KILLING FAMILY OF TENSORS AND HIDDEN SYMMETRIES

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

OKAN GÜNEL

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
PHYSICS

AUGUST 2022

Approval of the thesis:

KILLING FAMILY OF TENSORS AND HIDDEN SYMMETRIES

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ABSTRACT

KILLING FAMILY OF TENSORS AND HIDDEN SYMMETRIES

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August 2022, 69 pages

In this thesis, the symmetries of the spacetimes and symmetry related Killing family of tensors are investigated. The relations between the members of the Killing family of tensors are given. Conserved gravitational charges for the spacetimes are discussed. We calculated conserved charges for Myers-Perry spacetimes using the conserved Kastor-Traschen [1] current obtainable from Killing-Yano tensors in 4, 5, 6, 7, and 8 dimensions.

We also studied a Kerr-like metric for rotating spacetimes. We showed that they share a common (conformal) Killing-Yano tensor and that the Hamilton-Jacobi equation separates. We found Carter's fourth constant of motion [2] using the separability and using Killing tensors obtainable from the Killing-Yano tensor.

Keywords: Symmetries, Killing-Yano, Hamilton-Jacobi, conserved charges, separability

ÖZ

KILLING TENSÖR AİLESİ VE SAKLI SİMETRİLER

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Ağustos 2022 , 69 sayfa

Bu tezde, uzayzaman simetrileri ve bu simetrilerle bağdaştırılabilen Killing tensör ailesi incelenmiştir. Killing tensör ailesi bileşenleri arasındaki ilişkiler verilmiştir. Uzayzamanlarda korunan yükler tartışılmıştır. 4, 5, 6, 7 ve 8 boyutlu Myers-Perry uzayzamanlarında, Killing-Yano tensörlerden oluşturulan Kastor-Traschen [1] akımı kullanılarak korunan yükler bulunmuştur.

Ayrıca, dönme parametresine sahip Kerr benzeri uzayzamanlar için genel bir metrik çalışılmıştır. Bu metriklerin ortak bir (açıkorur) Killing-Yano tensörü olduğu ve Hamilton-Jacobi denkleminin ayrılabilir olduğu gösterilmiştir. Ayrılabilirlik ve Killing-Yano tensöründen elde edilebilecek olan Killing tensör kullanılarak, hareketin Carter sabiti [2] bulunmuştur.

Anahtar Kelimeler: Simetri, Killing-Yano, Hamilton-Jacobi, korunan yükler, ayrılabilirlik

Dedicated to my father.

ACKNOWLEDGMENTS

I am delighted to have the opportunity to thank some of the special people. I would like to express my appreciation to them.

First and foremost, I would like to express my deep gratitude to my supervisor Prof. Dr. B. Özgür Sarıoğlu, for his patient guidance, enthusiastic encouragement, and valuable critiques of this research work. I am grateful for his willingness to give time and complete support to me. Thank you so much.

I would also like to thank the examining committee members and Prof. Dr. Ulf Lindström for their valuable feedback.

Special thanks should be given to my amazing wife, Nilay Ayşe Günel, who inspired me in this journey. I am thankful for her endless support and help. I wish to express my deepest thanks to my mother and sister. They are very special to me.

Finally, I am thankful to my dear friends Önder Gezer, Ali Nuri Güneşdoğdu, Görkem Günay, Emre Ersoy, and Emre Cemal Gönen for their encouragement and help.

I acknowledge the financial support of the Scientific and Technological Research Council of Turkey (TÜBİTAK) through 2210-A National MSc Scholarship Program.

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LIST OF ABBREVIATIONS

KT	Kastor-Traschen
AD	Abbott-Deser
BL	Boyer-Lindquist
KS	Kerr-Schild
CKYT	Conformal Killing-Yano tensor
CCKYT	Closed conformal Killing-Yano tensor
PCKYT	Principle closed conformal Killing-Yano tensor

CHAPTER 1

INTRODUCTION

In physics, one tries to determine and analyze the motion of a single or a group of particles. These motions are determined through a set of differential equations. The symmetries play an important role while solving the differential equations. In general relativity, orbits of the particles are found by solving second-order nonlinear differential equations called geodesic equations. One can reduce the number of differential equations using symmetries or symmetry generators.

Some symmetries may be obvious and easy to see. However, there may be some hidden symmetries. The Killing family of tensors is useful to reveal those symmetries. A Killing vector is a symmetry generator of the spacetime. Killing vectors give first integrals of motion that are conserved along the geodesics. They can be generalized to higher rank symmetric or antisymmetric tensors, Killing and Killing-Yano tensors, respectively. Killing-Yano tensors can be used to get Killing tensors, and later, higher-order conserved quantities can be found by using Killing tensors. As Carter showed [2] for the Kerr metric, higher-order conserved quantities can be found from the separability of the Hamilton-Jacobi equation. In the following years, it was revealed that the existence of a Killing tensor for the Kerr metric led to the separability [3]. It was generalized to the charged particle case with the Kerr-Newman metric later by Hughston et al. [4]. In 1976, it was shown that the Dirac's equation in Kerr [5] and Kerr-Newman [6] spacetimes is also separable. Later, it was revealed that the Killing-Yano tensor is the reason behind the separability of Dirac's equation [7].

Conformal Killing vectors are the generators of the conformal transformations. Similar to the Killing vectors, conformal Killing vectors can be used to give the first integrals of motion for massless particles. Their generalization to higher ranks are called conformal Killing and conformal Killing-Yano tensors (CKYT). There is a duality between the closed conformal Killing-Yano tensors (CCKYT) and Killing-Yano tensors [8]. Using this duality, one can start from a CCKYT and construct different ranks of CCKYTs, Killing-Yano tensors, and Killing tensors. An extensive review of the relations within the Killing family of tensors, and hidden symmetries can be found in [9].

In addition to the first integrals of motion, the Killing family of tensors can be used for determining conserved quantities of the spacetime, e.g., total mass or angular momentum. In order to find exact conserved gravitational charges, one needs to find a covariantly closed current and integrate it over the whole space. Furthermore, one can find asymptotic conserved gravitational charges when the metric is divided into a background and a deviation that vanishes at the spatial infinity, using the Abbott-Deser (AD) construction [10]. In [10], a Killing vector is used with the energy-momentum tensor to construct a background covariantly conserved current. Kastor and Traschen [1] showed that similar to the AD construction, arbitrary rank Killing-Yano tensors also can be used to construct such currents to find background charges. In [11, 12], new conserved currents using the Killing-Yano tensors and CKYTs are given; however, these currents give exact charges instead of asymptotic charges.

The outline of the thesis is as follows. In the second chapter, we will give the definitions of Killing family of tensors with their properties and the relations between them. In the third chapter, we will show how Stokes' theorem can be used to get conserved charges by using covariantly conserved currents of arbitrary rank. In the fourth chapter, we will explicitly give Killing-Yano tensors of Myers-Perry blackholes in 4, 5, 6, 7, and 8 dimensions. We will also calculate conserved charges using these Killing-Yano tensors and KT currents. In the fifth chapter, we will work on a general Kerr-like metric and give (conformal) Killing and (conformal) Killing-Yano tensors. We will discuss the separability of the Hamilton-Jacobi equation and its relation with the Killing tensors. A discussion will be given in the final chapter.

1.1 Preliminaries

In this thesis, we work on spacetimes which are manifolds of finite dimension with metric compatible affine connections. Let g be the metric of the spacetime with the signature $(-,+,\ldots,+)$. It satisfies

$$\nabla_\mu g_{\nu\rho} = 0 \ , \tag{1.1}$$

for a metric compatible affine connection ∇. Throughout the thesis, torsion is assumed to be zero and connection coefficients are the symmetric Levi-Civita connections

$$\Gamma^\mu_{\ \nu\rho} = \frac{1}{2} g^{\mu\sigma}\left(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}\right) \ . \tag{1.2}$$

We will work in natural units with $G = c = \hbar = 1$. We will use square brackets and parantheses on indices to antisymmetrize and symmetrize them, respectively. Let T be an arbitrary tensor of rank-2. Then, the antisymmetrization and symmetrization of T can be done as

$$T_{[\mu\nu]} = \frac{1}{2!}\left(T_{\mu\nu} - T_{\nu\mu}\right) \ , \quad T_{(\mu\nu)} = \frac{1}{2!}\left(T_{\mu\nu} + T_{\nu\mu}\right) \ . \tag{1.3}$$

Following the notation of [13], a differential n-form ω in a D-dimensional $(D \geq n)$ manifold $\mathcal{M}$ is a totally antisymmetric tensor of type $(0, n)$ defined as

$$\omega = \frac{1}{n!}\omega_{\mu_1\ldots\mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} \ , \tag{1.4}$$

where x's are the coordinates on the manifold. The vector space of n-forms at a point p on the manifold $\mathcal{M}$ is denoted as $\Omega^n_p(\mathcal{M})$. The exterior derivative of n-form ω is a map $\Omega^n_p(\mathcal{M}) \to \Omega^{(n+1)}_p(\mathcal{M})$ defined as

$$d\omega = \frac{1}{n!}\partial_\nu\left(\omega_{\mu_1\ldots\mu_n}\right) dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} \ . \tag{1.5}$$

There is an isomorphism between $\Omega^n_p(\mathcal{M})$ and $\Omega^{(D-n)}_p(\mathcal{M})$ in a D-dimensional manifold defined by the Hodge star (dual) operation. Define the totally antisymmetric tensor ϵ by

$$\epsilon_{\mu_1\ldots\mu_D} = \begin{cases} +1, & \text{if } (\mu_1\mu_2\ldots\mu_m) \text{ is an even permutation of } (12\ldots m) \\ -1, & \text{if } (\mu_1\mu_2\ldots\mu_m) \text{ is an odd permutation of } (12\ldots m) \\ 0, & \text{otherwise .} \end{cases} \tag{1.6}$$

Hodge star of an n-form ω is

$$*\omega = \frac{\sqrt{|g|}}{n!(D-n)!}\omega_{\mu_1 \dots \mu_n} g^{\mu_1 \nu_1} \dots g^{\mu_n \nu_n} \epsilon_{\nu_1 \dots \nu_D} dx^{\nu_{n+1}} \wedge \dots \wedge dx^{\nu_D} \,, \tag{1.7}$$

where g is the determinant of the metric $g_{\mu\nu}$.

CHAPTER 2

KILLING FAMILY OF TENSORS

2.1 Killing Vectors

Let there be an infinitesimal transformation of coordinates

$$x'^{\mu} = x^{\mu} + \varepsilon\xi^{\mu}(x) \quad \text{where } |\epsilon| \ll 1 \ . \tag{2.1}$$

If the metric $g_{\mu\nu}$ is form invariant under this transformation, then it should satisfy

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x) \ . \tag{2.2}$$

Any transformation satisfying the form invariance of the metric is called an isometry. A form invariant metric is transformed under a coordinate transformation as follows:

$$g_{\mu\nu}(x) = \frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} g_{\rho\sigma}(x') \ . \tag{2.3}$$

Expanding (2.3) to the first order in ε, we get

$$\frac{\partial\xi^{\mu}(x)}{\partial x^{\rho}} g_{\mu\sigma}(x) + \frac{\partial\xi^{\nu}(x)}{\partial x^{\sigma}} g_{\rho\nu}(x) + \xi^{\mu}(x)\frac{\partial g_{\rho\sigma}(x)}{\partial x^{\mu}} = 0 \ . \tag{2.4}$$

This can be written in terms of $\xi_{\mu} = g_{\mu\nu}\xi^{\nu}$

$$\frac{\partial\xi_{\sigma}}{\partial x^{\rho}} + \frac{\partial\xi_{\rho}}{\partial x^{\sigma}} + \xi^{\mu}\left(\frac{\partial g_{\rho\sigma}}{\partial x^{\mu}} - \frac{\partial g_{\mu\sigma}}{\partial x^{\rho}} - \frac{\partial g_{\rho\mu}}{\partial x^{\sigma}}\right) = 0 \ , \tag{2.5}$$

$$\frac{\partial\xi_{\sigma}}{\partial x^{\rho}} + \frac{\partial\xi_{\rho}}{\partial x^{\sigma}} - 2\xi_{\mu}\Gamma^{\mu}{}_{\rho\sigma} = 0 \ . \tag{2.6}$$

In terms of covariant derivatives, the form invariance condition becomes

$$\nabla_{\rho}\xi_{\sigma} + \nabla_{\sigma}\xi_{\rho} = 0 \quad \text{or} \quad \nabla_{(\rho}\xi_{\sigma)} = 0 \ . \tag{2.7}$$

This equation is known as the Killing equation and any vector satisfying the equation is called a Killing vector [13]. Using the definition of the Lie derivative, the Killing equation can be written as

$$(\mathcal{L}_\xi g)_{\mu\nu} = 0\ . \tag{2.8}$$

This means that the local geometry does not vary as we move along the Killing vector ξ. Thus, the direction of the symmetry of a manifold is given by the Killing vectors. In order to find all the isometries of the metric, solving the Killing equation is enough. For the Minkowski metric, the symmetries are translations in space, time, and Lorentz transformations. However, finding all the Killing vectors for different metrics may be more complicated.

A linear combination of the Killing vectors also satisfies the Killing vector equation (2.7) and this is true for the whole Killing family of tensors. Let $\xi^{(i)}$ and $\xi^{(j)}$ be two vectors that satisfy (2.7). Then, their linear combination also satisfies

$$\begin{aligned} &\nabla_\rho \left(a\, \xi^{(i)}_\sigma + b\, \xi^{(j)}_\sigma\right) + \nabla_\sigma \left(a\, \xi^{(i)}_\rho + b\, \xi^{(j)}_\rho\right) = \\ &\qquad a\left(\nabla_\rho \xi^{(i)}_\sigma + \nabla_\sigma \xi^{(i)}_\rho\right) + b\left(\nabla_\rho \xi^{(j)}_\sigma + \nabla_\rho \xi^{(j)}_\rho\right) = 0\ . \end{aligned} \tag{2.9}$$

where a and b are arbitrary constants. Additionally, the commutator of two Killing vectors is also a Killing vector, but it is a linearly independent Killing vector. Let us use (2.8) with the commutator of two Killing vectors

$$\begin{aligned} \mathcal{L}_{[\xi^{(i)},\xi^{(j)}]} g =& \mathcal{L}_{\xi^{(i)}} \left(\mathcal{L}_{\xi^{(j)}} g\right) - \mathcal{L}_{\xi^{(j)}} \left(\mathcal{L}_{\xi^{(i)}} g\right) \\ =& 0\ . \end{aligned} \tag{2.10}$$

Thus, the commutator $\left[\xi^{(i)}, \xi^{(j)}\right]$ is also a Killing vector and the Killing vectors form a Lie algebra

$$\left[\xi^{(i)}, \xi^{(j)}\right] = \xi^{(k)}\ . \tag{2.11}$$

Killing vectors are related to the Riemann curvature tensor [14]

$$\nabla_\mu \nabla_\nu \xi^\rho = R^\rho{}_{\nu\mu\sigma} \xi^\sigma\ . \tag{2.12}$$

Consequently, a Killing vector ξ^μ is completely determined by the values of ξ^μ and its first covariant derivative, $\nabla_\nu \xi_\mu$, at any point on the manifold.

Killing vectors are conserved along geodesics. Let γ be a geodesic which is tangent to u^ν. Then, $\xi_\mu u^\mu$ is constant along the geodesic

$$u^\mu \nabla_\mu (\xi_\nu u^\nu) = u^\mu u^\nu \nabla_\mu \xi_\nu + u^\mu \xi_\nu \nabla_\mu u^\nu \ . \quad (2.13)$$

The second term vanishes when the geodesic equation $u^\mu \nabla_\mu u^\nu = 0$ is used

$$\begin{aligned} u^\mu \nabla_\mu (\xi_\nu u^\nu) &= u^\mu u^\nu \nabla_\mu \xi_\nu \\ &= u^\mu u^\nu \nabla_{(\mu} \xi_{\nu)} \ . \end{aligned} \quad (2.14)$$

Hence, a Killing vector ξ^ν satisfies

$$u^\mu \nabla_\mu (\xi_\nu u^\nu) = 0 \ . \quad (2.15)$$

For a test particle with mass m, $p^\nu = u^\nu m$ can be used to find a first integral of motion in terms of the momentum vector. In this fashion, every Killing vector is related to the existence of a conserved quantity along the geodesics [15].

2.1.1 Conformal Killing Vectors

Killing vectors can be extended to conformal Killing vectors. First, let us define conformal transformations. Conformal transformations are the transformations that change the metric such that

$$g'_{\mu\nu}(x') = e^{\varepsilon\alpha} g_{\mu\nu}(x) \ , \quad (2.16)$$

under an infinitesimal displacement $x'^\mu = x^\mu + \varepsilon \psi^\mu(x)$ where $\varepsilon \ll 1$. It can be written as

$$\frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} g_{\rho\sigma}(x') = e^{\varepsilon\alpha} g_{\mu\nu}(x) \ . \quad (2.17)$$

Expanding (2.17) to the first order in ε, we get

$$\frac{\partial \psi^\mu(x)}{\partial x^\rho} g_{\mu\sigma}(x) + \frac{\partial \psi^\nu(x)}{\partial x^\sigma} g_{\rho\nu}(x) + \psi^\mu(x) \frac{\partial g_{\rho\sigma}(x)}{\partial x^\mu} = \alpha g_{\rho\sigma}(x) \ . \quad (2.18)$$

We find α by multiplying (2.18) with the inverse metric

$$\alpha = \frac{2}{D} \frac{\partial \psi^\mu}{\partial x^\mu} + \frac{1}{D} \psi^\mu g^{\rho\sigma} \frac{\partial g_{\rho\sigma}}{\partial x^\mu} \ , \quad (2.19)$$

where D is the dimension of the manifold. In terms of the Lie derivatives, (2.18) becomes

$$\left(\mathcal{L}_\psi g\right)_{\mu\nu} = \alpha g_{\mu\nu} \; . \tag{2.20}$$

Using the covariant derivatives, we can write (2.20) as

$$\nabla_\mu \psi_\nu + \nabla_\nu \psi_\mu = \frac{2}{D} g_{\mu\nu} \nabla^\rho \psi_\rho \; . \tag{2.21}$$

This equation is called the conformal Killing equation and any vector satisfying the equation is called a conformal Killing vector [13]. The conformal Killing vectors are the infinitesimal generators of conformal transformations (2.16). Note that when $\alpha = 0$, the isometry generators are Killing vectors.

Conformal transformations change the scales of the objects on the manifold but preserves the shapes. Let us have two vectors ζ^μ and ϕ^μ. Then, we can define the angle θ between them as

$$\cos\theta = \frac{g_{\mu\nu}\zeta^\mu \phi^\mu}{\sqrt{g_{\rho\sigma}\zeta^\rho\zeta^\sigma g_{\tau\kappa}\phi^\tau\phi^\kappa}} \; . \tag{2.22}$$

The angle is invariant under the conformal transformations

$$\begin{aligned} \cos\theta' =& \frac{e^{\varepsilon\alpha} g_{\mu\nu}\zeta^\mu \phi^\mu}{\sqrt{e^{\varepsilon\alpha} g_{\rho\sigma}\zeta^\rho\zeta^\sigma e^{\varepsilon\alpha} g_{\tau\kappa}\phi^\tau\phi^\kappa}} \\ =& \cos\theta \; . \end{aligned} \tag{2.23}$$

Thus, the conformal transformations do not change the angles between vectors.

Similar to the Killing vectors, a linear combination of conformal Killing vectors is also a conformal Killing vector

$$\nabla_\mu \left(a\, \psi^{(i)}_\nu + b\, \psi^{(j)}_\nu\right) + \nabla_\nu \left(a\, \psi^{(i)}_\mu + b\, \psi^{(j)}_\mu\right) = \frac{2}{D} g_{\mu\nu} \nabla^\rho \left(a\, \psi^{(i)}_\rho + b\, \psi^{(j)}_\rho\right) \; , \tag{2.24}$$

where a and b are arbitrary constants. Rearranging this, we see that

$$a\left(\nabla_\mu \psi^{(i)}_\nu + \nabla_\nu \psi^{(i)}_\mu - \frac{2}{D} g_{\mu\nu}\nabla^\rho \psi^{(i)}_\rho\right) + b\left(\nabla_\mu \psi^{(j)}_\nu + \nabla_\nu \psi^{(j)}_\mu - \frac{2}{D} g_{\mu\nu}\nabla^\rho \psi^{(j)}_\rho\right) = 0 \; . \tag{2.25}$$

Hence, $a\,\psi^{(i)} + b\,\psi^{(j)}$ is also a conformal Killing vector if $\psi^{(i)}$ and $\psi^{(j)}$ are conformal Killing vectors.

Unlike Killing vectors, conformal Killing vectors are not conserved along all geodesics. Equation (2.14) for a conformal Killing vector is

$$\begin{aligned} u^\mu \nabla_\mu \left(\psi_\nu u^\nu\right) =& u^\mu u^\nu \nabla_{(\mu}\psi_{\nu)} \\ =& u^\mu u^\nu g_{\mu\nu} \frac{2}{D} \nabla^\rho \psi_\rho \ . \end{aligned} \tag{2.26}$$

ψ_ν is conserved along the geodesics only if $u^\mu u^\nu g_{\mu\nu} = 0$, which means the conformal Killing vectors are conserved only along null geodesics. Thus, they give rise to conserved quantities for massless particles.

2.2 Killing Tensors

Killing tensors are symmetric generalizations of Killing vectors to higher rank. Let $K_{\mu_1 \dots \mu_n}$ be a rank-n totally symmetric tensor

$$K_{(\mu_1 \dots \mu_n)} = K_{\mu_1 \dots \mu_n} \ . \tag{2.27}$$

K is a Killing tensor if it satisfies

$$\nabla_{(\nu} K_{\mu_1 \dots \mu_n)} = 0 \ . \tag{2.28}$$

Note that the metric $g_{\mu\nu}$ itself is a rank-2 Killing tensor because of the metric compatibility. Killing tensors give rise to the conserved quantities like the Killing vectors. Let u^ν be the tangent to a geodesic γ and satisfy $u^\mu \nabla_\mu u^\nu = 0$. Then, $K_{\mu_1 \dots \mu_n} u^{\mu_1} \dots u^{\mu_n}$ is a conserved quantity along the geodesic γ since it satisfies

$$u^\nu \nabla_\nu \left(K_{\mu_1 \dots \mu_n} u^{\mu_1} \dots u^{\mu_n}\right) = 0 \ . \tag{2.29}$$

Killing tensors can be obtained by symmetric products of Killing vectors. However, such Killing tensors do not give rise to new conserved quantities aside from those obtained by Killing vectors. Those Killing tensors will be called *reducible* Killing tensors. In 4-dimensions, Kerr metric and in higher dimensions, Myers-Perry metrics admit nontrivial Killing tensors.

