QUERY AGE OF INFORMATION IN COMMUNICATION NETWORKS

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MUHAMMED EMRULLAH ILDIZ

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submitted by **MUHAMMED EMRULLAH ILDIZ** in partial fulfillment of the requirements for the degree of **Master of Science in Electrical and Electronics Engineering Department, Middle East Technical University** by,

Prof. Dr. Halil Kalıpçılar
Dean, Graduate School of **Natural and Applied Sciences** ______________

Prof. Dr. İlkay Ulusoy
Head of Department, **Electrical and Electronics Engineering** ______________

Prof. Dr. Elif Uysal
Supervisor, **Electrical and Electronics Engineering, METU** ______________

Examining Committee Members:

Prof. Dr. Nail Akar
Electrical and Electronics Engineering, Bilkent University ______________

Prof. Dr. Elif Uysal
Electrical and Electronics Engineering, METU ______________

Prof. Dr. Tolga Mete Duman
Electrical and Electronics Engineering, Bilkent University ______________

Assist. Prof. Dr. Elif Tuğçe Ceran Arslan
Electrical and Electronics Engineering, METU ______________

Assist. Prof. Dr. Serkan Sarıtaş
Electrical and Electronics Engineering, METU ______________

Date: 22.08.2022

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Name, Surname: Muhammed Emrullah Ildız

Signature :

ABSTRACT

QUERY AGE OF INFORMATION IN COMMUNICATION NETWORKS

Ildız, Muhammed Emrullah
M.S., Department of Electrical and Electronics Engineering
Supervisor: Prof. Dr. Elif Uysal

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We study a pull-based status update communication model where a source node submits update packets to a channel with random transmission delay, at times requested by a remote destination node. The objective is to minimize the average query-age-of-information (QAoI), defined as the age of information (AoI) measured at query instants that occur at the destination side according to a stochastic arrival process. In reference to a push-based problem formulation defined in the literature where the source decides to update or wait at will, with the objective of minimizing the time average AoI at the destination, we name this problem the Pull-or-Wait (PoW) problem. We provide a comparison of the two formulations: (i) Under Poisson query arrivals and random transmission delay, an optimal policy that minimizes the time average AoI also minimizes the average QAoI, and these minimum values are equal; and (ii) the optimal average QAoI is shown to be less than or equal to the optimal time average AoI under the following two cases: (1) Periodic query arrivals and random transmission delay and (2) general query arrivals and constant transmission delay. We identify the PoW problem in the case of a single query as a stochastic shortest path (SSP) problem with uncountable state and action spaces, which has been not solved in previous literature. We derive an optimal solution for this SSP problem and use it

as a building block for the solution of the PoW problem under periodic query arrivals.

Keywords: Age of information, Communication Networks, Internet of things

ÖZ

HABERLEŞME AĞLARINDA SORGU ANI BİLGİ YAŞI

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Bir kullanıcı modulünün, güncelleme paketlerini kanal aracılığı ile rastgele bir kanal gecikmesine uğrayarak göndermesi için kaynak modulünden istemde bulunduğu bir güncelleme sistemi üzerinde çalışılmıştır. Amaç sorgu anlarındaki ortalama bilgi yaşını enküçüklemektir ve sorgu anları bir stokastik süreç ile belirlenmektedir. Arz güdümlü bir güncelleme sisteminde kaynak modulünün yeni bir güncelleme paketinin ne zaman gönderileceğine ortalama bilgi yaşını enküçüklediği daha önce çalışılmış bir probleme referansla yeni probleme Çek veya Bekle (PoW) problemi ismi verilmiştir. Bu iki problemin karşılaştırılması yapılmıştır: (i) Eğer sorgu anları Poisson süreç ile belirlenirse zamana göre ortalama bilgi yaşını enküçükleyen politika ayrıca sorgu anlarındaki ortalama bilgi yaşını da enküçüklemektedir ve iki ortalama en küçük bilgi yaşı birbirine eşittir; ve (ii) aşağıda verilecek iki durumda sorgu anlarındaki elde edilebilecek en iyi ortalama bilgi yaşı her zaman zamana göre elde edilebilecek en iyi ortalama bilgi yaşından daha küçüktür: (1) Sorgu anları periyodik ve kanal gecikmesi rastgele olduğunda ve (2) sorgu anları genel bir süreç ve kanal gecikmesi sabit olduğunda. PoW problemini tek sorgu anı için incelediğimizde bu problemin stokastik en kısa yol (SSP) problemleri sınıfından sayılamaz durum ve aksiyon uzayı sınıfına dahil olduğunu fark ettik ve bu sınıftaki problemlerin genel çözümü daha önce bulunamamıştır. Bu problemin çözümüne ulaştık ve bu çözümü periyodik sorgu anlarındaki PoW probleminin çözümünde kullandık.

Anahtar Kelimeler: Bilgi Yaşı, Haberleşme Ağları, Nesnelerin İnterneti

To the incompleteness of math

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AoI	Age of Information
QAoI	Query Age of Information
PoW Problem	Pull or Wait Problem
UoW Problem	Update or Wait Problem
SSP	Stochastic Shortest Path

CHAPTER 1

INTRODUCTION

1.1 Age of Information

The Internet of Things (IoT) paradigm has been gaining wide use in various sectors such as environmental monitoring [1], health and wellness [2], vehicular networks [3], smart cities [4], and so on. In many applications of these settings, a destination node seeks to have accurate information about a remote process measured by a sensor to utilize toward a computation. The received information packets by the destination node are not equally valuable: The value of the update packets highly depends on their timeliness.

As a metric to measure timeliness of update packets, the *age-of-information* (AoI), or simply *age*, has been introduced and studied in many different environments [5–7]. It is defined as the elapsed time since the generation of the latest received update packet. In other words, when the freshest information update packet available on the destination node at time t is generated at time $U(t)$, the age of information, $\Delta(t)$ on the destination node at time t is defined as follows: [8]

$$\Delta(t) = t - U(t) \tag{1.1}$$

To minimize AoI in a status update system, a sensor or a source node can generate an update packet any time by its own will and immediately send it to a destination node through a communication channel; this is referred to as the *generate-at-will* model [8–20]. This model was introduced in [9] and further studied in [10]. The problem formulation in [10] is concerned with the source generating updates judiciously, to minimize the overall time average AoI over a channel that imposes a random transmission delay. In this thesis, for brevity, we will refer to this formulation as the *Update-or-Wait (UoW)* problem. In the UoW problem, the source controls the age by determining the submission times of the update packets to the channel. The approach of minimizing the time average age of information as an objective models a destination node that continuously utilizes the update packets; however in many IoT scenarios the application running at the destination side will utilize the information updates at certain times, rather than continuously [21,22]. A policy that strives to keep the overall time average age at a minimum will not necessarily maintain minimal age at those utilization times.

1.2 Query Age Of Information

In this thesis, we define an extension of the UoW problem, which is referred to as the *Pull-or-Wait (PoW) problem.* In the PoW problem, the destination node requests an update packet from the source node in an effort to keep a low AoI at the next query instants, that are based on a stochastic arrival process. The *query-age-of-information* (QAoI) is defined as the age values measured at query instants. The goal of the destination in the PoW problem is to determine optimal request points to minimize QAoI, knowing only the statistics of the channel delay and the query arrival processes. The following simple example reveals the difference between the UoW and PoW problems.

1.3 An Interesting Example Comparing AoI and QAoI

Consider an IoT monitoring system that requires an update packet every 4 mseconds. Hence, the query instants are at times $4, 8, 12, \ldots$ The transmission delay of this channel is constant at 1.5 msec, but the requests for an update packet are assumed to arrive at the source node without any delay. The zero-wait policy is shown in [10] to be the optimal update policy for the UoW problem when the transmission delays are constant. The evolution of the age of information under the zero-wait policy is shown in Figure 1.1. This policy results in a time average age of information equal to 2.25

and performs one packet transmission per 1.5 msec. On the other hand, a reasonable policy, which is later shown to be an optimal policy, for the PoW problem is that the destination node requests update packets at times $2.5, 6.5, 10.5, \ldots$ as shown in Figure 1.1. As a result, this policy results in the average age of information at query instants equal to 1.5 and performs one packet transmission per query, 4 seconds.

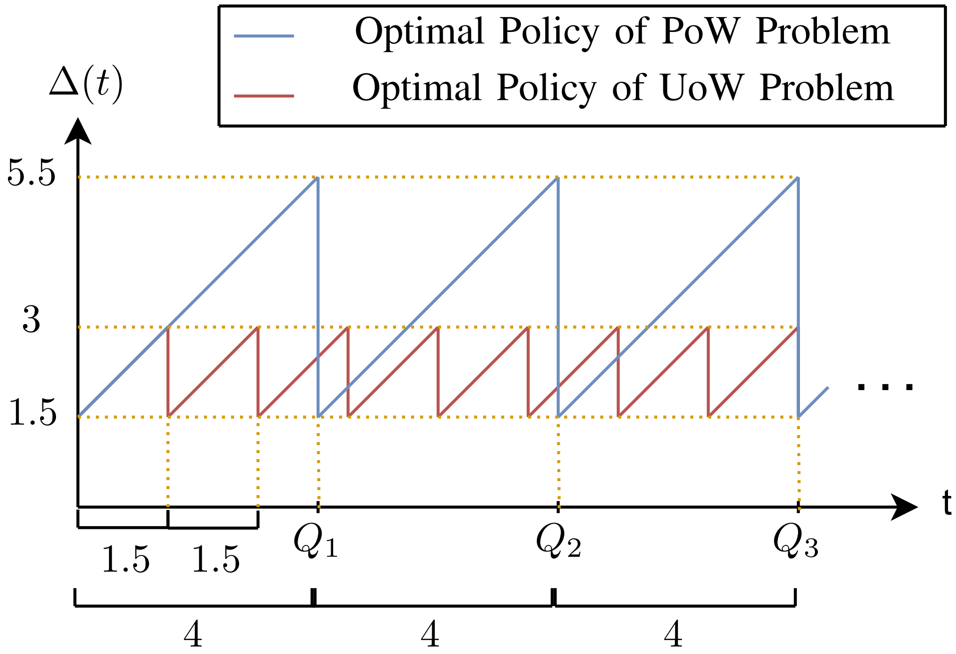

Figure 1.1: Evolution of the Age of Information under the optimal policies of the UoW and PoW formulations

Let us consider another IoT monitoring system in which the destination nodes utilize the update packets based on a stochastic process. Specifically, the interarrival times between utilization times *i.e.* queries are assumed to be either 6 msec or 7 msec with $1/2$ probability. The transmission delay from the source to the destination is 1.5 msec. The update policy that minimizes the time the average age of information is again the zero wait policy, which results in the average age of information equal to 2.25 msec. Additionally, the zero wait policy needs to send 2 update packets for every 3 msecs. An alternative update policy submits a new update packet to the channel at the time 4.5 msec after a query occurs. (In Section 3, this policy will be proved to be an optimal policy to minimize the expected query average age for this system.) This alternative update policy results in a query average age of information equal to 2 msec, and the expected transmission rate of this policy is $2/13$, which is far less than

$2/3$. As a result, minimizing the query age of information leads to better freshness with more contributive transmissions.

These two simple examples point out a crucial distinction between the UoW and PoW problems. The PoW formulation uses the knowledge about utilization time *i.e.* query instants to keep the AoI at the query instants much lower than that could be achieved in the UoW problem, while also reducing the number of transmissions. Hence, we observe that the two problems are distinct therefore the PoW problem calls for a comprehensive solution.

1.4 Contributions and Novelties

This thesis aims to answer the following questions: How to optimally request update packets to minimize the age of information upon query instants at the destination. Under what conditions is the PoW model significantly advantageous over the UoW model? The following are the key contributions of this thesis:

- We define the PoW problem as a direct extension of the UoW problem formulated in [10]. We show that under Poisson query arrivals, any optimal solution of the UoW problem is also an optimal solution of the PoW problem, achieving an equal age penalty. We prove that for periodic queries the optimal average age penalty of the PoW problem is always less than or equal to that of the UoW problem with the same power constraint.
- We identify the PoW problem for a single query, referred to as single query problem, as a stochastic shortest path problem with uncountable state and action spaces. To the best of our knowledge, this class of stochastic shortest path problems has not been solved in previous literature. We show the existence of a deterministic policy that solves this problem (Proposition 1) and characterize its first request point (Corollary 2). With the help of this characterization, we exhibit an explicit solution of the stochastic shortest path problem (Section 3.3).
- We employ the solution of the stochastic shortest path problem to construct a solution of the PoW problem under periodic query arrivals (Proposition 4).

- We expand the results in [23] by relaxing three aspects of the system model: Our analysis allows a general channel delay distribution; a general age penalty function; and does not require a discount factor in the objective function.

1.5 Related Work

AoI has attracted a remarkable amount of interest [6] and it has been studied under different formulations, such as enqueue-and-forward models [24–30], generate-at-will models [8,10–20], random access environments [31–35], and so on. Even though the age of information captures one semantic aspect of data, *i.e.* the freshness of information, it is not sufficient for all applications. For example, the optimal policy that minimizes the MSE in the remote estimation of a Wiener process over a random delay channel is distinct from the age optimal policy as shown in [36]. As a result, various suggestions for capturing the semantics of information have recently emerged [37–40]: the Age of Incorrect Information (AoII) extends the notion of fresh updates to that of fresh "informative" updates in [41–44]. Other metrics such as the Urgency of Information (UoI) and the Age of Changed Information (AoCI) have been proposed in [45] and [46], respectively.

The Query Age of Information (QAoI) is another metric that tries to capture the usefulness of an update packet with respect to an application more finely than the plain AoI. The QAoI is defined as the AoI measured at certain query instants, which represent the utilization times of the destination node in the application. This notion has been introduced in an independent set of works with different names such as Age upon Decision (AuD), Age of Effective Information (AoEI) [21,22,47–51]. The first works that suggest a pull-based communication model in the context of AoI are [47, 48], where a user proactively requests update packets from multiple servers, but the authors minimize the plain AoI and do not take utilization time into account. A series of works [21, 22, 49, 50] suggests AuD and studies a special case of the enqueue-and-forward model where a user utilizes upcoming update packets under a stochastic arrival process. This model leads the authors to measure the AoI at the utilization times. In [51], the authors study a multi-user information update system with Bernoulli update failures and suggest AoEI that measures the average AoI at the query instants.

The works that are most relevant to this work in this thesis are [10] and [23]. In [10], the authors consider a generate-at-will model to minimize time average AoI under a push-based communication model. We extend [10] to a pull-based communication model and modify the objective function with respect to the QAoI. In [23], the authors suggest the QAoI and study a similar pull-based communication model. Unlike the packet erasure channel that is considered in [23], we study more general channels that can have discrete, continuous, or mixed distributed transmission delays. In addition, we define an age penalty function $g(\Delta)$ to characterize the level of dissatisfaction for data staleness, where $g(.)$ can be any nonnegative, continuous, and nondecreasing function. This age penalty function enables us to simulate model-specific applications. Furthermore, we minimize the average age penalty at the query instants where there is no discount factor. In addition, we analytically compare the UoW and PoW problems under periodic and Poisson query arrival processes.

The rest of this thesis is organized as follows: In Section 2, we present the system models of the PoW and UoW problems. In Section 3, we formulate the PoW problem and analyze it. In Section 4, we compare the PoW problem with the UoW problem. In Section 5, we present numerical results to show the behavior of the solution in the PoW problem under different transmission delay processes. Finally, we conclude this thesis in Section 6 by summarizing our contributions and discussing future directions.

CHAPTER 2

SYSTEM MODEL AND PROBLEM FORMULATIONS

2.1 Pull or Wait (PoW) Problem

We consider a pull-based information update system depicted in Figure 2.1, where a destination node is interested in information updates generated by a source node. The destination node requests an update packet from the source node according to an update policy. The request arrives at the source node without any delay. When a request occurs, the source node immediately generates an update packet and submits it to the channel. The channel induces a random delay from the source node to the destination node. The destination node should not request a new update packet when the previously requested update packet has not arrived at the destination node, because this will incur an unnecessary waiting time in the queue.

The update packets delivered to the destination node are utilized toward a computation. In this information update system, we assume that the destination node possesses a query arrival process that represents the utilization time of the upcoming update packets received from the source node. The destination node aims to minimize the average AoI at the query instants. As the destination node can recognize past states of the query arrival process, it requests update packets from the source node by taking account of not only the random delays induced by the channel but also the past states of the query process.

Let the time that Update $j, j = 1, 2, \dots$ is requested from the source, and submitted to the communication channel be denoted by R_j. Update j is delivered to the destination node after a random transmission delay Y_j at time $D_j = R_j + Y_j$. Then, the destination node requests Update $j+1$ at time R_{j+1} after a waiting period

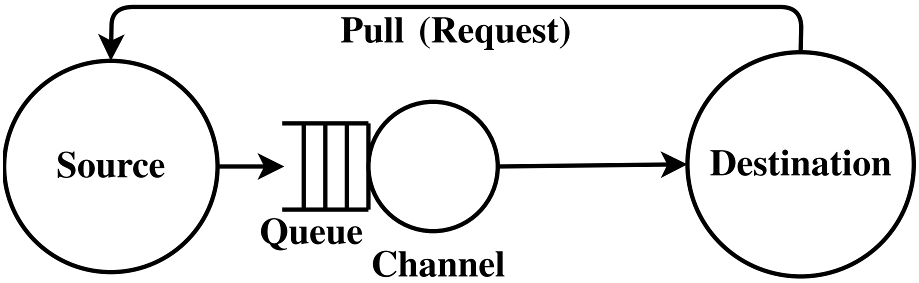

Figure 2.1: System Model of the PoW Problem

$Z_j \in [0, M]$. This implies that $R_{j+1} = D_j + Z_j$. We assume that the transmission delay process, $\{Y_j\}_{j=0}^{\infty}$, is i.i.d. and takes values in a bounded range such that $\Pr(Y_j \in [B_L, B_U]) = 1$ where $B_L > 0$. On the other side, the query arrives to the destination node at $\{Q_k, k = 1, 2, \ldots\}$ based on a stochastic process.

