In the fourth question, the teachers were given two tables, each including some numerical x and y values. Then, they were asked to decide whether those values belong to a linear function or a quadratic function. Some participants (*n*=11) calculated the first differences of the two functions, then decided whether they were linear or quadratic. For example, T7 wrote: “The first difference is constant, so the first one is linear. In the second one, the first difference is not constant. It is not linear.” Two teachers wrote standard forms of linear and quadratic functions, as $y = ax + b$ and $y = ax^2 + bx + c$, respectively. Then, they calculated a, b, and c coefficients and found the equations of both functions. Three participants sketched the graphs of both functions roughly to decide whether the given values belong to linear or quadratic functions.

4.1.1.2. Teachers’ Knowledge of Solving Quadratic Equations with One Unknown

There was one question that assessed and evaluated teachers’ knowledge of solving quadratic equations with one unknown. The teachers were asked to solve three quadratic equations and state some alternative methods for solving quadratic equations. Table 4.2 summarizes the teachers’ responses to this question.

Table 4.2. Teachers' (*n*=18) responses to the question related to CCK2— Teachers' knowledge of solving quadratic equations with one unknown

Answers	*f*	Teachers
Question 5: Solving quadratic equations with one unknown		
Using the quadratic formula (2 points)	17	T18, T17, T15, T14, T13, T12, T11, T9, T8, T7, T6, T5, T4, T2, T1, T3, T10.
Completing the square (2 points)	1	T16.
Follow-up question: Alternative ways for solving quadratic equations		
Algebra tiles	2	T9, T11.
Factorization	6	T9, T8, T13, T14, T12, T16, T17.
Change of variables	2	T9, T12.
Completing the square	3	T8, T4, T7.
The quadratic formula	1	T16.
None	7	T18, T6, T15, T2, T3, T10.

As seen in Table 4.2, almost all teachers (*n*=17) solved the equations using the quadratic formula. They first calculated the discriminant of the quadratic equation, and then used the quadratic formula to find the roots of the quadratic equation. Only one teacher (T16) solved the quadratic equations without using the quadratic formula. He used the method named "completing the square." A follow-up question also asked for alternative ways to solve quadratic equations other than those they had just used. The teachers stated various methods for solving quadratic equations such as algebra tiles, factorization, change of variables, and completing the square. Some teachers (*n*=7) did not suggest any alternative method. One teacher (T16), who had solved the equations by completing the square, suggested using the quadratic formula as an alternative method for solving quadratic equations.

4.1.1.3. Teachers' Knowledge of Sketching and Interpreting the Graphs of Quadratic Functions

Six items in the questionnaire assessed and evaluated the teachers' knowledge of sketching and interpreting the graphs of quadratic functions. Table 4.3 summarizes the teachers' responses to these items.

Table 4.3. Teachers' (n=18) responses to the questions related to CCK3—Teachers' knowledge of sketching and interpreting the graphs of quadratic functions

Answers	***f***	**Teachers**
Question 6: Defining the axis of symmetry of a quadratic function		
Structural description (2 points)	3	T16, T2, T7.
Procedural description (1 point)	11	T15, T10, T4, T12, T6, T14, T13, T11, T1, T9, T8.
Incorrect (0 point)	1	T17.
No answer (0 point)	3	T3, T5, T18.
Question 7: Defining the vertex of a quadratic function		
Structural description (2 points)	9	T2, T4, T7, T9, T11, T13, T14, T16, T18.
Procedural description (1 point)	5	T15, T1, T6, T8, T12, T17.
Incorrect (0 point)	1	T17
No answer (0 point)	3	T3, T5, T10.
Question 8: Defining the concavity of a quadratic function		
Structural description (2 points)	3	T4, T7, T16.
Procedural description (1 point)	7	T8, T9, T11, T15, T12, T13, T14.
Incorrect (0 point)	1	T2.
No answer (0 point)	7	T6, T5, T3, T17, T18, T10, T1.
Question 9: Finding some properties of a function and sketching the graph of it		
Drawing the correct graph (2 points)	18	All.
Question 10: Writing the quadratic function whose graph is given		
Finding the correct quadratic functions (2 points)	15	T15, T13, T6, T12, T4, T7, T9, T14, T18, T17, T16, T11, T8, T2, T10.
No answer (0 point)	3	T1, T3, T5.
Question 11: Determining the signs of the coefficients of a quadratic function by examining its graph		
Finding the signs of all the coefficients correctly (2 points)	17	All.
Finding one or two of the coefficients wrongly (1 point)	1	T18.

In the first one, the teachers were asked to explain what the axis of symmetry means for a quadratic function. Although the teachers were asked to explain the meaning of the axis of symmetry, most of them (n=11) described how to find the axis of symmetry. Their responses included: "It is the line $x = -b/2a$, passing through the vertex." (T6). Three teachers wrote a structural definition of the axis of symmetry.

For example, one of them stated: "It is the line separating the parabola into two symmetrical parts." (T16). One teacher (T17) incorrectly defined the axis of symmetry as "the apsis of the vertex".

Another question asked the teachers to explain what the vertex of a quadratic function means. As in the previous item, some participants (*n*=5) defined how to find the vertex, rather than explaining what it is. Their responses were like: "The vertex of a quadratic function is $(\frac{-b}{2a}, f(\frac{-b}{2a}))$." (T12). This statement was considered as a procedural description of the vertex, rather than a structural one since it is about how to find the vertex on a parabola, rather than explaining what it means for a quadratic function. One teacher (T17) wrote an incorrect definition of the vertex: "It is the ordinate of the maximum or the minimum point of a parabola." Half of the teachers (*n*=9) wrote a structural definition of the vertex. To illustrate, one of them stated: "The vertex is the maximum or minimum point of the quadratic function." (T13). This statement described what the vertex means for a quadratic function, rather than how to find it, or where it is located on a parabola.

Another question asked the teachers to explain what concavity means for quadratic functions. Few teachers (*n*=3) made structural descriptions of the concavity of a parabola by referring to its shape. For example, one of them wrote: "A parabola is concave down if it is ∩-shaped; concave up if it is ∪-shaped." (T4). Some teachers (*n*=7) made procedural descriptions of concavity by referring to its relationship with the leading coefficient of a quadratic function. For example, one of them, T8, wrote: "If $a > 0$, f is concave up; if $a < 0$, f is concave down." Another teacher, T12, wrote: "If f'' is positive, f is concave up; if f'' is negative, f is concave down."

In the next question, the teachers were asked to find some properties of a given quadratic function such as the vertex, the axis of symmetry, and *x*-intercept(s); then graph it. All of the teachers (*n*=18) found all the properties of the function and sketched its graph appropriately. Then, the teachers were given two parabolas and asked to find the corresponding quadratic functions. Most teachers (n=15) found the quadratic functions for both graphs correctly. They used different algebraic representations of quadratic functions. When the vertex was given in the graph, the teachers used the vertex form to obtain the quadratic function. When the x-intercepts were given, they used the intercept form to find the quadratic function.

Lastly, the teachers were given a graph and asked to comment on the signs of a, b, and c coefficients of the corresponding quadratic function. Almost all teachers (n=17) found the signs of the three coefficients correctly. While finding the sign of a, all the teachers used the same pattern. They checked the concavity of the graph and stated that $a > 0$ since the parabola is upwards." To determine the sign of b, two different approaches were observed. Most of the teachers (n=11) examined the sign of the apsis of the vertex, used the information that a is positive, and concluded that b is positive. For example, T3 wrote: "We know a is positive, $-b/2a$ is negative; so, b should be positive." (T3). However, some teachers (n=6) examined the sign of the sum of the roots to decide the sign of b. Since the roots were given in the graph, the teachers could easily comment on the sign of the sum of the roots. For example, T6 stated: "a is positive, and the sum of the roots $-b/2a$ is negative, so $b > 0$." There was also an incorrect response: "Since there are two roots, b should be positive." (T18). While determining the sign of c, most participants (n=15) examined the y-intercept. Since the ordinate of the y-intercept was on the lower side of the y-axis, they found that c should be negative. There were also a few teachers (n=3) who found the sign of c by checking the sign of the multiplication of the roots. For example, T9 stated: "a is positive. The multiplication of roots (c/a) is negative, so $c < 0$."

4.1.1.4. Teachers' Knowledge of Graphing Quadratic Functions Using Transformations

There were three items that assessed and evaluated teachers' knowledge of graphing quadratic functions using transformations. Table 4.4 summarizes the teachers' responses to these items.

Table 4.4. Teachers' (*n*=18) responses to questions related to CCK4— Teachers' knowledge of graphing quadratic functions using transformations

Answers	*f*	Teachers
Question 12: Explaining how to generate any quadratic function from the graph of $f(x) = x^2$		
Describing some of the transformations (1 point)	7	T17, T13, T7, T8, T6, T4, T12.
Describing all the transformations (2 points)	2	T2, T16.
No answer (0 point)	9	T1, T3, T5, T9, T10, T11, T14, T15, T18
Question 13: Comparing the width of the graphs of quadratic functions		
Examining the leading coefficients of quadratic functions (2 points)	7	T2, T4, T8, T14, T15, T17, T16.
Incorrect (0 point)	2	T9, T18.
No answer (0 point)	9	T7, T11, T13, T1, T3, T10, T5, T6, T12.
Question 14: Comparing the graphs of the quadratic functions $f(x) = x^2 - 5$ and $g(x) = (x - 5)^2$		
Comparing the transformations made onto $f(x) = x^2$ to obtain the two functions (2 points)	7	T17, T9, T7, T8, T6, T4, T2.
Comparing some characteristic of the quadratic functions (1 point)	6	T16, T18, T13, T11, T15, T14.
Incorrect (0 point)	1	T12.
No answer (0 point)	4	T1, T3, T5, T10.

Firstly, the teachers were asked to write their responses to a student's claim that it is possible to generate the graph of any quadratic function by applying some transformations on the graph of $f(x) = x^2$. Half of the teachers (*n*=9) stated that the student is right. However, their explanations were different from each other. For example, seven teachers stated that it is possible by making some translations on the graph of $f(x) = x^2$. One of them, T17, elaborately explained vertical and horizontal translations, and he wrote: "$y = f(x - a)$ is the translation along the *x*-axis a unit right; $y = f(x + a)$ is the translation along the *x*-axis a unit left. $y = f(x) - a$ and $y = f(x) + a$ are translations along the y-axis a unit below and above." Two teachers mentioned reflection, translations, and stretching as graph transformations. One of them, T2, stated: " The student is right. $f(x) = a(x - r)^2 + k$. We can first make horizontal and vertical translations. Then, reflect the graph according to the sign of a, then shrink or stretch it."

The second question asked the teachers to find the quadratic function generating the widest parabola, among the given four ones. Some teachers (*n*=7) correctly found the quadratic function that generated the widest parabola. For example, T17 wrote: "The smaller the $|a|$ becomes, the wider the parabola becomes. So, the answer is C." Two teachers suggested some incorrect strategies to decide the widest parabola. One of them, T9, calculated the difference of the roots and wrote: "$x_1 - x_2$=$\sqrt{\Delta}/|a|$. The answer is D, because $x_1 - x_2 = \sqrt{24}$, the biggest difference.*"* Another teacher, T18, established a relationship between the b coefficient and the width of the parabola, and stated: "The answer is A, because *b* is the biggest."

The last question about graph transformations is about comparing the graphs of the two functions $f(x) = x^2 - 5$ and $g(x) = (x - 5)^2$. Some teachers (*n*=7) made comparisons based on the transformations made on the quadratic function $y = x^2$. For example, T8 wrote: "Both of the functions can be obtained by applying some translations on $y = x^2$. $f(x) = x^2 - 5$ is obtained by translating $y = x^2$, 5 units below along the y-axis; whereas $g(x) = (x - 5)^2$ is obtained by translating $y = x^2$, 5 units right along the *x*-axis." There were also six teachers who compared the two functions based on their some characteristics without referring to any transformations. For example, T14 stated: "$f(x) = x^2 - 5$ intersects the *x*-axis at two different points, whereas $g(x) = (x-5)^2$ is tangent to the x-axis." There was also an incorrect response that T12 wrote: "$g(x) = (x-5)^2$ is parallel to the x-axis."

4.1.1.5. Teachers' Knowledge of Solving Real-Life Problems regarding Quadratic Functions

To assess and evaluate teachers' knowledge of solving real-life problems regarding quadratic functions, they were given a real-life problem and asked to solve it. Table 4.5 shows the teachers' responses to this question.

Table 4.5. Teachers' responses to questions related to CCK5— Teachers' knowledge of solving real-life problems regarding quadratic functions

Question 15: Solving a real-life problem that can be modeled by a quadratic function		
Answers	***f***	**Teachers**
Using an algebraic model (2 points)	1	T16.
Using a numerical approach (1 point)	2	T11, T2.
No answer (0 point)	15	T1, T3, T5, T10, T12, T4, T7, T9, T13, T14, T15, T18, T17, T8, T6.

In the question, there was a mathematical magazine whose price should be increased due to an increase in paper and production costs. The problem also stated that an increase in the selling price would cause a decrease in sales. The teachers were asked to suggest the new price that would yield the maximum profit. Most of the teachers (n=15) did not respond to this question. Only one teacher (T16) used an algebraic model to solve the problem. He defined a quadratic function that represented the income and calculated its vertex to find the maximum income. His solution was (as reproduced for readability):

Income: $(5,5).25000$

Income after the increase in the price: $g(x)$: $(5,5+\frac{1}{2}x).(25000-1250x)$

For $r=4,5$ the function has the maximum. $5,5+\frac{1}{2}.4,5=7,75$. So, the selling price should be 7,75 TL.

There were also some numerical approaches used by two teachers, without using a quadratic function. For example, the solution of T2 is presented below (as reproduced for readability):

$25000.5,5=137500$

$23750.6=142500$

$22500.6,5=146500$

$21250.7=148750$

$\underline{20000.7,5=150000}$

$\underline{18750.8=150000}$

$17500.8,5=148750$. So, I could suggest the selling price as 7,5 TL.

4.1.1.6. Teachers' Knowledge of Finding the Quadratic Functions with Given Points

There were two items in the questionnaire that assessed and evaluated teachers' knowledge of finding the quadratic functions passing through specific points. Table 4.6 summarizes the teachers' answers to these two items.

Table 4.6. Teachers' responses to questions related to CCK6— Teachers' knowledge of finding the quadratic function with given points

Answers	*f*	Teachers
Question 16: Finding the quadratic equation with its vertex and one point given		
Finding the correct quadratic function (2 points)	14	T6, T2, T8, T11, T16, T17, T18, T15, T14, T13, T9, T7, T4, T12.
No answer (0 point)	4	T1, T3, T5, T10.
Question 17: Finding the quadratic function with three points given		
Finding the correct quadratic function (2 points)	12	T12, T3, T7, T9, T14, T17, T16, T11, T8, T2, T4, T18.
No answer (0 point)	6	T1, T5, T10, T6, T13, T15.

In the first one, the teachers were asked to find the quadratic function given the vertex and one point on it. Most of the teachers (n=14) correctly found the quadratic function in the vertex form $y = a(x - r)^2 + k$. In the second one, the teachers were asked to determine the quadratic function whose arbitrary three points were given. This time, most teachers (n=12) used the standard form $y = ax^2 + bx + c$ to find the quadratic function. Teachers used different algebraic demonstrations of the quadratic functions.

4.1.1.7. Teachers' Knowledge of Finding the Intersection of Parabolas and Lines

There were one question and a follow-up question assessing and evaluating teachers' knowledge of finding the intersection of a line and a parabola. The teachers' responses to these items were summarized in Table 4.7.

Table 4.7. Teachers' answers to questions related to CCK7—Teachers' knowledge of finding the intersection of a parabola and a line

Answers	*f*	Teachers
Question 18: Explaining the conditions for the intersection of a parabola and a line		
Stating the three conditions for the intersection of a line and a parabola (2 points)	15	T15, T13, T6, T12, T4, T7, T9, T14, T18, T17, T16, T11, T8, T2, T10.
No answer (0 point)	3	T1, T3, T5.
Follow-up: Finding the intersection of a line and a parabola		
Correctly finding the point of intersection	15	T15, T13, T6, T12, T4, T7, T9, T14, T18, T17, T16, T11, T8, T2, T10.
No answer	3	T1, T3, T5.

Firstly, the teachers were asked to explain the conditions for a parabola $y = ax^2 + bx + c$ and a line $y = mx + n$ to intersect. Almost all teachers (*n*=15) correctly stated the conditions for the intersection of a line and a parabola. They equated the y-values of the parabola and the line; and obtained a quadratic equation. Then, they wrote similar statements to this one: "If $\Delta < 0$, they do not intersect. If $\Delta = 0$, the parabola is tangent to the line. If $\Delta > 0$, they intersect at two different points." (T14). In the follow-up of this question, the teachers were asked to find the points of intersection of a parabola and a line. Most of them (*n*=15) found the solution by using the same strategy they had explained previously. First, they obtained a new quadratic equation by equating the parabola and the line. Then, they calculated the discriminant of this new quadratic equation and stated similar statements like "the parabola and the line intersect at one point", or, "the parabola is tangent to the line."

4.1.2. Teachers' Specialized Content Knowledge of Quadratic Functions

For a general review of each teacher's SCK, the following graph is presented (Figure 4.2). The graph shows the teachers' scores from the SCK items of the questionnaire. There were 10 items (questions 19-28) in the questionnaire that assessed and evaluated teachers' SCK. The maximum score that can be taken from SCK items is 18 points. The teachers' scores range between 2 points (T6) and 17 points (T16).

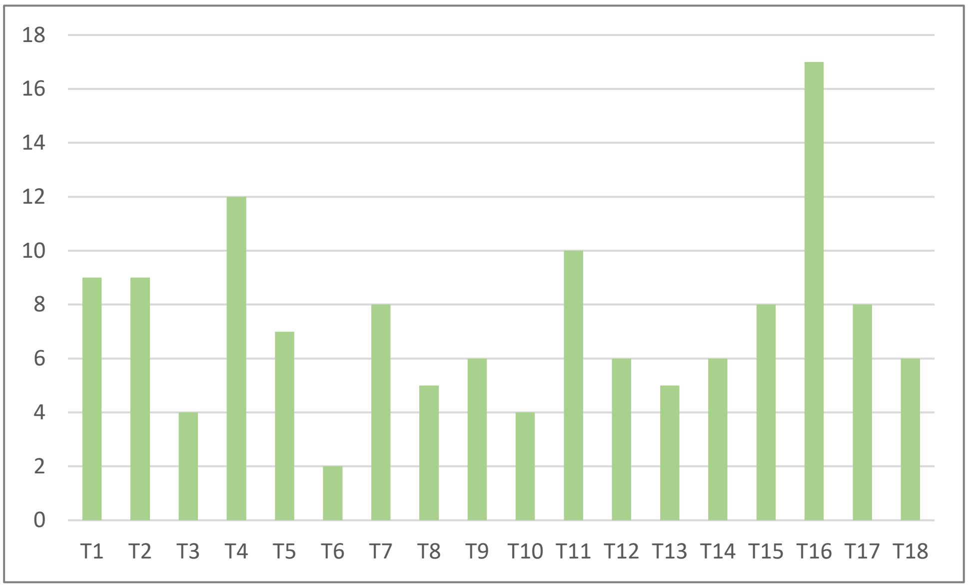

Figure 4.2. The teachers' scores on SCK items on the quadratic function concept questionnaire

For a detailed description of teachers' SCK, the results are presented under seven headings in the following sections.

4.1.2.1. Teachers' Knowledge of Explaining and Justifying Basic Formulas of Quadratic Functions

There were two items assessing and evaluating teachers' knowledge of explaining and justifying basic formulas of quadratic functions. The teachers' responses to these items were summarized in Table 4.8.

Table 4.8. Teachers' (*n*=18) responses to questions related to SCK1— teachers' knowledge of explaining and justifying basic formulas of quadratic functions

Answers	*f*	Teachers
Question 19: Stating and justifying the quadratic formula		
Algebraic justification only (1 point)	6	T18, T16, T4, T3, T1, T11.
No justification (0 point)	9	T14, T13, T12, T10, T9, T8, T2, T17, T15.
No answer (0 point)	3	T7, T6, T5.
Question 20: Solving a quadratic equation without using the quadratic formula		
Solving by completing the square (2 points)	11	T17, T1, T2, T3, T4, T5, T8, T11, T12, T13, T16.
Incorrect (0 point)	2	T10, T15.
No answer (0 point)	5	T6, T7, T9, T14, T18.

Firstly, the teachers were asked to state the quadratic formula, and explain how it is derived, both geometrically and algebraically. Some teachers (*n*=9) just wrote the quadratic formula as "$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$" without any justification. None of the teachers made a geometrical justification of the quadratic formula. However, six teachers made an algebraic justification. One of them, T11, wrote (as reproduced for readability):

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right) = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Then, the teachers were given a quadratic equation $x^2 - x + 1 = 0$ and they were asked to solve it without using the quadratic formula. Many teachers (*n*=11) solved the equation by completing the square method. To illustrate, the solution of T13 is presented below (as reproduced for readability):

$$x^2 - x + \frac{1}{4} - \frac{1}{4} + 1 = 0$$

$$\left(x - \frac{1}{2}\right)^2 = -\frac{3}{4}.$$

$x_1 = \frac{1+\sqrt{3}i}{2}$ and $x_2 = \frac{1-\sqrt{3}i}{2}$."

There was one teacher (T15) who did not solve the equation but stated: "The equation can be solved by factorization." This claim is not correct since the given equation cannot be factorized. There was also one teacher (T10) who attempted to solve the quadratic equation by using a third-order equation (as reproduced for readability):

$$x^3 + 1 = (x + 1)(x^2 - x + 1) = 0$$

$$x^3 = -1$$

$$x = -1,\ x = i,\ x = -i.$$

The above solution is incorrect since the numbers -1, i and $-i$ are not the roots of the given quadratic equation. Moreover, the use of a third-order equation is not one of the strategies for solving quadratic equations.

4.1.2.2. Teachers' Knowledge of Posing Real-Life Problems Regarding Quadratic Functions

For assessing and evaluating the teachers' knowledge of posing real-life problems regarding quadratic functions, they were asked to provide an example of a real-life problem that can be modeled and solved by a quadratic function. The teachers' responses were summarized in Table 4.9.

Table 4.9. Teachers' responses to questions related to SCK2— Teachers' knowledge of posing real-life problems regarding quadratic functions

Question 21: Stating a real-life problem about quadratic functions		
Answers	***f***	**Teachers**
Writing a problem statement (2 points)	1	T16
Writing a problem context (1 point)	10	T18, T15, T12, T11, T9, T8, T7, T4, T2, T1, T10.
No answer (0 point)	7	T14, T13, T6, T5, T3, T17.

Many teachers (*n*=10) wrote a problem context rather than the full statement of a real-life problem. These contexts included projectile motion, velocity-acceleration problems, and calculation of cost and profit-loss in economics. Only one teacher (T16) stated a problem regarding the maximum/minimum of quadratic functions: "Let the cost of a product be x TL. If the product is sold $x^2 - 5x + 14$ TL, what would be the minimum profit?"

4.1.2.3. Teachers' Knowledge of Recognizing Students' Incorrect Solutions Regarding Quadratic Functions

To assess and evaluate the teachers' knowledge of recognizing students' incorrect solutions regarding quadratic functions, they were given a problem related to quadratic functions with an incorrect student solution. They were asked to examine the solution and state whether it is correct or not, by explaining their reasons. The teachers' responses to this item were summarized in Table 4.10.

Table 4.10. Teachers' responses to questions related to SCK3— Teachers' knowledge of recognizing students' incorrect solutions regarding quadratic functions

Question 22: Examining a student's incorrect solution to a given problem regarding quadratic functions		
Answers	***f***	**Teachers**
Explaining all the incorrect steps (2 points)	6	T16, T14, T11, T4, T2, T17.
Explaining some of the incorrect steps (1 point)	9	T15, T13, T12, T10, T9, T7, T5, T3, T1.
No answer (0 point)	3	T6, T18, T8.

