EXCEPTIONAL LIE ALGEBRA $\mathfrak{g}_2$ AND ITS REPRESENTATIONS

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ABSTRACT

EXCEPTIONAL LIE ALGEBRA $\mathfrak{g}_2$ AND ITS REPRESENTATIONS

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In the classification of complex simple Lie algebras, there are five of them whose Dynkin diagrams are of exceptional type. The Lie algebra $\mathfrak{g}_2$ has the smallest dimension among these exceptional Lie algebras and together with its corresponding Lie group G_2, it plays an important role in differential geometry, mathematical physics, and modern string theory. In this thesis after a general introduction to Lie algebras, we show the classification of complex simple ones. Afterward, we give several constructions of the exceptional Lie algebra $\mathfrak{g}_2$ and investigate its fundamental representations.

Keywords: Lie algebra, Lie group, exceptional Lie algebra, exceptional Lie group

ÖZ

İSTİSNAİ $\mathfrak{g}_2$ LİE CEBİRİ VE TEMSİLLERİ

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Basit kompleks Lie cebirlerinin sınıflandırılmasında Dynkin diagramı sıradışı olan 5 tane Lie cebiri vardır. Bunlar arasında $\mathfrak{g}_2$ en küçük boyuta sahip olanıdır ve $\mathfrak{g}_2$ ye karşılık gelen Lie grubuyla beraber diferansiyel geometri, matematiksel fizik ve modern sicim kuramında önemli role sahiptir. Bu tezde genel bir giriş yaptıktan sonra basit kompleks Lie cebirlerinin sınıflandırmasını verdik. Daha sonra sıradışı $\mathfrak{g}_2$ Lie cebirinin farklı inşalarını ve temel temsillerini inceledik.

Anahtar Kelimeler: Lie grup, Lie cebiri, sıradışı Lie cebiri, sıradışı Lie grubu

To my family

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CHAPTER 1

INTRODUCTION

Lie groups and Lie algebras play an important role in many branches of mathematics. Among these, the most well-known branches can be given as the following:

• differential geometry (e.g. Chern-Weil theory),
• Riemannian geometry (e.g. holonomy and symmetric spaces),
• mathematical physics (e.g. gauge theory),
• algebraic topology (e.g. characteristic classes)
• low-dimensional topology (e.g. Chern-Simons theory),
• algebraic geometry (e.g. flag varieties),
• number theory (e.g. the Langlands program).

Lie theory was initiated by Norwegian mathematician Sophus Lie in the late nineteenth century. His ideas were to develop a theory of symmetries for differential equations, similar to the theory of Galois for algebraic equations. One big difference here was that the groups were not finite or discrete but they were continuous which can be said right now that they are also manifolds. His ideas were motivated by the work of Jacobi for differential equations and they were developed with a close collaboration with F. Klein.

Lie groups are smooth manifolds with a group structure where the group multiplication and inversion are smooth maps. They are very essential in geometry; in many cases they provide the first interesting examples due to the fact that they are the groups of symmetries of certain manifolds.

One of the fundamental results in the theory of Lie groups is that many properties of Lie groups can completely be determined by the corresponding Lie algebra, the tangent space of the Lie group at identity. (Conversely, Lie's third theorem states that every Lie algebra $\mathfrak{g}$ has a unique Lie group G which is connected and simply connected such that the Lie algebra of G is $\mathfrak{g}$.) Studying the Lie algebras reduce most of the problems in Lie groups to the linear algebra level. Furthermore they provide an important role in the big problem, the classification of Lie groups. Hence, by giving a complete classification of a large class of Lie algebras, namely complex semisimple Lie algebras, in 1888 a major improvement was done in Lie theory by Wilhelm Killing.

In the classification of complex semisimple Lie algebras, aside from a few classes of big families there are some exceptional Lie algebras, which are $\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$. The Lie algebra $\mathfrak{g}_2$, the main topic of this thesis, is 14-dimensional and it is the smallest in this list whereas $\mathfrak{f}_4$ is 52, $\mathfrak{e}_6$ is 78, $\mathfrak{e}_7$ is 133 and $\mathfrak{e}_8$ is 248 dimensional.

The Lie algebra $\mathfrak{g}_2$ was first realized by W. Killing. when he was trying to prove that there are only two families of complex simple Lie algebras, namely $\mathfrak{so}(n, \mathbb{C})$ and $\mathfrak{sl}(n, \mathbb{C})$. The reality was dramitacally surprising that there is actually one more family which is the symplectic algebra $\mathfrak{sp}(n, \mathbb{C})$ other than the five exceptional Lie algebras $\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$. Although Killing was able to make the correct classification, there were mistakes and missing parts in Killing's work, which was later completed by Elie Cartan [1], [2].

The Lie algebra $\mathfrak{g}_2$ together with its corresponding Lie group G_2 play a crucial role in Riemannian geometry and in theoretical physics. 7-dimensional manifolds whose holonomy groups are in G_2 are called G_2-manifolds. These special manifolds are very important in string theory. The mathematical structure of M-theory compactifications of four-dimensional space time determines the physics. These compactifications are 7-dimensional and they carry the structure of a G_2-manifold.

In this thesis, we mainly investigate complex simple Lie algebras, especially the exceptional Lie algebra $\mathfrak{g}_2$ and its several constructions. The organization of the thesis is as follows.

In Chapter 2, we first give the necessary background on Lie algebras, and then focus on complex semisimple Lie algebras, their equivalent formulations and basic properties.

In Chapter 3 starting with a motivational example, namely $\mathfrak{sl}(2, \mathbb{C})$ we summarize the classification of complex simple Lie algebras.

In Chapter 4, we give an outline of the representation of Lie algebras, first for an important example $\mathfrak{sl}(3, \mathbb{C})$ which is mainly used in the following chapter and then in general setting.

In the concluding chapter, namely Chapter 5, we give two constructions of the exceptional Lie algebra $\mathfrak{g}_2$. The first one is done by using the Dynkin diagram and its root system. The second is a recent construction by Wildberger [3] which basically uses combinatorics.

CHAPTER 2

PRELIMINARIES

2.1 Basic Concepts On Lie Algebras

Definition 2.1.1. A vector space L over a field $\mathbb{F}$, with a bilinear operation $L \times L \longrightarrow L$ denoted $[xy]$ and called the bracket or commutator of x and y, is called a Lie algebra over $\mathbb{F}$ if the following axioms are satisfied:

(L1) $[xx] = 0$ for any x in L.

(L2) $[x[yz]] + [y[zx]] + [z[xy]] = 0$ for x, y, z in L. (Jacobi identity)

Observe that for any x, y in L, $0 = [x+y, x+y] = [xx]+[xy]+[yx]+[yy] = [xy]+[yx]$ implies that $[xy] = -[yx]$.

In this thesis, we only study the finite dimensional Lie algebras over the complex field $\mathbb{C}$. Other than that we assume all the vector spaces to be finite dimensional over $\mathbb{C}$ unless otherwise stated explicitely.

Example 2.1.1. Any vector space V can be made into a Lie algebra trivially by defining $[xy] = 0$ for all $x, y \in V$. This Lie algebra is called **abelian**.

Example 2.1.2. Cross product on $\mathbb{R}^3$ defines a Lie algebra.

Example 2.1.3. Let V be an n dimensional vector space over $\mathbb{F}$, $\mathfrak{gl}(V)$ denotes the set of linear transformations from V to V with the multiplication $[f, g] = f \circ g - g \circ f$. Then $\mathfrak{gl}(V)$ is a Lie algebra, called the **general linear Lie algebra**. Fixing a basis for V, we may prefer to write the matrix representations of the elements of $\mathfrak{gl}(V)$ and denote it by $\mathfrak{gl}(n, \mathbb{F})$ with the bracket defined by $[M, N] = MN - NM$ for the $n \times n$ matrices M and N. We can show that the Jacobi identity

$$[M,[N,K]]+[N,[K,M]]+[K,[M,N]]=0$$

is satisfied.

Definition 2.1.2. A subspace K of L is called a **Lie subalgebra** if $[xy] \in K \; \forall x, y \in K$. A subspace I of L is called an **ideal** if $[xy] \in I$, for $x \in I, y \in L$.

Any subalgebra of $\mathfrak{gl}(V)$ is called a linear Lie algebra.

Remark We only have two sided ideals for Lie algebras since, $[xy] = -[yx]$.

Example 2.1.4. The set of trace 0 matrices form a subalgebra of $\mathfrak{gl}(n, \mathbb{F})$, called the **special linear algebra**, denoted, $\mathfrak{sl}(n, \mathbb{F})$. Indeed; for the matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathfrak{gl}(n, \mathbb{F})$, we have tr$(AB) = \sum_{i=1}^{n}(AB)_{ii} = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki} = \sum_{i=1}^{n}\sum_{k=1}^{n} b_{ki}a_{ik} = \sum_{k=1}^{n}\sum_{i=1}^{n} b_{ki}a_{ik} = \sum_{k=1}^{n}(BA)_{kk} =$ tr(BA). Therefore, tr$([A,B]) =$ tr$(AB - BA) =$ tr$(AB) -$ tr$(BA) = 0$. This shows that $\mathfrak{sl}(n, \mathbb{F})$ is actually an ideal of $\mathfrak{gl}(n, \mathbb{F})$.

Example 2.1.5. Among all, there are 4 families of linear Lie algebras which are fundamental to the classification.

$A_n = \mathfrak{sl}(n+1, \mathbb{C}) = \{A \in \mathfrak{gl}(n+1) \mid \text{tr}(A) = 0\}$ the special linear Lie algebra,

$B_n = \mathfrak{so}(2n+1, \mathbb{C}) = \{A \in \mathfrak{gl}(2n+1) \mid A + A^T = 0\}$ odd dimensional orthogonal algebra,

$C_n = \mathfrak{sp}(2n, \mathbb{C}) = \{A \in \mathfrak{gl}(2n) \mid J_nA + A^TJ_n = 0\}$, where $J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ symplectic algebra,

$D_n = \mathfrak{so}(2n, \mathbb{C}) = \{A \in \mathfrak{gl}(2n) \mid A + A^T = 0\}$ even dimensional orthogonal algebra.

Definition 2.1.3. A linear map $\phi : L \longrightarrow L'$ is called a **(Lie algebra) homomorphism** if $\phi([xy]) = [\phi(x)\phi(y)] \; \forall x, y \in L$

Definition 2.1.4. Two lie algebras L and L' are called **isomorphic** if there exists a Lie algebra homomorphism which is bijective.

Example 2.1.6. Let $\dim V = n$ and $\mathcal{B}$ be an ordered basis of V. Then the map $\Psi : \mathfrak{gl}(V) \longrightarrow \mathfrak{gl}(n, \mathbb{C})$ defined by $\Psi(T) = [T]_{\mathcal{B}}$ is an isomorphism of Lie algebras.

The restriction $\Psi_{|_{\mathfrak{sl}(V)}}$ establishes an isomorphism between $\mathfrak{sl}(V)$ and $\mathfrak{sl}(n, \mathbb{C})$.

Definition 2.1.5. Let L be a Lie algebra over $\mathbb{F}$. A homomorphism $\phi : L \longrightarrow \mathfrak{gl}(V)$ of Lie algebras where V is an $\mathbb{F}$-space is called a **representation** of L.

If we define $x.v = \phi(x)(v)$, a representation ϕ affords an L-module structure by the rules:

1. $x.v$ is linear in x and v.
2. $[xy].v = x.(y.v) - y.(x.v) \quad$ for $x, y \in L,\ v \in V$.

Conversely, if the action $x.v$ satisfies $1, 2$, then L-module structure affords a representation by the rule $\phi(x)(v) = x.v$. Therefore, studying representations and modules are equivalent.

Definition 2.1.6. Any linear map δ on a Lie algebra L is called a **derivation** if it satisfies $\delta([xy]) = [x\delta(y)] + [\delta(x)y]$ for $x, y \in L$.

Example 2.1.7 (Adjoint Representation)**.** Let x be any element of the Lie algebra L. We define the map ad$x : L \longrightarrow L$ by ad$x(y) = [xy]$. This map is linear but not a homomorphism. However, it is a derivation since, $[x[yz]] = [y[xz]] + [[xy]z]$ by Jacobi identity.

Now, we consider the map ad: $L \longrightarrow \mathfrak{gl}(L)$ defined by ad(x) =ad$x$. This map is a homomorphism, i.e, ad$[xy] = [$ad$x$,ad$y]$, called the **adjoint representation** of L.

$[\text{ad}x, \text{ad}y](z) = \text{ad}x\text{ad}y(z) - \text{ad}y\text{ad}x(z) = \text{ad}x([yz]) - \text{ad}y([xz]) = [x[yz]] - [y[xz]] = [x[yz]] + [[xz]y] = [[xy]z] = \text{ad}[xy](z)$.

Lemma 2.1.1. The set of derivations Der$L$ of L is a subalgebra of $\mathfrak{gl}(L)$. Moreover, ad$L$ is an ideal of Der$L$.

Example 2.1.8. A basis for $\mathfrak{sl}(2, \mathbb{C})$ is given by

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

It can be easily checked that $[xy] = h$, $[hx] = 2x$ and $[hy] = -2y$. This implies that h acts on the subspaces $\mathbb{C}x$ and $\mathbb{C}y$ by the bracket operator. In other words, We may write the $\mathbb{C}h$-module $\mathfrak{sl}(2, \mathbb{C})$ as a sum of 1-dimensional $\mathbb{C}h$-submodules, namely

$$\mathfrak{sl}(2, \mathbb{C}) = \mathbb{C}h \oplus \mathbb{C}x \oplus \mathbb{C}y.$$

Moreover, This observation can be generalized to $\mathfrak{sl}(n, \mathbb{C})$ for any n. Indeed, if $\mathfrak{h}$ is the subalgebra of diagonal matrices of $\mathfrak{sl}(n, \mathbb{C})$ then $\mathfrak{h}$ acts on the subspaces $\mathbb{C}E_{i,j}$ where $i \neq j$. Later, in chapter 3, we will see that any semisimple Lie algebra has a decomposition with this property.

Definition 2.1.7. If I and J are ideals of L then we define $[IJ] = \{\sum[x_i y_i] | x_i \in I, y_i \in J\}$.

Remark It is easy to show that $[IJ]$ is also an ideal of L by using Jacobi identity. In particular, $[LL]$ is an ideal of L.

Definition 2.1.8. A non-abelian Lie algebra L is called **simple** if it has no ideals except L and 0.

Example 2.1.9. $\mathfrak{sl}(V)$ is an ideal of $\mathfrak{gl}(V)$. So, $\mathfrak{gl}(V)$ is not simple.

We have two sequences of ideals of L :

(i) The **derived series** of L is defined to be the sequence of ideals $L^{(0)} = L$, $L^{(1)} = [LL]$, $L^{(2)} = [L^{(1)}L^{(1)}], \cdots\cdots, L^{(n)} = [L^{(n-1)}L^{(n-1)}]$.

(ii) The **lower central series** of L is defined to be the sequence of ideals $L^0 = L$, $L^1 = [LL]$, $L^2 = [LL^1], \cdots\cdots, L^n = [LL^{n-1}]$.

Definition 2.1.9. Let L denote a Lie algebra,

1. L is called **solvable** if $L^{(n)} = 0$ for some n.

2. L is called **nilpotent** if $L^n = 0$ for some n.

Remark 1 L is abelian $\iff L^{(1)} = L^1 = 0$, that is, any abelian algebra is nilpotent.

Remark 2 By induction, $L^{(n)} \subset L^n$ for any n, that is, any nilpotent algebra is solvable.

Definition 2.1.10. The center of the Lie algebra L is defined as $Z(L) = \{z \in L \mid [zx] = 0 \, \forall x \in L\}$. It is obviously an abelian ideal of L and L is abelian if and only if $Z(L) = L$.

Example 2.1.10. $E_{i,j}$ denotes the matrix with (i, j) entry 1 and other entries 0.

(i) $\mathfrak{d}(n, \mathbb{C})$ the subalgrebra of diagonal matrices of $\mathfrak{gl}(n, \mathbb{C})$ is abelian.

(ii) Let $\mathfrak{n}(n, \mathbb{C})$ be the subalgebra of strictly upper triangular matrices of $\mathfrak{gl}(n, \mathbb{C})$. Then, it is easy to show that $[\mathfrak{d}(n, \mathbb{C}), \mathfrak{n}(n, \mathbb{C})] = \mathfrak{n}(n, \mathbb{C})$. Indeed, if N is strictly upper triangular matrix, then for $i \geq j$, $(E_{i,i}.N)_{ij} = N_{ij} = 0$ and $(N \cdot E_{i,i})_{ij} = N_{ii} = 0$. Therefore, the i, j entry $(i \geq j)$ of the matrix $[E_{i,i}, N] = E_{i,i}N - NE_{i,i}$ is 0. Since any diagonal matrix is a linear combination of E_{ii}'s, we get $[\mathfrak{d}(n, \mathbb{C}), \mathfrak{n}(n, \mathbb{C})] \subset \mathfrak{n}(n, \mathbb{C})$. For the inclusion $\mathfrak{n}(n, \mathbb{C}) \subset [\mathfrak{d}(n, \mathbb{C}), \mathfrak{n}(n, \mathbb{C})]$, any $N \in \mathfrak{n}(n, \mathbb{C})$ is a linear combination of $E_{i,j}$'s where $i < j$. Then, $E_{i,j} = [E_{i,i}, E_{i,j}]$ finishes the assertion.

(iii) Let $L = \mathfrak{t}(n, \mathbb{C})$ be the subalgebra of upper triangular matrices of $\mathfrak{gl}(n, \mathbb{C})$. We can write $\mathfrak{t}(n, \mathbb{C}) = \mathfrak{d}(n, \mathbb{C}) \oplus \mathfrak{n}(n, \mathbb{C})$. Then by (ii), $[\mathfrak{t}(n, \mathbb{C}), \mathfrak{t}(n, \mathbb{C})] = \mathfrak{n}(n, \mathbb{C})$ and $L^2 = [LL^1] = [\mathfrak{t}(n, \mathbb{C}), \mathfrak{n}(n, \mathbb{C})] \subset [\mathfrak{d}(n, \mathbb{C}), \mathfrak{n}(n, \mathbb{C})] + [\mathfrak{n}(n, \mathbb{C}), \mathfrak{n}(n, \mathbb{C})] = \mathfrak{n}(n, \mathbb{C})$. Also, L^2 contains $\mathfrak{n}(n, \mathbb{C})$ since it contains $[\mathfrak{d}(n, \mathbb{C}), \mathfrak{n}(n, \mathbb{C})]$. This shows that, $L^i = \mathfrak{n}(n, \mathbb{C})$ for any i, and hence $\mathfrak{t}(n, \mathbb{C})$ is not nilpotent. But we will show later that $\mathfrak{t}(n, \mathbb{C})$ is solvable by using Engel's theorem.

Theorem 2.1.1. (a) Subalgebras and homomorphic images of a solvable Lie algebra are solvable.

(b) If I is a solvable ideal of a Lie algebra L such that L/I is solvable, then L is solvable.

(c) If I and J are solvable ideals of a Lie algebra L, then so is $I + J$.

Proof. (a) Let L be a solvable Lie algebra and M be a subalgebra of L. Using induction, we may show that $M^{(n)} \subset L^{(n)}$ which implies M is solvable. Now, for a surjective homomorphism $\phi : L \longrightarrow N$, we can show $\phi(L^{(n)}) = N^{(n)}$ by induction on n.

(b) Apply part (a) to canonical epimorphism.

(c) By second isomorphism theorem, ${(I+J)}/{J} \cong {I}/{(I \cap J)}$ where ${I}/{(I \cap J)}$ is solvable since it is a homomorphic image of I. Then, ${(I+J)}/{J}$ is solvable and hence by part (b), $I + J$ is solvable.

□

Remark Part (c) implies that the maximal solvable ideal is unique.

Theorem 2.1.2. (a) Subalgebras and homomorphic images of a nilpotent Lie algebra are nilpotent.

(b) For a Lie algebra L, if ${L}/{Z(L)}$ is nilpotent, then so is L.

(c) If a Lie algebra L is nilpotent and nonzero, then $Z(L) \neq 0$.

Proof. (a) Same argument in the previous theorem works.

(b) Consider the canonical epimorphism $\pi : L \longrightarrow {L}/{Z(L)}$. We can show that $\pi(L^i) = \pi(L)^i = \left({L}/{Z(L)}\right)^i$. Assume that $\left({L}/{Z(L)}\right)^n = 0$ for some n. Then, $\pi(L^n) = 0$ implies that $L^n \subset \text{Ker } \pi = Z(L)$. Therefore, $L^{n+1} = [L, L^n] \subset [L, Z(L)] = 0$.

(c) Let n be the smallest integer such that $L^n = 0$. Then, $L^n = [L, L^{n-1}] = 0$ implies that $L^{n-1} \subset Z(L)$ where $L^{n-1} \neq 0$. Therefore, $Z(L) \neq 0$.

□

Lemma 2.1.2. $[\mathfrak{sl}(n,\mathbb{C}), \mathfrak{sl}(n,\mathbb{C})] = \mathfrak{sl}(n,\mathbb{C})$. In particular, $\mathfrak{sl}(n,\mathbb{C})^n = \mathfrak{sl}(n,\mathbb{C})^{(n)} = \mathfrak{sl}(n,\mathbb{C})$ and so $\mathfrak{sl}(n,\mathbb{C})$ is not solvable.

Proof. Let $H_{i,j} = [E_{i,j}, E_{j,i}] = E_{i,i} - E_{j,j}$ for $i \neq j$. Then a basis for $\mathfrak{sl}(n,\mathbb{C})$ is $\{E_{i,j} \mid i \neq j\} \cup \{H_{i,i+1} = [E_{i,i+1}, E_{i+1,i}]\}$. In particular, $\{E_{i,j}, H_{i,j} \mid i \neq j\}$ spans

$\mathfrak{sl}(n, \mathbb{C})$.

The inclusion $\subset$ is obvious. Now, for $i \neq j$, $E_{i,i} - E_{j,j} = [E_{i,j}, E_{j,i}]$ and $E_{i,j} = 1/2[E_{i,i} - E_{j,j}, E_{i,j}] = 1/2[H_{i,j}, E_{i,j}]$. □

Lemma 2.1.3. $[\mathfrak{gl}(n, \mathbb{C}), \mathfrak{gl}(n, \mathbb{C})] = \mathfrak{sl}(n, \mathbb{C})$

Proof. $\text{tr}(AB - BA) = 0$ implies that $[\mathfrak{gl}(n, \mathbb{C}), \mathfrak{gl}(n, \mathbb{C})] \subset \mathfrak{sl}(n, \mathbb{C})$. Then, by the previous example $\mathfrak{sl}(n, \mathbb{C}) = [\mathfrak{sl}(n, \mathbb{C}), \mathfrak{sl}(n, \mathbb{C})] \subset [\mathfrak{gl}(n, \mathbb{C}), \mathfrak{gl}(n, \mathbb{C})]$. □

The previous two lemmas show that $\mathfrak{gl}(n, \mathbb{C})^n = \mathfrak{gl}(n, \mathbb{C})^{(n)} = \mathfrak{sl}(n, \mathbb{C})$ implying that $\mathfrak{gl}(n, \mathbb{C})$ is not solvable.

Remark If L is nilpotent, say $0 = L^n = [L[L[L...[LL]]]]$, then $[x_1[x_2[x_3....[x_n y]]]] = 0$. So, $(\text{ad}x)^n = 0$ for all $x \in L$. This means that every element of L is ad-nilpotent. Converse of this statement is also true but not that obvious. We will state without proving. It is referred as Engel's theorem:

Theorem 2.1.3 (Engel)**.** L is nilpotent if every element is ad-nilpotent.

As an application we can show that;

Example 2.1.11. (i) $\mathfrak{n}(n, \mathbb{C})$ is nilpotent: Any element is a linear combinations of $E_{i,j}$ with $i < j$. Then for $k < l$ we have $\text{ad}E_{i,j}(E_{k,l}) = [E_{i,j}, E_{k,l}] = \delta_{jk}E_{i,l} - \delta_{li}E_{k,j}$ which is equal to either $E_{i,l}$ or $E_{k,j}$, but then applying ad one more time, $[E_{i,j}, E_{i,l}] = 0$ and $[E_{i,j}, E_{k,j}] = 0$. By bilinearity of the bracket, we conclude that every element is ad-nilpotent and hence $\mathfrak{n}(n, \mathbb{C})$ is nilpotent (and solvable in particular) by Engel's theorem.

(ii) $\mathfrak{t}(n, \mathbb{C})$ is solvable: By the example 2.1.10, we know that

$$(\mathfrak{t}(n, \mathbb{C}))^{(1)} = [\mathfrak{t}(n, \mathbb{C}), \mathfrak{t}(n, \mathbb{C})] = \mathfrak{n}(n, \mathbb{C})$$

Then by induction, $(\mathfrak{t}(n, \mathbb{C}))^{(i+1)} = (\mathfrak{n}(n, \mathbb{C}))^{(i)}$ for any i. Since $\mathfrak{n}(n, \mathbb{C})$ is solvable by (i), then so is $\mathfrak{t}(n, \mathbb{C})$.

