^k = 1 - \frac{2P^k_{(0,T)}}{V^k_{(0,T)}}$

  Calculate the discounted final payoff for each simulation

  $DP[k] \leftarrow \exp\{-P^k_{(0,T)}\} \Big(\max(S^k(T) - K, 0) - \big(\frac{S_0}{B}\big)^{\xi^k} \max(S^k_B(T) - K, 0)\Big)$

end for

Calculate the Monte Carlo option price

$$C_{DO} = \frac{1}{n_r} \sum_{k=1}^{n_r} DP[k]$$

A.3 Pseudo-Code for GMMB Pricing via MC Approach Under RS

Algorithm 4 Monte Carlo Simulation for GMMB Pricing: Model 1

Require:

1: Initial fund value S_0, Guarantee level G, Maturity time T

2: Regime-switching interest rate vector r, Regime-switching volatility vector σ

3: Regime-switching mortality vector κ

4: Initial state of the Markov chain $X_0 = j$

5: Transition rate matrix of the Markov chain Q

6: Number of simulation paths n_r

Ensure: GMMB price P_{GMMB}

7: **for** $k = 1, 2, \ldots, n_r$ **do**

8: Simulate a Markov chain on which the parameters of the simulated path will depend on.

9: Find the states of the simulated chain.

10: Calculate the $P^k_{(t,T)}$,$V^k_{(t,T)}$ and $M^k_{(t,T)}$ given in Eq.(2.23), Eq.(2.24), Eq.(6.1).

11: Simulate the Markov-modulated fund value $S^k(T)$ at time T.

$$S^k(T) = S_0 \exp\left\{(P^k_{(0,T)} - V^k_{(0,T)}/2) + \sqrt{V^k_{(0,T)}}z\right\} \tag{A.4}$$

where z is a random number from the standard normal distribution.

12: Calculate the discounted final payoff for financial component

$DP[k] \leftarrow \exp\{-P^k_{(0,T)}\}\left(\max(S^k(T), G)\right)$

13: Multiply with mortality component

$MDP[k] \leftarrow \exp\{-M^k_{(0,T)}\}DP[k]$

14: **end for**

15: Calculate the Monte Carlo GMMB price

$$P_{GMBB} = \frac{1}{n_r}\sum_{k=1}^{n_r} MDP[k]$$

A.4 Sensitivity Analysis: Comparison with Finite Difference Method

In this section, the results of the proposed method are compared to the results of the finite difference method to show the validity of proposed method.

A.4.1 Regime-Switching European Option

Table A.1: Parameter set for the calculation of European option Greeks under RS

	Transition Rate Matrix			Model Parameters	
States	**State 1**	**State 2**	**State 3**	**Interest Rate**	**Volatility**
State 1	-2	1	1	0.1	0.15
State 2	1	-2	1	0.2	0.25
State 3	1	1	-2	0.3	0.35