Most teachers (*n*=15) noticed that the student's solution is incorrect. Some of them (*n*=6) identified all the incorrect steps in the student's solution and explained them in detail. For example, T17 stated: "The solution is incorrect. The parabola is downwards, the vertex gives the maximum, not the minimum. Also, the ordinate of the vertex gives the max/min value, not the apsis. The endpoints should be checked." (T17). However, half of the teachers (*n*=9) detected some of the student's errors and ignored some others. For example, one teacher (T1) stated: "The student is wrong because the parabola is downwards, the vertex gives the maximum." Another teacher (T10) wrote: "The student did not check the endpoints of the function." These teachers did not make an elaborate description of the student's errors.

4.1.2.4. Teachers' Knowledge of Understanding Students' Unusual Solutions Regarding Quadratic Functions

There were two items in the questionnaire assessing the teachers' knowledge of understanding students' unusual solutions regarding quadratic functions. In both of them, the teachers were given a question and a student's response to this question; then they were asked to comment on the student's solution. Table 4.11 summarizes the teachers' responses to these items.

Table 4.11. Teachers' responses to questions related to SCK4— Teachers' knowledge of understanding students' unusual solutions regarding quadratic equations and functions

Answers	*f*	Teachers
Question 23: Examining a student's solution to a task regarding quadratic equations		
Explaining the student's solution (2 points)	9	T16, T15, T11, T9, T7, T5, T4, T2, T17.
Only stating the solution is correct (0 point)	7	T18, T14, T13, T12, T10, T8, T1.
No answer (0 point)	2	T6, T3.
Question 24: Examining a student's solution to a task regarding quadratic functions		
Explaining the student's solution (2 points)	3	T16, T7, T15.
Only stating the solution is correct (0 point)	6	T17, T18, T12, T8, T5, T4.
Solving the problem using another approach (0 point)	6	T14, T11, T10, T9, T2, T1.
No answer (0 point)	3	T6, T7, T9, T14, T18.

In the first one, a quadratic equation and a student's solution to this equation were given. The student solved the quadratic equation by completing the square, without using the quadratic formula. The teachers were asked to examine the student's solution and decide whether it is correct or not, by explaining the reason for their answers. Some participants (*n*=7) stated that the solution is correct, without writing any explanation. Half of the teachers (*n*=9) stated that the solution is correct, and wrote the name of the student's approach as "completing the square". One teacher (T16) also wrote: "The student solved the equation by completing the square, without using the quadratic formula. This approach is my favorite while teaching quadratic equations. I care about my students understanding the origin of the formula."

In another question, the teachers were asked to examine a student's solution to a problem regarding quadratic polynomials, and decide whether the result is correct or incorrect by explaining their reason. In the problem, some information about the coefficients of a quadratic polynomial was given. Also, one of the roots of the polynomial was given. The question was to find the (unique) quadratic polynomial that satisfies the given conditions. The student found the quadratic polynomial by following some steps. Some teachers (n=6) stated that the solution is correct, without explaining why they thought so. Some other teachers (n=6) also stated that the solution is correct, and they justified the student's solution by finding the quadratic polynomial using a different approach. For example, the solution of one teacher (T10) is: "If one root is $7 + \sqrt{6}$, another is $7 - \sqrt{6}$. The sum of the roots is $\frac{-b}{a} = 14$, and the multiplication of the roots is $\frac{c}{a} = 43$. We know $a = 4$, hence $b = -56$ and $c = 172$. The students' solution is correct." Even though these teachers noticed that the student's solution is correct, they did not really engage in the student's approach. However, their focus was directly on the result, rather than the student's approach or what the student has thought while solving the problem. The teachers obtained the quadratic polynomial using their own approach and compared their results to the student's result. Only three teachers examined the student's approach. For example, one of them (T7) wrote: "The solution is correct. The student made some inverse operations. First, he wrote one of the roots as equal to *x*. Then, he squared the equation and found the result."

4.1.2.5. Teachers' Knowledge of Responding to Students' Why Questions About Quadratic Functions

Two items in the questionnaire assessed and evaluated teachers' knowledge of responding to students' why questions about quadratic functions and equations. The teachers' responses to these items were summarized below (Table 4.12).

Table 4.12. Teachers' responses to questions related to SCK5— Teachers' knowledge of responding to students' why questions about quadratic functions

Answers	***f***	**Teachers**
Question 25: Responding to a student's question about the effects of the translations on the coefficients of quadratic functions		
Making a correct explanation (2 points)	3	T16, T14, T1.
Incorrect (0 point)	13	T18, T17, T15, T12, T11, T10, T9, T8, T7, T5, T4, T3, T2.
No answer (0 point)	2	T13, T6.
Question 26: Responding to a student's question about dividing both sides of a quadratic equation by a variable		
Making a correct explanation (2 points)	17	T2, T11, T15, T16, T18, T1, T4, T5, T6, T7, T8, T9, T10, T12, T14, T13, T17.
No answer (0 point)	1	T3.

One of the questions was about the transformations made on the parabolas. The teachers were asked to respond to a student's question that why translating a parabola upwards and downwards changes only c while translating a parabola to the left and right changes both b and c in the quadratic function $y = ax^2 + bx + c$. Most of the teachers (n=13) failed to explain the reason for these interrelations. For example, one teacher, T3, stated: "While translating upwards and downwards, the roots do not change. So, only c changes. While translating it to the left and right, roots change. So, everything changes." This statement is not correct; because while translating a parabola upwards and downwards, the roots change. Another incorrect explanation was T18's, who stated: "While moving the parabola upwards and downwards, only c changes because the x value stays constant." Similarly, T12 wrote: "While moving up and down, only c changes because $x = 0$. While moving left and right, the roots change, then the sum and the multiplication of the roots change. So, b and c change." Another teacher (T9) stated: "The reason for this is that the vertical translation does not affect the roots. While translating left and right, the roots change; so both of the values change." Three teachers suggested a plausible explanation for the effects of the translations on the coefficients of the parabola. Their common idea was based on the location of the vertex. For example, one of them (T16) wrote:

> While translating upwards and downwards, the apsis of the vertex does not change. So, the sum of the roots stays constant but the roots change. So, the multiplication of the roots changes. Thus, b stays constant, and c changes. While translating left and right, both the sum of the roots and the multiplication of the roots change. Hence, b and c change.

In another question, the teachers were given an imaginary conservation between two students about the division of a quadratic equation by a variable, x. In the conservation, one student claimed that both sides of the equation cannot be divided by x. Another student responded, "If we can divide both sides by 3, why can't we divide by x?" The teachers were asked to state the most proper explanation for their students. Almost all teachers (n=17) provided plausible explanations. For example, T2 wrote: "I would say that an equation cannot be divided by x, because we can eliminate one of the roots, which is equal to 0." Another similar response was: "I would say that an equation cannot be divided by x, because we don't know the value of x. It might be equal to 0, and 0 cannot divide any number." (T17). As seen in these two responses, some teachers (n=7) provided an explanation based on the elimination of one root, while some (n=10) mentioned the division rule that 0 cannot divide any number.

4.1.2.6. Teachers' Knowledge of Finding an Example to Make a Specific Mathematical Point About Quadratic Functions

To assess and evaluate the teachers' knowledge of finding an example to make a specific mathematical point about quadratic functions, they were asked to state what kind of examples they would use in the classroom to emphasize the symmetrical property of a parabola. The summary of their responses is presented in Table 4.13.

Table 4.13. Teachers' responses to questions related to SCK6— Teachers' knowledge of finding an example to make a specific mathematical point about quadratic functions

Question 27: Stating examples to emphasize the symmetrical property of a parabola		
Answers	***f***	**Teachers**
Writing relevant examples (2 points)	3	T16, T18, T4.
Writing irrelevant examples (0 point)	12	T15, T14, T12, T11, T9, T8, T7, T5, T2 T17, T1, T10.
No answer (0 point)	3	T13, T6, T3.

Some teachers (n=12) made explanations that were not directly related to the symmetrical property of parabolas. One of them, T15, responded: "I would use geometrical examples." Another response was: "I show my students some symmetrical shapes such as a heart shape." Similarly, T11 stated: "I would show them butterfly shape as an example of symmetrical shape and make them understand what symmetrical means." Another teacher (T5) stated: "I would use a mirror." Some of them (n=3) suggested the use of mathematical software to emphasize the symmetrical property of a parabola. For example, T10 stated: "I would draw some parabolas using mathematical software and demonstrate the symmetrical property of parabolas on them." Three teachers proposed a different way of emphasizing the symmetrical property of a parabola. In his response to the questionnaire, one of them, T16, wrote: "I define r (the apsis of the vertex) as the half of the sum of the roots. I tell my students that the x-values that add up to $2r$ are symmetrical. For example, if $r = 5$, $f(1) = f(9)$ or $f(-5) = f(15)$. I want my students to notice this property."

4.1.2.7. Teachers' Knowledge of Modifying Tasks of Quadratic Functions

The teachers were asked one question to assess and evaluate their ability to modify tasks of quadratic functions. The summary of the teachers' responses to this question is shown in Table 4.14.

Table 4.14. Teachers' responses to questions related to SCK7— Teachers' knowledge of modifying tasks of quadratic functions

Question 28: Examining a given task about quadratic functions and modifying the task for their students		
Answers	***f***	**Teachers**
Making some reasonable modifications	7	T16, T14, T12, T11, T9, T8, T4.
Making no modification	4	T17, T15, T10, T1.
Making some unnecessary/irrelevant modifications	2	T18, T7.
No answer	5	T13, T6, T5, T3, T2.

The teachers were given a task about finding an unknown coefficient in a quadratic function and were asked two questions. Firstly, they were asked to examine the task and state whether their students could solve this task or not. Secondly, if they thought that this was an easy/difficult task for their students, they were asked to explain how they could modify it to be harder or easier. Four teachers stated that their students could solve the task; so they made no modifications. Half of the teachers (n=9) stated that the task could be hard for their students. So, they made some modifications to the task. For example, T18 wrote: "I would give extra information about the sign of the sum of the roots or the multiplication of the roots." Another teacher, T7, responded: "I would delete the statement "the distance between A and B is 3 units" and write "one of the roots is 3 more than the other"." However, these modifications do not seem to contribute to making the task easier, so they might be unnecessary. On the other hand, some teachers (n=7) made some plausible modifications. These included:

"I would give the numerical value of the b coefficient." (T8).

"I would give the sum of the roots as extra information." (T9).

"I would ask a very simple question like $f(x) = x^2 - mx + m + 3$ intersects the x-axis $at\ x = 3$, what is the value of m?" (T11).

"I would give the apsis of the vertex as extra information." (T12).

"I would change the problem as "$f(x) = x^2 - 5x + m - 1$ intersects the x-axis at two different points, A and B. If $|AB| = 3$ units, what is m?"" (T14).

4.1.3. Teachers' Horizon Content Knowledge of Quadratic Functions

For a general review of each teacher's HCK, the following graph is presented (Figure 4.3).

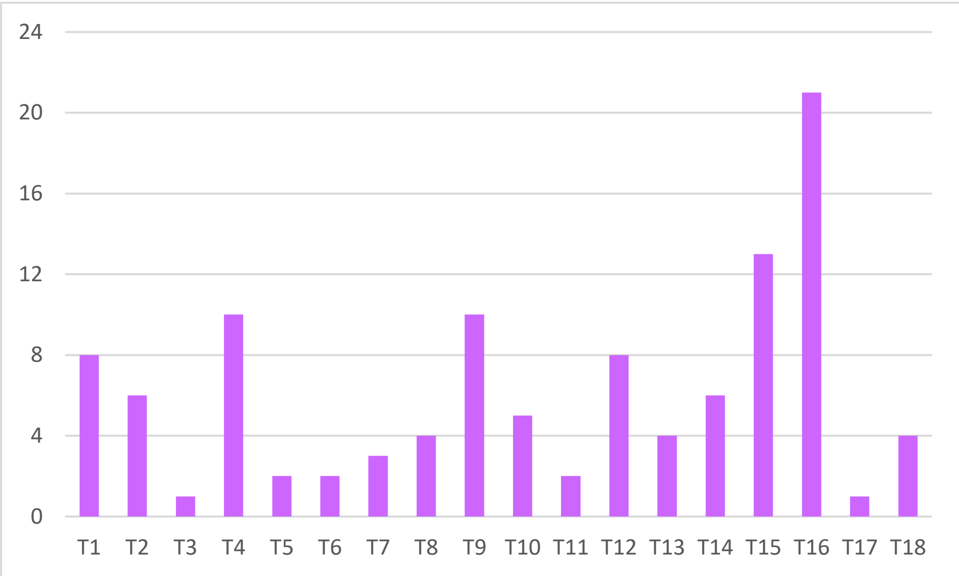

Figure 4.3. The teachers' scores on HCK items on the quadratic function concept questionnaire

The graph shows the teachers' scores from the HCK items on the questionnaire. The items from 29 to 40 evaluated the teachers' HCK. The maximum score that can be taken from the HCK items is 24 points. The teachers' scores range between 1 point (T3, T17) and 21 points (T16). For a detailed description of teachers' HCK, the results are presented under two headings in the following sections.

4.1.3.1. Teachers' Knowledge of How Quadratic Functions Are Related to Other Contents in the High School Curriculum

There were 6 items that assessed and evaluated the teachers' knowledge of how quadratic functions are related to other contents in the high school curriculum. Table 4.15 summarizes the teachers' responses to these items.

Table 4.15. Teachers' responses to questions related to HCK1—Teachers' knowledge of how quadratic functions are related to other contents in the high school curriculum

Answers	***f***	**Teachers**
Question 29: Explaining the relationship between the concavity of a parabola and the derivative		
Relating the concavity of the graph with the second derivative of the quadratic function (2 points)	1	T16
Irrelevant explanations (0 point)	11	T15, T12, T11, T9, T4, T2, T18, T6, T7, T1, T10.
No answer (0 point)	6	T17, T14, T13, T10, T5, T3.
Question 30: Comparing the graphs of quadratic functions and exponential functions		
Comparing the graphs of quadratic functions and exponential functions correctly(2 points)	4	T10, T12, T15, T16.
Incorrect (0 point)	7	T2, T4, T8, T13, T17, T3, T5.
No answer (0 point)	7	T1, T6, T7, T9, T11, T14, T18.
Question 31: Explaining the relationship between the vertex of a parabola and the derivative		
Explaining the relationship between the vertex and the derivative partially (1 point)	10	T18, T16, T15, T14, T12, T9, T13, T8, T7, T4.
Incorrect (0 point)	1	T2.
No answer (0 point)	7	T6, T5, T1, T17, T11, T10, T3.
Question 32: Relating quadratic functions to any concept from the physics course		
Writing any concept from the physics course related to quadratic functions (2 points)	7	T1, T4, T15, T16, T2, T12, T6.
No answer (0 point)	11	T3, T7, T8, T10, T11, T13, T17, T18, T9, T14, T5.
Question 33: Explaining the relationship between the golden ratio and quadratic equations		
Relating quadratic equations with the golden ratio (2 points)	5	T2, T8, T9, T15, T16.
Only stating the golden ratio (0 point)	4	T7, T14, T3, T5.
No answer (0 point)	9	T6, T10, T11, T13, T17, T18, T12, T4, T1.
Question 34: Determining whether the graph of $y = x^4$ is a parabola or not		
Stating that the graph of $y = x^4$ is not a parabola (2 points)	12	T1, T4, T7, T9, T10, T12, T13, T14, T15, T16, T18, T5.
Incorrect (0 point)	3	T8, T2, T17.
No answer (0 point)	3	T3, T6, T11.

In question 29, the teachers were asked to provide a plausible explanation for why the graph of the quadratic function $f(x) = ax^2 + bx + c$ is concave up if $a > 0$, concave down if $a < 0$. Only one teacher (T16) made a plausible explanation for this well-known fact of quadratic functions. He explained this fact referring to the second derivative of quadratic functions. Most teachers (n=11) wrote some irrelevant explanations that do not explain the reason for the aforementioned fact of quadratic functions. Their answers were like: "Sketching the graphs of several quadratic functions helps students understand the relationship between the sign of a, and the concavity of the graph." (T7). Similarly, T15 wrote: "I draw different parabolas and demonstrate to my students the change in the concavity of them, according to the sign of a." (T15).

The next question evaluated whether the teachers are able to compare the graphs of quadratic functions and the exponential functions. The teachers were asked to comment on a student's claim related to the patterns of the graphs of functions p and q, where p is an exponential function and q is a quadratic function. In the question, the student stated that after about $x = 3$, the quadratic function will always take greater values than the exponential function. Some teachers correctly (n=4) stated that the student's claim is false and made plausible explanations. One of them, T12, wrote: "The student is wrong because an exponential function eventually will get bigger than a quadratic function." (T12). There were also some teachers (n=7) who incorrectly stated that the student is right. For example, T3 stated: "When we examine the graphs of the two functions, we see that y-values of the function q is always greater than of the function p."

In another question, the teachers were asked to explain the relationship (if any) between the vertex of a quadratic function and the derivative of the function. One participant (T2) wrote an incorrect answer: "The first derivative of a function can be found by drawing tangents from the vertex of the function." Some teachers (n=10) explained the relationship between the vertex and the first derivative partially. They reported that the first derivative of the function is 0 at the vertex. Although this statement is correct, it is insufficient to explain the relationship between the vertex and the first derivative since it does not include information about the rate of change or the maximum-minimum points.

Then, the teachers were asked to state whether any concept from the physics course is related to quadratic functions. Seven teachers wrote some concepts from the physics course which might be related to quadratic functions. These are; free fall, projectile motion, and velocity-acceleration problems. Most teachers (n=11) did not respond to this question.

In the next question, the teachers were asked to explain how the golden ratio and quadratic equations are related. Half of the teachers (n=9) did not answer. Some teachers (n=4) only stated the numerical value of the golden ratio as $1 + \frac{\sqrt{5}}{2}$, without explaining its relation to quadratic equations. Five teachers explained the relationship between the golden ratio and quadratic equations. Their answers included: "The golden ratio is the positive root of the quadratic equation $\mathrm{x}^2 - \mathrm{x} - 1 = 0$." (T2).

Another question asked the teachers whether the graph of $y = x^4$ is a parabola or not. Most teachers correctly (n=12) stated that $y = x^4$ is not a parabola, but they did not state a reason for their response. Three teachers gave incorrect answers. Two of them stated that the graph of the function $y = x^4$ is a parabola. For example, T17 wrote: "It is a parabola because it is U-shaped." On the other hand, another teacher, T8, stated: "It is not a parabola because it is so wide. The arms of the parabolas are narrower."

4.1.3.2. Teachers' Knowledge of How Quadratic Functions Are Related to Advanced Mathematics

There were 6 items evaluating the teachers' knowledge of how quadratic functions are related to advanced mathematics. The summary of the teachers' responses is presented in Table 4.16.

Table 4.16. Teachers' responses to questions related to HCK2—Teachers' knowledge of how quadratic functions are related to advanced mathematics

Answers	***f***	**Teachers**
Question 35: Stating the reflection property of a parabola and its daily use		
Explaining the reflection property and its daily use correctly (2 points)	3	T16, T9, T4.
Incorrect (0 point)	6	T2, T18, T12, T7, T8, T14.
No answer (0 point)	9	T17, T15, T13, T11, T10, T6, T5, T3, T1.
Question 36: Explaining the relationship between a parabola and a hyperbola		
Explaining some differences between parabolas and hyperbolas (2 points)	3	T16, T2, T9.
Incorrect (0 point)	5	T8, T4, T12, T15, T18.
No answer (0 point)	10	T1, T3, T5, T6, T7, T10, T11, T13, T14, T17.
Question 37: Stating the fundamental theorem of algebra and its application to quadratic polynomials		
Applying the fundamental theorem of algebra to quadratic polynomials (2 points)	5	T1, T14, T15, T16, T4.
Incorrect (0 point)	1	T2.
No answer (0 point)	12	T3, T5, T6, T7, T8, T9, T10, T11, T12, T13, T17, T18.
Question 38: Choosing the most proper statement about a parabola		
Statement 1 is correct (0 point)	2	T1, T2.
Statement 2 is correct (0 point)	13	T12, T16, T7, T8, T9, T11, T14, T15, T17, T18, T3, T5, T6.
None (0 point)	2	T4, T13.
No answer (0 point)	1	T10.
Question 39: Defining a parabola and stating alternative definitions		
Describing the parabola as the graph of a quadratic function and stating the geometrical definition of the parabola (2 points)	1	T16.
Describing the parabola as the graph of a quadratic function only (1 point)	11	T1, T4, T8, T10, T11, T13, T14, T15, T17, T18, T3.
Incorrect (0 point)	4	T2, T7, T9, T12.
No answer (0 point)	2	T5, T6.
Question 40: Deciding whether a given shape is a parabola or not		
Distinguishing between a parabola and a catenary (2 points)	1	T16.
Stating that it is not a parabola without explanation (1 point)	5	T15, T9, T1, T12, T11.
Incorrect (0 point)	7	T18, T14, T13, T8, T4, T2, T3.
No answer (0 point)	5	T17, T10, T7, T6, T5.

Firstly, the teachers were asked to explain the reflection property of a parabola and its daily use. Half of the teachers (*n*=9) did not respond. Some teachers (*n*=6) made some explanations that were not related to the reflection property. One of them, T14, stated: “Arch bridges have a parabolic shape.” Another teacher, T12, stated: “I would tell my students that the vertex of a parabola is the axis of symmetry.” On the other hand, three teachers correctly stated the reflection property and its daily use. For example, one of them, T16, wrote: “A ray that is parallel to the axis of symmetry of the parabola is reflected and passes through the focus. It is used in the real-life in the construction of headlights and satellite dishes.”

The teachers were also asked to explain the relationship (if exists) between a parabola and a hyperbola. Most of the teachers (*n*=10) did not respond to this question. Few teachers (*n*=3) described the properties of a parabola and a hyperbola. One of them, T16, wrote: “They both are conic sections. A parabola is the set of points which are equidistant from a straight line and focus whereas a hyperbola is the set of points whose distances to two fixed points have a constant difference.” Some teachers (*n*=5) made some incorrect explanations as illustrated below:

> “A hyperbola is the symmetry of a parabola.” (T4)
>
> “A parabola is of the form $y = ax^2$, whereas a hyperbola is of the form $x = ay^2$.” (T15)
>
> “They both are the graphs of quadratic functions.” (T18)

In question 37, the teachers were asked to state the Fundamental Theorem of Algebra and how it applies to quadratic polynomials. Most teachers (*n*=12) did not respond to this question. One teacher (T2) gave an answer that was not directly related to this theorem and its application to quadratic polynomials. He drew some squares and represented a quadratic function by completing the square. Five teachers stated the theorem and its application to quadratic polynomials. For example, one of them, T14, stated: “A polynomial with degree n has n roots. Quadratic polynomials have two roots”. Another teacher wrote: “Quadratic equations have 2 roots. If the discriminant is less than 0, it has no roots." In the next question, the teachers were given two statements that were written by two students. These are:

Student 1: The graph of a quadratic function is a parabola.
Student 2: The graph of a quadratic function is called a parabola.

Then, they were asked to select the most correct statement with a justification for their answer. A few teachers (*n*=2) stated that the statement of Student 1 is the most correct. One of them, T1, who selected the first statement wrote: "The second statement is a definition, but a parabola cannot be defined. So Student 1 is correct." (T1). Another teacher, T2, wrote: "Student 1 is right because the other name for the parabolas is quadratic functions." Both of these explanations are incorrect. On the other hand, most of the teachers (*n*=13) thought that Student 2 is correct. Most of them (*n*=11) did not state any reason for their answer. Only one of them, T12, made an explanation: "The second statement is correct because it is a definition." There were also two teachers who stated that none of the statements is correct. One of them, T4, wrote that the correct statement should be: "The graph of a quadratic polynomial function is called a parabola." Another teacher, T13, stated that the correct statement should be: "The graph of a polynomial function $f(x) = ax^2 + bx + c,\ (a \neq 0,\ a, b, c \in \mathbb{R})$ is a parabola." The common idea in the previous two responses is based on adding the term *polynomial* before the word *function*.