(i) and (ii) imply that all the subalgebras of $\mathfrak{gl}(n, \mathbb{C})$ contained in $\mathfrak{t}(n, \mathbb{C})$ are solvable and those contained in $\mathfrak{n}(n, \mathbb{C})$ are nilpotent.

Lemma 2.1.4. $\mathfrak{sl}(n, \mathbb{C})$ is a simple Lie algebra for $n \geq 2$.

Proof. Let k be a nonzero ideal of $\mathfrak{sl}(n, \mathbb{C})$. Then it can be shown that $k = \mathfrak{sl}(n, \mathbb{C})$ by successive matrix operations on the elements of k. The details can be found in [4] or [5].

□

Lemma 2.1.5. If $f \in \mathfrak{gl}(V)$ is nilpotent , then ad$f$ is nilpotent.

Proof. Let λ_f and ρ_f be two endomorphisms defined as $\lambda_f(g) = fg$ and $\rho_f(g) = gf$. Then, λ_f and ρ_f commute and are nilpotent endomorphisms. So, we can use binomial theorem to conclude that ad$f = \lambda_f - \rho_f$ is nilpotent.

□

2.2 Semisimple Lie Algebras

Definition 2.2.1. Let L be a Lie algebra. Then,

(i) The unique maximal solvable ideal of L is called the **radical** of L, denoted by Rad$L$.

(ii) L is called **semisimple**, if Rad$L = 0$. In other words, L is semisimple if it does not contain any nonzero solvable ideal.

(iii) An element $x \in$ End(V) is called **semisimple** if the roots of its minimal polynomial are all distinct.

For any Lie algebra L it is easy to observe the followings.

(i) L is solvable if and only if Rad$L = L$.

(ii) If $0 \neq L$ is semisimple, then L is not solvable.

(iii) $L / \text{Rad}L$ is semisimple.

To understand the semisimple Lie algebras better, we need to study the solvable Lie algebras. We will give the theorem's of Lie and Cartan without proving.

Lemma 2.2.1. Let L be a solvable Lie algebra with a representation $\phi : L \longrightarrow \mathfrak{gl}(V)$. Then there is a common eigenvector of all $\phi(x)$, $x \in L$.

As a corollary of this lemma, one can prove the Lie's theorem using the induction on $\dim V$.

Theorem 2.2.1 (Lie's Theorem)**.** Let $\mathfrak{g}$ be a solvable Lie algebra and $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a representation. Then there exists a basis $\mathcal{B}$ of V such that $[\rho(x)]_{\mathcal{B}}$ is upper triangular for any $x \in \mathfrak{g}$.

Lie's theorem is actually a generalization of the triangular forms of linear operators in basic linear algebra.

Another corollaries of the lemma are the following.

Corollary 2.2.1. L is solvable if and only if $[LL]$ is nilpotent. In particular, $\mathrm{ad}_L x$ is nilpotent for any $x \in [LL]$.

Corollary 2.2.2. If L is solvable, then any irreducible representation is 1-dimensional.

Theorem 2.2.2 (Cartan's Criterion)**.** A Lie algebra L is solvable if and only if

$\mathrm{tr}(\mathrm{ad}x \cdot \mathrm{ad}y) = 0$ for all $x \in [LL]$ and $y \in L$.

The proof of Cartan's criterion uses the Jordan-Chevalley decomposition from the basic linear algebra.

2.2.1 Jordan Decomposition

(i) If $T_1, T_2 \in \mathrm{End}(V)$ are semisimple (diagonalizable) and commute, then they are simultaneously diagonalizable. That is, there is a basis $\mathcal{B}$ of V consisting of all eigenvectors of both T_1 and T_2. (See, [6]). Thus, $T_1 + T_2$ and $T_1 - T_2$ are semisimple.

(ii) If T is semisimple such that $T(W) \subset W$ where W is a subspace of V, then $T_{|W}$ is semisimple.

Theorem 2.2.3 (See, [7])**.** Let $T : V \longrightarrow V$ be a linear map where V is a finite dimensional vector space over $\mathbb{C}$. Then, the followings hold.

(i) There are unique linear maps T_s and T_n such that $T = T_s + T_n$, where T_s is semisimple, T_n is nilpotent and $T_s \circ T_n = T_n \circ T_s$.

(ii)There are polynomials $p, q \in \mathbb{C}[X]$ such that $p(T) = T_s$ and $q(T) = T_n$. In particular, T_s and T_n commute with any linear map commuting with T.

(iii) For the subspaces U and W of V satisfying $U \subset W \subset V$, if $T(W) \subset U$, then $T_s(W) \subset U$ and $T_n(W) \subset U$.

Lemma 2.2.2. Let T be a linear map from V to V (i.e, $T \in \text{End}(V)$) with Jordan decomposition $T = T_s + T_n$. Then, ad$T$ =adT_s+adT_n is the Jordan decomposition of ad$T$.

Proof. It can be shown that adT_s is semisimple and adT_n is nilpotent [7]. It remains to show that adT_s and adT_n commute, but this is obvious since one can write:

$[\text{ad}T_s, \text{ad}T_n] = \text{ad}[T_s, T_n] = \text{ad}0 = 0.$ □

Lemma 2.2.3. Der$L$ contains semisimple and nilpotent parts of all its elements.

2.2.2 The Killing Form

In this section, using the properties of the Killing form, we will give an equivalent conditions for semisimplicity. Also, the existence of the abstract Jordan decomposition depends on the results given in this section.

Definition 2.2.2. The symmetric bilinear form $\kappa : L \times L \longrightarrow \mathbb{C}$ defined by

$$\kappa(x, y) = \text{tr}(\text{ad}x \cdot \text{ad}y)$$

is called the **Killing form**.

Killing form has the associativity property:

$$\kappa([xy], z) = \kappa(x, [yz]) \tag{2.1}$$

This can be seen by:

$\kappa([xy],z)$ =tr(ad$[xy]$·ad$z$) =tr([ad$x$,ad$y$]·ad$z$) =tr(ad$x$·[ad$y$,ad$z$]) =tr(ad$x$·ad$[yz]$) $=$ $\kappa(x,[yz])$

Lemma 2.2.4. If κ is the Killing form of L and I is an ideal of L, then the Killing form κ_I of I coincides with $\kappa|_{I\times I}$.

Proof. For $x, y \in I$, ad$x$·ad$y$ maps L into I. So, its trace is equal to the trace of ad$x$·ad$y$ $|_I$. □

We can rewrite the Cartan's criterion in the language of Killing form as:

A Lie algebra L is solvable if and only if $\kappa(x,y) = 0$ for all $x \in [LL],\ y \in L$. This can be generalized to solvability of the ideals by the previous lemma.

Corollary 2.2.3 (Cartan's Criterion)**.** An ideal I of a lie algebra L is solvable if and only if $\kappa(x,y) = 0$ for all $x \in [II],\ y \in I$.

Proof. This is an immediate result of the lemma 2.2.4. □

Definition 2.2.3. The set $\{x \in L|\ \kappa(x,y) = 0 \text{ for all } y \in L\}$ is called the **radical** of the Killing form κ.

Radical is an ideal because of the associativity property of the Killing form.

Theorem 2.2.4. Finite dimensional Lie algebra L is semisimple if and only if it has no nonzero abelian ideal. In particular, $Z(L) = 0$ whenever L is semisimple.

Proof. ($\Rightarrow$): Since L is semisimple, Rad$L = 0$ and so, L has no non zero solvable ideals. This implies that L has no non zero abelian ideals.
($\Leftarrow$): Assume that Rad$L \neq 0$. Then, since Rad$L$ is solvable, Rad$L^{(n)} = 0$ for some $n > 0$ (Here, we take n to be the smallest such number).

$$\text{Rad}L^{(n)} = [\text{Rad}L^{(n-1)}, \text{Rad}L^{(n-1)}] = 0$$

which implies that Rad$L^{(n-1)}$ is non-zero, abelian.

□

Theorem 2.2.5 (Cartan's criterion of semisimplicity)**.** L is semisimple iff its Killing form is non-degenerate.

Proof. ($\Longrightarrow$): First note that Rad$L = 0$ since L is semisimple. Let K be the radical of κ. We have, $\kappa(x, y) = 0$ for all $x \in K,\ y \in L$. In particular, $\kappa(x, y) = 0$ for all $x \in K,\ y \in [KK]$. Since K is an ideal of L, it is solvable by the Cartan's criterion. This forces $K \subset$ Rad$L = 0$.

($\Longleftarrow$): We assume that the radical K of κ is 0. We need to show that L has no nonzero abelian ideal. In other words, it is enough to show that every abelian ideal of L is a subset of the radical K. Let I be an abelian ideal of L. For $x \in I$, $y \in L$, ad$x$·ad$y$ maps L into I. So, (ad$x$·ad$y$)^2 maps L into $[II] = 0$. This means that ad$x$·ad$y$ is a nilpotent endomorphism. But, nilpotent endomorphisms have trace 0. Therefore, $\kappa(x, y) =$tr(ad$x$·ad$y$) $= 0$ which means $x \in K$ as desired.

□

Theorem 2.2.6. If L is a semisimple Lie algebra, then there are ideals $L_1, ..., L_k \subset L$ so that

$$L = L_1 \oplus .... \oplus L_k$$

and each L_i is a simple Lie algebra. In fact, L_i's are the only simple ideals of L.

Proof. For any ideal I of L, $I^\perp = \{x \in L |\ \kappa(x, y) = 0 \ \text{ for all } \ y \in I\}$ is an ideal of L by the associativity of κ. For $x, y, z \in I \cap I^\perp$, $\kappa([xy], z) = 0$ since $[xy] \in I^\perp$ and $z \in I$. Therefore, $I \cap I^\perp$ is a solvable ideal of L and hence it must be 0. Moreover, we know that $\dim I + \dim I^\perp = \dim L$ since κ is a nondegenerate biliner form on L, and so $L = I \oplus I^\perp$. Now, by induction on the dimension of L, we can show that L has the desired decomposition.

If I is a simple ideal of L, then $[IL]$ is a nonzero ideal of I since $Z(L) \neq 0$. So, $[IL] = I$. Using the decomposition of L, we can write $[IL] = [IL_1] \oplus .... \oplus [IL_k] = I$ (This is a direct sum because each summand is in different L_i). Thus, only one of the summands is nonzero because each summand is an ideal of I and I is simple. If

$[IL_i] \neq 0$, then $[IL_i] = I$ and $[IL_i] = L_i$ since both I and L_i are simple. This concludes that $I = L_i$. □

Corollary 2.2.4. If L is a semisimple Lie algebra with the decomposition

$$L = I_1 \oplus .... \oplus I_k,$$

where I_i's are simple, then

1. $[LL] = L$, all ideals and homomorphic images of L are semisimple. By the theorem above, this shows that any ideal is $I = \sum_{i \in J} I_i$ for some $J \subset \{1, 2, ..., k\}$.
2. $L \cong \mathrm{ad}L$.
3. L is isomorphic to a subalgebra of $\mathfrak{gl}(L)$

Proof. (1) We observe that $[I_i, I_j] \subset I_i \cap I_j = 0$ if $i \neq j$ and $[I_i, I_i] = I_i$ since I_i is simple. Hence, $[LL] = [I_1, I_1] + [I_2, I_2] + \cdots + [I_k, I_k] = I_1 + I_2 + ... + I_k = L$.

Any ideal I is semisimple because solvable ideal of I is a solvable ideal of L. Now let ϕ be an epimorphism on L. Since Kerϕ is an ideal, (Ker$\phi)^\perp$ is also an ideal and we can write $L =$ Ker$\phi \bigoplus$(Ker$\phi)^\perp$ and get $\phi(L) \cong L/_{\text{Ker}\phi} \cong$ (Ker$\phi)^\perp$ which is semisimple because it is an ideal of L.

(2) The map ad: $L \longrightarrow$ ad(L) has kernel Ker ad$= Z(L) = 0$ and the image is ad$L$.

(3) This is obvious by part (2) since ad$L$ is a subalgebra of $\mathfrak{gl}(L)$.

□

Subalgebra of a semisimple Lie algebra need not be semisimple. Consider $\mathfrak{sl}(n, \mathbb{C})$ with the subalgebra $\mathfrak{n}(n, \mathbb{C})$ of strictly upper triangular matrices. $\mathfrak{n}(n, \mathbb{C})$ is a nilpotent and therefore a solvable Lie algebra. Hence Rad$(\mathfrak{n}(n, \mathbb{C})) = \mathfrak{n}(n, \mathbb{C}) \neq 0$.

Theorem 2.2.7. For a semisimple Lie algebra L, ad$L =$ Der$L$.

Theorem 2.2.8 (Abstract Jordan Decomposition)**.** Let L be a semisimple Lie algebra and $x \in L$. Then there are unique elements x_s and x_n in L with $x = x_s + x_n$, $[x_s x_n] = 0$ and ad$x =$adx_s+adx_n is the Jordan decomposition of ad$x$ in End(L).

Proof. Since L is semisimple, ad$L =$ Der$L$ and the map ad:$L \longrightarrow$ ad$L$ is an isomorphism of Lie algebras. Let ad$x$ has the Jordan decomposition ad$x = (\text{ad}x)_s + (\text{ad}x)_n$. Since ad$L$ contains semisimple and nilpotent parts of all its elements, $(\text{ad}x)_s$ and $(\text{ad}x)_n$ are in ad$L$. Then, we have $(\text{ad}x)_s =$adx_s and $(\text{ad}x)_n =$adx_n for some unique x_s and x_n in L since ad is one to one. So, ad$(x_s+x_n) =$ad$x$ which implies $x = x_s+x_n$ since ad is one to one. Also, $0 = [\text{ad}x_s$,ad$x_n] =$ad$[x_s x_n]$ implies that $[x_s x_n] = 0$.

□

Here, note that x_s is ad-semisimple (i.e, adx_s is semisimple in End(L)) and x_n is ad-nilpotent in End(L).

In the above theorem, x_s and x_n are called the semisimple and nilpotent parts of x respectively.

CHAPTER 3

CLASSIFICATION OF SEMISIMPLE LIE ALGEBRAS

The structures of semisimple Lie algebras are not so different than the structures of $\mathfrak{sl}(n,\mathbb{C})$. In fact, they have a similar module decompositions. Among all $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{sl}(2,\mathbb{C})$ has a special importance since we will actually see that any semisimple Lie algebra contains a copy of $\mathfrak{sl}(2,\mathbb{C})$. Therefore, any semisimple Lie algebra is a representation of $\mathfrak{sl}(2,\mathbb{C})$ and we begin with it.

3.1 Representations of $\mathfrak{sl}(2,\mathbb{C})$

Lemma 3.1.1. If L is semisimple and $\phi : L \longrightarrow \mathfrak{gl}(V)$ is a representation, then $\phi(L) \subset \mathfrak{sl}(V)$.

Proof. Since L is semisimple, then $[LL] = L$. So, $\phi(L) = \phi([LL]) \subset [\phi(L), \phi(L)] \subset [\mathfrak{gl}(V), \mathfrak{gl}(V)] = \mathfrak{sl}(V)$. □

Theorem 3.1.1 (Weyl's theorem)**.** If L is a semisimple Lie algebra , then any finite dimensional L-module is completely reducible. In other words, any finite dimensional L-module is a direct sum of irreducible L-modules.

Corollary 3.1.1. Let L be a subalgebra of $\mathfrak{gl}(V)$ (that is linear Lie algebra), then the abstract Jordan decomposition in $\mathfrak{gl}(L)$ coincides with the usual Jordan decomposition in $\mathfrak{gl}(V)$.

Corollary 3.1.2. Any representation ϕ of a semisimple Lie algebra preserves the Jordan decomposition. That is, if $x = x_s + x_n$ is the abstract Jordan decomposition of x, then $\phi(x) = \phi(x_s) + \phi(x_n)$ is the usual Jordan decomposition of $\phi(x)$.

L denotes $\mathfrak{sl}(2,\mathbb{C})$ throughout this chapter. It has a basis :

$$\left\{ x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

h is a semisimple element of L. Then $\phi(h)$ is diagonalizable in $\mathfrak{gl}(V)$. Then there is an ordered basis of V consisting of eigenvectors of $\phi(h)$, say $\mathcal{B} = \{e_1, e_2, ..., e_n\}$. We can write $[h.v]_{\mathcal{B}} = [\phi(h)(v)]_{\mathcal{B}} = diag[\lambda_1, \lambda_2, .., \lambda_n][v]_{\mathcal{B}}$ where λ_i' s are eigenvalues of $\phi(h)$. In other words, h acts diagonally on V. Now, let's consider the subspace $V_\lambda = \{v \in V : h.v = \phi(h)(v) = \lambda v\}$. $e_i \in V_{\lambda_i}$ and $V_\lambda = 0$ if λ is not an eigenvalue of $\phi(h)$. Then we can write $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$. Here, λ is called the **weight of h in** V if $V_\lambda \neq 0$ and V_λ is called the **weight space**.

Lemma 3.1.2. If $v \in V_\lambda$, then $x.v \in V_{\lambda+2}$ and $y.v \in V_{\lambda-2}$

Proof. $h.(x.v) = [hx].v + x.(h.v) = 2(x.v) + x.(\lambda v) = (\lambda + 2)(x.v)$.

$h.(y.v) = [hy].v + y.(h.v) = -2(y.v) + y.(\lambda v) = (\lambda - 2)(y.v)$

□

Since $\dim V < \infty$, there exists $\lambda \in \mathbb{C}$ such that $V_\lambda \neq 0$ and $V_{\lambda+2} = 0$. (Since there are finitely many nonzero V_λ, we can take the maximal eigenvalue). By the lemma, $x.v = 0$ for any $v \in V_\lambda$.

Definition 3.1.1. Any nonzero vector $v \in V_\lambda$ such that $x.v = 0$ is called a **maximal vector** of weight λ.

Lemma 3.1.3. Let V be an irreducible L- module and $v_0 \in V_\lambda$ is a maximal vector. Define $v_{-1} = 0$ and $v_i = \frac{1}{i!} y^i.v_0$. Then the actions of x, y, h are given by:

1. $h.v_i = (\lambda - 2i)v_i$
2. $y.v_i = (i + 1)v_{i+1}$
3. $x.v_i = (\lambda - i + 1)v_{i-1}$, $(i \geq 0)$

Proof. See, [7]. □

Conclusions: Different v_i's are linearly independent because (1) implies that different v_i's are eigenvectors corresponding to different eigenvalues $\lambda - 2i$. Therefore, there is an integer N such that $v_k = 0$ for all $k > N$. So we can define k to be the smallest integer such that $v_k \neq 0$ and $v_{k+1} = 0$. Now, the subspace of V with basis $(v_0, v_1, ..., v_k)$ is an L-submodule, but V is irreducible. Hence $\dim V = k + 1$ with basis $(v_0, v_1, .., v_k)$.

If we put $i = k + 1$ in the last lemma, we get $x.v_{k+1} = x.0 = 0 = (\lambda - k)v_k$ which implies that $\lambda = k = \dim V - 1$. Then the weight of a maximal vector is a nonnegative integer called the **highest weight** of V. $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda = V_\lambda \oplus V_{\lambda-2} \oplus .... \oplus V_{-\lambda}$. There are $\lambda + 1 = \dim V$ many nonzero weight space in this summand. This means that each weight space has dimension 1. So, we proved the theorem:

Theorem 3.1.2. If V is an irreducible L-module, then $V = \bigoplus_{\alpha} V_\alpha$, where α ranges over $\{k, k-2, ..., -(k-2), -k\}$, $k = \dim V - 1$ and $\dim V_\alpha = 1$.

Moreover, k is the weight of a maximal vector which is unique up to a scalar.

Up to isomorphism there is only one irreducible L-module of each dimension.

Corollary 3.1.3. Let V be any (finite dimensional) L- module, $(L = \mathfrak{sl}(2, \mathbb{C}))$. Then the eigenvalues of h on V are all integers, and each occurs along with its negative. Moreover, in any decomposition of V into direct sum of irreducible submodules, the number of summands is precisely $\dim V_0 + \dim V_1$.

Proof. By Weyl's theorem, any L-module is completely reducible. Write V as $V = V_1 \oplus V_2 \oplus .... \oplus V_n$ where each V_i is irreducible. Consider the weight λ of V. Observe that $V_\lambda = \bigoplus_{i=1}^{n} (V_i)_\lambda$. Then λ is a weight of h in some V_i. So, any eigenvalue is integer and it occurs with its negative. The number of summand is $\dim V_0 + \dim V_1$ because 0 or 1 is a weight of each V_i but not both of them.

$\square$

3.2 Root Space Decomposition

Recall the example 2.1.8. In general, we have the decomposition $\mathfrak{sl}(n,\mathbb{C}) = \mathfrak{h} \bigoplus_{i\neq j} \mathbb{C}E_{i,j}$, where $\mathfrak{h}$ is the subalgebra of trace 0 diagonal matrices and $\mathbb{C}E_{i,j}$'s are 1-dimensional $\mathfrak{h}$-submodules. This can be seen if we let

$h = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ with $\lambda_1 + \ldots + \lambda_n = 0$, then $hE_{i,j} = (\lambda_i - \lambda_j)E_{i,j}$.

Definition 3.2.1 (Cartan subalgebra)**.** A subalgebra H of a Lie algebra L is called a **Cartan subalgebra** if H is nilpotent and $H = N_L(H)$ where $N_L(H) = \{x \in L \mid [hx] \in H \ \ \forall h \in H\}$.

In the case of $\mathfrak{sl}(n,\mathbb{C})$, the subalgebra $\mathfrak{h}$ of diagonal matrices is clearly a Cartan subalgebra.

Remark $N_L(H)$ is the **normalizer** of H in L and it is the largest subalgebra of L in which H is an ideal.

Theorem 3.2.1. Every finite dimensional Lie algebra L has a Cartan subalgebra. Moreover, given any two Cartan subalgebras H_1 H_2 there exists $\theta \in \mathrm{Aut}(L)$ such that $\theta(H_1) = H_2$.

We will not prove this theorem for arbitrary Lie algebras. Instead, we will construct a Cartan subalgebra of a semisimple Lie algebra which will turn out to be a maximal toral subalgebra.

Definition 3.2.2. Let T be a subalgebra of a semisimple Lie algebra L. T is called **toral** if it contains only ad-semisimple elements.

Theorem 3.2.2. Every semisimple Lie algebra has a toral subalgebra.

Proof. For a semisimple Lie algebra L, Rad$L = 0$ which implies that L is not solvable and hence L is not nilpotent. Then by Engel's theorem, L contains elements which are not ad-nilpotent. Therefore, there exists elements $x \in L$ such that $x = s+n$

where ad$s$ is semisimple, ad$n$ is nilpotent and $s \neq 0$. Then there are nonzero subalgebras of L consisting of semisimple elements.

□

Theorem 3.2.3. Toral subalgebra of a semisimple Lie algebra is abelian.

Proof. Let T be a toral subalgebra of L, that is, all elements of T are ad-semisimple. We will show that ad$_T x = 0$ for any $x \in T$. Firstly, ad$x$ is diagonalizable for any $x \in T$. Then ad$_T x$ is also diagonalizable since it is just the restriction of ad$x$ to T. So, we need to show that ad$_T x$ has only eigenvalue 0. Assume that ad$_T x$ has eigenvalue $c \neq 0$, that is, $[xy] = cy$ for some $y \neq 0$. Then, ad$_T y(x) = -cy$ and $-cy$ is an eigenvector of ad$_T y$ belonging to 0. That is, ad$_T y$(ad$_T y(x)$)= 0. But since ad$_T y$ is diagonalizable, there is a basis of T consisting of eigenvectors of ad$_T y$, say $\{z_i : i = 1, 2, .., n\}$ and let t_i is an eigenvalue belonging to z_i. So, we can write $x \in T$ as a linear combination of these eigenvectors as $x = d_1 z_1 + d_2 z_2 + \cdots + d_k z_k$ where we assume $d_i \neq 0$ for $i = 1, 2, ..., k$. Then, we have ad$_T y(x) = d_1[yz_1] + \cdots + d_k[yz_k] = d_1 t_1 z_1 + \cdots + d_k t_k z_k$ and ad$_T y$(ad$_T y(x)$)= $d_1(t_1)^2 z_1 + \cdots + d_k(t_k)^2 z_k = 0$ which implies that $t_i = 0$ for $(i = 1, 2, .., k)$. This means that ad$_T y(x) = 0$ which is a contradiction because $-cy \neq 0$.

□

Theorem 3.2.4 (Cartan Decomposition)**.** Let H be a Cartan subalgebra of a semisimple Lie algebra L. Then L may be written as the direct sum of H with a number of 1-dimensional H-submodules. Moreover, such a decomposition is uniquely determined by H.