Nontrivial Killing tensors play an important role in separating the Hamilton-Jacobi equations and solving equations of motion for particles. Carter [2] showed that there is a fourth constant of motion in Kerr spacetime which can be obtained from the Killing tensor and demonstrated the separability of Hamilton-Jacobi equations. There is a strong connection between the separability and the existence of Killing tensors [3]. However, even if a spacetime admits a Killing tensor, that does not mean that the Hamilton-Jacobi equation separates. Chervonyi and Lunin [16] showed that this happens only if one uses a special set of coordinates.

2.2.1 Conformal Killing Tensors

Conformal Killing vectors can be extended to higher ranks and these tensors are called conformal Killing tensors. Let $Q_{\mu_1 \ldots \mu_n}$ be a rank-n totally symmetric tensor. Q satisfies the conformal Killing tensor equation which is a generalized version of the Killing tensor equation (2.28) [3]

$$\nabla_{(\nu} Q_{\mu_1 \ldots \mu_n)} = n g_{(\nu\mu_1} \bar{Q}_{\mu_2 \ldots \mu_n)} \ , \tag{2.30}$$

where $\bar{Q}$ can be found by tracing both sides. When $\bar{Q}$ vanishes, equation (2.30) reduces to the Killing tensor equation (2.28). A rank-2 conformal Killing tensor satisfies

$$\nabla_{(\rho} Q_{\mu\nu)} = 2 g_{(\rho\mu} \bar{Q}_{\nu)} \ . \tag{2.31}$$

Multiplying both sides with $g^{\rho\mu}$, we get

$$g^{\rho\mu} \left(\nabla_\rho Q_{\mu\nu} + \nabla_\mu Q_{\rho\nu} + \nabla_\nu Q_{\mu\rho} \right) = 2 \left(g^{\rho\mu} g_{\rho\mu} \bar{Q}_\nu + g^{\rho\mu} g_{\rho\nu} \bar{Q}_\mu + g^{\rho\mu} g_{\mu\nu} \bar{Q}_\rho \right) \ . \tag{2.32}$$

Since, ∇ is a metric compatible connection, we can solve it for $\bar{Q}$

$$\bar{Q}_\nu = \frac{1}{2\left(D+2\right)} \left(2\nabla_\mu Q^\mu{}_\nu + \nabla_\nu Q^\mu{}_\mu \right) \ . \tag{2.33}$$

For the higher rank conformal Killing tensors ($n > 2$), one should multiply with the inverse metric more than once to find the $\bar{Q}$.

Let u^ν be the tangent to a geodesic γ and satisfy $u^\mu \nabla_\mu u^\nu = 0$. Let us examine when $Q_{\mu_1 \dots \mu_n} u^{\mu_1} \dots u^{\mu_n}$ is a conserved quantity along the geodesic γ,

$$\begin{aligned} u^\nu \nabla_\nu \left(Q_{\mu_1 \dots \mu_n} u^{\mu_1} \dots u^{\mu_n} \right) =& u^{\mu_1} \dots u^{\mu_n} u^\nu \nabla_{(\nu} Q_{\mu_1 \dots \mu_n)} \\ =& n u^{\mu_1} \dots u^{\mu_n} u^\nu g_{(\nu\mu_1} \bar{Q}_{\mu_2 \dots \mu_n)} \\ =& n u^{(\mu_1} \dots u^{\mu_n} u^{\nu)} g_{\nu\mu_1} \bar{Q}_{\mu_2 \dots \mu_n} \ . \end{aligned} \tag{2.34}$$

For null geodesics, $u^{(\mu_1} \dots u^{\mu_n} u^{\nu)} g_{\nu\mu_1} = 0$. Therefore, conformal Killing tensors are conserved along the null geodesics and they give first integrals of motion for massless particles similar to the conformal Killing vectors.

2.3 Killing-Yano Tensors

Killing-Yano tensors are antisymmetric generalizations of Killing vectors to higher ranks. Let $f_{\mu_1 \dots \mu_n}$ be a rank-n totally antisymmetric tensor

$$f_{[\mu_1 \dots \mu_n]} = f_{\mu_1 \dots \mu_n} \ . \tag{2.35}$$

f is a Killing-Yano tensor if it satisfies [17]

$$\nabla_{(\nu} f_{\mu_1)\mu_2 \dots \mu_n} = 0 \ . \tag{2.36}$$

Equivalently, the covariant derivative of a Killing-Yano tensor is completely antisymmetric

$$\nabla_\nu f_{\mu_1 \dots \mu_n} = \nabla_{[\nu} f_{\mu_1 \dots \mu_n]} \ . \tag{2.37}$$

Killing-Yano tensors can be used for constructing Killing tensors. Let K be a product of Killing-Yano tensor f with itself

$$K^{\mu\nu} = f^{\mu\nu_2 \dots \nu_n} f^{\nu}{}_{\nu_2 \dots \nu_n} \ . \tag{2.38}$$

Let us take the covariant derivative of (2.38) and symmetrize the free indices

$$\begin{aligned} \nabla_{(\rho} K_{\mu\nu)} =& \nabla_{(\rho} \left(f_\mu{}^{\nu_2 \dots \nu_n} f_{\nu)\nu_2 \dots \nu_n} \right) \\ =& (-1)^{n-1} f^{\nu_2 \dots \nu_n}{}_{(\nu} \nabla_\rho f_{\mu)\nu_2 \dots \nu_n} + f_{(\mu}{}^{\nu_2 \dots \nu_n} \nabla_\rho f_{\nu)\nu_2 \dots \nu_n} \\ =& 0 \ , \end{aligned} \tag{2.39}$$

where in the last line, we used the definition of Killing-Yano tensors. Thus, K defined in (2.38) satisfies the Killing tensor equation (2.28), and Killing-Yano tensors can be considered as square roots of Killing tensors. Multipliying (2.36) with the inverse metric $g^{\nu\mu_1}$, and using metric compatibility, we see that Killing-Yano tensors are divergenceless

$$g^{\nu\mu_1}\nabla_{(\nu} f_{\mu_1)\mu_2\ldots\mu_n} = \nabla_\nu f^\nu{}_{\mu_2\ldots\mu_n} = 0 \; . \tag{2.40}$$

Killing-Yano tensors are related to the Riemann curvature tensor [1]

$$\nabla_\mu \nabla_\nu f_{\nu_1\ldots\nu_n} = (-1)^{n+1}\frac{(n+1)}{2} R^\rho{}_{\mu[\nu\nu_1} f_{\nu_2\ldots\nu_n]\rho} \; , \tag{2.41}$$

which is the generalised version of equation (2.12). Another useful identity is

$$f^{\mu\nu}\nabla_\mu G_{\nu\rho} = 0 \; , \tag{2.42}$$

where G is the Einstein tensor. This identity is reported in [11, 18] and also generalized to the higher ranks in [11].

2.3.1 Conformal Killing-Yano tensors

A CKYT of rank-n is a totally antisymmetric tensor on a D-dimensional manifold that satisfies

$$\nabla_\mu k_{\mu_1\ldots\mu_n} = \nabla_{[\mu} k_{\mu_1\ldots\mu_n]} + \frac{n}{D-n+1} g_{\mu[\mu_1}\nabla_{|\nu|} k^\nu{}_{\mu_2\ldots\mu_n]} \; . \tag{2.43}$$

CKYT is first introduced in [19] for rank-2 and [20] for rank-n. Equivalently, we can write

$$\nabla_{(\mu} k_{\mu_1)\ldots\mu_n} = \frac{1}{D-n+1} g_{\mu\mu_1}\nabla^\rho k_{\rho\mu_2\ldots\mu_n} - \frac{n-1}{D-n+1} g_{[\mu_2|(\mu}\nabla^\rho k_{|\rho|\mu_1)|\mu_3\ldots\mu_n]} \; . \tag{2.44}$$

The CKYT equation (2.43) is invariant under the Hodge duality. The Hodge duality takes the first term on the right hand side and puts it into the second term's form and vice versa. This means that the Hodge dual of a CKYT is also a CKYT. The proof can be found in various sources in the literature [21, 9]. Moreover, the wedge product of CKYT of rank-n with itself gives another CKYT of rank-$2n$ since $k_1 \wedge k_2$ also satisfies

(2.43) when k_1 and k_2 are CKYTs. Thus, CKYTs can be used to obtain higher rank CKYTs.

A CCKYT of rank-n is a totally antisymmetric closed tensor that satisfies the CKYT equation (2.43) and

$$\nabla_{[\mu}k_{\mu_1\dots\mu_n]} = 0 \ . \tag{2.45}$$

When the second term in the right hand side of (2.43) vanishes, the equation becomes the Killing-Yano equation. Hence, the Hodge dual of a CCKYT is a Killing-Yano tensor and vice versa [8].

As with Killing-Yano tensors, CKYTs can be used to obtain conformal Killing tensors

$$Q^{\mu\nu} = k^{\mu\nu_1\dots\nu_n}k^{\nu}{}_{\nu_1\dots\nu_n} \ . \tag{2.46}$$

Taking the covariant derivative of (2.46) and symmetrizing the indices, we get

$$\begin{aligned}\nabla_{(\sigma}Q_{\mu\nu)} =& k_{(\mu}{}^{\mu_2\dots\mu_n}\nabla_\sigma k_{\nu)\mu_2\dots\mu_n} + k_{(\nu}{}^{\mu_2\dots\mu_n}\nabla_\sigma k_{\mu)\mu_2\dots\mu_n} \\ =& 2k_{(\mu}{}^{\mu_2\dots\mu_n}\nabla_\sigma k_{\nu)\mu_2\dots\mu_n} \ .\end{aligned} \tag{2.47}$$

Let us multiply (2.44) with $k_\sigma{}^{\mu_2\dots\mu_n}$ and symmetrize the indices (σ, μ, μ_1)

$$\begin{aligned}k_{(\sigma}{}^{\mu_2\dots\mu_n}\nabla_\mu k_{\mu_1)\dots\mu_n} =& \frac{1}{D-n+1}g_{(\mu\mu_1}k_{\sigma)}{}^{\mu_2\dots\mu_n}\nabla^\rho k_{\rho\mu_2\dots\mu_n} \\ &- \frac{n-1}{D-n+1}g_{[\mu_2|(\mu}k_\sigma{}^{\mu_2\dots\mu_n}\nabla^\rho k_{|\rho|\mu_1)|\mu_3\dots\mu_n]} \ .\end{aligned} \tag{2.48}$$

Note that when we lower the index of $k_\sigma{}^{\mu_2\dots\mu_n}$ with the metric in the last term of (2.48), we get indices which should be symmetrized and antisymmetrized at the same time. Hence, the last term of (2.48) vanishes

$$k_{(\sigma}{}^{\mu_2\dots\mu_n}\nabla_\mu k_{\mu_1)\dots\mu_n} = \frac{1}{D-n+1}g_{(\mu\mu_1}k_{\sigma)}{}^{\mu_2\dots\mu_n}\nabla^\rho k_{\rho\mu_2\dots\mu_n} \ . \tag{2.49}$$

Substituting this to (2.47), we get

$$\nabla_{(\sigma}Q_{\mu\nu)} = \frac{2}{D-n+1}g_{(\mu\nu}k_{\sigma)}{}^{\mu_2\dots\mu_n}\nabla^\rho k_{\rho\mu_2\dots\mu_n} \ . \tag{2.50}$$

Thus, the conformal Killing tensor equation (2.30) is satisfied by (2.46) with the $\bar{Q}$

$$\bar{Q}_\sigma = \frac{1}{D-n+1}k_\sigma{}^{\mu_2\dots\mu_n}\nabla^\rho k_{\rho\mu_2\dots\mu_n} \ . \tag{2.51}$$

Note that if we multiply k with another CKYT, say h, the conformal Killing tensor equation (2.30) is still satisfied. In other words, two different CKYTs can also be used to obtain conformal Killing tensors.

2.3.2 Killing-Yano Tower

A principal conformal Killing-Yano tensor (PCKYT) is a nondegenerate rank-2 totally antisymmetric CCKYT

$$\nabla_\mu k_{\mu_1\mu_2} = \frac{2}{D-1} g_{\mu[\mu_1} \nabla_{|\nu|} k^{\nu}{}_{\mu_2]} \ . \tag{2.52}$$

Since PCKYTs are closed ($dk = 0$), we can find a Killing-Yano potential b such that

$$k = db \ , \tag{2.53}$$

where b is a 1-form.

The Hodge duality of conformal Killing-Yano tensors goes deeper. We can create a structure called a *Killing-Yano tower* by using the Hodge duality property of CCKYTs. Starting from a Killing-Yano potential, one can find a PCKYT (or a rank-2 CCKYT) by taking an exterior derivative. In a D-dimensional manifold, PCKYT can be wedged with itself several times to get different rank CCKYTs. One can obtain different conformal Killing tensors for each CCKYT by (2.46).

The Hodge dual of those CCKYTs give different rank Killing-Yano tensors starting from rank-$(D-2)$ to 1 or 2 depending on evenness or oddness of the dimension D, respectively. Different Killing tensors for each Killing-Yano tensors can be obtained by using (2.38). To conclude, using a Killing-Yano potential, one can create CCKYTs, Killing-Yano tensors, conformal Killing tensors and Killing tensors.

CHAPTER 3

CONSERVED QUANTITIES

3.1 Linearized General Relativity

In this section, we will divide the spacetime into two parts. A general metric $g_{\mu\nu}$ can be written as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \ , \tag{3.1}$$

where $\bar{g}$ is the background metric and h is a deviation from the background that has components assumed to be small compared to the background metric. If the background is chosen as Minkowski metric, $\bar{g}$ is η. Using the relation $g_{\mu\nu}g^{\nu\sigma} = \delta_\mu^{\ \sigma}$, the inverse of the metric becomes

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2) \ , \tag{3.2}$$

where $\mathcal{O}(h^2)$ represents terms of order h^2 or higher. At this point, it is important to point out that if the deviation can be constructed by $h_{\mu\nu} = k_\mu k_\nu$, where k is a null vector with respect to both g and $\bar{g}$, i.e. $k_\mu k^\mu = 0$, the inverse metric can be exactly written as

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu} \ . \tag{3.3}$$

To proceed, we need the Christoffel symbols and curvature tensors. After linearization, we can write the Christoffel symbols in the following form

$$\Gamma^\mu{}_{\nu\rho} = \bar{\Gamma}^\mu{}_{\nu\rho} + \left(\Gamma^\mu{}_{\nu\rho}\right)_L + \mathcal{O}(h^2) \ . \tag{3.4}$$

Substituting the metric into the definition of the Christoffel symbols (1.2), we get

$$\begin{aligned}\Gamma^{\mu}{}_{\nu\rho} =& \frac{1}{2}\left(\bar{g}^{\mu\rho} - h^{\mu\rho}\right)\left[\partial_\nu\left(\bar{g}_{\sigma\rho} + h_{\sigma\rho}\right) + \partial_\rho\left(\bar{g}_{\nu\sigma} + h_{\nu\sigma}\right) - \partial_\sigma\left(\bar{g}_{\nu\rho} + h_{\nu\rho}\right)\right] + \mathcal{O}(h^2) \\ =& \bar{\Gamma}^{\mu}{}_{\nu\rho} + \frac{1}{2}\bar{g}^{\mu\rho}\left[\partial_\nu h_{\sigma\rho} + \partial_\rho h_{\nu\sigma} - \partial_\sigma h_{\nu\rho}\right] \\ & - \frac{1}{2}h^{\mu\rho}\left[\partial_\nu \bar{g}_{\sigma\rho} + \partial_\rho \bar{g}_{\nu\sigma} - \partial_\sigma \bar{g}_{\nu\rho}\right] + \mathcal{O}(h^2)\ .\end{aligned} \tag{3.5}$$

The linearized Christoffel symbols become

$$\left(\Gamma^{\mu}{}_{\nu\rho}\right)_L = \frac{1}{2}\bar{g}^{\mu\rho}\left[\partial_\nu h_{\sigma\rho} + \partial_\rho h_{\nu\sigma} - \partial_\sigma h_{\nu\rho}\right] - \frac{1}{2}h^{\mu\rho}\left[\partial_\nu \bar{g}_{\sigma\rho} + \partial_\rho \bar{g}_{\nu\sigma} - \partial_\sigma \bar{g}_{\nu\rho}\right]\ . \tag{3.6}$$

For the Minkowski background, the second term vanishes since components of the Minkowski metric are constants. We can further simplify the linearized Christoffel symbols by using the covariant derivative associated with the background metric

$$\bar{\nabla}_\rho h_{\mu\nu} = \partial_\rho h_{\mu\nu} - \bar{\Gamma}^{\sigma}{}_{\rho\mu} h_{\sigma\nu} - \bar{\Gamma}^{\sigma}{}_{\rho\nu} h_{\mu\sigma}\ , \tag{3.7}$$

and the metric compatibility for the background

$$\partial_\rho \bar{g}_{\mu\nu} - \bar{\Gamma}^{\sigma}{}_{\rho\mu}\bar{g}_{\sigma\nu} - \bar{\Gamma}^{\sigma}{}_{\rho\nu}\bar{g}_{\mu\sigma} = 0\ . \tag{3.8}$$

Then, the linearized Christoffel symbols can be written in terms of the covariant derivatives of the background

$$\left(\Gamma^{\mu}{}_{\nu\rho}\right)_L = \frac{1}{2}\bar{g}^{\mu\rho}\left[\bar{\nabla}_\nu h_{\sigma\rho} + \bar{\nabla}_\rho h_{\nu\sigma} - \bar{\nabla}_\sigma h_{\nu\rho}\right]\ . \tag{3.9}$$

Let us continue with linearizing the Riemann tensor. The Riemann tensor for a Levi-Civita connection is

$$R^{\rho}{}_{\mu\nu\sigma} = \partial_\nu \Gamma^{\rho}{}_{\sigma\mu} - \partial_\sigma \Gamma^{\rho}{}_{\nu\mu} + \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\sigma\mu} - \Gamma^{\rho}{}_{\sigma\lambda}\Gamma^{\lambda}{}_{\nu\mu}\ . \tag{3.10}$$

We try to get it in the form

$$R^{\rho}{}_{\mu\nu\sigma} = \bar{R}^{\rho}{}_{\mu\nu\sigma} + \left(R^{\rho}{}_{\mu\nu\sigma}\right)_L + \mathcal{O}(h^2)\ . \tag{3.11}$$

Substituting (3.5) to (3.10), we get

$$\begin{aligned}R^{\rho}{}_{\mu\nu\sigma} = \bar{R}^{\rho}{}_{\mu\nu\sigma} + \partial_\nu\left(\Gamma^{\rho}{}_{\sigma\mu}\right)_L - \partial_\sigma\left(\Gamma^{\rho}{}_{\nu\mu}\right)_L + \bar{\Gamma}^{\rho}{}_{\nu\kappa}\left(\Gamma^{\kappa}{}_{\mu\sigma}\right)_L + \left(\bar{\Gamma}^{\rho}{}_{\nu\kappa}\right)_L\Gamma^{\kappa}{}_{\mu\sigma} \\ - \bar{\Gamma}^{\rho}{}_{\sigma\kappa}\left(\Gamma^{\kappa}{}_{\mu\nu}\right)_L - \left(\bar{\Gamma}^{\rho}{}_{\nu\kappa}\right)_L\Gamma^{\kappa}{}_{\mu\sigma} + \mathcal{O}(h^2)\ .\end{aligned} \tag{3.12}$$

In terms of the covariant derivative of the background, linearized Riemann tensor becomes

$$(R^{\rho}{}_{\mu\nu\sigma})_L = \bar{\nabla}_\nu \left(\Gamma^{\rho}{}_{\mu\sigma}\right)_L - \bar{\nabla}_\sigma \left(\Gamma^{\rho}{}_{\mu\nu}\right)_L \ . \tag{3.13}$$

3.2 Conserved Charges from Covariantly Conserved Currents

Firstly, let us define integration on an orientable manifold. Following [13], the integral of an r-form $\alpha = \alpha(x)dx^1 \wedge \ldots \wedge dx^r$ over the region U is defined as

$$\int_{\mathcal{M}} \alpha = \pm \int_{\mathcal{M}} \alpha(x) dx^1 ... dx^r \ , \tag{3.14}$$

where $+$ is chosen if and only if the coordinate basis $(\partial_1, \ldots, \partial_r)$ has the same orientation as U.

Stokes' theorem states that the integral of a closed form $d\omega$ over the manifold $\mathcal{M}$ is equal to the integral of ω over the boundary of the manifold $\mathcal{M}$ [13]

$$\int_{\mathcal{M}} d\omega = \int_{\partial\mathcal{M}} \omega \ . \tag{3.15}$$

We shall use Stokes' theorem to obtain conserved charges in the following sections with conserved currents satisfying $d * j = 0$, and show that we can find conserved charges.