In the next question, the teachers were asked to define a parabola and state some alternative definitions for it. The purpose of this question was to examine whether the teachers have any idea about the geometrical definition of a parabola as the question that asked about the parabola-hyperbola relationship. Some teachers (*n*=4) wrote some incorrect statements like:

"A parabola is a quadratic function." (T2)
"A parabola is the graph of a quadratic equation." (T9)
"A parabola is a quadratic equation. (T12)"

Most of the teachers (*n*=11) defined a parabola as the graph of a quadratic function and did not suggest any alternative definitions. Only one teacher, T16, presented an alternative definition. He wrote: "A parabola is the graph of a quadratic function. Alternative definition: A parabola is the set of points that are equidistant from both the directrix and the focus."

The next question evaluated whether the teachers could distinguish a parabola from a catenary. For this purpose, the teachers were shown a figure (the shape of a uniform flexible chain) and asked to state whether that shape is a parabola or not. Some participants (*n*=7) stated that it is a parabola, without further explanation, while some (*n*=5) stated that it was not a parabola. For example, T15 wrote: "I would say that it resembles a parabola, but it is not." Only one teacher, T16, mentioned a *catenary*, which is a curve formed by a wire, rope, or chain hanging freely from two points that are not in the same vertical line. He wrote: "I would say that it is a catenary."

4.1.4. Summary of Teachers' Subject Matter Knowledge of Quadratic Functions

Teachers' SMK of quadratic functions was discussed in the previous sections, on the basis of their CCK, SCK, and HCK. In analyzing the results of the questionnaire, the teachers' responses to CCK, SCK, and HCK items were discussed separately. The graph in the Figure 4.4 summarizes the teachers' overall performance on the quadratic function concept questionnaire. It also enables the reader to compare an individual teacher's scores on each dimension of the questionnaire. The maximum scores of each dimension in the questionnaire were not equal (36 points for CCK items, 18 points for SCK items, and 24 points for HCK items) as the number of items included in these dimensions were not equal. Thus, the teachers' scores on each dimension were modified to be out of 100 points to make a more meaningful comparison between teachers' performances on each dimension. As shown in Figure 4.4, the total score of the questionnaire is 300 points.

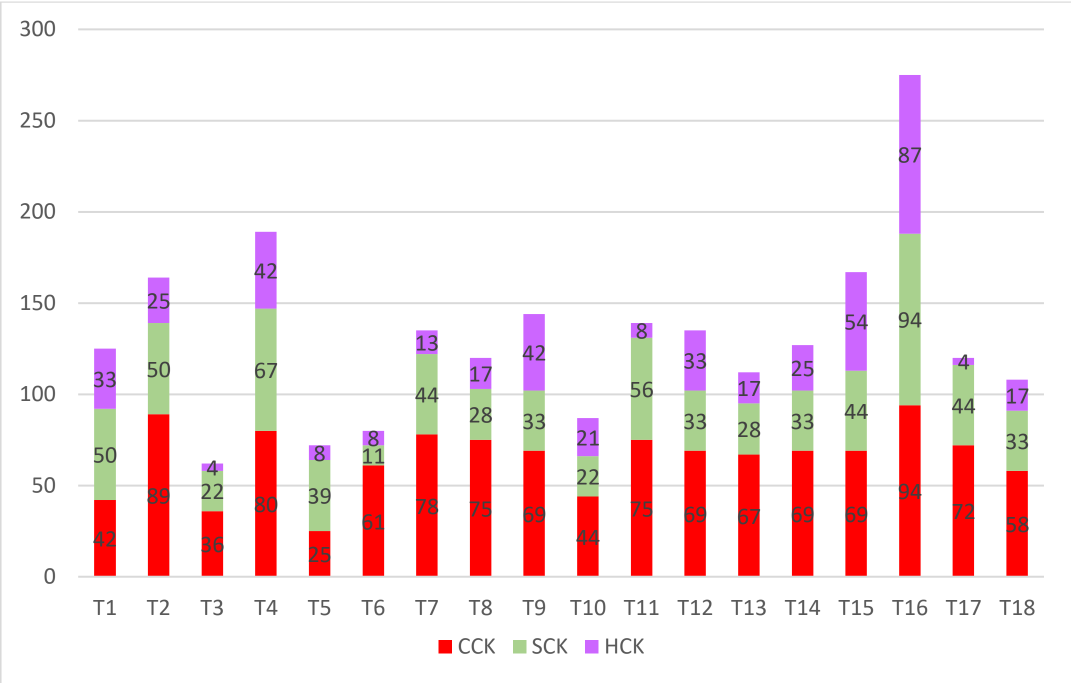

Figure 4.4. The teachers' scores on CCK, SCK, and HCK items

In the graph, each dimension is represented by a different color. The purple area represents the teachers' HCK scores, the green area represents the teachers' SCK scores, and the red area represents the teachers' CCK scores. However, it is not intended to say that these dimensions are disjoint. Although this study attempted to differentiate and measure each dimension, it is an undeniable fact that these sub-dimensions of teacher knowledge interact with each other. Thus, it might be useful to note that the purpose of this graph is to present a general picture of teachers' SMK and compare their scores on CCK, SCK, and HCK items in the questionnaire. As it can be seen in Figure 4.4, the majority of teachers' performance on the CCK items are remarkably better than their performances on the SCK and HCK items. Moreover, teachers' performances on the HCK items are the lowest among the three dimensions for most of the participants.

4.2. Contribution of Subject Matter Knowledge to Student Learning Outcomes: The Case of Can

In this section, the case of Can was presented. The data obtained from the questionnaire, the interview, and the classroom observation were combined and triangulated. Can is the teacher who was referred to as “T17” in the first phase of the study. He has 19 years of teaching experience, and he was teaching at an Anatolian High School when this study was conducted.

His overall performance on the quadratic function concept questionnaire was moderate. He performed better in the questions related to CCK, however, he failed to solve questions that require a deeper and more connected understanding of the mathematical concepts. His scores from the HCK items were extremely lower than his scores from the CCK items. His performance on SCK items was moderate. These will elaborately be discussed in the following sections.

4.2.1. Can’s Subject Matter Knowledge of Quadratic Functions

The descriptions of Can’s subject matter knowledge were developed from his responses to the questionnaire (see Appendix A), the follow-up interview (see Appendix C), and classroom observations. These descriptions are presented under three headings in the following sections.

4.2.1.1. Can’s Common Content Knowledge of Quadratic Functions

For an elaborate discussion of Can’s CCK, the results are presented under seven headings that indicate the sub-dimensions of teachers’ CCK.

Can's conception of quadratic equations and functions

Can defined both quadratic functions and quadratic equations referring to their algebraic representations. When he was asked to define a quadratic equation, he wrote: "A quadratic equation is $ax^2 + bx + c = 0$ where a, b, c are real numbers and $a \neq 0$." Similarly, he defined a quadratic function: "A quadratic function is $f(x) = ax^2 + bx + c$ where a, b, c are real numbers and $a \neq 0$." When he was asked to distinguish quadratic functions, quadratic equations, and quadratic polynomials, he stated: "An equation involves an equality. The exponents of quadratic polynomials must be natural numbers."

In the questionnaire, when the teachers were asked to decide whether two given tables containing x and y values indicate a linear or a quadratic function, he calculated the first differences and stated: "The first one is linear since first differences are constant. The second one is quadratic since the first differences are not constant". In the interview, the researcher asked him:

> **Researcher:** In the 4th question, you have stated that the values in the first table belong to a linear function, the values in the second table belong to a quadratic function. Could you explain how did you decide it?
>
> **Can:** I examined the differences. Four, four, four, four. The first one is linear. In the second table, there is no linear increase or decrease. Here is 3, here is 2, here is 5. Since the differences were different, I said that this was a quadratic function.
>
> **Researcher:** Okay. You examined the differences. Does a non-constant difference always indicate a quadratic function?
>
> **Can:** Hmm… I did not think about that. I don't know.

As it can be seen in his words, Can did not have any idea about the constant second differences of quadratic functions. He examined only the first differences of the two functions and thought that if the first difference is constant, the function is linear, and if it is non-constant, the function is quadratic.

Can's knowledge of solving quadratic equations with one unknown

In the questionnaire, he solved the quadratic equations by using the quadratic formula. In the follow-up, the teachers were required to state alternative methods to solve quadratic equations. He suggested factorization as an alternative way of solving quadratic equations. During his instruction he used the quadratic formula and the factorization for solving quadratic equations. He never used completing the square method to solve a quadratic equation.

Can's knowledge of sketching and interpreting the graphs of quadratic functions

He made some incorrect or structural descriptions of the concepts like the vertex, the axis of symmetry, and the concavity of quadratic functions, in his responses to the questionnaire. For example, he defined the axis of symmetry as "the apsis of the vertex". This is not a correct definition of the axis of symmetry since it is a line that separates the parabola into two symmetrical parts, not a single point. In the follow-up interview, when he was asked to find the axis of symmetry of the function $g(x) = -6x^2+12x+5$, he calculated the r-value (i.e., the apsis of the vertex) and wrote "$r = 1$" as the axis of the symmetry of the function $g(x)$.

He defined the vertex as "the ordinate of the maximum or the minimum point of a parabola". This is also an incorrect definition since the vertex is the point (r, k) in the coordinate plane. In the follow-up interview, when he was asked to find the vertex of the function $f(x) = 3x^2 + 9x + 6$, he found the ordinate of the vertex and wrote "37/4" as the vertex of the function. This response is consistent with his definition of the vertex in the questionnaire. He did not write a definition for concavity in the questionnaire.

The teachers were also asked to find some properties of a quadratic function such as the axis of symmetry, the vertex, the *x*-intercepts, and the *y*-intercept, then sketch the graph of it. Like all the other teachers, he found all the properties and then drew the correct graph. While finding the axis of symmetry, he wrote “$r = -1$” as the axis of symmetry, rather than writing “$x = -1$”. In the next question, two graphs were given and the teachers were asked to find the quadratic functions. He correctly found the quadratic functions. When the vertex was given in the graph, he used the vertex form of quadratic functions. When the *x*-intercepts were given, he used the intercept form of quadratic functions. He used different algebraic forms of quadratic functions according to the nature of the task. In connection with this, in the interview the researcher asked him:

> **Researcher:** In the 10th question, you have written the first function in the vertex form, the second function in the intercept form. Do you use these different algebraic demonstrations in your classroom instruction?
>
> **Can:** We use the vertex form in the questions about translations because we tell the translations as adding a constant to the inside or outside of the function. We say that if we add inside, the function moves along the *x*-axis; if we add outside, the function moves along the *y*-axis.
>
> **Researcher:** In what cases do you use the standard form?
>
> **Can:** We use the standard form if the graph is not given, and three arbitrary points are given. We move based on the types of questions. Maybe, we might do wrong since we give importance to the types of questions, not the concept of the parabola.

During the classroom instruction, as he told in the interview, he used different algebraic forms of quadratic functions. To illustrate this, an example from his instruction is given below (Figure 4.5).

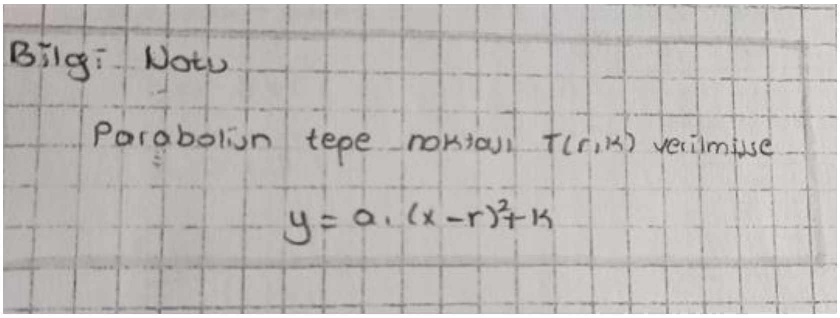

Figure 4.5. A section from Can's instruction

Lastly, the teachers were shown a graph. They were asked to comment on the signs of the coefficients of the corresponding quadratic function. The response of Can is presented below (Figure 4.6).

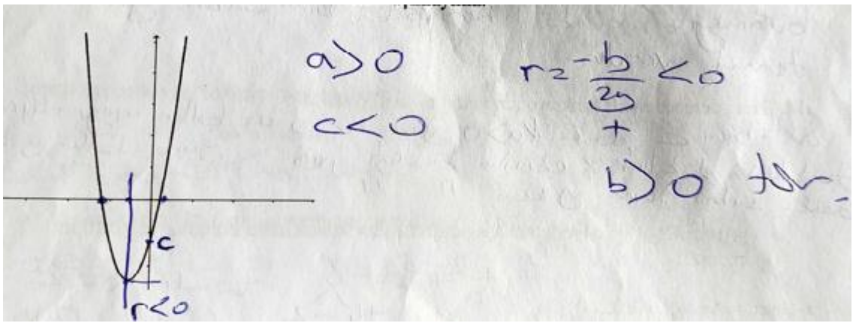

Figure 4.6. Can's response to question 11

As seen in Figure 4.6, he first determined the sign of a. He determined the sign of b by using the information that r is negative. He determined the sign of c by writing the ordinate of the y-intercept as c in the graph. In the interview, the following conservation also reveals the pattern that he used to determine the signs of the coefficients.

> **Researcher:** In the 11th question, you wrote that $a > 0$, $b > 0$, and $c < 0$. How did you determine the signs?
>
> **Can:** The parabola is upwards, so $a > 0$. We know a is positive, the vertex $(-b/2a)$ is negative; so b must be greater than 0. Since the ordinate of the y-intercept is negative, c must be negative.

Can's knowledge of graphing quadratic functions using transformations

When he was asked to explain how to generate the graph of any quadratic function from the graph of $f(x) \ = x^2$, he stated that it can be done by vertical and horizontal translations (question 12). In question 13, he properly found the quadratic function that generates the widest parabola among the given four quadratic functions. He stated: "The smaller the $|a|$ gets, the wider the parabola becomes. So, the answer is C." During classroom instruction, he did not tell his students about this property of the parabolas. When he was asked to compare the graphs of the functions $f(x) = x^2 - 5$ and $g(x) = (x - 5)^2$ without drawing their graphs, he compared the functions by explaining the transformations made on $y = x^2$ (question 14). He stated: "$f(x)$ is the translation of $y = x^2$, 5 units below along the y-axis and $g(x)$ is the translation of x^2, 5 units right along the *x*-axis. Their shape is the same." As it can be seen in his words, he knows the shape is conserved during vertical or horizontal translations.

Can's knowledge of solving real-life problems regarding quadratic functions

Can did not solve the real-life problem of quadratic functions in the questionnaire (question 15). In the interview, the researcher asked him to read the question again and think about it.

> **Researcher:** In the questionnaire, you did not solve the question 15. Could you examine the question again and think about how it can be solved?
>
> **Can:** Hmm… Let me look at the question (examines the question and his answer). Well, I have tried to calculate the loss for each 0,5 cent. I multiplied 1250 and 0,5 and found that each 0,5 cent increase in the price causes 625 dollars loss.
>
> **Researcher:** Well, you did not continue.
>
> **Can:** Yes, I don't know how to move on. I am stuck here.

As it can be seen in his above words, he could not write a quadratic function for calculating the new income. He just made some numerical operations that were not enough to find solution for the problem.

Can's knowledge of finding the quadratic function with given points

Can correctly found the quadratic functions when some specific points on them were given. In the first one, the vertex and an arbitrary point on the function were given. In this case, he used the vertex form and found the quadratic function correctly. In the second one, three arbitrary points on the function were given. This time, he used the standard form to find the quadratic function. His solution is presented below (Figure 4.6). As shown in Figure 4.7, he used the standard form and calculated the coefficients a, b, and c correctly.

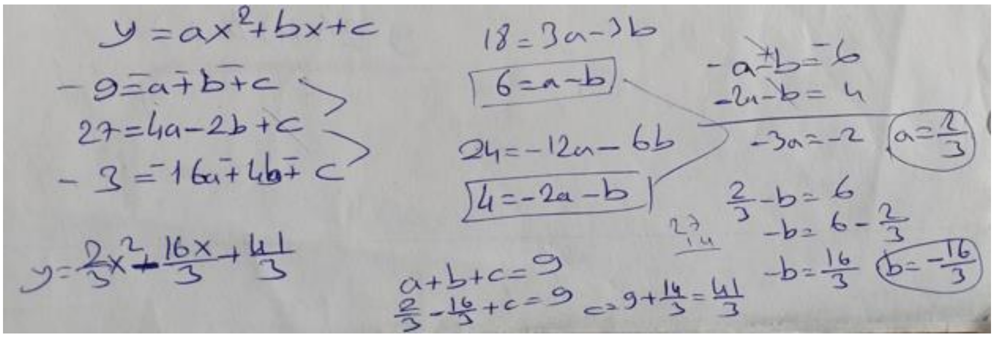

Figure 4.7. Can's response to question 17

Can's knowledge of finding the intersection of a parabola and a line

Can correctly explained the intersection of parabolas and lines. In question 18, he identified three conditions for the intersection of a parabola and a line. He stated: "

$ax^2 + bx + c = mx + n$. We examine the discriminant of this new equation. If Δ<0, they do not intersect; if Δ>0, they intersect at two points; if Δ=0, the parabola is tangent to the line." Then, he found the intersection of the line $y = 11x - 13$ and the parabola $y = 2x^2 + 3x - 5$ as shown in Figure 4.8.

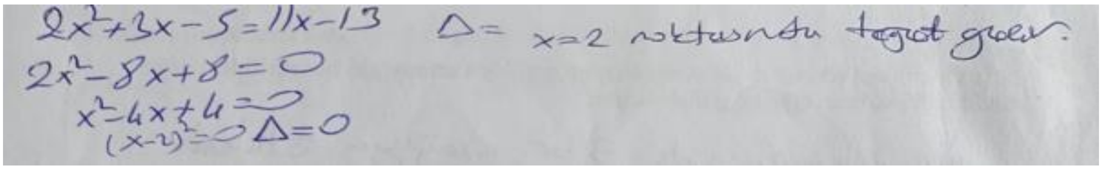

Figure 4.8. Can's response to the follow-up of question 18

During his instruction, Can told his students this content and solved several examples of finding the intersection of a parabola and a line. One of them is presented below (Figure 4.9).

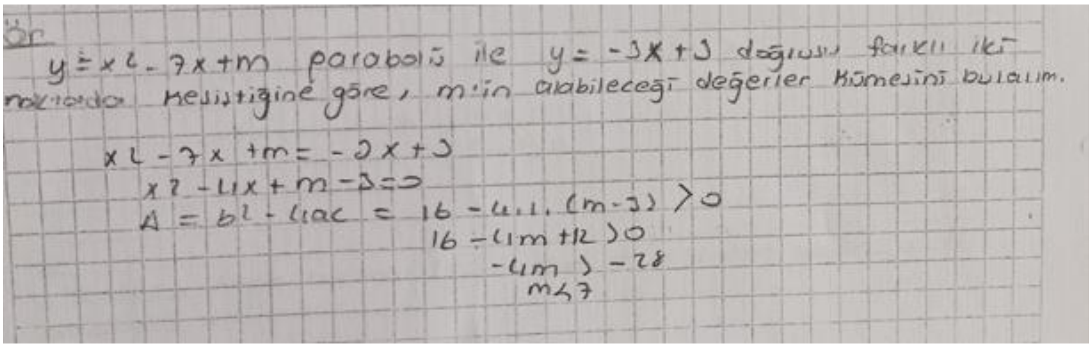

Figure 4.9. A section from Can's instruction

4.2.1.2. Can's Specialized Content Knowledge of Quadratic Functions

For a detailed description of Can's SCK, the results are presented under seven headings that indicate the sub-dimensions of teachers' SCK in the present study.

Can's knowledge of explaining and justifying basic formulas of quadratic functions

In the questionnaire, the teachers were asked to state and justify the quadratic formula both geometrically and algebraically (question 19). He did not make any algebraic or geometrical justification; he just stated the quadratic formula. Then, the teachers were asked to solve the quadratic equation $x^2 - x + 1 = 0$ without using the quadratic formula (question 20). He correctly solved the equation by completing the square. He did not use this method in the classroom to make an algebraic justification of the quadratic formula or to solve quadratic equations. In the interview, the researcher asked him:

> **Researcher:** In question 19, you said that you would justify the quadratic formula by drawing the graph. I could not get what you meant. How do you justify the quadratic formula on the graph?
>
> **Can:** I mean if a parabola has two x-intercepts, the discriminants must be greater than zero.
>
> **Researcher:** Do you think that this is the justification of the quadratic formula?
>
> **Can:** Actually no. I explain why the discriminant is zero for perfect square functions.
>
> **Researcher:** Do your students ask about where the quadratic formula has come from?
>
> **Can:** Usually no. It is easier for them to memorize the formula rather than to prove it. We have time restrictions and so we cannot engage in proofs. In the book which was given by the government, there exist some proofs but we skip them.

As seen in the above conservation, he stated that he does not have time for the justifications or proofs of formulas. During the classroom instruction, as he also stated in the interview, he did not tell his students the justification of the quadratic formula. He introduced completing the square method shortly and found the vertices of two functions by his method. Then, he moved on without using the completing the square method in any part of his instruction.

Can's knowledge of posing real-life problems regarding quadratic functions

Can did not answer the question in the questionnaire which asked them to write a real-life problem they might use during classroom instruction (question 21). In the interview, the researcher asked him whether he uses real-life problems in his instruction:

> **Researcher:** In question 21, you were asked to provide an example of a real-life problem you share with or ask to your students that can be modeled and solved by a quadratic functions. You did not answer. Do you use this kind of problems during your instruction?
>
> **Can:** I generally do not use, I cannot. In the textbooks, there are real-life problems but we do not use them. While beginning unit on parabola, there are some examples of the Eifel Tower, the Bosphorus Bridge, and the satellite dishes as daily examples of parabolic curves. We tell students these examples, but we fail to solve real-life problems. This is our biggest weakness.
>
> **Researcher:** Why do you think so?
>
> **Can:** Because we have arithmetical thinking rather than algebraic thinking. We only make quantitative operations; we do not follow the new education system which is based on non-routine problems. As secondary mathematics teachers, I think we do not completely know what a non-routine problem is.

The classroom observation is consistent with his above words. While introducing quadratic functions, he shortly mentioned arch bridges and satellite dishes as daily examples of parabolas. Then, he did not solve any real-life problems about quadratic functions during his instruction.

Can's knowledge of recognizing students' incorrect solutions regarding quadratic functions

When the teachers were asked to describe the incorrect steps in a student's solution to a problem about the vertex and maximum/minimum points of a quadratic function (question 22), he wrote: "The student is wrong because the parabola is downwards, the vertex does not give the minimum. Also, the ordinate of the vertex gives the max/min value, not the apsis. The endpoints should be checked." As seen in his response, he recognized all the mistakes in the student's solution and explained them clearly.

Can's knowledge of understanding students' unusual solutions regarding quadratic functions

The teachers were given two problems, each together with a student's solution about quadratic functions and equations. In the first one (question 23), there was a quadratic equation, which can be factorized. However, the student solved that equation by completing the square method. He responded: "The student solved the equation by completing the square. The result is correct." As he stated, he knew the method of completing the square for solving quadratic equations, as a secondary mathematics teacher. However, he does not prefer to use this method in his instructional practice.