At any time t, let $U(t)$ denote the generation time of the update packet that has been most recently received by the destination node. Consequently,

$$\Delta(t) = t - \max\{R_j \colon D_j \leq t\} \tag{2.1}$$

We also introduce an age penalty function, $g(\Delta)$, that represents the level of dissatisfaction for data staleness or the need for a new information update. This function is defined as $g \colon [0, \infty) \to [0, \infty)$ and it is continuous, nonnegative, and nondecreasing. Our goal is to minimize the average age penalty at the time of queries by controlling the sequence of waiting periods, $(Z_0, Z_1, \ldots)$. Let $\pi = (Z_0, Z_1, \ldots)$ denote an update policy. A causal update policy determines the waiting period Z_j based on the sequence $(Z_i)_{i=0}^{j-1}$, the random processes $\{Y_j\}_{j=0}^{\infty}$, $\{Q_k\}_{k=1}^{\infty}$, and their realizations before D_j. Let Π be the set of all causal update policies. Then, the objective function is defined as the following:

$$\bar{h}_{opt} = \min_{\pi \in \Pi} \limsup_{n \to \infty} \frac{E\left[\sum_{k=1}^{n} g\big(\Delta(Q_k)\big)\right]}{n} \tag{2.2}$$

$$\text{s.t. } \liminf_{n \to \infty} \frac{1}{n} E\left[\sum_{j=1}^{n} (Y_j + Z_j)\right] \geq \frac{1}{f_{max}} \tag{2.3}$$

Throughout the thesis, we refer to this problem as the **Pull or Wait (PoW) problem**. We refer to the objective function of the PoW problem as the **query average age**

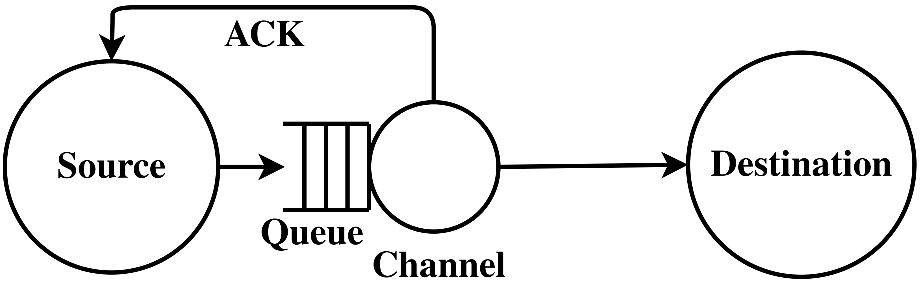

Figure 2.2: System Model of the UoW Problem

penalty.

2.2 Update or Wait (UoW) Problem

In the system model that was studied in [10] and that is depicted in Figure 2.2, the source node generates update packets and sends them to the destination node through the channel. Different from the system model of the PoW problem, the destination node does not request an update packet. Instead, the source node submits update packets to the channel seeking to minimize the time average age penalty at the destination node. Therefore, the objective function is the following:

$$\bar{g}_{opt} = \min_{\pi \in \Pi} \limsup_{n \to \infty} \frac{E\left[\int_0^{D_n} g\big(\Delta(t)\big)dt\right]}{E[D_n]} \tag{2.4}$$

$$\text{s.t. } \liminf_{n \to \infty} \frac{1}{n} E\left[\sum_{j=1}^{n} (Y_j + Z_j)\right] \geq \frac{1}{f_{max}} \tag{2.5}$$

Throughout the thesis, we refer to this problem as the **Update or Wait (UoW) problem**. We refer to the objective function of the UoW problem as the **time average age penalty**.

CHAPTER 3

PROBLEM FORMULATION AND ANALYSIS

In this chapter, we formulate the PoW problem without the power constraint (2.3). We first analyze the PoW problem under a specific case of single query. Let $Q > 0$ be the time at which the query occurs. For this case, Problem (2.2) reduces to:

$$\bar{h}_{opt}^{one}(Q) = \min_{\pi \in \Pi} E\big[g(\Delta(Q))\big] \tag{3.1}$$

Henceforth, we will refer to Problem (3.1) as the single query problem. As we will show in the rest of this section, the solution of the single query problem will be a building block of the solution of the PoW problem, given in (2.2), under periodic query arrivals.

The single query problem belongs to the class of stochastic shortest path problems with uncountable state and action spaces. The state of the problem at stage j is the pair of the remaining time from the delivery point of Update j until the query and the current age at the delivery point of Update j, $(Q - D_j, \Delta(D_j))$[1]. The random disturbance and the control action at stage j are Y_j and Z_j, respectively. The absorbing state occurs at stage j when $Q - D_j \leq 0$. State transitions that do not end in the absorbing state are costless. The cost of reaching the absorbing state from a state $(Q - D_j, \Delta(D_j))$ where $Q - D_j > 0$ is $g(Q - D_j + \Delta(D_j))$. This problem class is introduced in [52] for a finite state space, compact action space, a transition kernel that is continuous for all actions, under the assumption that an optimal

[1] It is shown in Proposition 1 that there exists an optimal policy of the single query problem in which Z_j is determined as a function of $Q - D_j$ and $\Delta(D_j)$. As a result, the single query problem can be minimized in the set of deterministic policies. When Z_j is determined as a function of $Q - D_j$ and $\Delta(D_j)$, the pair $(Q - D_j, \Delta(D_j)), j \geq 0$ forms a Markov chain because $\Delta(D_j) = Y_j$, Y_j's are i.i.d., and $Q - D_{j+1} = Q - D_j - Y_j - Z_j$.

policy must be proper (*i.e.* reachability of the termination state in a finite expected time). [53] relaxes the assumptions of [52] such that the state and action spaces are arbitrary, the transition kernel does not need to be continuous, but the space of the random disturbance is countable. A related problem class is introduced by [54] as transient Markov decision problems with solutions that are *transient policies* (similar, but not identical, to *proper policies*), general state and action spaces, and continuous transition kernel. [55] further relaxes the assumptions of [54] to the existence of non-transient policies, but keeps the assumption about the continuity of the transition kernel [55, Assumption 1b]. None of these results are directly applicable to the single query problem because in our problem the random disturbance Y_j may not come from a countable set and the transition kernel is not restricted to be continuous especially when the random disturbance Y_j has a mixed distribution.

In the rest of this section, we will show the existence of a deterministic optimal policy for the single query problem, and characterize its first request point in Section 3.1. With the help of this characterization, we will reformulate the PoW problem under periodic query arrivals in terms of the single query problem in Section 3.2. Finally, we will provide a complete solution of the single query problem in Section 3.3, which concludes the solution of the PoW problem in (2.2) under periodic query arrivals.

3.1 Existence of a Deterministic Optimal Policy for the Single Query Problem

In this section, we first show that there exists an optimal policy, π_1^{opt}, for the single query problem, that is a deterministic policy. Then, we define the *border point* of π_1^{opt} for a query arriving at time Q, denoted as $Q^{BP} \in [Q - 3B_U, Q - B_U]$. We prove that Q^{BP} is an optimal request point under the policy π_1^{opt} for every delivery point D_j satisfying $D_j < Q - 3B_U$. This property will help us transform the solution of the single query problem into a solution of the PoW problem under periodic query arrivals.

At any delivery point D_j, an optimal update policy seeks to find a request point R_{j+1} to minimize the expected age penalty at the query. To express the expected age penalty at the query in terms of a request point R_j, we define the G_R^{π} function. In addition to the G_R^π function, we define the G_D^π function to express the expected age penalty at the query in terms of a delivery point D_j as the following:

Definition 1. *For a given query Q, let R_j and D_j be any request and delivery points, respectively. $G_R^\pi\colon [0,\infty)\times[0,\infty)\to[0,\infty)$ and $G_D^\pi\colon [0,\infty)\times[0,\infty)\to[0,\infty)$ are defined as follows:*

$$G_R^\pi\Big(Q-R_j,\Delta(R_j)\Big) = E\Big[g(\Delta(Q))\Big|\pi \text{ is applied}, R_j \text{ is a request point}, \text{AoI at } R_j \text{ is } \Delta(R_j)\Big] \tag{3.2}$$

$$G_D^\pi\Big(Q-D_j,\Delta(D_j)\Big) = E\Big[g(\Delta(Q))\Big|\pi \text{ is applied}, D_j \text{ is a delivery point}, \text{AoI at } D_j \text{ is } \Delta(D_j)\Big] \tag{3.3}$$

These expectations are taken over the possible transmission delays and the waiting period decisions by the policy $\pi\in\Pi$.

It will be shown in Proposition 1 that the information of the remaining time until the query $Q-D_j$ and the AoI at the delivery point $\Delta(D_j)$ are sufficient statistics to determine an optimal waiting period. This implies that the minimization of the single query problem can be performed by only considering the set of causal policies that determines the waiting period Z_j based on $Q-D_j$ and $\Delta(D_j)$. Therefore, there is no need to explicitly provide the sequences of $(Y_i)_{i=0}^{j}$ and $(Z_i)_{i=0}^{j-1}$ for the functions G_R^π and G_D^π.

The two functions have a chain relationship with each other. When the destination node requests an update packet from the source node at R_j, Update j is delivered to the destination node after a random transmission delay Y_j at time $D_j = R_j + Y_j$. Hence, $\Delta(D_j) = Y_j$. If the delivery occurs before the query i.e. $Q-R_j-Y_j\geq 0$, the expected age penalty can be represented with the function G_D^π. If $Q-R_j-Y_j<0$, the AoI at the query is $Q-R_j+\Delta(R_j)$ for sure. This relationship can be written as follows:

$$G_R^\pi\Big(Q-R_j, \Delta(R_j)\Big)$$
$$= E\Big[G_D^\pi(Q - R_j - Y_j, Y_j)\Big| Y_j \leq Q - R_j\Big] \times \Pr(Y_j \leq Q - R_j) \qquad (3.4)$$
$$+ g\big(Q - R_j + \Delta(R_j)\big) \times \Pr(Y_j > Q - R_j)$$

This expectation is taken over possible transmission delays.

On the other hand, when the update packet is delivered to the destination node at D_j, the destination node waits for a duration Z_j to request a new update packet. Hence, the request point is $Q - D_j - Z_j$, and the AoI at the request point is $\Delta(D_j) + Z_j$. When the request point is before the query i.e, $Q - D_j - Z_j \geq 0$, the expected age penalty at the query can be represented with the function G_R^π. When $Q - D_j - Z_j < 0$, the AoI at the query is $Q - D_j + \Delta(D_j)$ for sure. This relationship can also be written as follows:

$$G_D^\pi\Big(Q - D_j, \Delta(D_j)\Big)$$
$$= E\Big[G_R^\pi(Q - D_j - Z_j, \Delta(D_j) + Z_j)\Big| Z_j \leq Q - D_j\Big] \times \Pr(Z_j \leq Q - D_j) \qquad (3.5)$$
$$+ g\big(Q - D_j + \Delta(D_j)\big) \times \Pr(Z_j > Q - D_j)$$

This expectation is taken over possible waiting periods that are determined by the policy π in order to take randomized policies into account.

Now, we move on to obtain a deterministic optimal policy of the single query problem. The optimal age penalty in this problem can be achieved in a special subset of Π. In the next proposition, we prove this in detail.

Definition 2.

- *A policy $\pi \in \Pi$ is said to be a stationary and deterministic policy if there exists decision function $z : [0, \infty) \times [0, \infty) \to [0, M]$ such that $Z_j = z(Y_j, Q - D_j)$ for $j = 0, 1, \ldots$*
- *The set of all stationary and deterministic policies is denoted as Π_{SD}.*

Proposition 1. *If the transmission delay process $\{Y_j\}_{j=0}^{\infty}$ is i.i.d. such that* $\Pr(Y_j \in$

$[B_L, B_U]) = 1$, $M < \infty$, *and the penalty function* g *is continuous, non-negative, and non-decreasing, then there exists a deterministic update policy that is optimal for the single query problem.*

Proof. In the proof, we need to use the extended version of the functions G_R^π and G_D^π that must include the sequences of $(Y_i)_{i=0}^{j}$ and $(Z_i)_{i=0}^{j-1}$ in order to cover all possible causal update policies. Hence, they are $G_D^\pi(Q - D_j, \Delta(D_j), (Y_i)_{i=0}^{j}, (Z_i)_{i=0}^{j-1})$ and $G_R^\pi(Q - R_j, \Delta(R_j), (Y_i)_{i=0}^{j-1}, (Z_i)_{i=0}^{j-1})$. Let us map each $Q - D_j$ to a natural number n satisfying $(n-1)B_L \leq Q - D_j < nB_L$. We perform discrete induction on n. The proposition is first proved for every j, $(Y_i)_{i=0}^{j}$, and $(Z_i)_{i=0}^{j-1}$ that satisfy $(n-1)B_L \leq Q - D_j < nB_L$ when $n = 1$. Then, the proposition is assumed to be correct when $n = 2, 3, \ldots, K$ where K is an arbitrary natural number. Finally, it is proved when $n = K + 1$. The detailes are given in the Appendix A. □

According to the previous proposition, there exists a deterministic optimal update policy $\pi_1^{opt} \in \Pi_{SD}$ that decides waiting periods based on the values of $Q - D_j$ and $\Delta(D_j)$ for every j, $(Y_i)_{i=0}^{j}$, and $(Z_i)_{i=0}^{j-1}$. Interestingly, for some specific values of $Q - R_j$, the expected age penalty at the query may not depend on the value of $\Delta(R_j)$. For example, when the destination node is supposed to request an update packet from the source node before $Q - B_U$, the requested update packet must reach the destination node before the query. This is because the transmission delay can be at most B_U. Therefore, the AoI at the request point cannot affect the expected age penalty at the query. The next proposition proves this in detail.

Proposition 2. *If the elapsed time since a request point until the query is greater than* B_U, *then the AoI at the request point does not affect the expected age penalty at the query under a deterministic policy.*

Proof. This proposition is an immediate result of (3.4). If $Q - R_j \geq B_U$, then $Y_j \leq Q - R_j$ for sure. Therefore, (3.4) becomes

$$G_R^\pi\Big(Q - R_j, \Delta(R_j)\Big) = E\Big[G_D^\pi\big(Q - R_j - Y_j, Y_j\big)\Big] \tag{3.6}$$

As the transmission delay process is i.i.d. and $\Delta(R_j) = Y_{j-1} + Z_{j-1}$, $\Delta(R_j)$ does not affect $G_D^{\pi}(Q-R_j-Y_j, Y_j)$ when $Q-R_j$ is given. Hence, the proof is completed. Note that this property is valid for every $\pi \in \Pi_{SD}$. □

As a result of previous proposition, we can modify the function $G_R^{\pi_1^{opt}}$ when π_1^{opt} is a deterministic optimal policy and $Q-R_j$ is greater than or equal to B_U. Hence, for every request point R_j and its AoI $\Delta(R_j)$ satisfying $Q-R_j \geq B_U$, we redefine the $G_R^{\pi_1^{opt}}$ function with one argument as the following:

$$G_R^{\pi_1^{opt}}(Q-R_j) = G_R^{\pi_1^{opt}}\Bigg(Q-R_j, \Delta(R_j)\Bigg) \tag{3.7}$$

For a given query Q and a deterministic optimal policy π_1^{opt}, let us define its border point Q^{BP} that satisfies the following:

$$G_R^{\pi_1^{opt}}(Q-Q^{BP}) = \inf_{R_j:\, R_j \leq Q-B_U} G_R^{\pi_1^{opt}}(Q-R_j) \tag{3.8}$$

In the next proposition, we show the existence of a border point. Then, we specify one of these points as the border point.

Proposition 3. *Let D_j^* be a specific delivery point satisfying $D_j^* = Q-3B_U$ and $Y_j^* = B_L$. The request point R_{j+1}^* determined by a deterministic optimal policy π_1^{opt} is a border point for the query Q. We designate R_{j+1}^* as the 'selected' border point.*

Proof. In the proof, we first prove that the request must occur by the time $Q-B_U$ i.e, $R_{j+1}^* \leq Q-B_U$. This ensures that R_{j+1}^* is in the intended interval of (3.8). Then, we show that the optimal request point, R_{j+1}^*, attains the infimum in (3.8). The detailed proof is given in Appendix B. □

In the rest, for brevity, we will refer to the selected border point as the border point. The exact location, Q^{BP}, of the border point depends on the exact time of the query and the optimal policy π_1^{opt}. This is because the border point is specified as the request point that is determined by π_1^{opt} when the delivery point is $Q-3B_U$ and the age at the delivery point is B_L. Hence, the border point can be considered as a function of a query Q and a deterministic optimal policy π_1^{opt}. Nevertheless, there is a special property of the border point concerning the relation between Q and Q^{BP}, proved in the following corollary:

Corollary 1. *The time duration between a query and its border point does not depend on the exact time of the query for a given deterministic optimal policy.*

Proof. This corollary is an immediate result of Proposition 1 and the definition of R^*_{j+1}. The request point R^*_{j+1} determined by a deterministic optimal policy π_1^{opt} is the border point when $D^*_j = Q - 3B_U$ and $\Delta(D^*_j) = Y^*_j = B_L$ regardless of the exact time of the query. The optimal waiting period at D^*_j is solely determined by π_1^{opt} based on $Q - D^*_j$ and $\Delta(D^*_j)$ by Proposition 1. As Q changes, $Q - D^*_j$ and $\Delta(D^*_j)$ do not change. Hence, Z^*_j does not change. As $R^*_{j+1} = Q - 3B_U + Z^*_j$, the proof is completed. □

We next prove in Lemma 1 that if a delivery point occurs before $Q - 3B_U$, then it is optimal to wait until the border point to place a request.

Lemma 1. *Let Q^{BP} be the border point of a query Q and a deterministic optimal policy π_1^{opt}. Then, for any delivery point D_j satisfying $D_j < Q - 3B_U$, the border point Q^{BP} is an optimal request point under the policy π_1^{opt}.*

Proof. To reach contradiction, suppose that the claim is false. Then, there exists a delivery point $D_j \in [0, Q - 3B_U)$ and an AoI at the delivery $\Delta(D_j)$ such that the request point R_{j+1} determined by a deterministic optimal policy π_1^{opt} satisfies the following: $G_R^{\pi_1^{opt}}(Q - Q^{BP}) > G_R^{\pi_1^{opt}}(Q - R_{j+1}, \Delta(R_{j+1}))$[2]. By (3.8), R_{j+1} cannot be in the interval $[0, Q - B_U)$. By Lemma 5 that is given in the proof of Proposition 1, R_{j+1} cannot be in the interval $[Q - B_U, Q]$ as well. This completes the proof. □

Corollary 2. *There exists a deterministic optimal policy π_1^{opt} for a given query Q satisfying $Q > 3B_U$ such that the first request point is the border point.*

Proof. This is an immediate result of Lemma 1 and the designation of the border point in Proposition 3. □

[2] As R_{j+1} can be in the interval $[Q - B_U, Q]$, the $G_R^{\pi_1^{opt}}$ function should be written with AoI argument.