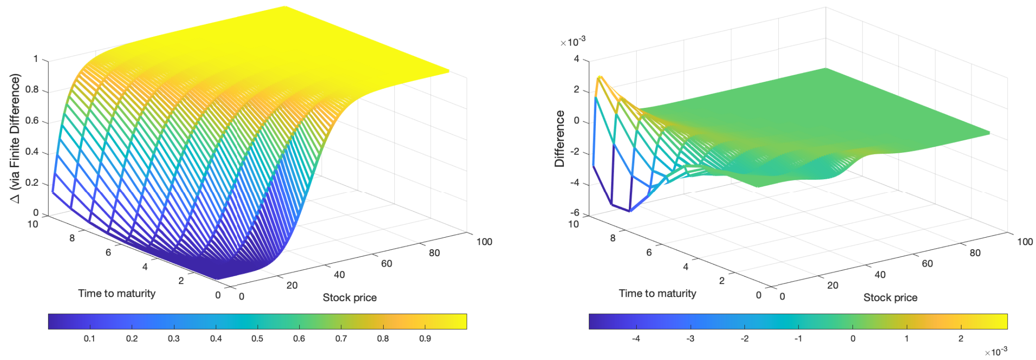

Figure A.1: Comparison with Finite Difference: Delta

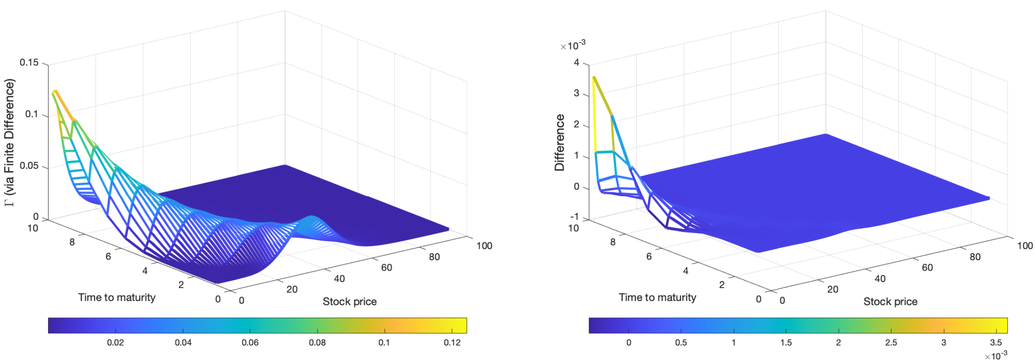

Figure A.2: Comparison with Finite Difference: Gamma

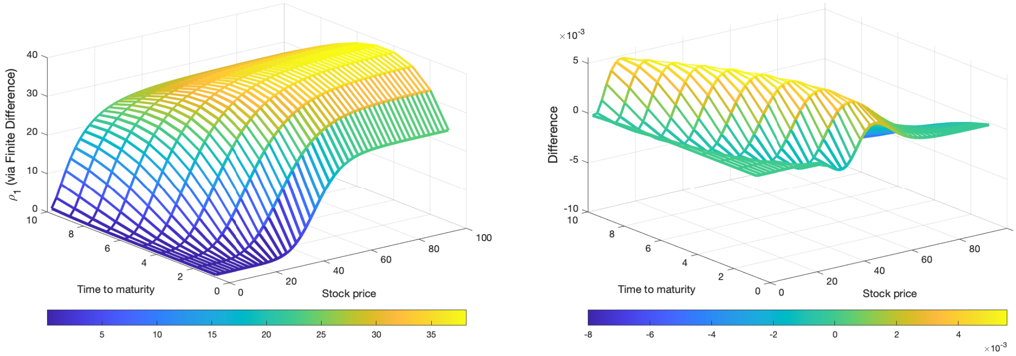

Figure A.3: Comparison with Finite Difference: ρ_1

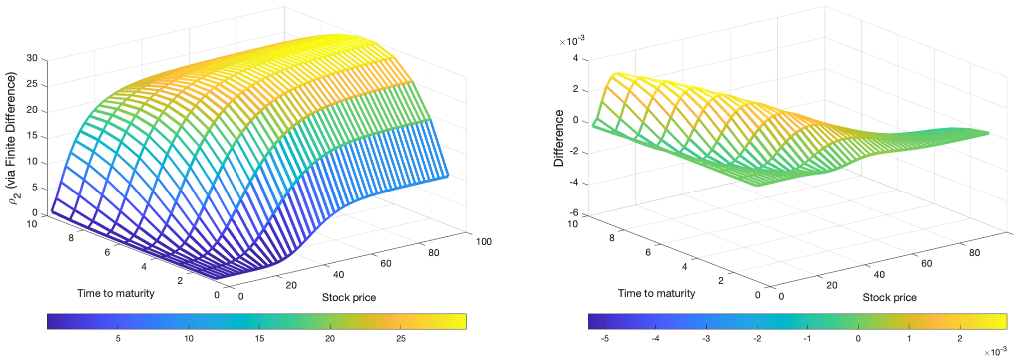

Figure A.4: Comparison with Finite Difference: ρ_2