In the second one (question 24), the question was to find the (unique) quadratic polynomial, with some information about the coefficients and one of the roots were given. He stated that the solution is correct without explaining why he thought so. Thus, in the interview, the researcher asked him to express why he thought so. This is illustrated below:

> **Researcher:** In question 24, you have written that the student is right. Could you explain why did you think so?

Can: If one of the roots is $7 + \sqrt{6}$, another one must be $7 - \sqrt{6}$.

Researcher: Yes.

Can: We can find the solution since two roots are known. We can use the sum and multiplication of the roots to find the coefficients.

In fact, the aim of that question was to lead teachers to analyze the student's approach. However, he focused on finding the result by his own approach rather than analyzing the student's solution. He examined the question and explained how to find the solution without paying attention to the student's solution strategy.

Can's knowledge of responding to students' why questions about quadratic functions

In the questionnaire, two questions were asked to understand teachers' ability to respond to students' why questions. In the first one (question 25), they were asked to respond to a student's question that asked why translating a parabola upwards and downwards changes only *c*, while translating a parabola left and right changes both b and c in $f(x) = ax^2 + bx + c$. In the questionnaire, he did not present a plausible explanation for this question. He said: "While translating upwards and downwards, only the ordinate value changes. The ordinate value only affects *c*." In the interview, the researcher asked him:

Researcher: Could you explain your response to question 25? You said while translating a parabola upwards and downwards, only the ordinate value changes and the ordinate value only affects c.

Can: Hmm…I use the vertex form. When we add values to k, the parabola goes up. When we subtract values from k, the parabola goes down. The width of the parabola does not change. So, only c changes.

Researcher: How do you explain why both b and c change while translating the parabola horizontally?

Can: *c* changes because of *r*. In fact, *r* depends on *b*. When I write

$(x-r)^2+k$, for each number I add to r, the parabola moves right or left, in opposite direction with the sign of the number which was added.

The above conversation indicated that Can did not present a plausible explanation to explain the relationship between the coefficients and the translations made on the parabolas.

Can's knowledge of finding an example to make a specific mathematical point about quadratic functions

In the questionnaire, when he was asked what kind of examples, he would use in the classroom to emphasize the symmetrical property of parabolas (question 27), he wrote: "I would draw parabolas on Geogebra." In the interview, the researcher asked him to explain his response in detail, as illustrated below:

> **Researcher:** You said you would use Geogebra to emphasize the symmetrical property of a parabola. How would you do this? What kind of examples can you use?
>
> **Can:** As I said, I would draw some parabolas and find their vertices. To be honest, I had never had an extra effort to emphasize the symmetrical property. Of course, I say that parabolas are symmetrical shapes; but I mean I did not think about a specific example to highlight the importance of the symmetrical property.

During the classroom observation, Can used some examples which might help to emphasize symmetrical property, as presented below. (Figure 4.10). He did not use a mathematical software during his instruction. As seen in Figure 4.10, he wrote that $f(r+m)=f(r-m)$ and solved some examples about this property. However, while teaching this content, he did not underline the symmetrical property.

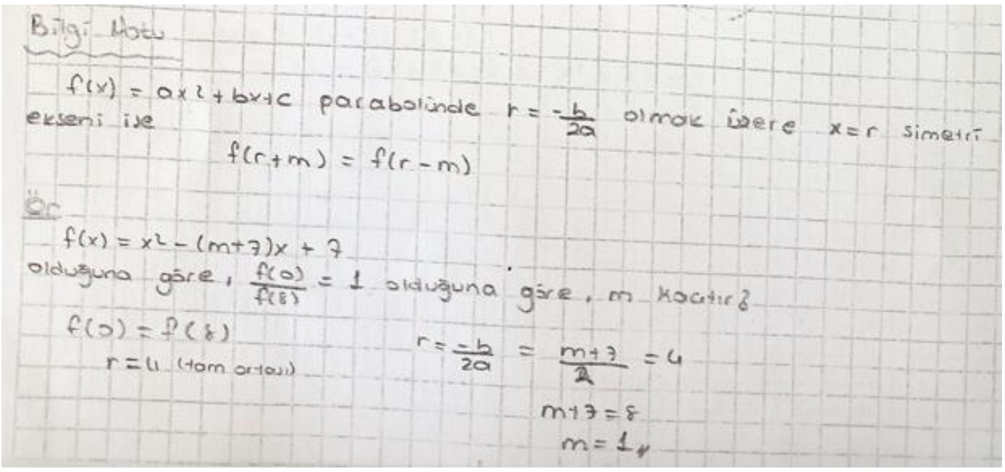

Figure 4.10. A section from Can's instruction

Can's knowledge of modifying tasks regarding quadratic functions

In the questionnaire, when the teachers were asked to modify a task considering their students, he did not make any change in the question. He told that his students could easily solve the task and find the correct result.

4.2.1.3. Can's Horizon Content Knowledge of Quadratic Functions

For an elaborate discussion of Can's HCK regarding quadratic functions and equations, the results are presented under two headings that include Can's knowledge of: how quadratic functions are related to other contents in the high school curriculum and how quadratic functions are related to advanced mathematics.

Can's knowledge of how quadratic functions are related to other contents in the high school curriculum

Based on the questionnaire results and the interview, Can's knowledge of the relationship between quadratic functions and other contents in the high school curriculum was extremely weak. Can presented no correct answer to the items in the questionnaire that evaluated his knowledge of how quadratic functions are related to other contents in the high school curriculum (questions 29-34). He gave either incorrect or no answer to all of those items. In question 29, he did not explain the relationship between the derivative and the concavity of a parabola, as illustrated below:

> **Researcher:** How can you explain why the graph of a quadratic function is concave down if $a < 0$, and concave up if $a > 0$?
>
> **Can:** I give this to my students as a rule. I say if $a > 0$ the arms of the parabola opens up, if $a < 0$ the arms of the parabola opens down.
>
> **Researcher:** Well, did you think why this is so?
>
> **Can:** No. This is a well-known rule. When we draw the graph, we can see it easily.

In the next question (question 30), the teachers were asked to compare the graphs of exponential and quadratic functions. He stated: "According to the graph, q is always greater than p." In the interview, the researcher asked him the same question again and required him to explain his answer.

> **Researcher:** In question 30, you have stated that the quadratic function q will always take greater values than the exponential function p. Could you explain why did you think so?
>
> **Can:** I said that q is always greater because exponential function grows faster.
>
> **Researcher:** But q is not exponential, q is a quadratic function.
>
> **Can:** I supposed that q is exponential. I meant to say that exponential function grows faster. For example, think about 2^x. When x gets bigger, it grows faster, and the arms of the graph approach the y-axis faster.
>
> **Researcher:** Well, I get it. You misunderstood the notations (p and q) of the functions. You say that exponential function eventually gets bigger.
>
> **Can:** Yes.

The above conservation indicates that Can knows that the exponential function grows faster than the quadratic function. In the next question, he did not explain the relationship between the vertex of a quadratic function and the derivative. In the interview, the researcher asked him the same question again (question 31).

Researcher: Is there a relationship between the vertex and derivative?

Can: Yes, but in grade 12.

Researcher: How?

Can: In grade 12, a third-order equation is given. Its derivative becomes a second-order equation. When the minimum value is to be found, we use r.

Also, the fact that the first order derivative is the slope, is told in grade 12.

Researcher: Well, I do not mean their curricular relationship. I am asking how do yourself associate them conceptually?

Can: I cannot give a certain answer to this question. They are always related.

When he was asked to determine whether $y = x^4$ is a parabola or not (question 34), he wrote: "It is a parabola, because it is U-shaped." Another question was about the relationship between the golden ratio and quadratic equations (question 33). In the questionnaire, Can did not answer this question. So, in the interview, the researcher asked the same question again, as illustrated below:

Researcher: In question 33, you were asked to explain (if any) the relationship between the golden ratio and quadratic equations. You did not write anything. Do you have an idea about their relationship?

Can: Golden ratio is not a parabolic curve. I have no idea about their relation. Maybe, there is, but I don't know.

The teachers were also asked to tell what kind of examples they would provide their students to emphasize the relationship between any concept from the physics course and quadratic functions (question 32). Can did not respond to this question. In the interview, the researcher asked him:

> **Researcher:** In question 32, you were asked whether quadratic functions are related to any concept from the physics course, you did not answer. Could you give some examples from physics course which might be related to quadratic functions?
>
> **Can:** Sometimes, students say that this is similar to projectile motion; but they do not make a connection between this concept and parabolas.
>
> **Researcher:** Do you associate them during your classroom instruction?
>
> **Can:** No, I don't.

During his instruction, he did not emphasize the connection between quadratic functions and any other content in the high school curriculum. Based on the questionnaire results, the classroom observation, and the interview, Can has poor knowledge of how quadratic functions are related to other contents in the high school curriculum.

Can's knowledge of how quadratic functions are related to advanced mathematics

In the questionnaire, Can did not answer the question that asked the reflection property and its daily use (question 35). In the interview, the researcher asked him:

> **Researcher:** In question 35, you were asked to explain reflection property of a parabola. Could you explain what this property is and where it is used in daily life?
>
> **Can:** I have no idea about the reflection property.

In another question, Can did not explain the relationship between a hyperbola and a parabola (question 36). He also stated that he has never heard about the fundamental theorem of algebra, in his response to question 37. So, the researcher asked him the same question in the interview. This is illustrated below:

> **Researcher:** In the questionnaire, you were asked to state the fundamental theorem of algebra and its application to quadratic polynomials. You wrote that you have never heard this theorem.
>
> **Can:** I have no idea about this theorem.

When the teachers were asked to define a parabola and give alternative definitions (question 39), he wrote: "The graph of a quadratic function is called a parabola." In the interview, the researcher asked him to state any alternative definitions:

> **Researcher:** You defined a parabola as the graph of a quadratic function. Do you know any alternative definitions?
>
> **Can:** I don't know. Maybe there is, we can investigate. I only know that definition. According to me, a parabola is the graph of a quadratic function.

As he confirmed in the questionnaire, he considers a parabola as the graph of a quadratic function. He is not aware of the geometrical definition of a parabola, which is related to a point (focus) and a line (directrix). In connection with the previous question, the teachers were asked to select the most correct statement among the given two ones, which are presented below (question 38).

> Statement 1: The graph of a quadratic function is a parabola.
>
> Statement 2: The graph of a quadratic function is called a parabola.

In the questionnaire, Can wrote that the second statement is correct, without further explanation. In the interview, the researcher asked him:

> **Researcher:** In question 38, you selected the second statement as the most correct? Could you explain why?
>
> **Can:** I selected the second statement because it is a definition.

In the last question of the questionnaire, which asked the teachers whether a given curve is a parabola or not, he had no answer (question 40). So, the researcher asked him:

Researcher: You did not respond to the last question. Have you ever heard the term catenary?

Can: No.

Researcher: Do you think that this shape is a parabola? What is required to be a parabola?

Can: Parabolas have two symmetrical roots. They are symmetrical shapes.

Researcher: Isn't this shape (the shape in question 40) symmetrical?

Can: It looks symmetrical. It can be a parabola.

Researcher: If a shape is symmetrical, is it enough criteria to become a parabola?

Can: I think yes.

On the basis of the questionnaire results, the classroom observation, and his responses to the interview, Can's knowledge of how quadratic functions and equations are related to advanced mathematics is fairly limited.

4.2.2. The contribution of Can's Subject Matter Knowledge of Quadratic Functions to Student Learning Outcomes

A total of 23 students were in Can's class. Three of them were absent on the day the questionnaire was administered. So, 20 students were administered the quadratic function concept test (see Appendix C), which provided the data for interpreting his students' learning outcomes of quadratic functions. The responses of 3 students were excluded from the analysis, since they did not respond any questions in the test. Thus, the responses of 17 students were analyzed to evaluate students' learning outcomes regarding quadratic functions. A summary of the results of Can's students' performance on the test is presented below (Figure 4.11).

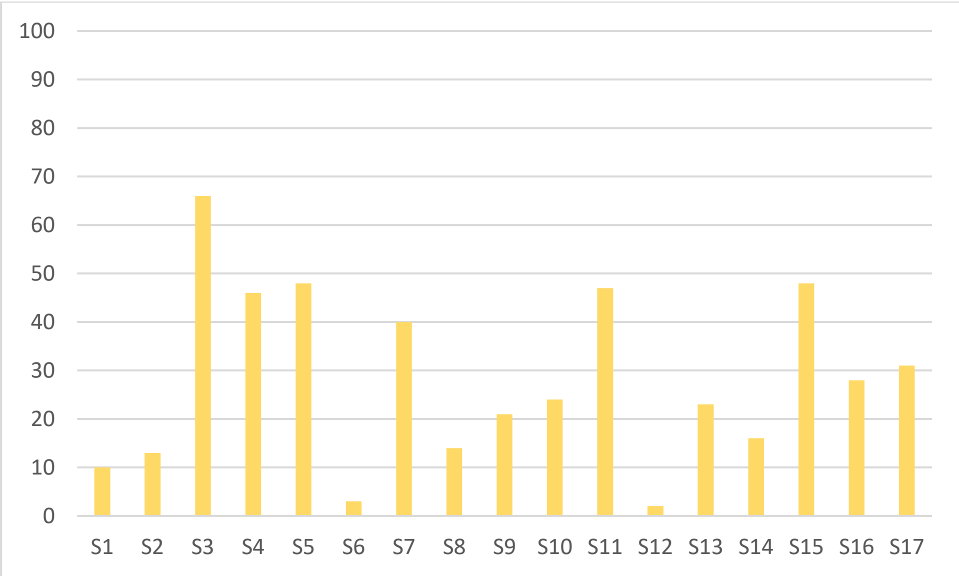

Figure 4.11. Can's students' scores on the quadratic function concept test

As seen in Figure 4.11, Can's students' test scores range between a minimum of 2 and a maximum of 66 out of 100 points. The average score of the students on the test is 28.2. In general, his students' performance on the test is very limited. The students' performance is discussed in detail in the next sections based on the objectives of the mathematics curriculum regarding quadratic functions.

4.2.2.1. Finding the Vertex, x-Intercepts, the y-Intercept, and the Axis of Symmetry

In the first question of the quadratic function concept test, the students were asked to find *x*-intercepts, the *y*-intercept, vertex, and axis of symmetry of the graph of the function $f(x) = x^2 + 2x - 8$ and then graph it. While they were finding *x*-intercepts, some students (*n*=7) factorized the quadratic equation and correctly found *x*-intercepts as -4 and 2. Some students (*n*=4) used the quadratic formula to find the *x*-intercepts of the quadratic function. Six students did not answer this part of the question. None of the students used completing the square method to solve the quadratic equation. When the students were asked to find the *y*-intercept, two of them found it by calculating $f(0)$ and correctly wrote $(0, -8)$ as the *y*-intercept. Many students (*n*=10) wrote only "-8" as the *y*-intercept, without using the proper notation as (x, y). Four students had no answer. There was only one student who found the *y*-intercept incorrectly as 0. She wrote -4 and 2 (the roots of the quadratic equation) for x and obtained $y = 0$.

When the students were asked to find the vertex, some students (*n*=8) found it correctly as the point (r, k). They firstly found r by using the formula. Then, six of them found k by calculating $f(r)$, while one of them used the formula "$k = \frac{4ac - b^2}{4a}$". Two students wrote the apsis of the vertex (r) as the vertex, whereas seven students wrote the ordinate of the vertex (k) as the vertex. When the students were asked to find the axis of symmetry, none of them correctly wrote it as a line equation. There were five students who were aware of the interrelation between the apsis of the vertex and the axis of symmetry. They wrote the axis of symmetry as "$r = -1$." Two participants wrote the vertex $(-1, -9)$ as the axis of symmetry. Two participants wrote some irrelevant numbers without explaining how they found these numbers. Eight participants did not respond. When they were asked to graph the function, only four students correctly sketched the graph. Six students did not sketch any graphs whereas seven of them sketched incorrect graphs.

In another question, the students were asked to find c in the function $f(x) = x^2 + bx + c$ with its vertex given. Most students (*n*=13) correctly found c. Firstly, they found b by using the formula for r. Twelve of them found c by using that $f(r) = k$. Their solutions were like: "$-b/2a = 2$, $b = -4$, $f(r) = k$, $f(2) = 6$, $2^2 - 8 + c = 6$, $c = 10$." Unlike the others, one student used the formula for finding the ordinate of the vertex: "$-b/2a = 2, b = -4$; $k = \frac{4ac - b^2}{4a} = 6$; $4c = 40, c = 10$." Three students did not respond to this question, whereas one student found an incorrect result.

The result suggested some evidence that teachers' content knowledge of quadratic functions interact with student learning outcomes. First of all, Can solved the the quadratic equations in the questionnaire by using the quadratic formula and he suggested "factorization" as an alternative strategy for solving quadratic equations. During his instruction, he used factorization for solving the quadratic equations which can be factorized; he used the quadratic formula for those which cannot be factorized. He shortly mentioned completing the square method, but he did not use this method for finding the roots of a quadratic equation. As so Can, his students used the quadratic formula and factorization to find the *x*-intercepts of a quadratic function. None of his students used completing the square method for finding the roots of a quadratic equation. This finding provides evidence of the relationship between teachers' subject matter knowledge and students' learning outcomes regarding solving quadratic equations or finding the *x*-intercepts of a quadratic function.

Secondly, half of the students wrote the ordinate of the vertex, as the vertex, as Can did in his response to the questionnaire. Can defined the vertex as "it is the ordinate of the maximum or the minimum point of a parabola" and found the vertex of the function $f(x) = 3x^2 + 9x + 6$ as $\frac{37}{4}$, which represents the ordinate of the vertex. This finding also provides evidence of the relationship between teachers' content knowledge and students' learning outcomes regarding finding the vertex of a quadratic function.

Thirdly, Can's students failed to find the axis of symmetry of a parabola. Some of them (n=5) wrote the axis of symmetry as "$r = -1$", as Can did. In the questionnaire, he defined the axis of symmetry as "the apsis of the vertex" and found the axis of symmetry of the function $g(x) = -6x^2 + 12x + 5$ as "$r = 1$." As so Can, his students perceive the axis of symmetry as the apsis of the vertex, rather than a line passing through the vertex. Thus, an interaction could be made between teacher knowledge and students' learning outcomes regarding finding the axis of symmetry of a parabola.

When compared to the first question, Can's students performed better on the second question, which was about finding an unknown coefficient of a quadratic function whose vertex is given. During his instruction, Can solved similar kinds of questions. One of them is illustrated below (Figure 4.12).

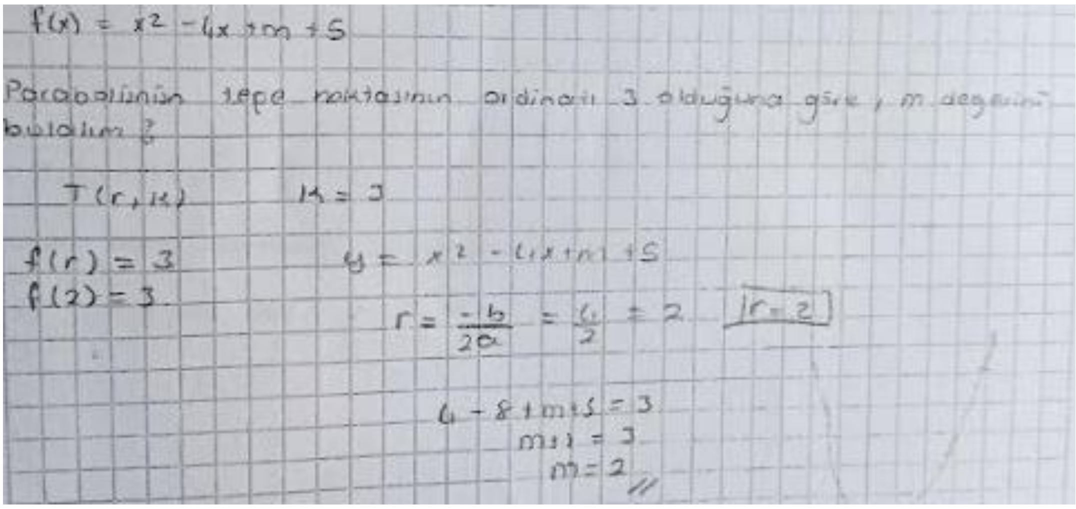

Figure 4.12. A section from Can's instruction

4.2.2.2. Associating the Vertex with the Maximum or the Minimum of a Quadratic Function

In the third question, the students were asked to find the minimum of a quadratic function. Seven students correctly associated the minimum of the function with the ordinate of the vertex. Six of them calculated r firstly; then found $f(r)$ as the minimum. Their responses were like: "$T(r,k)$, $r = -b/2a = -1$, $f(-1) = k = 2$." One participant directly used the formula for k, without calculating r, and made a calculation error while applying the formula $k = \frac{4ac-b^2}{4a}$. Five students gave some incorrect answers. Three of them calculated $f(0)$ as the minimum of the function.

One student tried to find the x-intercepts to find the minimum. Another student found the discriminant (Δ) of the quadratic equation and stated that "Δ is the minimum". Five participants did not answer this question. Similarly, in the fourth question, the students were asked to find the maximum value of a given function. Seven students correctly associated the maximum of the function with the ordinate of the vertex, k. Six of them calculated r firstly; then found $f(r)$ as the minimum: "$r = -b/2a = 2$, $k = f(2) = -4 + 8 + 6 = 10$". As in the previous question, one student directly used the formula for k, without calculating r, and made a calculation error while applying the formula. Two students found "$f(1) = 9$" as the maximum, which is incorrect. Eight students did not respond to this question.

Most of Can's students performed poorly in associating the vertex of a parabola with the minimum or the maximum of the function. During his instruction, although he solved several problems about finding the vertex of a quadratic function, he did not solve problems about the minimum or the maximum of quadratic functions and did not associate the vertex with the maximum or the minimum of a quadratic function. He solved several questions about finding the vertex of a parabola, but he did not solve problems that asked to find the minimum or the maximum of a quadratic function. Thus, his students' poor performance in finding the maximum or the minimum of a quadratic function might be associated with his instructional practice.

4.2.2.3. Commenting on the Effect of the Change in the Coefficients on the Graph of the Function

In the fifth question of the quadratic function concept test, none of Can's students provided a correct answer. In the question, the students were asked to comment on two cases about how the graph of the quadratic function $y = 2x^2$ changes depending on the leading coefficient. Four students did not respond to this question, whereas the remaining (n=16) made incorrect explanations. For example, nine of them made some vertical or horizontal translations on the graph. The response of a student who made a vertical translation on the graph is illustrated in Figure 4.13.

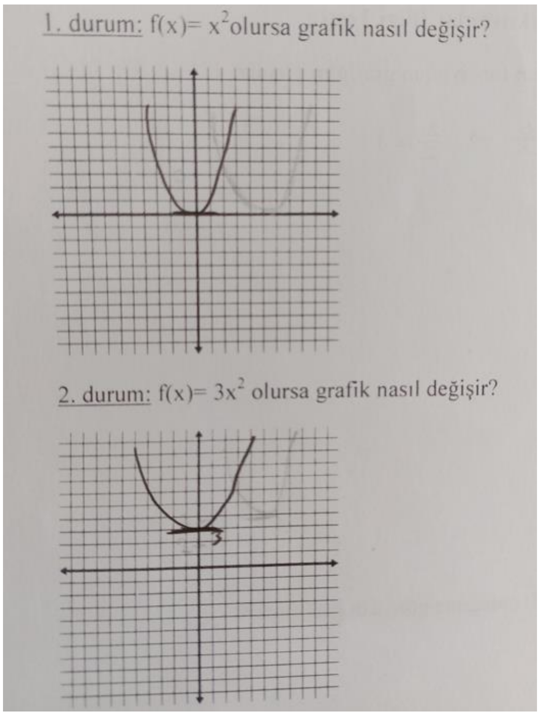

Figure 4.13. An example from Can's students' responses to question 5

A few students (n=3) tried to explain the relationship between the leading coefficient and the width of the graph, but they expressed it wrongly. Their explanations were like: "the arms of the parabola become larger, if |a| gets larger." Accordingly, one of them sketched the following graph shown in Figure 4.14.