Corollary 3. *If $Q > 3B_U$, $\bar{h}_{opt}^{one}(Q)$ is independent of the exact time of the query Q. In other words, we can define $\bar{h}_{opt}^{one}$ as the following:*

$$\bar{h}_{opt}^{one} = \bar{h}_{opt}^{one}(Q) = G_R^{\pi_1^{opt}}(Q - Q^{BP}) \tag{3.9}$$

where π_1^{opt} is a deterministic optimal policy and Q^{BP} is their border.

Proof. From Corollary 2, there exists a deterministic optimal policy π_1^{opt} whose first request point is the border for a given query Q satisfying $Q > 3B_U$. This means that $\bar{h}_{opt}^{one}(Q) = G_R^{\pi_1^{opt}}(Q - Q^{BP})$. Furthermore, the time duration between $Q - Q^{BP}$ does not change when Q is shifted by Corollary 1. Hence, the expected age penalty at the border point for any $Q > 3B_U$ is the same because the destination node can request an update packet at the border point under an optimal policy. As a result, we can define $\bar{h}_{opt}^{one} = \bar{h}_{opt}^{one}(Q)$. This completes the proof. □

Thus far, we have shown the existence of an optimal policy π_1^{opt} that has two important properties:

- π_1^{opt} is a deterministic optimal policy that decides the waiting period at D_j solely based on the $Q - D_j$ and $\Delta(D_j)$.
- The first request point of the policy π_1^{opt} is in the interval $[Q - 3B_U, Q - B_U]$.

These two properties enable us to transform the optimal update policy of the single query problem into an optimal update policy of the PoW problem under periodic query arrivals.

3.2 Periodic Sequence of Queries

In this section and next section, we assume that the query arrival process $\{Q_k\}_{k=1}^{\infty}$ is deterministic and periodic with T. Let $Q_k = kT$, for $k = 1, 2, \ldots$. Furthermore, we assume that $T > 4B_U$.[3] Based on these assumptions, we construct an optimal update

[3] Considering the delay in many practical communication links is expected to be much lower than the query period for typical applications, this assumption is not restrictive for many practical cases of interest.

policy π^{opt} for a periodic sequence of queries in the next proposition. Then, we point out the properties of the update policy π^{opt} based on the next proposition.

Proposition 4. *If the transmission delay process* $\{Y_j\}_{j=0}^{\infty}$ *is i.i.d. such that* $\Pr(Y_j \in [B_L, B_U]) = 1$ *and the query arrival process,* $\{Q_k\}_{k=1}^{\infty}$*, is deterministic and periodic with* $T > 4B_U$*, then* $\bar{h}_{opt}$ *is equal to* $\bar{h}_{opt}^{one}$*.*

Proof. It is clear that $\bar{h}_{opt}^{one} \leq \bar{h}_{opt}$. Otherwise, it would contradict the optimal solution of the single query problem. Therefore, it is enough to construct an update policy π^{opt} achieving $\bar{h}_{opt}^{one}$ of expected age penalty for the periodic sequence of queries.

Let π_1^{opt} be the optimal policy of the single query problem characterized in Corollary 2. Let Q_i^{BP} be the border point of Q_i and π_1^{opt}. From the starting point, π^{opt} can follow π_1^{opt} between $[0, Q_1]$. This can be performed because π_1^{opt} decides to wait until Q_1^{BP} and $Q_1^{BP} \geq Q_1 - 3B_U > 0$. From Corollary 3, the expected age penalty at Q_1 is $G_R^{\pi_1^{opt}}(Q_1 - Q_1^{BP})$. As the policy π^{opt} follows π_1^{opt} until the point Q_1, the channel must be idle before $Q_1 + B_U$ as the transmission delay can be at most B_U. When the channel is idle, π^{opt} can follow π_1^{opt} again, but this time the policy is performed for the query Q_2. The act of following the policy π_1^{opt} is possible because $Q_2^{BP} \geq Q_2 - 3B_U > Q_1 + B_U$. Hence, the expected age penalty at Q_2 is $G_R^{\pi_1^{opt}}(Q_2 - Q_2^{BP})$ by Corollary 3. For the remaining queries $Q_3, Q_4, \ldots$, it can be replicated similar to Q_2. Then, the expected age penalty at every query Q_k is $G_S^{\pi_1^{opt}}(Q_k - Q_k^{BP})$. From Corollary 1, all of the expected age penalties are equal to $\bar{h}_{opt}^{one}$. □

The previous proposition allows us to decouple the immediate next query from the set of all the queries while constructing an optimal policy π^{opt} for the PoW problem under periodic query arrivals. As a result, the update policy π^{opt} takes only the immediate next query into account. This decoupling property enables us to solve the PoW problem without a discount factor. The next corollary presents another result of the decoupling property.

Corollary 4. *Let* $A_j = Q - T\left\lfloor \frac{D_j}{T} \right\rfloor$ *that represents the remaining time until the next query at a delivery point* D_j*. The update policy* π^{opt} *constructed in Proposition 4 is a stationary and deterministic policy, which is a function of* $\Delta(D_j) = Y_j$ *and* A_j*.*

Proof. The update policy π^{opt} is a repetitive employment of the update policy π_1^{opt}, that is characterized in Corollary 2. Therefore, π^{opt} possesses all the properties of π_1^{opt}. As π_1^{opt} is solely determined based on $Q - D_j$ and $\Delta(D_j)$ by Proposition 1, π^{opt} is stationary and deterministic function of A_j and $\Delta(D_j) = Y_j$. □

Note that we prove in Corollary 4 that the constructed policy π^{opt} is a stationary and deterministic policy, which is a function of A_j and Y_j. We also show in Proposition 4 that the optimal update policy for the PoW problem under periodic query arrivals turns out to *myopic* in the sense that at any delivery point, the decision about the optimal waiting time does not depend on future queries other than the immediate next one. Therefore, what remains to solve the PoW problem is to find an optimal policy for the single query problem, and apply it at each consecutive query interval.

3.3 Explicit Solution of PoW Problem

In the previous section, we exploited the decoupling property Proposition 4 to show that one can construct a solution of the PoW problem under periodic query arrivals through employing a sequence of deterministic policies that each solve the single query problem. In this section, we provide an explicit solution of the single query problem by generating a sequence of update policies that are solutions of stochastic shortest path problems with finite state and action spaces obtained by quantization. Then, we show that the sequence of update policies converges to an optimal policy of the single query problem with increasingly fine quantization. The quantization argument is given next.

We divide the real line interval $[0, Q]$ into N equal sub-intervals, and define two new transmission delay processes:

1. Upper Quantized Transmission Delay Process: If a transmission delay Y_j occurs with a probability in a transmission delay process, the transmission delay is quantized to $\frac{Q}{N}\left\lceil \frac{Y_j}{Q/N} \right\rceil$ with the same probability in its upper quantized transmission delay process. In other words, for every $m \in \mathbb{N}$, we have the following:

$$\Pr\left(Y_j^{upp} = m\frac{Q}{N}\right) = \Pr\left(Y_j \in \left((m-1)\frac{Q}{N}, m\frac{Q}{N}\right]\right) \tag{3.10}$$

2. Lower Quantized Transmission Delay Process: If a transmission delay Y_j occurs with a probability in a transmission delay process, the transmission delay is quantized to $\frac{Q}{N}\left\lfloor\frac{Y_j}{Q/N}\right\rfloor$ with the same probability in its lower quantized transmission delay process. In other words, for every $m \in \mathbb{N}$, we have the following:

$$\Pr\left(Y_j^{low} = (m-1)\frac{Q}{N}\right) = \Pr\left(Y_j \in \left[(m-1)\frac{Q}{N}, m\frac{Q}{N}\right)\right) \tag{3.11}$$

Even though the transmission delays are quantized, an optimal policy can determine waiting periods in the real interval $[0, M]$. Hence, the state space is still an uncountable set. The next proposition allows us to restrict the state space to a finite set.

Proposition 5. *When a quantization on the transmission delay is performed for any number of sub-intervals N, there exists an optimal update policy whose request points are in the set $\left\{0, \frac{Q}{N}, \frac{2Q}{N}, \ldots, Q\right\}$.*

Proof. The proof is given in Appendix C. □

The state and action spaces for lower and upper quantizations of a transmission delay process becomes finite because the ages at the delivery points are quantized and the possible delivery points form a finite set as a result of Proposition 5. Then, we can define the spaces of A_j, Y_j, and Z_j as follows:

Definition 3. *For a given query Q, let us define the following sets:*

- $\mathcal{A}^N = \left\{0, \frac{Q}{N}, \frac{2Q}{N}, \ldots, Q\right\}$
- $\mathcal{Z}^N = \left\{0, \frac{Q}{N}, \frac{2Q}{N}, \ldots, \frac{\left\lfloor\frac{M}{Q/N}\right\rfloor Q}{N}\right\}$
- $\mathcal{Y}^N = \left\{\frac{\left\lfloor\frac{B_L}{Q/N}\right\rfloor Q}{N}, \frac{\left(\left\lfloor\frac{B_L}{Q/N}\right\rfloor+1\right) Q}{N}, \ldots, \frac{\left\lceil\frac{B_U}{Q/N}\right\rceil Q}{N}\right\}$

Up to now, we have only analyzed the optimal update policy for quantized transmission delays. The next proposition puts an upper and a lower bound to the optimal expected age penalty for an unquantized transmission delay process. Furthermore, it proposes an update policy whose expected age penalty lays between the upper and lower bounds with the help of characterization in Section 3.1.

Proposition 6. *For any given transmission delay process and the number of subintervals N, the following hold:*

(i) There exists an update policy for an unquantized transmission delay process whose expected age penalty is less than or equal to the optimal age penalty for the upper quantized transmission delay process.

(ii) The optimal expected age penalty for lower quantization of a transmission delay process is less than or equal to the optimal expected age penalty for the unquantized transmission delay process.

Proof. For the proof of (i), we construct an update policy for an unquantized transmission delay process whose expected age penalty is less than or equal to the optimal expected age penalty for the upper quantized transmission delay process. There exists an optimal update policy for the upper quantized transmission delay process by Proposition 1. Let π_1^{opt} be a deterministic optimal policy that is characterized in Corollary 2. Let $z^{opt}(.,.)$ be the decision function of the update policy π_1^{opt}. The constructed optimal policy determines R_{j+1} for $j \geq 0$ as the following:

$$R_{j+1} = Q - \frac{Q}{N}\left\lceil \frac{A_j}{Q/N} \right\rceil + z^{opt}\left(\frac{Q}{N}\left\lceil \frac{\Delta(Y_j)}{Q/N} \right\rceil, \frac{Q}{N}\left\lceil \frac{A_j}{Q/N} \right\rceil \right) \tag{3.12}$$

Now, let us prove that this constructed policy gives the desired expected age penalty. Let $(Y_i)_{i=1}^{J}$ where J is an arbitrary natural number be a transmission delay sequence from the unquantized transmission delay process when an update packet at the border point is requested. The correspondence of the transmission delay sequence on the upper quantized transmission delay process is $\left(\frac{Q}{N}\left\lceil \frac{Y_i}{Q/N} \right\rceil\right)_{i=1}^{J}$. If the constructed policy follows the steps above, then the request points are the same for $(Y_i)_{i=1}^{J}$ and $\left(\frac{Q}{N}\left\lceil \frac{Y_i}{Q/N} \right\rceil\right)_{i=1}^{J}$. Thus, for any D_j where $1 \leq j \leq J$, the AoI in the interval $\Big[Q - D_j, \frac{Q}{N}\left\lceil \frac{(Q-D_j)}{Q/N} \right\rceil\Big)$ is smaller for the unquantized transmission delay process. For every point outside this interval, the AoI will be the same for both of the transmission delay processes. This is valid for every transmission delay sequence $(Y_i)_{i=1}^J$, hence the expected age penalty for the unquantized transmission delay process is less than or equal to the optimal expected age penalty for the upper quantized transmission delay process.

The proof of (ii) is similar to the previous part. Let π_1^{opt} be a deterministic optimal policy for the unquantized transmission delay process. By Proposition 1, π_1^{opt} can find the optimal waiting period for every $\Delta(D_j)$ and $Q - D_j$. If the destination nodes follow the same update policy π_1^{opt} for the lower quantized transmission delay process, the obtained expected age penalty is less than or equal to the optimal expected age penalty for the unquantized transmission delay process. This completes the proof. □

The optimal update policy for the upper quantization of a transmission delay process enables us to construct an update policy for the transmission delay process. The expected age penalty resulting from this constructed update policy is proved to lays between the optimal expected age penalties of the upper and lower quantized transmission delay processes. Furthermore, we show in the next proposition that the upper and lower bounds converge to each other as N increases. Thus, we can find an update policy whose expected age penalty is arbitrarily close to the optimal expected age penalty for any transmission delay process and age penalty function.

Proposition 7. *For $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that the difference between optimal expected age penalties of upper and lower quantizated transmission delay processes is less than ϵ if the quantization is performed with $N \geq N_1$ sub-intervals.*

Proof. The proof is provided in Appendix D. □

Propositions 6 and 7 employ optimal solutions of the upper and lower quantized transmission delay processes while constructing an update policy for the unquantized transmission delay process. Hence, the remaining part of this section is to solve the stochastic shortest path problem for quantized transmission delay processes. When the transmission delay process is quantized, the problem turns out to be a stochastic shortest path problem with finite state and action spaces as a result of Proposition 5. This problem class can be solved by the value iteration method given the explicit cost of each action in each state [56].

To provide an explicit cost of each action in each state, we again use the function G_R^{π}. We prove in Proposition 1 that there exists a deterministic policy $\pi_1^{opt} = z(Y_j, A_j)$ that is optimal for a given transmission delay process $\{Y_i\}$. Then, the following can be obtained by incorporating (3.5) into (3.4):

$$\begin{aligned} &G_R^{\pi_1^{opt}}\Big(Q - R_j, \Delta(R_j)\Big) \\ &= E\Big[G_R^{\pi_1^{opt}}(Q - R_j - Y_j - Z_j, Y_j + Z_j)\Big| Y_j + Z_j \le Q - R_j\Big] \\ &\qquad\qquad \times \Pr(Y_j + Z_j \le Q - R_j) \\ &+ g\big(Q - R_j + \Delta(R_j)\big) \times \Pr(Y_j + Z_j > Q - R_j) \end{aligned} \tag{3.13}$$

where $Z_j = z(Y_j, A_j)$.

The single query problem is explicitly solved in Algorithm 1. In this algorithm, the functions ${}^{upper}G_R^{\pi^{opt}}$ and ${}^{lower}G_R^{\pi^{opt}}$ denote the expected age penalties for the upper and lower quantized transmission delays, respectively. These functions are recursively calculated by using (3.13) similar to the value iteration method. This calculation is performed through the loop in $\mathcal{A}^N$ with ascending order. The optimal waiting time for a pair $(Y_j, A_j) \in \mathcal{Y}^N \times \mathcal{A}^N$ is determined by minimizing the function G_R^{π} in the set $\mathcal{Z}^N$. Note that the set $\mathcal{A}^N, \mathcal{Y}^N$, and $\mathcal{Z}^N$ is employed in the algorithm as if they are arrays.

The output of Algorithm 1 is a decision function of Y_j and A_j that characterizes an optimal update policy of the single query problem for the upper quantized transmission delay process. An optimal policy of the single query problem for the unquantized transmission delay process is constructed by an optimal update policy for the upper quantized transmission delay process as it is shown in Proposition 6(i). Then, the constructed update policy is applied to each consecutive query interval, which is optimal for the PoW problem under periodic query arrivals.

Algorithm 1 is more general and efficient than the algorithm obtained in [23]. Algorithm 1 is more general because it gives an optimal policy for every bounded transmission delay process in a continuous domain whereas the algorithm in [23] is restricted to the time slotted systems and packet erasure channels. Additionally, different from [23], we introduce a continuous penalty function that can model a wide class of applications. Algorithm 1 is more efficient than the algorithm obtained in [23] because we first decouple the queries and then apply a value iteration algorithm whereas the algorithm in [23] applies the value iteration algorithm to directly infinitely many queries, which significantly increases the complexity.

Algorithm 1 Solution of the Single Query Problem

given tolerance ϵ and sufficiently large N
repeat
for $i = 1$ to length($\mathcal{A}^N$) **do**
  for $j = 1$ to length($\mathcal{Y}^N$) **do**
    for $k = 1$ to length($\mathcal{Z}^N$) **do**
      Using (3.13), calculate

$$
{}^{upper}G_R^{\pi^{opt}}\Big(\mathcal{A}^N(i) - \mathcal{Z}^N(k), \mathcal{Y}^N(j) + \mathcal{Z}^N(k)\Big),
$$
$$
{}^{lower}G_R^{\pi^{opt}}\Big(\mathcal{A}^N(i) - \mathcal{Z}^N(k), \mathcal{Y}^N(j) + \mathcal{Z}^N(k)\Big)
$$

    end for

$$
{}^{upper}z\Big(\mathcal{Y}^N(j), \mathcal{A}^N(i)\Big) = \max\left\{ \arg\min_{x \in \mathcal{Z}^N} {}^{upper}G_R^{\pi^{opt}}\Big(\mathcal{A}^N(i) - x, \mathcal{Y}^N(j) + x\Big) \right\}
$$

$$
{}^{lower}z\Big(\mathcal{Y}^N(j), \mathcal{A}^N(i)\Big) = \max\left\{ \arg\min_{x \in \mathcal{Z}^N} {}^{lower}G_R^{\pi^{opt}}\Big(\mathcal{A}^N(i) - x, \mathcal{Y}^N(j) + x\Big) \right\}
$$

  end for
end for
$N = 2N$
until ${}^{upper}G_R^{\pi^{opt}}(Q) - {}^{lower}G_R^{\pi^{opt}}(Q) < \epsilon$
return ${}^{upper}z$

CHAPTER 4

COMPARISON BETWEEN POW AND UOW PROBLEMS

In this chapter, we compare these two problems. First, we share a case in which the two problems are equivalent. Then, we show the superity of the PoW problem over UoW problem under the following two cases: (i) The transmission delay is bounded and the query arrival process is periodic; and (2) the transmission delay is constant and the query arrival process is an i.i.d. general process.