3.2.1 Rank-1 Conserved Currents

Let us have a covariantly conserved current $\nabla_\mu j^\mu = 0$ on a D-dimensional manifold. The current j can be written as $j = j_\mu dx^\mu$. Following the discussion in [14], let us define

$$\omega := *j = \frac{\sqrt{|g|}}{(D-1)!} j^\mu \epsilon_{\mu\nu_1 \ldots \nu_{D-1}} dx^{\nu_1} \wedge ... \wedge dx^{\nu_{D-1}} \ . \tag{3.16}$$

Then the exterior derivative of ω is

$$d\omega = d * j = \frac{1}{(D-1)!} \partial_{\nu_D} (\sqrt{|g|} j^\mu \epsilon_{\mu\nu_1 \ldots \nu_{D-1}}) dx^{\nu_D} \wedge dx^{\nu_1} \wedge ... \wedge dx^{\nu_{D-1}} \ . \tag{3.17}$$

To use the covariant conservation, we need a relation between the exterior derivative and the covariant derivative. Using the relation

$$\nabla_\mu j^\mu = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}j^\mu) \; , \tag{3.18}$$

$d\omega$ becomes

$$d\omega = \frac{1}{(D-1)!}\sqrt{|g|}\,(\nabla_{\nu_D} j^\mu)\,\epsilon_{\mu\nu_1\ldots\nu_{D-1}} dx^{\nu_D}\wedge dx^{\nu_1}\wedge \ldots \wedge dx^{\nu_{D-1}} \; . \tag{3.19}$$

Here, we used $\nabla_\mu \epsilon_{\nu_1..\nu_D} = 0$. Any D-form can be written as a function h multiplied by the invariant volume element

$$(d\omega)_{\nu_1\ldots\nu_D} = D\sqrt{|g|}\epsilon_{\mu[\nu_1\ldots\nu_{D-1}}\nabla_{\nu_D]}j^\mu = h\sqrt{|g|}\epsilon_{[\nu_1\ldots\nu_D]} \; . \tag{3.20}$$

In order to evaluate the function h, contract the indices with $\epsilon^{\nu_1\ldots\nu_D}$

$$D\sqrt{|g|}\epsilon^{\nu_1\ldots\nu_D}\epsilon_{\mu[\nu_1\ldots\nu_{D-1}}\nabla_{\nu_D]}j^\mu = h\sqrt{|g|}\epsilon^{\nu_1\ldots\nu_D}\epsilon_{\nu_1\ldots\nu_D} \; , \tag{3.21}$$

$$D!\sqrt{|g|}\delta^{\nu_D}_\mu \nabla_{\nu_D} j^\mu = D! h\sqrt{|g|} \; , \tag{3.22}$$

$$h = \nabla_\mu j^\mu \; . \tag{3.23}$$

Then, $d\omega$ can be written as

$$d\omega = \sqrt{|g|}\,(\nabla_\mu j^\mu)\,\frac{\epsilon_{\nu_1\ldots\nu_D}}{D!} dx^{\nu_D}\wedge dx^{\nu_1}\wedge \ldots \wedge dx^{\nu_{D-1}} \; . \tag{3.24}$$

The integral of $d\omega$ can be used in Stokes' theorem

$$\int_{\mathcal{M}} d\omega = \int_{\mathcal{M}} \sqrt{|g|}\,(\nabla_\mu j^\mu)\,\epsilon_{\nu_1\ldots\nu_D} dx^{\nu_1}\ldots dx^{\nu_D} \tag{3.25}$$

$$= \int_{\partial\mathcal{M}} \omega = \int_{\partial\mathcal{M}} \sqrt{|g|} j^\mu \epsilon_{\mu\nu_1\ldots\nu_{D-1}} dx^{\nu_1}\ldots dx^{\nu_{D-1}} \; . \tag{3.26}$$

At this point, one should check if the boundary is integrable. Let n^μ be a timelike normal vector to the boundary of the manifold $\mathcal{M}$, it is said to be hypersurface orthogonal if it satisfies the hypersurface orthogonality condition (Frobenius theorem) [14]

$$n_{[\mu}\nabla_\nu n_{\rho]} = 0 \; . \tag{3.27}$$

We can find the induced metric of the boundary $\partial\mathcal{M}$ by $g_{\mu\nu} = g^{(\partial\mathcal{M})}_{\mu\nu} - n_\mu n_\nu$ ($-$ sign is chosen since the normal vector to the hypersurface must be timelike in order to talk about conservation in *time*). For the boundary or the submanifold to be integrable, the induced metric $g^{(\partial\mathcal{M})}_{\mu\nu}$ should be nondegenerate [14]. Then, we can define the volume element on the boundary. The volume element on a manifold $\mathcal{M}$ is

$$\epsilon = \sqrt{|g|}\epsilon_{\nu_1\ldots\nu_D} dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_D} \ . \tag{3.28}$$

On the submanifold $\partial\mathcal{M}$ of dimension $(D-1)$, the volume element becomes

$$\hat{\epsilon} = \sqrt{|g^{(\partial\mathcal{M})}|}\hat{\epsilon}_{\nu_1\ldots\nu_{D-1}} dy^{\nu_1} \wedge \ldots \wedge dy^{\nu_{D-1}} \ . \tag{3.29}$$

where $g^{(\partial\mathcal{M})}$ is the metric of the submanifold and y's are the coordinates on the submanifold.

In the submanifold's coordinates, equation (3.28) is

$$\epsilon = \sqrt{|g^{(\partial\mathcal{M})}|}\epsilon_{\mu\nu_1\ldots\nu_{D-1}} dz \wedge dy^{\nu_1} \wedge \ldots \wedge dy^{\nu_{D-1}} \ , \tag{3.30}$$

where z is assumed to be the coordinate normal to the submanifold. Then, contraction of ϵ with n^μ should give $\hat{\epsilon}$ in y coordinates

$$\sqrt{|g|}n^\mu \epsilon_{\mu\nu_1\ldots\nu_{D-1}} = \sqrt{|g^{(\partial\mathcal{M})}|}\hat{\epsilon}_{\nu_1\ldots\nu_{D-1}} \ . \tag{3.31}$$

The right hand side of (3.26) in the submanifold's coordinates becomes

$$\int_{\partial\mathcal{M}} \sqrt{|g^{(\partial\mathcal{M})}|} j^\mu n_\mu \hat{\epsilon}_{\nu_1\ldots\nu_{D-1}} dy^{\nu_1} \ldots dy^{\nu_{D-1}} = \int_{\partial\mathcal{M}} \sqrt{|g^{(\partial\mathcal{M})}|} j^\mu n_\mu d^{D-1}y \ . \tag{3.32}$$

Since the current is covariantly closed, the right hand side of (3.25) gives zero. If we divide $\partial\mathcal{M}$ into two parts such that $\partial\mathcal{M} = \Sigma_+ \cup \Sigma_-$ (note that Σ_+ and Σ_- have opposite orientations), we get

$$\int_{\Sigma_+} \sqrt{|g^{(\partial\mathcal{M})}|} j^\mu n_\mu d^{D-1}y - \int_{\Sigma_-} \sqrt{|g^{(\partial\mathcal{M})}|} j^\mu n_\mu d^{D-1}y = 0 \ . \tag{3.33}$$

Hence, the integrant does not change over spacelike hypersurfaces. In other words, we can define the integral as a conserved charge

$$Q = \int_\Sigma \sqrt{|g^{(\Sigma)}|} j^\mu n_\mu d^{D-1}y \ . \tag{3.34}$$

The discussion is not over yet. If Σ is a manifold with a boundary and the current can be written as a divergence of another antisymmetric tensor, say charge density ℓ, Stokes' theorem can be applied once more[1]. Let $j^\mu = \nabla_\nu \ell^{\nu\mu}$ with $\ell^{\nu\mu} = \ell^{[\nu\mu]}$ then the charge Q at spatial infinity is

$$Q = \int_\Sigma \sqrt{|g^{(\Sigma)}|}\nabla_\nu \ell^{\nu\mu} n_\mu d^{D-1}y = \int_{\partial\Sigma} \sqrt{|g^{(\partial\Sigma)}|} r_\nu n_\mu \ell^{\nu\mu} d^{D-2}z \, , \tag{3.35}$$

where $\partial\Sigma$ is the boundary of Σ, z's are coordinates on $\partial\Sigma$, $g^{(\partial\Sigma)}$ is the induced metric on $\partial\Sigma$ and r^μ is the spacelike unit normal to $\partial\Sigma$.

3.2.2 Rank-2 Conserved Currents

For an antisymmetric conserved current $\nabla_{\mu_1} j^{\mu_1\mu_2} = 0$, where $j^{\mu_1\mu_2} = \nabla_\rho \ell^{\rho\mu_1\mu_2}$ with $\ell^{\rho\mu_1\mu_2} = \ell^{[\rho\mu_1\mu_2]}$, we can find a conserved charge similar to what we did in the previous section. Let us start by defining

$$\omega := *j = \frac{\sqrt{|g|}}{2!(D-2)!} j^{\mu_1\mu_2} \epsilon_{\mu_1\mu_2\nu_1\ldots\nu_{D-2}} dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_{D-2}} \, . \tag{3.36}$$

The exterior derivative of ω is

$$d\omega = d{*}j = \frac{1}{(D-2)!2!} \partial_\sigma (\sqrt{|g|} j^{\mu_1\mu_2} \epsilon_{\mu_1\mu_2\nu_1\ldots\nu_{D-2}}) dx^\sigma \wedge dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_{D-2}} \, . \tag{3.37}$$

We can generalize equation (3.18) to any rank as the long as the tensor j is antisymmetric [22]

$$\nabla_{\mu_1} j^{\mu_1\mu_2} = \frac{1}{\sqrt{|g|}} \partial_{\mu_1} \left(\sqrt{|g|} j^{\mu_1\mu_2} \right) \, . \tag{3.38}$$

After changing the partial derivative to the covariant one, $d\omega$ becomes

$$d\omega = \frac{\sqrt{|g|}}{(D-2)!2!} \epsilon_{\mu_1\mu_2\nu_1\ldots\nu_{D-2}} \left(\nabla_\sigma j^{\mu_1\mu_2}\right) dx^\sigma \wedge dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_{D-2}} \, . \tag{3.39}$$

In the previous section, we used the invariant volume element at this stage. However, now $d\omega$ is not a volume form. To overcome this, we can construct a $(D-1)$-dimensional hypersurface Σ, and work on it to get a volume form on the hypersurface.

[1] How it is applicable for a rank-2 tensor is shown in subsection 3.2.2

This hypersurface must be integrable since we will integrate $d\omega$ over it. Assuming the metric on $\mathcal{M}$ can be divided into two parts

$$g_{\mu\nu} = \gamma_{\mu\nu} + N_\mu N_\nu \ , \tag{3.40}$$

where γ is the nondegenerate induced metric on the $(D-1)$-dimensional hypersurface, and assuming the normal vector N^μ satisfies the hypersurface orthogonality condition (3.27), we can define the volume element on the hypersurface [14]. The volume element on the manifold $\mathcal{M}$ is

$$\epsilon = \sqrt{|g|}\epsilon_{\nu_1\ldots\nu_D} dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_D} \ . \tag{3.41}$$

On the hypersurface, the volume element is

$$\hat{\epsilon} = \sqrt{|\gamma|}\epsilon_{\nu_1\ldots\nu_{D-1}} dy^{\nu_1} \wedge \ldots \wedge dy^{\nu_{D-1}} \ . \tag{3.42}$$

The coordinates on the manifold are denoted by x, and the coordinates on the hypersurface are denoted by y. With z being the coordinate normal to the hypersurface, the contraction of ϵ with N^ν should give $\hat{\epsilon}$ in y coordinates

$$\sqrt{|g|}N^\nu \epsilon_{\nu\nu_1\ldots\nu_{D-1}} = \sqrt{|\gamma|}\hat{\epsilon}_{\nu_1\ldots\nu_{D-1}} \ . \tag{3.43}$$

Let us rewrite equation (3.39) in the hypersurface's coordinates

$$d\omega = \frac{\sqrt{|\gamma|}}{(D-2)!2!} N_{\mu_1}\hat{\epsilon}_{\mu_2\nu_1\ldots\nu_{D-2}} \left(\nabla_\sigma j^{\mu_1\mu_2}\right) dy^\sigma \wedge dy^{\nu_1} \wedge \ldots \wedge dy^{\nu_{D-2}} \ . \tag{3.44}$$

To change the index of the covariant derivative, we can use the invariant volume element on the hypersurface as we did in the previous section

$$(dw)_{\nu_1\ldots\nu_{D-2}\sigma} = \frac{(D-1)\sqrt{|\gamma|}}{2!} N_{\mu_1}\hat{\epsilon}_{\mu_2[\nu_1\ldots\nu_{D-2}}\nabla_{\sigma]} j^{\mu_1\mu_2} = h\sqrt{|\gamma|}\hat{\epsilon}_{[\sigma\nu_1\ldots\nu_{D-2}]} \ . \tag{3.45}$$

Contract both sides with $\epsilon^{\nu_1\ldots\nu_{D-2}\sigma}$

$$\frac{(D-1)!}{2!} N_{\mu_1}\delta^\sigma_{\mu_2}\nabla_\sigma j^{\mu_1\mu_2} = h(D-1)! \ , \tag{3.46}$$

$$h = \frac{1}{2!} N_{\mu_1}\nabla_{\mu_2} j^{\mu_1\mu_2} \ . \tag{3.47}$$

That means we can write $d\omega$ as

$$d\omega = \frac{\sqrt{|\gamma|}}{(D-1)!2!} N_{\mu_1} \hat{\epsilon}_{\sigma\nu_1 \ldots \nu_{D-2}} \left(\nabla_{\mu_2} j^{\mu_1\mu_2}\right) dy^\sigma \wedge dy^{\nu_1} \wedge \ldots \wedge dy^{\nu_{D-2}} \ . \tag{3.48}$$

Before integrating $d\omega$, let us write ω in the hypersurface's coordinates

$$\omega = \frac{\sqrt{|\gamma|}}{(D-2)!2!} j^{\mu_1\mu_2} N_{\mu_1} \hat{\epsilon}_{\mu_2\nu_1 \ldots \nu_{D-2}} dy^{\nu_1} \wedge \ldots \wedge dy^{\nu_{D-2}} \ . \tag{3.49}$$

Assume that, we can further write

$$g_{\mu\nu} = h_{\mu\nu} + N_\mu N_\nu - n_\mu n_\nu \ , \tag{3.50}$$

where n^μ is a timelike unit normal and $h_{\mu\nu}$ is the induced metric on the $(D-2)$-dimensional boundary of the hypersurface, $\partial\Sigma$, satisfying the integrability conditions mentioned above. Then w becomes

$$\omega = \frac{\sqrt{|h|}}{(D-2)!2!} j^{\mu_1\mu_2} N_{\mu_1} n_{\mu_2} \bar{\epsilon}_{\nu_1 \ldots \nu_{D-2}} dz^{\nu_1} \wedge \ldots \wedge dz^{\nu_{D-2}} \tag{3.51}$$

in the coordinates of $\partial\Sigma$.

We can use Stokes' theorem by integrating $d\omega$ on the hypersurface

$$\int_\Sigma d\omega = \int_\Sigma \frac{\sqrt{|\gamma|}}{(D-1)!2!} N_{\mu_1} \hat{\epsilon}_{\sigma\nu_1 \ldots \nu_{D-2}} \left(\nabla_{\mu_2} j^{\mu_1\mu_2}\right) dy^\sigma dy^{\nu_1} \ldots dy^{\nu_{D-2}} \tag{3.52}$$

$$= \int_{\partial\Sigma} \frac{\sqrt{|h|}}{(D-2)!2!} N_{\mu_1} n_{\mu_2} \bar{\epsilon}_{\nu_1 \ldots \nu_{D-2}} j^{\mu_1\mu_2} dz^{\nu_1} \ldots dz^{\nu_{D-2}} \tag{3.53}$$

$$= \int_{\partial\Sigma} \frac{\sqrt{|h|}}{2!} N_{\mu_1} n_{\mu_2} j^{\mu_1\mu_2} d^{D-2}z \ . \tag{3.54}$$

If we divide $\partial\Sigma$ into two parts with opposite orientations and use the covariant conservation, it is evident that the integral does not change over spacelike hypersurfaces. Thus, we can define it as a charge

$$Q = \int_{\bar{\Sigma}} \frac{\sqrt{|h|}}{2!} N_{\mu_1} n_{\mu_2} j^{\mu_1\mu_2} d^{D-2}z \ . \tag{3.55}$$

Before imposing the potential for the current, we have seen that if we take the integral on $(D-1)$-dimensional hypersurface, then we can use Stokes' theorem for rank-2 conserved currents. Hence, if we can write the current as $j^{\mu\nu} = \nabla_\rho l^{\rho\mu\nu}$, we can calculate the charge at the boundary of $\bar{\Sigma}$ with the induced metric $g^{(\partial\bar{\Sigma})}_{\mu\nu} = h_{\mu\nu} - r_\mu r_\nu$ using the potential ℓ

$$Q = \int_{\partial\bar{\Sigma}} \frac{\sqrt{|g^{(\partial\Sigma)}|}}{2!} N_{\mu_1} n_{\mu_2} r_{\mu_3} \ell^{\mu_1\mu_2\mu_3} d^{D-3}u \ , \tag{3.56}$$

where we assumed $\bar{\Sigma}$ is a submanifold with a boundary, r^μ is a spacelike normal to the boundary and $\partial\bar{\Sigma}$ is integrable.

3.2.3 Rank-m Conserved Currents

The keypoint of the previous discussion is to get a $d\omega$ which can be written as a function multiplied by the invariant volume element. However, if we define ω as $*j$ where j is a rank-m ($m \neq 1$) antisymmetric current, $d\omega$ is not a volume form. So one can take the integral on the hypersurface of dimension $(D - m + 1)$ in the manifold $\mathcal{M}$. Then, $d\omega$ is a volume form for the hypersurface. Let us start by assuming that the metric associated with the manifold $\mathcal{M}$ can be decomposed as

$$g_{\mu\nu} = g^{(\Sigma)}_{\mu\nu} + N^{(1)}_\mu N^{(1)}_\nu + ... + N^{(m-1)}_\mu N^{(m-1)}_\nu \ , \tag{3.57}$$

where Σ is the integrable hypersurface of dimension $(D-m+1)$, $g^{(\Sigma)}_{\mu\nu}$ is the induced metric on the hypersurface and $N^{(1)}, ..., N^{(m-1)}$ are the spacelike normal vectors of Σ. Here, we assume that the induced metric is nondegenerate, the normal vectors $N^{(1)}, ..., N^{(m-1)}$ satisfy the hypersurface orthogonality condition (3.27), and are mutually orthogonal

$$N^{(i)\mu} N^{(j)}_\mu = 0 \tag{3.58}$$

where i and j runs from 1 to $(m-1)$.

The volume element on the manifold $\mathcal{M}$ is

$$\epsilon = \sqrt{|g|}\epsilon_{\nu_1...\nu_D} dx^{\nu_1} \wedge ... \wedge dx^{\nu_D} \ . \tag{3.59}$$

The volume element on the hypersurface Σ of dimension $(D - m + 1)$ becomes

$$\hat{\epsilon} = \sqrt{|g^{(\Sigma)}|}\hat{\epsilon}_{\nu_1...\nu_{D-m+1}} dy^{\nu_1} \wedge ... \wedge dy^{\nu_{D-m+1}} \ , \tag{3.60}$$

where y's are the coordinates on the hypersurface. In these coordinates, equation (3.59) can be written as

$$\epsilon = \sqrt{|g^{(\Sigma)}|}\epsilon_{\mu_1...\mu_{m-1}\nu_1...\nu_{D-m+1}} dz^{\mu_1} \wedge \ldots \wedge dz^{\mu_{m-1}} \wedge dy^{\nu_1} \wedge \ldots \wedge dy^{\nu_{D-m+1}} \ . \tag{3.61}$$

z's are the coordinates associated with normals to the hypersurface. Then, contraction of ϵ with the normal vectors to the hypersurface, $N^{(i)}$, should give $\hat{\epsilon}$ in y coordinates

$$N^{(1)\mu_1} \dots N^{(m-1)\mu_{m-1}} \epsilon_{\mu_1 \dots \mu_{m-1} \nu_1 \dots \nu_{D-m+1}} \sqrt{|g|} = \hat{\epsilon}_{\nu_1 \dots \nu_{D-m+1}} \sqrt{|g^{(\Sigma)}|} \,, \qquad (3.62)$$

or

$$\epsilon_{\mu_1 \dots \mu_{m-1} \nu_1 \dots \nu_{D-m+1}} \sqrt{|g|} = N^{(1)}_{\mu_1} \dots N^{(m-1)}_{\mu_{m-1}} \hat{\epsilon}_{\nu_1 \dots \nu_{D-m+1}} \sqrt{|g^{(\Sigma)}|} \,. \qquad (3.63)$$

Generalization of the previous discussion to rank-m is necessary to work with anti-symmetric current tensors of arbitrary rank. Let us start with writing the current j as $j = j_{\mu_1 \dots \mu_m} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_m}$. Hodge dual of j is defined to be ω

$$\omega := *j = \frac{\sqrt{|g|}}{(D-m)!m!} j^{\mu_1 \dots \mu_m} \epsilon_{\mu_1 \dots \mu_m \nu_1 \dots \nu_{D-m}} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_{D-m}} \,. \qquad (3.64)$$

The exterior derivative of ω is

$$d\omega = d*j = \frac{1}{(D-m)!m!} \partial_\sigma (\sqrt{|g|} j^{\mu_1 \dots \mu_m} \epsilon_{\mu_1 \dots \mu_m \nu_1 \dots \nu_{D-m}}) dx^\sigma \wedge dx^{\nu_1} \wedge \dots \wedge dx^{\nu_{D-m}} . \qquad (3.65)$$

We can use (3.18) for rank-m case

$$\nabla_\mu j^{\mu \nu_1 \dots \nu_{m-1}} = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} j^{\mu \nu_1 \dots \nu_{m-1}}) \,. \qquad (3.66)$$

Then, $d\omega$ is

$$d\omega = \frac{\sqrt{|g|}}{(D-m)!m!} \epsilon_{\mu_1 \dots \mu_m \nu_1 \dots \nu_{D-m}} \left(\nabla_\sigma j^{\mu_1 \dots \mu_m} \right) dx^\sigma \wedge dx^{\nu_1} \wedge \dots \wedge dx^{\nu_{D-m}} \,. \qquad (3.67)$$

Using (3.63), we can transform $d\omega$ to hypersurface's coordinates and it becomes

$$d\omega = \frac{\sqrt{|g^{(\Sigma)}|}}{(D-m)!m!} N^{(1)}_{\mu_1} \dots N^{(m-1)}_{\mu_{m-1}} \hat{\epsilon}_{\mu_m \nu_1 \dots \nu_{D-m}} \left(\nabla_\sigma j^{\mu_1 \dots \mu_m} \right) dy^\sigma \wedge dy^{\nu_1} \wedge \dots \wedge dy^{\nu_{D-m}} . \qquad (3.68)$$

$d\omega$ is a volume element on the hypersurface Σ of dimension $(D-m+1)$. Similar to what we did in the rank-1 case, we can write it as a function h multiplied by the invariant volume element of Σ

$$\begin{aligned}
(dw)_{\nu_1 \dots \nu_{D-m+1}} &= \\
&= \frac{(D-m+1)\sqrt{|g^{(\Sigma)}|}}{m!} N^{(1)}_{\mu_1} \dots N^{(m-1)}_{\mu_{m-1}} \hat{\epsilon}_{\mu_m [\nu_1 \dots \nu_{D-m}} \nabla_{\sigma]} j^{\mu_1 \dots \mu_m} \\
&= h \sqrt{|g^{(\Sigma)}|} \hat{\epsilon}_{[\nu_1 \dots \nu_{D-m} \sigma]} \,. \qquad (3.69)
\end{aligned}$$

Contract both sides with $\epsilon^{\nu_1 \ldots \nu_{D-m}\sigma}$

$$\frac{(D-m+1)!}{m!} N^{(1)}_{\mu_1} \ldots N^{(m-1)}_{\mu_{m-1}} \delta^{\sigma}_{\mu_m} \nabla_\sigma j^{\mu_1 \ldots \mu_m} = h(D-m+1)! \,, \tag{3.70}$$

$$h = \frac{1}{m!} N^{(1)}_{\mu_1} \ldots N^{(m-1)}_{\mu_{m-1}} \nabla_{\mu_m} j^{\mu_1 \ldots \mu_m} \,. \tag{3.71}$$