Definition 3.2.3. Let H be a Cartan subalgebra of a semisimple Lie algebra L. A nonzero functional
$\alpha : H \longrightarrow \mathbb{C}$ is called a **root of** L **with respect to** H if
$L_\alpha = \{x \in L : [hx] = \alpha(h)x \ , \ \forall h \in H\} \neq 0.$

The set of roots will be denoted by Φ.

Theorem 3.2.5. Let H be a maximal toral subalgebra of a semisimple Lie algebra L. Then L has a direct sum decomposition

$$L = L_0 \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

Proof. Since H is abelian, $[\text{ad}h_1\text{ad}h_2]$ =ad$[h_1h_2]$ =ad$0 = 0$. This means that the elements of $\text{ad}_L H$ commutes. Therefore, the elements of $\text{ad}_L H$ are simultaneously diagonalizable. That is, there is a basis $\mathcal{B} = \{e_i\}$ of L consisting of eigenvectors of ad$h$ for any $h \in H$. This is equivalent to saying that L is a direct sum of subspaces L_α defined above.

□

$L_0 = C_L(H)$ and $H \subset C_L(H)$ obviously. Later, we will prove that $H = C_L(H)$.

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

This decomposition is called the **root space decomposition**.

Theorem 3.2.6. For any $\alpha, \beta \in H^*$, $[L_\alpha L_\beta] \subset L_{\alpha+\beta}$. If $\alpha \neq 0$, then every element of L_α is ad-nilpotent. If $\alpha + \beta \neq 0$, then $\kappa(L_\alpha, L_\beta) = 0$

Proof. Let $x \in L_\alpha$ and $y \in L_\beta$. Then, $[h[xy]] = [[hx]y] + [x[hy]] = [\alpha(h)x, y] + [x, \beta(h)y] = \alpha(h)[x, y] + \beta(h)[x, y] = (\alpha + \beta)(h)[x, y]$. Therefore, $[xy] \in L_{\alpha+\beta}$.

If $\alpha \neq 0$ and $x \in L_\alpha$, then for $y \in L_\beta$ we have $(\text{ad}x)^n(y) \in L_{n\alpha+\beta}$. There is a number n_β such that $n_\beta\alpha + \beta$ is not a root. In other words, for every $\beta \in H^*$ and $y \in L_\beta$, there is n_β such that $(\text{ad}x)^{n_\beta}(y) = 0$. Let n =max$\{n_\beta \ : \ \beta \in H^*\}$, then $(\text{ad}x)^n(y) = 0$ for any $y \in L$.

Now, assume that $\alpha+\beta \neq 0$. Then there exists, $h \in H$ such that $(\alpha+\beta)(h) \neq 0$. Using the associativity of κ we have, $(\alpha+\beta)(h)\kappa(x, y) = \alpha(h)\kappa(x, y)+\beta(h)\kappa(x, y) = \kappa([hx], y) + \kappa(x, [hy]) = -\kappa([xh], y) + \kappa(x, [hy]) = -\kappa(x, [hy]) + \kappa(x, [hy]) = 0$. So, $\kappa(x, y) = 0$.

□

Corollary 3.2.1. Let L be a semisimple Lie algebra. Then, the restriction of κ to $L_0 = C_L(H)$ is nondegenerate.

Proof. Let $\kappa(y, L_0) = 0$, where $y \in L_0$. If $\alpha \in \Phi$, then $\alpha + 0 \neq 0$ implies that $\kappa(y, x) = 0$ for any $x \in L_\alpha$. Thus, $\kappa(y, x) = 0$ for any $x \in L$. But κ is nondegenerate on L. Hence, $y = 0$. □

Theorem 3.2.7. Maximal toral subalgebra H is self-centralizing.

Proof. We will proceed in steps given in the book [7].

Let H be a toral subalgebra of L and let $C = C_L(H)$. Obviously, $H \subset C$.

If $x \in C$ is semisimple, then $H + \mathbb{C}x$ is toral because ad$(h + cx)$ =ad$h$ + c.ad$x$ which is a sum of commuting semisimple elements and hence semisimple. Then by maximality of H, $H = H + \mathbb{C}x$. So, $x \in H$. Now, for $x \in C$ with the abstract Jordan decomposition $x = x_s + x_n$, since ad$x$ maps H into 0, then so do adx_s and adx_n. Thus, x_s and x_n are in C.

Now we'll show that $\kappa_{|_H}$ is nondegenerate. Assume that $h \in H$ with $\kappa(h, H) = 0$. For $x \in C$, $\kappa(h, x_s) = 0$ since $x_s \in H$ and $\kappa(h, x_n)$ =tr(ad$h$·adx_n)= 0 because ad$h$ and adx_n commute and adx_n is nilpotent. Therefore, $\kappa(h, C) = 0$ which implies $h = 0$ by the corollary.

If $x \in C$ is semisimple, then $x \in H$ and so ad$_C x = 0$. If x is a nilpotent element of C, then ad$x$ is nilpotent and so ad$_C x$ is nilpotent. For any $x \in C$ with $x = x_s + x_n$, ad$_C x$ =ad$_C x_s$+ad$_C x_n$ =ad$_C x_n$ is nilpotent. By Engel's theorem, C is nilpotent.

Next, C can be shown to be abelian. Last step is to show that $C = H$. If every element of C is semisimple then $C = H$. If not, say $x \in C$ is not semisimple. Then, $x_n \neq 0$ and $x_n \in C$. Since C is abelian, for any $y \in C$, adx_n and ad$y$ commute and $\kappa(x_n, y)$ =tr(adx_n.ad$y$) $= 0$. This is a contradiction because $\kappa_{|_C}$ is non degenerate by the corollary. □

Corollary 3.2.2. Maximal toral subalgebra of a semisimple Lie algebra is a Cartan subalgebra.

Proof. Let H be a maximal toral subalgebra. We know that H is abelian. Let $0 \neq x \in N_L(H) - H$, then $x \in L_\alpha$ for some $\alpha \in \Phi$ and $[hx] = \alpha(h)x \in H$ for any $h \in H$

since $x \in N_L(H)$. Now, as H is abelian, $0 = [h[hx]] = (\alpha(h))^2 x$ for any $h \in H$. This implies that $\alpha = 0$ which is a contradiction since $0 \notin \Phi$.

□

Corollary 3.2.3. Let H be a Cartan subalgebra of a semisimple Lie algebra L, then for any $\alpha \in H^*$ there is a unique element $t_\alpha \in H$ such that $\kappa(t_\alpha, h) = \alpha(h)$ for all $h \in H$.

Proof. Define $\phi : H \longrightarrow H^*$ by $\phi(x) = f_x$, where $f_x(y) = \kappa(x, y)$. ϕ is linear, because κ is bilinear. Moreover, ϕ is 1-1, since κ is nondegenerate on H. Therefore, ϕ is also onto, since H and H^* are finite dimensional. Hence for every $\alpha \in H^*$ there exists unique $t_\alpha \in H$ such that $\phi(t_\alpha) = \alpha$ and $\alpha(h) = \phi(t_\alpha)(h) = \kappa(t_\alpha, h)$ for every $h \in H$. □

Theorem 3.2.8.

1. Φ spans H^*
2. If α is a root, then so is $-\alpha$.
3. Let $\alpha \in \Phi$, $x \in L_\alpha$, $y \in L_{-\alpha}$. Then $[xy] = \kappa(x, y)t_\alpha$.
4. If $\alpha \in \Phi$, then $[L_\alpha, L_{-\alpha}]$ is one dimensional, with basis t_α.
5. $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$, for $\alpha \in \Phi$.
6. If $\alpha \in \Phi$ and $0 \neq x_\alpha \in L_\alpha$, then $\exists y_\alpha \in L_{-\alpha}$ such that $x_\alpha, y_\alpha, h_\alpha = [x_\alpha, y_\alpha]$ span a subalgebra $\mathfrak{s}_\alpha$ of L isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.
7. $h_\alpha = \dfrac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$; $\;h_\alpha = -h_{-\alpha}$

Proof. (1) Let $< \Phi >= W$. Consider $W^\perp = \{h \in H : \alpha(h) = 0 \ \ \forall \alpha \in W\}$. By using duality and nondegeneracy of κ on H, one can prove that dim$H^* = \dim W + \dim(W^\perp)$. Details can be found in [6]. Assume that, $W \neq H^*$. Then dim$W^\perp \neq 0$. So, there exists nonzero $h \in H$ such that $\alpha(h) = 0$ for all $\alpha \in \Phi$. This implies that $[h, L_\alpha] = 0$. Then $[h, L] = 0$. So, $h \in Z(L)$, but $Z(L) = 0$ since L has no nonzero abelian ideal, being semisimple. But h is assumed to be nonzero. Therefore, we must have $W =< \Phi >= H^*$.

(2) Suppose that $-\alpha$ is not a root, that is, $L_{-\alpha} = 0$. So, $\kappa(L_\alpha, L_{-\alpha}) = 0$. But we also have that $\kappa(L_\alpha, L_\beta) = 0$ for $\beta \neq -\alpha$ by the theorem 3.2.6. Therefore, $\kappa(L_\alpha, L) = 0$ contradicts the nondegeneracy of the killing form.

(3) For any $h \in H$, we have $\kappa(h, [xy]) = \kappa([hx], y) = \kappa(\alpha(h)x, y) = \alpha(h)\kappa(x, y) = \kappa(t_\alpha, h)\kappa(x, y) = \kappa(\kappa(x, y)t_\alpha, h) = \kappa(h, \kappa(x, y)t_\alpha)$ so that we have :

$\kappa(h, \kappa(x, y)t_\alpha) = \kappa(h, [xy])$. Then by nondegeneracy of κ, $[xy] = \kappa(x, y)t_\alpha$.

(4) For $0 \neq x \in L_\alpha$, $\kappa(x, L_{-\alpha}) \neq 0$, since κ is nondegenerate. Then there is nonzero $y \in L_{-\alpha}$ with $\kappa(x, y) \neq 0$. Then by (3), $[xy] \neq 0$ and hence $[L_\alpha, L_{-\alpha}] \neq 0$. Then by (3), again it must have a basis t_α.

(5) Suuppose $\alpha(t_\alpha) = 0$. So, we have $[t_\alpha x] = 0 = [t_\alpha y]$ for all $x \in L_\alpha$, $y \in L_{-\alpha}$. As in (4), we can find $x \in L_\alpha, y \in L_{-\alpha}$ such that $\kappa(x, y) \neq 0$. We may assume $\kappa(x, y) = 1$. Then by (3), $[xy] = t_\alpha$. The elements x, y, t_α generates a 3-dimensional subalgebra S which is solvable since $S^{(1)} =< t_\alpha >$ and $S^{(2)} = 0$. Then, $\text{ad}_L t_\alpha$ is nilpotent by 2.2.1. It is also semisimple since $t_\alpha \in H$. Then $t_\alpha = 0$ which is a contradiction.

(6) We are given $0 \neq x_\alpha \in L_\alpha$. As in the proof (4), we can find nonzero $y_\alpha \in L_{-\alpha}$ such that $\kappa(x_\alpha, y_\alpha) \neq 0$. By multiplying y_α with a suitable scalar, we can assume that $\kappa(x_\alpha, y_\alpha) = \dfrac{2}{\kappa(t_\alpha, t_\alpha)}$. Now define $h_\alpha = \dfrac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$. Then we have the multiplications

$[x_\alpha, y_\alpha] = h_\alpha$ by (3) and $[h_\alpha, x_\alpha] = \dfrac{2}{\alpha(t_\alpha)}[t_\alpha, x_\alpha] = \dfrac{2}{\alpha(t_\alpha)}\alpha(t_\alpha)x_\alpha = 2x_\alpha$.

$[h_\alpha, y_\alpha] = \dfrac{2}{\alpha(t_\alpha)}[t_\alpha, y_\alpha] = \dfrac{2}{\alpha(t_\alpha)}(-\alpha)(t_\alpha)y_\alpha = -2y_\alpha$.

(7) $-\alpha(h) = -\kappa(t_\alpha, h) = \kappa(-t_\alpha, h)$ and also, $-\alpha(h) = (-\alpha)(h) = \kappa(t_{-\alpha}, h)$. So, $\kappa(-t_\alpha, h) = \kappa(t_{-\alpha}, h)$ for all $h \in H$. By nondegeneracy of κ on H, we see that $-t_\alpha = t_{-\alpha}$. Then,

$$h_{-\alpha} = \frac{2t_{-\alpha}}{-\alpha(t_{-\alpha})} = \frac{-2t_\alpha}{-\alpha(-t_\alpha)} = \frac{-2t_\alpha}{\alpha(t_\alpha)} = -h_\alpha$$

□

Remark 1 t_α is defined uniquely by the rule $\alpha(H) = \kappa(t_\alpha, H)$ by the corollary 3.2.3.

From now on, H_α is defined as $H_\alpha = \dfrac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$ so that H_α is the unique element in $[L_\alpha, L_{-\alpha}]$ with $\alpha(H_\alpha) = 2$.

Remark 2 Notice also that there may be more than one $y_\alpha \in L_{-\alpha}$ for each $x_\alpha \in L_\alpha$. But the next lemma shows that this is not the case because $\dim L_{-\alpha} = 1$.

Remark 3 3-dimensional semisimple Lie algebra over $\mathbb{C}$ has a 3-dimensional subalgebra which is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. Therefore, any 3-dimensional semisimple Lie algebra is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

Now using the results of section 3.1 and the theorem 3.2.8, we can prove the following theorem which characterizes the root spaces.

Lemma 3.2.1. (i) dim$L_\alpha = 1$ for any root α. So, $\mathfrak{s}_\alpha = L_\alpha + L_{-\alpha} + [L_\alpha, L_{-\alpha}]$.

(ii) If α is a root then $c\alpha$ is also a root if and only if $c = \pm 1$.

Theorem 3.2.9. (i) If α and β are roots, then $\beta(h_\alpha)$ is an integer called the **Cartan integer**, and $\beta - \beta(h_\alpha)\alpha$ is also a root.

(ii) For the roots α, β with $\beta \neq \pm\alpha$ if r, q are the largest integers such that $\beta - r\alpha$ and $\beta + q\alpha$ are roots, then $\beta + k\alpha$ is a root for any k with $-r < k < q$ and $\beta(h_\alpha) = r - q$. The roots $\beta + k\alpha$ form a string called the **α-string through** β.

(iii) If α, β and $\alpha + \beta$ are roots, then $[L_\alpha L_\beta] = L_{\alpha+\beta}$.

***Proof*.** If $\alpha = \pm\beta$ then $\beta(h_\alpha) = \pm 2$ and $\beta - \beta(h_\alpha)\alpha = \pm\alpha$ which is a root. Let $\beta \neq \pm\alpha$ be roots. Define $K = \sum_{i \in \mathbb{Z}} L_{\beta + i\alpha}$. K is an $\mathfrak{s}_\alpha$-submodule of L. Weight spaces are one dimensional by part a, and the weights are $(\beta + i\alpha)(h_\alpha) = \beta(h_\alpha) + 2i$ which is always even or always odd depending on $\beta(h_\alpha)$. So, by the analysis of the representations of $\mathfrak{sl}(2, \mathbb{C})$, K is irreducible. The highest and lowest weights are $\beta(h_\alpha) + 2q$ and $\beta(h_\alpha) - 2r$ respectively where q and r are the greatest integers such that $\beta + q\alpha$ and $\beta - r\alpha$ are roots. Since the weights of K form an arithmetic progression with difference 2, roots $\beta + i\alpha$ form an unbroken string of roots.

If $[x_\alpha, x_\beta] = 0$, then x_β is a highest weight vector of K with weight $\beta(h_\alpha)$. But, $\alpha + \beta$ is a root, so that h_α acts on $L_{\alpha+\beta}$ by the eigenvalue $\beta(h_\alpha) + 2$. Therefore, $\beta(h_\alpha)$ is not the highest weight which leads to contradiction. Thus, $0 \neq [L_\alpha, L_\beta] \subset L_{\alpha+\beta}$ implies $[L_\alpha, L_\beta] = L_{\alpha+\beta}$ because each root space is one dimensional. $\square$

Example 3.2.1. We will find the root space decomposition of $L = \mathfrak{sl}(3, \mathbb{C})$. Let $E_{i,j}$ denote the 3×3 matrix with i, j entry is 1 and all other entries are 0.

Consider the subalgebra $\mathfrak{h}$ of diagonal elements of L which are $H = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$ where $a_1 + a_2 + a_3 = 0$. A basis for $\mathfrak{h}$ is

$$\left\{ H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}$$

Observe that $\mathfrak{h}$ is a toral subalgebra. Actually, it is a maximal toral subalgebra. This can be seen by letting $\mathfrak{h}'$ to be any toral subalgebra containin $\mathfrak{h}$. Then for any $H \in \mathfrak{h}$ $[H, E_{i,j}]$ is nonzero if $i \neq j$ and hence $\mathfrak{h}'$ can not be self-centralizing.

To find the root space decomposition of $\mathfrak{sl}(3, \mathbb{C})$, we introduce the functionals α_i on $\mathfrak{h}$.

$$\alpha_i \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} = a_i \text{ where } a_1 + a_2 + a_3 = 0.$$

The element $E_{i,j}$ is in the root space $L_{\alpha_i - \alpha_j}$. To show this, let

$$H = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$$

We can write $H = a_1 E_{1,1} + a_2 E_{2,2} + a_3 E_{3,3}$ and so

$[H, E_{1,2}] = a_1[E_{1,1}, E_{1,2}] + a_2[E_{2,2}, E_{1,2}] + a_3[E_{3,3}, E_{1,2}] = a_1 E_{1,2} - a_2 E_{1,2} = (a_1 - a_2)E_{1,2} = (\alpha_1 - \alpha_2)(H)E_{1,2}$ so that $E_{1,2} \in L_{\alpha_1 - \alpha_2}$. Similarly, each $E_{i,j} \in L_{\alpha_i - \alpha_j}$ if $i \neq j$. Therefore, L has a root space decomposition

$$L = \mathfrak{h} \oplus \bigoplus_{i \neq j} L_{\alpha_i - \alpha_j}$$

3.3 Root Systems

Semisimple Lie algebras can be classified by their root systems. In fact, it is enough to classify the irreducible root systems which correspond to simple Lie algebras. This is because, decomposition into simple ideals and the decomposition of the corresponding root system into irreducible root systems are compatible. To each irreducible root system there is a unique connected Dynkin diagram. We conclude the classification by the classification of connected Dynkin diagrams.

Throughout this section Φ denotes the set of roots of a semisimple Lie algebra L corresponding to a Cartan subalgebra H.

Definition 3.3.1. $(\alpha, \beta) := \kappa(t_\alpha, t_\beta)$ defines a bilinear form on H^*.

We will restrict this bilinear form to some euclidean space and get a real valued inner product space.

Lemma 3.3.1. $\kappa(h_\alpha, h_\alpha)\kappa(t_\alpha, t_\alpha) = 4$ for any root α.

Proof. By theorem 3.2.8, we have $h_\alpha = [x_\alpha y_\alpha] = \kappa(x_\alpha, y_\alpha)t_\alpha$ and then $2 = \alpha(h_\alpha) = \kappa(x_\alpha, y_\alpha)\alpha(t_\alpha) = \kappa(x_\alpha, y_\alpha)\kappa(t_\alpha, t_\alpha)$. On the other hand,

$$\kappa(h_\alpha, h_\alpha) = [\kappa(x_\alpha, y_\alpha)]^2.\kappa(t_\alpha, t_\alpha).$$

Combining these, we get $\kappa(h_\alpha, h_\alpha)\kappa(t_\alpha, t_\alpha) = [\kappa(x_\alpha, y_\alpha).\kappa(t_\alpha, t_\alpha)]^2 = 4.$ □

Lemma 3.3.2. For any roots α and β, $\kappa(t_\alpha, t_\beta)$ is a rational number.

Proof. First observe that all the eigenvalues $\lambda(h_\alpha)$ and $\lambda(h_\beta)$of adh_α and adh_β are integer. Then the trace of adh_α·adh_β which is $\kappa(h_\alpha, h_\beta)$ is an integer. Now, using the theorem 3.2.8 again, we write $\kappa(t_\alpha, t_\beta) = \dfrac{\kappa(t_\alpha, t_\alpha)\kappa(t_\beta, t_\beta)}{4}\kappa(h_\alpha, h_\beta)$. But $\kappa(t_\alpha, t_\alpha)$ is rational because of the previous lemma. Therefore, $\kappa(t_\alpha, t_\beta)$ is a rational number. □

Theorem 3.3.1. Suppose that the roots $\alpha_1, ..., \alpha_k \in \Phi$ form a basis for H^* and let ${H_\mathbb{R}}^*$ be the real vector space spanned by $\{\alpha_1, ..., \alpha_k\}$. Then, ${H_\mathbb{R}}^*$ does not depend on the choice of the basis $\{\alpha_1, ..., \alpha_k\}$.

Proof. By the lemma above, it can be shown that any root in Φ is a $\mathbb{Q}$-linear combination of the roots α_i.

□

Theorem 3.3.2. The bilinear form (,) is a real valued inner product on ${H_\mathbb{R}}^*$. In other words, ${H_\mathbb{R}}^*$ is a Euclidean space.

Proof. Restricting the bilinear form on H^* to ${H_\mathbb{R}}^*$, we get a symmetric bilinear form ${H_\mathbb{R}}^* \times {H_\mathbb{R}}^* \longrightarrow \mathbb{R}$. We need to show that it is positive definite. Let $\alpha \in {H_\mathbb{R}}^*$. $(\alpha, \alpha) = \kappa(t_\alpha, t_\alpha)$ =tr(adt_α·adt_α) which is equal to the sum of the squares of the eigenvalues of adt_α. The eigenvalues are $\beta(t_\alpha) = \kappa(t_\alpha, t_\beta) \in \mathbb{Q}$ where $\beta \in \Phi$ so that (α, α) is a non-negative real number. Moreover, $(\alpha, \alpha) = 0$ implies that $\beta(t_\alpha) = 0$ for any $\beta \in \Phi$. Then, $[t_\alpha x] = \beta(t_\alpha)x = 0$ for $x \in L_\beta$. Thus, $[t_\alpha x] = 0$ for any $x \in L$. So, $t_\alpha \in Z(L) = 0$ since L is semisimple. Therefore, $\alpha = 0$.

□

$\dim H = \dim H^* = \dim {H_\mathbb{R}}^* = l$ is called the **rank** of L and $\dim L = l + |\Phi|$

We proved that Φ has the following properties:

1. Φ is finite, spans ${H_\mathbb{R}}^*$ and does not contain 0.
2. If $\alpha \in \Phi$, then the only multiples of α in Φ are $\pm\alpha$.
3. If $\alpha \in \Phi$, $\sigma_\alpha(\beta) = \beta - \dfrac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha$ is in Φ for any $\beta \in \Phi$.
4. If $\alpha, \beta \in \Phi$, then $\dfrac{2(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$.

We denote $\dfrac{2(\beta, \alpha)}{(\alpha, \alpha)}$ by $\langle \beta, \alpha \rangle$. Then, $\langle \beta, \alpha \rangle = \beta(H_\alpha)$ by definition.

Definition 3.3.2. Any subset of a euclidean space E satisfying these is called a **root system**.

Theorem 3.3.3 ([7]). Let L be semisimple Lie algebra with a Cartan subalgebra H and a root space Φ. Then Φ is a root system in the euclidean space ${H_{\mathbb{R}}}^*$.

Definition 3.3.3. Two root systems are equivalent if there is a vector space isomorphism between them.

Theorem 3.3.4. There is a 1-1 correspondence between the set of equivalence classes of root systems and the set of isomorphism classes of finite dimensional complex semisimple Lie algebras.

The proof is not easy. Humphreys [7] first establishes a one to one correspondence between the pairs (L, H) and the root systems. Then, he shows that the equivalence class of the root system of a pair (L, H) does not depend on the Cartan subalgebra H.

Therefore, to classify the complex semisimple Lie algebras we classify the root systems.

Lemma 3.3.3 (Finiteness Lemma). If $\alpha, \beta \in \Phi$ with $\beta \neq \pm\alpha$. Then

$$\langle \beta, \alpha \rangle . \langle \alpha, \beta \rangle \in \{0, 1, 2, 3\}.$$

Proof. $\langle \beta, \alpha \rangle . \langle \alpha, \beta \rangle = \dfrac{4(\alpha, \beta)^2}{(\alpha, \alpha)(\beta, \beta)} = 4cos^2\theta \in \mathbb{Z}^+$ and θ is not an integral multiple of π since $\beta \neq \pm\alpha$. So, the possibilities for $4cos^2\theta$ are $0, 1, 2, 3$. □

Corollary 3.3.1. For a pair of roots α, β with $\|\beta\| \geq \|\alpha\|$ and $\beta \neq \pm\alpha$, the only possibilities are:

Table 3.1: Angle between the roots

$<\alpha,\beta>$	$<\beta,\alpha>$	θ	$\frac{\|\beta\|^2}{\|\alpha\|^2}$
0	0	$\frac{\pi}{2}$	undetermined
1	1	$\frac{\pi}{3}$	1
-1	-1	$\frac{2\pi}{3}$	1
1	2	$\frac{\pi}{4}$	2
-1	-2	$\frac{3\pi}{4}$	2
1	3	$\frac{\pi}{6}$	3
-1	-3	$\frac{5\pi}{6}$	3

Let Φ be any fixed root system in the Euclidean space E. For any $\alpha \in E$, there is a reflection σ_α in E defined by $\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$. σ_α reflects the vectors in E through the hyperplane $P_\alpha = \{\beta \in E \mid (\beta,\alpha) = 0\}$.