4.1 Poisson Query Arrival Process

In this section, we prove that PoW problem is equivalent to UoW problem when the query arrival process is Poisson.

Proposition 8. *Let the query arrival process of a PoW problem be a Poisson process. For any transmission delay process, the optimal update policy that solves the UoW problem also solves the PoW problem with the same transmission delay process. Moreover, the optimal time average and query average age penalties are equal.*

Proof. The proof is provided in Appendix E, and it is based on the 'Poisson arrivals see time averages' property exhibited by the query process. □

4.2 General Transmission Delay Process

In this section, we assume that the transmission delay process is i.i.d. and takes values in a bounded interval. On the other hand, the query arrival process is periodic with any period T.

It was proven in [10] that an optimal solution of the UoW problem can be found in special subset of the set Π, which was defined as follows:

Definition 4.

- *A policy $\pi \in \Pi$ is said to be a stationary and deterministic policy if there exists a decision function $z : [0, \infty) \rightarrow [0, M]$ such that $Z_j = z(Y_j)$ for all $j = 0, 1, \ldots$.*

- *The set of all stationary and deterministic policies is denoted as Π_{SD}.*

For a given f_{max} and a transmission delay process $\{Y_j\}$, let $\pi^{opt} \in \Pi$ be an optimal update policy for the UoW problem. It was proven in [10] that the objective function in (2.4) attains its limit under the stationary and deterministic policies. Hence, we can define the function $g_{opt} \colon [B_L, B_U] \rightarrow \mathbb{R}^+$ as follows:

$$g_{opt}(Y_0) = \lim_{n \to \infty} \frac{E_{\mathcal{Y}}\big[\int_0^{D_n} g(\Delta(t))dt \big| Y_0, \pi^{opt}\big]}{E[D_n]} \tag{4.1}$$

where the expectation is taken with respect to transmission delay sequences given that the update policy π^{opt} is performed. Let us define a function $f \colon ([0, T], \Pi_{SD}) \rightarrow \Pi$ in an effort to construct update policies based on a stationary and deterministic update policy.

$$f(x, \pi^{in}) = \pi^{out} \tag{4.2}$$

where π^{in} is a stationary deterministic policy in which $Z_j = z(Y_j)$, $z : [0, \infty) \rightarrow [0, M]$. Then, π^{out} is a causal policy in which $Z_j = z(Y_j)$ with the same z function for $j > 0$ and $Z_0 = z(Y_0) + x$. Note that determination of Z_j for $j > 0$, *i.e.* π^{out}, is a function of Y_j, which makes the policy stationary and deterministic for $j > 0$.

Lemma 2. *Let Π^{PoW} be the image of the function $f(x, \pi^{opt})$ where $x \in [0, T]$. The objective function of the PoW problem attains its limit for every policy in the set Π^{PoW}.*

Proof. The proof is provided in Appendix F. □

As a result of previous proposition, let us define the function $h\colon [B_L, B_U] \times \Pi^{PoW} \to \mathbb{R}^+$ as follows:

$$h(Y_0, \pi) = \lim_{n\to\infty} \frac{E_{\mathcal{Y}}\big[\sum_{k=1}^{n} g(\Delta(Q_k))\big|Y_0, \pi\big]}{n} \tag{4.3}$$

where the expectation is taken with respect to possible transmission delay sequences given that $\pi \in \Pi^{PoW}$ is performed. In the rest of the section, we seek to establish a relationship between the functions $h(Y_0, \pi)$ and $g_{opt}(Y_0)$ such that an average of $h(Y_0, \pi)$ for some $\pi \in \Pi^{PoW}$ is equivalent to $g_{opt}(Y_0)$.

Let $\Delta_{\pi,\mathbf{Y}}(t)$ denote AoI at time t, when a stationary and deterministic policy π is performed on a transmission delay sequence $\mathbf{Y} = (Y_0, Y_1, \dots)$. When the performed policy π is a stationary and deterministic policy which is function of Y_j, $\Delta_{\pi,\mathbf{Y}}(t)$ is a function of t. Then, it is obvious from the definition of the function f in (4.2) that

$$\Delta_{\pi^{opt},\mathbf{Y}}(t) = \Delta_{f(x,\pi^{opt}),\mathbf{Y}}(t+x), \qquad for\ t > M + B_U \tag{4.4}$$

As (4.4) holds for every transmission delay sequence, we can take expectation on transmission delay sequences. Thus, the following equation holds for every $t > M + B_U$:

$$E_{\mathcal{Y}}\big[\Delta_{\pi^{opt},\mathbf{Y}}(t)\big] = E_{\mathcal{Y}}\big[\Delta_{f(x,\pi^{opt}),\mathbf{Y}}(t+x)\big] \tag{4.5}$$

Let m be a natural number such that $(m+1)T > M + T + B_U$. Then, we can obtain the following:

$$\begin{aligned} h\big(Y_0, f(x,\pi^{opt})\big) &= \lim_{n\to\infty} \frac{E_{\mathcal{Y}}\big[\sum_{k=1}^{n} g\big(\Delta_{f(x,\pi^{opt}),\mathbf{Y}}(Q_k)\big)\big|Y_0\big]}{n} \\ &\overset{(a)}{=} \lim_{n\to\infty} \frac{E_{\mathcal{Y}}\big[\sum_{k=m+1}^{n} g\big(\Delta_{f(x,\pi^{opt}),\mathbf{Y}}(Q_k)\big)\big|Y_0\big]}{n-m} \\ &\overset{(b)}{=} \lim_{n\to\infty} \frac{E_{\mathcal{Y}}\big[\sum_{k=m+1}^{n} g\big(\Delta_{\pi^{opt},\mathbf{Y}}(Q_k - x)\big)\big|Y_0\big]}{n-m} \end{aligned} \tag{4.6}$$

where (a) follows from the properties of limit and (b) follows from (4.5).

Let us divide the interval $[0, T]$ into small intervals with length δ. Then, we obtain the following:

$$
\begin{aligned}
&\frac{\sum_{i=1}^{T/\delta} h\Big(Y_0, f\big((i-1)\delta, \pi^{opt}\big)\Big)}{T/\delta} \\
&\qquad \overset{(a)}{=} \sum_{i=1}^{T/\delta} \lim_{n\to\infty} \frac{E_{\mathcal{Y}}\Big[\sum_{k=m+1}^{n} g\big(\Delta_{\pi^{opt},\mathbf{Y}}\big(Q_k-(i-1)\delta\big)\big)\Big]}{(n-m)T/\delta} \\
&\qquad \overset{(b)}{=} \lim_{n\to\infty} \sum_{i=1}^{T/\delta} E_{\mathcal{Y}}\Bigg[\frac{\sum_{k=m+1}^{n} g\big(\Delta_{\pi^{opt},\mathbf{Y}}\big(Q_k-(i-1)\delta\big)\big)}{(n-m)T/\delta}\Bigg] \\
&\qquad \overset{(c)}{=} \lim_{n\to\infty} E_{\mathcal{Y}}\Bigg[\frac{\sum_{i=Q_m/\delta+1}^{Q_n/\delta} g\big(\Delta_{\pi^{opt},\mathbf{Y}}\big(i\delta\big)\big)}{\big(Q_n-Q_m\big)/\delta}\Bigg]
\end{aligned}
\tag{4.7}
$$

where (a) follows from (4.6), (b) follows from interchanging the order of the limit and summation by Lebesgue's Dominated Convergence Theorem as all of the terms are upper bounded by $g(B_u + M)$, and (c) follows from exchanging summation and expectation.

As δ goes to 0, we can obtain the following:

$$
\begin{aligned}
&\lim_{\delta\to 0}\frac{\sum_{i=1}^{T/\delta} h\big(Y_0, f\big((i-1)\delta, \pi^{opt}\big)\big)}{T/\delta} \\
&\qquad \overset{(a)}{=} \lim_{\delta\to 0}\lim_{n\to\infty} E_{\mathcal{Y}}\Bigg[\frac{\sum_{i=Q_m/\delta+1}^{Q_n/\delta} g\big(\Delta_{\pi^{opt},\mathbf{Y}}\big(i\delta\big)\big)}{\big(Q_n-Q_m\big)/\delta}\Bigg| Y_0\Bigg] \\
&\qquad \overset{(b)}{=} \lim_{n\to\infty}\lim_{\delta\to 0} E_{\mathcal{Y}}\Bigg[\frac{\sum_{i=Q_m/\delta+1}^{Q_n/\delta} g\big(\Delta_{\pi^{opt},\mathbf{Y}}\big(i\delta\big)\big)}{\big(Q_n-Q_m\big)/\delta}\Bigg| Y_0\Bigg] \\
&\qquad \overset{(c)}{=} \lim_{n\to\infty} E_{\mathcal{Y}}\Bigg[\lim_{\delta\to 0}\frac{\sum_{i=Q_m/\delta+1}^{Q_n/\delta} g\big(\Delta_{\pi^{opt},\mathbf{Y}}\big(i\delta\big)\big)}{\big(Q_n-Q_m\big)/\delta}\Bigg| Y_0\Bigg] \\
&\qquad \overset{(d)}{=} \lim_{n\to\infty} E_{\mathcal{Y}}\Bigg[\frac{\int_{Q_m}^{Q_n} g(\Delta_{\pi^{opt},\mathbf{Y}}(t))dt}{\big(Q_n-Q_m\big)}\Bigg| Y_0\Bigg] \\
&\qquad \overset{(e)}{=} \lim_{n\to\infty} E_{\mathcal{Y}}\Bigg[\frac{\int_{0}^{Q_n} g(\Delta_{\pi^{opt},\mathbf{Y}}(t))dt}{Q_n}\Bigg| Y_0\Bigg] \\
&\qquad \overset{(f)}{=} \lim_{n\to\infty} E_{\mathcal{Y}}\Bigg[\frac{\int_{0}^{D_n} g(\Delta_{\pi^{opt},\mathbf{Y}}(t))dt}{D_n}\Bigg| Y_0\Bigg] \overset{(g)}{=} g_{opt}(Y_0)
\end{aligned}
\tag{4.8}
$$

In (4.8), (a) follows from (4.7). (b) follows from Moore Osgood Theorem as the term with the expectation is proved to be uniformly convergent in Appendix G. (c) follows from Lebesgue's Dominated Convergence Theorem as $g(\Delta_{\pi^{opt},\mathbf{Y}}(.))$ is upper bounded by $g(B_U + M)$. (d) follows from the Riemann Integration that is proved in [57, Theorem 6.10]. (e) and (f) are obtained from the following facts: (i) $[0, Q_m]$ is a bounded interval. (ii) $B_L < \Delta_{\pi^{opt},\mathbb{Y}}(t) < M + B_U$ for all $t \in \mathbb{R}^+$. (iii) Let D_i be the closest delivery point to a query Q_k. Then $|D_i - Q_k| < (B_U + M)/2$. (g) follows from (4.1).

As a result, we establish the intended relationship between the functions $h(Y_0, \pi)$ and $g_{opt}(Y_0)$. Now, let us prove the fundamental result of this section:

Theorem 1. *If the transmission delay process* $\{Y_j\}_{j=0}^{\infty}$ *is i.i.d. such that* $\Pr(Y_j \in [B_L, B_U]) = 1$ *and the query arrival process* $\{Q_k\}_{k=1}^{\infty}$ *is periodic, then* $\bar{h}_{opt} \leq \bar{g}_{opt}$ *with the same transmission power constraint* f_{max}*, for every period* T*.*

Proof. We are going to prove that for every starting point of $Y_0 \in [B_L, B_U]$, there exists $x \in [0, T]$ such that $h(Y_0, f(x, \pi^{opt})) \leq g_{opt,Y_0}$ where π^{opt} the optimal update policy for the UoW problem. Suppose that this is not true. Then, there exists $Y_0 \in [B_L, B_U]$ such that for all $x \in [0, T]$, $h(Y_0, f(x, \pi^{opt})) > g_{opt,Y_0}$. Furthermore, $g(\Delta_{\pi^{opt},\mathbf{Y}}(t))$ is lower semi-continuous with respect to t because g is continuous and non-decreasing. As g is uniformly continuous on the interval $[B_L, B_U + M]$ and bounded in this compact interval; lower semi-continuity of $E_{\mathcal{Y}}[g(\Delta_{f(x,\pi^{opt}),\mathbf{Y}}(Q_k))]$ with respect to x can be easily shown by its definition. Then, $h(Y_0, f(x, \pi^{opt})$ turns out to be sum of countable lower semi-continuous functions with respect to x. Countable sum of lower semi-continuous functions is lower semi-continuous when they are lower bounded [58, Chapter 2]. As the variable x is in a compact set $[0, T]$, the function attains $h(Y_0, f(x, \pi^{opt})$ its infimum. Hence, there exists $C > 0$ such that (4.9) is satisfied.

$$h(Y_0, f(x, \pi^{opt})) \geq g_{opt,Y_0} + C \tag{4.9}$$

$$\lim_{\delta \to 0} \frac{\sum_{i=1}^{T/\delta} h\big(Y_0, f\big((i-1)\delta, \pi^{opt}\big)\big)}{T/\delta} \geq g_{opt,Y_0} + C \tag{4.10}$$

(4.10) can be obtained as a result of (4.9). Then, (4.10) contradicts (4.8). Therefore, for every $Y_0 \in [B_L, B_U]$, there exists $x \in [0, T]$ such that $h(Y_0, f(x, \pi^{opt})) \leq g_{opt,Y_0}$. □

Corollary 5. *Theorem 1 is valid when there is no power constraint in the objectives of both PoW and UoW problems.*

Proof. It is proven in Theorem 1 that there exists $\pi \in \Pi^{PoW}$ such that the query average age penalty of the PoW problem under the policy π is less than or equal to the optimal time average age penalty of the UoW problem. When there are no power constraints in the objectives of both PoW and UoW problems, then we can construct such a set Π^{PoW} from the optimal policy of the UoW problem. Therefore, the proof is still valid. □

4.3 General Query Arrival Process

In this section, we assume that the transmission delays are constant whereas the query arrival process can be any i.i.d. process.

Again, it was shown in [10] that there exists a solution of the UoW problem in the set of stationary and deterministic policies. Suppose that $\pi^{opt} \in \Pi_{SD}$ is an optimal policy for a transmission delay time Y. Then, the waiting times determined by π^{opt} will be a constant Z. In that case, the AoI will be a function whose value linearly rises from Y to $2Y + Z$ and returns to Y with period $Y + Z$. Let us define the function $mod : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ to express the AoI.

$$\begin{aligned} mod(x, y) &= x - \left\lfloor \frac{x}{y} \right\rfloor \times y \\ \Delta(t) &= mod(t, Y + Z) + Y \end{aligned} \tag{4.11}$$

Then, $\bar{g}_{opt}$ can be expressed as:

$$g_{opt} = \frac{\int_{Y}^{2Y+Z} g(t))dt}{Y + Z} \tag{4.12}$$

Let us define $h : \Pi \to \mathbb{R}^+$ as:

$$h(\pi) = \limsup_{n\to\infty} \frac{E\left[\sum_{k=1}^{n} g\big(\Delta(Q_k)\big)\middle|\pi\right]}{n} \tag{4.13}$$

In the rest of the section, we will categorize the interarrival times, $\{Q'_k\}$, into two classes similar to [59]. If Q'_k belongs to class 1, we will show that PoW problem can attain a better age penalty than the optimal age penalty of the UoW problem just by changing the first waiting time Z_0. If Q'_k belongs to class 2, we will show that the optimal age penalty of the PoW problem is equal to that of the UoW problem. These two classes are defined as follows:

Definition 5.

- *If there exists a* $\beta \in \mathbb{R}$ *such that* $\Pr(Q'_k \in \{\beta, 2\beta, 3\beta, \dots\} = 1)$ *and* $\beta = \frac{p}{q}(Y + Z) \quad (p, q \in \mathbb{N})$*, then the distribution of the* Q'_k *is said to be class 1.*
- *If the distribution of* Q'_k *is not class 1, it is said to be class 2.*

It should be noted that the classification of the query interarrival times depends on the stationary and deterministic policy applied.

4.3.1 Class 1 Interarrival Times Between Queries

Let us define a function $f : \mathbb{R}^+ \cup \{0\} \times \Pi_{SD} \to \Pi$ that changes the first waiting time Z_0 as follows:

$$f(x, \pi^{in}) = \pi^{out} \tag{4.14}$$

In the above equation, π^{in} is a stationary and deterministic policy with deterministic function z. π^{out} determines the waiting times by $Z_0 = z(Y) + x$, $Z_j = z(Y) \colon j > 0$. Then, π^{out} policy is also a stationary and deterministic policy for $Z_j \colon j > 0$. Let Π^{PoW} denote the image set of the function f.

Let $\Delta_\pi(t)$ denote the change of the AoI over time. Then, the definition of the function f implies that the below holds for every $\pi \in \Pi_{SD}$.

$$\Delta_\pi(t) = \Delta_{f(x,\pi)}(t + x), \quad \text{if } t > Y + x \tag{4.15}$$

Since the AoI is periodic, as stated in (4.11), and $\{Q'_k\}$'s form a i.i.d. process, the QAoI $\Delta(Q_k)$ can be expressed as a Markov chain. Two important properties of this Markov chain are specified in the following lemma.