Thus, $d\omega$ can be written as

$$d\omega = \frac{\sqrt{|g^{(\Sigma)}|}\hat{\epsilon}_{\nu_1 \ldots \nu_{D-m}\sigma}}{(D-m+1)!m!} N^{(1)}_{\mu_1} \ldots N^{(m-1)}_{\mu_{m-1}} \left(\nabla_{\mu_m} j^{\mu_1 \ldots \mu_m}\right) dy^\sigma \wedge dy^{\nu_1} \wedge \cdots \wedge dy^{\nu_{D-m}}. \tag{3.72}$$

We can also write ω in the hypersurface's coordinates using (3.63)

$$\omega = \frac{\sqrt{|g^{(\Sigma)}|}}{(D-m)!m!} j^{\mu_1 \ldots \mu_m} N^{(1)}_{\mu_1} \ldots N^{(m-1)}_{\mu_{m-1}} \hat{\epsilon}_{\mu_m \nu_1 \ldots \nu_{D-m}} dy^{\nu_1} \wedge \ldots \wedge dy^{\nu_{D-m}} \,. \tag{3.73}$$

If we integrate $d\omega$, we can use Stokes' theorem (see (3.15))

$$\int_\Sigma d\omega = \int_\Sigma \frac{\sqrt{|g^{(\Sigma)}|} N^{(1)}_{\mu_1} \ldots N^{(m-1)}_{\mu_{m-1}}}{(D-m+1)!m!} \left(\nabla_{\mu_m} j^{\mu_1 \ldots \mu_m}\right) \hat{\epsilon}_{\nu_1 \ldots \nu_{D-m+1}} dy^{\nu_1} \ldots dy^{\nu_{D-m+1}} \tag{3.74}$$

$$= \int_{\partial\Sigma} \frac{\sqrt{|g^{(\Sigma)}|}}{(D-m)!m!} j^{\mu_1 \ldots \mu_m} N^{(1)}_{\mu_1} \ldots N^{(m-1)}_{\mu_{m-1}} \hat{\epsilon}_{\mu_m \nu_1 \ldots \nu_{D-m}} dy^{\nu_1} \ldots dy^{\nu_{D-m}} \tag{3.75}$$

$$= \int_{\partial\Sigma} \frac{\sqrt{|g^{(\partial\Sigma)}|}}{(D-m)!m!} j^{\mu_1 \ldots \mu_m} N^{(1)}_{\mu_1} \ldots N^{(m-1)}_{\mu_{m-1}} n_{\mu_m} \bar{\epsilon}_{\nu_1 \ldots \nu_{D-m}} dz^{\nu_1} \ldots dz^{\nu_{D-m}} \tag{3.76}$$

$$= \int_{\partial\Sigma} \frac{\sqrt{|g^{(\partial\Sigma)}|}}{m!} j^{\mu_1 \ldots \mu_m} N^{(1)}_{\mu_1} \ldots N^{(m-1)}_{\mu_{m-1}} n_{\mu_m} d^{D-m}z \,, \tag{3.77}$$

where $g^{(\partial\Sigma)}$ is the metric of the boundary of the hypersurface Σ defined by the relation $g^{(\Sigma)}_{\mu\nu} = g^{(\partial\Sigma)}_{\mu\nu} - n_\mu n_\nu$ and z's are the coordinates on the boundary. Note that n^μ is a timelike normal to the hypersurface Σ.

A covariantly conserved current satisfies $\nabla_\mu j^{\mu\nu_1 \ldots \nu_{m-1}} = 0$. As a result, the integration in (3.74) is zero. Therefore, we can define the integral in (3.77) as a conserved charge

$$Q = \int_{\bar{\Sigma}} \frac{\sqrt{|g^{(\partial\Sigma)}|}}{m!} j^{\mu_1 \ldots \mu_m} N^{(1)}_{\mu_1} \ldots N^{(m-1)}_{\mu_{m-1}} n_{\mu_m} d^{D-m}z \,. \tag{3.78}$$

$\bar{\Sigma}$ is the hypersurface of constant time with the induced metric $g^{(\partial\Sigma)}$. We can further impose $j^{\mu_1\cdots\mu_m} = \nabla_\nu \ell^{\nu\mu_1\cdots\mu_m}$ if the current can be written as the covariant derivative of a potential ℓ with $\ell^{\nu\mu_1\cdots\mu_m} = \ell^{[\nu\mu_1\cdots\mu_m]}$, and $\bar{\Sigma}$ is a hypersurface with a boundary. Then, using Stokes' theorem on $\bar{\Sigma}$, we get one less integral to take

$$Q = \int_{\bar{\Sigma}} \frac{\sqrt{|g^{(\partial\Sigma)}|}}{m!} \nabla_\nu \ell^{\nu\mu_1\cdots\mu_m} N^{(1)}_{\mu_1} \dots N^{(m-1)}_{\mu_{m-1}} n_{\mu_m} d^{D-m} z \tag{3.79}$$

$$= \int_{\partial\bar{\Sigma}} \frac{\sqrt{|g^{(\partial\bar{\Sigma})}|}}{m!} \ell^{\nu\mu_1\cdots\mu_m} N^{(1)}_{\mu_1} \dots N^{(m-1)}_{\mu_{m-1}} n_{\mu_m} r_\nu d^{D-m-1} u \;, \tag{3.80}$$

where $g^{(\partial\bar{\Sigma})}$ is the metric of the boundary of the manifold $\bar{\Sigma}$ defined by the relation $g^{(\partial\Sigma)}_{\mu\nu} = g^{(\partial\bar{\Sigma})}_{\mu\nu} + r_\mu r_\nu$.

3.2.4 Background Charges

In this section, we are going to show that a conserved charge can be found in a different manner when the metric can be divided into two parts, one is the background, and the other is a deviation

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \;, \tag{3.81}$$

where $h_{\mu\nu}$ need not to be small everywhere but should have a sufficient fall-off rate at spatial infinity such that spacetime is characterized by $\bar{g}_{\mu\nu}$ at spatial infinity.

In most cases, the background is chosen as Minkowski space, and in some cases, it is chosen as de-Sitter or anti de-Sitter spaces. Let $(j)_L$ be a covariantly closed vector with respect to the background metric

$$\bar{\nabla}_\mu (j)^\mu_L = 0 \;, \tag{3.82}$$

where $\bar{\nabla}$ is the covariant derivative associated with the background metric $\bar{g}$. If we integrate it over the D-dimensional manifold $\mathcal{M}$, we get

$$\int_{\mathcal{M}} d^D x \sqrt{|\bar{g}|} \bar{\nabla}_\mu (j)^\mu_L \;. \tag{3.83}$$

We can use (3.18) for the background

$$\bar{\nabla}_\mu (j)^\mu_L = \frac{1}{\sqrt{|\bar{g}|}} \partial_\mu (\sqrt{|\bar{g}|} (j)^\mu_L) \;. \tag{3.84}$$

Then, the integral becomes

$$\int_{\mathcal{M}} d^D x \partial_\mu (\sqrt{|\bar{g}|}(j)^\mu_L) \, . \tag{3.85}$$

Using Stokes' theorem, we can write the integral on the $(D-1)$-dimensional boundary. Let n be a timelike normal vector to the boundary of the manifold $\mathcal{M}$

$$\bar{g}_{\mu\nu} = \gamma_{\mu\nu} - n_\mu n_\nu \, . \tag{3.86}$$

After invoking Stokes' theorem, the integral becomes

$$\int_{\mathcal{M}} d^D x \partial_\mu (\sqrt{|\bar{g}|}(j)^\mu_L) = \int_{\partial\mathcal{M}} d^{D-1} x \sqrt{|\gamma|} n_\mu (j)^\mu_L \, . \tag{3.87}$$

Since the current is conserved (3.82), we can define it as a conserved charge

$$Q = \int_\Sigma d^{D-1} x \sqrt{|\gamma|} n_\mu (j)^\mu_L \, . \tag{3.88}$$

If the current $(j)_L$ can be written as $(j)^\nu_L = \bar{\nabla}_\mu \bar{\ell}^{\mu\nu}$ with $\bar{\ell}^{\mu\nu} = \bar{\ell}^{[\mu\nu]}$, and t is the coordinate associated with the timelike vector n^μ's direction, then we can use Stokes' theorem once more

$$\begin{aligned} Q = \int_\Sigma d^{D-1} x \sqrt{|\gamma|} n_\mu \bar{\nabla}_\nu \bar{\ell}^{\nu\mu} &= \int_\Sigma d^{D-1} x \delta^t_\mu \partial_\nu (\sqrt{|\bar{g}|} \bar{\ell}^{\nu\mu}) \\ &= \int_{\partial\Sigma} d^{D-2} x \sqrt{|\bar{\gamma}|} n_\mu r_\nu \bar{\ell}^{\nu\mu} \, . \end{aligned} \tag{3.89}$$

as long as there is a spacelike normal vector r^μ which does not vanish anywhere on the manifold. The charge Q can be calculated at the boundary of Σ using the following decomposition of $\bar{g}$

$$\bar{g}_{\mu\nu} = \bar{\gamma}_{\mu\nu} - n_\mu n_\nu + r_\mu r_\nu \, , \tag{3.90}$$

where $\bar{\gamma}$ is the induced metric on the boundary of Σ.

Let us have a linearized antisymmetric rank-m covariantly conserved current with respect to the background metric

$$\bar{\nabla}_\mu \left(j^{\mu\nu_1\ldots\nu_{m-1}}\right)_L = \bar{\nabla}_\mu \bar{\nabla}_\rho \bar{\ell}^{\rho\mu\nu_1\ldots\nu_{m-1}} = 0 \, , \tag{3.91}$$

where $\bar{\ell}$ is the totally antisymmetric potential for the linearized current satisfying $\bar{\ell}^{\rho\mu\nu_1\ldots\nu_{m-1}} = \bar{\ell}^{[\rho\mu\nu_1\ldots\nu_{m-1}]}$. In order to find the conserved charges, we should integrate over $(D-m-1)$-dimensional hypersurfaces if we have a rank-$(m+1)$ potential for the rank-m current. However, we can integrate over $(D-2)$-dimensional hypersurfaces if we contract the potential with $(m-1)$ vectors and they satisfy

$$\bar{\nabla}_\mu \bar{\nabla}_\rho \left(\bar{\ell}^{\rho\mu\nu_1\ldots\nu_{m-1}} x^{(1)}_{\nu_1} \ldots x^{(m-1)}_{\nu_{m-1}} \right) = 0 \, . \tag{3.92}$$

Expanding the derivatives, we get

$$\begin{aligned} &\left(\bar{\nabla}_\mu \bar{\nabla}_\rho \bar{\ell}^{\rho\mu\nu_1\ldots\nu_{m-1}} \right) x^{(1)}_{\nu_1} \ldots x^{(m-1)}_{\nu_{m-1}} \\ &+\bar{\nabla}_\mu \left(\bar{\ell}^{\rho\mu\nu_1\ldots\nu_{m-1}} \right) \sum_{i=1}^{m-1} x^{(1)}_{\nu_1} \ldots (\bar{\nabla}_\rho x^{(i)}_{\nu_i}) \ldots x^{(m-1)}_{\nu_{m-1}} + (\mu \leftrightarrow \rho) \\ &+\bar{\ell}^{\rho\mu\nu_1\ldots\nu_{m-1}} \sum_{i=1}^{m-1} \sum_{j\neq i}^{m-1} x^{(1)}_{\nu_1} \ldots (\bar{\nabla}_\rho x^{(i)}_{\nu_i}) \ldots (\bar{\nabla}_\mu x^{(j)}_{\nu_j}) \ldots x^{(m-1)}_{\nu_{m-1}} \\ &+\bar{\ell}^{\rho\mu\nu_1\ldots\nu_{m-1}} \sum_{i=1}^{m-1} x^{(1)}_{\nu_1} \ldots \left(\bar{\nabla}_\mu \bar{\nabla}_\rho x^{(i)}_{\nu_i} \right) \ldots x^{(m-1)}_{\nu_{m-1}} = 0 \, . \end{aligned} \tag{3.93}$$

The first term in (3.93) is zero from (3.91). Since $\bar{\ell}$ is antisymmetric, and the other terms in the second and third lines are symmetric on μ and ρ indices, they also vanish. Finally, the last term should be zero, and that could be the condition on the vectors x^i. However, if the background metric is flat, then the last term is symmetric on μ and ρ indices, and it vanishes too

$$\bar{\ell}^{\rho\mu\nu_1\ldots\nu_{m-1}} \sum_{i=1}^{m-1} x^{(1)}_{\nu_1} \ldots \bar{\nabla}_{(\mu} \bar{\nabla}_{\rho)} (x^{(i)}_{\nu_i}) \ldots x^{(m-1)}_{\nu_{m-1}} = 0 \, . \tag{3.94}$$

Thus, if the background metric is flat, we can contract $\bar{\ell}$ with any vector, and it can still be used to find conserved charges. A reasonable choice for the vectors x^i is as follows. Assuming that the background metric can be decomposed as

$$\begin{aligned} \bar{g}_{\mu\nu} =& \gamma_{\mu\nu} - n_\mu n_\nu + r_\mu r_\nu \\ =& \bar{\gamma}_{\mu\nu} - n_\mu n_\nu + r_\mu r_\nu + \sum_{i=1}^{m-1} x^i_\mu x^i_\nu \, . \end{aligned} \tag{3.95}$$

Then, using the equation (3.89), we can define the charge as

$$Q = \int_{\partial\Sigma} d^{D-2}x \sqrt{|\gamma|} n_\mu r_\rho x^{(1)}_{\nu_1} \ldots x^{(m-1)}_{\nu_{m-1}} \bar{\ell}^{\rho\mu\nu_1\ldots\nu_{m-1}} \, , \tag{3.96}$$

where $\partial\Sigma$ is the $(D-2)$-dimensional boundary with the induced metric γ. Hence, we can integrate on $(D-2)$-dimensional space to calculate the conserved charges instead of $(D-m-1)$-dimensional space if the background is flat.

3.3 Conserved Currents

3.3.1 KT Current

Rank-n Killing-Yano tensors can be used to construct rank-n conserved currents. Kastor and Traschen [1] showed that

$$
\begin{aligned}
j^{\mu_1\dots\mu_n} =& N_n\, \delta^{\mu_1\dots\mu_n\sigma_1\sigma_2}_{\nu_1\dots\nu_n\rho_1\rho_2}\, f^{\nu_1\dots\nu_n}\, R_{\sigma_1\sigma_2}{}^{\rho_1\rho_2} \\
=& -\frac{(n-1)}{4}\, R^{[\mu_1\mu_2}{}_{\nu\rho}\, f^{\mu_3\dots\mu_n]\nu\rho} + (-1)^{n+1}\, R_\rho{}^{[\mu_1}\, f^{\mu_2\dots\mu_n]\rho} - \frac{1}{2n}\, R\, f^{\mu_1\dots\mu_n} ,
\end{aligned}
\tag{3.97}
$$

satisfies

$$
\nabla_\nu j^{\mu_1\dots\mu_n} = 0 , \tag{3.98}
$$

where the symbol $\delta^{\mu_1\dots\mu_n}_{\nu_1\dots\nu_n}$ is the generalised Kronecker delta

$$
\delta^{\mu_1\dots\mu_n}_{\nu_1\dots\nu_n} = \delta^{[\mu_1}_{\nu_1}\dots\delta^{\mu_n]}_{\nu_n} . \tag{3.99}
$$

Assuming that the metric can be asymptotically split into two parts

$$
g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} , \tag{3.100}
$$

where $\bar{g}$ is the background, and h is a deviation which has small components compared to g at the spatial infinity, and $\bar{f}$ is a Killing-Yano tensor for the background,

$$
\bar{\nabla}_\mu \bar{f}_{\nu_1\dots\nu_n} + \bar{\nabla}_{\nu_1}\bar{f}_{\mu\nu_2\dots\nu_n} = 0 , \tag{3.101}
$$

the linearized version of the current (3.97) is

$$
(j^{\mu_1\dots\mu_n})_L = N_n\, \delta^{\mu_1\dots\mu_n\sigma_1\sigma_2}_{\nu_1\dots\nu_n\rho_1\rho_2}\, \bar{f}^{\nu_1\dots\nu_n} \left(\bar{R}_{\sigma_1\sigma_2\kappa}{}^{[\rho_1}\, h^{\rho_2]\kappa} + 2\,\bar{\nabla}_{\sigma_1}\bar{\nabla}^{\rho_2} h_{\sigma_2}{}^{\rho_1}\right) , \tag{3.102}
$$

and satisfies

$$
\bar{\nabla}_{\mu_1}(j^{\mu_1\dots\mu_n})_L = 0 . \tag{3.103}
$$

Here $\bar{\nabla}$ is the covariant derivative and $\bar{R}^\mu{}_{\nu\rho\sigma}$ is the Riemann curvature tensor associated with the background metric $\bar{g}_{\mu\nu}$.

It is shown that (3.102) can be used to construct a conserved asymptotic charge similar to AD charge construction [10] for asymptotically transverse flat backgrounds

[1], and asymptotically (A)dS backgrounds [23]. Later, a general condition on the background curvature for the KT current to have asymptotic charges is given in [11] as

$$\bar{f}^{[\mu_1 \dots \mu_n}\, \bar{R}_{\nu_1 \nu_2 \rho}{}^{\nu_1}\, h^{\nu_2]\rho} + (-1)^n 2 h_{\nu_2}{}^{[\nu_2} \bar{R}_{\rho}{}^{\nu_1}{}_{\nu_1}{}^{\mu_1}\, \bar{f}^{\mu_2 \dots \mu_n]\rho} = 0\,. \tag{3.104}$$

When (3.104) is satisfied, $(j)_L$ can be written as the divergence of a totally antisymmetric potential $\bar{\ell}$ with $\bar{\ell}^{\mu\nu_1\dots\nu_n} = \bar{\ell}^{[\mu\nu_1\dots\nu_n]}$

$$(j^{\nu_1 \dots \nu_n})_L = \bar{\nabla}_\mu \bar{\ell}^{\mu\nu_1\dots\nu_n}\ . \tag{3.105}$$

The potential $\bar{\ell}$ for a rank-n Killing-Yano tensor is given as

$$\begin{aligned}\bar{\ell}^{\nu\mu_1\dots\mu_n} = 2N_n\, \delta^{\mu_1\dots\mu_n\nu\rho}_{\sigma_1\dots\sigma_n\eta_1\eta_2}\, \bar{f}^{\sigma_1\dots\sigma_n}\, \bar{\nabla}^{\eta_2}\, h_\rho{}^{\eta_1} \\ - \frac{1}{2n}\Big(h\, \bar{\nabla}^\nu\, \bar{f}^{\mu_1\dots\mu_n} - (n+1)\, h^{\rho[\nu}\, \bar{\nabla}_\rho\, \bar{f}^{\mu_1\dots\mu_n]}\Big)\ .\end{aligned} \tag{3.106}$$

Then, a conserved charge can be found by integrating the potential at spatial infinity

$$Q = \int_{\partial\Sigma} d^{D-3}x n_{[\mu} x_\nu r_{\rho]} \sqrt{|\bar{g}^{(\partial\Sigma)}|}\bar{\ell}^{\mu\nu\rho}\ . \tag{3.107}$$

3.3.2 Cotton Current

Killing-Yano tensors and CKYTs can be used to construct Cotton currents introduced in [12]. The Cotton tensor in $D \geq 3$ dimensions is defined as

$$C_{\mu\nu\rho} = 2\nabla_{[\rho} R_{\nu]\mu} - \frac{1}{D-1} g_{\mu[\nu}\nabla_{\rho]} R\ , \tag{3.108}$$

where $R_{\mu\nu}$ is the Ricci tensor, and R is the curvature scalar. The Cotton tensor is antisymmetric on two indices, traceless on all index pairs, and divergenceless. It satisfies

$$\begin{aligned} C_{\mu\nu\rho} &= C_{\mu[\nu\rho]}\ , \\ C_{[\mu\nu\rho]} &= 0\ , \\ \nabla^\mu C_{\mu\nu\rho} &= 0\ . \end{aligned} \tag{3.109}$$

It is proportional to the divergence of the Weyl tensor $W_{\sigma\mu\nu\rho}$ in $D > 3$ dimensions [12]

$$C_{\mu\nu\rho} = \frac{D-2}{D-3} \nabla_\sigma W^\sigma{}_{\mu\nu\rho}\ . \tag{3.110}$$

In 3-dimensional manifolds, $C_{\mu\nu\rho} = 0$ is the necessary and sufficient condition for the conformal flatness [13]. Two conserved currents can be constructed using the Cotton tensor [12]

$$j^\mu = C^{\mu\nu\rho} f_{\nu\rho} \,, \qquad J^\mu = C^{\mu\nu\rho} k_{\nu\rho} \,. \tag{3.111}$$

f and k are rank-2 Killing-Yano tensor and CKYT, respectively. Both currents satisfy

$$\nabla_\mu j^\mu = 0 \,, \qquad \nabla_\mu J^\mu = 0 \,. \tag{3.112}$$

Using the identity (2.42), and divergenceless property (2.40), current j can be written as

$$j^\mu = \nabla_\nu \left(\frac{(D-2)}{(D-1)} f^{\mu\nu} R \right) \,. \tag{3.113}$$

Thus, a potential ℓ for the current j satisfying $j^\nu = \nabla_\mu \ell^{\mu\nu}$ can be written as

$$\ell^{\mu\nu}_{(f)} = \frac{(D-2)}{(D-1)} f^{\mu\nu} R \,. \tag{3.114}$$

A potential ℓ for the current J is also given in [12].

$$\ell^{\mu\nu}_{(k)} = \frac{2\,(D-4)}{(D-3)} G^{\rho[\mu} k_\rho{}^{\nu]} + \frac{2\,(D-2)^2}{(D-1)\,(D-3)} \nabla^{[\mu} \bar{k}^{\nu]} + \left(\frac{D-2}{D-1} \right) R k^{\mu\nu} \,, \tag{3.115}$$

where G is the Einstein tensor and $\bar{k}$ is defined as $\bar{k}_\mu := \nabla_\nu k^\nu{}_\mu$.

CHAPTER 4

KT CHARGES FOR MYERS-PERRY METRICS

In this chapter, we will use the Killing-Yano tensors of Myers-Perry metrics to construct conserved charges. We will first divide the spacetime into the flat background and the deviation. Then, using the Killing-Yano tensors of the background, we will calculate the potentials (3.106) for the KT current. Finally, we will obtain asymptotic conserved charges using (3.96) for 4, 5, 6, 7, and 8 dimensions.