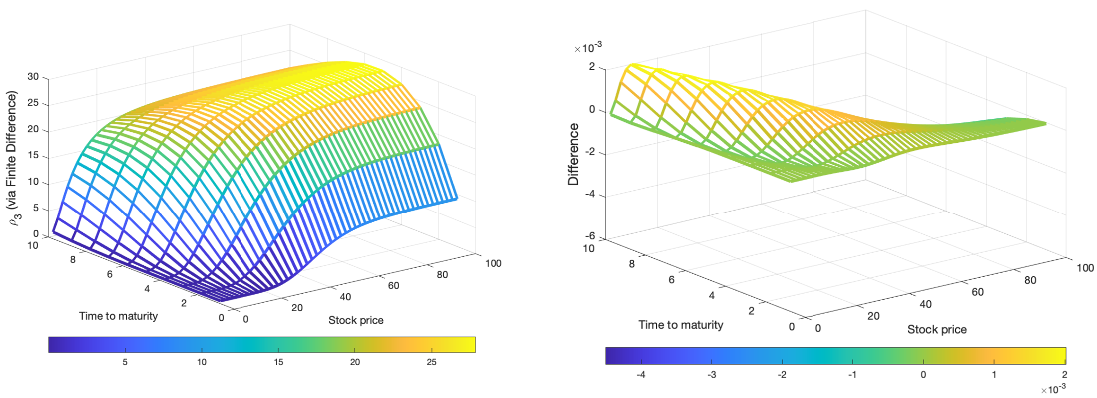

Figure A.5: Comparison with Finite Difference: ρ_3

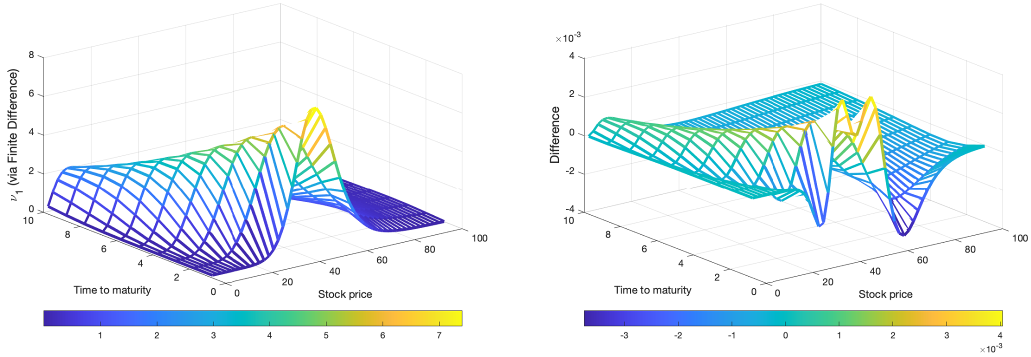

Figure A.6: Comparison with Finite Difference: ν_1

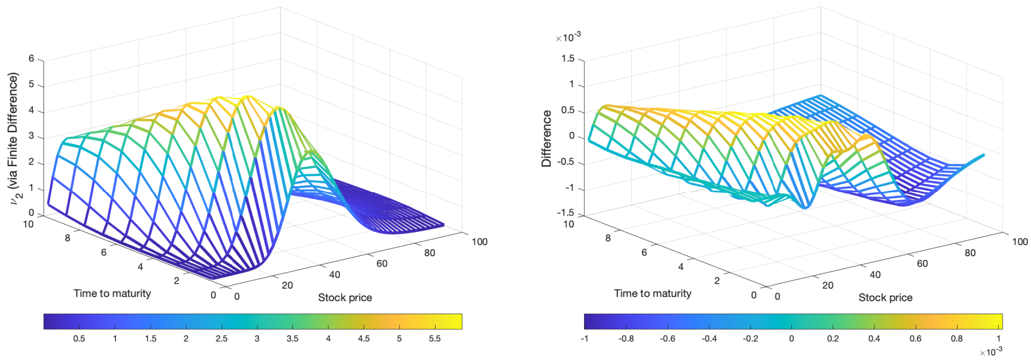

Figure A.7: Comparison with Finite Difference: ν_2

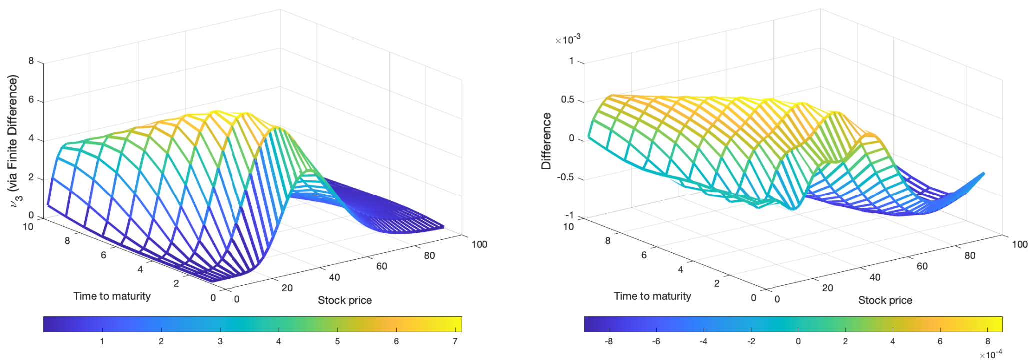