Lemma 3.3.4. Let $\alpha \neq \pm\beta$. If $(\alpha,\beta) > 0$, then $\alpha - \beta$ is a root.

Proof. $(\alpha,\beta) > 0$ implies that $\langle\alpha,\beta\rangle$ is positive. Then by the corrollary, $\langle\alpha,\beta\rangle = 1$ or $\langle\beta,\alpha\rangle = 1$. Then either $\sigma_\alpha(\beta) = \beta - \alpha$ or $\sigma_\beta(\alpha) = \alpha - \beta$. In any case, $\alpha - \beta$ is a root. $\square$

Definition 3.3.4. The subgroup of automorphisms of E generated by the reflections $\{\sigma_\alpha \mid \alpha \in \Phi\}$ is called the **Weyl group**.

Weyl groups are very important not only to classify the root systems but also to understand the representations of a given Lie algebra. For the properties of the Weyl group, consult [7] or [8].

If the rank is 2, the only possible root systems are the followings.

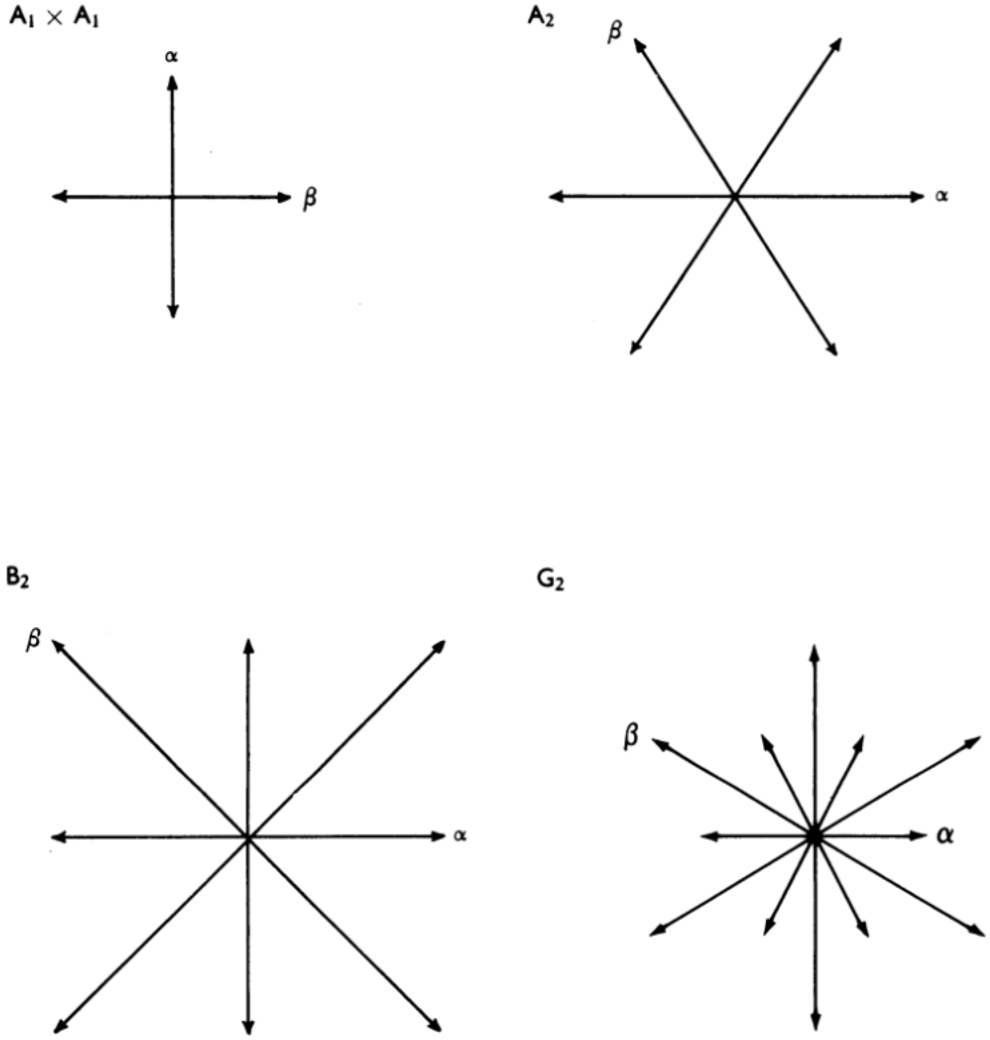

Figure 3.1: Root systems of rank 2

Example 3.3.1. We will find the root system of $L = \mathfrak{sl}(3, \mathbb{C})$. By the example 3.2.1, the roots of $L = \mathfrak{sl}(3, \mathbb{C})$ are

$$\{\alpha_1 - \alpha_2, \alpha_1 - \alpha_3, \alpha_2 - \alpha_3, \alpha_2 - \alpha_1, \alpha_3 - \alpha_1, \alpha_3 - \alpha_2\}.$$

Clearly, $\Delta = \{\alpha_1 - \alpha_2, \alpha_3 - \alpha_1\}$ is a base for the root system of $\mathfrak{sl}(3, \mathbb{C})$. Observe that $[E_{1,2}, E_{2,1}] = E_{1,1} - E_{2,2} = H_{1,2} \in L_{\alpha_1 - \alpha_2}$ and $(\alpha_1 - \alpha_2)(H_{1,2}) = 2$ which implies that $H_{\alpha_1 - \alpha_2} = H_{1,2}$. Similarly $H_{\alpha_i - \alpha_j} = H_{i,j}$.

$\langle \alpha_1 - \alpha_2, \alpha_3 - \alpha_1 \rangle = (\alpha_1 - \alpha_2)(H_{\alpha_3 - \alpha_1}) = -1$ and $\langle \alpha_3 - \alpha_1, \alpha_1 - \alpha_2 \rangle = (\alpha_3 - \alpha_1)(H_{\alpha_1 - \alpha_2}) = -1$. Similar argument shows that

$\langle \alpha_1 - \alpha_2, \alpha_2 - \alpha_3 \rangle = -1$ and $\langle \alpha_2 - \alpha_3, \alpha_1 - \alpha_2 \rangle = -1$

$\langle \alpha_3 - \alpha_1, \alpha_2 - \alpha_3 \rangle = -1$ and $\langle \alpha_2 - \alpha_3, \alpha_3 - \alpha_1 \rangle = -1$

Therefore, all non-proportional roots have the same length which implies that the root system of $\mathfrak{sl}(3, \mathbb{C})$ is of type A_2.

Definition 3.3.5. A subset Δ of Φ is called a **base** if Δ is a basis of E and each root β can be written as $\beta = \sum k_\alpha \alpha$, $\alpha \in \Delta$ with integral coefficients k_α all nonnegative or nonpositive. Any root in Δ is called **positive simple**. $\beta \in \Phi$ is positive (respectively negative) if all k_α are positive (respectively negative). Any root in $-\Delta$ is called **negative simple**.

Moreover, any base defines a partial order on E by $\mu \prec \lambda$ if $\lambda - \mu$ is a sum of positive roots or $\mu = \lambda$.

Lemma 3.3.5. For any $\alpha \in \Phi$, define $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. If Φ is a root system with base Δ, then $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$ is also a root system in the same Euclidean space with base $\Delta^\vee = \{\alpha^\vee \mid \alpha \in \Delta\}$. $\alpha^\vee$ is called a **coroot**.

Let $\gamma \in E$, define $\Phi^+(\gamma) = \{\alpha \in \Phi \mid (\gamma, \alpha) > 0\}$. An element $\gamma \in E$ is called **regular** if $\gamma \in E - \bigcup\limits_{\alpha \in \Phi} P_\alpha$. If $\gamma \in E$ is regular, then $\Phi = \Phi^+(\gamma) \bigcup -\Phi^+(\gamma)$.

$\alpha \in \Phi^+(\gamma)$ is called **decomposable** with respect to γ if $\alpha = \beta_1 + \beta_2$ where $\beta_i \in \Phi^+(\gamma)$.

Theorem 3.3.5. The set of all indecomposable roots with respect to any regular element is a base of Φ. Moreover, any base is a set of all indecomposable roots with respect to some regular element. In particular, any root system Φ has a base.

The connected components of $E - \bigcup_{\alpha \in \Phi} P_\alpha$ are called **(open) Weyl chambers**.

Theorem 3.3.6. There is a one to one correspondence between Weyl chambers and the bases.

Proof. Let $\mathfrak{C}(\gamma)$ denote the Weyl chamber containing γ. $\mathfrak{C}(\gamma) = \mathfrak{C}(\gamma')$ if and only if $\Phi^+(\gamma) = \Phi^+(\gamma')$ if and only if $\Delta(\gamma) = \Delta(\gamma')$. Therefore, there is a one to one correspondence between the Weyl chambers and bases. □

Definition 3.3.6 (Fundamental Weyl chamber)**.** Let Δ be a base, then $\Delta = \Delta(\gamma)$ for some regular γ. The fundamental Weyl chamber associated to Δ is defined to be $\mathfrak{C}(\Delta) = \mathfrak{C}(\gamma)$.

Remark This definition is well defined because if $\Delta = \Delta(\gamma^{'})$, then $\mathfrak{C}(\gamma^{'}) = \mathfrak{C}(\gamma)$.

Lemma 3.3.6. Fundamental Weyl chamber $\mathfrak{C}(\Delta)$ is the set of all $\gamma \in E$ with $(\gamma, \alpha) > 0$ for any $\alpha \in \Delta$. In other words, all the vectors in E making acute angle with every positive simple root.

Definition 3.3.7 (Cartan matrix)**.** Let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ be a base for the root system Φ. The matrix $(< \alpha_i, \alpha_j >)$ is called the Cartan matrix.

Definition 3.3.8. The **Dynkin diagram** of a root system of rank l is defined to be the graph with l vertices labelled with the simple roots α_i and with edges given as follows:

- between the vertices labelled by α_i and α_j, we draw $\langle \alpha_i, \alpha_j \rangle . \langle \alpha_j, \alpha_i \rangle \in \{0, 1, 2, 3\}$ many lines.

- If α_i and α_j have different lengths and are not orthogonal then we draw an arrow pointing from the longer root to the shorter root.

Definition 3.3.9. A root system Φ is **irreducible** if it can not be written as a union of two proper subsets which are orthogonal.

Theorem 3.3.7. To classify a simple Lie algebra, it is enougt to classify irreducible root systems or equivalently the connected Dynkin diagrams.

We state the theorem of classification of root systems without proving.

Theorem 3.3.8. If Φ is any irreducible root system of rank l, the only possibilities for its Dynkin diagram are the following:

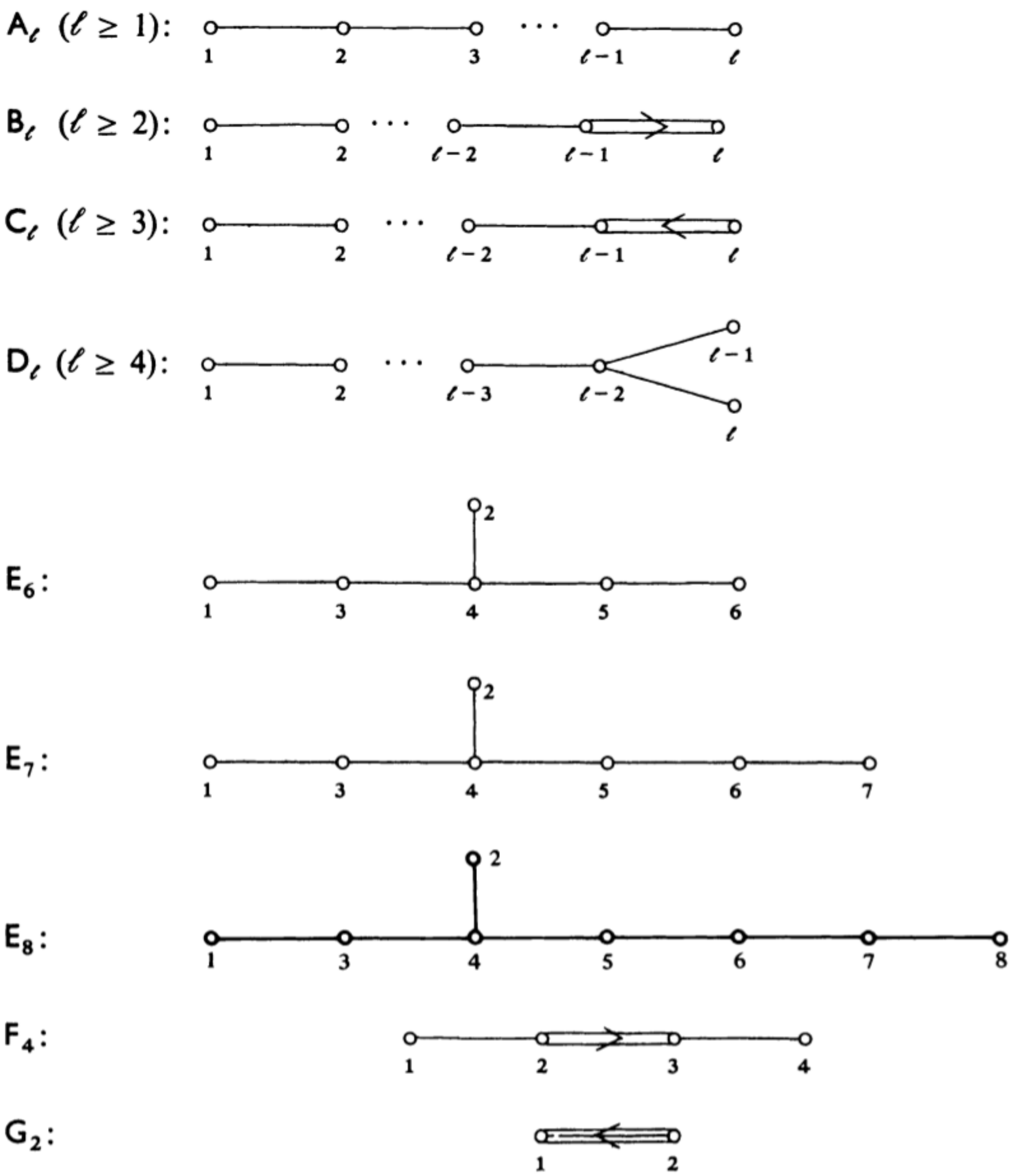

Figure 3.2: Classification

Actually, for each Dynkin diagram in the figure 3.2, there exists an irredducible root system with this Dynkin diagram.

CHAPTER 4

REPRESENTATIONS OF LIE ALGEBRAS

Representation of a Lie algebra is important because it leads some unfamiliar construction of the Lie algebra itself. We will see an example at the end of the thesis. The Lie algebra $\mathfrak{g}_2$ can be constructed using its standard representation without much of knowledge about Lie theory. The method used in classifying representations of $\mathfrak{sl}(3,\mathbb{C})$ will give a strategy to classify the representations for an arbitrary Lie algebra hence we start with that.

4.1 Representations of $\mathfrak{sl}(3,\mathbb{C})$

Let $L = \mathfrak{sl}(3,\mathbb{C})$. By the example 3.2.1, the root space decomposition of L is

$$L = \mathfrak{h} \oplus \bigoplus_{i \neq j} L_{\alpha_i - \alpha_j}$$

where $\mathfrak{h}$ is the set of diagonal matrices of trace 0.

Let $\phi : L \longrightarrow \mathfrak{gl}(V)$ be any representation of L. Since H_1 and H_2 are commuting semisimple elements, then $\phi(H_1)$ and $\phi(H_2)$ are also commuting semisimple elements. So, they are simultaneously diagonalizable. Therefore, there is a basis of V consisting of eigenvectors of both $\phi(H_1)$ and $\phi(H_2)$. Let $\mathcal{B} = \{e_1, e_2, ..., e_n\}$ be such a basis. This means that each e_i is in $V_\alpha = \{v \in V : \; Hv = \alpha(H)v \;\; \forall H \in \mathfrak{h}\}$ for some $\alpha \in \mathfrak{h}^*$. Then, any L-module V (Sometimes we will refer as representation) has a decomposition $\bigoplus V_\alpha$ where α ranges over $\mathfrak{h}^*$. If $V_\alpha \neq 0$, then the eigenvalue α is called a weight and the eigenspace V_α is called a weight space.

If we consider the adjoint representation of L then this decomposition is just the root space decomposition. We will use the root space decomposition to understand any representation of L.

The functionals $\alpha_i - \alpha_j$ are the eigenvalues of the adjoint representation of L other than the eigenvalue 0 with multiplicity 2 since the dimension of $L_0 = \mathfrak{h}$ is 2. To describe the representations we use weight diagrams.

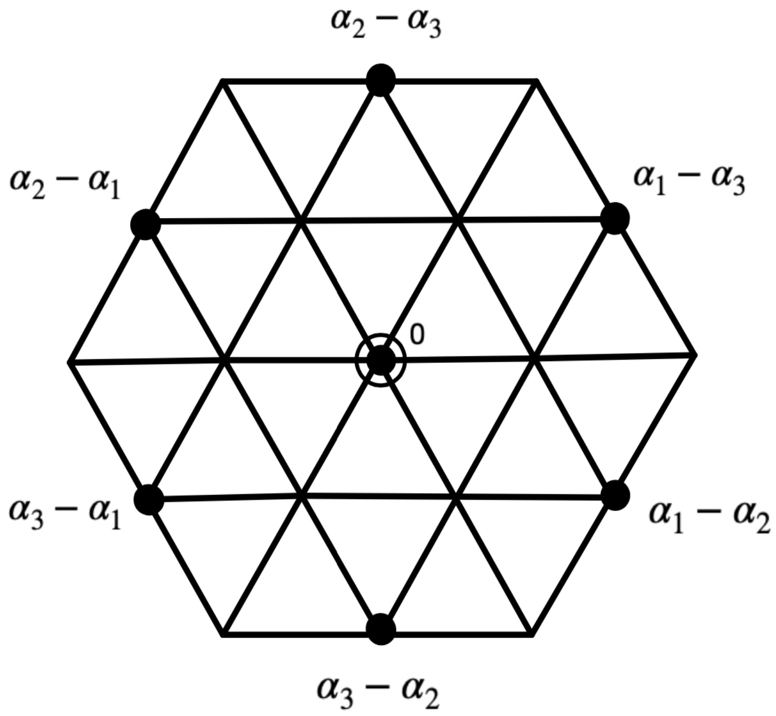

Figure 4.1: Adjoint representation

In the figure 4.1, the weights of the adjoint representation are shown with dots and circles around dots show the multiplicity of the weights.

Lemma 4.1.1. Let $x \in L_\alpha$ and $v \in V_\beta$. Then $x.v \in V_{\alpha+\beta}$.

Proof. $H.(x.v) = x.(H.v) + [Hx].v = x.\beta(H)v + \alpha(H)x.v = (\alpha+\beta)(H)(x.v)$ for any $H \in \mathfrak{h}$. $\square$

Lemma 4.1.2. If V is an irreducible L-module then the difference of the eigenvalues of h in V is a $\mathbb{Z}$-linear combination of $\alpha_i - \alpha_j$'s, where $1 \leq i, j \leq 3$.

Proof. Let $V_\beta \neq 0$. Define $W = \bigoplus_{c,d\in\mathbb{Z}} V_{\beta+c(\alpha_1-\alpha_2)+d(\alpha_1-\alpha_3)}$. If $w \in W$, then for some $c, d \in \mathbb{Z},\ w \in V_{\beta+c(\alpha_1-\alpha_2)+d(\alpha_1-\alpha_3)}$. In other words, w has weight $\beta + c(\alpha_1 - \alpha_2) + d(\alpha_1 - \alpha_3)$. Now, if $x \in L_{\alpha_i-\alpha_j}$, (we can write $\alpha_i - \alpha_j = (-1)(\alpha_1 - \alpha_i) + (\alpha_1 - \alpha_j)$) then $x.w$ has weight $\beta + c(\alpha_1 - \alpha_2) + d(\alpha_1 - \alpha_3) + (-1)(\alpha_1 - \alpha_i) + (\alpha_1 - \alpha_j)$ which implies that $x.w \in W$. Therefore, W is a submodule of an irreducible module V. Thus, $W = V$. Since $\alpha_i - \alpha_j$ is a linear combination of $\alpha_1 - \alpha_2$ and $\alpha_1 - \alpha_3$, we are done.

□

Remark The vectors $\alpha_i - \alpha_j$ generate a lattice $\wedge_R$, called the root lattice.

Lemma 4.1.3. Let V be an irreducible representation of L. Then there is $v \in V$ such that $v \in V_\alpha$ for some α and $E_{1,2}.v = E_{1,3}.v = E_{2,3}.v = 0$.

Definition 4.1.1. In any representation an element which is killed by $E_{1,2}$, $E_{1,3}$ and $E_{2,3}$ is called a highest weight vector.

Lemma 4.1.4. Let V be an irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$, and $v \in V$ a highest weight vector. Then V is generated by the images of v under successive applications of the three operators $E_{2,1}$, $E_{3,1}$, $E_{3,2}$.

Proof. Let W be a subspace of V spanned by the images of v under the subalgebra of $\mathfrak{sl}(3,\mathbb{C})$ generated by $E_{2,1}$, $E_{3,1}$, $E_{3,2}$. It is enough to show that W is an L-submodule of V. This will conclude that $W = V$ by irreducibility of V. To do this we first check that $E_{1,2}$, $E_{2,3}$ carry W into itself. Then more generally, if w_n denotes any word of length n or less in the letters $E_{2,1}$ and $E_{3,2}$ and take W_n to be the vector space spanned by the vectors $w_n(v)$. Then we may show that W is the union of the spaces W_n since $E_{3,1} = [E_{3,2}, E_{2,1}]$. This proves the lemma.

□

Corollary 4.1.1. If α is a weight of a highest weight vector then the dimensions of $V_{\alpha+n(\alpha_2-\alpha_1)}$ and $V_{\alpha+n(\alpha_2-\alpha_1)}$ are at most 1.

Proof. Let $v \in V_\alpha$. The successive actions of $E_{2,1}$, $E_{3,1}$, $E_{3,2}$ on v will never lie in V_α, but it will generate all V. This implies that $V_{\alpha+n(\alpha_2-\alpha_1)}$ and $V_{\alpha+n(\alpha_2-\alpha_1)}$ are spanned by the vectors $(E_{2,1})^n(v)$ and $(E_{3,2})^n(v)$. So the result follows.

□

The proof of the lemma 4.1.4 actually shows

Theorem 4.1.1. If V is any representation of $\mathfrak{sl}(3,\mathbb{C})$ and $v \in V$ a highest weight vector, then the subrepresentation W of V generated by the images of v by successive applications of the three operators $E_{2,1}, E_{3,2}$ and $E_{3,1}$ is irreducible.

As a corollary of this theorem: If V is irreducible and there are two linearly independent highest weight vectors then we will get two irreducible submodules, which will be equal to V itself by successive applications of the elements $E_{i,j}$ where $i > j$. Then these two vectors can only differ by a scalar.

4.1.1 Description of the weight diagrams

Assume that V is an irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$, α is the highest weight and $v \in V_\alpha$. By the results above, all eigenvalues are in one-third of the plane as shown in the figure 4.2.

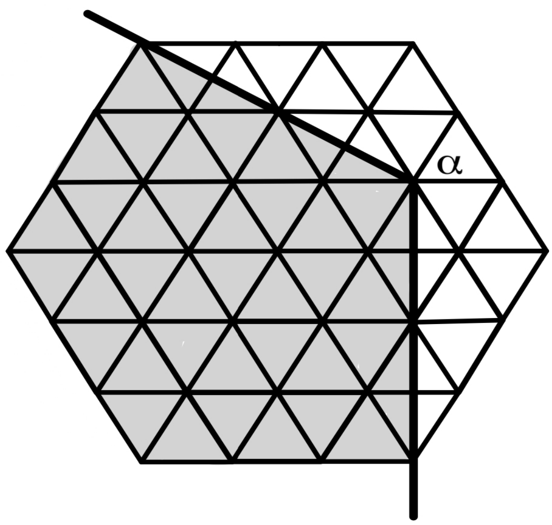

Figure 4.2: Border lines

${E_{2,1}}^m(v) = 0$ for some m.