Lemma 3. *If the policy $\pi \in \Pi^{PoW}$ is applied, the markov chain expressing the AoI at query instants occurring after $2Y + M$, i.e. $Q_k > 2Y + M$, has q states where each state communicates with all other states.*

Proof. The definition of Π^{PoW} implies there exists a $\pi^{in} \in \Pi_{SD}$ such that $f(x, \pi^{in}) = \pi$. Let Z be the waiting time determined by policy π^{in}. Without loss of generality, assume that β is the greatest real number satisfying $\beta = \frac{p}{q}(Y + Z)$. Further assume, p and q are relatively prime natural numbers. Definition of β implies that Q_k should be a positive integer multiple of β. Furthermore, $q \times \beta = p \times (Y + Z)$. Since $\Delta(t)$ is a periodic function with period $(Y + Z)$, there are atmost q states $\Delta(Q_k)$ can take. These states are the following: $mod(\beta + x, Y + Z) + Y, mod(2\beta + x, Y + Z) + Y, \ldots, mod(q\beta + x, Y + Z) + Y$. If we prove that these q states communicate with each other, we prove there cannot be less than q states since it shows that every state is accessible from the initial state. Assume we are at state m. Let's check whether all states are accessible from state m. Since β is the greatest possible number, either $\Pr(Q'_k = \beta) > 0$ is true or there exists relatively prime numbers $p_1, q_1 \in \mathbb{N}$ such that $\Pr(Q'_k = p_1\beta), \Pr(Q'_k = q_1\beta) > 0$. If $\Pr(Q'_k = \beta) > 0$, after $n - m + q$ queries, state n can be reached with a positive probability. Else if $\Pr(Q'_k = p_1\beta), \Pr(Q'_k = q_1\beta) > 0$ state n can be reached after $a + b$ queries since there exists $a, b, c \in \mathbb{N}$ such that $p_1a + q_1b = n - m + qc$ where p_1 and q_1 are relatively prime numbers. Consequently, all states are accessible from each other which concludes the proof. □

Corollary 6. *Let $c_1, c_2, \ldots, c_q$ denote the AoI of the q states. This Markov chain reaches to a steady state where the steady state probabilities are denoted by $p_1, p_2, \ldots, p_q$. Then we have,*

$$h(\pi) = \sum_{i=1}^{q} g(c_i) \times p_i \tag{4.16}$$

Proof. Lemma 3 shows that the Markov chain is ergodic and unichain. Then, this Markov chain reaches to one and only one steady state [60, Chapter 4.3.1] and these steady state probabilities are equal to the limiting fraction of time in each state. □

Now, we share the main contribution of this section.

Theorem 2. *Let $\pi^{opt} \in \Pi_{SD}$ be the optimal policy of the UoW problem under the constraint f_{max}. Assume that the distribution of the interarrival times of the queries are class 1 when the waiting times are determined by the policy π^{opt}. Then, there exists a $x \in [0, Y+Z]$ such that the average age obtained from the PoW problem when the policy $f(x, \pi^{opt})$ is applied is less than or equal to the average age obtained from the UoW problem. Furthermore, policy $f(x, \pi^{opt})$ satisfies the condition f_{max}.*

Proof. Assume it is not true. Hence for all $x \in [0, Y+Z]$, policy $f(x, \pi^{opt})$ is greater than g_{opt}. Then,

$$\frac{\int_0^{Y+Z} h(f(t, \pi^{opt}))dt}{Y+Z} > g_{opt} \tag{4.17}$$

On the other hand,

$$\begin{aligned} \frac{\int_0^{Y+Z} h(f(t, \pi^{opt}))dt}{Y+Z} &\overset{(a)}{=} \frac{\sum_{i=1}^{q} p_i \times \int_0^{Y+Z} g(mod(i\beta + t, Y+Z) + Y)dt}{Y+Z} \\ &\overset{(b)}{=} \frac{\sum_{i=1}^{q} p_i \times \int_Y^{2Y+Z} g(t)dt}{Y+Z} \\ &\overset{(c)}{=} \frac{\int_Y^{2Y+Z} g(t))dt}{Y+Z} \\ &\overset{(d)}{=} g_{opt} \end{aligned} \tag{4.18}$$

where (a) can be obtained from 6; (b) can be obtained from the age penalty function of the states shown in the proof of Lemma 3; (c) can be obtained form the fact that the sum of p_i's are equal to 1 since p_i's are the steady state probabilities; finally, (d) can be obtained from (4.12).

Then, (4.17) contradicts with (4.18). Hence, there exists a $x \in [0, Y+Z]$ such that the average age obtained from the PoW problem using policy $f(x, \pi^{opt})$ is less than or equal to the optimal age obtained from the UoW problem. Since policy $f(x, \pi^{opt})$ only affects the first waiting time and π^{opt} satisfies the condition on f_{max}, $f(x, \pi^{opt})$ also satisfies the condition f_{max}. This concludes the proof. □

Theorem 2 shows that when the inerarrival times between queries i.e. $\{Q'_k\}$, is class 1, the optimal average age can be obtained from the PoW problem is less than that of the UoW problem.

4.3.2 Class 2 Interarrival Times Between Queries

Say that policy $\pi^{opt} \in \Pi_{SD}$ is an optimal solution for the UoW problem under the constraint f_{max}. Let the interarrival times of the queries be class 2 when Z is the waiting time determined by the policy π^{opt} and the transmission delays are constant. In this subsection, we show that the $h(\pi^{opt})$ is equal to the g_{opt}. For this purpose, we define the following:

Definition 6. *If there exist $a \leq c < d \leq b$ for every $c, d \in \mathbb{R}$ such that the below conditions are satisfied, the real series $(x_1, x_2, x_3, \dots)$ is equidistributed in the closed interval $[a, b]$.*

$$\lim_{n \to \infty} \frac{\left|\{x_1, x_2, \dots, x_n\} \cap [c, d]\right|}{n} = \frac{d - c}{b - a} \tag{4.19}$$

Here, $|\{x_1, x_2, \dots, x_n\} \cap [c, d]|$ designates the number of elements of x_j in the interval $[c, d]$.

Now, we share the main contribution of this subsection:

Theorem 3. *Let $\pi^{opt} \in \Pi_{SD}$ be the optimal policy of the UoW problem under the constraint f_{max}. Also, let the interarrival times be class 2 when the waiting times are determined by π^{opt}. Then, the optimal age obtained from the PoW problem with policy π^{opt} is equal to that of the UoW problem.*

Proof. Consider the series whose elements are $g(\Delta(Q_1)), g(\Delta(Q_2)), g(\Delta(Q_3)), \dots$. If we apply the elements of these series as the policy π^{opt}, corresponding time average age penalties form the series with elements $g(mod(Q_1, Y+Z)+Y), g(mod(Q_2, Y+Z)+Y), g(mod(Q_3, Y+Z)+Y), \dots$ due to (4.11). Since the inerarrival times of the queries are i.i.d. for this series, its elements are equidistributed in the closed interval $[g(Y), g(2Y+Z)]$ [59, Theorem 2]. Then,

$$h(\pi^{opt}) = \frac{\int_{Y}^{2Y+Z} g(t))dt}{Y+Z} = g_{opt} \tag{4.20}$$

□

Theorem 3 shows the goal of the section for class 2 inerarrival times of the queries $\{Q'_k\}$.

Corollary 7. *Theorems 2 and 3 are valid when there is no power constraint in the objectives of both PoW and UoW problems.*

Proof. The proof is the same as the proof of Corollary 5. □

Theorems 2 and 3 complete the contribution of this section, where we showed the superiority of PoW problem over UoW problem when the query arrival process is general and the transmission delays are constant.

CHAPTER 5

NUMERICAL RESULTS

Throughout the section, we exhibit the behavior of the average age penalties for the PoW and UoW problems under different transmission delay processes. To be consistent with our system model which assumes finite valued transmission delay, we will utilize truncated versions of certain transmission delay distributions such as exponential and log-normal distributions. Specifically, we truncate the values to start at 0.01 and go up to a maximum value chosen such that the cumulative distribution of the transmission delay at this value is 0.95. We choose $T = 4B_U$.

We compare three different update policies: the zero-wait policy, the optimal policy of the UoW problem found in [10], and the optimal policy of the PoW problem found in Algorithm 1. The optimal solutions of the UoW problem and the PoW problem are referred to as UoW-optimal policy and PoW-optimal policy, respectively. The average age penalty of the PoW-optimal policy is calculated by averaging the age penalties at the query instants. The average age penalties of the zero-wait policy and UoW-optimal policy are calculated as time average age penalties. Perhaps surprisingly, in all of our simulations, the time-average AoI and QAoI are identical for the zero-wait and UoW-optimal policies. The reason is, in all of our examples $X_j = Y_j + Z_j$ obeys the 'Case 1 i.i.d.' random variable definition in [59]. Case 1 random variables are all the random variables except the cases that there exists $\beta \in \mathbb{R}$ such that $\Pr(X_j \in \{k\beta : \ k \in \mathbb{N}\}) = 1$ or $\Pr(X_j = 0) = 1$. The proof for the equivalence of the time-average AoI and QAoI is subject to our future works. Note that the random variable X_j under the PoW-optimal policy may not be an i.i.d. random variable, that is why the PoW-optimal policy can result in a lower age than the time-average age of the UoW-optimal policy.

Table 5.1: Lower bounds on the query average age with i.i.d. truncated exponential distributed service times

$\frac{Q}{N}$ \ λ	1	1.2	1.4	1.6	1.8	2
0.16	1.297	1.097	0.897	0.792	0.702	0.624
0.08	1.359	1.159	0.958	0.854	0.763	0.684
0.04	1.391	1.191	0.99	0.885	0.795	0.715
0.02	1.407	1.207	1.006	0.901	0.811	0.731

Table 5.2: Upper bounds on the query average age with i.i.d. truncated exponential distributed service times

$\frac{Q}{N}$ \ λ	1	1.2	1.4	1.6	1.8	2
0.16	1.457	1.257	1.057	0.952	0.862	0.784
0.08	1.439	1.239	1.038	0.934	0.843	0.764
0.04	1.431	1.231	1.03	0.925	0.835	0.755
0.02	1.427	1.227	1.026	0.921	0.831	0.751

Table 5.3: Number of G_R^π calculations to determine an optimal update policy when service times are i.i.d. truncated exponential distribution and the penalty function is identity.

$\frac{Q}{N}$ \ λ	1	1.2	1.4	1.6	1.8	2
0.16	7×10^4	5×10^4	3×10^4	2×10^4	2×10^4	1×10^4
0.08	6×10^5	4×10^5	2×10^5	2×10^5	1×10^5	1×10^5
0.04	5×10^6	3×10^6	2×10^6	1×10^6	1×10^6	8×10^5
0.02	4×10^7	3×10^7	1×10^7	1×10^7	7×10^6	6×10^6

Tables 5.1, 5.2, and 5.3 illustrate the change in lower bounds of the query average age, upper bounds of the query average age, and the number of calculations to find an optimal policy, respectively for different numbers of sub-intervals N under i.i.d. truncated exponentially distributed services times. Observing the tables 5.1 and 5.2, we detect that the upper bound is much stricter than the lower bound. This is also the case for the other transmission delay processes such as log-normal, beta, uniform distributions. Even though the number of calculations is exponentially increasing as the number of sub-intervals N increases, the upper bounds of the query average age are rapidly converging. It means that reaching a satisfactory approximate solution for the PoW problem does not require an excessive number of G_R^π calculations. As a result, we decide to present only the upper bound of the query average age to avoid

confusion in the following figures.

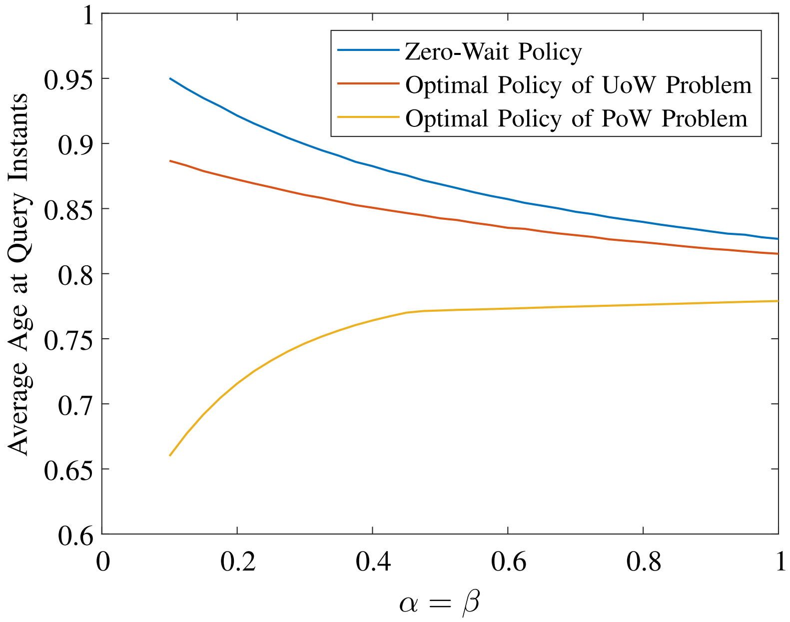

Figure 5.1: Average age at query instants with i.i.d. beta distributed services times. The optimal policies of the PoW problems are found with $\frac{Q}{N} = 0.05$.

Figures 5.1, 5.2, and 5.3 illustrate the behavior of the average ages under i.i.d. beta distributed service times with equal α, β parameters, i.i.d. truncated log-normal distributed service times, i.i.d. truncated Pareto distributed service times, respectively. When $\alpha = \beta = 1$, the Beta distribution becomes a uniform distribution between 0 and 1. As $\alpha = \beta$ approaches 0, it approaches a bimodal distribution concentrated around 0 and 1 with probability close to 0.5 each. Interestingly, as α and β increase, the average ages of the zero-wait policy and the UoW-optimal policy decrease whereas the average age of the PoW-optimal policy increases even though the mean of the beta distribution is constant, $\frac{\alpha}{\alpha+\beta} = \frac{1}{2}$. The benefit of using the PoW-optimal policy is pronounced when the transmission delay is bi-modal distributed. The log-normal distribution is a heavy-tailed distribution especially for large σ. We observe in Figure 5.2 that the PoW-optimal policy performs better than the other policies in heavy-tailed distribution as well. On the other hand, as α goes to ∞, the Pareto distribution converges to the dirac delta function $\delta(t - x_m)$. We choose $x_m = 1$ which leads that UoW-optimal policy is equivalent to the zero wait policy for $\alpha \geq 3$ [10, Theorem 5]. We observe in Figure 5.3 that PoW-optimal policy performs well as the transmission delay distribution approaches the dirac delta function.

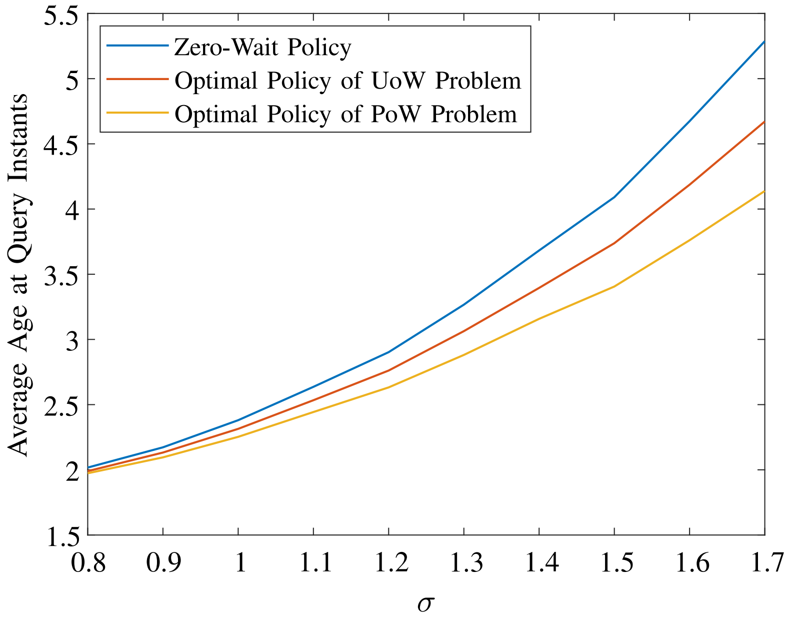

Figure 5.2: Average age at query instants with i.i.d. truncated log-normal distributed services times with parameters (σ, μ) where $\mu = 0$. The optimal policies of the PoW problems are found with $\frac{Q}{N} = 0.2$.

Figure 5.4 exhibits the behavior of the average age penalties for different α when the age penalty function $g(x) = e^{\alpha x} - 1$ and service times are exponentially distributed with $\lambda = 1$. This nonlinear age penalty function represents destination nodes that demand very fresh update packets and harshly penalize stale update packets. In the figure, we observe that the PoW-optimal policy works much better than the other policies especially for high α values. It means that the pull-based communication model is beneficial to utilize when the destination node demands very fresh update packets.

Up to now, we have not put any constraint on the number of transmissions for the policies. Figures 5.5 and 5.6 illustrate the behavior of the average ages under truncated i.i.d. exponential distributed service times and Pareto distributed service times, respectively, when the number of transmissions in the UoW-optimal policy is constrained by the number of transmissions made by the PoW-optimal policy. We observe that the average age of the PoW-optimal policy is much lower than the average age of the UoW-optimal policy for an equal number of transmissions. This implies that in a practical situation, applying the PoW solution can be significantly more energy-efficient, for the same age performance.

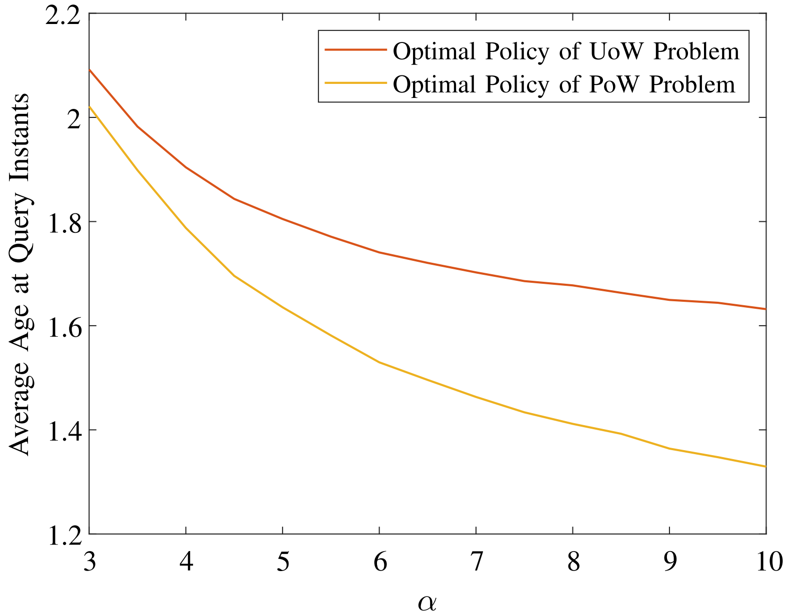

Figure 5.3: Average age at query instants with i.i.d. truncated Pareto distributed services times with (x_m, α) where $x_m = 1$. The optimal policies of the PoW problems are found with $\frac{Q}{N} = 0.05$. Note that the optimal policy of the UoW problem is equivalent to the zero wait policy when $x_m = 1$ and $\alpha \geq 3$.