Figure A.8: Comparison with Finite Difference: ν_3

A.4.2 Regime-Switching GMMB

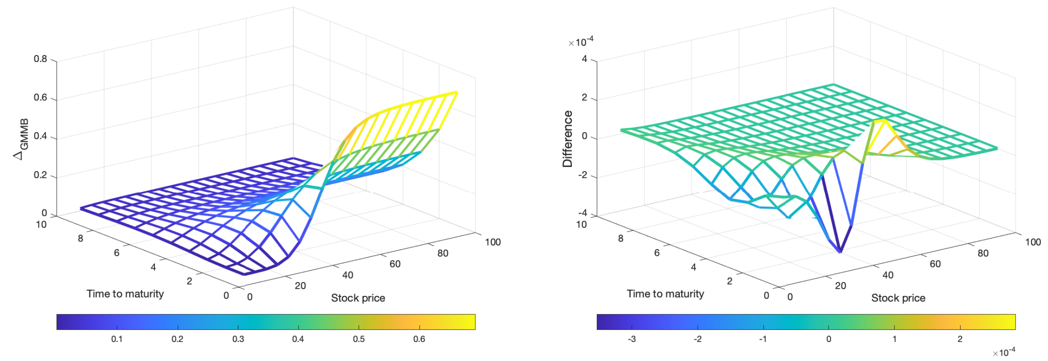

Figure A.9: Comparison with Finite Difference: Delta

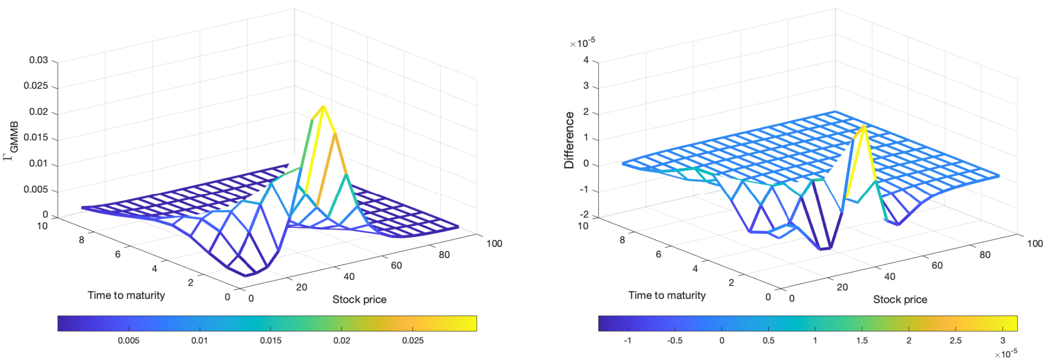

Figure A.10: Comparison with Finite Difference: Gamma

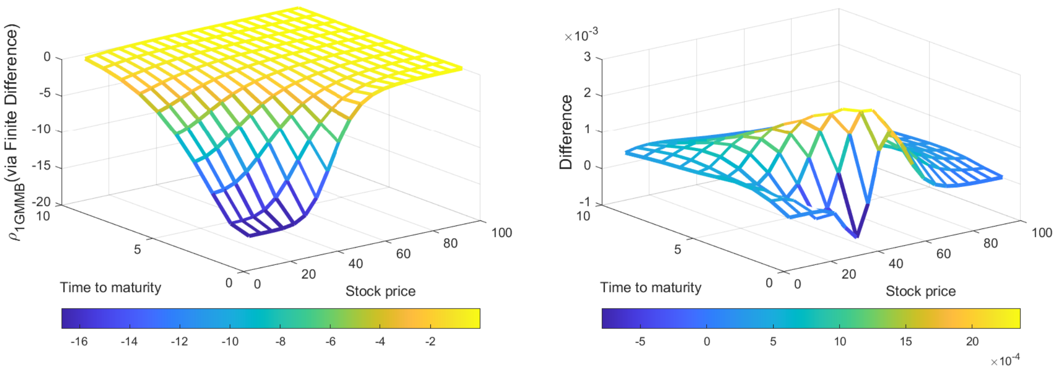

Figure A.11: Comparison with Finite Difference: ρ_1