In theorem 3.2.8 we proved that the elements $E_{1,2}$, $E_{2,1}$ and $H_{1,2} = [E_{1,2}, E_{2,1}]$ span a subalgebra $s_{\alpha_1-\alpha_2}$ isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. Then, $W = \bigoplus_n V_{\alpha+n(\alpha_2-\alpha_1)}$ is a representation of $s_{\alpha_1-\alpha_2}$. This implies that the eigenvalues of $H_{1,2}$ on W are integers and symmetric with respect to 0. $s_{\alpha_1-\alpha_2}$ carry V_α in the direction of $\alpha_2 - \alpha_1$ as in the figure:

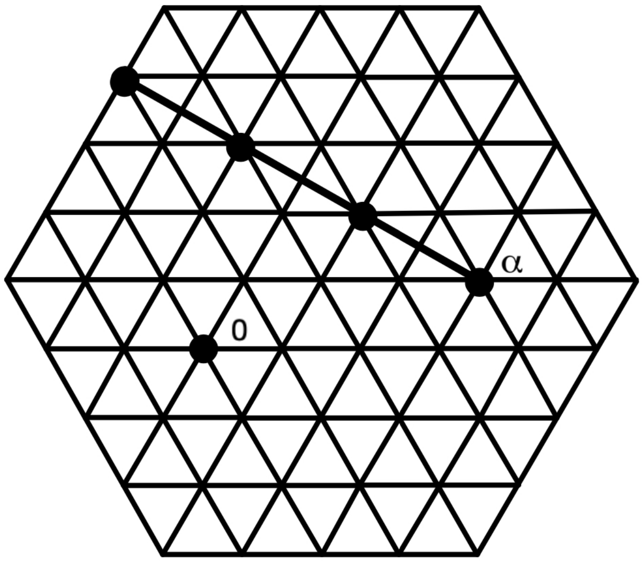

Figure 4.3: The eigenvalues in the direction of $\alpha_2 - \alpha_1$

The dots in the diagram must be symmetric with respect to line $\langle H_{1,2}, L\rangle = 0$.

Indeed, if we consider the subalgebra $s_{\alpha_i-\alpha_j}$, where $i \neq j$ the eigenvalues of $H_{i,j}$ are symmetric about the line $\langle H_{i,j}, L\rangle = 0$.

Therefore the set of weights of V are bounded by a hexagon symmetric with respect to the lines $\langle H_{i,j}, L\rangle = 0$. For the detailed explanations see [8].

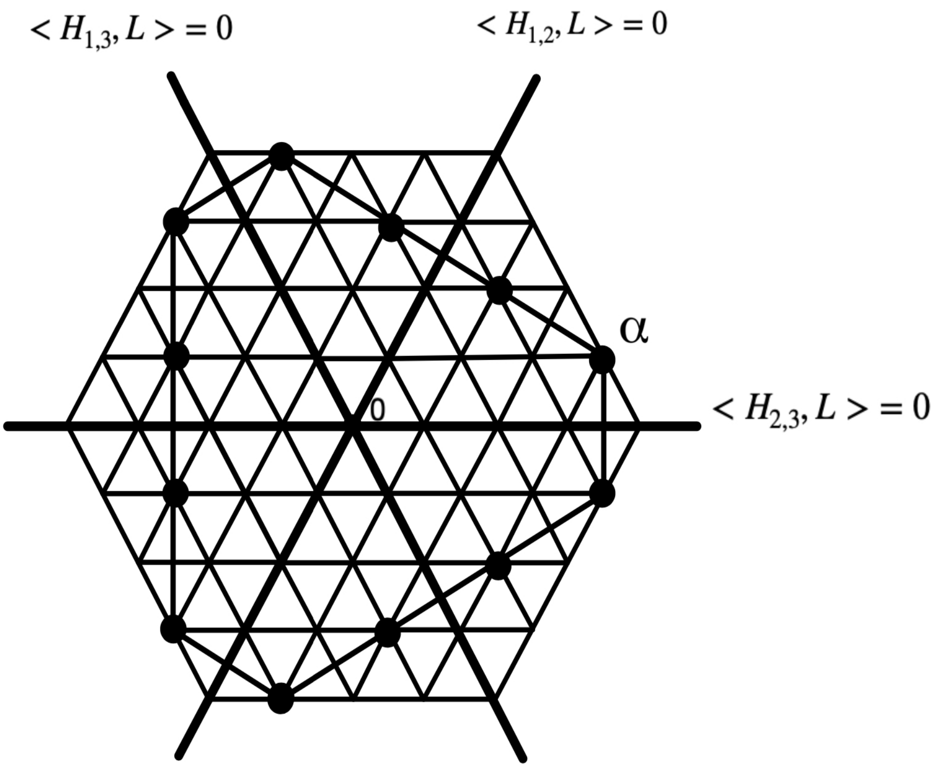

Figure 4.4: Weight hexagon

Theorem 4.1.2. All the weights of any irreducible representation of $L = \mathfrak{sl}(3, \mathbb{C})$ lie in the lattice $\wedge_W \subset \mathfrak{h}^*$ generated by the α_i and be congruent modulo the root lattice $\wedge_R$ generated by $\alpha_i - \alpha_j$.

Combining with the previous results we get the complete set of weights.

Theorem 4.1.3. Let V be any irreducible representation of $L = \mathfrak{sl}(3, \mathbb{C})$.Then the set of weights of V is the set of linear functionals congruent to α modulo the root lattice $\wedge_R$ and lying in the hexagon with vertices the images of α under the group generated by reflections in the lines $\langle H_{i,j}, L\rangle = 0$.

This completes the possible configurations of the weights. Actually, for each configuration of weights there is exactly one irreducible representation.

Hence we see that any highest weight vector is of the form $a\alpha_1 - b\alpha_3$.

4.1.2 Examples

Example 4.1.1 (Standard Representation)**.** Standard representation of $L = \mathfrak{sl}(3, \mathbb{C})$ is given by the map

$\rho : \mathfrak{sl}(3, \mathbb{C}) \longrightarrow \mathfrak{gl}(V)$, where $V = \mathbb{C}^3$ and $\rho(x)(v) = x.v$ is just the matrix multiplication. Let

$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ be the standard basis elements of $\mathbb{C}^3$. Then

$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \alpha_1 \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

More generally, $H.e_i = \alpha_i(H)e_i$ and hence $e_i \in V_{\alpha_i}$

$V_\alpha = \{v \in \mathbb{C}^3 : H.v = \alpha(H)v, \forall H \in \mathfrak{h}\}$

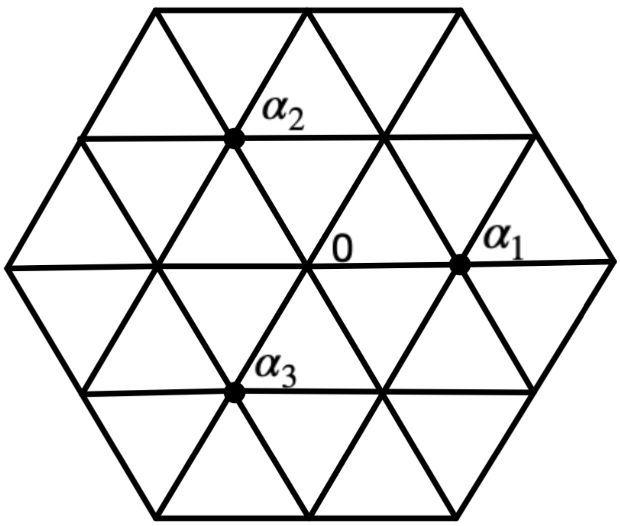

Figure 4.5: Standard representation

Example 4.1.2 (Dual representation)**.** Dual representation V^* of V is given by the rule $(x.f)(v) = -f(x.v)$. See [9]. Since

$$(H.e_1^*)(e_1) = -e_1^*(H.e_1) = -e_1^*(\alpha_1(H)e_1) = -\alpha_1(H)e_1^*(e_1) = -\alpha_1(H)$$

$(H.e_1^*)(e_2) = 0$ and $(H.e_1^*)(e_3) = 0$. Hence, $H.e_1^* = -\alpha_1(H)e_1^*$. This implies that $-\alpha_1$ is a weight of V^* with weight vector e_1^*.

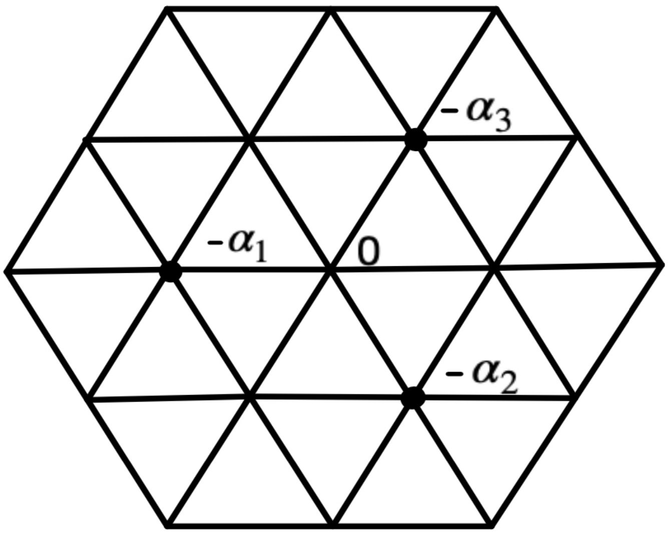

Figure 4.6: Dual representation

The weights of Sym^2V are pairwise sums of the weights of V. That is, the weights are $2\alpha_1$, $2\alpha_2$, $2\alpha_3$, $\alpha_1 + \alpha_2$, $\alpha_1 + \alpha_3$, $\alpha_2 + \alpha_3$.

4.2 Tensors, Exterior Powers and Symmetric Powers

Let V be a vector space with basis $\mathcal{B} = \{e_1, e_2, \ldots, e_m\}$.

Then $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : 1 \leq i_j \leq m\}$ is a basis for $V^{\otimes n}$.

We can construct the exterior power $\bigwedge^n V$ as the quotient space of $V^{\otimes n}$ by the subspace generated by $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ with two of the vectors are equal. Then a basis for $\bigwedge^n V$ is $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} : 1 \leq i_1 < i_2 < \cdots < i_n \leq m\}$.

Similarly, we can construct $\mathrm{Sym}^n V$ as the quotient space of $V^{\otimes n}$ by the subspace generated by all $v_1 \otimes v_2 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}$ where $\sigma \in \mathfrak{S}_n$. Then a basis for $\mathrm{Sym}^n V$ is $\{e_{i_1} \cdot e_{i_2} \cdot \ldots \cdot e_{i_n} : 1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq m\}$.

Definition 4.2.1. Let V and W be representations of a Lie algebra L (L-modules in other words), then $V \bigotimes W$ can be made a representation by the rule

$$x.(v \otimes w) = x.v \otimes w + v \otimes x.w$$

More generally, the action of L on $V^{\otimes n} = V \bigotimes V \bigotimes \cdots \bigotimes V$ is given as

$$x.(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = x.v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_1 \otimes x.v_2 \otimes \cdots \otimes v_n + \cdots + v_1 \otimes v_2 \otimes \cdots \otimes x.v_n$$

The action of L on $\mathrm{Sym}^n V$ and $\bigwedge^n V$ are defined similarly.

Example 4.2.1. As an example, consider the standard representation V of $\mathfrak{sl}(2, \mathbb{C})$. Its weight decomposition is $V = V_{-1} \oplus V_1$. $H.v = v$ for $v \in V_1$ and $H.v = -v$ for $v \in V_{-1}$. $H.(v.w) = (H.v).w + v.(H.w)$. If, $v \in V_1$ and $w \in V_{-1}$, then $H.(v.w) = v.w + v(-w) = 0$ so that 0 is an eigenvalue for $\mathrm{Sym}^2 V$. Similarly, 2 and -2 are the other eigenvalues for $\mathrm{Sym}^2 V$. Actually, the weight decomposition of $\mathrm{Sym}^2 V$ is $\mathrm{Sym}^2 V = V_{-2} \bigoplus V_0 \bigoplus V_2$ which is exactly the 3-dimensional irreducible representation of V.

Remark: The weights of the $\bigwedge^n V$ are the sums of n distinct weights of V. The weights of $\mathrm{Sym}^n V$ are the sums of n weights chosen from m weights of V with repeatation. The weights of $V^{\otimes n}$ are the sums of n weights chosen from m weights of V with repeatation and with multiplicities. These can be verified by considering the basis elements given above.

Consider $\mathfrak{sl}(3, \mathbb{C})$, let V be the standard representation and V^* be the dual represen-tataion of V. The weights of $\mathrm{Sym}^n V$ and $\mathrm{Sym}^n V^*$ occur with multiplicity 1. So, $\mathrm{Sym}^n V$ and $\mathrm{Sym}^n V^*$ are always irreducible. Denote these representations as

$\mathrm{Sym}^n V = \Gamma_{n,0}$ and $\mathrm{Sym}^n V^* = \Gamma_{0,n}$.

Lemma 4.2.1. If V and W are representations having highest weight vectors v and w with weights α and β respectively, then $v \otimes w$ is a highest weight vector of $V \otimes W$ with weight $\alpha + \beta$.

Proof. $H \cdot (v \otimes w) = H \cdot v \otimes w + v \otimes H \cdot w = \alpha(H) v \otimes w + \beta(H) v \otimes w = (\alpha + \beta)(H) v \otimes w$ implies that $v \otimes w$ is a weight vector of weight $\alpha + \beta$. It is actually a highest weight vector since $E_{1,2} \cdot (v \otimes w) = E_{1,2} \cdot v \otimes w + v \otimes E_{1,2} \cdot w = 0 \otimes w + v \otimes 0 = 0$ and $E_{1,3} \cdot (v \otimes w) = E_{2,3} \cdot (v \otimes w) = 0$ similarly. □

Lemma 4.2.2. There exists a unique irreducible representation of $\mathfrak{sl}(3, \mathbb{C})$ with highest weight $a\alpha_1 - b\alpha_3$ for any $a, b \in \mathbb{N}$.

Proof. Notice that α_1 is the highest weight vector of V and $-\alpha_3$ is the highest weight vector of V^*. So, $a\alpha_1$ and $-b\alpha_3$ are the highest weights of $\Gamma_{a,0}$ and $\Gamma_{0,b}$ respectively. Then, $a\alpha_1 - b\alpha_3$ is a highest weight of $\Gamma_{a,0} \bigotimes \Gamma_{0,b}$ by the lemma. Then by 4.1.1, there is an irreducible subrepresentation $\Gamma_{a,b}$ of $\Gamma_{a,0} \bigotimes \Gamma_{0,b}$ with highest weight $a\alpha_1 - b\alpha_3$.

□

Example 4.2.2. Consider the standard representation $V = \mathbb{C}^3$ of $\mathfrak{sl}(3, \mathbb{C})$. The weights are $\alpha_1, \alpha_2, \alpha_3$ with weight vectors e_1, e_2, e_3 respectively.

The weights of $\bigwedge^2 V$ are $\{\alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \alpha_2 + \alpha_3\}$ with weight vectors $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$

The weights of $\mathrm{Sym}^2 V$ are $\{2\alpha_1, 2\alpha_2, 2\alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \alpha_2 + \alpha_3\}$ with weight vectors $\{e_1^2, e_2^2, e_3^2, e_1 \cdot e_2, e_1 \cdot e_3, e_2 \cdot e_3\}$.

The weights of $V^{\otimes 2}$ are $\{2\alpha_1, 2\alpha_2, 2\alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \alpha_2 + \alpha_3\}$ with weight vectors $\{e_1 \otimes e_1, e_2 \otimes e_2, e_3 \otimes e_3, e_1 \otimes e_2, e_1 \otimes e_3, e_2 \otimes e_3\}$ where the multiplicity of the weights $2\alpha_1, 2\alpha_2, 2\alpha_3$ are 2.

4.3 Representations in General setting

Any representation of a semisimple Lie algebra is completely reducible.

Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ and V be any representation of $\mathfrak{g}$. That is, we have a homomorphism $\phi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$.

Since $\mathfrak{h}$ consists of commuting semisimple elements, then $\phi(\mathfrak{h})$ consists of commuting semisimple endomorphisms. So, the elements of $\phi(\mathfrak{h})$ are simultaneously diagonalized. Then there is a basis of V consisting of eigenvectors of all elements of $\phi(\mathfrak{h})$. Let $\{e_1, e_2, \dots e_n\}$ be a basis for V consisting of eigenvectors of all $\phi(H)$, where $H \in \mathfrak{h}$. Then for each j we can write $H \cdot e_j = \lambda_j(H)e_j$ for some $\lambda_j \in \mathfrak{h}^*$. λ_j is called an eigenvalue of the action of H. Thus, V can be written as $V = \bigoplus V_\alpha$ where α runs over $\mathfrak{h}^*$ and $V_\alpha = \{v \in V : \ Hv = \alpha(H)v \ \ \forall H \in \mathfrak{h}\}$. The eigenvalues α existing in this decomposition are called the weights and V_α are called weight spaces. The dimension of V_α is called the multiplicity of α.

To describe the representations we use the weight diagrams

In the rest of the section $\mathfrak{g}$, $\mathfrak{h}$ and V will be as above unless otherwise specified.

Lemma 4.3.1. For any root β of $\mathfrak{g}$, if $x \in \mathfrak{g}_\beta$ and $v \in V_\alpha$, then $x \cdot v \in V_{\alpha+\beta}$

Proof. $H\cdot(x\cdot v) = x\cdot(H\cdot v)+[H,x]\cdot v = x\cdot\alpha(H)v+\beta(H)x\cdot v = (\alpha+\beta)(H)(x\cdot v)$ for any $H \in \mathfrak{h}$. □

Lemma 4.3.2. Let V be an irreducible representation of $\mathfrak{g}$. All the weights of V are congruent to each other modulo the root lattice $\wedge_R$.

Proof. The same proof applies as in the irreducible representations of $\mathfrak{sl}(3, \mathbb{C})$. Indeed, $W = \bigoplus_{\beta\in\wedge_R} V_{\alpha+\beta}$ is a subrepresentation of V.

□

Now, for any root α there is a subalgebra $\mathfrak{s}_\alpha \cong \mathfrak{sl}(2, \mathbb{C})$ of $\mathfrak{g}$ with basis elements $X_\alpha, Y_\alpha, H_\alpha = [X_\alpha, Y_\alpha]$. V is also a representation of $\mathfrak{s}_\alpha$ and so the eigenvalues of H_α are integers. Therefore, every eigenvalue of a representation of $\mathfrak{g}$ takes integer values on all H_α. Accordingly, define $\wedge_W := \{\beta \in \mathfrak{h}^* \mid \beta(H_\alpha) \in \mathbb{Z} \ \ \forall\alpha \in \Phi\}$ which contains all the weights of all representations of $\mathfrak{g}$.

Lemma 4.3.3. $\wedge_W = \{\beta \in \mathfrak{h}^* \mid \beta(H_\alpha) \in \mathbb{Z} \ \ \forall\alpha \in \Delta\}$

Proof. Let $\beta(H_\alpha) = \langle\beta, \alpha\rangle \in \mathbb{Z}$ for any $\alpha \in \Delta$. We will show that $\langle\beta, \alpha\rangle \in \mathbb{Z}$ for any Φ. Let $\alpha \in \Phi$ be arbitrary element. $\langle\beta, \alpha\rangle = 2(\beta, \alpha)/(\alpha, \alpha) = (\beta, 2\alpha/(\alpha, \alpha)) = (\beta, \alpha^\vee)$

$\alpha^\vee = m_1{\alpha_1}^\vee + m_2{\alpha_2}^\vee + \ldots + m_n{\alpha_n}^\vee$

$\langle\beta, \alpha\rangle = (\beta, \alpha^\vee) = m_1(\beta, {\alpha_1}^\vee) + \ldots + m_n(\beta, {\alpha_n}^\vee) = m_1 \langle\beta, \alpha_1\rangle + \ldots m_n \langle\beta, \alpha_n\rangle$

□

Lemma 4.3.4. $\wedge_W$ is actually a lattice.

Proof. Let $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a base for Φ. Then, the vectors $2\alpha_i/(\alpha_i, \alpha_i) = {\alpha_i}^\vee$ form a basis for the Euclidean space $H^*_\mathbb{R}$. Let $\omega_1, \omega_2, \ldots, \omega_n$ be the dual basis vectors with respect to the inner product, i.e, $(\omega_i, {\alpha_j}^\vee) = 2(\omega_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{ij}$. This means that $\omega_i(H_{\alpha_j}) = \delta_{ij} \in \mathbb{Z}$ and so $\omega_i \in \wedge_W$. Then, $\sigma_{\alpha_i}(\omega_j) = \omega_j - \delta_{ij}\alpha_i$. Let $\lambda \in \wedge_W$ with $\lambda(H_{\alpha_i}) = \langle\lambda, \alpha_i\rangle = m_i \in \mathbb{Z}$. Write $m_i = m_1 \langle\omega_1, \alpha_i\rangle + \ldots + m_n \langle\omega_n, \alpha_i\rangle$, $\langle\lambda - (m_1\omega_1 + \ldots + m_n\omega_n), \alpha_i\rangle = 0$. Therefore, $(\lambda - (m_1\omega_1 + \ldots + m_n\omega_n), \alpha_i) = 0$. By the non-degeneracy, we get $\lambda = m_1\omega_1 + \ldots + m_n\omega_n$. □

This proof also shows that the weight lattice lies in the Euclidean space derived from the Cartan subalgebra.

Then we use the the symmetry of the eigenvalues of H_α to conclude that the set of weights of any representation of $\mathfrak{g}$ is invariant under the Weyl group. Moreover, the multiplicities are preserved. We state this as a lemma.

Lemma 4.3.5. The set of weights of any representation of $\mathfrak{g}$ is invariant under the Weyl group.

Proof. Let $V = \bigoplus V_\beta$. Fix a root α. V is also an $\mathfrak{s}_\alpha$-module. Then $V_{[\beta]} = \bigoplus\limits_{n\in\mathbb{Z}} V_{\beta+n\alpha}$ is an $\mathfrak{s}_\alpha$-submodule of V.

By the analysis of irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$, nonzero weights differ by 2. Note that the weight of the vector v in $V_{\beta+n\alpha}$ is $\beta(H_\alpha) + 2n$ since $H_\alpha.v = (\beta + n\alpha)(H_\alpha).v = (\beta(H_\alpha) + n\alpha(H_\alpha)).v = (\beta(H_\alpha) + 2n).v$. This implies that the nonzero summands are the weight spaces $V_{\beta+n\alpha}$, where $-r \leq n \leq q$ for some positive integers

r, q. By the symmetry of weights, $\beta(H_\alpha) - 2r + \beta(H_\alpha) + 2q = 0$. So, $\beta(H_\alpha) = r - q$. For the action of the Weyl group

$\sigma_\alpha(\beta - k\alpha) = \beta - k\alpha - (\beta - k\alpha)(H_\alpha)\alpha = \beta - k\alpha - \beta(H_\alpha)\alpha + 2k\alpha = \beta + (k - \beta(H_\alpha))\alpha = \beta + (k + q - r)\alpha$. So, σ_α reverses the string of weights appearing in $V_{[\beta]}$.

Note that σ_α generates the Weyl group of $\mathfrak{s}_\alpha$. Thus the set of weights of any representation of $\mathfrak{g}$ is invariant under the Weyl group of $\mathfrak{g}$.

□

Next, we order the roots by choosing a linear functional l on the root lattice with no kernel in the root lattice. We let $R^+ = \{\alpha \mid l(\alpha) > 0\}$ and $R^- = \{\alpha \mid l(\alpha) < 0\}$ be positive and negative roots respectively. This is equivalent to choosing a base which also determines the associated Weyl chamber. Associated base Δ is the set of positive roots which can not be written as a sum two positive roots. Recall the definition 3.3.5 that we call $\alpha \in \Delta$ positive simple root and $\alpha \in -\Delta$ negative simple root. Moreover, the fundamental Weyl chamber $\mathfrak{C}(\Delta)$ is the set of $\gamma \in E$ such that $(\gamma, \alpha) > 0$ for any $\alpha \in \Delta$.

Definition 4.3.1. Any nonzero $v \in V$ is called a highest weight vector if $x \cdot v = 0$ for any $x \in \mathfrak{g}_\alpha$ and any $\alpha \in R^+$.

Now we state a fundamental result related to the highest weight.

Theorem 4.3.1. Any representation of $\mathfrak{g}$ has a highest weight vector. Moreover, successive applications of the root spaces $\mathfrak{g}_\alpha$, where $\alpha \in R^-$ to a highest weight vector generates an irreducible representation. Lastly, any irreducible representation contains a unique highest weight vector up to scalars.