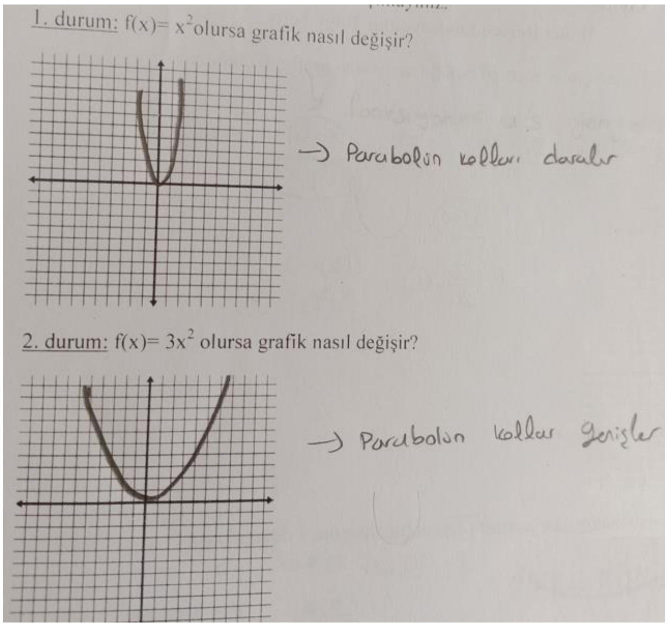

Figure 4.14. An example from Can's students' responses to question 5

Based on their performance on the quadratic function concept test, Can's students' performance in associating the coefficients of a quadratic function with its graph was fairly limited. On the other hand, the teachers were asked to find the widest parabola among the given ones. Can stated: "The smaller the |a| becomes, the wider the parabola becomes." However, during his instruction, he did not solve exercises about this property of the parabolas. He introduced the translation and symmetry of functions in general and solved some questions about these contents. Although he solved several questions about function transformations, he did not tell the stretching of parabolas (multiplying a quadratic function by a constant). He wrote on the board: "If the function $y = f(x)$ is multiplied by a constant k as $y = k.f(x)$, all the y-values in the range of the function are multiplied by k." Then, he applied this property to the linear functions as illustrated in Figure 4.15.

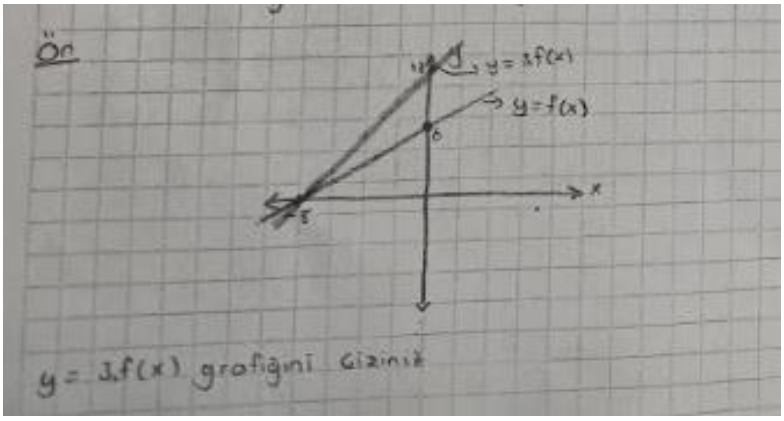

Figure 4.15. A section from Can's instruction

During his instruction, he did not solve any example related to the stretching of parabolas. Thus, students' low performance in commenting on the effect of the change in the leading coefficient on the shape of the parabola might be associated with his classroom instruction.

4.2.2.4. Finding the Quadratic Function Given Three Points or Two Points that One of Them is the Vertex

The students were asked two questions regarding this objective of the mathematics curriculum. In the first one (question 6), they were asked to find the quadratic function with given two points such that one of them is the vertex. Only three students could find the quadratic function. They used the vertex form of quadratic functions. For example, one of them responded: "$f(x) = a(x - r)^2 + k$, $f(x) = a(x + 2)^2 + 2$, $f(1) = 11$, $9a + 2 = 11$, $a = 1$, $f(x) = (x + 2)^2 + 2$." However, most of them (*n*=14) failed to find the quadratic function. Eleven students had no answer. Two of them tried to find the quadratic function; one of them used the intercept form whereas another one used the standard form. The solution of the student who used the intercept form was: "$y = a(x - x_1).(x - x_2)$, $y = a(x - 1).(x + 2)$, $y = -2a$, $y = -2(x - 1).(x + 2)$." Another student who tried to find the quadratic function by using the standard form wrote: "$f(x) = ax^2 + bx + c$,

$c = 11,\ f(-2) = 2,\ 4a - 2b + 11 = 2,\ 4a - 2b = -9,\ r = -\frac{b}{2a},\ b = 4a,\ 4a - 8a = -9,\ a = 9/4$."

In the second one (question 7), the students were asked to find the quadratic function whose three points such that one of them is on the y-axis were given. None of the students could find the quadratic function. Most of them (n=12) did not answer, whereas five students found an incorrect result. Three of them used the intercept form and considered the given two points as the x-intercepts. For example, one of them wrote: "$y = a.(x - x_1).(x - x_2),\ y = 4.(x + 1).(x - 2),\quad y = 4x^2 - 4x - 8$." One of them used the vertex form; and thought that one of the given points is the vertex: "$f(x) = a(x - r)^2 + k, y = a(x - 2)^2 + 6; y = x^2 - 4x + 10$."

When the above responses are examined, Can's students had some difficulty in finding the quadratic function whose some points were given. In the questionnaire that was administered to the teachers, there were similar questions that asked to find the quadratic functions with some points given (questions 16&17). Can found the quadratic function by using the intercept form, when the vertex was given; he used the standard form when three arbitrary points were given, as most of the teachers did. However, his students failed to solve similar questions. During his classroom instruction, he also solved several questions about this content as illustrated in Figure 4.16.

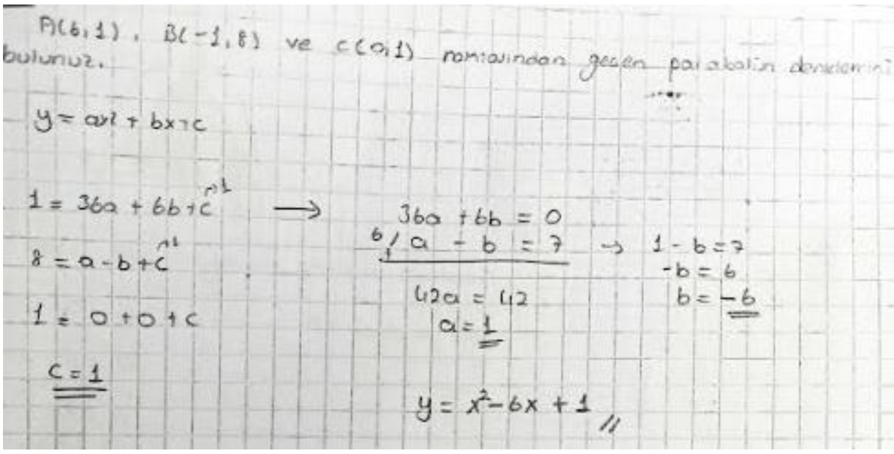

Figure 4.16. A section from Can's instruction

4.2.2.5. Investigating the Intersection of a Line and a Parabola

When the students were asked to examine the intersection of a line and a parabola (question 8), approximately half of the students (*n*=8) gave the correct answer. They equated the line equation and quadratic function, and obtained a new quadratic equation. Then, they investigated the discriminant of this new quadratic equation. Their solutions were like: "$x^2 + 5x + 2 = 3x + 1$, $x^2 + 2x + 1 = 0$, $\Delta = b^2 - 4ac$=0. They are tangent." Two students did not respond whereas seven students gave incorrect answers. Some of them (*n*=3) investigated the discriminant of the given quadratic equation and wrote: "$f(x) = x^2 + 5x + 2, \Delta = b^2 - 4ac = 17 > 0$, they have two points of intersection." Four students made some incorrect explanations. For example, one student responded: "They do not intersect because the line $y = 3x + 1$ is not a quadratic line."

When compared to other questions in the test, the students performed better on this question regarding investigating the intersection of a line and a parabola. In the questionnaire, the teachers were also asked to find the intersection of a line and a parabola (question 18). As most of the teachers did, Can investigated the discriminant of the new quadratic equation and easily found the intersection of the line and the parabola. During his instruction, he solved several problems about this content, as illustrated in Figure 4.17.

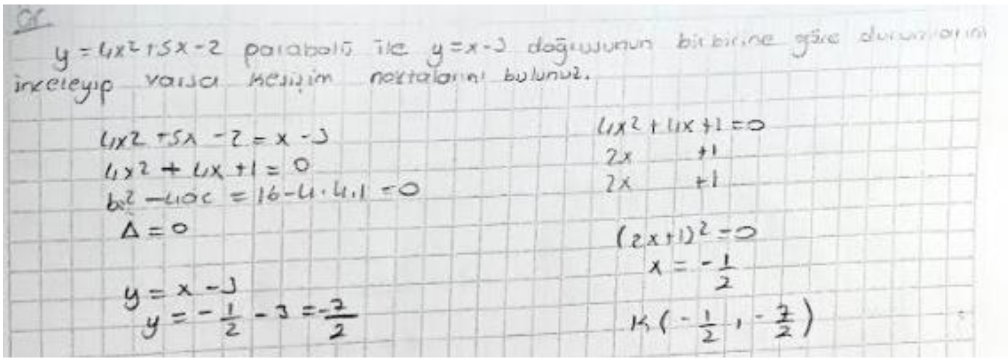

Figure 4.17. A section from Can's instruction

4.2.2.6. Solving Problems that can be Modeled by Quadratic Functions

In question 9, the students were asked to find the maximum area of a rectangle with its perimeter given. The students were expected to solve this question by using a quadratic model. However; none of Can's students constructed a quadratic model. Some of them (*n*=3) found the correct answer by trying some numbers, which adds up to 18, that is the half of the perimeter of the rectangle. Their solutions were like:

$a + b = 18$
1.17=17
2.18=36
.
.
$8.10 = 80$
$9.9 = 81$ (maximum area)

Six participants also followed similar steps and replaced numerical values to find the maximum area. However, they found the result as 80 since they thought that "if the dimensions would be $9\,x\,9$, it would be a square, not a rectangle". Thus, they selected the dimensions that would yield the maximum area as $8\,x\,10$. There was also one student who tried to find the solution by determining a ratio between the dimensions of the rectangle. He determined a $3:2$ ratio between the long side and the short side of the rectangle and made some operations that yielded an incorrect result. Seven participants did not respond to this question.

In question 10, the students were given a quadratic function that represented the height of a ball that was hit by someone playing football. The question was to find at what time the ball reaches 3 meters above the ground. Only two participants correctly solved the problem. To illustrate, one of them wrote: "$h(t) = -t^2 + 4t = 3$, $-t^2 + 4t - 3 = 0$, $t_1 = 1$, and $t_2 = 3$." Seven students found incorrect results. Some of them (n=5) calculated $h(3)$ whereas some of them (n=2) calculated the maximum of the function. Eight students did not respond to this question.

As it can be seen in their responses, Can's student's ability to solve real-life problems about quadratic functions was fairly limited. During classroom instruction, Can never solved real-life problems regarding quadratic functions, as he stated in the interview. Furthermore, Can could not solve the real-life problem in the questionnaire that required a mathematical model of quadratic functions. Thus, Can's students' inability to solve real-life problems about quadratic functions might be related to his inadequate subject matter knowledge. Moreover, in the interview, when the researcher asked him whether he emphasized the relationship between quadratic functions and some other concepts, he said that he did not tell these kinds of interrelations between the concepts. Thus, his students' low performance in solving real-life problems about quadratic functions might also be related to his low HCK.

4.3. Contribution of Subject Matter Knowledge to Student Learning Outcomes: The Case of Ahmet

In this section, the case of Ahmet was presented. The data obtained from the questionnaire, interview, and classroom observation were combined and triangulated.

Ahmet is the teacher who was called "T16" in the previous sections. He had 22 years of teaching experience and he was teaching at an Anatolian high school when this study was conducted.

He had a very good performance on the quadratic function concept questionnaire. His scores on the CCK, SCK, and HCK items were the highest among all the participating teachers. These will elaborately be discussed in the following sections.

4.3.1. Ahmet's Subject Matter Knowledge of Quadratic Functions

The descriptions of Ahmet's content knowledge were developed from his responses to the questionnaire (see Appendix A), the follow-up interview (see Appendix C), and classroom observations. These descriptions are presented under three headings in the following sections.

4.3.1.1. Ahmet's Common Content Knowledge of Quadratic Functions

For an elaborate discussion of Ahmet's CCK, the results are presented under seven headings that indicate the sub-dimensions of teachers' CCK identified in the present study.

Ahmet's conception of quadratic equations and functions

When he was asked to define a quadratic equation (question 1), he stated: "A quadratic equation ($ax^2 + bx + c = 0$) is a tool for finding the x-intercepts of a quadratic function." Similarly, when he was asked to define a quadratic function (question 2), he wrote: "A quadratic function ($ax^2 + bx + c = 0$) generates a parabola." When he was asked to distinguish quadratic functions, quadratic equations and quadratic polynomials (question 3), he focused on their geometrical aspects. He stated: “As I stated above, quadratic functions generate parabolas and the intersection of the parabola with the *x*-axis can be determined by a quadratic equation.”

Unlike Can, Ahmet underlined the geometrical aspects of quadratic functions and equations, rather than stating their demonstrations. When he was asked to determine whether the *x* and *y* values that were given in the tables belong to a linear or a quadratic function (question 4), he checked the first differences, as Can did. In the interview, the researcher asked him:

> **Researcher:** In the 4th question, you have stated that the values in the first table belong to a linear function, the values in the second table belong to a quadratic function. Could you explain how did you decide it?
>
> **Ahmet:** I examined the first differences.
>
> **Researcher:** How do you guarantee that it is quadratic if the first differences are non-constant?
>
> **Ahmet:** Hmm… According to the table, there are two roots. So we can generate a quadratic equation from those values.

Ahmet’s knowledge of solving quadratic equations with one unknown

In the questionnaire, when the teachers were asked to solve some quadratic equations (question 5), Ahmet was the only teacher who solved all of them by completing the square. His solutions are presented in Figure 4.18.

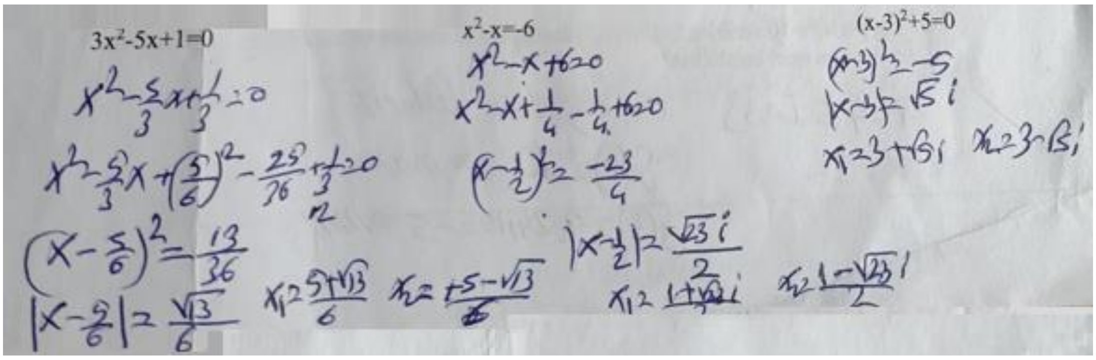

Figure 4.18. Ahmet's solution to question 5

In the follow-up, he stated the quadratic formula as an alternative solution method. In the interview, the researcher asked him:

> **Researcher:** You solved the quadratic equations by completing the square. Do you use this method in the classroom?
>
> **Ahmet:** Yes, of course. I frequently use it. This is my favorite solution method.

As he stated in the interview, he used completing the square method for solving quadratic equations several times.

Ahmet's knowledge of sketching and interpreting the graphs of quadratic functions

Unlike Can, he made structural descriptions of the vertex and the axis of symmetry. For example, he defined the axis of symmetry (question 6) as "the line according to which the parabola is symmetrical". Similarly, he defined the vertex of a parabola (question 7) as "the maximum or the minimum point of a quadratic function depending on the sign of the leading coefficient". In his response to question 8 that asked to define the concavity, he wrote: "It tells us about the direction of the parabola." In question 9, he properly found the vertex, the x-intercepts, the y-intercept, graph orientation, and the minimum value of the given quadratic function and sketched the graph of it accurately. He also correctly found the equations of two functions whose graphs were given (question 10). As so Can, he used multiple algebraic demonstrations of quadratic functions. He wrote the first function in the vertex form and the second function in the intercept form. In the interview, the researcher asked:

> **Researcher:** In the 10th question, you have written the first function in the vertex form, the second function in the intercept form. Do you use these different algebraic demonstrations in your classroom instruction?
>
> **Ahmet:** Yes, of course. I show my students some graphs and explain how to write the equations. For example, I say that if we know the x-intercepts, we use this form $f(x) = a.(x - x_1).(x - x_2)$.

His above words were supported by classroom observation. In the classroom, he emphasized the use of multiple algebraic demonstrations as shown in Figure 4.19.

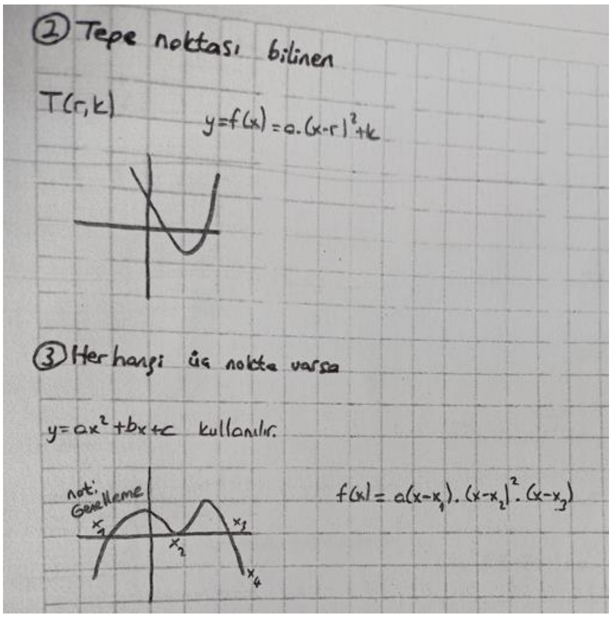

Figure 4.19. A section from Ahmet's instruction

Lastly, the teachers were given a parabola and asked to comment on the signs of the coefficients of the corresponding quadratic function (question 11). Ahmet's response is presented below (Figure 4.20).

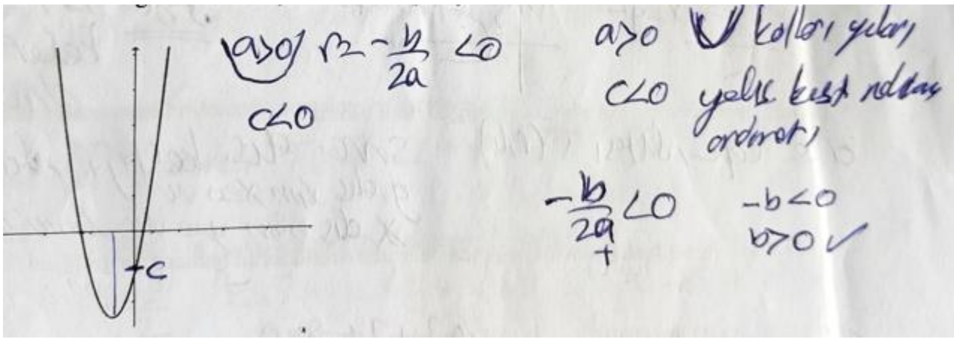

Figure 4.20. Ahmet's response to question 11

When the Figure 4.20 is examined, he firstly determined the sign of the leading coefficient, a. Then, as so Can, he determined the sign of b by examining the sign of the apsis of the vertex, r. He determined the sign of c by noting that it is the ordinate of the point where the parabola cuts the y-axis. In the interview, the researcher asked him to explain how he determined the signs of the coefficients:

> **Researcher:** In the eleventh question, you wrote that $a > 0$, $b > 0$, and $c < 0$. How did you determine?
>
> **Ahmet:** The parabola is upwards, so $a > 0$. The vertex $(-b/2a)$ is negative; so $b > 0$. Since the ordinate of the y-intercept is negative, c must be negative.

Ahmet's knowledge of graphing quadratic functions using transformations

When he was asked to explain how to generate the graph of any quadratic function from the graph of $f(x) = x^2$ (question 12), he wrote: "We can do it by reflections and translations." In question 13, he could find the widest parabola among the given four parabolas. He wrote: "For $f(x) = ax^2 + bx + c$, the larger the $|a|$ becomes, the arms of the parabola is getting closer to the y-axis." As so Can, during his instructional practice, he did not tell this property. When he was asked to compare the graphs of the functions $f(x) = x^2 - 5$ and $g(x) = (x - 5)^2$ (question 14), he wrote some properties of them: "Both of them are concave up. $g(x)$ is a perfect square and thus tangent to the x-axis. $f(x)$ intersects the x-axis at two different points.” Unlike Can, he compared the functions in terms of their some characteristics. He did not describe the transformations made on the function $f(x) = x^2$.

Ahmet’s knowledge of solving real life problems regarding quadratic functions

Ahmet was the only participant who properly constructed a quadratic model and solved the problem stated in question 15. His original solution is presented below (Figure 4.21). During his instruction, he solved several real-life problems.

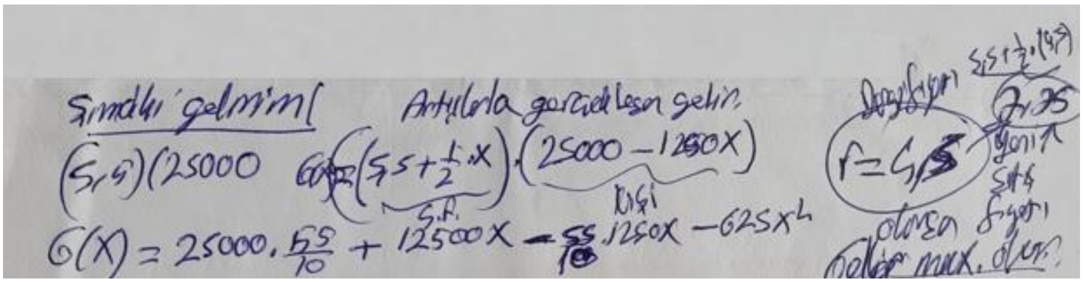

Figure 4.21. Ahmet’s solution to question 15

Ahmet’s knowledge of finding the quadratic functions with given points

As so Can, Ahmet correctly found the quadratic functions with some points given. In the first one, when the vertex and another point on the parabola were given; he used the vertex form to find the quadratic function (question 16). In the second one, when three arbitrary points on the parabola were given, he used the standard form to find the quadratic function. His solution to question 16 is presented below (as reproduced for readability):

$$
\begin{aligned}
&y = a(x-r)^2 + k \\
&y = a(x+\frac{1}{4})^2 + \frac{11}{4} \\
&5 = a(-1+\frac{1}{4})^2 + \frac{11}{4} \\
&a = 4. \\
&y = 4(x+\frac{1}{4})^2 + \frac{11}{4}
\end{aligned}
\tag{4.4}
$$

During his instruction, he also emphasized the use of different algebraic demonstrations. He told his students that if three arbitrary points are given, it is better to use the standard form. On the other hand, he stated that if the vertex is known, it is much practical to use the vertex form to find the quadratic function.