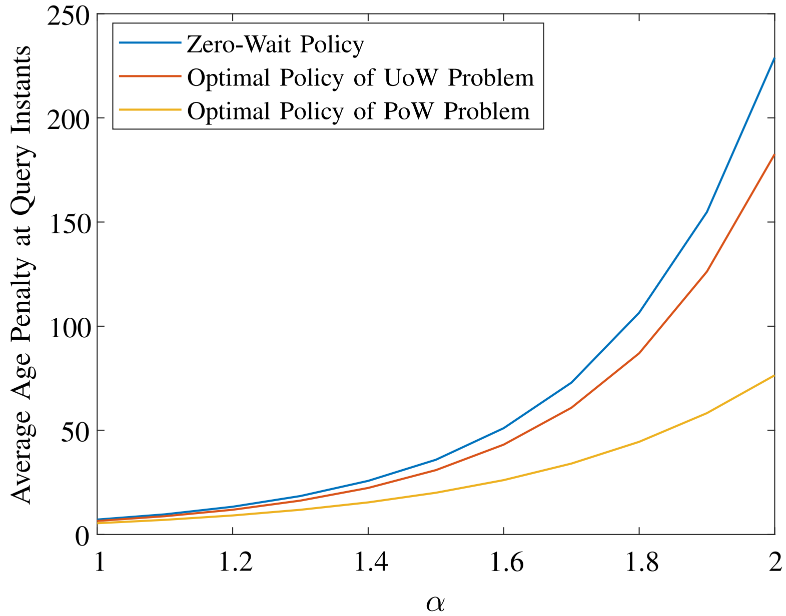

Figure 5.4: Average age penalty at query instants with the age penalty function $g(x) = e^{\alpha x} - 1$ and i.i.d. truncated exponential distributed services times where $\lambda = 1$. The optimal policies of the PoW problems are found with $\frac{Q}{N} = 0.05$.

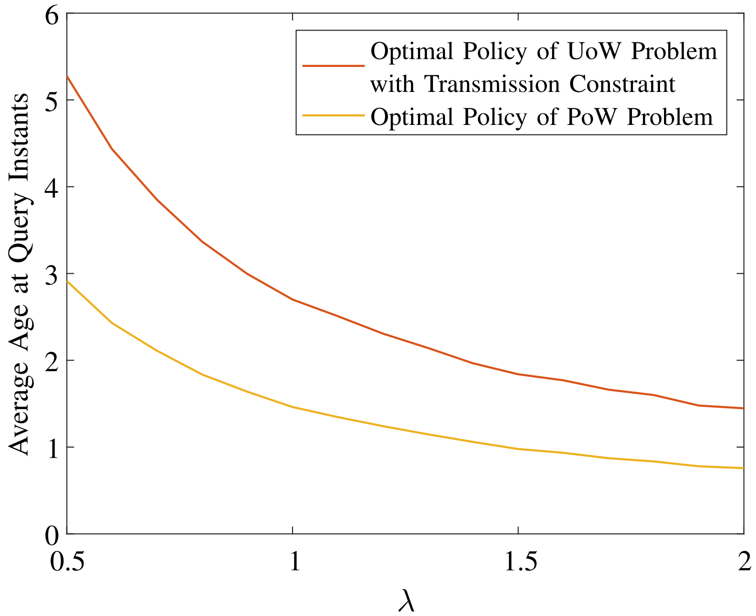

Figure 5.5: Average age at query instants with i.i.d. truncated exponential distributed services times with the parameter λ when the optimal policy of the UoW problem is constrained to transmit the same number of update packets as the optimal policy of PoW the problem. The optimal policies of the PoW problems are found with $\frac{Q}{N} = 0.05$.

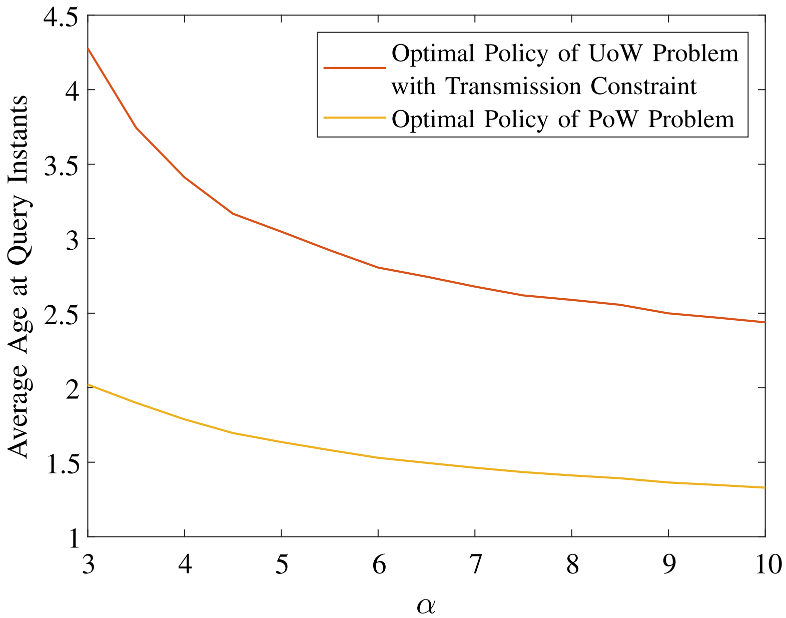

Figure 5.6: Average age at query instants with i.i.d. truncated Pareto distributed services times with the parameters (x_m, α) where $x_m = 1$, when the optimal policy of the UoW problem is constrained to transmit the same number of update packets as the optimal policy of the PoW problem. The optimal policies of the PoW problems are found with $\frac{Q}{N} = 0.05$.

CHAPTER 6

CONCLUSION

We studied the optimal control of the status update system in which the destination node requests the source node to submit an update packet to the channel. We defined a continuous, non-decreasing, and non-negative penalty function to represent the level of dissatisfaction on data staleness. While solving the PoW problem, we first identified the PoW problem under the single query case as a stochastic shortest path problem with uncountable state and action spaces. For this specific SSP problem, we obtained an optimal policy. Using the solution of the SSP problem, we found out an optimal policy for the PoW problem under periodic query arrival processes. Furthermore, we provided an analytical comparison between the UoW and PoW problems: (i) An optimal policy that minimizes the UoW problem also minimizes the PoW problem under Poisson query arrivals. Furthermore, their average age penalties are equivalent. (ii) The optimal query average age penalty under periodic query arrivals is always less than or equal to the optimal time average age penalty. An interesting by product is that for a large class of distributions, the QAoI achieved by Zero-Wait and the UoW-optimal policies are identical to the time-average AoI achieved by these policies, and both are remarkably higher than the QAoI achieved by the PoW-optimal policy, even when the former two are allowed an unconstrained number of transmissions. For the same number of tranmissions, the PoW-optimal result achieves a more significant lowering of QAoI, which in turn implies the potential energy efficiency of a PoW-optimal solution for a desired Query AoI performance.

Future directions for this work include the general solution of the PoW problem (i.e., for general query arrival processes, and delay processes with memory), and exhibiting the superiority of the result to those obtained by previous push-based solutions.

REFERENCES

[1] G. Mois, S. Folea, and T. Sanislav, "Analysis of three iot-based wireless sensors for environmental monitoring," *IEEE Transactions on Instrumentation and Measurement*, vol. 66, no. 8, pp. 2056–2064, 2017.

[2] M. Hassanalieragh, A. Page, T. Soyata, G. Sharma, M. Aktas, G. Mateos, B. Kantarci, and S. Andreescu, "Health monitoring and management using internet-of-things (iot) sensing with cloud-based processing: Opportunities and challenges," in *2015 IEEE International Conference on Services Computing*, pp. 285–292, 2015.

[3] H. Li, Y. Liu, Z. Qin, H. Rong, and Q. Liu, "A large-scale urban vehicular network framework for iot in smart cities," *IEEE Access*, vol. 7, pp. 74437–74449, 2019.

[4] A. Zanella, N. Bui, A. Castellani, L. Vangelista, and M. Zorzi, "Internet of things for smart cities," *IEEE Internet of Things Journal*, vol. 1, no. 1, pp. 22–32, 2014.

[5] S. Kaul, R. Yates, and M. Gruteser, "Real-time status: How often should one update?," in *2012 Proceedings IEEE INFOCOM*, pp. 2731–2735, 2012.

[6] R. D. Yates, Y. Sun, D. R. Brown, S. K. Kaul, E. Modiano, and S. Ulukus, "Age of information: An introduction and survey," *IEEE Journal on Selected Areas in Communications*, vol. 39, no. 5, pp. 1183–1210, 2021.

[7] Y. Sun, I. Kadota, R. Talak, and E. Modiano, *Age of Information: A New Metric for Information Freshness*. Morgan Claypool, 2019.

[8] R. D. Yates, "Lazy is timely: Status updates by an energy harvesting source," in *2015 IEEE International Symposium on Information Theory (ISIT)*, pp. 3008–3012, 2015.

[9] B. T. Bacinoglu, E. T. Ceran, and E. Uysal-Biyikoglu, "Age of information un-

der energy replenishment constraints," in *2015 Information Theory and Applications Workshop (ITA)*, pp. 25–31, 2015.

[10] Y. Sun, E. Uysal-Biyikoglu, R. D. Yates, C. E. Koksal, and N. B. Shroff, "Update or wait: How to keep your data fresh," *IEEE Transactions on Information Theory*, vol. 63, pp. 7492–7508, Nov 2017.

[11] X. Wu, J. Yang, and J. Wu, "Optimal status update for age of information minimization with an energy harvesting source," *IEEE Transactions on Green Communications and Networking*, vol. 2, no. 1, pp. 193–204, 2018.

[12] A. Arafa, J. Yang, S. Ulukus, and H. V. Poor, "Age-minimal transmission for energy harvesting sensors with finite batteries: Online policies," *IEEE Transactions on Information Theory*, vol. 66, no. 1, pp. 534–556, 2020.

[13] S. Feng and J. Yang, "Age of information minimization for an energy harvesting source with updating erasures: Without and with feedback," *IEEE Transactions on Communications*, vol. 69, no. 8, pp. 5091–5105, 2021.

[14] A. Arafa, J. Yang, S. Ulukus, and H. V. Poor, "Using erasure feedback for online timely updating with an energy harvesting sensor," in *2019 IEEE International Symposium on Information Theory (ISIT)*, pp. 607–611, 2019.

[15] B. T. Bacinoglu, E. T. Ceran, and E. Uysal-Biyikoglu, "Age of information under energy replenishment constraints," in *2015 Information Theory and Applications Workshop (ITA)*, pp. 25–31, 2015.

[16] B. T. Bacinoglu and E. Uysal-Biyikoglu, "Scheduling status updates to minimize age of information with an energy harvesting sensor," in *2017 IEEE International Symposium on Information Theory (ISIT)*, pp. 1122–1126, 2017.

[17] S. Feng and J. Yang, "Optimal status updating for an energy harvesting sensor with a noisy channel," in *IEEE INFOCOM 2018 - IEEE Conference on Computer Communications Workshops (INFOCOM WKSHPS)*, pp. 348–353, 2018.

[18] S. Feng and J. Yang, "Minimizing age of information for an energy harvesting source with updating failures," in *2018 IEEE International Symposium on Information Theory (ISIT)*, pp. 2431–2435, 2018.

[19] B. T. Bacinoglu, Y. Sun, E. Uysal, and V. Mutlu, "Optimal status updating with a finite-battery energy harvesting source," *Journal of Communications and Networks*, vol. 21, no. 3, pp. 280–294, 2019.

[20] E. T. Ceran, D. Gündüz, and A. György, "Average age of information with hybrid arq under a resource constraint," *IEEE Transactions on Wireless Communications*, vol. 18, no. 3, pp. 1900–1913, 2019.

[21] Z. Bao, Y. Dong, Z. Chen, P. Fan, and K. B. Letaief, "Age-optimal service and decision processes in internet of things," *IEEE Internet of Things Journal*, vol. 8, no. 4, pp. 2826–2841, 2021.

[22] Y. Dong, Z. Chen, S. Liu, P. Fan, and K. B. Letaief, "Age-upon-decisions minimizing scheduling in internet of things: To be random or to be deterministic?," *IEEE Internet of Things Journal*, vol. 7, no. 2, pp. 1081–1097, 2020.

[23] F. Chiariotti, J. Holm, A. E. Kalωr, B. Soret, S. K. Jensen, T. B. Pedersen, and P. Popovski, "Freshness on demand: Optimizing age of information for the query process," in *ICC 2021 - IEEE International Conference on Communications*, pp. 1–6, 2021.

[24] C. Kam, S. Kompella, G. D. Nguyen, and A. Ephremides, "Effect of message transmission path diversity on status age," *IEEE Transactions on Information Theory*, vol. 62, no. 3, pp. 1360–1374, 2016.

[25] C. Kam, S. Kompella, G. D. Nguyen, J. E. Wieselthier, and A. Ephremides, "On the age of information with packet deadlines," *IEEE Transactions on Information Theory*, vol. 64, no. 9, pp. 6419–6428, 2018.

[26] M. Costa, M. Codreanu, and A. Ephremides, "On the age of information in status update systems with packet management," *IEEE Transactions on Information Theory*, vol. 62, no. 4, pp. 1897–1910, 2016.

[27] L. Huang and E. Modiano, "Optimizing age-of-information in a multi-class queueing system," in *2015 IEEE International Symposium on Information Theory (ISIT)*, pp. 1681–1685, 2015.

[28] R. D. Yates and S. K. Kaul, "The age of information: Real-time status updating by multiple sources," *IEEE Transactions on Information Theory*, vol. 65, no. 3, pp. 1807–1827, 2019.

[29] A. M. Bedewy, Y. Sun, and N. B. Shroff, "Optimizing data freshness, throughput, and delay in multi-server information-update systems," in *2016 IEEE International Symposium on Information Theory (ISIT)*, pp. 2569–2573, 2016.

[30] A. M. Bedewy, Y. Sun, and N. B. Shroff, "Age-optimal information updates in multihop networks," in *2017 IEEE International Symposium on Information Theory (ISIT)*, pp. 576–580, 2017.

[31] O. T. Yavascan and E. Uysal, "Analysis of slotted aloha with an age threshold," *IEEE Journal on Selected Areas in Communications*, vol. 39, no. 5, pp. 1456–1470, 2021.

[32] H. Chen, Y. Gu, and S.-C. Liew, "Age-of-information dependent random access for massive iot networks," in *IEEE INFOCOM 2020 - IEEE Conference on Computer Communications Workshops (INFOCOM WKSHPS)*, pp. 930–935, 2020.

[33] Z. Jiang, B. Krishnamachari, X. Zheng, S. Zhou, and Z. Niu, "Timely status update in massive iot systems: Decentralized scheduling for wireless uplinks," *arXiv preprint arXiv:1801.03975*, 2018.

[34] R. D. Yates and S. K. Kaul, "Status updates over unreliable multiaccess channels," in *2017 IEEE International Symposium on Information Theory (ISIT)*, pp. 331–335, IEEE, 2017.

[35] X. Chen, K. Gatsis, H. Hassani, and S. S. Bidokhti, "Age of information in random access channels," *arXiv preprint arXiv:1912.01473*, 2019.

[36] Y. Sun, Y. Polyanskiy, and E. Uysal, "Sampling of the wiener process for remote estimation over a channel with random delay," *IEEE Transactions on Information Theory*, vol. 66, no. 2, pp. 1118–1135, 2020.

[37] E. Uysal, O. Kaya, A. Ephremides, J. Gross, M. Codreanu, P. Popovski, M. Assaad, G. Liva, A. Munari, T. Soleymani, B. Soret, and K. H. Johansson, "Semantic communications in networked systems," 2021.

[38] J. Bao, P. Basu, M. Dean, C. Partridge, A. Swami, W. Leland, and J. A. Hendler, "Towards a theory of semantic communication," in *2011 IEEE Network Science Workshop*, pp. 110–117, 2011.

[39] B. Güler, A. Yener, and A. Swami, "The semantic communication game," *IEEE Transactions on Cognitive Communications and Networking*, vol. 4, no. 4, pp. 787–802, 2018.

[40] P. Popovski, O. Simeone, F. Boccardi, D. Gündüz, and O. Sahin, "Semantic-effectiveness filtering and control for post-5g wireless connectivity," *Journal of the Indian Institute of Science*, vol. 100, pp. 435–443, Apr. 2020.

[41] A. Maatouk, S. Kriouile, M. Assaad, and A. Ephremides, "The age of incorrect information: A new performance metric for status updates," *IEEE/ACM Transactions on Networking*, vol. 28, no. 5, pp. 2215–2228, 2020.

[42] Y. Chen and A. Ephremides, "Minimizing age of incorrect information for unreliable channel with power constraint," 2021.

[43] S. Kriouile and M. Assaad, "Minimizing the age of incorrect information for real-time tracking of markov remote sources," 2021.

[44] C. Kam, S. Kompella, and A. Ephremides, "Age of incorrect information for remote estimation of a binary markov source," in *IEEE INFOCOM 2020 - IEEE Conference on Computer Communications Workshops (INFOCOM WKSHPS)*, pp. 1–6, 2020.

[45] X. Zheng, S. Zhou, and Z. Niu, "Urgency of information for context-aware timely status updates in remote control systems," *IEEE Transactions on Wireless Communications*, vol. 19, no. 11, pp. 7237–7250, 2020.

[46] W. Lin, X. Wang, C. xu, X. Sun, and X. Chen, "Average age of changed information in the internet of things," in *2020 IEEE Wireless Communications and Networking Conference (WCNC)*, pp. 1–6, 2020.

[47] Y. Sang, B. Li, and B. Ji, "The power of waiting for more than one response in minimizing the age-of-information," in *GLOBECOM 2017 - 2017 IEEE Global Communications Conference*, pp. 1–6, 2017.