4.1 Myers-Perry and Kerr-Schild (KS) Form

In even $(2n+2)$ dimensions, Myers-Perry metric [24] is

$$ds^2 = -dt^2 + \frac{mr}{FR}(dt + \sum_{i=1}^{n} a_i \mu_i^2 d\phi_i)^2 + \frac{FRdr^2}{R - mr} + \sum_{i=1}^{n}(r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2) + r^2 d\alpha^2, \tag{4.1}$$

subject to constraint

$$\alpha^2 + \sum_{i=1}^{n} \mu_i^2 = 1 \ . \tag{4.2}$$

In odd $(2n+1)$ dimensions

$$ds^2 = -dt^2 + \frac{mr^2}{FR}(dt + \sum_{i=1}^{n} a_i \mu_i^2 d\phi_i)^2 + \frac{FRdr^2}{R - mr^2} + \sum_{i=1}^{n}(r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2), \tag{4.3}$$

subject to constraint

$$\sum_{i=1}^{n} \mu_i^2 = 1 \ . \tag{4.4}$$

F and R functions are given as

$$F = 1 - \sum_{k=1}^{n} \frac{a_k^2 \mu_k^2}{r^2 + a_k^2}, \quad R = \prod_{k=1}^{n}(r^2 + a_k^2) \ . \tag{4.5}$$

These coordinates are referred to as Boyer-Lindquist (BL) coordinates since, in 4 dimensions, $a \to -a$ and $m \to 2m$ gives Kerr metric in BL coordinates. In KS form, the Myers-Perry metric can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + hk_\mu k_\nu \ . \tag{4.6}$$

Here, k_μ is a null vector with respect to both g and η. Thus, it satisfies

$$k_\mu k_\nu g^{\mu\nu} = k_\mu k_\nu \eta^{\mu\nu} = 0 \ . \tag{4.7}$$

The inverse metric $g^{\mu\nu}$ can be written exactly as

$$g^{\mu\nu} = \eta^{\mu\nu} - hk^\mu k^\nu \ . \tag{4.8}$$

k differs in even and odd dimensions. In the following discussion, let us set the parameter ε to 1 for even dimensions, and to 0 for odd dimensions

$$k_\mu dx^\mu = dt + \sum_{i=1}^{n} \frac{r(x^i dx^i + y^i dy^i) + a_i(x^i dy^i - y^i dx^i)}{r^2 + a_i^2} + \varepsilon \frac{zdz}{r} \ , \tag{4.9}$$

$$h_{odd} = \frac{mr^2}{FR}, \quad h_{even} = \frac{mr}{FR} \ . \tag{4.10}$$

F is also modified

$$F = 1 - \sum_{i=1}^{n} a_i^2 \frac{(x^i)^2 + (y^i)^2}{(r^2 + a_i^2)^2} \ . \tag{4.11}$$

In this form, r is not a coordinate but a function of (x^i, y^i, z). It is determined from the nullness condition of k_μ (4.7)

$$\sum_{i=1}^{n} \frac{(x^i)^2 + (y^i)^2}{r^2 + a_i^2} + \varepsilon \frac{z^2}{r^2} = 1 \ . \tag{4.12}$$

Coordinate transformation from KS form (t, x^i, y^i, z) to Eddington-like coordinates $(\bar{t}, r, \bar{\phi}_i, \mu_i)$ [25] is given by the following relations:

$$x^i = \mu_i \sqrt{r^2 + a_i^2} \cos\left[\bar{\phi}_i - \arctan\frac{a_i}{r}\right] \ , \tag{4.13}$$

$$y^i = \mu_i \sqrt{r^2 + a_i^2} \sin\left[\bar{\phi}_i - \arctan\frac{a_i}{r}\right] \ , \tag{4.14}$$

$$z = r\alpha = r\sqrt{1 - \sum_{i=1}^{n} \mu_i^2} \ . \tag{4.15}$$

Note that the final z transformation does not exist in odd dimensions. In order to get BL coordinates (t, r, ϕ_i, μ_i), one more transformation is needed. For odd dimensions,

$$dt = d\bar{t} - \frac{mr^2}{F - mr^2} dr \ , \tag{4.16}$$

$$d\phi_i = d\bar{\phi}_i + \frac{F}{F - mr^2} \frac{a_i}{r^2 + a_i^2} dr \ , \tag{4.17}$$

and for even dimensions,

$$dt = d\bar{t} - \frac{mr}{F - mr} dr \ , \tag{4.18}$$

$$d\phi_i = d\bar{\phi}_i + \frac{F}{F - mr} \frac{a_i}{r^2 + a_i^2} dr \ . \tag{4.19}$$

4.1.1 Killing-Yano Tensor in KS Coordinates

We found that the PCKYT in Cartesian like coordinates (KS coordinates) is in a simple form

$$h_{\mu\nu} dx^\mu \wedge dx^\nu = \sum_{i=1}^{n} \left(-a_i dx^i \wedge dy^i + x^i dt \wedge dx^i + y^i dt \wedge dy^i \right) + \varepsilon z dt \wedge dz \ . \tag{4.20}$$

Transforming the coordinates from KS to BL by transformations given in the previous section, h becomes

$$h_{\mu\nu} dx^\mu \wedge dx^\nu = \sum_{i=1}^{n} a_i \mu_i d\mu_i \wedge \left[a_i dt + \left(r^2 + a_i^2 \right) d\phi_i \right] + r dr \wedge \left[dt + \sum_{i=1}^{n} a_i \mu_i^2 d\phi_i \right] , \tag{4.21}$$

which is the known PCKYT of Myers-Perry metric [26]. To get the Killing-Yano tensor of rank-$(D-2)$ in KS coordinates, we will take the Hodge dual of PCKYT given in equation (4.20)

$$\begin{aligned} f = *h = \sum_{i=1}^{n} (& -a_i dx^1 \wedge dy^1 \wedge ... \wedge \hat{dx^i} \wedge \hat{dy^i} \wedge ... \wedge dx^n \wedge dy^n \wedge dz \wedge dt \\ & + x^i dx^1 \wedge dy^1 \wedge ... \wedge \hat{dx^i} \wedge dy^i \wedge ... \wedge dx^n \wedge dy^n \wedge dz \\ & + y^i dx^1 \wedge dy^1 \wedge ... \wedge dx^i \wedge \hat{dy^i} \wedge ... \wedge dx^n \wedge dy^n \wedge dz) \\ & + z dx^1 \wedge dy^1 \wedge ... \wedge dx^n \wedge dy^n \ . \end{aligned} \tag{4.22}$$

The differential forms with a hat on them are omitted from the wedge products. The form of f in KS coordinates is simpler than that in the BL coordinates. Even though (4.22) looks complicated, for each a_i, coordinates except the rotation plane are wedged, and for remaining terms, like x^i, other spatial coordinates are wedged considering cyclicity. Note that, for odd dimensions, z term does not contribute, and there is no dz term in wedge products.

4.2 Myers-Perry Metric in Ellipsoidal Coordinates

Another useful coordinate system for Myers-Perry metrics is the ellipsoidal coordinates. Transformations from BL coordinates (t, r, μ_i, ϕ_i) to ellipsoidal coordinates (t, r, x_i, ϕ_i) in even dimensions are given as

$$(a_i\mu_i)^2 = \frac{1}{c_i^2}\prod_{k=1}^{n}(a_i^2 + x_k), \quad \text{where } c_i^2 = \prod_{k=1(\neq i)}^{n}(a_i^2 - a_k^2) \,. \tag{4.23}$$

Note that, when the rotation parameters a_i vanish, the coordinate transformation is not well defined. Following [16], the Myers-Perry metric in the ellipsoidal coordinates can be written as

$$ds^2 = -(e^t)^2 + (e^r)^2 + \sum_{i=1}^{n}[(e^{x_i})^2 + (e^{\phi_i})^2] \,, \tag{4.24}$$

with the frames

$$\begin{aligned} e^t &= \sqrt{\frac{R - mr}{FR}}\left[dt + \sum_{k=1}^{n}\frac{G_k}{a_k c_k^2}d\phi_k\right], \quad e^{x_i} = \sqrt{\frac{-(r^2 - x_i)d_i}{4x_i H_i}}dx_i \,, \\ e^{\phi_i} &= \sqrt{\frac{H_i}{d_i(r^2 - x_i)}}\left[dt + \sum_{k=1}^{n}\frac{G_k(r^2 + a_k^2)}{a_k c_k^2(x_i + a_k^2)}d\phi_k\right], \quad e^r = \sqrt{\frac{FR}{R - mr}}dr \,, \end{aligned} \tag{4.25}$$

$$\begin{aligned} d_i &= \prod_{k=1(\neq i)}^{n}(x_i - x_k), \quad H_i = \prod_{k=1}^{n}(x_i + a_k^2), \quad G_i = \prod_{k=1}^{n}(x_k + a_i^2) \,, \\ R &= \prod_{k=1}^{n}(r^2 + a_k^2), \quad FR = \prod_{k=1}^{n}(r^2 - x_k), \quad c_i^2 = \prod_{k=1(\neq i)}^{n}(a_i^2 - a_k^2) \,. \end{aligned} \tag{4.26}$$

In odd dimensions, the coordinate transformations are modified

$$\mu_i^2 = \frac{1}{c_i^2}\prod_{k=1}^{n-1}(a_i^2 + x_k) \,. \tag{4.27}$$

The frames in ellipsoidal coordinates are also modified for odd dimensions

$$e^t = \sqrt{\frac{R - mr^2}{FR}} \left[dt + \sum_{k=1}^{n} \frac{a_k G_k}{c_k^2} d\phi_k \right] , \quad e^{x_i} = \sqrt{\frac{(r^2 - x_i) d_i}{4H_i}} dx_i ,$$

$$e^{\phi_i} = \sqrt{-\frac{H_i}{x_i d_i (r^2 - x_i)}} \left[dt + \sum_{k=1}^{n} \frac{G_k a_k (r^2 + a_k^2)}{c_k^2 (x_i + a_k^2)} d\phi_k \right] , \quad e^r = \sqrt{\frac{FR}{R - mr^2}} dr , \tag{4.28}$$

$$d_i = \prod_{k=1(\neq i)}^{n-1} (x_i - x_k), \quad H_i = \prod_{k=1}^{n} (x_i + a_k^2), \quad G_i = \prod_{k=1}^{n-1} (x_k + a_i^2) ,$$

$$R = \prod_{k=1}^{n} (r^2 + a_k^2), \quad FR = r^2 \prod_{k=1}^{n-1} (r^2 - x_k), \quad c_i^2 = \prod_{k=1(\neq i)}^{n} (a_i^2 - a_k^2) . \tag{4.29}$$

Moreover, there is one additional frame that was not present in even dimensions:

$$e^{\psi} = \sqrt{\frac{\prod_{k=1}^{n} a_k^2}{r^2 \prod_{k=1}^{n-1} (-x_k)}} \left[dt + \sum_{k=1}^{n} \frac{G_k (r^2 + a_k^2)}{c_k^2 a_k} d\phi_k \right] . \tag{4.30}$$

In the ellipsoidal coordinates, the PCKYT h is the same for even and odd dimensions

$$h = r e^r \wedge e^t + \sum_{i=1}^{n-1} \sqrt{-x_i} e^{x_i} \wedge e^{\phi_i} . \tag{4.31}$$

In order to obtain Killing-Yano tensors, the discussion in subsection 2.3.2 can be used, since we know the PCKYT. The wedge product of a PCKYT with itself ($h \wedge .. \wedge h$) is also a CCKYT. The Hodge dual of a CCKYT gives a Killing-Yano tensor. Thus, the Killing-Yano tensor of rank-$(D-2)$ is

$$f^{(D-2)} = *h . \tag{4.32}$$

The lower rank Killing-Yano tensors can be found by

$$f^{(D-2k)} = * \left(\wedge h^k \right) , \tag{4.33}$$

where $\wedge h^k$ means that h is wedged with itself k times. For example, in even dimensions, Killing-Yano tensor of rank-$2n$ is

$$f^{(2n)} = r \prod_{k=1}^{n} [e^{x_k} \wedge e^{\phi_k}] + e^t \wedge e^r \left[\sum_{i=1}^{n} (-\sqrt{-x_i}) \prod_{k=1(\neq i)}^{n} (e^{x_k} \wedge e^{\phi_k}) \right] . \tag{4.34}$$

We also have the volume form, which is the simplest Killing-Yano tensor

$$f^{(D)} = e^t \wedge e^r \wedge \prod_{k=1}^{n} (e^{x_k} \wedge e^{\phi_k}) \wedge e^{\psi} \ , \tag{4.35}$$

where e^{ψ} is wedged only in odd dimensions.

4.2.1 Killing-Yano Tensors in Even Dimensions

4.2.1.1 4 Dimensions (Kerr Metric)

In $D = 4$, the Myers-Perry metric is equivalent to the Kerr metric with the modifications $m \to 2m$ and $a \to -a$. The Killing-Yano tensor of rank-2 is

$$f^{(2)} = re^x \wedge e^{\phi} - \sqrt{-x_i} e^t \wedge e^r \ . \tag{4.36}$$

The functions (4.26) and the frames in the ellipsoidal coordinates are

$$H = (x + a^2) = G, \quad R = (r^2 + a^2), \quad FR = (r^2 - x) \ , \tag{4.37}$$

$$\begin{aligned} e^t &= \sqrt{\frac{(r^2 + a^2) - mr}{(r^2 - x)}} \left[dt + \frac{(x + a^2)}{a} d\phi \right], \quad e^r = \sqrt{\frac{(r^2 - x)}{(r^2 + a^2) - mr}} dr \ , \\ e^{\phi} &= \sqrt{\frac{(x + a^2)}{(r^2 - x)}} \left[dt + \frac{(r^2 + a^2)}{a} d\phi \right], \quad e^x = \sqrt{\frac{-(r^2 - x)}{4x(x + a^2)}} dx \ , \end{aligned} \tag{4.38}$$

and we set $c = d = 1$. Wedge product of these frames are

$$e^t \wedge e^r = dt \wedge dr + \frac{(x + a^2)}{a} d\phi \wedge dr \ , \tag{4.39}$$

$$e^x \wedge e^{\phi} = \sqrt{\frac{-1}{4x}} \left[dx \wedge dt + \frac{(r^2 + a^2)}{a} dx \wedge d\phi \right] \ . \tag{4.40}$$

Substituting these to the Killing-Yano tensor, we get

$$\begin{aligned} f^{(2)} = \sqrt{\frac{-r^2}{4x}} &\left[dx \wedge dt + \frac{(r^2 + a^2)}{a} dx \wedge d\phi \right] \\ &- \sqrt{-x} \left[dt \wedge dr + \frac{(x + a^2)}{a} d\phi \wedge dr \right] \ . \end{aligned} \tag{4.41}$$

One can use equation (4.23) to turn back to μ

$$x = a^2 \mu^2 - a^2 \ , \tag{4.42}$$

$$dx = 2a^2\mu d\mu \; . \tag{4.43}$$

Thus, the Killing-Yano tensor becomes

$$\begin{aligned} f^{(2)} = &\sqrt{\frac{r^2}{(1-\mu^2)}}\mu\left[ad\mu\wedge dt + (r^2+a^2)d\mu\wedge d\phi\right] \\ &+\sqrt{a^2-a^2\mu^2}\left[dt\wedge dr + a\mu^2 d\phi\wedge dr\right] \; . \end{aligned} \tag{4.44}$$

Substituting $\mu = \sin\theta$ and $d\mu = \cos\theta d\theta$, also letting $a \to -a$, we get the Killing-Yano tensor for Kerr metric in BL coordinates

$$f^{(2)} = ar\sin\theta dt\wedge d\theta + r(r^2+a^2)\sin\theta d\theta\wedge d\phi - a\cos\theta dt\wedge dr + a^2\cos\theta\sin^2\theta d\phi\wedge dr. \tag{4.45}$$

4.2.1.2 6 Dimensions

In $D = 6$, there are two Killing-Yano tensors obtainable from the PCKYT (4.31). The first one is the dual of h which is a Killing-Yano tensor of rank-4

$$f^{(4)} = re^{x_1}\wedge e^{\phi_1}\wedge e^{x_2}\wedge e^{\phi_2} - \sqrt{-x_1}e^t\wedge e^r\wedge e^{x_2}\wedge e^{\phi_2} - \sqrt{-x_2}e^t\wedge e^r\wedge e^{x_1}\wedge e^{\phi_1} \; . \tag{4.46}$$

The second one is the Killing-Yano tensor of rank-2

$$f^{(2)} = r(-\sqrt{-x_1}e^{x_2}\wedge e^{\phi_2} - \sqrt{-x_2}e^{x_1}\wedge e^1) + \sqrt{x_1 x_2}e^t\wedge e^r \; . \tag{4.47}$$

The frames (4.25) in 6 dimensions are as follows

$$\begin{aligned} H_i = (x_i+a_1^2)(x_i+a_2^2), \quad G_i = (x_1+a_i^2)(x_2+a_i^2), \\ R = (r^2+a_1^2)(r^2+a_2^2), \quad FR = (r^2-x_1)(r^2-x_2) \; , \end{aligned} \tag{4.48}$$

$$\begin{aligned} e^t = &\sqrt{\frac{(r^2+a_1^2)(r^2+a_2^2)-mr}{(r^2-x_1)(r^2-x_2)}}\left[dt + \frac{(x_1+a_1^2)(x_2+a_1^2)}{a_1(a_1^2-a_2^2)}d\phi_1\right. \\ &\left.+\frac{(x_1+a_2^2)(x_2+a_2^2)}{a_2(a_2^2-a_1^2)}d\phi_2\right] \; , \end{aligned}$$

$$\begin{aligned} e^{\phi_1} = &\sqrt{\frac{(x_1+a_1^2)(x_1+a_2^2)}{(x_1-x_2)(r^2-x_1)}}\left[dt + \frac{(x_2+a_1^2)(r^2+a_1^2)}{a_1(a_1^2-a_2^2)}d\phi_1\right. \\ &\left.+\frac{(x_2+a_2^2)(r^2+a_2^2)}{a_2(a_2^2-a_1^2)}d\phi_2\right] \; , \end{aligned}$$

$$e^{\phi_2} = \sqrt{\frac{(x_2+a_1^2)(x_2+a_2^2)}{(x_2-x_1)(r^2-x_2)}}\left[dt + \frac{(x_1+a_1^2)(r^2+a_1^2)}{a_1(a_1^2-a_2^2)}d\phi_1 \right.$$
$$\left. + \frac{(x_1+a_2^2)(r^2+a_2^2)}{a_2(a_2^2-a_1^2)}d\phi_2\right] \ ,$$

$$e^r = \sqrt{\frac{(r^2-x_1)(r^2-x_2)}{(r^2+a_1^2)(r^2+a_2^2)-mr}}dr \ ,$$
$$e^{x_1} = \sqrt{\frac{-(r^2-x_1)(x_1-x_2)}{4x_1(x_1+a_1^2)(x_1+a_2^2)}}dx_1 \ , \quad e^{x_2} = \sqrt{\frac{-(r^2-x_2)(x_2-x_1)}{4x_2(x_2+a_1^2)(x_2+a_2^2)}}dx_2 \ . \tag{4.49}$$

The wedge products of these frames are

$$e^t \wedge e^r = dt\wedge dr + \frac{(x_1+a_1^2)(x_2+a_1^2)}{a_1(a_1^2-a_2^2)}d\phi_1\wedge dr + \frac{(x_1+a_2^2)(x_2+a_2^2)}{a_2(a_2^2-a_1^2)}d\phi_2\wedge dr \ , \tag{4.50}$$

$$e^{x_1}\wedge e^{\phi_1} = \sqrt{\frac{-1}{4x_1}}\left[dx_1\wedge dt + \frac{(x_2+a_1^2)(r^2+a_1^2)}{a_1(a_1^2-a_2^2)}dx_1\wedge d\phi_1 \right.$$
$$\left. + \frac{(x_2+a_2^2)(r^2+a_2^2)}{a_2(a_2^2-a_1^2)}dx_1\wedge d\phi_2\right] \ , \tag{4.51}$$

$$e^{x_2}\wedge e^{\phi_2} = \sqrt{\frac{-1}{4x_2}}\left[dx_2\wedge dt + \frac{(x_1+a_1^2)(r^2+a_1^2)}{a_1(a_1^2-a_2^2)}dx_2\wedge d\phi_1 \right.$$
$$\left. + \frac{(x_1+a_2^2)(r^2+a_2^2)}{a_2(a_2^2-a_1^2)}dx_2\wedge d\phi_2\right] \ . \tag{4.52}$$

Let us calculate $f^{(2)}$ as an example

$$\begin{aligned} f^{(2)} =& \frac{r}{2}dt\wedge\left[\sqrt{\frac{x_2}{x_1}}dx_1 + \sqrt{\frac{x_1}{x_2}}dx_2\right] + \sqrt{x_1x_2}dt\wedge dr \\ &+ \frac{r}{2}d\phi_1\wedge\left[\sqrt{\frac{x_2}{x_1}}\frac{(x_2+a_1^2)(r^2+a_1^2)}{a_1(a_1^2-a_2^2)}dx_1 + \sqrt{\frac{x_1}{x_2}}\frac{(x_1+a_1^2)(r^2+a_1^2)}{a_1(a_1^2-a_2^2)}dx_2\right] \\ &+ \frac{r}{2}d\phi_2\wedge\left[\sqrt{\frac{x_2}{x_1}}\frac{(x_2+a_2^2)(r^2+a_2^2)}{a_2(a_2^2-a_1^2)}dx_1 + \sqrt{\frac{x_1}{x_2}}\frac{(x_1+a_2^2)(r^2+a_2^2)}{a_2(a_2^2-a_1^2)}dx_2\right] \\ &+ \sqrt{x_1x_2}\left[\frac{(x_1+a_1^2)(x_2+a_1^2)}{a_1(a_1^2-a_2^2)}d\phi_1\wedge dr + \frac{(x_1+a_2^2)(x_2+a_2^2)}{a_2(a_2^2-a_1^2)}d\phi_2\wedge dr\right] \ . \end{aligned} \tag{4.53}$$

The following relations can be used to transform coordinates to/from ellipsoidal coordinates x_i:

$$a_1^2\mu_1^2 = \frac{(x_1+a_1^2)(x_2+a_1^2)}{a_1^2-a_2^2} \ , \tag{4.54}$$

$$a_2^2\mu_2^2 = \frac{(x_1 + a_2^2)(x_2 + a_2^2)}{a_2^2 - a_1^2} \ . \tag{4.55}$$

In the flat, nonrotating background, the wedge dual of the PCKYT (4.21) with itself vanishes. Thus, one can only obtain a rank-$(D-2)$ Killing-Yano tensor for the background of the Myers-Perry spacetimes. In $D = 6$, the rank-4 Killing-Yano tensor of the background is

$$f^{(4)} = r^5 \sin^3\theta_2 \sin\theta_1 \cos\theta_1 d\phi_1 \wedge d\phi_2 \wedge d\theta_1 \wedge d\theta_2 \ , \tag{4.56}$$

in the BL coordinates with the direction cosines chosen as $\mu_1 = \sin\theta_2 \sin\theta_1$, and $\mu_2 = \sin\theta_2\cos\theta_1$.