Definition 4.3.2. $\alpha \in R^+$ is called simple if it can not be written as a sum of two positive roots. Similarly, $\alpha \in R^-$ is called simple if it can not be written as a sum of two negative roots.

Corollary 4.3.1. Any irreducible representation V is generated by the images of its highest weight vector under successive applications of root spaces $\mathfrak{g}_\alpha$ where α is a negative simple root.

Lemma 4.3.6. Vertices of the convex hull of the weights of an irreducible representation V are the images of the highest weight α under the Weyl group.

Proof. Any vertex adjacent to α is $\alpha + k\beta$ for some $\beta \in R^-$. Then, $V_{[\alpha]} = \bigoplus V_{\alpha+n\beta}$ is an $\mathfrak{s}_\beta$-submodule of V. Then, $k > 0$ and the weights of the nonzero summands in V are $\alpha, \alpha+\beta, \ldots, \alpha+k\beta$ since α is the highest weight. Corresponding string of eigenvalues for the action of H_β are $\alpha(H_\beta), \alpha(H_\beta)+2, \ldots, \alpha(H_\beta)+2k$, which must be symmetric about 0. Therefore, $\alpha(H_\beta) = -k$ and the adjacent vertex must be $\alpha - \alpha(H_\beta)\beta = \sigma_\beta(\alpha)$.

So any adjacent vertex to α is of the form $\sigma_\beta(\alpha)$ for some $\beta \in R^-$. Other vertices are given by the successive applications of the reflections σ_γ.

□

Corollary 4.3.2. The highest weight of an irreducible representation lies on the closure of the fundamental Weyl chamber.

Proof. Let α be the highest weight of V. We saw in the proof of lemma 4.3.6 that $\alpha(H_\gamma) \leq 0$ for any $\gamma \in R^-$. So, we may write $\alpha(H_\gamma) \geq 0$ or $\langle \alpha, \gamma \rangle \geq 0$ for any $\gamma \in R^+$. Therefore, α belongs to the closure of the fundamental Weyl chamber associated to the base given by the ordering.

□

This can be declared as α makes an acute or right angle with any positive root. We denote the closure of the fundamental Weyl chamber by $\mathfrak{W}$.

Corollary 4.3.3. The set of weights is exactly the weights which are congruent to α modulo $\wedge_R$ and lie in the convex hull of the images of α under the Weyl group.

Proof. Follows immediately by the previous results. □

Lemma 4.3.7. Two irreducible representations with the same highest weight are isomorphic.

Proof. Let V and W be irreducible representations with highest weight α. Let $v \in V_\alpha$ and $w \in W_\alpha$. That is, $\mathfrak{g}_\beta.v = \mathfrak{g}_\beta.w = 0$ for any $\beta \in R^+$. Then, $\mathfrak{g}_\beta.(v+w) = 0$, and hence $v + w$ is a highest weight vector in $V + W$ with weight α since $H.(v+w) = \alpha(H)(v+w)$. Therefore, by 4.3.1 $v+w$ generates an irreducible subrepresentation U of $V + W$. But then the projections $\Pi_1 : U \to V$ and $\Pi_2 : U \to W$ are nonzero module homomorphisms. This forces $U \cong V \cong W$ by Schur's lemma.

□

We denote the unique irreducible representation up to isomorphism with highest weight α by Γ_α and the set of isomorphism classes of irreducible representations of $\mathfrak{g}$ by $\mathfrak{Irr}(\mathfrak{g})$.

The next theorem gives all possible highest weights for any irreducible representation

Theorem 4.3.2. There exists a bijection $\Psi : \mathfrak{Irr}(\mathfrak{g}) \longrightarrow \mathfrak{W} \cap \wedge_W$ given by

$$\Psi(\Gamma_\alpha) = \alpha$$

Proof. Ψ is well defined since the highest weight must lie in $\mathfrak{W}$ by the corollary 4.3.2. Ψ is one to one by the lemma 4.3.7. Ψ is onto but to prove it we need to use Verma modules which can be generalized to the infinite dimensional representations. See, [10].

□

4.3.1 Fundamental Weights

The generators ω_i of the weight lattice $\wedge_W$ defined in 4.3.4 are clearly in the closure of the fundamental Weyl chamber associated to the ordering of the roots. Hence any ω_i is actually a weight, called a fundamental weight. Moreover, any highest weight is a nonnegative $\mathbb{Z}$ linear combination of ω_i's in a unique way.

If $\alpha = a_1\omega_1 + \ldots + a_n\omega_n$, we write $\Gamma_\alpha = \Gamma_{a_1\omega_1+\ldots+a_n\omega_n} = \Gamma_{a_1,a_2,\ldots,a_n}$ for the irreducible representation of weight α.

Let C be the Cartan matrix of the semisimple Lie algebra $\mathfrak{g}$ of rank n and let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be the positive simple roots. There exists unique $H_{\alpha_i} \in [\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{-\alpha_i}]$ with $\alpha_i(H_{\alpha_i}) = 2$. We can write

$$\alpha_i = \alpha_i(H_{\alpha_1})\omega_1 + \ldots + \alpha_i(H_{\alpha_n})\omega_n = [\alpha_i(H_{\alpha_1}), \ldots, \alpha_i(H_{\alpha_n})][\omega_1, \ldots, \omega_n]^t = \sum_{j=1}^{n} \alpha_i(H_{\alpha_j})\omega_j$$

which is the multiplication of the i-th row of the Cartan matrix with the column matrix $[\omega_1, \ldots, \omega_n]^t$. Therefore,

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = C \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix}$$

where C is the Cartan matrix of $\mathfrak{g}$. Then, the fundamental weights are given by

$$\begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix} = C^{-1} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

CHAPTER 5

THE EXCEPTIONAL LIE ALGEBRA $\mathfrak{g}_2$

5.1 Construction of $\mathfrak{g}_2$ by Dynkin diagram

The Dynkin diagram of type G_2 is shown in the figure 5.1.

Figure 5.1: Dynkin diagram G_2

Corresponding to the Dynkin diagram G_2 we will find a Lie algebra called $\mathfrak{g}_2$.

G_2 has two simple roots α_1, α_2 such that $\langle \alpha_1, \alpha_2 \rangle \langle \alpha_2, \alpha_1 \rangle = 3$. Then the angle between α_1 and α_2 is $\dfrac{5\pi}{6}$ with $\|\alpha_2\| = \sqrt{3}\,\|\alpha_1\|$ by the table 3.1 because the angle between two simple roots must be obtuse. The Weyl group permutes the roots. In fact, the Weyl group $\mathcal{W}$ of the root system of G_2 is generated by $\{\sigma_{\alpha_1}, \sigma_{\alpha_2}\}$ which is just the dihedral group of order 12.

Then the root system for G_2

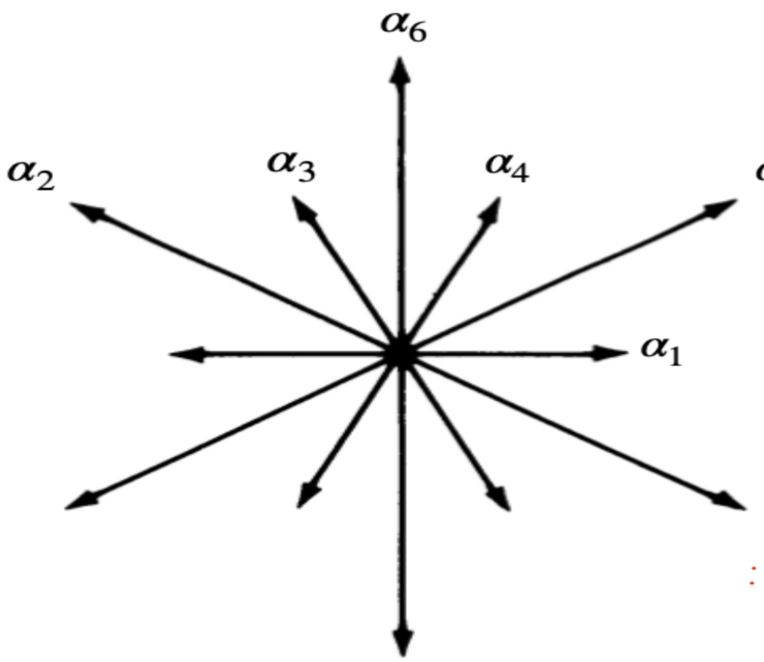

Figure 5.2: Roots of G_2

We first assume that there is some Lie algebra, called $\mathfrak{g}_2$ corresponding to this root system. Then, it has a Cartan decomposition, say $\mathfrak{g}_2 = L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$ where α_1, α_2 are the simple roots. Set $\alpha_1 + \alpha_2 = \alpha_3$, $\alpha_4 = 2\alpha_1 + \alpha_2$, $\alpha_5 = 3\alpha_1 + \alpha_2$, $\alpha_6 = \alpha_2 + \alpha_5$, $\beta_i = -\alpha_i$. So, we have 12 roots with α_i's are positive and β_i's are negative.

By theorem 3.2.8 (6), there exists $X_1 \in L_{\alpha_1}$ and $Y_1 \in L_{\beta_1}$ such that $X_1, Y_1, H_1 = [X_1, Y_1]$ span a subalgebra s_{α_1} isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ where $H_1 = \dfrac{2t_{\alpha_1}}{\kappa(t_{\alpha_1}, t_{\alpha_1})}$.

Similarly, there exists $X_2 \in L_{\alpha_2}$ and $Y_2 \in L_{\beta_2}$ such that $X_2, Y_2, H_2 = [X_2, Y_2]$ span a subalgebra s_{α_2} isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ where $H_2 = \dfrac{2t_{\alpha_2}}{\kappa(t_{\alpha_2}, t_{\alpha_2})}$.

Now, we define $X_3 = [X_1, X_2], X_4 = [X_1, X_3], X_5 = [X_1, X_4], X_6 = [X_2, X_5]$. We also define Y_1, Y_2, Y_3, Y_4 similarly. Then, we can easily observe that $X_i \in L_{\alpha_i}$ and $Y_i \in L_{\beta_i}$. Therefore, the elements $H_1, H_2, X_1, ..., X_6, Y_1, ..., Y_6$ form a basis for $\mathfrak{g}_2$ because the rank is 2 and $\dim L_\alpha = 1$ for any $\alpha \in \Phi$.

If $\alpha_i + \alpha_j$ is not a root, then $L_{\alpha_i+\alpha_j} = 0$ and so, $[X_i, X_j] = 0$,

If $\beta_i + \beta_j$ is not a root, then $L_{\beta_i+\beta_j} = 0$ and so, $[Y_i, Y_j] = 0$,

If $\alpha_i + \beta_j$ is not a root, then $L_{\alpha_i+\beta_j} = 0$ and so, $[X_i, Y_j] = 0$.

$$\alpha_1(H_1) = \alpha_1\Big(\frac{2t_{\alpha_1}}{\kappa(t_{\alpha_1}, t_{\alpha_1})}\Big) = 2\frac{\kappa(t_{\alpha_1}, t_{\alpha_1})}{\kappa(t_{\alpha_1}, t_{\alpha_1})} = 2\frac{(\alpha_1, \alpha_1)}{(\alpha_1, \alpha_1)} = 2$$

$$\alpha_1(H_2) = \alpha_1\Big(\frac{2t_{\alpha_2}}{\kappa(t_{\alpha_2}, t_{\alpha_2})}\Big) = 2\frac{\kappa(t_{\alpha_1}, t_{\alpha_2})}{\kappa(t_{\alpha_2}, t_{\alpha_2})} = 2\frac{(\alpha_1, \alpha_2)}{(\alpha_2, \alpha_2)} =< \alpha_1, \alpha_2 >= -1$$

$$\alpha_2(H_1) = \alpha_2\Big(\frac{2t_{\alpha_1}}{\kappa(t_{\alpha_1}, t_{\alpha_1})}\Big) = 2\frac{\kappa(t_{\alpha_2}, t_{\alpha_1})}{\kappa(t_{\alpha_1}, t_{\alpha_1})} = 2\frac{(\alpha_2, \alpha_1)}{(\alpha_1, \alpha_1)} =< \alpha_2, \alpha_1 >= -3$$

$$\alpha_2(H_2) = \alpha_2\Big(\frac{2t_{\alpha_2}}{\kappa(t_{\alpha_2}, t_{\alpha_2})}\Big) = 2\frac{\kappa(t_{\alpha_2}, t_{\alpha_2})}{\kappa(t_{\alpha_2}, t_{\alpha_2})} = 2\frac{(\alpha_2, \alpha_2)}{(\alpha_2, \alpha_2)} = 2$$

$[X_1, X_5] = [X_1, X_6] = 0$, $[X_2, X_3] = [X_2, X_4] = 0$,
$[X_3, X_4] = \text{ad}X_3(X_4) = \text{ad}[X_1, X_2](X_4) = \text{ad}X_1\text{ad}X_2(X_4) - \text{ad}X_2\text{ad}X_1(X_4) = \text{ad}X_1(0) - \text{ad}X_2[X_1, X_4] = 0 - \text{ad}X_2(X_5) = -[X_2, X_5] = -X_6$
$[X_3, X_5] = [X_3, X_6] = 0$, $[X_4, X_5] = [X_4, X_6] = 0$, $[X_5, X_6] = 0$.
$[Y_i, Y_j]$ can be computed similarly.

$[X_1, Y_1] = H_1$, $[X_1, Y_2] = 0$

$[X_1, Y_3] = \text{ad}X_1(Y_3) = \text{ad}X_1\text{ad}Y_1(Y_2) = \text{ad}[X_1, Y_1](Y_2) + \text{ad}Y_1\text{ad}X_1(Y_2) = \text{ad}H_1(Y_2) + 0 = [H_1, Y_2] = \beta_2(H_1)Y_2 = -\alpha_2(H_1)Y_2 = 3Y_2$.

$[X_1, Y_4] = \text{ad}X_1(Y_4) = \text{ad}X_1\text{ad}Y_1(Y_3) = \text{ad}[X_1, Y_1](Y_3) + \text{ad}Y_1\text{ad}X_1(Y_3) = [H_1, Y_3] + \text{ad}Y_1(3Y_2) = \beta_3(H_1)Y_3 + 3[Y_1, Y_2] = (\beta_1 + \beta_2)(H_1)Y_3 + 3Y_3 = (-2+3)Y_3 + 3Y_3 = 4Y_3$

$[X_1, Y_5] = \text{ad}X_1(Y_5) = \text{ad}X_1\text{ad}Y_1(Y_4) = \text{ad}[X_1, Y_1](Y_4) + \text{ad}Y_1\text{ad}X_1(Y_4) = [H_1, Y_4] + \text{ad}Y_1(4Y_3) = \beta_4(H_1)Y_4 + 4[Y_1, Y_3] = (\beta_1 + \beta_3)(H_1)Y_4 + 4Y_4 = (-2+1)Y_4 + 4Y_4 = 3Y_4$

$[X_1, Y_6] = 0$

$[X_2, Y_1] = 0$, $[X_2, Y_2] = H_2$,

$[X_2, Y_3] = \text{ad}X_2\text{ad}Y_1(Y_2) = \text{ad}[X_2, Y_1](Y_2) + \text{ad}Y_1\text{ad}X_2(Y_2) = 0 + [Y_1, H_2] = -[H_2, Y_1] = -\beta_1(H_2)Y_1 = \alpha_1(H_2)Y_1 = -Y_1$

$[X_2, Y_4] = \text{ad}X_2\text{ad}Y_1(Y_3) = \text{ad}[X_2, Y_1](Y_3) + \text{ad}Y_1\text{ad}X_2(Y_3) = 0 + [Y_1, -Y_1] = 0$ (Actually, it is zero because $\alpha_2 + \beta_4$ is not a root.)

$[X_2, Y_5] = 0$,

$[X_2, Y_6] = \text{ad}X_2\text{ad}Y_2(Y_5) = \text{ad}[X_2, Y_2](Y_5) + \text{ad}Y_2\text{ad}X_2(Y_5) = [H_2, Y_5] + 0 = \beta_5(H_2)Y_5 = -(\alpha_1 + \alpha_1 + \alpha_1 + \alpha_2)(H_2)Y_5 = -(-3 + 2)Y_5 = Y_5$.

$[X_3, Y_1] = \text{ad}X_3(Y_1) = \text{ad}[X_1, X_2](Y_1) = \text{ad}X_1\text{ad}X_2(Y_1) - \text{ad}X_2\text{ad}X_1(Y_1) = 0 - [X_2, H_1] = [H_1, X_2] = \alpha_2(H_1)X_2 = -3X_2$,

$[X_3, Y_2] = \text{ad}X_3(Y_2) = \text{ad}[X_1, X_2](Y_2) = \text{ad}X_1\text{ad}X_2(Y_2) - \text{ad}X_2\text{ad}X_1(Y_2) = [X_1, H_2] - 0 = -[H_2, X_1] = -\alpha_1(H_2)X_1 = X_1$,

$[X_3, Y_3] = \text{ad}X_1\text{ad}X_2(Y_3) - \text{ad}X_2\text{ad}X_1(Y_3) = \text{ad}X_1(-Y_1) - \text{ad}X_2(3Y_2) = -H_1 - 3H_2$,

$[X_3, Y_4] = \text{ad}X_1\text{ad}X_2(Y_4) - \text{ad}X_2\text{ad}X_1(Y_4) = \text{ad}X_1(0) - \text{ad}X_2(4Y_3) = -4[X_2, Y_3] = -4(-Y_1) = 4Y_1$,

$[X_3, Y_5] = 0$,

$[X_3, Y_6] = \text{ad}X_1\text{ad}X_2(Y_6) - \text{ad}X_2\text{ad}X_1(Y_6) = [X_1, Y_5] - \text{ad}X_2(0) = 3Y_4$,

$[X_4, Y_1] = \text{ad}X_1\text{ad}X_3(Y_1) - \text{ad}X_3\text{ad}X_1(Y_1) = [X_1, -3X_2] - [X_3, H_1] = -3X_3 + \alpha_3(H_1)X_3 = -3X_3 + (2 - 3)X_3 = -4X_3$,

$[X_4, Y_2] = 0$,

$[X_4, Y_3] = \text{ad}X_1\text{ad}X_3(Y_3) - \text{ad}X_3\text{ad}X_1(Y_3) = \text{ad}X_1(-H_1 - 3H_2) - \text{ad}X_3(3Y_2) = [X_1, -H_1 - 3H_2] - 3[X_3, Y_2] = [H_1, X_1] + 3[H_2, X_1] - 3[X_3, Y_2] = \alpha_1(H_1)X_1 + 3\alpha_1(H_2)X_1 - 3X_1 = 2X_1 - 3X_1 - 3X_1 = -4X_1$,

$[X_4, Y_4] = \text{ad}X_1\text{ad}X_3(Y_4) - \text{ad}X_3\text{ad}X_1(Y_4) = \text{ad}X_1(4Y_1) - \text{ad}X_3(4Y_3) = 4H_1 - 4(-H_1 - 3H_2) = 8H_1 + 12H_2$,

$[X_4, Y_5] = \mathrm{ad}X_1\mathrm{ad}X_3(Y_5) - \mathrm{ad}X_3\mathrm{ad}X_1(Y_5) = 0 - \mathrm{ad}X_3(3Y_4) = -3[X_3, Y_4] = -12Y_1$,

$[X_4, Y_6] = \mathrm{ad}X_1\mathrm{ad}X_3(Y_6) - \mathrm{ad}X_3\mathrm{ad}X_1(Y_6) = [X_1, 3Y_4] - [X_3, 0] = 3[X_1, Y_4] = 12Y_3$,

$[X_5, Y_1] = \mathrm{ad}X_1\mathrm{ad}X_4(Y_1) - \mathrm{ad}X_4\mathrm{ad}X_1(Y_1) = [X_1, -4X_3] - [X_4, H_1] = -4X_4 + \alpha_4(H_1)X_4 = -4X_4 + (\alpha_1 + \alpha_1 + \alpha_2)(H_1)X_4 = -4X_4 + (2 + 2 - 3)X_4 = -3X_4$,

$[X_5, Y_2] = \mathrm{ad}X_1\mathrm{ad}X_4(Y_2) - \mathrm{ad}X_4\mathrm{ad}X_1(Y_2) = 0 - 0 = 0$,

$[X_5, Y_3] = 0$,

$[X_5, Y_4] = \mathrm{ad}X_1\mathrm{ad}X_4(Y_4) - \mathrm{ad}X_4\mathrm{ad}X_1(Y_4) = [X_1, 8H_1 + 12H_2] - [X_4, 4Y_3] = -8\alpha_1(H_1)X_1 - 12\alpha_1(H_2)X_1 + 16X_1 = -16X_1 + 12X_1 + 16X_1 = 12X_1$,

$[X_5, Y_5] = \mathrm{ad}X_1\mathrm{ad}X_4(Y_5) - \mathrm{ad}X_4\mathrm{ad}X_1(Y_5) = [X_1, -12Y_1] - [X_4, 3Y_4] = -12H_1 - 3(8H_1 + 12H_2) = -36H_1 - 36H_2$,

$[X_5, Y_6] = \mathrm{ad}X_1\mathrm{ad}X_4(Y_6) - \mathrm{ad}X_4\mathrm{ad}X_1(Y_6) = [X_1, 12Y_3] - [X_4, 0] = 12(3Y_2) = 36Y_2$,

$[X_6, Y_1] = 0$,

$[X_6, Y_2] = \mathrm{ad}X_2\mathrm{ad}X_5(Y_2) - \mathrm{ad}X_5\mathrm{ad}X_2(Y_2) = 0 - [X_5, H_2] = [H_2, X_5] = \alpha_5(H_2)X_5$,

$[X_6, Y_3] = \mathrm{ad}X_2\mathrm{ad}X_5(Y_3) - \mathrm{ad}X_5\mathrm{ad}X_2(Y_3) = [X_2, 0] - [X_5, -Y_1] = [X_5, Y_1] = -3X_4$,

$[X_6, Y_4] = \mathrm{ad}X_2\mathrm{ad}X_5(Y_4) - \mathrm{ad}X_5\mathrm{ad}X_2(Y_4) = [X_2, 12X_1] - [X_5, 0] = -12X_3$,

$[X_6, Y_5] = \mathrm{ad}X_2\mathrm{ad}X_5(Y_5) - \mathrm{ad}X_5\mathrm{ad}X_2(Y_5) = [X_2, -36H_1 - 36H_2] - [X_5, 0] = 36([H_1, X_2] + [H_2, X_2]) = 36(\alpha_2(H_1) + \alpha_2(H_2))X_2 = 36(-3 + 2)X_2 = -36X_2$,

$[X_6, Y_6] = \mathrm{ad}X_2\mathrm{ad}X_5(Y_6) - \mathrm{ad}X_5\mathrm{ad}X_2(Y_6) = [X_2, 36Y_2] - [X_5, Y_5] = 36H_2 - (-36H_1 - 36H_2) = 36H_1 + 72H_2$.

Therefore, we get the multiplication table for $\mathfrak{g}_2$.

	H_2	X_1	Y_1	X_2	Y_2	X_3	Y_3	X_4	Y_4	X_5	Y_5	X_6	Y_6
H_1	0	$2X_1$	$-2Y_1$	$-3X_2$	$3Y_2$	$-X_3$	Y_3	X_4	$-Y_4$	$3X_5$	$-3Y_5$	0	0
H_2		$-X_1$	Y_1	$2X_2$	$-2Y_2$	X_3	$-Y_3$	0	0	$-X_5$	Y_5	X_6	$-Y_6$
X_1			H_1	X_3	0	X_4	$3Y_2$	X_5	$4Y_3$	0	$3Y_4$	0	0
Y_1				0	Y_3	$3X_2$	Y_4	$4X_3$	Y_5	$3X_4$	0	0	0
X_2					H_2	0	$-Y_1$	0	0	X_6	0	0	Y_5
Y_2						$-X_1$	0	0	0	0	Y_6	X_5	0
X_3							$-H_1$ $-3H_2$	$-X_6$	$4Y_1$	0	0	0	$3Y_4$
Y_3								$4X_1$	$-Y_6$	0	0	$3X_4$	0
X_4									$8H_1$ $+12H_2$	0	$-12Y_1$	0	$12Y_3$
Y_4										$-12X_1$	0	$12X_3$	0
X_5											$-36H_1$ $-36H_2$	0	$36Y_2$
Y_5												$36X_2$	0
X_6													$36H_1$ $+72H_2$

Figure 5.3: Multiplication table for $\mathfrak{g}_2$

This table can be modified with nicer one by multiplying some of the basis elements by a scalar. We want that $[X_i, Y_i]$ will give the distinguished element H_i of s_{α_i} such that $\alpha_i(H_i) = 2$. For this purpose, we divide X_4 and Y_4 by 2, X_5, X_6, Y_5, Y_6 by 6 and X_5, Y_3 by -1. The new table is

	H_2	X_1	Y_1	X_2	Y_2	X_3	Y_3	X_4	Y_4	X_5	Y_5	X_6	Y_6
H_1	0	$2X_1$	$-2Y_1$	$-3X_2$	$3Y_2$	$-X_3$	Y_3	X_4	$-Y_4$	$3X_5$	$-3Y_5$	0	0
H_2		$-X_1$	Y_1	$2X_2$	$-2Y_2$	X_3	$-Y_3$	0	0	$-X_5$	Y_5	X_6	$-Y_6$
X_1			H_1	X_3	0	$2X_4$	$-3Y_2$	$-3X_5$	$-2Y_3$	0	Y_4	0	0
Y_1				0	$-Y_3$	$3X_2$	$-2Y_4$	$2X_3$	$3Y_5$	$-X_4$	0	0	0
X_2					H_2	0	Y_1	0	0	$-X_6$	0	0	Y_5
Y_2						$-X_1$	0	0	0	0	Y_6	$-X_5$	0
X_3							H_1+3H_2	$-3X_6$	$2Y_1$	0	0	0	Y_4
Y_3								$-2X_1$	$3Y_6$	0	0	$-X_4$	0
X_4									$2H_1+3H_2$	0	$-Y_1$	0	$-Y_3$
Y_4										X_1	0	X_3	0
X_5											H_1+H_2	0	$-Y_2$
Y_5												X_2	0
X_6													H_1+2H_2

Figure 5.4: New multiplication table for $\mathfrak{g}_2$

5.1.1 Relation between $\mathfrak{g}_2$ and $\mathfrak{sl}(3, \mathbb{C})$

In the previous section we assumed that there is a Lie algebra with the root system G_2. We constructed the multiplication table under this assumption. Now, we have to check that this multiplication table of brackets defines a Lie algebra. We need to check that the Jacobi identity is satisfied. However, there are $\binom{14}{3}$ different triples from the basis and hence it is not reasonable to check all triples from the basis. We will use the description of $\mathfrak{sl}(3, \mathbb{C})$ and its representations to understand the brackets in $\mathfrak{g}_2$.