[48] F. Li, Y. Sang, Z. Liu, B. Li, H. Wu, and B. Ji, "Waiting but not aging: Optimizing information freshness under the pull model," *IEEE/ACM Transactions on Networking*, vol. 29, no. 1, pp. 465–478, 2021.

[49] Y. Dong, Z. Chen, S. Liu, and P. Fan, "Age of information upon decisions," in *2018 IEEE 39th Sarnoff Symposium*, pp. 1–5, 2018.

[50] Y. Dong and P. Fan, "Age upon decisions with general arrivals," in *2018 9th IEEE Annual Ubiquitous Computing, Electronics Mobile Communication Conference (UEMCON)*, pp. 825–829, 2018.

[51] B. Yin, S. Zhang, Y. Cheng, L. X. Cai, Z. Jiang, S. Zhou, and Z. Niu, "Only those requested count: Proactive scheduling policies for minimizing effective age-of-information," in *IEEE INFOCOM 2019 - IEEE Conference on Computer Communications*, pp. 109–117, 2019.

[52] D. P. Bertsekas and J. N. Tsitsiklis, "An analysis of stochastic shortest path problems," *Mathematics of Operations Research*, vol. 16, no. 3, pp. 580–595, 1991.

[53] D. P. Bertsekas, "Proper policies in infinite-state stochastic shortest path problems," *IEEE Transactions on Automatic Control*, vol. 63, no. 11, pp. 3787–3792, 2018.

[54] S. R. Pliska, "On the transient case for markov decision chains with general state spaces," in *Dynamic Programming and its Applications* (M. L. Puterman, ed.), pp. 335–349, Academic Press, 1978.

[55] H. W. James and E. J. Collins, "An analysis of transient markov decision processes," *Journal of Applied Probability*, vol. 43, no. 3, pp. 603–621, 2006.

[56] D. P. Bertsekas, *Dynamic Programming and Optimal Control*, vol. 2. MA, USA: Athena Scientific, 3rd ed., 2005.

[57] W. Rudin, *Principles of Mathematical Analysis*. New York, NY, USA: McGraw-Hill, 3rd ed., 1976.

[58] W. Rudin, *Real and Complex Analysis*. New York: McGraw-Hill Book Co., 3 ed., 1987.

[59] H. Robbins, "On the equidistribution of sums of independent random variables," *Proceedings of the American Mathematical Society*, vol. 4, no. 5, pp. 786–799, 1953.

[60] R. G. Gallager, *Stochastic Processes: Theory for Applications*. Cambridge University Press, 2013.

APPENDIX A

PROOF OF PROPOSITION 1

Proof. We first prove that $Q - D_j$ and $\Delta(D_j)$ are sufficient statistics to obtain an optimal Z_j for every j, $(Y_i)_{i=0}^{j}$, and $(Z_i)_{i=0}^{j-1}$. We perform induction on $Q - D_j$. Let us map each $Q - D_j$ to a natural number n such that $(n-1)B_L \leq Q - D_j < nB_L$. If $n = 1$, then $Q - D_j < B_L$. For every waiting period Z_j, the age penalty at the query is constant because a new update cannot arrive until the query. Then, the age penalty at the query is $g(Q - D_j + \Delta(D_j))$. Thus, if $n = 1$, $Q - D_j$ and $\Delta(D_j)$ are sufficient statistics to obtain an optimal Z_j for every j, $(Y_i)_{i=0}^{j}$, and $(Z_i)_{i=0}^{j-1}$. Let us assume that $Q - D_j$ and $\Delta(D_j)$ are sufficient statistics to obtain an optimal Z_j for $n = 2, 3, \ldots K$ where K is an arbitrary natural number. Let Π_K be the set of all causal waiting policies such that if $\pi \in \Pi_K$; then π determines waiting times at delivery points $D_j : Q - D_j < KB_L$ solely based on $Q - D_j$ and $\Delta(D_j)$, for the delivery points $D_j : Q - D_j \geq KB_L$, the waiting policy may not determine the waiting time based on $Q - D_j$ and $\Delta(D_j)$. Due to the induction assumption, the single query problem can be minimized in the set of Π_K. Let us prove that the single query problem can be minimized in the set of Π_{K+1} as well. For every $\pi \in \Pi_K$, we can obtain the following:

$$
\begin{aligned}
&G_R^{\pi}\bigg(Q - D_j - Z_j, \Delta(D_j) + Z_j, (Y_i)_{i=0}^{j}, (Z_i)_{i=0}^{j}\bigg) \\
&\overset{(a)}{=} E\bigg[G_D^{\pi}\bigg(Q - D_j - Z_j - Y_{j+1}, Y_{j+1}, (Y_i)_{i=0}^{j+1}, (Z_i)_{i=0}^{j}\bigg)\bigg| Y_{j+1} + Z_j \leq Q - D_j\bigg] \\
&\qquad\qquad\qquad \times \Pr\bigg(Y_{j+1} + Z_j \leq Q - D_j\bigg) \\
&\quad + g\bigg(Q - D_j + \Delta(D_j)\bigg) \times \Pr\bigg(Y_{j+1} + Z_j > Q - D_j\bigg)
\end{aligned}
\tag{A.1}
$$

where (a) follows from (3.4). $Q - D_{j+1} = Q - D_j - Z_j - Y_{j+1} < KB_L$ as $Y_{j+1} \geq B_L$ and $Q - D_j < (K+1)B_L$. This means that we can exploit the induction assumption in the RHS of (A.1) to claim that $(Y_i)_{i=0}^{j}$ and $(Z_i)_{i=0}^{j-1}$ does not affect the value of the term with expectation given $Q - D_{j+1} = Q - D_j - Y_{j+1} - Z_j$ and $\Delta(D_{j+1}) = Y_{j+1}$. This is because $\pi \in \Pi_K$. In the term with penalty function, only $Q - D_j$ and $\Delta(D_j)$ appear. This means that the optimal control problem of choosing an optimal Z_j at the delivery point D_j does not depend on $(Y_i)_{i=0}^{j}$ and $(Z_i)_{i=0}^{j-1}$. This completes the induction. Once the single query problem can be minimized in the set of $\bigcup_{K=1}^{\infty} \Pi_K$, it is easy to show that the calculation of the functions G_D^{π} and G_R^{π} can be performed by only knowing $Q - D_j$ and $\Delta(D_j)$ for every $\pi \in \bigcup_{K=1}^{\infty} \Pi_K$. The proof can be performed with a similar induction.

From now on, we can omit $(Y_i)_{i=0}^{j}$ and $(Z_i)_{i=0}^{j-1}$ from G_R^{π} and G_D^{π}. For the part related to the existence of a deterministic optimal policy, we construct a deterministic optimal policy by performing another induction on $Q - D_j$. Before move on to the induction, we state some simple observation.

Lemma 4. *Let us assume that there exists a deterministic optimal policy π_1^{opt}.*

(i) Let $h \colon \mathbb{R} \to \mathbb{R}$ such that $h(\epsilon) = max_{x \in [0, M+B_U]} g(x+\epsilon) - g(x)$. Then, we can obtain the following for every $t_1, t_2 \in \mathbb{R}$

$$0 \leq G_R^{\pi_1^{opt}}(t_1, t_2 + \epsilon) - G_R^{\pi_1^{opt}}(t_1, t_2) \leq h(\epsilon) \tag{A.2}$$

(ii) If $f(x) = G_R^{\pi_1^{opt}}(Q - D_j - x, \Delta(D_j))$ is a lower semi-continuous function for a given $Q - D_j$ and $\Delta(D_j)$, then $f'(x) = G_R^{\pi_1^{opt}}(Q - D_j - x, \Delta(D_j) + x)$ is a lower semi-continuous function as well.

(iii) If $f(x) = G_R^{\pi_1^{opt}}(Q - D_j - x, \Delta(D_j))$ is a lower semi-continuous function for every $Q - D_j$ and $\Delta(D_j)$ satisfying $Q - D_j < C$, where C is an arbitrary real number, then $f''(x) = G^{\pi_1^{opt}}(Q - D_i - Y_{i+1} - x, Y_{i+1})$ is a lower semi-continuous function as well for every $Q - D_i$ and $\Delta(D_i)$ satisfying $Q - D_i < C$.

(iv) For every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\Pr(t_1 < Y_j \leq t_1 + \delta) < \epsilon \tag{A.3}$$

where t_1 is a given real number satisfying $t_1 \in [B_L, B_U]$.

(v) For every $Q - D_j$, $\Delta(D_j)$, and $\epsilon > 0$, there exists $\delta > 0$ such that

$$G_D^{\pi_1^{opt}}(Q - D_j - Z_j - Y_{j+1}, Y_{j+1}) - g(Q - D_j + \Delta(D_j) - \delta) < \epsilon \tag{A.4}$$

Proof. (i) It follows from (3.4) and the facts that the penalty function g is continuous and non-decreasing.

(ii) It follows from the definition of lower semi-continuity and Lemma 4(i)

(iii) It follows from Lemma 4(ii) and the fact that π_1^{opt} is a deterministic optimal policy.

(iv) The transmission delay is measurable on Borel algebra on the real line.

(v) It follows from (3.5) and the fact that the penalty function g is continuous and non-decreasing.

□

The idea which will be proven by the induction is that $f(x) = G_R^{\pi_1^{opt}}(Q - D_j - x, \Delta(D_j))$ is a lower semi-continuous function. From Lemma 4(ii), $f'(x) = G_R^{\pi_1^{opt}}(Q - D_j - x, \Delta(D_j) + x)$ is a lower semi-continuous function as well. Therefore, it attains its infimum for every $Q - D_j$ and $\Delta(D_j)$ due to the extension of Extreme Value Theorem to semi-continuity. Then, this infimum point can be determined as the waiting time at the delivery point D_j. This policy is a deterministic optimal policy that decides the waiting periods solely based on $Q - D_j$ and $\Delta(D_j)$. Now, let us move on to the induction. When $Q - D_j < B_L$, then all waiting periods result in the same age penalty. This means that there exists a deterministic optimal policy π_1^{opt} for $n = 1$. Additionally, $f(x) = G_R^{\pi_1^{opt}}(Q - D_j - x, \Delta(D_j))$ is lower semi-continuous for every D_j satisfying $Q - D_j < B_L$. Let us assume for $n = 2$ that $f(x) = G_R^{\pi_1^{opt}}(Q - D_j - x, \Delta(D_j))$ is lower semi-continuous for every D_j satisfying $B_L \leq Q - D_j < 2B_L$. Note that the superscript π_1^{opt} refers in the definition of the function f that the deterministic optimal policy π_1^{opt} is performed starting with $(j+2)^{th}$ request because $(j+1)^{th}$ request has already determined as $D_j + x$. The delivery point D_{j+1} must satisfy $Q - D_{j+1} < (2-1)B_L$ in which there exists a deterministic optimal policy. As $f(x) = G_R^{\pi_1^{opt}}(Q - D_j - x, \Delta(D_j))$ is lower semi-continuous for $n = 2$, $f'(x) = G_R^{\pi_1^{opt}}(Q - D_j - x, \Delta(D_j) + x)$ is a lower semi-continuous function as well by Lemma 4(ii). Hence, the function f' attains its minimum for every $B_L \leq Q - D_j < 2B_L$ and $\Delta(D_j)$. Therefore, there exists a deterministic optimal policy for $n = 2$ as well. Similar to the transition from $n = 1$ to $n = 2$, let us assume one by one that the function $f(x) = G_R^{\pi_1^{opt}}(Q - D_j - x, \Delta(D_j))$ is lower semi-continuous and there exists a deterministic optimal policy for $n = 2, 3, \ldots, K$ where K is an arbitrary natural number. Let us prove that $f(x) = G_R^{\pi_1^{opt}}(Q - D_j - x, \Delta(D_j))$ is a lower semi-continuous function for $n = K + 1$. To reach contradiction, suppose that the claim is false. Then, there exists $Q - D_j, \Delta(D_j)$, and x_0 satisfying $KB_L \leq Q - D_j < (K + 1)B_L$ such that $f(x) = G_R^{\pi_1^{opt}}(Q - D_j - x, \Delta(D_j))$ is not lower semi-continuous at x_0. Hence, there exist either an increasing or a decreasing sequence (x_n) and $C > 0$ such that $\lim_{n \to \infty} x_n = x_0$ and $f(x_n) - f(x_0) < -C$ for every $n \in \mathbb{N}$.

If (x_n) is a increasing sequence, then we obtain the following by (3.4):

$$\begin{aligned} f(x_n) - f(x_0) = {} & A \times \Pr(Y_{j+1} \leq Q - D_j - x_0) \\ & + B \times \Pr(Q - D_j - x_0 < Y_{j+1} \leq Q - D_j - x_n) \\ & + C \times \Pr(Q - D_j - x_n < Y_{j+1}) \end{aligned} \tag{A.5}$$

where A, B, and C are the following:

$$\begin{aligned} A = {} & E\Big[G_D^{\pi_1^{opt}}\big(Q - D_j - x_n - Y_{j+1}, Y_{j+1}\big) \\ & - G_D^{\pi_1^{opt}}\big(Q - D_j - x_0 - Y_{j+1}, Y_{j+1}\big) \Big| Y_{j+1} \leq Q - D_j - x_0 \Big] \end{aligned} \tag{A.6}$$

$$\begin{aligned} B = {} & g(Q - D_j + \Delta(D_j)) - E\Big[G_D^{\pi_1^{opt}}\big(Q - D_j - x_n - Y_{j+1}, Y_{j+1}\big) \Big| \\ & Q - D_j - X_n < Y_{j+1} \leq Q - D_j - x_0 \Big] \end{aligned} \tag{A.7}$$

$$C = g(Q - D_j - x_n + \Delta(D_j)) - g(Q - D_j - x_0 + \Delta(D_j)) \tag{A.8}$$

From the induction assumption and Lemma 4(iii), A can be arbitrarily small. B is upper bounded by $g(M + B_U)$ and the multipliers of B in (A.5) can be arbitrarily small by Lemma 4(iv). C can be arbitrarily small due to the continuity of the penalty function g. Therefore, there exists x_n such that $f(x_n) - f(x) \geq -C$, which is a contradiction.

If (x_n) is a decreasing sequence, then an equation similar to (A.5) can be written. The terms that are similar to A and C can be analyzed similarly. The term that is similar B can be analyzed with the help of Lemma 4(v). After the analysis, a similar contradiction can be achieved.

As a result, the function $f(x)$ is lower semi-continuous. From Lemma 4(ii), the function $f'(x)$ is lower semi-continuous for every $Q - D_j$ and $\Delta(D_j)$ satisfying $KB_L \leq Q - D_j < (K + 1)B_L$. Thus, the function attains its infimum, and the infimum point can be determined as a deterministic optimal waiting period Z_j, which completes the induction. □

APPENDIX B

PROOF OF PROPOSITION 3

Proof. We start this proof with a lemma:

Lemma 5. *For any delivery point $D_j \in [0, Q - 2B_U]$ and its AoI $\Delta(D_j)$, an optimal request point R_{j+1} must be until $Q - B_U$ i.e. $R_{j+1} \leq Q - B_U$.*

Proof. Let us assume that this lemma is not true: There exist a delivery point $D_j \in [0, Q - 2B_U]$ and its AoI at the delivery $\Delta(D_j)$ such that an optimal request point is $R_{j+1} > Q - B_U$. Let this policy follows π^{opt} and let $R^* = R_{j+1}$. We will show that there exists $\pi^{modified}$ such that $G_D^{\pi^{modified}}(Q - D_j, \Delta(D_j)) \leq G_D^{\pi^{opt}}(Q - D_j, \Delta(D_j))$. Let $\pi^{modified}$ determine $R_{j+1}^{mod} = D_j$ and $R_{j+2}^{mod} = R^*$. As the time duration between D_j and R^* is greater than B_U, $\pi^{modified}$ can determine R_{j+2}^{mod} as R^* regardless of the transmission delay of the $(j+1)^{th}$ update. After the request at R^*, let $\pi^{modified}$ imitate π^{opt}. This means that $G_R^{\pi^{opt}}(Q - t_1, t_2) = G_R^{\pi^{modified}}(Q - t_1, t_2)$ for every $t_1 \geq R^*$ and $t_2 \in [B_L, B_U + M]$. As a result of the modification, we can state that

$$\begin{aligned} G_D^{\pi^{modified}}\left(Q - D_j, \Delta(D_j)\right) &\overset{(a)}{=} G_R^{\pi^{modified}}\left(Q - R_{j+2}^{mod}, \Delta(R_{j+2}^{mod})\right) \\ &\overset{(b)}{=} G^{\pi^{opt}}\left(Q - R^*, \Delta(R_{j+2}^{mod})\right) \end{aligned} \tag{B.1}$$

where (a) follows from the decision of $\hat{R}_{j+1}$ and $\hat{R}_{j+2}$, and (b) follows from the fact that $\pi^{modified}$ imitates π^{opt} starting from the point R^*.