4.2.1.3 8 Dimensions

In $D = 8$, there are three Killing-Yano tensors. The Killing-Yano tensor of rank-6 is

$$\begin{aligned} f^{(6)} &= r e^{x_1} \wedge e^{\phi_1} \wedge e^{x_2} \wedge e^{\phi_2} \wedge e^{x_3} \wedge e^{\phi_3} - \sqrt{-x_1} e^t \wedge e^r \wedge e^{x_2} \wedge e^{\phi_2} \wedge e^{x_3} \wedge e^{\phi_3} \\ &- \sqrt{-x_2} e^t \wedge e^r \wedge e^{x_1} \wedge e^{\phi_1} \wedge e^{x_3} \wedge e^{\phi_3} - \sqrt{-x_3} e^t \wedge e^r \wedge e^{x_1} \wedge e^{\phi_1} \wedge e^{x_2} \wedge e^{\phi_2} \ . \end{aligned} \tag{4.57}$$

The Killing-Yano tensor of rank-4 is

$$\begin{aligned} f^{(4)} &= -r(\sqrt{-x_1} e^{x_2} \wedge e^{\phi_2} \wedge e^{x_3} \wedge e^{\phi_3} \\ &\quad + \sqrt{-x_2} e^{x_1} \wedge e^{\phi_1} \wedge e^{x_3} \wedge e^{\phi_3} + \sqrt{-x_3} e^{x_1} \wedge e^{\phi_1} \wedge e^{x_2} \wedge e^{\phi_2}) \\ &+ e^t \wedge e^r \wedge (\sqrt{x_1 x_2} e^{x_3} \wedge e^{\phi_3} + \sqrt{x_1 x_3} e^{x_2} \wedge e^{\phi_2} + \sqrt{x_2 x_3} e^{x_1} \wedge e^{\phi_1}) \ . \end{aligned} \tag{4.58}$$

The Killing-Yano tensor of rank-2 is

$$f^{(2)} = -\sqrt{-x_1 x_2 x_3} e^t \wedge e^r + r\sqrt{x_2 x_3} e^{x_1} \wedge e^{\phi_1} + r\sqrt{x_1 x_3} e^{x_2} \wedge e^{\phi_2} + r\sqrt{x_1 x_2} e^{x_3} \wedge e^{\phi_3} . \tag{4.59}$$

Frames (4.25) in 8 dimensions are modified such that

$$\begin{aligned} &H_i = (x_i + a_1^2)(x_i + a_2^2)(x_i + a_3^2) \ , \quad G_i = (x_1 + a_i^2)(x_2 + a_i^2)(x_3 + a_i^2) \ , \\ &R = (r^2 + a_1^2)(r^2 + a_2^2)(r^2 + a_3^2) \ , \quad FR = (r^2 - x_1)(r^2 - x_2)(r^2 - x_3) \ , \end{aligned} \tag{4.60}$$

$$e^t = \sqrt{\frac{(r^2+a_1^2)(r^2+a_2^2)(r^2+a_3^2)-mr}{(r^2-x_1)(r^2-x_2)(r^2-x_3)}}[dt + \frac{(x_1+a_1^2)(x_2+a_1^2)(x_3+a_1^2)}{a_1(a_1^2-a_2^2)(a_1^2-a_3^2)}d\phi_1 + \frac{(x_1+a_2^2)(x_2+a_2^2)(x_3+a_2^2)}{a_2(a_2^2-a_1^2)(a_2^2-a_3^2)}d\phi_2 + \frac{(x_1+a_3^2)(x_2+a_3^2)(x_3+a_3^2)}{a_3(a_3^2-a_1^2)(a_3^2-a_2^2)}d\phi_3] \, ,$$

$$e^r = \sqrt{\frac{(r^2-x_1)(r^2-x_2)(r^2-x_3)}{(r^2+a_1^2)(r^2+a_2^2)(r^2+a_3^2)-mr}}dr \, , \tag{4.61}$$

$$e^{\phi_1} = \sqrt{\frac{(x_1+a_1^2)(x_1+a_2^2)(x_1+a_3^2)}{(x_1-x_2)(x_1-x_3)(r^2-x_1)}}\left[dt + \frac{(x_2+a_1^2)(x_3+a_1^2)(r^2+a_1^2)}{a_1(a_1^2-a_2^2)(a_1^2-a_3^2)}d\phi_1 + \frac{(x_2+a_2^2)(x_3+a_2^2)(r^2+a_2^2)}{a_2(a_2^2-a_1^2)(a_2^2-a_3^2)}d\phi_2 + \frac{(x_2+a_3^2)(x_3+a_3^2)(r^2+a_3^2)}{a_3(a_3^2-a_1^2)(a_3^2-a_1^2)}d\phi_3\right] \, ,$$

$$e^{x_1} = \sqrt{\frac{-(r^2-x_1)(x_1-x_2)(x_1-x_3)}{4x_1(x_1+a_1^2)(x_1+a_2^2)(x_1+a_3^2)}}dx_1 \, ,$$

where $i = 1, 2, 3$. We will not give frames $e^{\phi_2}, e^{\phi_3}, e^{x_2}, e^{x_3}$ since e^{ϕ_1} and e^{x_1} gives enough insight about them. The relations for the coordinate transformation between BL and ellipsoidal coordinates are

$$\begin{aligned} a_1^2\mu_1^2 &= \frac{(x_1+a_1^2)(x_2+a_1^2)(x_3+a_1^2)}{(a_1^2-a_2^2)(a_1^2-a_3^2)} \, , \\ a_2^2\mu_2^2 &= \frac{(x_1+a_2^2)(x_2+a_2^2)(x_3+a_2^2)}{(a_2^2-a_1^2)(a_2^2-a_3^2)} \, , \\ a_3^2\mu_3^2 &= \frac{(x_1+a_3^2)(x_2+a_3^2)(x_3+a_3^2)}{(a_3^2-a_1^2)(a_3^2-a_2^2)} \, . \end{aligned} \tag{4.62}$$

For the flat nonrotating metric, only the Killing-Yano tensor of rank-6 survives. Choosing $\mu_1 = \sin\theta_3 \sin\theta_2 \sin\theta_1$, $\mu_2 = \sin\theta_3 \sin\theta_2 \cos\theta_1$, and $\mu_3 = \sin\theta_3 \cos\theta_2$, it becomes

$$f^{(6)} = r^7 \cos\theta_1 \cos\theta_2 \sin\theta_1 \sin^3\theta_2 \sin^5\theta_3 \, d\phi_1 \wedge d\phi_2 \wedge d\phi_3 \wedge d\theta_1 \wedge d\theta_2 \wedge d\theta_3 \, , \tag{4.63}$$

in the BL coordinates.

4.2.2 Killing-Yano Tensors in Odd Dimensions

4.2.2.1 5 Dimensions

In $D = 5$, we can find a rank-3 Killing-Yano tensor by taking the Hodge dual of the PCKYT (4.31)

$$f^{(3)} = *h = -re^{\phi} \wedge e^x \wedge e^{\psi} + \sqrt{-x}e^{\psi} \wedge e^t \wedge e^r \ . \tag{4.64}$$

The frames (4.28) are

$$\begin{aligned} d = 1, \quad H = (x + a_1^2)(x + a_2^2), \quad G_i = (x + a_i^2) \ , \\ R = (r^2 + a_1^2)(r^2 + a_2^2), \quad FR = r^2(r^2 - x), \quad c_1^2 = (a_1^2 - a_2^2) = -c_2^2 \ , \end{aligned} \tag{4.65}$$

$$\begin{aligned} e^t &= \sqrt{\frac{R - mr}{r^2(r^2 - x)}} \left[dt + \frac{a_1(x + a_1^2)}{(a_1^2 - a_2^2)} d\phi_1 + \frac{a_2(x + a_2^2)}{(a_2^2 - a_1^2)} d\phi_2 \right] \ , \\ e^r &= \sqrt{\frac{r^2(r^2 - x)}{R - mr}} dr \ , \quad e^x = \sqrt{\frac{(r^2 - x)}{4(x + a_1^2)(x + a_2^2)}} dx \ , \\ e^{\phi} &= \sqrt{-\frac{(x + a_1^2)(x + a_2^2)}{x(r^2 - x)}} \left[dt + \frac{a_1(r^2 + a_1^2)}{(a_1^2 - a_2^2)} d\phi_1 + \frac{a_2(r^2 + a_2^2)}{(a_2^2 - a_1^2)} d\phi_2 \right] \ , \\ e^{\psi} &= \sqrt{-\frac{a_1^2 a_2^2}{xr^2}} \left[dt + \frac{(x + a_1^2)(r^2 + a_1^2)}{(a_1^2 - a_2^2)a_1} d\phi_1 + \frac{(x + a_2^2)(r^2 + a_2^2)}{(a_2^2 - a_1^2)a_2} d\phi_2 \right] \ , \end{aligned} \tag{4.66}$$

where $i = 1, 2$. Let us calculate the Killing-Yano tensor. The wedge products of these frames are

$$\begin{aligned} e^{\psi} \wedge e^t \wedge e^r = &-\frac{a_1 a_2}{r\sqrt{-x}} \left[\frac{a_1^2(x + a_1^2) - (x + a_1^2)(r^2 + a_1^2)}{(a_1^2 - a_2^2)a_1} dt \wedge d\phi_1 \wedge dr \right. \\ &+ \frac{a_2^2(x + a_2^2) - (x + a_2^2)(r^2 + a_2^2)}{(a_2^2 - a_1^2)a_2} dt \wedge d\phi_2 \wedge dr \\ &\left. - (x + a_1^2)(x + a_2^2) \frac{a_2^2(r^2 + a_1^2) - a_1^2(r^2 + a_2^2)}{(a_1^2 - a_2^2)^2 a_1 a_2} d\phi_1 \wedge d\phi_2 \wedge dr \right] \\ = &\frac{1}{(a_1^2 - a_2^2)} \frac{r}{\sqrt{-x}} [-a_2(x + a_1^2) dt \wedge d\phi_1 \wedge dr + a_1(x + a_2^2) dt \wedge d\phi_2 \wedge dr \\ &+ (x + a_1^2)(x + a_2^2) d\phi_1 \wedge d\phi_2 \wedge dr] \ , \end{aligned} \tag{4.67}$$

$$
\begin{aligned}
e^{\psi} \wedge e^{\phi_1} \wedge e^{x} = \frac{a_1 a_2}{2xr} \Bigg[& \left(\frac{(x+a_1^2)(r^2+a_1^2)}{(a_1^2-a_2^2)a_1} - \frac{a_1(r^2+a_1^2)}{(a_1^2-a_2^2)} \right) dt \wedge d\phi_1 \wedge dx \\
& + \left(\frac{(x+a_2^2)(r^2+a_2^2)}{(a_2^2-a_1^2)a_2} - \frac{a_2(r^2+a_2^2)}{(a_2^2-a_1^2)} \right) dt \wedge d\phi_2 \wedge dx \\
& -(r^2+a_1^2)(r^2+a_2^2) \frac{a_2^2(x+a_1^2) - a_1^2(x+a_2^2)}{(a_1^2-a_2^2)^2 a_1 a_2} d\phi_1 \wedge d\phi_2 \wedge dx \Bigg] \\
= \frac{1}{2r(a_1^2-a_2^2)} & [a_2(r^2+a_1^2) dt \wedge d\phi_1 \wedge dx - a_1(r^2+a_2^2) dt \wedge d\phi_2 \wedge dx \\
& + (r^2+a_1^2)(r^2+a_2^2) d\phi_1 \wedge d\phi_2 \wedge dx] \, .
\end{aligned}
\tag{4.68}
$$

The Killing-Yano tensor becomes

$$
\begin{aligned}
f^{(3)} = & -\frac{1}{2(a_1^2-a_2^2)} [a_2(r^2+a_1^2) dt \wedge d\phi_1 \wedge dx - a_1(r^2+a_2^2) dt \wedge d\phi_2 \wedge dx \\
& \qquad + (r^2+a_1^2)(r^2+a_2^2) d\phi_1 \wedge d\phi_2 \wedge dx] \\
& + \frac{r}{(a_1^2-a_2^2)} [-a_2(x+a_1^2) dt \wedge d\phi_1 \wedge dr + a_1(x+a_2^2) dt \wedge d\phi_2 \wedge dr \\
& \qquad + (x+a_1^2)(x+a_2^2) d\phi_1 \wedge d\phi_2 \wedge dr] \, .
\end{aligned}
\tag{4.69}
$$

Letting $\mu_1 = \sin\theta$, and $\mu_2 = \cos\theta$, coordinates can be transformed from ellipsoidal coordinates to BL coordinates by

$$
\mu_1^2 = \frac{(a_1^2+x)}{(a_1^2-a_2^2)} = \sin^2\theta \, , \tag{4.70}
$$

$$
\mu_2^2 = -\frac{(a_2^2+x)}{(a_1^2-a_2^2)} = \cos^2\theta \, . \tag{4.71}
$$

The coordinate x and the differential form dx in terms of θ are

$$
x = \sin^2\theta(a_1^2-a_2^2) - a_1^2 = -\cos^2\theta(a_1^2-a_2^2) - a_2^2 \, , \tag{4.72}
$$

$$
dx = 2\sin\theta\cos\theta(a_1^2-a_2^2) d\theta \, . \tag{4.73}
$$

Substitution to the Killing-Yano tensor finally yields

$$
\begin{aligned}
f^{(3)} = & -\sin\theta\cos\theta \left[a_2(r^2+a_1^2) dt \wedge d\phi_1 - a_1(r^2+a_2^2) dt \wedge d\phi_2 \right. \\
& \left. +(r^2+a_1^2)(r^2+a_2^2) d\phi_1 \wedge d\phi_2 \right] \wedge d\theta - r \left[a_2 \sin^2\theta dt \wedge d\phi_1 \right. \\
& \left. + a_1 \cos^2\theta dt \wedge d\phi_2 - \sin^2\theta\cos^2\theta(a_1^2-a_2^2) d\phi_1 \wedge d\phi_2 \right] \wedge dr \, .
\end{aligned}
\tag{4.74}
$$

4.2.2.2 7 Dimensions

In $D = 7$, the PCKYT (4.31) is

$$h = re^r \wedge e^t + \sqrt{-x_1} e^{x_1} \wedge e^{\phi_1} + \sqrt{-x_2} e^{x_2} \wedge e^{\phi_2} \,. \tag{4.75}$$

The rank-5 Killing-Yano tensor can be found by taking the Hodge dual of h

$$\begin{aligned} f^{(5)} &= *h \\ &= (re^{\phi_1} \wedge e^{x_1} \wedge e^{\phi_2} \wedge e^{x_2} + e^t \wedge e^r (\sqrt{-x_1} e^{x_2} \wedge e^{\phi_2} + \sqrt{-x_2} e^{x_1} \wedge e^{\phi_1})) \wedge e^{\psi}. \end{aligned} \tag{4.76}$$

There is also another CCKYT

$$h \wedge h = re^t \wedge e^r (\sqrt{-x_1} e^{\phi_1} \wedge e^{x_1} + \sqrt{-x_2} e^2 \wedge e^{x_2}) - \sqrt{x_1 x_2} e^{\phi_1} \wedge e^{\phi_2} \wedge e^{x_1} \wedge e^{x_2} \,, \tag{4.77}$$

which gives the rank-3 Killing-Yano tensor

$$f^{(3)} = *(h \wedge h) = r(\sqrt{-x_1} e^{\phi_2} \wedge e^{x_2} + \sqrt{-x_2} e^{\phi_1} \wedge e^{x_1}) \wedge e^{\psi} - \sqrt{x_1 x_2} e^t \wedge e^r \wedge e^{\psi}. \tag{4.78}$$

The frames (4.28) are

$$\begin{aligned} R = (r^2 + a_1^2)(r^2 + a_2^2)(r^2 + a_3^2), \quad H_j = (x_j + a_1^2)(x_j + a_2^2)(x_j + a_3^2), \\ d_1 = -d_2 = x_1 - x_2, \quad FR = r^2(r^2 - x_1)(r^2 - x_2), \quad G_i = (x_1 + a_i^2)(x_2 + a_i^2), \\ c_1^2 = (a_1^2 - a_2^2)(a_1^2 - a_3^2), \quad c_2^2 = (a_2^2 - a_1^2)(a_2^2 - a_3^2), \quad c_3^2 = (a_3^2 - a_1^2)(a_3^2 - a_2^2) \,. \end{aligned} \tag{4.79}$$

$$\begin{aligned} e^t &= \sqrt{\frac{R - mr}{r^2(r^2 - x_1)(r^2 - x_2)}} \left[dt + \frac{a_1(x_1 + a_1^2)(x_2 + a_1^2)}{(a_1^2 - a_2^2)(a_1^2 - a_3^2)} d\phi_1 \right. \\ &\quad \left. + \frac{a_2(x_1 + a_2^2)(x_2 + a_2^2)}{(a_2^2 - a_1^2)(a_2^2 - a_3^2)} d\phi_2 + \frac{a_3(x_1 + a_3^2)(x_2 + a_3^2)}{(a_3^2 - a_1^2)(a_3^2 - a_2^2)} d\phi_3 \right] \,, \\ e^r &= \sqrt{\frac{r^2(r^2 - x_1)(r^2 - x_2)}{R - mr}} dr \,, \\ e^{\phi_1} &= \sqrt{-\frac{(x_1 + a_1^2)(x_1 + a_2^2)(x_1 + a_3^2)}{x_1(x_1 - x_2)(r^2 - x_1)}} \left[dt + \frac{a_1(x_2 + a_1^2)(r^2 + a_1^2)}{(a_1^2 - a_2^2)(a_1^2 - a_3^2)} d\phi_1 \right. \\ &\quad \left. + \frac{a_2(x_2 + a_2^2)(r^2 + a_2^2)}{(a_2^2 - a_1^2)(a_2^2 - a_3^2)} d\phi_2 + \frac{a_3(x_2 + a_3^2)(r^2 + a_3^2)}{(a_3^2 - a_1^2)(a_3^2 - a_2^2)} d\phi_3 \right] \,, \end{aligned}$$

$$
\begin{aligned}
e^{x_1} &= \sqrt{\frac{(x_1-x_2)(r^2-x_1)}{4(x_1+a_1^2)(x_1+a_2^2)(x_1+a_3^2)}}dx_1 \ , \\
e^{\phi_2} &= \sqrt{-\frac{(x_2+a_1^2)(x_2+a_2^2)(x_2+a_3^2)}{x_2(x_2-x_1)(r^2-x_2)}}\left[dt+\frac{a_1(x_1+a_1^2)(r^2+a_1^2)}{(a_1^2-a_2^2)(a_1^2-a_3^2)}d\phi_1\right. \\
&\left.+\frac{a_2(x_1+a_2^2)(r^2+a_2^2)}{(a_2^2-a_1^2)(a_2^2-a_3^2)}d\phi_2+\frac{a_3(x_1+a_3^2)(r^2+a_3^2)}{(a_3^2-a_1^2)(a_3^2-a_2^2)}d\phi_3\right] \ , \\
e^{x_2} &= \sqrt{\frac{(x_2-x_1)(r^2-x_2)}{4(x_2+a_1^2)(x_2+a_2^2)(x_2+a_3^2)}}dx_2 \ , \\
e^{\psi} &= \sqrt{\frac{a_1^2a_2^2a_3^2}{x_1x_2r^2}}\left[dt+\frac{(x_1+a_1^2)(x_2+a_1^2)(r^2+a_1^2)}{(a_1^2-a_2^2)(a_1^2-a_3^2)a_1}d\phi_1\right. \\
&\left.+\frac{(x_1+a_2^2)(x_2+a_2^2)(r^2+a_2^2)}{(a_2^2-a_1^2)(a_2^2-a_3^2)a_2}d\phi_2+\frac{(x_1+a_3^2)(x_2+a_3^2)(r^2+a_3^2)}{(a_3^2-a_1^2)(a_3^2-a_2^2)a_3}d\phi_3\right] \ ,
\end{aligned} \tag{4.80}
$$

where $i = 1, 2, 3$ and $j = 1, 2$. The coordinate transformations between ellipsoidal coordinates and BL coordinates are

$$
\begin{aligned}
\mu_1^2 &= \frac{(x_1+a_1^2)(x_2+a_1^2)}{(a_1^2-a_2^2)(a_1^2-a_3^2)} \ , \\
\mu_2^2 &= \frac{(x_1+a_2^2)(x_2+a_2^2)}{(a_2^2-a_1^2)(a_2^2-a_3^2)} \ , \\
\mu_3^2 &= \frac{(x_1+a_3^2)(x_2+a_3^2)}{(a_3^2-a_1^2)(a_3^2-a_2^2)} \ .
\end{aligned} \tag{4.81}
$$

The rank-3 Killing-Yano tensor (4.78) vanishes in the flat nonrotating 7-dimensional background. Letting $\mu_1 = \sin\theta_1 \sin\theta_2$, $\mu_2 = \cos\theta_1 \sin\theta_2$, and $\mu_3 = \cos\theta_2$, the rank-5 Killing-Yano tensor (4.76) becomes

$$
f^{(5)} = r^6 \cos\theta_1 \cos\theta_2 \sin\theta_1 \sin^3\theta_2 \, d\phi_1 \wedge d\phi_2 \wedge d\phi_3 \wedge d\theta_1 \wedge d\theta_2 \ , \tag{4.82}
$$

in the flat nonrotating background.

4.3 KT Charges

In this section, conserved charges obtained from (3.106) and (3.96) for Myers-Perry black holes are calculated with the help of the Mathematica software [27]. We found that, from the Killing-Yano tensors obtainable from (4.21), only rank-$(D-2)$ Killing-Yano tensors have nonvanishing components for the D-dimensional flat background.

To simplify the results, let K_D be a constant defined as

$$K_D := \frac{(D-1)\pi\,\Omega_{D-2}}{2(D-2)^2}\ , \tag{4.83}$$

where $\Omega_D = 2\pi^{D/2}/\Gamma(\frac{D}{2})$ is the area of a unit D-sphere.