Observe that the root diagram for G_2 consists of two nested hexagons which is the same picture if we combine the weight diagrams of the standard representation, its dual and the adjoint representation of $\mathfrak{sl}(3, \mathbb{C})$. [1]

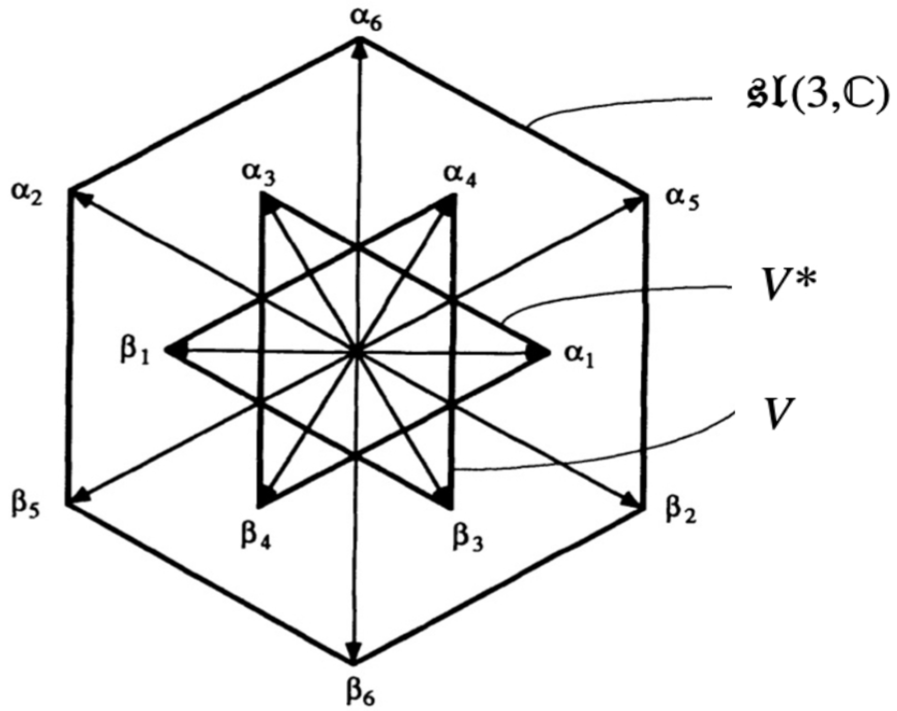

Figure 5.5: $\mathfrak{g}_2$ as the union of the weight diagrams

Consider the subalgebra $\mathfrak{g}_0$ spanned by the ordered basis

$$\{H_5, H_2, X_5, Y_5, X_2, Y_2, X_6, Y_6\}.$$

[1] The figures 5.3, 5.4 and 5.5 are taken from [8].

The multiplication table for $\mathfrak{g}_0$ is the same with $\mathfrak{sl}(3,\mathbb{C})$ if we take the ordered basis $\{H_{1,2}, H_{2,3}, E_{1,2}, E_{2,1}, E_{2,3}, E_{3,2}, E_{1,3}, E_{3,1}\}$ for $\mathfrak{sl}(3,\mathbb{C})$. So, $\mathfrak{g}_0 \cong \mathfrak{sl}(3,\mathbb{C})$

Let $V = \mathbb{C}\{X_4, Y_1, Y_3\}$ and $W = \mathbb{C}\{Y_4, X_1, X_3\}$. Then obviously, $\mathfrak{g}_2 = \mathfrak{g}_0 \oplus V \oplus W$ as a vector space decomposition. V and W are not Lie algebras with respect to the bracket operator, but we will show that they are irreducible representations of $\mathfrak{g}_0$, namely V corresponds to the standard representation of $\mathfrak{sl}(3,\mathbb{C})$ and W corresponds to the dual representation of V. First, we recall that

Let e_1, e_2, e_3 be the standard basis elements of $\mathbb{C}^3$ which are the column matirices. Then the standard representation of $\mathfrak{sl}(3,\mathbb{C})$ on $\mathbb{C}^3$ is given by the matrix multiplication. That is, $E_{i,j}e_k = e_i\delta_{jk}$. Then we can easily compute the dual representation on $(\mathbb{C}^3)^*$ with basis elements ${e_1}^*, {e_2}^*, {e_3}^*$ by the rule $(x \cdot f)(v) = -f(x \cdot v)$ which yields $E_{i,j}.{e_k}^*(e_t) = -{e_k}^*(E_{i,j}e_t) = -\delta_{jt}\delta_{ik}$ so that $E_{i,j}.{e_k}^* = -{e_j}^*\delta_{ik}$.

We consider the following isomorphisms

We identify the basis elements of $\mathfrak{g}_0$ with the basis elements of $\mathfrak{sl}(3,\mathbb{C})$ respecting the order, which gives an isomorphism of Lie algebras.

The elements X_4, Y_1, Y_3 are identified with the column matrices e_1, e_2, e_3 of $\mathbb{C}^3$ and the elements Y_4, X_1, X_3 are identified with the dual basis elements ${e_1}^*, {e_2}^*, {e_3}^*$ of $(\mathbb{C}^3)^*$ respecting the given order.

For instance, $[X_6, Y_3] = X_4$ by the multiplication table and the corresponding matrix multiplication (action of $\mathfrak{sl}(3,\mathbb{C})$ on the standard representation $\mathbb{C}^3$) is $E_{1,3}e_3 = e_1$.

Similarly, the action of $\mathfrak{g}_0$ on W by the commutator can be seen by identifying $\mathfrak{g}_0$ with $\mathfrak{sl}(3,\mathbb{C})$ and W with $(\mathbb{C}^3)^*$.

For instance, $[X_5, Y_4] = -X_1$ and the corresponding matrix multiplication is
$E_{1,2} \cdot {e_1}^* = -{e_2}^*$.

$[V, V] \subset W$ and $[W, W] \subset V$. Also, the bracket in V corresponds to the multiplications:

$e_1 \cdot e_2 = -2{e_3}^*$

$e_1 \cdot e_3 = 2{e_2}^*$

$e_2 \cdot e_3 = -2{e_1}^*$

We use the isomorphism between V^* and $\wedge^2 V$ given by

${e_1}^* = e_2 \wedge e_3$

${e_2}^* = e_3 \wedge e_1$

$e_3^* = e_1 \wedge e_2$

So that the brackets $[v_1, v_2]$ on V corresponds to $-2 \cdot v_1 \wedge v_2$ whereas, the brackets on W corresponds to $[\phi, \psi] = 2 \cdot \phi \wedge \psi$.

Lastly, the brackets of the form $[v, w]$ where $v \in V$, $w \in W$ corresponds to the formula

$$[e_i, {e_j}^*] = 3E_{i,j} - \delta_{ij} I \tag{5.1}$$

which can also be described by

$$[v_1, \phi](v_2) = 3\phi(v_2)v_1 - \phi(v_1)v_2$$

for $v_1, v_2 \in V$, $\phi \in W$. $[v, \phi] \in \mathfrak{sl}(3, \mathbb{C}) \subset \mathfrak{gl}(V)$.

The following lemma is very useful when we check the Jacobi identities.

Lemma 5.1.1. $[v, \phi]$ is characterized by $\kappa([v, \phi], Z) = 18\phi(Z \cdot v)$ for all $Z \in \mathfrak{sl}(3, \mathbb{C})$ where κ is the Killing form of $\mathfrak{sl}(3, \mathbb{C})$. Writing, $[v, \phi] = 18 \cdot v * \phi$ we get $\kappa(v * \phi, Z) = \phi(Z \cdot v)$.

Proof. The Killing form κ of $\mathfrak{sl}(3, \mathbb{C})$ satisfies $\kappa(X, Y) = 6.\mathrm{tr}(X \cdot Y)$ [8] Combining with 5.1i we get

$\kappa([e_i, {e_j}^*], Z) = 6\mathrm{tr}(3E_{i,j}Z) = 18Z_{j,i} = -18(Z \cdot {e_j}^*)(e_i) = 18{e_j}^*(Z \cdot e_i)$. By linearity of the Killing form and the bracket, the result follows. $\square$

Now, we are ready to check the Jacobi identity.

5.1.2 Existence of $\mathfrak{g}_2$

We will chose 3 elements from $\mathfrak{g}_2$ to check the Jacobi identity. Let (a, b, c) represents that Jacobi identity is satisfied if we choose a elements from $\mathfrak{g}_0$, b elements from V and c elements from W. So, we need to check 10 cases, namely

$(3,0,0),\ (2,1,0),\ (2,0,1),\ (1,2,0),\ (1,0,2),\ (1,1,1),\ (0,3,0),\ (0,0,3),\ (0,2,1),\ (0,1,2)$.

1. $(3,0,0)$ satisfied because $\mathfrak{g}_0 \cong \mathfrak{sl}(3,\mathbb{C})$.

2. $(2,1,0)$ and $(2,0,1)$ follows from the observation that V is the sandard representation and W is its dual. For instance let $X, Y \in \mathfrak{g}_0$ and $v \in V$, the Jacobi identity for the case $(2,1,0)$ is equivalent to show that

$$[X,[Y,v]] + [Y,[v,X]] + [v,[X,Y]] = 0$$

$$\iff X.Y.v - [Y,[X,v]] - [[X,Y],v] = 0$$

$$\iff X.Y.v - Y.X.v - [X,Y].v = 0$$

   which is obviously true. The case $(2,0,1)$ is similar

3. $(1,2,0)$ and $(1,0,2)$ are true by the action of $\mathfrak{g}_0$ to the exterior power $\bigwedge^2 V$ or $\bigwedge^2 W$. Let $X \in \mathfrak{g}_0$ and $v_1, v_2 \in V$, the Jacobi identity for the case $(1,2,0)$ is equivalent to

$$[X,[v_1,v_2]] + [v_1,[v_2,X]] + [v_2,[X,v_1]] = 0$$

$$\iff -2X.(v_1 \wedge v_2) + [[X,v_2],v_1] - [[X,v_1],v_2] = 0$$

$$\iff -2X.(v_1 \wedge v_2) - 2(X.v_2) \wedge v_1 + 2(X.v_1) \wedge v_2 = 0$$

$$\iff -2X.(v_1 \wedge v_2) + 2v_1 \wedge (X.v_2) + 2(X.v_1) \wedge v_2 = 0$$

   which is obviously true by the action on the exterior power. The lattter case is similar.

4. $(1,1,1)$: Let $Z \in \mathfrak{g}_0$, $v \in V$ and $f \in V^*$. The Jacobi identity is

$$[Z,[v,f]] + [v,[f,Z]] + [f,[Z,v]] = 0$$

$$\Longleftrightarrow$$

$$[Z,[v,f]] = [v,[Z,f]] + [[Z,v],f]$$

$$\Longleftrightarrow$$

$$[Z, v * f] = v * (Z \cdot f) + (Z \cdot v) * f \tag{5.2}$$

The Killing form is nondegenerate on $\mathfrak{g}_0$, then 5.2 is equivalent to

$$\kappa(Y,[Z,v * f]) = \kappa(Y, v * (Z \cdot f)) + \kappa(Y,(Z \cdot v) * f) \text{ for any } Y \in \mathfrak{g}_0$$

The Killing form is associative by equaton 2.1. So, the left hand side becomes

$$\kappa(Y,[Z,v * f]) = \kappa([Y,Z], v * f) = f([Y,Z] \cdot v)$$

The right hand side is

$$(Z \cdot f)(Y \cdot v) + f(Y \cdot (Z \cdot v))$$

Therefore, we need to show that

$$f([Y,Z] \cdot v) = (Z \cdot f)(Y \cdot v) + f(Y \cdot (Z \cdot v))$$

This is obvious since

$f([Y,Z] \cdot v) = f(Y \cdot (Z \cdot v) - Z \cdot (Y \cdot v)) = f(Y \cdot (Z \cdot v) - f(Z \cdot (Y \cdot v))) = f(Y \cdot (Z \cdot v) + (Z \cdot f)(Y \cdot v)$

5. $(0,3,0)$ and $(0,0,3)$: Let $u, v, w \in V$, the Jacobi identity for the case $(0,3,0)$ can be read as $[u[vw]] + [v[wu]] + [w[uv]] = 0$. This is true if and only if

$$u * [vw] + v * [wu] + w * [uv] = 0$$

$$\iff$$

$$\kappa(u * [vw] + v * [wu] + w * [uv], Z) = 0 \; \forall Z \in \mathfrak{g}_0.$$

$$\iff$$

$$[vw](Z \cdot u) + [wu](Z \cdot)v + [uv](Z \cdot w) = 0 \; \forall Z \in \mathfrak{g}_0.$$

$$\iff$$

$$v \wedge w \wedge (Z \cdot u) + w \wedge u \wedge (Z \cdot v) + u \wedge v \wedge (Z \cdot w) = 0 \; \forall Z \in \mathfrak{g}_0.$$

$$\iff$$

$$u \wedge v \wedge (Z \cdot w) + u \wedge (Z \cdot v) \wedge w + (Z \cdot u) \wedge v \wedge w = Z \cdot (u \wedge v \wedge w) = 0$$
$$\forall Z \in \mathfrak{g}_0.$$

The last equality $Z \cdot (u \wedge v \wedge w) = 0$ holds because $\bigwedge^3 V = \mathbb{C}$ is one dimensional and any semisimple Lie algebra over $\mathbb{C}$ acts trivially on $\bigwedge^3 V = \mathbb{C}$.

6. $(0, 2, 1)$: Let $u, v \in V$, $\phi \in V^*$. The Jacobi identity is

$$[v, [u, \phi]] + [u, [\phi, v]] + [\phi, [v, u]] = 0.$$

$$\iff -[[u, \phi], v] + [[v, \phi], u] + 4 \cdot (v \wedge u) \wedge \phi = 0$$
$$\iff -[u, \phi] \cdot v + [v, \phi] \cdot u = -4 \cdot (v \wedge u) \wedge \phi = -4[\phi(v)u - \phi(u)v]$$
$$\iff \phi(u)v - 3\phi(v)u + 3\phi(u)v - \phi(v)u = -4[\phi(v)u - \phi(u)v]$$

which is obvious.

Note the identity $-4[\phi(v)u - \phi(u)v] = -[u, \phi] \cdot v + [v, \phi] \cdot u$. Apply ψ to both sides

$$-4[\phi(v)\psi(u) - \phi(u)\psi(v)] = -\psi([u, \phi] \cdot v) + \psi([v, \phi] \cdot u) \tag{5.3}$$

7. $(0,1,2)$: Let $v \in V$, $\phi, \psi \in V^*$. The Jacobi identity holds if and only if $[v,[\phi,\psi]] = [[v,\phi],\psi] - [[v,\psi],\phi]$. The left hand side is $-4 \cdot v \wedge (\phi \wedge \psi) = -4(\psi(v)\phi) - \phi(v)\psi)$. So, we need to show that $-4(\psi(v)\phi) - \phi(v)\psi) = [v,\phi] \cdot \psi - [v,\psi] \cdot \phi$ in V^*. Equivalently, $-4(\psi(v)\phi(u)) - \phi(v)\psi(u)) = \phi([v,\psi] \cdot u) - \psi([v,\phi] \cdot u)$ for any $u \in V$.

   By equation 5.3, the left hand side is

   $\psi([u,\phi] \cdot v) - \psi([v,\phi] \cdot u) = \psi([u,\phi] \cdot v) - \psi([v,\phi] \cdot u)$

   So, the Jacobi identity becomes

   $\psi([u,\phi] \cdot v) - \psi([v,\phi] \cdot u) = \phi([v,\psi] \cdot u) - \psi([v,\phi] \cdot u)$. Then multiplying both sides with 18, it becomes

   $\kappa([v,\psi],[u,\phi]) - \kappa([u,v],[v,\phi]) = \kappa([u,\phi],[v,\psi]) - \kappa([u,\psi],[v,\phi])$.

   The last statement is true since the Killing form is symmetric.

As a result the multiplication table given by 5.4 actually defines a Lie algebra.

5.1.3 Representations of $\mathfrak{g}_2$

The Cartan matrix of $\mathfrak{g}_2$ is given by

$$C = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

the inverse is

$$C^{-1} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

The fundamental weights are then $\omega_1 = 2\alpha_1 + \alpha_2$ and $\omega_2 = 3\alpha_1 + 2\alpha_2$ which are again roots, namely $\omega_1 = \alpha_4$ and $\omega_2 = \alpha_6$. In particular, the weight lattice and the root lattice coincide.

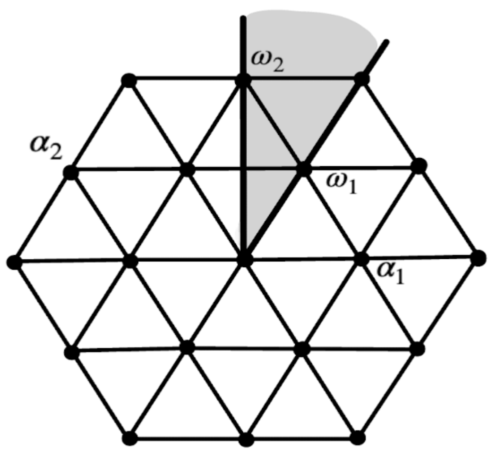

Figure 5.6: Weight lattice of $\mathfrak{g}_2$

Consider the irreducible representation $\Gamma_{1,0}$ with highest weight ω_1. Vertices of the convex hull of the weights can be found by applying the Weyl group to the highest weight ω_1. Weyl group is generated by the reflections through the hyperplanes orthogonal to a simple root [7]. So we get the vertices of the hexagon.

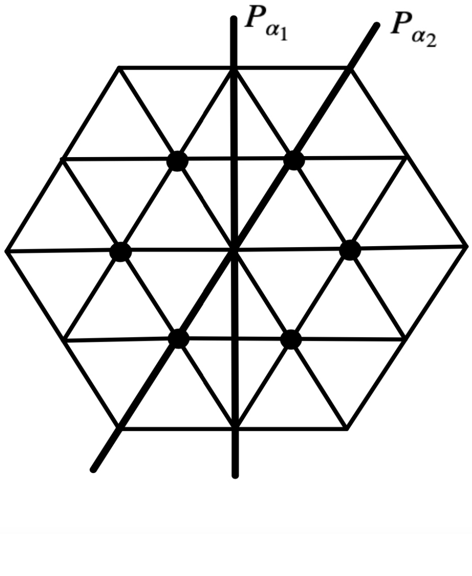

Figure 5.7: Vertices of the standard representation of $\mathfrak{g}_2$

Then adding simple negative roots to ω_1 in the convex hull of the weights we get all the weights of $\Gamma_{1,0}$ as in the figure 5.8. The multiplicity of 0 is 1, because there is only one way of getting 0 adding the simple negative roots to ω_1. (This corresponds to appyling negative roots to generate the representation)

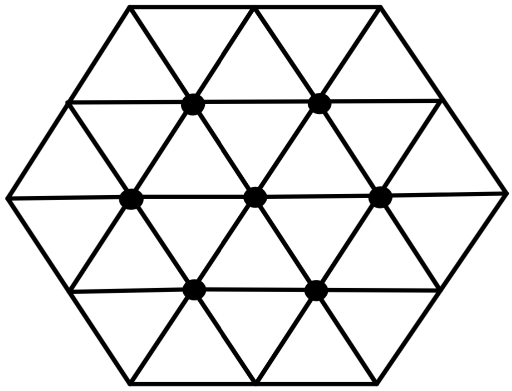

Figure 5.8: Weight diagram of the standard representation of $\mathfrak{g}_2$

Now, consider $\Gamma_{0,1}$. We first find the vertices of the convex hull of weights as we did for $\Gamma_{1,0}$. Then using the weight strings we can plot all the weights in the diagram as

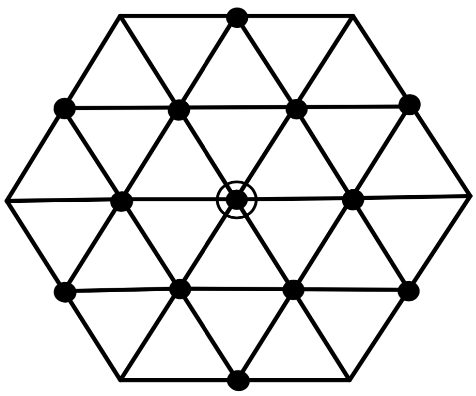

Figure 5.9: Weight diagram of $\Gamma_{0,1}$

The multiplicity of 0 is 2 because there are two ways from ω_2 to 0. This is just the adjoint representation since the dots are all the roots of $\mathfrak{g}_2$.

Example 5.1.1. $\bigwedge^2 V$ has 21 weights which can be found by adding distinct weights.

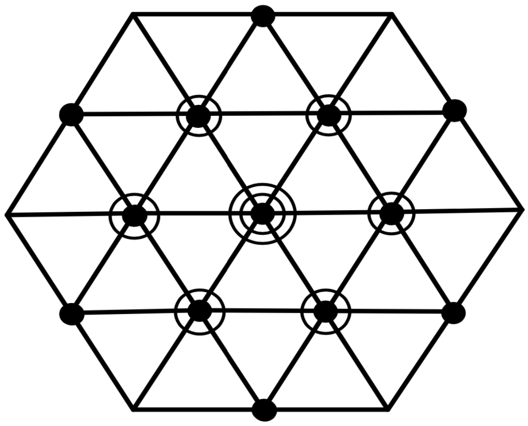

Figure 5.10: Weight diagram of the exterior square

By the diagram we see that $\bigwedge^2 V \cong \Gamma_{0,1} \bigoplus V$. In particular, $\Gamma_{0,1} \subset \bigwedge^2 V \subset V \bigotimes V$.

Corollary 5.1.1. Every irreducible representation of $\mathfrak{g}_2$ is contained in some tensor power of the standard representation V.

Proof. By 4.2.2, $\mathrm{Sym}^a V$ has a highest weight vector of weight $a\omega_1$ and $\mathrm{Sym}\Gamma_{0,1}$ has a highest weight vector of weight $b\omega_2$. So, $\mathrm{Sym}^a V \bigotimes \mathrm{Sym}^b \Gamma_{0,1}$ has a highest weight vector of weight $a\omega_1 + b\omega_2$. This vector generates $\Gamma_{a,b}$. Therefore, $\Gamma_{a,b} \subset \mathrm{Sym}^a V \bigotimes \mathrm{Sym}^b \Gamma_{0,1} \subset V^{\bigotimes a} \bigotimes (V \bigotimes V)^{\bigotimes b} \subset V^{\bigotimes (a+2b)}$.

□

Other two important examples are $\mathrm{Sym}^2 V$ and $\bigwedge^3 V$. Using the weight diagrams it can be proven that $\mathrm{Sym}^2 V \cong \Gamma_{2,0} \bigoplus \mathbb{C}$ and $\bigwedge^3 V \cong \Gamma_{2,0} \bigoplus V \bigoplus \mathbb{C}$. See, [8] for the details. These decompositions have nice results.

Corollary 5.1.2. The standard representation of $\mathfrak{g}_2$ embeds $\mathfrak{g}_2$ into $\mathfrak{so}(7, \mathbb{C})$.