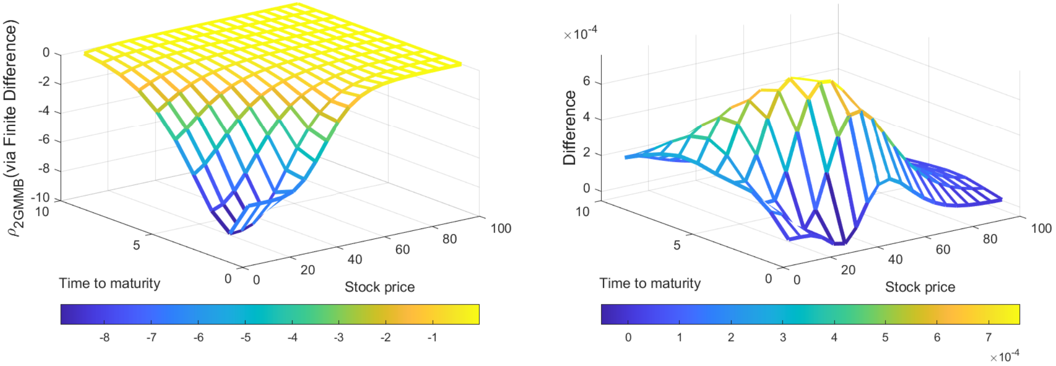

Figure A.12: Comparison with Finite Difference: ρ_2

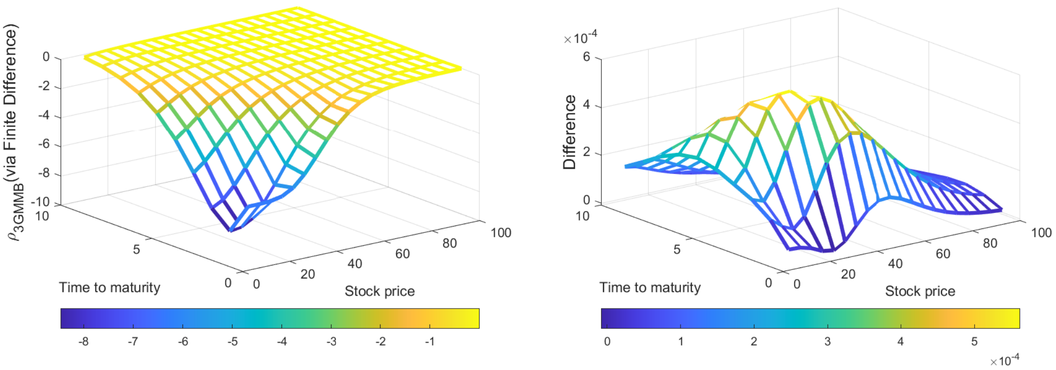

Figure A.13: Comparison with Finite Difference: ρ_3

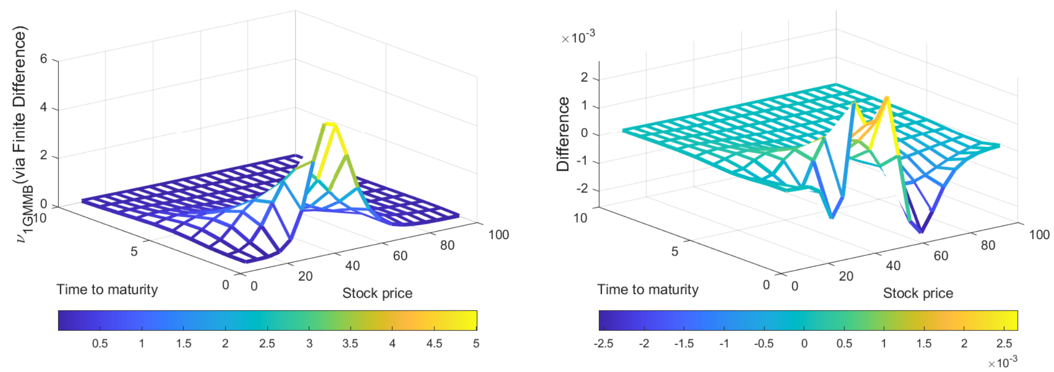

Figure A.14: Comparison with Finite Difference: ν_1

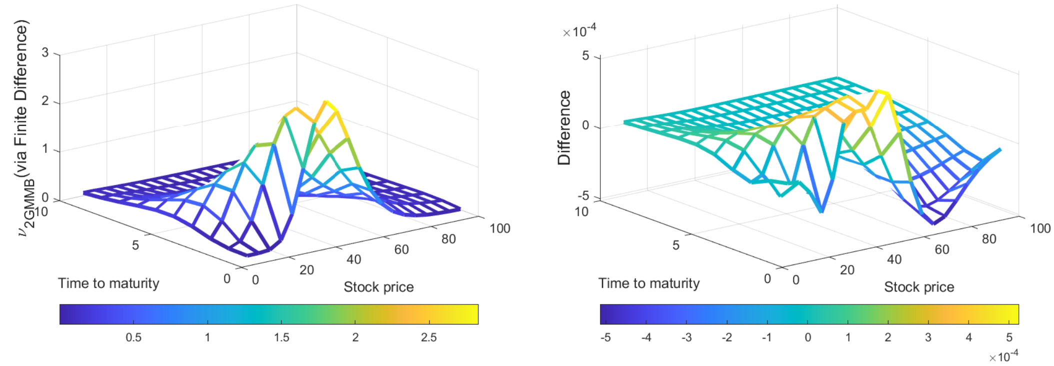

Figure A.15: Comparison with Finite Difference: ν_2