Ahmet's knowledge of finding the intersection of a parabola and a line

In question his response to question 18, he identified three conditions for the intersection of a line and a parabola. His solution was (as reproduced for readability):

$$
\begin{aligned}
&ax^2 + bx + c = mx + n. \\
&ax^2 + (b-m)x + c - n = 0 \\
&\Delta = (b-m)^2 - 4a(c-n) \\
&\Delta > 0 \text{ (two points of intersection)} \\
&\Delta = 0 \text{ (one point of intersection)} \\
&\Delta < 0 \text{ (no point of intersection)}
\end{aligned}
\tag{4.5}
$$

In the follow-up, the teachers were asked to find the point(s) of the intersection of a given line and a parabola. He correctly found their point of intersection as seen in his original solution presented in Figure 4.22.

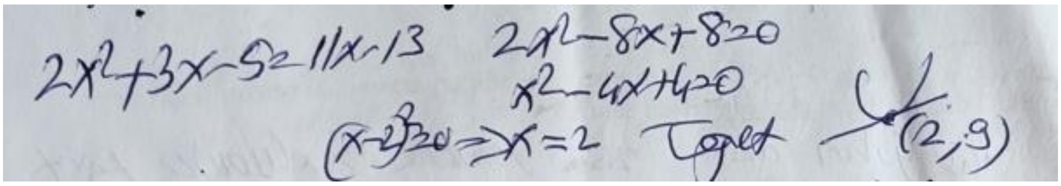

Figure 4.22. Ahmet's response to the follow-up of question 18

During the classroom instruction, as so Can, he told the intersection of a parabola and a line. This is illustrated in Figure 4.23.

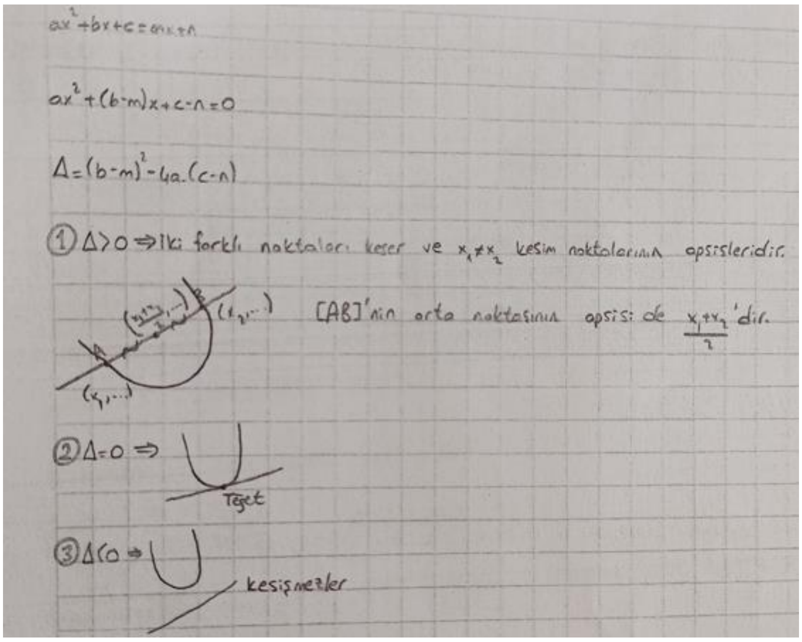

Figure 4.23. A section from Ahmet's instruction

4.3.1.2. Ahmet's Specialized Content Knowledge of Quadratic Functions

For a detailed description of Can's SCK, the results are presented under seven headings that indicate the sub-dimensions of teachers' SCK in the present study.

Ahmet's knowledge of explaining and justifying basic formulas of quadratic functions

In the questionnaire, the teachers were asked to state and derive the quadratic formula both geometrically and algebraically (question 19). Ahmet made only an algebraic justification of the formula. Then, he correctly solved the quadratic equation $x^2 - x + 1 = 0$ by completing the square (question 20). In the interview, the researcher asked:

> **Researcher:** You made an algebraic justification of the quadratic formula. Do you know any geometrical justification?
>
> **Ahmet:** Yes, we can use rectangles and squares but I don't show it geometrically in the classroom. I use geometrical demonstrations while teaching mathematical identities. However, for quadratic functions, I want my students to notice that they could solve quadratic equations by completing the square.

As he said in the interview, he told his students completing the square method and solved several exercises about this method as illustrated in Figure 4.24. He gives much importance to the use of completing the square method while solving quadratic equations.

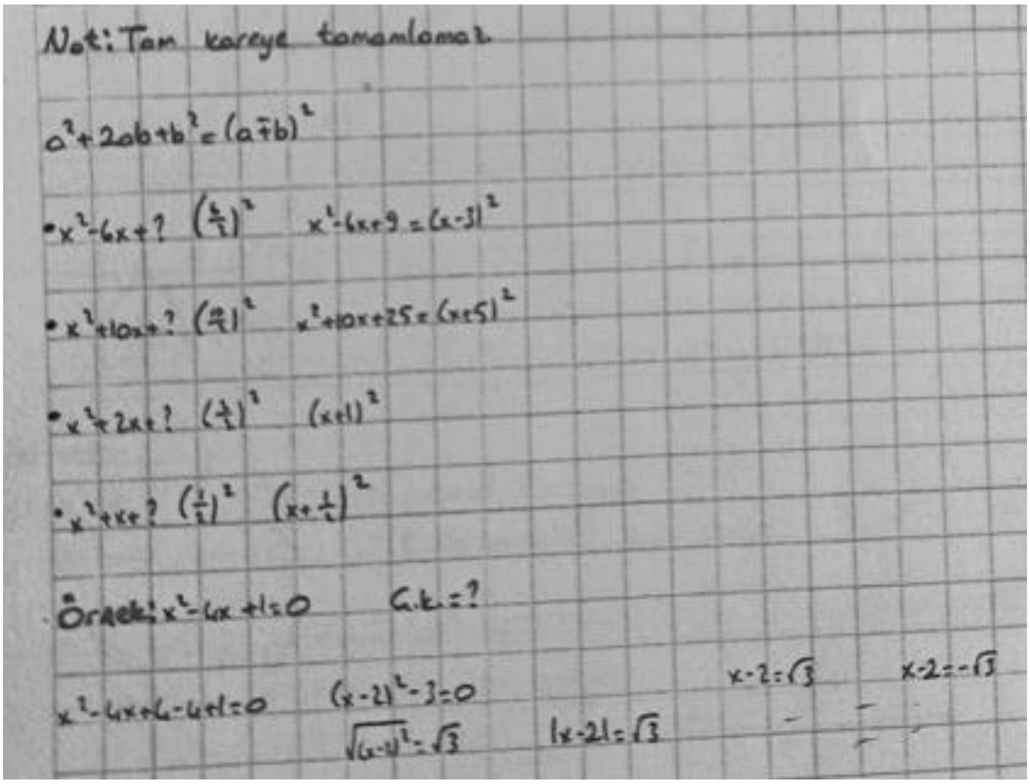

Figure 4.24. A section from Ahmet's instruction

Ahmet's knowledge of posing real-life problems regarding quadratic functions

In the questionnaire, when the participants were asked what kind of real-life examples they would use while teaching quadratic functions and equations (question 21), he stated a problem: "Let the cost of a product be *x* TL. If the product is sold $x^2 - 5x + 14$ TL, what would be the minimum profit?" In the interview, the researcher asked:

> **Researcher:** In question 21, you have written a profit-loss problem as example of real life problems regarding quadratic functions. Do you use this kind of problems as a part of your instruction? If yes, how often do you use?
>
> **Ahmet:** At the beginning, I say that parabolas are related to the construction of arch bridges. At the end, I solve maximum-minimum problems that are mostly related to profit-loss or the maximum areas of a rectangle.

His above words were confirmed by the classroom observation. As he stated, he mentioned arch bridges while introducing quadratic functions. In the final, he solved several problems about quadratic functions, as presented in Figure 4.25.

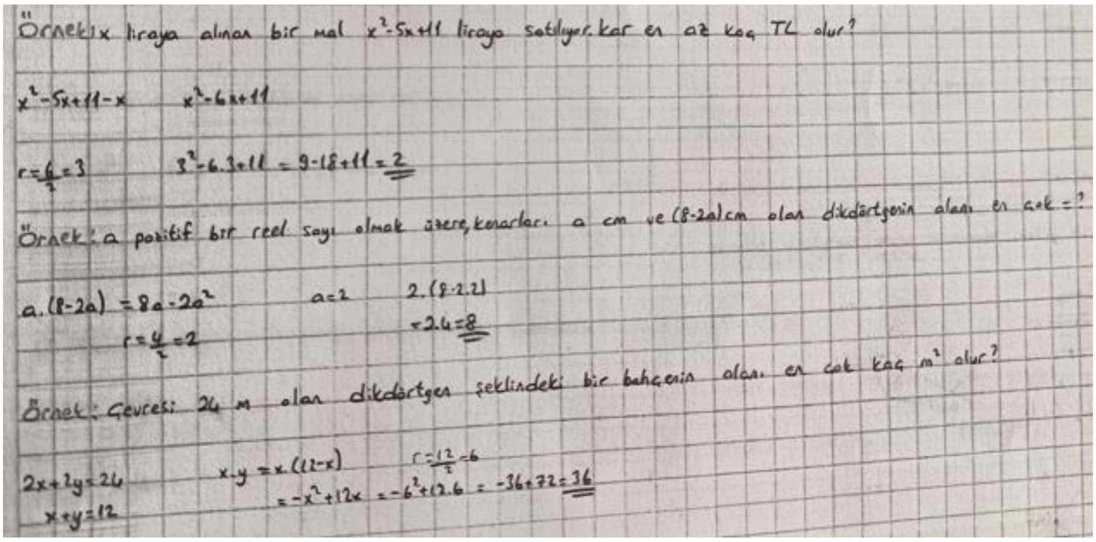

Figure 4.25. A section from Ahmet's instruction

Ahmet's knowledge of recognizing students' incorrect solutions regarding quadratic functions

As so Can, Ahmet identified all the errors in a given student solution (question 22). He wrote: "It is wrong. $k = f(r)$ gives maximum if $a > 0$, and gives minimum if $a > 0$. Here $a = -3$, so the vertex gives maximum. The student did not notice this. Also, the student did not check the values of the function at the endpoints."

Ahmet's knowledge of understanding students' unusual solutions regarding quadratic functions

The teachers were given two problems, each together with a student's solution. In the first one (question 23), there was a quadratic equation that can be factorized and the student solved the equation by completing the square method. Ahmet's comment on this solution was: “The student solved the equation by completing the square, without using the quadratic formula. This approach is my favorite while teaching quadratic equations. I give importance to my students to comprehend the origin of the formula.” In the questionnaire, the researcher asked him

> **Researcher:** You stated that completing the square method is your favorite approach to solve quadratic equations and you gave importance to your students understanding this approach. Why do you care so much about completing the square, rather than using the quadratic formula, which might generally be more practical?
>
> **Ahmet:** Because it is necessary in analytic geometry to understand the equation of a circle. I care about understanding the origins of the formulas. I do not prefer my students to memorize all the formulas. I advise them to avoid overloading the mass of information into their brains.
>
> **Researcher:** How is it related to analytic geometry? Could you explain?
>
> **Ahmet:** For example, if the equation of a circle is not given, they could find it by completing the square by using the analytics of the circle. They can find the radius and the center of the circle by completing the square. Mathematics is like a loop. Students should understand the connection between topics. They should have mathematical literacy. They should make sense of what they have learned. As I said before, for example, knowing that the roots of a quadratic equation are the *x*-intercepts of a parabola has vital importance.

In another item (question 24), the question was to find the (unique) quadratic polynomial with given some information about the coefficients and one of the roots. Unlike Can, Ahmet focused on the student’s solution process rather than the result. He wrote: “The solution is correct. The student went from the result (the root) to the beginning. He/she wrote the root of the polynomial equal to *x* and took the square to get rid of the radical. Then, the student multiplied the expression by 4 since the leading coefficient is 4.”

Ahmet's knowledge of responding to students' why questions about quadratic functions

In question 25, the teachers were asked to provide a plausible reason for why translating a parabola upwards and downwards changes only c while translating a parabola left and right changes both b and c. Ahmet responded:

> While translating upwards and downwards, the apsis of the vertex does not change. So, the sum of the roots stays constant, and thus b stays constant. As the roots change, the multiplication of the roots also changes. So, c changes. While translating left and right, both the sum of the roots and the multiplication of the roots change. Thus, both b and c change.

In the interview, the researcher asked:

> **Researcher:** In the questionnaire, you explained the change in the coefficients while translating parabolas. Do you tell your students about these changes?
>
> **Ahmet:** Yes. I introduce the parabola by drawing the graph of $y = ax^2$. Then, I draw $y = ax^2$+k, and say that if k is positive we will move the parabola k units above the y-axis, if k is negative we will move it k units below the y-axis. Translation of the parabola is the basis for translating all the functions. We can apply this on all the functions. This tells us how to draw $f(x) - r$ when $f(x)$ is given. As I said before, since we will encounter this in the next sections, I show them these translations.
>
> **Researcher:** You move on by considering the next concepts.
>
> **Ahmet:** Yes. This is valid for all function translations.

As he said in the interview, he introduced the graphs of quadratic functions step by step. This is illustrated below in Figures 4.26 and 4.27.

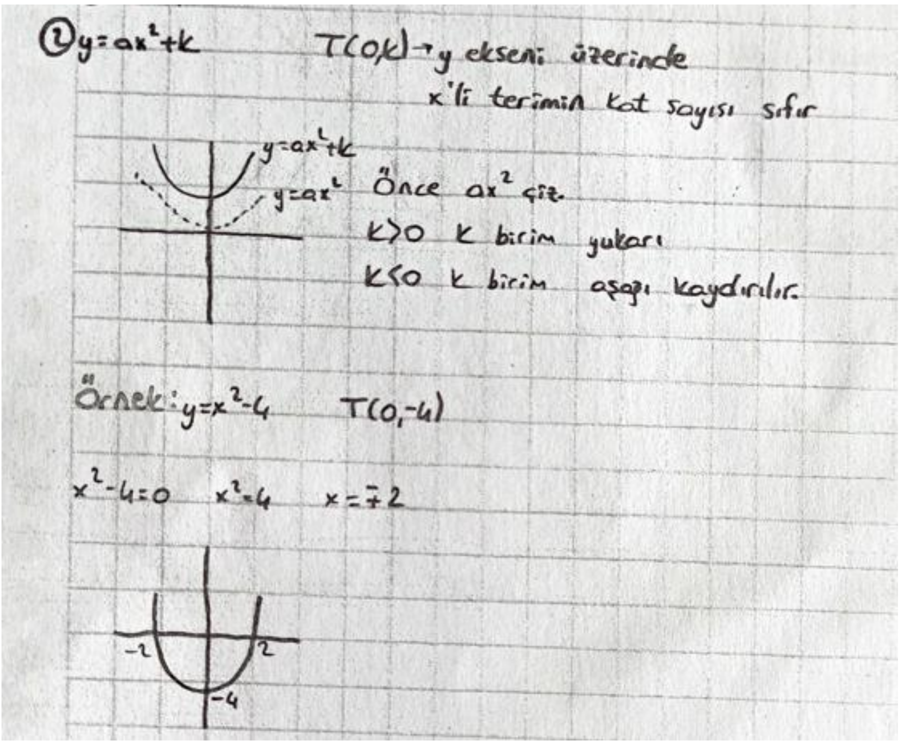

Figure 4.26. A section from Ahmet's instruction

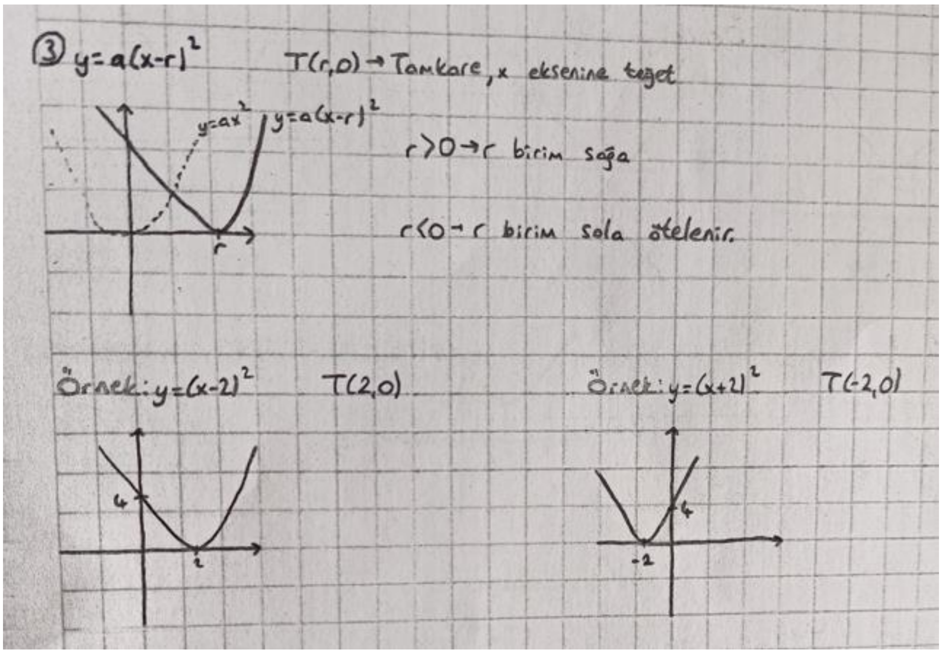

Figure 4.27. A section from Ahmet's instruction

Ahmet's knowledge of finding an example to make a specific mathematical point about quadratic functions

When he was asked what kind of examples he would use in the classroom to emphasize the symmetrical property of parabolas (question 27), he wrote: "I am defining r (the apsis of the vertex) as the half of the sum of the roots. I tell my students that the x-values that sum up to $2r$ are symmetrical. For example if $r = 5$, $f(1) = f(9)$ or $f(-5) = f(15)$. I want my students to notice this property." In the interview, the researcher asked him:

> **Researcher:** In question 27, you have stated that you define a as the half of the sum of the roots of a quadratic equation. Could you explain how you do this?
>
> **Ahmet:** (draws a parabola with $r = 5, f(0) = 4$) For example, I give my students this graph and I want my students to find $f(10)$. Students find the quadratic function and then calculate $f(10)$ as 4 but I want them to notice this short way without finding the equation. In short, I use symmetrical property here. I always ask this kind of questions. I want my students to notice $f(1) = f(9)$ or $f(100) = f(-90)$.

As he stated in the questionnaire and the interview, during his instruction, he defined r as the half of the sum of the roots. Then, he defined the line $x = r$ as the axis of symmetry. This is illustrated in Figure 4.28.

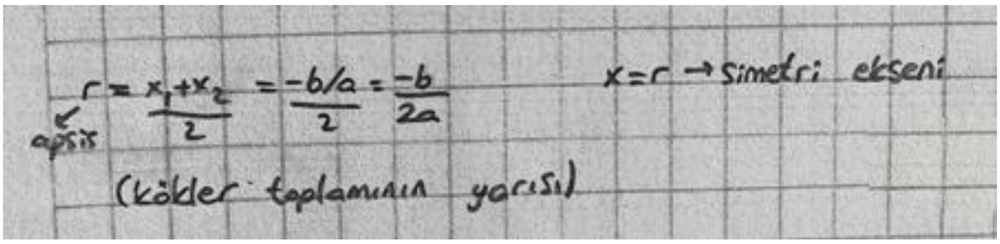

Figure 4.28. Section from Ahmet's instruction

Ahmet's knowledge of modifying tasks regarding quadratic functions

In question 28, the teachers were asked to determine whether their students could solve a task regarding quadratic functions. They were also asked to make some modifications to the task if they think that their students could not solve it. Ahmet made some modifications to the task. He replaced one statement in the task "…given that the distance between A and B is 3 units…" with the statement "…given that the apsis of the midpoint of A and B is 2…". In the interview, the researcher asked him:

> **Researcher:** In question 28, you have changed the question by giving the apsis of the midpoint of A and B. Could you explain why?
>
> **Ahmet:** I think, some of my students can find the solution by using the roots difference formula. However, if the apsis of the vertex was given, the task would be easier and most of my students can solve it.

As it can be seen in his above words, he thought that some students could solve the task, and changed one statement in the task to make the task easier for most of his students.

4.3.1.3. Ahmet's Horizon Content Knowledge of Quadratic Functions

For an elaborate discussion of Ahmet's HCK regarding quadratic functions, the results are presented under two headings that include Ahmet's knowledge of: how quadratic functions are related to other contents in the high school curriculum and how quadratic functions are related to advanced mathematics.

Ahmet's knowledge of how quadratic functions are related to other contents in the high school curriculum

Based on the questionnaire results and the interview, Ahmet's knowledge of the relationship between quadratic functions and other contents in the high school curriculum was strong. Unlike Can, he responded to all the questions regarding the connection between quadratic functions and other contents such as the derivative, exponential functions, etc. (questions 29-34). In question 29, he wrote that the second derivative can be used to explain the relationship between the sign of the leading coefficient and the concavity of the parabola. In the interview, the researcher asked him:

> **Researcher:** You wrote that we can use the second derivative to explain the relationship between the leading coefficient and the concavity of the parabola. Could you explain how?
>
> **Ahmet:** For the function $f(x) = ax^2 + bx + c$, $f''(x) = 2a$ is always constant. If $a > 0$, then $f''(x) > 0$, so the graph is concave up. If $a < 0$, then $f''(x) < 0$, so the graph is concave down. There is no inflection point to change the concavity of the graph.

On the basis of his response to the questionnaire (question 30), he is also aware that exponential functions (with base greater than 1) grow faster than quadratic functions. Then, he explained the association between the vertex and the first derivative (question 31) as: “The vertex is a local extremum point. At the vertex, the first derivative is zero (the slope of the tangent line is zero).” As regarding to the relationship between the quadratic functions and the physics course (question 32), he stated: “I would give the example of projectile motion.” Ahmet has the knowledge of the relationships between the quadratic function and some other mathematical concepts as well as its relationship with interdisciplinary areas such as the physics.

In the next item (question 33), he explained the relationship between the golden ratio and quadratic equations. He stated: “It is $1 + \frac{\sqrt{5}}{2}$, which is the positive root of the quadratic equation $x^2 - x - 1 = 0$.” Lastly, when the teachers were asked to decide whether the graph of $y = x^4$ is a parabola or not (question 34), he responded: “It is not a parabola because it is not of the second order.” This explanation is correct since all the parabolas can be modeled by a quadratic function.

Ahmet’s knowledge of how quadratic functions are related to advanced mathematics

Ahmet’s knowledge of the relationships between quadratic functions and advanced mathematics was strong, on the basis of his responses to the questionnaire and the interview. For example, when the participants were asked to describe (if exists) the relationship between a parabola and a hyperbola (question 36), he wrote: “A parabola is the set of points which are equidistant from a straight line and a focus whereas a hyperbola is the set of points whose distances to two fixed points have a constant difference.” Then, in the interview the researcher asked him:

> **Researcher:** You wrote the definitions of a parabola and a hyperbola in your response to the questionnaire. What would you say about these two concepts?
>
> **Ahmet:** They are both conic sections. I can say this. Their graphs are of course different. The definition of a hyperbola does not take place in the current high school mathematics curriculum.

He also gave some examples of the daily use of the reflection property of parabolas (question 35). He stated: “It is used in real-life, for example, in the construction of headlights and satellite dishes.” In the interview, the researcher asked him:

> **Researcher:** You stated that the reflection property is used in the construction of headlights and satellite dishes. Could you explain what this property is?
>
> **Ahmet:** Reflection property says that any ray parallel to the axis of the parabola will reflect and pass through the focus of the parabola. The logic behind the headlights and satellite dishes is this property. I do not know exactly, the engineers must know and use it better.