Proof. Let $\rho : \mathfrak{g}_2 \longrightarrow \mathfrak{sl}(7, \mathbb{C})$ be the corresponding homomorphism for the standard representation V.

Say $\phi : \mathrm{Sym}^2 V \longrightarrow \Gamma_{2,0} \bigoplus \mathbb{C}$ be an isomorphism of $\mathfrak{g}_2$-modules and $\pi_2 : \Gamma_{2,0} \bigoplus \mathbb{C} \longrightarrow \mathbb{C}$ be the projection which is a module homomorphism. So, we have a nonzero module homorphism $\pi_2 \circ \phi : \mathrm{Sym}^2 V \longrightarrow \mathbb{C}$.

Define $B : V \times V \longrightarrow \mathbb{C}$ by $B(v_1, v_2) = \pi_2 \circ \phi(v_1 \cdot v_2)$ which is obviously a symmetric bilinear form. Moreover, $B(x.v_1, v_2) + B(v_1, x, v_2) = \pi_2 \circ \phi((x.v_1) \cdot v_2) + \pi_2 \circ \phi(v_1 \cdot (x.v_2)) = \pi_2 \circ \phi((x.v_1) \cdot v_2 + v_1 \cdot (x.v_2)) = \pi_2 \circ \phi(x.(v_1 \cdot v_2)) = x.\pi_2 \circ \phi(v_1 \cdot v_2)$ is zero because any semisimple Lie algebra acts trivially on any one dimensional space. Therefore, the action of $\mathfrak{g}_2$ on V preserves a symmetric bilinear form (quadratic form). In other words, the standard representation embeds $\mathfrak{g}_2$ into $\mathfrak{so}(7, \mathbb{C})$. □

Similarly, as a corollary of the isomorphism $\bigwedge^3 V \cong \Gamma_{2,0} \bigoplus V \bigoplus \mathbb{C}$, one can prove that the action of $\mathfrak{g}_2$ on V preserves a skew-symmetric trilinear form on V. Even more is actually true.

Theorem 5.1.1. $\mathfrak{g}_2$ is isomorphic to the algebra of endomorphisms of a seven dimensional vector space V preserving a general skew-symmetric cubic form on V.

5.2 A combinatorial Construction of $\mathfrak{g}_2$

In this section we follow the construction made by Wildberger, [3]. It uses the 7-dimensional representation of $\mathfrak{g}_2$, which must be the standard representation mentioned in the previous chapter. Actually, the smallest non-trivial representation of $\mathfrak{g}_2$ is 7-dimensional.

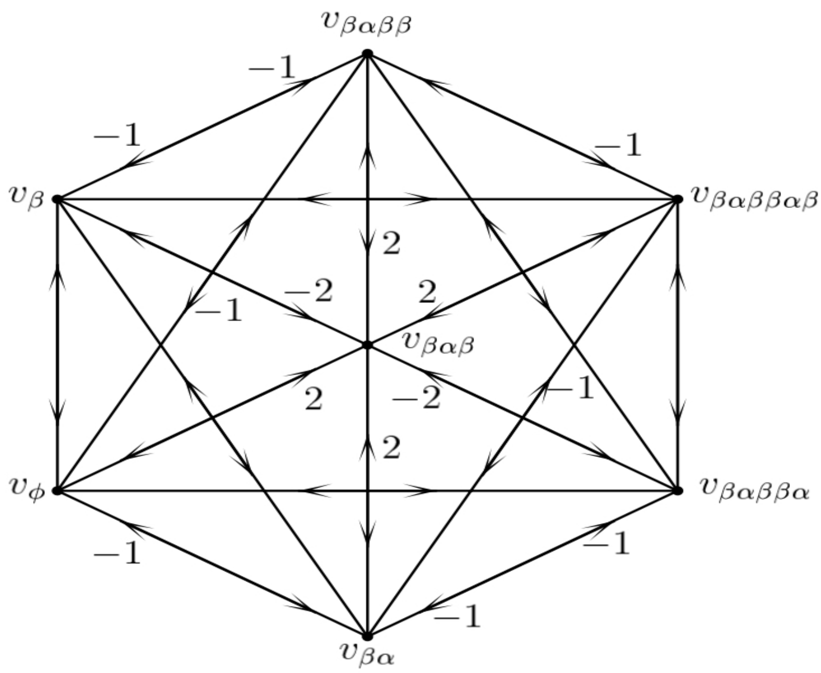

Figure 5.11: G_2 hexagon

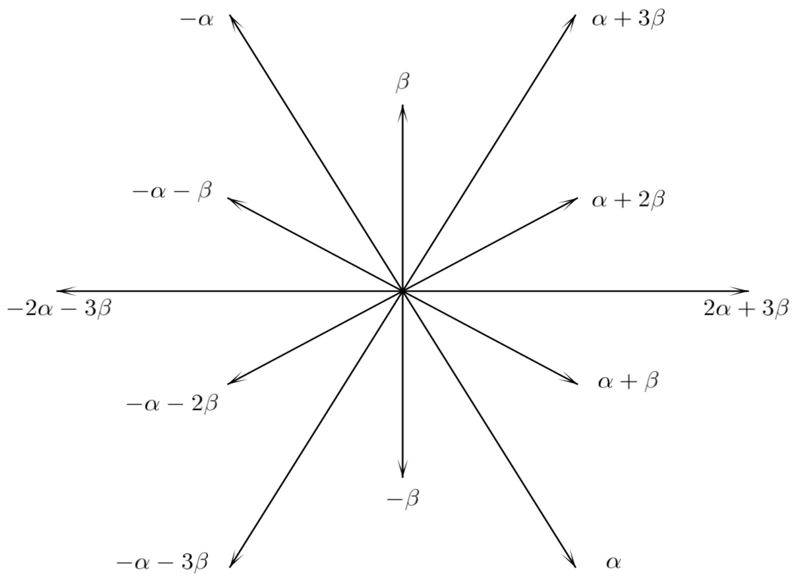

Figure 5.12: Root diagram of G_2

We will consider the G_2 hexagon and the root system in the same plane. For a root γ define an operator X_γ on the vertices of the G_2 hexagon as:

$X_\gamma : v \to nw$ if there is an edge from v to w of direction γ and weight n. For example there is an edge from v_β to $v_{\beta\alpha\beta\beta}$ in the direction of $\gamma = \alpha + 2\beta$ with weight -1. So, $X_{\alpha+2\beta}(v_\beta) = -v_{\beta\alpha\beta\beta}$. Corresponding to 12 roots of G_2 we have 12 operators. Define $XY := Y \circ X$ and $[X, Y] = XY - YX$. Define also the operators $H_\gamma = [X_{-\gamma}, X_\gamma]$.

Theorem 5.2.1. The operators X_γ, H_γ span the Lie algebra $\mathfrak{g}_2$.

5.2.1 Mutation and Numbers Games

By playing mutation and numbers games on the dynkin diagram of G_2 we will get the root system of G_2 and the G_2 hexagon.

Consider the directed graph G_2 and the integer valued functions on the vertices of the directed graph G_2. We will denote the mutation of the function f at the vertex v as f^v and define it to be the function $f^v(w) = f(w)$ if $v \neq w$ and $f^v(v) = -f(v) + \sum_w n_w f(w)$ where w is a neighbor of v and n_w is the number of directed edge from w to v.

Now, arbitrary sequence of the mutations of a delta function is called a root. To find the roots of G_2 we consider two delta functions δ_α and δ_β where δ_α sends α to 1 and β to 0. δ_β is defined similarly. Consider the mutation of δ_α at α. ${\delta_\alpha}^\alpha(\beta) = 0$ and ${\delta_\alpha}^\alpha(\alpha) = -\delta_\alpha(\alpha) + \delta_\alpha(\beta) = -1$. Therefore, ${\delta_\alpha}^\alpha = -\delta_\alpha$.

${\delta_\alpha}^\beta(\alpha) = 1$ and ${\delta_\alpha}^\beta(\beta) = -\delta_\alpha(\beta) + 3\delta_\alpha(\alpha) = 3$. Therefore, ${\delta_\alpha}^\beta = \delta_\alpha + 3\delta_\beta$. We can also check that ${\delta_\beta}^\alpha = \delta_\alpha + \delta_\beta$. In other words, if the delta function δ_β is mutated at the vertex α, we get the root $\delta_\alpha + \delta_\beta$ which will be written as $\alpha + \beta$. Applying mutation games, we get 12 roots as $R(G_2) = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(\alpha+2\beta), \pm(\alpha+3\beta), \pm(2\alpha+3\beta)\}$.

The dual mutation of the function f at v is denoted as $\hat{f}^v$ and defined by $\hat{f}^v(v) = -f(v)$, $\hat{f}^v(w) = f(w) + n_w f(v)$ where n_w is the number of the directed edge from v to w. Note that if w is not a neighborhood of v then $\hat{f}^v(w) = f(w)$.

Numbers game is defined to be a successive applications of dual mutations to a delta function on the vertices of a directed graph. If we restrict ourselves to dual mutations at the vertices where the function take positive value, then we get the G_2 hexagon.

As an example, consider the delta function δ_β. $\hat{\delta_\beta}^\alpha(\alpha) = -\delta_\beta(\alpha) = 0$ and $\hat{\delta_\beta}^\alpha(\beta) = \delta_\beta(\beta) + 3\delta_\beta(\alpha) = 1$. So, $\hat{\delta_\beta}^\alpha = \delta_\beta$.

$\hat{\delta_\beta}^\beta(\beta) = -\delta_\beta(\beta) = -1$ and $\hat{\delta_\beta}^\beta(\alpha) = \delta_\beta(\alpha) + \delta_\beta(\beta) = 1$. So, $\hat{\delta_\beta}^\beta = \delta_\alpha - \delta_\beta$.

If we denote $n\delta_\alpha + m\delta_\beta$ as (n, m), then we have the diagram of positive dual mutations

$$(0,1) \xrightarrow{\hat{\beta}} (1,-1) \xrightarrow{\hat{\alpha}} (-1,2) \xrightarrow{\hat{\beta}} (1,-2) \xrightarrow{\hat{\alpha}} (-1,1) \xrightarrow{\hat{\beta}} (0,-1)$$

Then we get the change sequence $\beta\alpha\beta^2\alpha\beta$ or we write $\beta\alpha\beta\beta\alpha\beta$.

All possible sequences are the vertices of the G_2 hexagon. In other words, they are ideals of the multiset $P = \{\alpha, \alpha, \beta, \beta, \beta, \beta\}$.

Multiset is a set where each element can be seen more than once. The number of occurrences of an element will be called the multiplicity of the element.

Consider the multiset $P = \{\alpha, \alpha, \beta, \beta, \beta, \beta\}$ with the linear order by $\beta < \alpha < \beta < \beta < \alpha < \beta$. Now, P is a partially ordered multiset. Recall that $I \subset P$ is called an **ideal** if $x \in I$ and $y \leq x$ implies that $y \in I$. The ideals of P are then $\emptyset, \beta, \beta\alpha, \beta\alpha\beta, \beta\alpha\beta\beta, \beta\alpha\beta\beta\alpha, \beta\alpha\beta\beta\alpha\beta$ which will turn out to be all the vertices of the G_2 hexagon. For the ideals u and v construct an edge connecting u and v of direction γ from v to u if $\sum_{x \in u} x - \sum_{x \in v} x = \gamma$. Now we need to define the weight of this directed edge.

A subset C of a partially ordered set is called **convex** if $a, b \in C$ with $a \leq z \leq b$ implies that $z \in C$.

Any convex subset of P is called a **layer**. For a positive root γ, layer L is called γ-**layer** if $\sum_{v \in L} v = \gamma$. The set of γ-layers is denoted as $\mathcal{L}_\gamma$. It is ordered by the rule $L_1 \leq L_2$ if $L_1 \subset I(L_2)$, where $I(S)$ is the ideal generated by $S \subset P$.

Example 5.2.1. Consider $\mathcal{L}_{\alpha+2\beta}$, the elements are $L_0 = \beta\alpha\beta$, $L_1 = \alpha\beta\beta$, $L_2 = \beta\beta\alpha$, $L_3 = \beta\alpha\beta$. $I(L_0) = L_0$, $I(L_1) = \beta\alpha\beta\beta$, $I(L_2) = \beta\alpha\beta\beta\alpha$, $I(L_3) = \beta\alpha\beta\beta\alpha\beta$

This implies that $L_0 \leq L_1 \leq L_2 \leq L_3$.

The minimal element is denoted as L_m and maximal element is denoted as L_M.

Next we define the parity and dual parity of the elements of $\mathcal{L}_{\alpha+2\beta}$ as $\epsilon(L) = (-1)^n$ where n is the number of α, β interchanges to pass from L to L_m.

The dual parity is defined as $\hat{\epsilon}(L) = (-1)^n$ where n is the number of α, β interchanges to pass from L to L_M.

Alternatively, there is only one order preserving map between two γ-layer. This map is a permutation.

Define σ_L as the unique order preserving permutation between L and L_m.

$\hat{\sigma}_L$ as the unique order preserving permutation between L and L_M.

Then it is clear that $\epsilon(L) = sgn(\sigma_L)$ and $\hat{\epsilon}(L) = sgn(\hat{\sigma}_L)$.

5.2.2 A ladder of γ-layers

A ladder of γ-layers of size k is defined to be a sequence of disjoint γ-layers $L_1, L_2, ..., L_k$ such that $L_1 \cup L_2 \cup \cdots \cup L_k$ is again a layer.

Define the functions

$s(L) =$maximum size of a ladder starting at L,

$f(L) =$maximum size of a ladder finishing at L.

Now we can define the weight and dual weight of a γ-layer L as

$w(L) = \epsilon(L)s(L)$, $\hat{w}(L) = \hat{\epsilon}(L)f(L)$.

5.2.3 G_2 hexagon and the layers

Vertices of G_2 hexagon which are $v_\emptyset, v_\beta, v_{\beta\alpha}, v_{\beta\alpha\beta}, v_{\beta\alpha\beta\beta}, v_{\beta\alpha\beta\beta\alpha}, v_{\beta\alpha\beta\beta\alpha\beta}$ correspond to the ideals $\emptyset, \beta, \beta\alpha, \beta\alpha\beta, \beta\alpha\beta\beta, \beta\alpha\beta\beta\alpha, \beta\alpha\beta\beta\alpha\beta$ of P.

Edges of G_2 correspond to layers as follows:

the number of γ-layers and the edges in the direction of γ are equal.

The edge from v_I to v_J where $I \subset J$ is associated to the layer $J - I$.

For example Consider $\mathcal{L}_{\alpha+2\beta}$.

The edge from $v_\emptyset$ to $v_{\beta\alpha\beta}$ corresponds to the layer $L_0 = \beta\alpha\beta$.

The edge from $v_{\beta\alpha\beta}$ to $v_{\beta\alpha\beta\beta\alpha\beta}$ corresponds to the layer $L_3 = \beta\alpha\beta$.

In the G_2 hexagon the weight of the edge from $v_\emptyset$ to $v_{\beta\alpha\beta}$ is 2 and the corresponding layer L_0 has weight $w(L_0) = \epsilon(L)s(L) = 2$. So, there is also correspondence between the weights. Moreover, the weights of the edges from v_J to v_I where $I \subset J$ is the same with the dual weight of the layer $J - I$.

As an example the edge from $v_{\beta\alpha\beta}$ to $v_\emptyset$ has the weight 1 and $\hat{w}(L_0) = 1$.

5.2.4 The 7-dimensional Representation of $\mathfrak{g}_2$

We obtained the G_2 hexagon and the root system of G_2 by the mutation and numbers games. Now, using the set of positive roots $\{\alpha, \beta, \alpha+\beta, \alpha+2\beta, \alpha+3\beta, 2\alpha+3\beta\}$, we will define linear operators X_γ, Y_γ on a complex vector space V spanned by the vertices of the G_2 hexagon. Then, brackets of these operators will span the Lie algebra $\mathfrak{g}_2$. For this construction we will not need the Lie theory, but to verify that this is the Lie algebra $\mathfrak{g}_2$ we need to remember the Cartan decomposition of $\mathfrak{g}_2$.

The complex vector space V is spanned by $\{v_I\}$ where I is an ideal of P. Define operators X_L, Y_L on V by

$X_L(v_I) = v_{I\cup L}$ if $I \cup L$ is an ideal and $I \cap L = \emptyset$, 0 otherwise.

$Y_L(v_I) = v_{I-L}$ if $L \subset I$ and $I - L$ is an ideal, 0 otherwise.

For a positive root γ define X_γ, Y_γ and H_γ on V by

$$X_\gamma = \sum_{L\in\mathcal{L}_\gamma} w(L)X_L$$

$$Y_\gamma = \sum_{L \in \mathcal{L}_\gamma} \hat{w}(L) Y_L$$

$$H_\gamma = [X_\gamma, Y_\gamma]$$

It can be observed easily that the operators X_γ and Y_γ correspond to the edges of the G_2 hexagon as follows: X_γ sends v_I to nv_J if there exists an edge from v_I to v_J of weight n in the direction of γ.

A set $\{X_\gamma\}$ of operators on a vector space is defined to be a bracket set if a bracket of two elements is a multiple of some element in the set.

If $\{X_\gamma | \gamma \in \Gamma\}$ is a bracket set, then obviously it spans a Lie algebra.

Theorem 5.2.2. $\{X_\gamma, Y_\gamma, H_\gamma | \gamma \in R^+\}$ is a bracket set spanning a Lie algebra $\mathfrak{g} \cong \mathfrak{g}_2$.

Proof. To check all the commutation relations we first consider the positive root triples $\{\gamma, \gamma', \gamma''\}$ where $\gamma + \gamma' = \gamma''$. There are exactly 6 root triples, one of which is $\{\alpha, \beta, \alpha + \beta\}$. Every positive root triple $\{\gamma, \gamma', \gamma''\}$ leads to 6 commutation relations namely, $[X_\gamma, X_{\gamma'}]$, $[X_{\gamma+\gamma'}, Y_{\gamma'}]$, $[X_{\gamma+\gamma'}, Y_\gamma]$, $[Y_\gamma, Y_{\gamma'}]$, $[Y_{\gamma+\gamma'}, Y_{\gamma'}]$, $[Y_{\gamma+\gamma'}, Y_\gamma]$. Checking last 3 of them on the G_2 hexagon corresponds to checking the first 3 of them because the weights and edges of the G_2 hexagon is symmetric with respect to its center. So , we only need to check the first 3 relations. Also, this positive root triple determines a several triangles in the G_2 hexagon with sides having directions $\gamma, \gamma', \gamma''$. To check any one of these three commutation relations, it is enough to check the consistency of the weights on each triangles determined by this root triple.

As an example consider the root triple $\{\alpha, \beta, \alpha + \beta\}$. The 6 relations are $[X_\alpha, X_\beta]$, $[X_{\alpha+\beta}, Y_\beta]$, $[X_{\alpha+\beta}, Y_\alpha]$, $[Y_\alpha, Y_\beta]$, $[Y_{\alpha+\beta}, Y_\beta]$, $[Y_{\alpha+\beta}, Y_\alpha]$. Enough to check $[X_\alpha, X_\beta]$, $[X_{\alpha+\beta}, Y_\beta]$, $[X_{\alpha+\beta}, Y_\alpha]$.

Next, we check the pairs of roots which are at right angles. All of these result in 0.

Lastly, we check the relations $[H_\gamma, X_{\gamma'}]$ and $[H_\gamma, Y_{\gamma'}]$.

$H_\gamma = [X_\gamma, Y_\gamma]$ is a scalar operator. So, $[H_\gamma, X_\gamma]$ and $[H_\gamma, Y_\gamma]$ are multiples of X_γ

and Y_γ. $[X_{\gamma'}, H_\gamma] = -[X_\gamma, [Y_\gamma, X_{\gamma'}]] - [Y_\gamma, [X_{\gamma'}, X_\gamma]]$ is a multiple of $X_{\gamma'}$. Similar argument holds for Y operators. This concludes that the set $\{X_\gamma, Y_\gamma, H_\gamma | \ \gamma \in R^+\}$ is a bracket set.

We will show that any H_γ is a linear combination of H_α and H_β.

The matrix representations of H_α and H_β are the diagonal matrices :

$$[H_\alpha]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad [H_\beta]_{\mathcal{B}} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Observe that last three diagonal entries are just the negatives of the first three and the fouth diagonal entry is 0. Also, the first entry is just the sum of the second and third entries. This motivates us to show that this is true for any H_γ.

$H_\gamma(v_{\beta\alpha\beta}) = 0$. Now, if we take a symmetry of any vertex with respect to the center the image will change sign since the weights are the same but the directions are reversed. So, $H_\gamma(v_\emptyset) = -H_\gamma(v_{\beta\alpha\beta\beta\alpha\beta})$, $H_\gamma(v_\beta) = -H_\gamma(v_{\beta\alpha\beta\beta\alpha})$ and $H_\gamma(v_{\beta\alpha}) = -H_\gamma(v_{\beta\alpha\beta\beta})$. Then, H_γ is determined by the values $H_\gamma(v_\emptyset)$, $H_\gamma(v_\beta)$, $H_\gamma(v_{\beta\alpha})$. Moreover, $H_\gamma(v_\emptyset) = H_\gamma(v_\beta) + H_\gamma(v_{\beta\alpha})$. This can be observed simply by multiplying the weights in the directions γ and $-\gamma$. Therefore, H_γ is determined by the values $H_\gamma(v_\beta)$, $H_\gamma(v_{\beta\alpha})$ and the matrix representation satisfies the conditions mentioned above. H_α and H_β are linearly independent and hence any H_γ is a linear combination of them. Note that $[H_\gamma, H_{\gamma'}] = 0$ since H_γ is a scalar operator for any $\gamma \in R^+$.

Therefore, the span of the bracket set is a 14 dimensional Lie algebra, say $\mathfrak{g}$. Then $h = span\{H_\alpha, H_\beta\}$ is a Cartan subalgebra of $\mathfrak{g}$. Recall that a Cartan subalgebra is a nilpotent subalgebra which is self-normalizing. Nilpotency is obvious since h is abelian. It is self-normalizing since $[X_\gamma, H]$ and $[Y_\gamma, H]$ are multiples of X_γ and Y_γ for any $H \in h$. Moreover, the roots of $\mathfrak{g}$ with respect to h are the same with the roots of $\mathfrak{g}_2$.

H_γ corresponds to the data $f_\gamma = (H_\gamma(v_\beta), H_\gamma(v_{\beta\alpha}))$. We assume that the G_2 hexagon and the root system of G_2 share a common origin. We consider f_γ as a linear functional on this plane and we assume that $H_\gamma(v_\beta) = f_\gamma(-\alpha - \beta)$, $H_\gamma(v_{\beta\alpha}) = f_\gamma(-\beta)$.

For example $f_\alpha = (-1, 1)$ and $f_\beta = (1, -2)$. This ensures that $[H_\alpha, X_\gamma] = f_\alpha(\gamma)X_\gamma$ and $[H_\alpha, Y_\gamma] = -f_\alpha(\gamma)Y_\gamma$. Define a functional $\bar{\gamma}$ on h as $\bar{\gamma}(H_\alpha) = f_\alpha(\gamma)$ and $\bar{\gamma}(H_\beta) = f_\beta(\gamma)$. Then $X_\gamma \in \mathfrak{g}_{\bar{\gamma}}$. So, the roots of $\mathfrak{g}$ are the same with the roots of $\mathfrak{g}_2$. This implies that $\mathfrak{g} \cong \mathfrak{g}_2$. □

This completes the construction of $\mathfrak{g}_2$.

REFERENCES

[1] I. Agricola, “Old and new on the exceptional group g_2,” *Notices of the American Mathematical Society*, pp. 922–929, 2008.

[2] T. Hawkins, *Emergence of the theory of Lie groups: An essay in the history of mathematics 1869–1926*. Springer Science & Business Media, 2012.

[3] N. Wildberger, “A combinatorial construction of g_2,” *Journal of Lie Theory*, vol. 13, no. 1, pp. 155–165, 2003.

[4] J. Stillwell, *Naive Lie theory*. Springer Science & Business Media, 2008.

[5] R. W. Carter, R. W. Carter, G. B. Segal, and I. G. MacDonald, *Lectures on Lie groups and Lie algebras*, vol. 32. Cambridge University Press, 1995.

[6] T. S. Blyth and E. F. Robertson, *Further linear algebra*. Springer Science & Business Media, 2013.

[7] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, vol. 9. Springer Science & Business Media, 2012.

[8] W. Fulton and J. Harris, *Representation theory: a first course*, vol. 129. Springer Science & Business Media, 2013.

[9] B. Hall, *Graduate Texts in Mathematics: Lie Groups, Lie Algebras and Representations (v. 222)*. Springer, 2003.

[10] A. Kirillov Jr, *An introduction to Lie groups and Lie algebras*, vol. 113. Cambridge University Press, 2008.