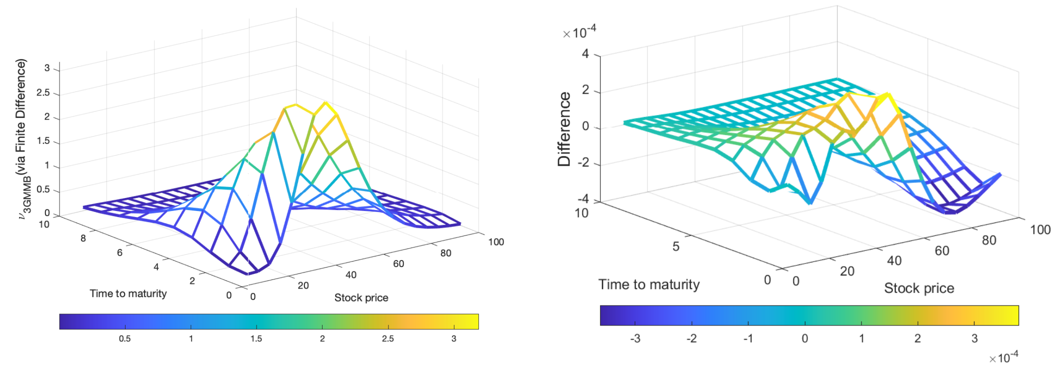

Figure A.16: Comparison with Finite Difference: ν_3

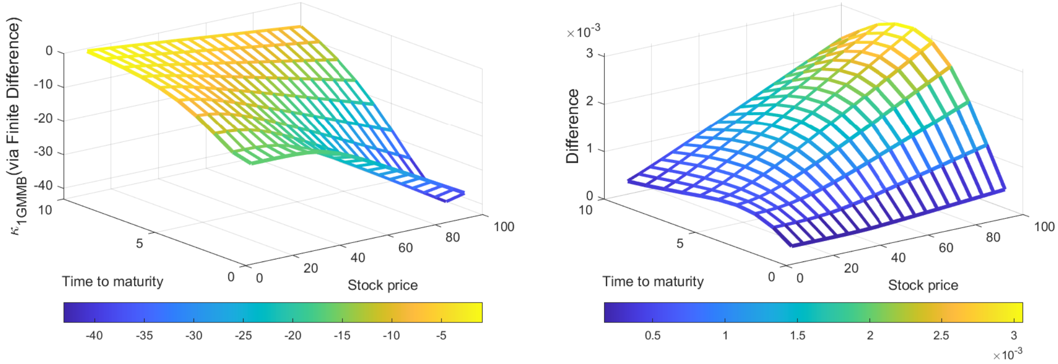

Figure A.17: Comparison with Finite Difference: κ_1

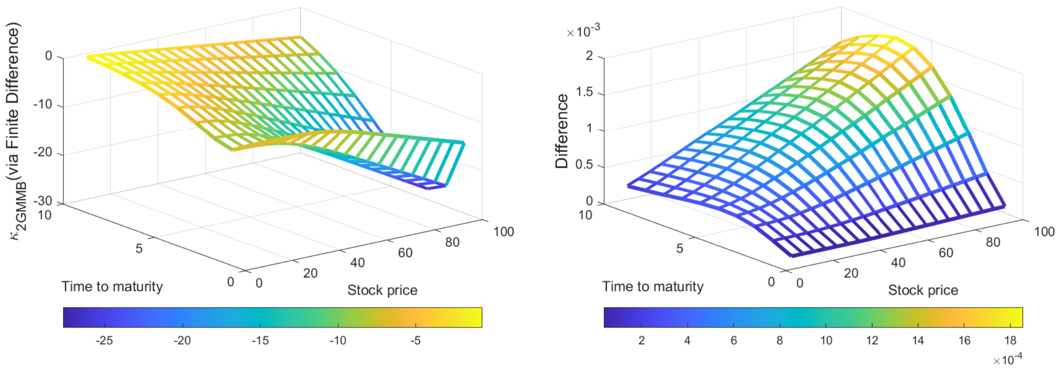

Figure A.18: Comparison with Finite Difference: κ_2

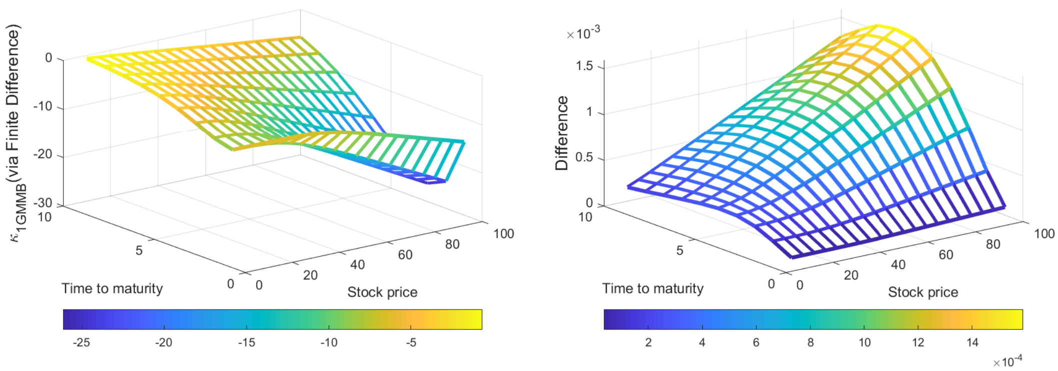

Figure A.19: Comparison with Finite Difference: κ_3

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PROJECTS

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Uğur Ö., Tekin Ö. Aladağlı E. E., Aydoğan B., Enginar O., Project Supported by Higher Education Institutions, A Real-Time Prototype for Algorithmic Trading and Financial Tools, (DAP-705-2018-2783), 2018