In question 37, he explained the fundamental theorem of algebra and its application to quadratic equations. He stated: “Any polynomial of degree n has n roots. So, a quadratic equation has 2 roots.” Then, the participants were also required to select the most correct statement among the given two ones, which are presented below (question 38).

Statement 1: The graph of a quadratic function is a parabola.

Statement 2: The graph of a quadratic function is called a parabola.

In the questionnaire, Ahmet selected the second statement without further explanation. In the interview, the researcher asked him:

> **Researcher:** In question 38, you selected the second statement as the most correct? Could you explain why?
>
> **Ahmet:** The graph of a quadratic function defines a parabola like a first-degree function defines a line. In this case, we call them parabolas. We do not use the conic definition; it was removed from the curriculum.

In question 39, he defined a parabola as "the graph of a quadratic function" and stated an alternative definition as "the set of points that are equidistant from both the directrix and the focus". He was the only teacher who stated the geometrical definition of a parabola. Even though he does not use this definition in his instruction because it is not included in the curriculum, he knows the geometrical definition of a parabola and its interrelation with hyperbolas.

The last question was about distinguishing a parabola from a catenary (question 40). In the question, there was a figure which had the shape of a uniform flexible chain. Of the 18 teachers, only Ahmet stated that the shape is a catenary. In the interview, the researcher asked him:

> **Researcher:** In question 40, you have stated that this shape is a catenary. Could you explain why?
>
> **Ahmet:** Yes. I think it resembles a parabola at first sight, but I think that is different. Who discovered the catenaries? Hmmm… Was he Leibniz? I am not sure.
>
> **Researcher:** Do you know the equations of catenaries?
>
> **Ahmet:** Catenaries have different equations, but I don't know exactly. I will investigate it.

4.3.2. The contribution of Ahmet's Subject Matter Knowledge of Quadratic Functions to Student Learning Outcomes

A total of 28 students were enrolled in Ahmet's course. All the students in his class were administered the quadratic function concept test which provided the data for interpreting Ahmet's students' learning outcomes of quadratic functions (see Appendix C). A summary of the results of Ahmet's students' performance on the test is presented below (Figure 4.29).

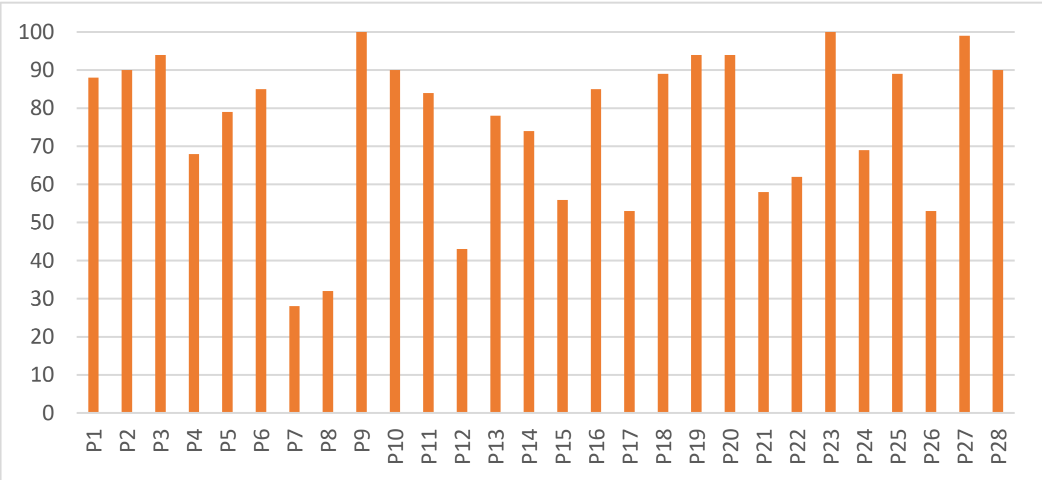

Figure 4.29. Ahmet's students' scores on the quadratic function concept test

Figure 4.29 shows that Ahmet's students' test scores range between a minimum of 28 and a maximum of 100 points. The average test score of the students is 75,8. In general, his students' performance on the test is good. The students' performance is discussed in detail in the next sections based on the objectives of the mathematics curriculum regarding quadratic functions.

4.3.2.1. Finding the Vertex, x-Intercepts, the y-Intercept, and Axis of Symmetry

In the first question of the quadratic function concept test, the students were asked to find the x-intercept(s), the y-intercept, the vertex, and the axis of symmetry of the parabola generated by $f(x) = x^2 + 2x - 8$, and graph it. While finding x-intercepts, unlike Can's students, some of them (n=9) used completing the square method. For example, one student wrote: "$f(x) = (x + 1)^2 - 9 = 0,\ x + 1 = 3,\ x + 1 = -3,$ $x_1 = 2, x_2 = -4$." Fourteen students used factorization, whereas five students did not answer. None of Ahmet's students used the quadratic formula for finding the x-intercepts of the quadratic function. During his instruction, unlike Can, Ahmet emphasized the importance of completing the square method and used this method for solving quadratic equations and finding the x-intercepts of quadratic functions.

When they were asked to find the y-intercept, almost all the students (n=27) correctly found it. Fifteen of them properly that the y-intercept is $(0, -8)$, whereas twelve of them wrote only the number "-8" as the y-intercept.

While finding the vertex, most students (n=25) found it correctly as the point $(-1, -9)$. They all firstly found r by using the formula; then, they found k, by finding $f(r)$. Only two students found an incorrect point as the vertex. They calculated $f(2)$ as the vertex.

When the students were asked to find the axis of symmetry, most of them (n=26) were aware of the interrelation between the apsis of the vertex and the axis of symmetry. However, they had a problem with the use of mathematical terminology. Most of them (n=14) wrote "$r = -1$" as the axis of symmetry. Twelve students correctly stated that the axis of symmetry is the line $x = -1$. Two students wrote an irrelevant number as the axis of symmetry. When they were asked to graph the function, most students (n=26) sketched the graph correctly. Only two students sketched incorrect graphs.

There is some evidence for the contribution of Ahmet's SMK to student learning outcomes. To illustrate, in the questionnaire, he was the only participant who solved the quadratic equations by completing the square method. During his instruction, he frequently used completing the square method, for finding the x-intercepts of a quadratic function. Unlike Can's students, some of his students (n=9) used completing the square method for finding the x-intercepts of a quadratic function. None of his students used the quadratic formula, whereas half of them used the factorization method. When his students were asked to find the vertex of a quadratic function, almost all of them (n=26) correctly found the vertex as the point (r, k). Most of them (n=17) calculated firstly r, then found $k = f(r)$. Unlike Can's students, Ahmet's students noticed that the vertex is the point (r, k), not the "k" value. Moreover, approximately half of Ahmet's students were aware that the axis of symmetry is a line, not a point as they wrote "$x = -1$" as the axis of symmetry of the given function. In the questionnaire, Ahmet defined the axis of symmetry as "a line separating the parabola into two symmetrical parts", and found the axis of symmetry of the function $g(x) = -6x^2 + 12x + 5$ as "$x = 1$". However, as so Can's students, half of Ahmet's students wrote "$r = -1$" as the axis of symmetry. In the second question, which asked to find a missing coefficient in a quadratic function whose vertex is given, most students found the correct result. As so Can, during his instruction, Ahmet solved similar kinds of questions.

4.3.2.2. Associating the Vertex with the Maximum or the Minimum of a Function

When the students were asked to find the minimum of a given function (question 3) and the maximum of a given function (question 4), all of Ahmet's students correctly associated the minimum and the maximum of a quadratic function with the ordinate of the vertex, k. In both questions, they calculated firstly r, and then found k as $f(r)$. For example, one of his student's response to question 4 is: "$r = -\frac{b}{2a} = 2,\ k = h(2) = -4 + 8 + 6 = 10$."

Ahmet’s students’ performance on finding the minimum or the maximum of a parabola was good. During his instruction, Ahmet solved several problems about the maximum and minimum of quadratic functions, as illustrated in Figure 4.30.

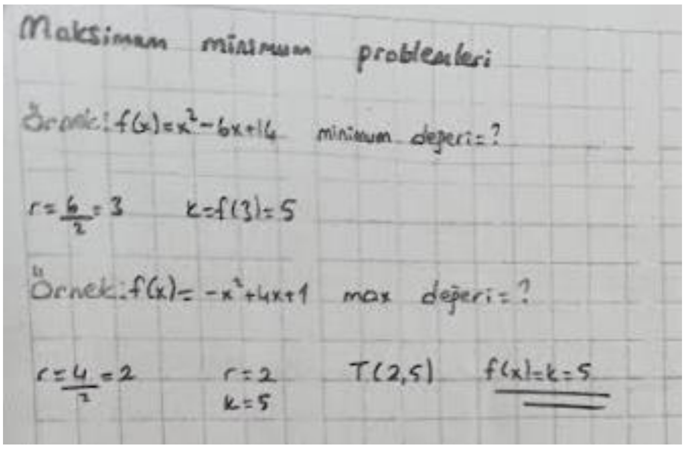

Figure 4.30. A section from Ahmet’s instruction

4.3.2.3. Commenting on the Effect of the Change in the Coefficients on the Graph of the Function

In the fifth question, the students were asked to comment on two cases about how the graph of a quadratic function changes when the leading coefficient a gets smaller or larger. Most students (*n*=20) provided a reasonable explanation for the change in the graph as the leading coefficient changes. One of the students’ responses is presented in Figure 4.31. Three students did not answer this question whereas five students wrote incorrect statements and drew incorrect graphs. For example, in the first case, one of them stated: “*x* values increase, *y* values do not change, and the vertex increases.”

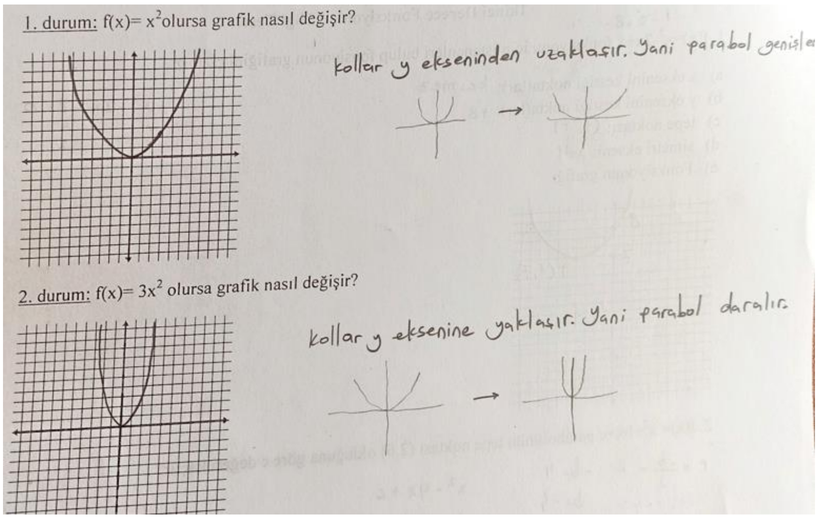

Figure 4.31. An example from Ahmet's students' responses to question 5

Most of Ahmet's students performed well in explaining the relationship between the leading coefficient of a quadratic function and its graph. Based on the questionnaire results, it can also be said that Ahmet also has knowledge of the relationship between the change in the coefficients and the graph of a quadratic function. During the classroom instruction, he emphasized the interrelation between algebraic and graphical representations of quadratic functions. Thus, Ahmet's content knowledge might contribute to his instruction positively, and his instruction might positively have affected his students' learning outcomes about interpreting the graphs of quadratic functions.

4.3.2.4. Finding the Quadratic Function Given Three Points or Two Points That One of Them is the Vertex

In question 6, the students were asked to find the quadratic function whose vertex and one point were given. Most of them (*n*=22) were able to find the quadratic function by using the vertex form. For example, one of them responded: "$f(x) = a(x-r)^2 + k,\ \ f(x) = a(x+2)^2 + 2,\ \ f(1) = 11,\ \ a = 1,\ \ f(x) = a(x+2)^2$." Three students did not respond. A few students (*n*=3) used the vertex form and correctly found a, too. However; they also wrote the quadratic equation in the standard form $y = ax^2 + bx + c$, and tried to calculate the other coefficients b and c. They made calculation errors and came up with an incorrect equation.

In question 7, the students were asked to find the quadratic function whose three points such that one of them is the y-intercept was given. Thirteen students found the quadratic function by using the standard form. Since the y-intercept was given, they easily found c in the quadratic function $y = ax^2 + bx + c$. Then, they substituted the two given points in the function, and obtained two equations. Finally, they found a and b coefficients by solving both equations. The students who found an incorrect answer (n=10) tried to find the quadratic function by using the intercept form. They considered the given two points as the x-intercepts.

Most of Ahmet's students (n=22) were able to find the quadratic function when the vertex is given. However, when they were given three points, not the vertex, approximately half of them (n=15) had difficulty in finding the quadratic function. All of those who found incorrect results used the intercept form, as so Can's students. In the questionnaire, as Can and most teachers did, Ahmet used the intercept form when the vertex is given, and he used the standard form when three points were given. During his classroom instruction, as so Can, Ahmet solved several questions about finding the quadratic function whose some points were given, as shown in Figure 4.32. However, the students in two cases performed better when the vertex is given. When three points were given, the students of both teachers had some difficulties in writing the quadratic function.

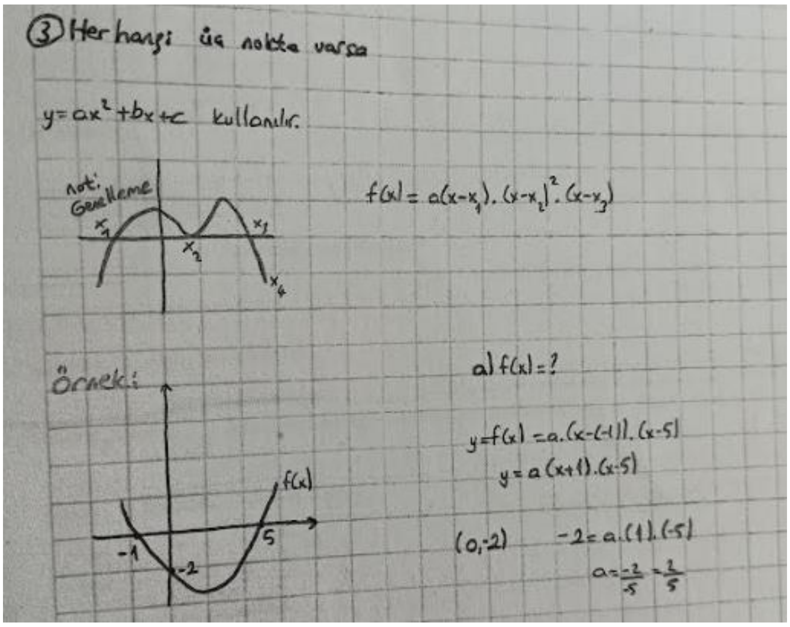

Figure 4.32. A section from Ahmet's instruction

4.3.2.5. Investigating the Intersection of a Line and a Parabola

Most of his students (*n*=24) correctly examined the intersection of a given line and a parabola. The majority of them (*n*=18) investigated the discriminant of the new quadratic equation that they obtained by equating the *y*- values of two functions. They wrote that they are tangent to each other since the discriminant equals to 0. Some students (*n*=6) did not calculate the discriminant of the new quadratic equation. They stated that the parabola and the line are tangent to each other since the quadratic equation is a perfect square. These students established a relationship between the discriminant of a quadratic equation and its form (being a perfect square or not). Three students did not respond to this question whereas one student had an incorrect answer. He calculated the discriminant of the given quadratic equation.

In the questionnaire, the teachers were also asked to find the intersection of a line and a parabola. As most teachers and Can did, Ahmet obtained a new quadratic equation and investigated its discriminant. Furthermore, as so Can, he solved several problems about investigating the intersection of a parabola and a line in the classroom, as illustrated in Figure 4.33.

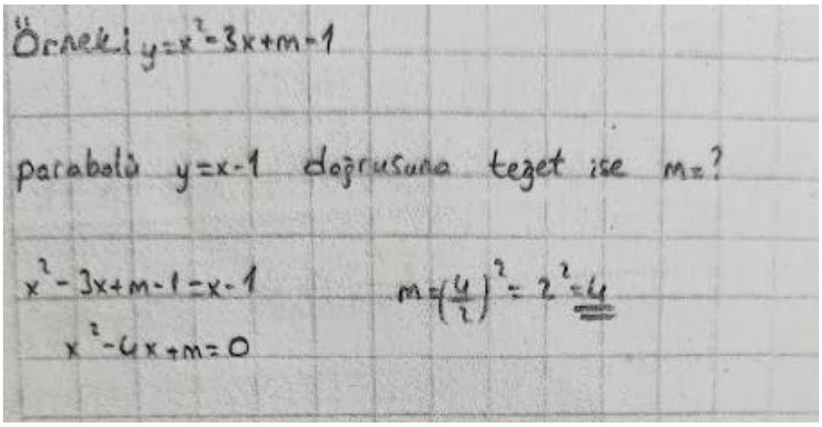

Figure 4.33. Section from Ahmet's instruction

4.3.2.6. Solving Problems That Can Be Modeled by Quadratic Functions

In question 9, the students were asked to find the maximum area of a rectangle with its perimeter given. Twelve students correctly found the result by constructing a quadratic model. One of the student's solution is presented below (Figure 4.34). There were also some students (n=7) who found the correct result without constructing a quadratic model, by trying some numbers for the dimensions. Similarly, three students replaced several numerical values for the dimensions of the rectangle. However, they eliminated the case $9x9$ since they thought that it would be a square, not a rectangle.

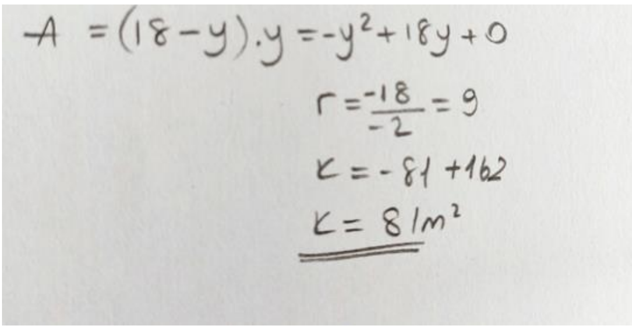

Figure 4.34. An example from Ahmet's students' responses to question 9

In question 10, the students were given a quadratic function, which represents the height of a ball that was hit by someone playing football. The question was to find at what time the ball reaches 3 meters above the ground. Most students (*n*=15) correctly found the solution. For example, one of them wrote: $h(t) = -t^2 + 4t = 3$, $-t^2 + 4t - 3 = 0$, $t_1 = 1$, $t_2 = 3$. Four participants found only one of the roots $t = 3$ or $t = 1$. Some students (*n*=6) found incorrect results. Two of them calculated the value of $h(3)$ whereas four of them found the maximum of the function. Three students did not respond.

CHAPTER 5

CONCLUSION AND DISCUSSION

The primary purposes of this study were to identify high school mathematics teachers' SMK in three dimensions, CCK, SCK, and HCK, and to examine its contribution to student learning outcomes on quadratic functions. There were two research questions:

1. As regarding to quadratic functions, what SMK do secondary mathematics teachers have?
   d) As regarding to quadratic functions, what CCK do secondary mathematics teachers have?
   e) As regarding to quadratic functions, what SCK do secondary mathematics teachers have?
   f) As regarding to quadratic functions, what HCK do secondary mathematics teachers have?
2. How do CCK, SCK, and HCK contribute to the instructional practice and thus, student learning outcomes regarding quadratic functions?

To answer the first research question about teachers' SMK of quadratic functions, a questionnaire was administered to 18 high school mathematics teachers. Two case studies were carried out to address the second research question that investigates the contribution of teachers' SMK of quadratic functions to instructional practice, and thus student learning outcomes. Two teachers from the participants of the first part of the study were selected for the second part of the study, which included interviews and classroom observations. In this chapter, the main findings of the study were concluded and discussed. This chapter also included implications of the study, recommendations for future studies, and limitations of the study.

5.1. Teachers' Subject Matter Knowledge of Quadratic Functions

The teachers' SMK of quadratic functions was discussed under three headings: teachers' CCK, SCK, and HCK.

5.1.1. Teachers' Common Content Knowledge of Quadratic Functions

In this study, teachers' CCK of quadratic functions includes seven components teachers' conceptions of quadratic equations and functions, teachers' knowledge of solving quadratic equations with one unknown, teachers' knowledge of sketching and interpreting the graphs of quadratic functions, knowledge of graphing quadratic functions using transformations, teachers' knowledge of solving real-life problems regarding quadratic functions, teachers' knowledge of finding the quadratic function with given points, and teachers' knowledge of finding the intersection of a parabola and a line.

The result showed that although teachers defined quadratic functions, quadratic equations, and quadratic polynomials in their algebraic forms, most of them have limited knowledge of the interrelation between these concepts. Most of the teachers examined the first differences for the given (x, y) ordered pairs to decide whether the function is linear or quadratic; however, none of the teachers mentioned the constant second differences of quadratic functions. The findings also indicated that the majority of teachers used the quadratic formula for solving a quadratic equation. When they were asked to define some basic elements of a parabola, like the axis of symmetry, vertex, or the concavity of a parabola, the majority made procedural descriptions rather than structural descriptions. For example, most of them described the axis of symmetry as the line $x = -b/2a$, passing through the vertex. Likewise, some teachers defined the vertex as the point $(\frac{-b}{2a}, f(\frac{-b}{2a})$. Some teachers described how to find the type of the concavity of a parabola based on the leading coefficient, whereas some associated the concavity with the second derivative of the function. All the teachers correctly sketched the graph of a given quadratic function.

When the participants were given a parabola and asked to find the corresponding quadratic function, they used multiple algebraic forms of quadratic functions. That is, when the vertex was given in the graph, they used the vertex form; and when the x-intercepts were given, they used the intercept form. When the participants were given a graph and asked to comment on the signs of a, b, and c coefficients of the corresponding quadratic function, all of them examined the shape of the parabola and stated that $a > 0$ since the parabola is upwards. To determine the sign of b, two different approaches were observed. Most teachers examined the sign of the apsis of the vertex to determine the sign of b. However, some participants examined the sign of the sum of the roots to find the sign of b. While determining the sign of c, most participants examined the y-intercept whereas some participants found the sign of c by examining the multiplication of the roots.

The result also indicated that teachers mostly stated translation as a graph transformation. They did not mention other types of transformations such as reflection, stretching or shrinking of parabolas. Only a few teachers stated reflection and stretching as graph transformations. The majority of teachers did not explain how the graph of any quadratic function can be obtained from the graph of the function $y = x^2$. The teachers were also asked to compare two functions without graphing them; many participants noticed that their shape was the same. However, some teachers compared the graphs in terms of their x-intercepts, the y-intercept, and the vertices without referring to the transformations and the conservation of the shape. Moreover, many participants did not explain the effect of the coefficients on the shape of a parabola. When the teachers were asked to determine the widest parabola among the given ones, few of them correctly stated the relationship between the absolute value of the leading coefficient and the width of the parabola.

The findings also revealed that teachers have limited knowledge of solving real-life problems about quadratic functions. That is, most of the teachers could not construct a mathematical model to solve the real-life problem in the questionnaire. Some teachers attempted to solve the problem without forming a quadratic model, trying some numerical values to obtain the result, but they were unsuccessful. Teachers' performance in finding the quadratic function with some points given was good. When the vertex was known, they used the vertex form; and when the three points were known, they used the standard form to obtain the quadratic function. Lastly, the majority of teachers found the intersection of a parabola and a line correctly.