In each dimension, we divided the spacetime into the background metric $\bar{g}$ and a deviation h, where the background is the Minkowski metric in spherical coordinates (t, r, θ, ϕ)

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}\ . \tag{4.84}$$

For $D = 4$, Kerr metric, the Killing-Yano tensor (4.45) is taken at the limit $a \to 0$ to get the Killing-Yano tensor $\bar{f}$ of the flat background

$$\bar{f} = r^3 d\theta \wedge d\phi\ . \tag{4.85}$$

We can write the background metric as

$$\bar{g}_{\mu\nu} = -n_\mu n_\nu + r_\mu r_\nu + \gamma_{\mu\nu}\ . \tag{4.86}$$

$\gamma_{\mu\nu}$ is the metric on S^2 of radius r. $n^\mu = (-1, 0, 0, 0)$ and $r^\mu = (0, 1, 0, 0)$ are timelike and spacelike normal vectors to S^2, respectively. KT potential (3.106) has two relevant components

$$\bar{\ell}^{tr\theta} = \frac{am\sin\theta\,(a^2\cos^2\theta + 3r^2)}{2r\,(r^2 + a^2\cos^2\theta))^2}, \quad \bar{\ell}^{r\theta\phi} = \frac{mr}{\sin\theta\,(r^2 + a^2\cos^2\theta)^2}\ . \tag{4.87}$$

$\bar{\ell}^{tr\theta}$ gives a conserved charge at the spatial infinity $r \to \infty$. x^μ is chosen as $(0, 0, 1/r, 0)$ such that the background metric can be written as

$$\bar{g}_{\mu\nu} = -n_\mu n_\nu + r_\mu r_\nu + x_\mu x_\nu + \bar{\gamma}_{\mu\nu}\ . \tag{4.88}$$

Then the conserved charge can be found by using (3.96)

$$\begin{aligned} Q_{4D} = \int_{S^2} \sqrt{|\gamma|} n_\mu r_\nu x_\rho \bar{\ell}^{\mu\nu\rho} d^2x = & \int_0^{2\pi}\int_0^{\pi} \left(r^3 \sin\theta \bar{\ell}^{tr\theta}\right)_{r\to\infty} d\theta d\phi \\ = & \frac{3}{2} am\pi^2 = 2amK_4\ . \end{aligned} \tag{4.89}$$

The total mass differs in the Kerr and the 4-dimensional Myers-Perry metrics. To be consistent with the calculations in the higher dimensions, let $m \to m/2$. Then the KT charge for the 4-dimensional Myers-Perry metric is

$$Q_{4D} = amK_4 \ . \tag{4.90}$$

For $D = 5$, the rank-3 Killing-Yano tensor (4.74) is taken at the limit $a_1, a_2 \to 0$ and it becomes $r^4 \sin\theta \cos\theta \, d\theta \wedge d\phi_1 \wedge d\phi_2$. There are three relevant components of KT potential (3.106)

$$\begin{aligned}
\bar{\ell}^{tr\theta\phi_1} =& \frac{a_2 m \cot\theta \left((a_2^2 - a_1^2)\cos^2\theta - a_2^2 - 2r^2 \right)}{3r^3 \left((a_1^2 - a_2^2)\cos^2\theta + a_2^2 + r^2 \right)^2} \ , \\
\bar{\ell}^{tr\theta\phi_2} =& \frac{a_1 m \tan\theta \left((a_1^2 - a_2^2)\cos^2\theta + a_2^2 + 2r^2 \right)}{3r^3 \left((a_1^2 - a_2^2)\cos^2\theta + a_2^2 + r^2 \right)^2} \ , \\
\bar{\ell}^{r\theta\phi_1\phi_2} =& \frac{m \csc\theta \sec\theta \left((a_1^2 - a_2^2)\cos^2\theta + a_2^2 + 3r^2 \right)}{6r^3 \left((a_1^2 - a_2^2)\cos^2\theta + a_2^2 + r^2 \right)^2} \ .
\end{aligned} \tag{4.91}$$

Similarly, we write the background metric as

$$\bar{g}_{\mu\nu} = -n_\mu n_\nu + r_\mu r_\nu + \gamma_{\mu\nu} \ . \tag{4.92}$$

$\gamma_{\mu\nu}$ is the metric on S^3 of radius r. The last component of $\bar{\ell}$ does not give a conserved charge since the contraction of $\bar{\ell}$ with the timelike normal vector $n^\mu = (-1, 0, 0, 0, 0)$ vanishes. We choose different normal vectors to get different charges. Let $x^{(1)\mu} = (0, 0, 1/r, 0, 0)$, $x^{(2)\mu} = (0, 0, 0, (1/r)\sin\theta, 0)$, and $x^{(3)\mu} = (0, 0, 0, 0, (1/r)\cos\theta)$. Then, we can obtain two conserved charges at the spatial infinity $r \to \infty$

$$\begin{aligned}
Q_{5D}^{(1)} &= \int_{S^3} \sqrt{|\gamma|} n_\mu r_\nu x_\rho^{(1)} x_\sigma^{(2)} \bar{\ell}^{\mu\nu\rho\sigma} d^3x \ , \\
Q_{5D}^{(2)} &= \int_{S^3} \sqrt{|\gamma|} n_\mu r_\nu x_\rho^{(1)} x_\sigma^{(3)} \bar{\ell}^{\mu\nu\rho\sigma} d^3x \ .
\end{aligned} \tag{4.93}$$

$$\begin{aligned}
Q_{5D}^{(1)} =& \int_0^{2\pi} \int_0^{2\pi} \int_0^{\pi/2} \left(r^5 \sin^2\theta \cos\theta \bar{\ell}^{tr\theta\phi_1} \right)_{r\to\infty} d\theta d\phi_1 d\phi_2 \\
=& -\frac{2(2\pi)^2}{3^2} a_2 m = -a_2 m K_5 \ ,
\end{aligned} \tag{4.94}$$

$$\begin{aligned}
Q_{5D}^{(2)} =& \int_0^{2\pi} \int_0^{2\pi} \int_0^{\pi/2} \left(r^5 \sin\theta \cos^2\theta \bar{\ell}^{tr\theta\phi_2} \right)_{r\to\infty} d\theta d\phi_1 d\phi_2 \\
=& \frac{2(2\pi)^2}{3^2} a_1 m = a_1 m K_5 \ .
\end{aligned} \tag{4.95}$$

In higher dimensions ($D > 5$), the background metric only admits rank-2 CCKYT. Thus, only rank-($D-2$) Killing Yano tensors are used. We will not give the nonvanising components of rank-($D-1$) KT potentials for ($D > 5$). The charges and the nonvanishing components of KT potential from which the charges are obtained are given in table 4.1.

Table 4.1: KT Charges in 4, 5, 6, 7, and 8 Dimensional Myers-Perry Spacetimes

Dimensions	Charges	Components of KT Potential	Result
4	Q_{4D}	$\ell^{tr\theta}$	$K_4 am$
5	$Q_{5D}^{(1)}$ $Q_{5D}^{(2)}$	$\ell^{tr\phi_1\theta}$ $\ell^{tr\theta\phi_2}$	$K_5 a_2 m$ $K_5 a_1 m$
6	$Q_{6D}^{(1)}$ $Q_{6D}^{(2)}$	$\ell^{tr\phi_1\theta_1\theta_2}$ $\ell^{tr\phi_2\theta_2\theta_1}$	$K_6 a_2 m$ $K_6 a_1 m$
7	$Q_{7D}^{(1)}$ $Q_{7D}^{(2)}$ $Q_{7D}^{(3)}$	$\ell^{tr\phi_1\phi_2\theta_1\theta_2}$ $\ell^{tr\phi_3\phi_1\theta_1\theta_2}$ $\ell^{tr\phi_2\phi_3\theta_1\theta_2}$	$K_7 a_3 m$ $K_7 a_2 m$ $K_7 a_1 m$
8	$Q_{8D}^{(1)}$ $Q_{8D}^{(2)}$ $Q_{8D}^{(3)}$	$\ell^{tr\phi_1\phi_2\theta_1\theta_2\theta_3}$ $\ell^{tr\phi_3\phi_1\theta_1\theta_2\theta_3}$ $\ell^{tr\phi_2\phi_3\theta_1\theta_2\theta_3}$	$K_8 a_3 m$ $K_8 a_2 m$ $K_8 a_1 m$

We obtained the angular momentum components of the Myers-Perry blackholes upto 8 dimensions and we expect the results would be similar and follow the obvious pattern in higher dimensions. We can conclude that the asymptotic KT charge gives the different components of the angular momentum for Myers-Perry blackholes. An important observation is that one Killing-Yano tensor is enough to get the different components of the total angular momentum. If we used the Killing vectors ∂_{ϕ_i} in D dimensions ($2n+2$ or $2n+1$), we should have made n AD charge calculations to get the same result.

CHAPTER 5

KERR-LIKE SPACETIMES

5.1 Metric and Killing Family of Tensors

The metrics we are going to work with have a similar form to the Kerr metric in BL coordinates

$$\begin{aligned} ds^2 = &-\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dt^2 - \frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}dtd\phi \\ &+\left(\frac{\left(r^2+a^2\right)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \ , \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} \Sigma(r,\theta) &= r^2 + a^2\cos^2\theta \ , \\ \Delta(r) &= r^2 + a^2 + F(r) \ , \end{aligned} \tag{5.2}$$

with the electromagnetic potentials

$$A = p\frac{G(\theta)}{\Sigma}\left(adt - \left(r^2+a^2\right)d\phi\right) - q\frac{R(r)}{\Sigma}\left(dt - a\sin^2\theta d\phi\right) \ , \tag{5.3}$$

$$A^* = q\frac{G(\theta)}{\Sigma}\left(adt - \left(r^2+a^2\right)d\phi\right) + p\frac{R(r)}{\Sigma}\left(dt - a\sin^2\theta d\phi\right) \ . \tag{5.4}$$

q is the electric, and p is the magnetic charge. The electromagnetic field strength tensor F and the dual one P can be obtained by taking the exterior derivative of the potentials

$$F = dA \ , \quad P = dA^* \ . \tag{5.5}$$

There are several solutions found in the literature in the given form. $F_K(r) = -2mr$, and $F_{KN}(r) = -2mr + q^2$ gives the Kerr [28] and Kerr-Newman [29, 30] metrics, respectively. $F_{KN}(r) = -2mr + q^2 + p^2$ is the dyonic solution of the Kerr-Newman metric. Also, setting $F_{GD}(r) = -2mr + \left(q^2+p^2\right)\left(1 - \beta\left(r^2+a^2\right)\right)^2$ gives Garcìa-Dìaz's [31] nonlinear electrodynamics solution with the notation given in [32].

The functions in electromagnetic potentials are $G_{GD}(\theta) = \cos\theta(1 - \beta a^2 \sin^2\theta)$ and $R_{GD}(r) = r(1 - \beta(r^2 + a^2))$ for Garcìa-Dìaz's metric. Kerr-Newman metric and its dyonic version can be obtained by setting $\beta = 0$. Kerr metric satisfies the vacuum Einstein equations $G_{\mu\nu} = 0$. (Dyonic) Kerr-Newman metric satisfies the Einstein equations $G_{\mu\nu} = \kappa(T_{KN})_{\mu\nu}$ with the energy-momentum tensor $(T_{KN})_{\mu\nu}$

$$(T_{KN})_{\mu\nu} = F_\mu{}^\sigma F_{\nu\sigma} + P_\mu{}^\sigma P_{\nu\sigma} \; . \tag{5.6}$$

These spacetimes admit two Killing vectors

$$\xi^{(1)} = \partial_t, \quad \xi^{(2)} = \partial_\phi \; . \tag{5.7}$$

In KS form [33], metric (5.1) can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h k_\mu k_\nu \; . \tag{5.8}$$

k_μ is a null vector with respect to both g and η, and is

$$k_\mu dx^\mu = dt + \frac{r(xdx + ydy) - a(xdy - ydx)}{r^2 + a^2} + \frac{zdz}{r} \; . \tag{5.9}$$

h is found to be

$$h = \frac{-r^2 F(r)}{r^4 + a^2 z^2} \; . \tag{5.10}$$

In this form, r is not a coordinate but a function of (x, y, z). It is determined from

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1 \; . \tag{5.11}$$

The coordinate transformation from KS form (t, x, y, z) to Eddington-like coordinates [25] $(\bar{t}, r, \theta, \bar{\phi})$ is given by the following relations

$$x = \sqrt{r^2 + a^2} \sin\theta \cos\left[\bar{\phi} + \arctan\frac{a}{r}\right] \; , \tag{5.12}$$

$$y = \sqrt{r^2 + a^2} \sin\theta \sin\left[\bar{\phi} + \arctan\frac{a}{r}\right] \; , \tag{5.13}$$

$$z = r\cos\theta \; , \tag{5.14}$$

$$\bar{t} = t \; . \tag{5.15}$$

In order to get BL coordinates (t, r, θ, ϕ), one more transformation is needed

$$d\bar{t} = dt - \frac{F(r)}{\Delta(r)} dr \ , \tag{5.16}$$

$$d\bar{\phi} = d\phi + \frac{a}{\Delta(r)} dr \ . \tag{5.17}$$

These spacetimes share a common Killing-Yano tensor f and a CCKYT k independent of the function $F(r)$:

$$\begin{aligned} f = -a\cos\theta dt \wedge dr + &ar\sin\theta dt \wedge d\theta \\ &-a^2\cos\theta\sin^2\theta dr \wedge d\phi + r(r^2+a^2)\sin\theta d\theta \wedge d\phi \ , \end{aligned} \tag{5.18}$$

$$\begin{aligned} k = -r dt \wedge dr - &a^2 \sin\theta\cos\theta dt \wedge d\theta \\ &-ar\sin^2\theta dr \wedge d\phi - a(r^2+a^2)\sin\theta\cos\theta d\theta \wedge d\phi \ . \end{aligned} \tag{5.19}$$

A Killing tensor can be constructed from the Killing-Yano tensor (5.18)

$$K^{\mu\nu} = f^{\mu\rho} f^{\nu}{}_{\rho} \ . \tag{5.20}$$

The Killing tensor K is

$$K^{\mu\nu} = \begin{bmatrix} a^2 - a^2 \frac{F(r)\left(r^2+a^2\right)\cos^2\theta}{\Delta(r)\Sigma(r,\theta)} & 0 & 0 & a - a^3 \frac{F(r)\cos^2\theta}{\Delta(r)\Sigma(r,\theta)} \\ 0 & -a^2\cos^2\theta \frac{\Delta(r)}{\Sigma(r,\theta)} & 0 & 0 \\ 0 & 0 & \frac{r^2}{\Sigma(r,\theta)} & 0 \\ a - a^3 \frac{F(r)\cos^2\theta}{\Delta(r)\Sigma(r,\theta)} & 0 & 0 & \frac{r^2}{\sin^2\theta\Sigma(r,\theta)} + \frac{a^4\cos^2\theta}{\Delta(r)\Sigma(r,\theta)} \end{bmatrix} . \tag{5.21}$$

Similarly, we can construct a conformal Killing tensor from the CCKYT (5.19)

$$Q^{\mu\nu} = k^{\mu\rho} k^{\nu}{}_{\rho} \ . \tag{5.22}$$

The conformal Killing tensor Q is

$$Q^{\mu\nu} = \begin{bmatrix} r^2 + a^2\sin^2\theta - \frac{F(r)\left(r^2+a^2\right)r^2}{\Delta(r)\Sigma(r,\theta)} & 0 & 0 & a - a\frac{r^2F(r)}{\Delta(r)\Sigma(r,\theta)} \\ 0 & -\frac{r^2\Delta(r)}{\Sigma(r,\theta)} & 0 & 0 \\ 0 & 0 & \frac{a^2\cos^2\theta}{\Sigma(r,\theta)} & 0 \\ a - a\frac{r^2F(r)}{\Delta(r)\Sigma(r,\theta)} & 0 & 0 & \frac{a^2}{\sin^2\theta\Sigma(r,\theta)} - \frac{a^4+a^2F(r)}{\Delta(r)\Sigma(r,\theta)} \end{bmatrix} . \tag{5.23}$$

5.2 Separability of Hamilton-Jacobi Equation

The Lagrangian for a particle with mass μ, electric charge e, and magnetic charge g is

$$L = \frac{1}{2} g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} + eA_{\mu}\dot{x}^{\mu} + gA^{*}_{\mu}\dot{x}^{\mu} \ . \tag{5.24}$$

where $\dot{x}^{\mu} = \frac{dx^{\mu}}{d\lambda}$ with λ as an affine parameter related to the proper time by $\tau = \mu\lambda$. Since the metric is itself a Killing tensor, $g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$ is a conserved quantity along the geodesics

$$g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = -\mu^2 \ . \tag{5.25}$$

This is the first conserved quantity; the rest-mass of the particle. Canonical momentum can be obtained from the Lagrangian

$$\begin{aligned} p_{\mu} &= \frac{\partial L}{\partial \dot{x}^{\mu}} \\ &= g_{\mu\nu}\dot{x}^{\nu} + eA_{\mu} + gA^{*}_{\mu} \ . \end{aligned} \tag{5.26}$$

Then, the Hamiltonian of the particle is

$$H = \frac{1}{2} g^{\mu\nu}\left(p_{\mu} - eA_{\mu} - gA^{*}_{\mu}\right)\left(p_{\nu} - eA_{\nu} - gA^{*}_{\nu}\right) \ . \tag{5.27}$$

Since the Lagrangian does not explicitly depend on the affine parameter λ, the Hamiltonian is conserved. If we write the Hamiltonian in terms of the generalized coordinates, it simply is

$$H = \frac{1}{2} g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = -\frac{1}{2}\mu^2 \ . \tag{5.28}$$

The equations of motion can be obtained from the Lagrangian (5.24)

$$\frac{D^2 x^{\mu}}{d\tau^2} = \frac{e}{\mu}\frac{dx^{\nu}}{d\tau}F^{\mu}{}_{\nu} + \frac{g}{\mu}\frac{dx^{\nu}}{d\tau}P^{\mu}{}_{\nu} \ . \tag{5.29}$$

The metric (5.1) possesses two Killing vectors:

$$\xi^{(t)} = \partial_t, \quad \xi^{(\phi)} = \partial_{\phi} \ . \tag{5.30}$$

From the symmetries of the metric, we can obtain two constants of motion. The first one arises from the conservation of energy, and the second one arises from the conservation of angular momentum. As shown in [4], we can also write the conserved quantities associated with the Killing vectors and tensors by using canonical momentum for the charged particles, i.e. $p^{\mu}\nabla_{\mu}\left(\xi^{\nu}p_{\nu}\right) = 0$. Thus, conserved quantities associated with the Killing vectors are

$$p_t = -E, \quad p_{\phi} = \Phi \; . \tag{5.31}$$

Up to now, we have obtained three constants of motion. In order to solve the equations of motion, we need to find another conserved quantity that cannot be found from the obvious symmetries of the metric. In order to find it, Carter [2] used the separability of the Hamilton-Jacobi equation. Following Carter's discussion, we can write a separable action for the Lagrangian (5.24) as

$$S = \frac{1}{2}\mu^2\lambda - Et + \Phi\phi + S_{\theta}(\theta) + S_r(r) \; . \tag{5.32}$$

The Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial \lambda} + \frac{1}{2}g^{\mu\nu}\left(\frac{\partial S}{\partial x^{\mu}} - eA_{\mu} - gA^{*}_{\mu}\right)\left(\frac{\partial S}{\partial x^{\nu}} - eA_{\nu} - gA^{*}_{\nu}\right) = 0 \; . \tag{5.33}$$

Writing it explicitly,

$$\begin{aligned}
\mu^2 + & \frac{\left[\left(r^2+a^2\right)^2 - a^2\sin^2\theta\Delta(r)\right]}{\Delta(r)\Sigma^3(r,\theta)}\left[E\Sigma(r,\theta) + (gq+ep)\,aG(\theta) + (gp-eq)\,R(r)\right]^2 \\
& - \frac{2a(\Delta(r) - (r^2+a^2))}{\Delta(r)\Sigma^3(r,\theta)}\left[E\Sigma(r,\theta) + (gq+ep)\,aG(\theta) + (gp-eq)\,R(r)\right] \\
& \qquad \left[\Phi\Sigma(r,\theta) + (gq+ep)\left(r^2+a^2\right)G(\theta) + (gp-eq)\,a\sin^2\theta R(r)\right] \\
& + \frac{\Delta(r) - a^2\sin^2\theta}{\sin^2\theta\Delta(r)\Sigma^3(r,\theta)}\Big[\Phi\Sigma(r,\theta) + (gq+ep)\left(r^2+a^2\right)G(\theta) \\
& \qquad + (gp-eq)\,a\sin^2\theta R(r)\Big]^2 \\
& + \frac{\Delta(r)}{\Sigma(r,\theta)}\left(\frac{dS_r}{dr}\right)^2 + \frac{1}{\Sigma(r,\theta)}\left(\frac{dS_{\theta}}{d\theta}\right)^2 = 0 \; .
\end{aligned} \tag{5.34}$$

After using the definitions (5.2) and a careful investigation, equation (5.34) can be written as

$$
\begin{aligned}
\mu^2 r^2+&\Delta(r)\left(\frac{dS_r}{dr}\right)^2-E^2\frac{\left(r^2+a^2\right)^2}{\Delta(r)}-\Phi^2\frac{a^2}{\Delta(r)}+2aE\Phi\frac{\left(r^2+a^2\right)}{\Delta(r)}\\
&-2\left(eq-gp\right)\frac{R(r)}{\Delta(r)}\left(\Phi a-E\left(r^2+a^2\right)\right)-\left(eq-gp\right)^2\frac{R^2(r)}{\Delta(r)}\\
=-\mu^2 a^2\cos^2\theta-&\left(\frac{dS_\theta}{d\theta}\right)^2-E^2a^2\sin^2\theta-\frac{\Phi^2}{\sin^2\theta}+2aE\Phi\\
&-2\left(ep+gq\right)G(\theta)\left(\frac{\Phi}{\sin^2\theta}-Ea\right)-\left(ep+gq\right)^2\frac{G^2(\theta)}{\sin^2\theta}=\mathcal{K}\ .
\end{aligned}
\tag{5.35}
$$