On the other hand, as π^{opt} determines the request point R_{j+1} as R^*, we can state that

$$G_D^{\pi^{opt}}\left(Q - D_j, \Delta(D_j)\right) = G_R^{\pi^{opt}}\left(Q - R^*, \Delta(R_{j+1})\right) \tag{B.2}$$

As $\Delta(R_{j+1}) > \Delta(R_{j+2}^{mod})$, and $Q-R^* < B_U$, we can say that $G_R^{\pi^{opt}}(Q-R^*, \Delta(R_{j+1}))$ $> E[G^{\pi^{opt}}(Q-R^*, \Delta(R_{j+2}^{mod}))]$[1] by (3.4). As a result of (B.1) and (B.2), $G_D^{\pi^{modified}}(Q-D_j, \Delta(D_j)) < G_D^{\pi^{opt}}(Q-D_j, \Delta(D_j))$ that contradicts with the fact that π^{opt} is the optimal policy. Hence, there is no such D_j, which completes the proof. □

As a result of Lemma 5, $R_{j+1}^* \leq Q - B_U$. From Proposition 2, AoI at R_{j+1}^* does not affect the age penalty at the query. Next, we prove that there is no $R_{j+1} \in [0, Q-B_U]$ such that $G_R^{\pi_1^{opt}}(Q-R_{j+1}^*) > G_R^{\pi_1^{opt}}(Q-R_{j+1})$. If there existed such $R_{j+1} \in [Q-3B_U, Q-B_U]$, then the destination node would determine the optimal request point for the delivery point D_j^* as R_{j+1}. Hence, we can state the following for every delivery point D_j and its transmission delay Y_j satisfying $D_j \in [Q-3B_U, Q-2B_U]$ and $Y_j \in [B_L, B_U]$:

$$G_R^{\pi^{opt}}(Q-R_{j+1}^*) \leq G_D^{\pi^{opt}}(Q-D_j, Y_j) \tag{B.3}$$

On the other hand, such R_{j+1} cannot be in the interval $[0, Q-3B_U]$ as well. This statement is proved by induction. Similar to the proof of Proposition 1, R_{j+1} is mapped to a natural number n if it satisfies $(n-1)B_L \leq Q-3B_U-R_{j+1} < nB_L$. It is true for $n=1$ *i.e.* such R_{j+1} cannot be in the interval $0 \leq Q-3B_U-R_{j+1} < B_L$ because of the following:

$$\begin{aligned} G_R^{\pi_1^{opt}}\left(Q-R_{j+1}\right) &\overset{(a)}{=} E\left[G_D^{\pi_1^{opt}}\left(Q-R_{j+1}-Y_{j+1}, Y_{j+1}\right)\right] \\ &\overset{(b)}{\geq} G_R^{\pi_1^{opt}}\left(Q-R_{j+1}^*\right) \end{aligned} \tag{B.4}$$

where (a) follows from (3.4), and (b) follows from (B.3). Let us assume that the induction statement is true for $n = 2, 3, \ldots, K$ where K is an arbitrary natural number. This statement assumes the following for every request point R_{j+1} satisfying $0 \leq Q-3B_U-R_{j+1} < KB_L$:

$$G_R^{\pi^{opt}}(Q-R_{j^*+1}) \leq G_R^{\pi^{opt}}(Q-R_{j+1}) \tag{B.5}$$

Let us prove the induction statement for $n = K+1$. For every R_{j+1} satisfying

[1] If there exists $x < B_U$ such that $\Pr(Y_j \in (x, B_U]) = 0$, then B_U can be shifted to x. Thus, we can assume that $\Pr(Y_j \in (x, B_U]) > 0$ for every $x < B_U$. As a result, we can claim that $G_R^{\pi^{opt}}(Q-R^*, \Delta(R_{j+1}))$ is strictly greater than $E[G^{\pi^{opt}}(Q-R^*, \Delta(R_{j+2}^{mod}))]$.

$KB_L \leq Q - 3B_U - R_{j+1} < (K+1)B_L$, we have the following:

$$\begin{aligned} G_R^{\pi_1^{opt}}\big(Q - R_{j+1}\big) &\overset{(a)}{=} E\Big[G_D^{\pi_1^{opt}}\big(Q - R_{j+1} - Y_{j+1}, Y_{j+1}\big)\Big] \\ &\overset{(b)}{\geq} G_R^{\pi_1^{opt}}\big(Q - R^*_{j+1}\big) \end{aligned} \tag{B.6}$$

where (a) follows from (3.4), and (b) follows from (B.5). This implies that the induction is completed.

As a result, for every $R_{j+1} \in [0, Q - B_U]$, we have the following:

$$G_R^{\pi_1^{opt}}(Q - R^*_{j+1}) \leq G_R^{\pi_1^{opt}}(Q - R_{j+1}) \tag{B.7}$$

It means that R^*_{j+1} attains its infimum value on the interval $[0, Q - B_U]$. This completes the proof. □

APPENDIX C

PROOF OF PROPOSITION 5

Proof. We perform a similar induction included in the proof of Proposition 1. Let us map each $Q - D_j$ to a natural number n that satisfies $(n-1)B_L \leq Q - D_j < nB_L$. If $n = 1$, the request point does not affect the expected age penalty at the query. Thus, requesting at the query is an optimal request point that proves the proposition statement for $n = 1$. Let us assume that the optimal request point is in the set $\{0, \frac{N}{Q}, \frac{2N}{Q}, \ldots, Q\}$ when a delivery occurs at time D_j satisfying $(n-1)B_L \leq Q - D_j < nB_L$ for $n = 1, 2, \ldots, K$ where K is an arbitrary natural number. Let us prove that the optimal request point is in the set $\{0, \frac{N}{Q}, \frac{2N}{Q}, \ldots, Q\}$ when a delivery occurs at time D_j satisfying $KB_L \leq R - D_j < (K+1)B_L$. Let us assume the inverse. There exists a delivery point D_j such that $KB_L \leq Q - D_j < (K+1)B_L$ and the is no optimal request point in the set $\{0, \frac{N}{Q}, \frac{2N}{Q}, \ldots, Q\}$. As there exists an optimal policy from Proposition 1, there exists an optimal request point $R_{j+1} \notin \{0, \frac{N}{Q}, \frac{2N}{Q}, \ldots, Q\}$. For every quantized transmission delay Y_{j+1}, the next delivery point satisfies $Q - D_{j+1} < KB_L$. If $Q - D_{j+1} > 0$, the optimal next request point should be in the set $\{0, \frac{N}{Q}, \frac{2N}{Q}, \ldots, Q\}$ due to the induction assumption. Instead of requesting at R_{j+1}, if the request was performed at $R_{j+1}^{mod} = \frac{\left\lceil \frac{R_{j+1}}{N/Q} \right\rceil N}{Q}$, there would be two cases based on the transmission delay Y_{j+1}. For every $m \in \mathbb{N}$; if $D_{j+1}^{mod} > m\frac{Q}{N}$, then $D_{j+1} > m\frac{Q}{N}$; if $D_{j+1}^{mod} < m\frac{Q}{N}$, then $D_{j+1} < m\frac{Q}{N}$ because of the quantized transmission delay process, where $m\frac{Q}{N}$ represents the possible next request point or the query. Therefore, requesting an update packet at R_{j+1}^{mod} is optimal given that R_{j+1} is an optimal request point. This conclusion contradicts with the assumption. Hence, there exists an optimal request point in the set $\{0, \frac{N}{Q}, \frac{2N}{Q}, \ldots, Q\}$ for every delivery point, which completes the proof. □

APPENDIX D

PROOF OF PROPOSITION 7

Proof. Let $\delta = \frac{1}{2}\max_{x\in[0,M+B_U]} g(x+\epsilon) - g(x)$. Let N_1 be a natural number satisfying $N_1 > \frac{3Q^2}{4\delta B_L}$. There exists a deterministic optimal policy π_1^{opt} whose first request point is the border point for lower quantization of the transmission delay with $N \geq N_1$ by Corollary 2. We construct an update policy π_1^{mod} for upper quantization of the transmission delay with N by utilizing π_1^{opt}. Let the border point corresponding to the lower quantized transmission delay and π_1^{opt} be Q^{BP}. Let π_1^{mod} pull its first request at $R_1^{mod} = Q^{BP} - \frac{Q}{N} \times \frac{Q-Q^{BP}}{B_L}$. Note that $R_1^{mod} - Q^{BP} < \delta$ as $Q^{BP} \geq Q - 3B_U > \frac{Q}{4}$ and $N \geq N_1$. Let $(Y_j)_{j=1}^J$ be an arbitrary transmission delay sequence from the unquantized transmission delay process where $\sum_{j=1}^J Y_j > Q$. Let $(Y_j^{upp})_{j=1}^J$ and $(Y_j^{low})_{j=1}^J$ be the sequences that correspond to upper and lower quantized of $(Y_j)_{j=1}^J$, respectively. Let $(Z_j)_{j=1}^J$ be the waiting time sequences that is causally determined by π_1^{opt} based on $(Y_j^{low})_{j=1}^J$. If π_1^{mod} determines the waiting periods the same as π_1^{opt} after the first request point *i.e.* $R_{j+1}^{mod} = R_j^{mod} + Y_j^{upp} + Z_j, j \geq 1$, then the difference between age penalties under π_1^{opt} and π_1^{mod} is less than ϵ. This is because $0 < R_j - R_j^{mod} < \delta$ for every j where R_j is the j^{th} request point under π_1^{opt}. As this is valid for every transmission delay sequence $(Y_j)_{j=1}^J$, its expected difference is less than ϵ. This completes the proof. $\square$

APPENDIX E

PROOF OF PROPOSITION 8

Proof. Let $\Pi_{Pois}^{UoW}, \Pi_{Pois}^{PoW}$ be the sets of optimal causal policies for the UoW and PoW problems, respectively, for a given transmission delay process under a Poisson query arrival process. We prove in this proof that $\Pi_{Pois}^{UoW} \subset \Pi_{Pois}^{PoW}$ and $\Pi_{Pois}^{PoW} \subset \Pi_{Pois}^{UoW}$ for every transmission delay process, which completes the first part of the proposition.

Let N_t be a Poisson counting process with a parameter λ. Then, N_t has the stationary and independent increments property. By Taylor expansion, we can state that

$$\Pr(N_{t+\delta} - N_t = 1) = \lambda\delta e^{-\lambda\delta} = \lambda\delta + o(\delta) \tag{E.1}$$

Let us divide the time interval $[0, Q_n]$ into small interval with length δ. Let $\mathbb{P}$ be the partition that consists of these small intervals. Then, an upper Darboux sum can be derived as follows:

$$\begin{aligned} E\Big[\sum_{k=1}^{n} g(\Delta(Q_k))\Big] &\leq E\Big[\sum_{j=0}^{Q_n/\delta} P(N_{t+\delta} - N_t = 1) \times \sup_{t:\, t\in[0,\delta)} g\big(\Delta(j\delta + t)\big)\Big] \\ &= E\Big[\sum_{j=0}^{Q_n/\delta} (\lambda\delta + o(\delta)) \times \sup_{t:\, t\in[0,\delta)} g\big(\Delta(j\delta + t)\big)\Big] \\ &= U(g(\Delta), \mathbb{P}) \end{aligned} \tag{E.2}$$

Similar to the upper Darboux sum, a lower Darboux sum can be derived as follows:

$$\begin{aligned} E\bigg[\sum_{k=1}^{n} g(\Delta(Q_k))\bigg] &\geq E\bigg[\sum_{j=0}^{Q_n/\delta} P(N_{t+\delta} - N_t = 1) \times \inf_{t:\, t\in[0,\delta)} g\big(\Delta(j\delta + t)\big)\bigg] \\ &= E\bigg[\sum_{j=0}^{Q_n/\delta} (\lambda\delta + o(\delta)) \times \inf_{t:\, t\in[0,\delta)} g\big(\Delta(j\delta + t)\big)\bigg] \\ &= L(g(\Delta), \mathbb{P}) \end{aligned} \tag{E.3}$$

Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be the partitions consisted of small intervals with length δ in which there is no delivery and there is a delivery, respectively. This means that $\mathbb{P} = \mathbb{P}_1 \cup \mathbb{P}_2$ and $\mathbb{P}_1 \cap \mathbb{P}_2 = \emptyset$. Then, there exists $\delta_1 > 0$ such that the partition $\mathbb{P}_1$ is organized with δ_1 length small intervals and $U(g(\Delta), \mathbb{P}_1) - L(g(\Delta), \mathbb{P}_1) < \epsilon/2$ since $g(\Delta(.))$ is continuous on the partition $\mathbb{P}_1$. Inside any interval in $\mathbb{P}_2$, the supremum point is less than or equal to $g(B_U + M)$ while the infimum point is greater than or equal to $g(B_L)$. On the other hand, the number of small intervals in $\mathbb{P}_2$ can be at most Q_n/B_L. Therefore if the partition $\mathbb{P}_2$ is organized with small intervals with length δ_2 equal to $\frac{\epsilon \times B_L}{3Q_n \times (g(M+B_U) - g(B_L))}$, then $U(g(\Delta), \mathbb{P}_2) - L(g(\Delta), \mathbb{P}_2) \leq \epsilon/3 < \epsilon/2$. As a result, if the partition $\mathbb{P}$ is organized with $\delta = \min(\delta_1, \delta_2)$ length small intervals, then $U(g(\Delta), \mathbb{P}) - L(g(\Delta), \mathbb{P}) < \epsilon$ because $U(g(\Delta), \mathbb{P}_1) + U(g(\Delta), \mathbb{P}_2) = U(g(\Delta), \mathbb{P})$ and $L(g(\Delta), \mathbb{P}_1) + L(g(\Delta), \mathbb{P}_2) = L(g(\Delta), \mathbb{P})$. Hence, for every $n \in \mathbb{N}$, we have proved the following by [57, Theorem 6.6]:

$$E\bigg[\sum_{k=1}^{n} g(\Delta(Q_k))\bigg] = E\bigg[\lambda \int_0^{Q_n} g(\Delta(t))dt\bigg] \tag{E.4}$$

Note that λ is just a constant and (E.4) holds for every $n \in \mathbb{N}$. Then, we can obtain the following:

$$\begin{aligned} \limsup_{n\to\infty} \frac{E\bigg[\sum\limits_{k=1}^{n} g(\Delta(Q_k))\bigg]}{n} &\overset{(a)}{=} \limsup_{n\to\infty} E\left[\frac{\int_0^{Q_n} g(\Delta(t))dt}{Q_n}\right] \\ &\overset{(b)}{=} \limsup_{n\to\infty} E\left[\frac{\int_0^{D_n} g(\Delta(t))dt}{D_n}\right] \end{aligned} \tag{E.5}$$

where (a) follows from $E[Q_n] = n/\lambda$. (b) can be shown by using the following two facts: (i) Let D_i be the closest delivery point to a query Q_k. Then, $|D_i - Q_k| < (B_U + M)/2$. (ii) the function $g(\Delta)$ has upper and lower bounds. As a result of (E.5), minimizing (2.2) and (2.4) are equivalent, which implies that Π_{Pois}^{UoW} and Π_{Pois}^{PoW} are equivalent. Furthermore, their average age penalties are equivalent by (E.5). $\square$

APPENDIX F

PROOF OF LEMMA 2

Proof. In this proof, under a stationary and deterministic policy $\pi \in \Pi^{PoW}$, we show that the limit a_n exists as n goes to ∞ where a_n is the following:

$$a_n = \frac{E\big[\sum_{k=1}^{n} g(\Delta(Q_k))\big|Y_0\big]}{n} \tag{F.1}$$

Y_0 is given in the expectation, $Z_0 = z(Y_0) + x$ where x is constant; hence Z_0 is constant. Let $X_j = Y_j + Z_j$ for $j \in \mathbb{N}$. The transmission delays are i.i.d., and the update policy π is a stationary and deterministic policy, which is a function of Y_j for $j > 0$; thus X_j is i.i.d. for $j > 0$. The probabilities of X_j can be calculated based on the probabilities of Y_j and the policy π.

The request points can be represented as $R_{j+1} = Z_0 + \sum_{i=1}^{j} X_i$ for $j \geq 1$ and $R_1 = Z_0$. Let the stopping time $\tau_k = \min\{\tau_k : R_{\tau_k} > Q_k\}$. The modulo operation is defined as the following:

$$mod(R_{\tau_k}, T) = R_{\tau_k} - T \times \max\{k : Tk < R_{\tau_k}\} \tag{F.2}$$

We can construct a Markov chain whose states are $\{mod(R_{\tau_k}, T), k \geq 1\}$.
$E[g(\Delta(Q_k))|mod(R_{\tau_k}, T)]$ can be calculated from the conditional expectation of X_j given Y_j, independent of k. Throughout the proof, we consider X_j in two different scenarios similar to [59].

The first scenario is $\Pr(X_j \in \{k\beta\colon B_L \leq k\beta \leq B_U + M \text{ and } k \in \mathbb{N}\}) = 1$ such that β is a rational multiple of T. Hence, the Markov chain has a finite number of states. Let these states be $1, 2, \ldots, N$ These states communicate with each other. Therefore, it has a steady state distribution [60, Section 4.3.1] and the limiting time-average fraction of time spent in each state can be calculated from [60, Theorem 7.2.6]. Let these fractions be $p_1, \ldots, p_N$. Then all the subsequences of a_n goes to the same value equal to $\sum_{i=1}^{N} E[g(\Delta(Q_k))|mod(R_{\tau_k}, T) = i^{th}\text{state}] \times p_i$. All the subsequences of a_n goes to the same limit, so the limit of a_n exists.

The second scenario is all the random variables X_j except the previous scenario. Given $R_j = mT + a$ where $m \in \mathbb{N}$ and $a < T$, the probability of which R_j is a stopping time is equal to $\Pr(X_{j-1} > a)$. From [59, Theorem 1 and 2], R_j is equidistributed in modulo T with probability 1. Due to the equidistriution, the limiting time-average fraction of time spent in the state of $mod(R_{\tau_k}, T) = a$ exists and it is proportional to $\Pr(X_j > a)$. Once the limiting time-average fraction exists, all the subsequences of a_n defined in (F.1) goes to the same value similar to the second scenario. Thus, the limit of a_n exists. This completes the proof. $\square$

APPENDIX G

PROOF OF UNIFORM CONVERGENCE

Proof. Let $f_k \colon \mathbb{N} \to \mathbb{R}$, $k \in \mathbb{N}$ be a function such that

$$f_k(n) = \frac{\sum_{i=0}^{Q_n/2^{-k}} g\big(\Delta_{\pi^{opt},\mathbf{Y}}\big(i \times 2^{-k}\big)\big)}{Q_n/2^{-k}} \tag{G.1}$$

Let $f \colon \mathbb{N} \to \mathbb{R}$ be a function such that

$$f(n) = \frac{\int_0^{Q_n} g\big(\Delta_{\pi^{opt},\mathbf{Y}}(t)\big) dt}{Q_n} \tag{G.2}$$

If we prove that $f_k \to f$ uniformly for every $\{Y_j\}$ and $\{Z_j\}$ sequence providing that $Y_j \in [B_L, B_U]$ and $Z_j \in [0, M]$, we can ignore the expectation since it is uniformly convergent for every possible sequence. Then, the proof is completed.

Let $M_k \in \mathbb{R}$ be

$$M_k = \sup_{n \in \mathbb{N}} |f_k(n) - f(n)| \tag{G.3}$$

As penalty function g is non-decreasing and $\Delta_{\pi^{opt},\mathbf{Y}}(.) < B_U + M$, then $M_k \leq g(M + B_U)$. Furthermore, as k goes to infinity, M_k approaches 0 due to the continuity of the penalty function, g. As a result, f_k is uniformly convergent to f by [57, Theorem 7.9]. This completes the proof. □