We separated the r and θ-dependent parts of the Hamilton-Jacobi equation. Thus, we can conclude that both parts must be equal to a separation constant $\mathcal{K}$. This constant is called the Carter constant. We can work on both sides separately to find the full action. Let us define two functions

$$
\begin{aligned}
\Theta(\theta):=&-\mathcal{K}-\mu^2a^2\cos^2\theta-E^2a^2\sin^2\theta-\frac{\Phi^2}{\sin^2\theta}+2aE\Phi\\
&-2\left(ep+gq\right)G(\theta)\left(\frac{\Phi}{\sin^2\theta}-Ea\right)-\left(ep+gq\right)^2\frac{G^2(\theta)}{\sin^2\theta}\ ,
\end{aligned}
\tag{5.36}
$$

$$
\begin{aligned}
T(r):=&\mathcal{K}-\mu^2r^2+E^2\frac{\left(r^2+a^2\right)^2}{\Delta(r)}+\Phi^2\frac{a^2}{\Delta(r)}-2aE\Phi\frac{\left(r^2+a^2\right)}{\Delta(r)}\\
&+2\left(eq-gp\right)\frac{R(r)}{\Delta(r)}\left(\Phi a-E\left(r^2+a^2\right)\right)+\left(eq-gp\right)^2\frac{R^2(r)}{\Delta(r)}\ .
\end{aligned}
\tag{5.37}
$$

Then, S_r and S_θ can be found by integrating these functions

$$
\begin{aligned}
S_\theta(\theta)&=\int_\theta\sqrt{\Theta}d\theta\ ,\\
S_r(r)&=\int_r\frac{\sqrt{T}}{\Delta(r)}dr\ .
\end{aligned}
\tag{5.38}
$$

The total action becomes

$$
S=\frac{1}{2}\mu^2\lambda-Et+\Phi\phi+\int_\theta\sqrt{\Theta}d\theta+\int_r\frac{\sqrt{T}}{\Delta(r)}dr\ . \tag{5.39}
$$

Carter proposed another method [34] to get the fourth constant of motion. Let the Hamiltonian be in the form

$$
H=\frac{1}{2}\frac{H_r(r)+H_\mu(\mu)}{U_r(r)+U_\mu(\mu)}\ , \tag{5.40}
$$

where $U_r(r)$ and $U_\mu(\mu)$ are functions of a single coordinate, and $H_r(r)$ is independent of other coordinates and independent of p_μ, and $H_\mu(\mu)$ is independent of other coordinates and independent of p_r. Then, one can construct a conserved quantity

$$\mathcal{K} = \frac{U_r H_\mu - U_\mu H_r}{U_r + U_\mu} \,. \tag{5.41}$$

Since $\mathcal{K}$ commutes with the Hamiltonian, it is a conserved quantity and can be used as the fourth constant of motion. The functions $H_r(r)$, $H_\theta(\theta)$, $U_r(r)$, and $U_\theta(\theta)$ for the metric (5.1) are

$$\begin{aligned}
H_r =& \Delta(r)p_r^2 - \frac{E^2\left(r^2+a^2\right)^2}{\Delta(r)} - \Phi^2\frac{a^2}{\Delta(r)} + 2aE\Phi\frac{(r^2+a^2)}{\Delta(r)} \\
&- 2(eq-gp)\frac{R(r)}{\Delta(r)}\left(\Phi a - E\left(r^2+a^2\right)\right) - (eq-gp)^2\frac{R^2(r)}{\Delta(r)} \,, \\
H_\theta =& p_\theta^2 + E^2a^2\sin^2\theta + \frac{\Phi^2}{\sin^2\theta} - 2aE\Phi \\
&+ 2(ep+gq)G(\theta)\left(\frac{\Phi}{\sin^2\theta} - Ea\right) + (ep+gq)^2\frac{G^2(\theta)}{\sin^2\theta} \,, \\
U_r =& r^2, \quad U_\theta = a^2\cos^2\theta \,.
\end{aligned} \tag{5.42}$$

If we substitute the Hamiltonian to (5.41) in terms of the separation constant (5.35), we get

$$\mathcal{K} = \frac{U_r(\mathcal{K} - \mu^2 U_\theta) - U_\theta(-\mathcal{K} + \mu^2 U_r)}{U_r + U_\theta} = \mathcal{K} \,. \tag{5.43}$$

Thus, both constants are equal to each other. Using (5.41), the Carter constant is

$$\begin{aligned}
\mathcal{K} =& -a^2\cos^2\theta\frac{\Delta(r)}{\Sigma(r,\theta)}p_r^2 + \frac{r^2}{\Sigma(r,\theta)}p_\theta^2 + + \left(\frac{r^2}{\sin^2\theta} + \frac{a^4\cos^2\theta}{\Delta(r)}\right)\frac{\Phi^2}{\Sigma(r,\theta)} \\
&+ \left(r^2a^2\sin^2\theta + \frac{\left(r^2+a^2\right)^2 a^2\cos^2\theta}{\Delta(r)}\right)\frac{E^2}{\Sigma(r,\theta)} - 2aE\Phi\left(1 - \frac{a^2\cos^2\theta F(r)}{\Delta(r)\Sigma(r,\theta)}\right) \\
&+ 2r^2(ep+gq)\frac{G(\theta)}{\Sigma(r,\theta)}\left(\frac{\Phi}{\sin^2\theta} - Ea\right) + r^2(ep+gq)^2\frac{G^2(\theta)}{\sin^2\theta\Sigma(r,\theta)} \\
&+ 2a^2\cos^2\theta(eq-gp)\frac{R(r)}{\Delta(r)\Sigma(r,\theta)}\left(\Phi a - E\left(r^2+a^2\right)\right) \\
&+ a^2\cos^2\theta(eq-gp)^2\frac{R^2(r)}{\Delta(r)\Sigma(r,\theta)} \,.
\end{aligned} \tag{5.44}$$

There is also a simpler method if the spacetime is known to admit a Killing tensor K. Aside from the conserved quantities obtained from the Killing vectors, there is another conserved quantity $\bar{\mathcal{K}}$ [4]

$$\bar{\mathcal{K}} = K_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = K^{\mu\nu}\left(p_\mu - eA_\mu - gA^*_\mu\right)\left(p_\nu - eA_\nu - gA^*_\nu\right) \ . \tag{5.45}$$

If we use (5.21), the two constants $\mathcal{K}$ and $\bar{\mathcal{K}}$ are equal to each other

$$\bar{\mathcal{K}} = \mathcal{K} \ . \tag{5.46}$$

Thus, one can use the second-order terms in momenta in (5.44) to extract the components of the Killing tensor.

For the massless and chargeless particles, Carter constant (5.35) becomes

$$\begin{aligned}\Delta(r)\left(\frac{dS_r}{dr}\right)^2 - E^2\frac{\left(r^2+a^2\right)^2}{\Delta(r)} - \Phi^2\frac{a^2}{\Delta(r)} + 2aE\Phi\frac{\left(r^2+a^2\right)}{\Delta(r)} \\ = -\left(\frac{dS_\theta}{d\theta}\right)^2 - E^2a^2\sin^2\theta - \frac{\Phi^2}{\sin^2\theta} + 2aE\Phi = \mathcal{Q} \ .\end{aligned} \tag{5.47}$$

The Hamiltonian can be written in the form of (5.40) with the functions $H_r(r)$, $H_\theta(\theta)$, $U_r(r)$, and $U_\theta(\theta)$

$$\begin{aligned}H_r =& \Delta(r)p_r^2 - \frac{E^2\left(r^2+a^2\right)^2}{\Delta(r)} - \Phi^2\frac{a^2}{\Delta(r)} + 2aE\Phi\frac{\left(r^2+a^2\right)}{\Delta(r)} = \mathcal{Q} \ , \\ H_\theta =& p_\theta^2 + E^2a^2\sin^2\theta + \frac{\Phi^2}{\sin^2\theta} - 2aE\Phi = -\mathcal{Q} \ , \\ U_r =& r^2, \quad U_\theta = a^2\cos^2\theta \ .\end{aligned} \tag{5.48}$$

Then, the following is a Carter-like constant for the null geodesics and is a symmetrized version of the quantity $\mathcal{Q}$ that is defined in (5.47)

$$\begin{aligned}\mathcal{Q} &= \frac{U_\theta H_\theta - U_r H_r}{U_r + U_\theta} \\ &= \frac{U_r\mathcal{Q} + U_\theta\mathcal{Q}}{U_r + U_\theta} = \mathcal{Q} \ .\end{aligned} \tag{5.49}$$

After substituting (5.48) into (5.49), we get

$$\begin{aligned}\mathcal{Q} = -r^2\frac{\Delta(r)}{\Sigma(r,\theta)}p_r^2 + \frac{E^2r^2\left(r^2+a^2\right)^2}{\Sigma(r,\theta)\Delta(r)} - 2aE\Phi\left(1 - \frac{r^2F(r)}{\Sigma(r,\theta)\Delta(r)}\right) \\ + a^2\cos^2\theta\frac{p_\theta^2}{\Sigma(r,\theta)} + E^2a^4\frac{\sin^2\theta\cos^2\theta}{\Sigma(r,\theta)} + \frac{\Phi^2a^2}{\Sigma(r,\theta)}\left(\frac{r^2}{\Delta(r)} + \cot^2\theta\right) \ .\end{aligned} \tag{5.50}$$

Since the conformal Killing tensors are conserved along the null geodesics, one can construct a conserved quantity by using (5.23)

$$\bar{\mathcal{Q}} = Q^{\mu\nu} p_\mu p_\nu \ . \tag{5.51}$$

These two constants are equal to each other

$$\mathcal{Q} = \bar{\mathcal{Q}} \ . \tag{5.52}$$

While we can directly use the conformal Killing tensor to solve the null geodesics, we can also use the Carter constant (5.35) and take the massless limit ($\mu \to 0$). In both cases, we end up with the same differential equations for the null geodesics.

5.3 Cotton Charges

In this section, we will calculate Cotton charges [12] given in the subsection 3.3.2. Since the metric (5.1) admits both Killing-Yano tensor and CCKYT, we can construct both of the Cotton currents.

Cotton charge can be found by using the equation (3.34) with the Cotton currents given in equation (3.111). Let the timelike normal n^μ be

$$n^\mu = \left(\sqrt{\frac{\Sigma}{\Delta - a^2 \sin^2\theta}}, 0, 0, 0\right) . \tag{5.53}$$

The metric can be written as

$$g_{\mu\nu} = h_{\mu\nu} - n_\mu n_\nu \ , \tag{5.54}$$

where $h_{\mu\nu}$ is the induced metric on the 3-dimensional spacelike hypersurface. The determinant of the induced metric is

$$\sqrt{|h|} = \frac{\sin\theta \left(r^2 + a^2\cos^2\theta\right)^{3/2}}{\sqrt{r^2 + a^2\cos^2\theta + F(r)}} \ . \tag{5.55}$$

The two Cotton charges can be obtained by integrating over the space with the induced metric $h_{\mu\nu}$

$$\begin{aligned} Q^{(f)} &= \int n_\mu j^\mu \sqrt{|h|} dr d\theta d\phi \ , \\ Q^{(k)} &= \int n_\mu J^\mu \sqrt{|h|} dr d\theta d\phi \ . \end{aligned} \tag{5.56}$$

First, let us work on $Q^{(f)}$ using the current j. We find that the θ integral vanishes

$$\int_0^\pi n_\mu j^\mu \sqrt{|h|} d\theta = 0 \; . \tag{5.57}$$

Thus, we cannot obtain a Cotton charge for the metric (5.1) using the Killing-Yano tensor (5.18)

$$Q^{(f)} = 0 \; . \tag{5.58}$$

Let us move on to the charge $Q^{(k)}$. We calculated the θ integral from 0 to π, and ϕ integral from 0 to 2π. The result is

$$\begin{aligned} Q^{(k)} = -\frac{4}{3}\pi \int_r \Bigg(& 4\tan^{-1}\left(\frac{a}{r}\right) \frac{\left[2\left(r^2+a^2\right)+F(r)\right]\left[6F(r)-6rF'(r)+r^2F''(r)\right]}{arF(r)} \\ & - \frac{2\tan^{-1}\left(\frac{a}{\sqrt{r^2+F(r)}}\right)\Delta(r)}{aF(r)\sqrt{r^2+F(r)}} \big[F(r)\left(rF^{(3)}(r)-4F''(r)+24\right) \\ & +4r^2F''(r)-24rF'(r)\big] - 2F''(r)\Bigg) dr \; . \end{aligned} \tag{5.59}$$

For the Kerr metric ($F_K(r) = -2mr$), $Q_K^{(k)}$ vanishes. For the dyonic Kerr-Newman metric ($F_{KN}(r) = -2mr + p^2 + q^2$), $Q_{KN}^{(k)}$ is

$$\begin{aligned} Q_{KN}^{(k)} = -\frac{32\pi}{a}(p^2+q^2) \int_{r_+}^\infty \Bigg(& \frac{\tan^{-1}\left(\frac{a}{r}\right)\left(2a^2+2r^2-2mr+p^2+q^2\right)}{r\left(p^2+q^2-2mr\right)} \\ & - \frac{2\left(r^2+a^2-2mr+p^2+q^2\right)\tan^{-1}\left(\frac{a}{\sqrt{r^2-2mr+p^2+q^2}}\right)}{\left(p^2+q^2-2mr\right)\sqrt{r^2-2mr+p^2+q^2}} \Bigg) dr \; , \end{aligned} \tag{5.60}$$

where $r_+ := m + \sqrt{m^2 - a^2 - p^2 - q^2}$. Unfortunately, we could not take the r integral. This result is the same as the one calculated in [12]. It is important to point out that the Cotton charge is proportional to the total charge of the spacetime (p^2+q^2). At the limit where the rotation parameter a is zero (when the Kerr metric reduces to the dyonic Reissner-Nordström metric [14]), the charge is

$$Q_{RN}^{(k)} = \frac{32\pi(p^2+q^2)}{m+\sqrt{m^2-p^2-q^2}} \; . \tag{5.61}$$

Since we can calculate the integral as $a \to 0$, we can try to calculate it when $a \ll r$.

Let us denote the integrand with I

$$I=-\frac{32\pi}{a}(p^2+q^2)\Bigg(\frac{\tan^{-1}\left(\frac{a}{r}\right)\left(2a^2+2r^2-2mr+p^2+q^2\right)}{r\left(p^2+q^2-2mr\right)} \\ -\frac{2\left(r^2+a^2-2mr+p^2+q^2\right)\tan^{-1}\left(\frac{a}{\sqrt{r^2-2mr+p^2+q^2}}\right)}{\left(p^2+q^2-2mr\right)\sqrt{r^2-2mr+p^2+q^2}}\Bigg)\,. \quad (5.62)$$

We should expand the integral by using the fundamental theorem of calculus for small a. However, the integrals are not simplified. Instead, we can expand the integrand and the integration limits for small a and then, integrate. Since it would be the same with starting with a slowly rotating Kerr-Newman metric and integrating from the perturbed outer horizon to the infinity, we prefer to follow this path. Expanding the integrand around $a=0$, we get

$$I=32\pi(p^2+q^2)\left(\frac{1}{r^2}-\frac{a^2}{3r^4}+\frac{4a^2}{3}\frac{1}{r^2\left(r^2+p^2+q^2-2mr\right)}\right)+\mathcal{O}\left(a^4\right)\,. \quad (5.63)$$

Now, we can take the r-integral

$$Q^{(k)}_{KN}=\int_{r_+}^{\infty}Idr=-32\pi(p^2+q^2)\Bigg[\frac{1}{r}-\frac{a^2}{9r^3} \\ -\frac{4a^2}{3}\Bigg(-\frac{1}{(p^2+q^2)r}+(2m^2-p^2-q^2)\ln\left|\frac{r-m-\sqrt{m^2-p^2-q^2}}{r-m+\sqrt{m^2-p^2-q^2}}\right| \\ +m\sqrt{m^2-p^2-q^2}\ln\left|\frac{r^2}{r^2-2mr+p^2+q^2}\right|\Bigg)+\mathcal{O}(a^4)\Bigg]_{r_+}^{\infty}\,. \quad (5.64)$$

The integral vanishes at the limit $r\to\infty$. The outer horizon r_+ is also expanded around $a=0$

$$r_+=m+\sqrt{m^2-p^2-q^2}-\frac{a^2}{2}\frac{1}{\sqrt{m^2-p^2-q^2}}+\mathcal{O}(a^4)\,. \quad (5.65)$$

Substituting (5.65) into the integrand and keeping terms with the second order of a, we get the Cotton charge for the dyonic Kerr-Newman metric with small rotation

$$
\begin{aligned}
Q_{KN}^{(k)} \approx 32\pi \left(p^2+q^2\right) \Bigg[& \frac{1}{m+\sqrt{m^2-p^2-q^2}} - \frac{a^2}{9\left(m+\sqrt{m^2-p^2-q^2}\right)^3} \\
+ \frac{a^2}{6\left(p^2+q^2\right)^2\sqrt{m^2-p^2-q^2}} & \Bigg(5\left(p^2+q^2\right) - 2m^2 + 2m\sqrt{m^2-p^2-q^2} \\
& - 16m\sqrt{m^2-p^2-q^2}\ln\left(m+\sqrt{m^2-p^2-q^2}\right) \\
& + 8\left(2m^2-p^2-q^2\right)\ln\left(2m\sqrt{m^2-p^2-q^2}\right) \\
& - 8\left(2m^2-p^2-q^2-2m\sqrt{m^2-p^2-q^2}\right)\ln(a)\Bigg)\Bigg].
\end{aligned}
\tag{5.66}
$$

The charge reduces to the (5.61) as $a \to 0$. For Garcìa-Dìaz's metric ($F_{GD}(r) = -2mr + \left(q^2+p^2\right)\left(1-\beta\left(r^2+a^2\right)\right)^2$), the exact expression of $Q_{GD}^{(k)}$ is even more complicated. There are terms of order r^2 in the integral. Thus, one would expect the charge to diverge. Since the whole expression is too long and complicated, instead, let us look at the charge as $a \to 0$

$$
Q_{GD}^{(k)} = -\frac{32\pi(p^2+q^2)}{3}\int_{r_+}^{\infty}\frac{(\beta r^2+3)}{r^2}dr = \left[\frac{32}{3}\pi(p^2+q^2)\left(\beta r - \frac{3}{r}\right)\right]_{r_+}^{\infty}. \tag{5.67}
$$

The charge diverges as long as $\beta \neq 0$. Therefore, we cannot obtain Cotton charges for Garcìa-Dìaz's metric with the Killing-Yano tensor (5.18) and CKYT (5.19). Nevertheless, we have seen that the Cotton charges are related to the spacetime's total electric and magnetic charges.

CHAPTER 6

CONCLUSION

In this thesis, we reviewed the properties of Killing, Killing-Yano, conformal Killing, and conformal Killing-Yano tensors. We showed how to construct conserved charges from covariantly conserved currents and calculated asymptotic conserved charges with the KT current for Myers-Perry spacetimes. For the spacetimes with a Kerr-like rotation parameter, we showed that the Hamilton-Jacobi equation separates and we gave the Killing family of tensors.

We started giving the definitions of Killing and conformal Killing vectors and then generalized them to higher rank tensors. We emphasized the importance of (conformal) Killing tensors and their relations to the symmetries of the spacetime. Even though higher rank tensors are not directly related to the symmetries, they reveal hidden symmetries which are essential while working on the geodesics. We investigated the properties of the Killing family of tensors and showed how to construct the structure called the Killing-Yano tower. If the Killing-Yano potential is given, one can construct a PCKYT. Different rank Killing-Yano tensors can be obtained using the PCKYT and higher rank CCKYTs. Additionaly, different (conformal) Killing tensors can be obtained by multiplying these (conformal) Killing-Yano tensors.

In chapter 3, we used Stokes' theorem to construct conserved charges from covariantly conserved current vectors. Afterward, we generalized the discussion to higher rank antisymmetric conserved currents. For higher rank currents, one should integrate on hypersurfaces instead of the whole space in order to use Stokes' theorem, and these hypersurfaces must be integrable submanifolds. We also showed that a conserved charge can be constructed using the background if the spacetime behaves as flat or (A)dS at the spatial infinity [10]. We also gave three examples of conserved currents [1, 12] obtained by using Killing-Yano and CKYTs.

In the next chapter, we gave the Myers-Perry metrics [24] and their PCKYT in BL, KS and ellipsoidal coordinates. We found the PCKYT of the Myers-Perry metrics (4.20) in a compact form in KS coordinates. Afterwards, we calculated the Killing-Yano tensors in 4, 5, 6, 7, and 8 dimensions by using the Killing-Yano tower structure. We noticed that only rank-$(D-2)$ Killing-Yano tensors do not vanish in the flat background which can be easily interpreted from the PCKYT (4.21). As the rotation and mass parameters go to zero, the PCKYT is left with only one component, $rdr\wedge dt$. Since the wedge product of $rdr \wedge dt$ with itself vanishes, one cannot get any Killing-Yano tensor of rank lower than $(D-2)$ for the flat background.

For the rank-$(D-2)$ Killing-Yano tensors of the flat background, we calculated KT potentials [1] (3.106) in 4, 5, 6, 7, and 8 dimensions using the Mathematica software [27]. Nonvanishing components of the potentials are integrated over the background at the spatial infinity to calculate the asymptotic gravitational KT charges of Myers-Perry spacetimes. We found that KT charges correspond to angular momentums of the Myers-Perry black holes upto 8 dimensions. We presume that it will follow the same pattern in higher dimensions

$$Q_D^{(i)} \sim a_i m \,, \tag{6.1}$$

where $i = 1, \ldots, n$ with the dimension $D = 2n+2$ or $D = 2n+1$.

In the last chapter, we studied a metric that looks like the Kerr metric in BL coordinates with additional unknown terms given as $F(r)$. Kerr(-Newman) [28, 29, 30], and Garcìa-Dìaz [31] metrics can be obtained by setting $F(r)$ to different functions. A significant property of these spacetimes is that they admit the same Killing-Yano and CKYTs in addition to the two Killing vectors ∂_t and ∂_ϕ. Consequently, we constructed a Killing tensor (and a conformal Killing tensor) and discussed the separability of the Hamilton-Jacobi equations. We found Carter's fourth constant [2] for these spacetimes and one can solve the equations of motion for a test particle with mass $/mu$, electric charge e, and magnetic charge g. Moreover, we calculated the Cotton currents [12] for the metric. The current with the Killing-Yano tensor did not give a conserved charge when integrated, however, we got an expression for generic $F(r)$ using the current with CKYT. For the Kerr metric, the Cotton charge vanished. We could not take the integral for Kerr-Newman and Garcìa-Dìaz's metrics but we calculated the charge as the rotation parameter $a \rightarrow 0$. While Reissner-Nordström metric's Cotton charge is found as

$$Q_{(k)} = \frac{32\pi(p^2+q^2)}{m+\sqrt{m^2-p^2-q^2}} \ , \tag{6.2}$$

the Cotton charge of Garcìa-Dìaz's nonrotating metric diverged. However, we can still conclude that the Cotton charges are related to the total electric and magnetic charges of the spacetime.

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