Although there were some differences between their solution strategies or approaches to solving problems, secondary mathematics teachers' performance on CCK items was good. They were able to explain basic facts or procedures about quadratic functions, use correct mathematical notations, and correctly solve simple questions about quadratic functions.

The results of this study coincide with the study of Bansilal et al. (2014) that investigated secondary mathematics teachers' CCK. In their study, the teachers were given the graph of a function $f(x)$, and they were asked to find the maximum of $1 - f(x)$. They were expected to make some transformations on the vertex of the given parabola. Bansilal et al. (2014) reported that most secondary teachers failed to make these transformations. However, they did not have problems finding the *x*-intercepts of a parabola. Likewise, in the present study, most teachers did not have a problem finding the *x*-intercepts of a parabola; however, they have limited knowledge of explaining graph transformations. In another study, Ubah and Bansilal (2018) explored how pre-service teachers found the algebraic equation for a quadratic function given in the graphical form. Ubah and Bansilal (2018) reported that although some of the participants could find the equation using one method; most failed to find the equation using two methods and the most common method function was using the intercept form of quadratic functions. On the contrary, in the current study, teachers used more than one method (using the intercept form and using the standard form). In the study of Ubah and Bansilal (2018), many pre-service teachers could not even write the correct algebraic form of a quadratic function (Ubah, & Bansilal, 2018). On the contrary, in the present study, secondary teachers did not have difficulty writing the algebraic form of a quadratic function given in the graphical form.

This study also confirms the findings of the study by Aziz et al. (2018) that investigated pre-service secondary mathematics teachers' views on distinguishing quadratic functions and quadratic equations. Like the participants in Aziz et al.'s (2018) study, the teachers in the current study wrote differences between quadratic equations and quadratic functions mostly based on their standard forms. The findings of the current study also revealed that some teachers compared quadratic functions and quadratic equations based on their main characteristics and based on geometrical aspects. Aziz et al. (2018) also reported that factorization and the quadratic formula were the two methods the participants commonly used to solve quadratic equations; and a few participants used completing the square method for solving a quadratic equation. Similarly, in this study, the quadratic formula and factorization was the most common method for solving a quadratic equation; only one teacher used completing the square method to determine the roots of a quadratic equation.

5.1.2. Teachers' Specialized Content Knowledge of Quadratic Functions

In this study, teachers' SCK is discussed under seven components including teachers' knowledge of: explaining and justifying basic formulas of quadratic functions, posing problems regarding quadratic functions, recognizing students' incorrect solutions regarding quadratic functions, understanding students' unusual solutions regarding quadratic functions, responding to students' why questions about quadratic functions, finding an example to make a specific mathematical point about quadratic functions, and modifying tasks regarding quadratic functions.

The result showed that the majority of teachers' performances on SCK items were lower than their performances on CCK items. Although teachers used the quadratic formula, the majority of them did not justify it. Some teachers made an algebraic justification of the formula by completing the square; none of the participants made a geometrical justification. When the teachers were asked to state a real-life problem about quadratic functions, the majority of them did not state a problem. Most teachers wrote a problem context rather than the full statement of the problem, including projectile motion and velocity-acceleration problems from the physics course. The teachers were also asked to recognize a student's incorrect solution. Some participants noticed all the incorrect steps, whereas half of them recognized some of the incorrect steps. The comments that the teachers made about the student's solutions were incomplete.

The findings also indicated that the majority of teachers could not explain the effect of the translations on the coefficients of the quadratic functions. That is, most teachers failed to identify the association between the shape of the parabola, the coefficients, and the location of the vertex. When they were asked to explain why a quadratic function $ax^2 + bx + c$ cannot be divided by the variable x, almost all of them explained the reason. The teachers were also asked to state examples to underline the symmetrical property of a parabola. The majority of teachers made some incorrect or irrelevant explanations. Only a few teachers provided a plausible example to emphasize the symmetrical property during instructional practice. Lastly, the teachers were given a task of quadratic functions and asked to explain whether their students could solve it or not. Many teachers made some reasonable changes to the task since they thought it might be hard for their students.

In this study, teacher have limited SCK for teaching quadratic functions. Similarly, Zembat (2013) reported that teachers have quite limited understanding of the core mathematical ideas, analyzing the students' work, in the assessment of understanding mathematical ideas, and making curricular decisions. Zembat (2013) suggested that teachers should improve their SCK to fill the gap between where they are and where they need to be.

5.1.3. Teachers' Horizon Content Knowledge of Quadratic Functions

In this study, teachers' HCK was analyzed in two dimensions: the knowledge of how quadratic functions are related to other contents in the high school curriculum and the knowledge of how quadratic functions are related to advanced mathematics.

The result indicated that although teachers know basic facts, operations or procedures about quadratic functions, the majority of them were unable to connect quadratic functions with other content in the high school curriculum. Most teachers did not explain the relationship between the concavity of a parabola and the second derivative of the quadratic function. Furthermore, the majority of teachers could not compare the graphs of parabolas with the graphs of exponential functions. Although more than half of the teachers were able to explain the relationship between the vertex of a parabola and the first derivative partially, none of them mentioned the rate of change or the maximum-minimum points of the quadratic function. In addition, few teachers explained the connection between the golden ratio and quadratic equations. Lastly, the teachers were asked to relate the concept of the quadratic function with any concept from the physics course. Some participants stated that the quadratic function is related to free fall, projectile motion, and velocity-acceleration problems from the physics course, whereas most of them did not give any examples.

The results also revealed that the teachers' knowledge of how quadratic functions relate to advanced mathematics is limited. When the teachers were asked to compare a parabola and a hyperbola, only a few defined a hyperbola and explained their relationship. Most of them did not state the fundamental theorem of algebra and its application to quadratic polynomials. Furthermore, almost all the teachers did not state the geometrical definition of a parabola as a conic section. In addition, most of the participants did not explain the term *catenary*, which has a shape that looks like a parabola but is somehow different. This finding coincides with the study of Miheso-O'Connor Khakasa and Berger (2016) who reported that teachers were uncomfortable with engaging in responses that require HCK. Miheso-O'Connor Khakasa and Berger (2016) also reported that teachers have limited knowledge of when and how to use the advanced mathematical knowledge.

5.2. The Contribution of Teachers' Subject Matter Knowledge of Quadratic Functions to Student Learning Outcomes

The second phase of the study revealed the contribution of teachers' SMK of the quadratic function concept and student learning outcomes. The data suggested evidence of that teachers' SMK of quadratic functions contributed to student learning outcomes. The study also revealed that teachers' SMK of quadratic functions affected their instructional practices, and their instructional practices interacted with students' performance.

The study showed that the teachers' SMK of quadratic functions influenced their instructional practices. Can's instruction was mostly based on the procedural aspects of the quadratic functions rather than the conceptual aspects. He solved the quadratic equations by the quadratic formula or factorization. He did not use the completing the square method. He did not give much importance to justifying basic formulas of quadratic functions, such as the quadratic formula. He sometimes used incorrect notation for mathematical equations or incorrect explanations for some properties. For example, he defined the vertex as the ordinate of a parabola's maximum or minimum point. Also, he defined the axis of symmetry as the apsis of the vertex and found the axis of symmetry of a function as $r = 1$ rather than a line equation that $x = 1$. The exercises he solved in the classroom mostly required procedural knowledge. He showed many examples of finding the vertex of a quadratic function during his instruction. However, he did not solve problems about the maximum or minimum of a parabola. He failed to explain the effect of the translations of parabolas on the coefficients of the functions. During his instructional practice, he did not emphasize the transformations of quadratic functions. Also, he failed to solve the real-life problem in the questionnaire. Accordingly, he did not solve any real-life problems about quadratic functions in the classroom. However, he was good at using multiple representations of functions. In his instructional practice, he solved several questions about finding the equation of a quadratic function. He emphasized using the intercept form if the vertex is given, and he told his students to use the standard form if the y-intercept and two other points are given. He also solved several problems about finding the intersection of a parabola and a line.

On the other hand, unlike Can's instructional practice, Ahmet's instruction involved more detailed explanations, justifications of the mathematical rules, and connections between mathematical concepts. In contrast to Can, Ahmet focused primarily on the conceptual aspects of quadratic functions as well as the procedural aspects during his instruction. His performance on the quadratic function concept questionnaire was clearly better than the other participating teachers. He had the highest scores on the CCK, SCK, and HCK items of the questionnaire. His SMK was stronger than Can and also the other teachers in the first phase of the study. He was the only teacher who solved the quadratic equations by completing the square. He emphasized justifying basic formulas or properties of quadratic functions and explaining the connections to higher mathematical ideas. He made structural definitions of the concepts such as the vertex and the axis of symmetry that are crucial to understanding quadratic functions. Unlike Can, he solved several real-life problems about quadratic functions, including maximum-minimum problems. He was also the only participant who solved the real-life problem in the questionnaire by forming a quadratic model. Like Can, he solved several questions about finding the equation of a quadratic function and the intersection of a parabola and a line. As Can did, he emphasized multiple representations of quadratic functions during his instruction. He could also explain the effect of the translations of parabolas on the coefficients of the functions. In his instructional practice, he introduced the graphs of quadratic functions step by step, first introducing the quadratic function $y = ax^2$, and obtaining other quadratic functions by making translations over this function. Thus, his instruction seems more planned and connected since he cares about conceptual knowledge rather than procedural.

The influence of teacher knowledge on the quality of the instructional practice is consistent with those reported in the literature (e.g., Hatisaru, 2013; Sánchez & Llinares, 2003). Hatisaru's (2013) study found that teachers' KCS influenced the quality of their instruction regarding the function concept. Similarly, Sánchez and Llinares (2003) reported that pre-service teachers' ways of knowing the subject matter affected their pedagogical reasoning, i.e., what they considered important for students and which representations they use in the classroom.

The findings also indicated a relationship between instructional practice and student learning outcomes. When the teachers finished the lessons on quadratic functions, the majority of Can's students did not find basic elements of the quadratic function like the *x*-intercepts, the vertex, and the axis of symmetry and did not sketch the graph. In addition, none of them used completing the square method for solving a quadratic equation. However, the majority of Ahmet's students correctly found the *x*-intercepts, the vertex, and the axis of symmetry of a given function and sketched its correct graph. Additionally, some used completing the square to find the *x*-intercepts of the function. Unlike Can's students, none of his students used the quadratic formula to find the roots of a quadratic equation. Moreover, when the students were asked to find the minimum or the maximum of a quadratic function, most of Can's students did not find it. In contrast, all of Ahmet's students could correctly find the maximum or the minimum of a quadratic function. Ahmet's students were able to associate the vertex with the minimum or the maximum of a quadratic function.

None of Can's students could comment on the effect of the change in the leading coefficient on the parabola. They did not explain the basic interrelation between the leading coefficient and the parabola. However, most of Ahmet's students provided an explanation for the interrelation between the leading coefficient and the parabola. Unlike Can's students, they were aware that when the absolute value of the leading coefficient gets larger, the parabola becomes narrower.

Can's students' performances in investigating the intersection of a line and a parabola were relatively better than when compared to the other questions in the test. Many students correctly commented on the intersection of a line and a parabola examining the discriminant of the new quadratic equation that they obtained by equating the *y*-values of their standard forms. Likewise, the majority of Ahmet's students examined the intersection of a parabola and a line. Some of them did not calculate the discriminant of the new quadratic equation, rather, they stated that the parabola is tangent to the line since the new quadratic equation is a perfect square. This shows that some of Ahmet's students understand the relationship between the discriminant of a quadratic equation and its algebraic form (being a perfect square). During his instructional practice, Ahmet mostly used completing the square method and emphasized the vertex form of quadratic functions. Furthermore, the majority of Can's students could not solve real-life problems about quadratic functions. Ahmet's students' performances were clearly better in solving real-life problems regarding quadratic functions than Can's students. These findings provide strong evidences for the contribution of instructional practice to the student learning outcomes.

When the students were asked to find the quadratic function given its vertex and one point, only a few of Can's students found it correctly, using the vertex form. In Ahmet's class, the majority of students could find the quadratic function by using the vertex form. The students were also asked to find the quadratic function given the *y*-intercept and two points. In this case, none of Can's students found the quadratic function; approximately half of Ahmet's students found the quadratic function using the standard form. Although the majority of Ahmet's students found the quadratic function when the vertex was given, some of them did not find the quadratic function when the *y*-intercept and two points were given. In both cases, the students' performance was better in finding the quadratic function when the vertex is given.

Educators seem to have a consensus that instructional practice affects students' performance (Gençtürk, 2012; Hatisaru, 2013; Hatisaru, & Erbaş, 2017; Ibeawuchi, 2010; Shechtman et al., 2010). The NCTM (2000) reported that "students learn mathematics through the experiences teachers provide. Thus, students' understanding of mathematics, their ability to use it to solve problems, and their confidence in, and disposition toward, mathematics are all shaped by the teaching they encounter in school" (p. 16-17). The data also revealed that the performance of students of the teacher with strong MKT was better than those of teachers who had relatively low MKT for teaching quadratic functions. This is consistent with the literature (Callingham et al., 2016, Tchoshanov et al., 2017). Callingham et al.'s (2016) study reported that students of teachers who had a strong PCK performed better than the students of teachers who had weak PCK in the questionnaire. Tchoshanov et al. (2017) also provided evidence for the relationship between teachers' mathematics content knowledge and student performance at the lower secondary level.

The result also indicated the existence of some mediating factors for the relationship among teacher knowledge, instructional practice, and student learning. For example, during his instructional practice, although Can solved several questions about finding the quadratic function with some points (the vertex, the y-intercept, or any point) given, his students had some difficulty in finding the quadratic function especially when the y-intercept and two points given. This finding might be explained by some mediating factors such as teacher beliefs, inherent complexities of the function concept, the students' academic background, and the students' difficulties in arithmetic (Hatisaru & Erbaş, 2017; Hill, Blunk et al., 2008; Shechtman et al., 2010).

5.3. Implications

This study highlights secondary mathematics teachers' SMK of quadratic functions and its contribution to instructional practice and student learning outcomes regarding quadratic functions. The results have both methodological and practical implications for mathematics educators, secondary mathematics teachers, and policymakers.

First of all, research confirms that teachers' content knowledge is critical for student learning (Tchoshanov et al., 2017). Although its critical role in student learning, the findings of this study revealed that some aspects of the teachers' SMK of quadratic functions are inadequate, which might adversely affect student learning. Teachers' performance on CCK items were better than their performance on SCK and HCK items. That is, secondary mathematics teachers in this study have adequate knowledge of basic facts or procedures of the quadratic function concept, which is expected for them to teach the content. However, their SMK is limited to basic facts or procedures. Most teachers did not explain the relationship between quadratic functions and other concepts in the secondary mathematics curriculum or in the advanced mathematics. Teacher education programs or professional development programs might be organized to enhance pre-service teachers' and in-service teachers' SCK and HCK regarding a specific content.

The mathematics educators in faculties of education may design some method courses regarding specific learning areas such as geometry, functions, numbers, or algebra. This might give the pre-service teachers the opportunity to develop their SMK of a specific content, understanding all the definitions, representations, teaching methods, the use of mathematical terminology, and the use of mathematical models. Likewise, professional development programs might also be designed to enhance practicing teachers' SMK of a specific content. The findings suggested evidence of the link between teachers' SMK and student learning outcomes regarding quadratic functions, mediated by instructional practice.

Secondly, the questions that were asked to teachers in this study were beyond basic facts or procedures about quadratic functions and they addressed many other aspects of knowledge that a teacher might possess to yield an effective teaching. Some questions were about the daily use of quadratic functions whereas some were about explaining basic formulas of quadratic functions. There were also some questions that addressed students' solutions. Engaging in these questions might give the teachers opportunity for self-assessment and allow them to evaluate the extent of their own SMK regarding a specific content. Thus, teachers might have an opportunity to reflect their own strengths and weaknesses about this particular content and make plan to improve themselves for effective teaching.

Another implication of this study is related to the specification of teacher knowledge for teaching quadratic functions. This study contributed to clarifying what a secondary mathematics teacher should know for teaching quadratic functions. For this purpose, the key components of teachers' CCK, SCK, and HCK were identified. This specification of teacher knowledge regarding a specific-content might contribute to the field of mathematics education. Mathematics educators should extend this literature examining teacher knowledge and specifying the key aspects of teacher knowledge for teaching any other content from other mathematical domains such as algebra, numbers, etc.

5.4. Recommendations for Future Studies

This study helps to understand the complex relationship between teachers' SMK, their instructional practices, and student learning outcomes regarding a specific content, the quadratic function. There is much more to be learned about this relationship. In this study, no student achievement data is used. Whether and what changes occur in students' learning outcomes with different academic backgrounds might be an important question to investigate. Thus, this study should be replicated with different teachers and students including the student achievement data. Data obtained from these studies will contribute to understanding how students' academic backgrounds mediate the influence of teachers' SMK on student learning outcomes of a specific content.

This study found that secondary teachers' SMK – especially SCK and HCK - regarding the quadratic function concept is limited. Although the participants of the current study included teachers in various high school types (i.e., science high schools, Anatolian high schools, etc.), this study did not examine teachers' SMK based on the school types. Further studies should compare teachers' SMK of quadratic functions based on the school types (i.e., one teacher from science high school and one teacher from a vocational high schools).

The present study was limited to quadratic functions. More aspects of the polynomial functions of higher degrees should be added to the results of this study. The results of these studies might contribute to getting a clear picture of teachers' SMK for polynomial functions.

The present study used Ball et al.'s (2008) model to investigate teachers' SMK and its contribution to student learning. Further studies should investigate the PCK of the model, which includes KCS, KCT, and the knowledge of curriculum. These studies should investigate which dimensions of teacher knowledge have the most influence on student learning outcomes of specific mathematical content. For example, they should examine whether SMK or PCK is more influential on student performance.

In the present study, two different cases were investigated. One of the teachers got the highest scores on CCK, SCK, and HCK dimensions of the questionnaire that indicates he has a balanced distribution of three dimensions of SMK. Another teacher got an intermediate score from the CCK items, relatively low score from the SCK items and the lowest score from the HCK items, that indicates he has an unbalanced distribution of three dimensions of SMK. Further studies should investigate different cases with different teachers. For example, T5 in this study might be a different case with low CCK score, relatively higher SCK score, and the lowest HCK score.

5.5. Limitations of the Study

This study has some limitations in terms of the participants, the instruments, and the procedure for data collection. First of all, this study distinguished and measured secondary mathematics teachers' CCK, SCK, and HCK of quadratic functions. Distinguishing between sub-domains of teacher knowledge has been a concern for many researchers (Hill et al., 2008; Howell, 2012). As Ball et al. (2008) stated "it is not always easy to discern where one of our categories divides from the next" (Ball et al., 2008, p. 403). Thus, trying to distinguish the sub-domains was a challenge throughout the study. To overcome this problem, the researcher created a table of specifications that identified the key components of CCK, SCK, HCK (see Table 3.1). Moreover, the researcher frequently consulted the experts in mathematics education to discuss ambiguous cases. However, there might be still some small ambiguities with matching the items in the questionnaire with a sub-domain of teacher knowledge. An item that aimed to evaluate teachers' SCK might also include any piece of knowledge from CCK or HCK. These three dimensions cannot be not strictly separated from each other. This is the biggest limitation of the present study.

Secondly, a total of 18 volunteer teachers accepted to participate in the first phase of this study. Also, the study was conducted in a particular setting with two teachers and their students, and included a detailed description of two cases. Even though the present study does not aim to generalize the results to other secondary mathematics teachers and their students, the number of participants might be a limitation since a limited number of teachers may not represent a variety of perspectives of teacher knowledge regarding quadratic functions. Thus, the results of the present study may not necessarily be generalizable to other teachers and students in different school settings.

In this study, no student achievement data is used. The contribution of teacher knowledge to student performance cannot be exactly known without student achievement data. However, it is evident from the previous research that teacher knowledge strongly and positively affects student performance. In this study, SMK dimension of teacher knowledge that has been linked to gains in student performance is investigated. Thus, the present study helps to identify those links to some extent.

Another limitation of the study is the absence of video recording during teachers' instructional practice. The notebook of one student and the observation notes of the researcher were used to analyze the instructional practice of teachers. In this study, some elements of teachers' instructional practice were analyzed: the examples that teachers use, the use of multiple representations, making connections among mathematical concepts, making justifications of formulas, and using real-life problems while teaching quadratic functions. Some elements of teachers' instructional practice were not included in the analysis, i.e., responding to students' why questions, and analyzing the students' solutions while teaching quadratic functions in the classroom.

Lastly, during classroom observations, the researcher's presence may have influenced the teachers' instructional practices. To minimize this, the researcher participated in two classes before the instruction on quadratic functions started and established a close relationship with the teacher and the students to make them feel more comfortable. Thus, the teacher and the students were more likely to accept the presence of the researcher as a part of their classroom environment.

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APPENDICES

APPENDIX A. THE QUADRATIC FUNCTION CONCEPT QUESTIONNAIRE

Question 1: What is a quadratic equation?

Question 2: What is a quadratic function?

Question 3: How are quadratic equations, quadratic functions, and quadratic polynomials interrelated? Could you explain the differences or similarities between them?

Question 4: Decide whether the following table of values belongs to either linear or quadratic functions:

x	-3	-2	-1	0	1	2	3
y	14	10	6	2	-2	-6	10

x	-3	-2	-1	0	1	2	3
y	3	0	-1	0	3	8	15

Question 5: Solve the quadratic equations below.

a. $3x^2 - 5x + 1 = 0$ **b.** $x^2 - x = -6$ **c.** $(x - 3)^2 + 5 = 0$

Follow-up: What other ways (if any) are there to solve it?

Question 6: What does the *axis of symmetry* mean for a quadratic function?

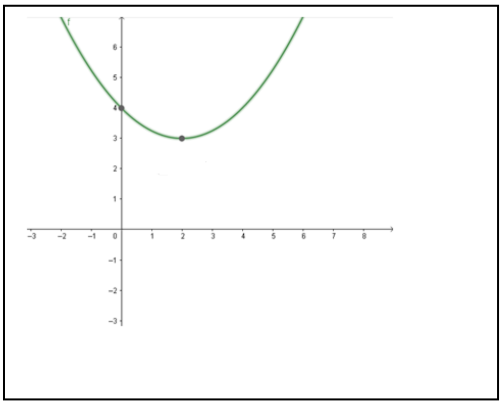
Question 7: What does the *vertex* of a quadratic function mean?

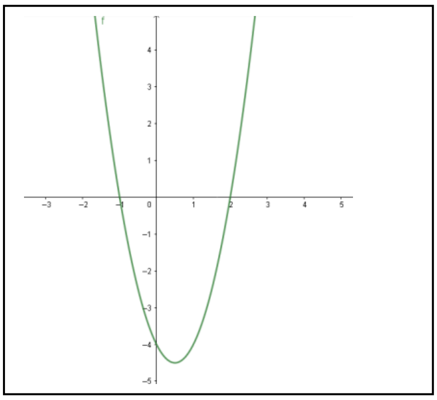
Question 8: What does *concavity* of a quadratic function mean?

Question 9: Find the following properties of the function $g(t) = t^2 + 2t - 8$, and then graph it.

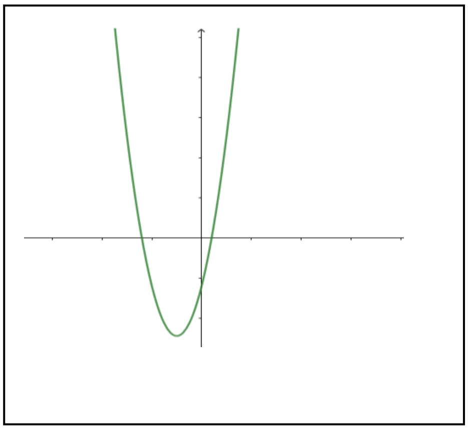
Graph orientation: up/down
x-intercept(s):
y-intercept:
Axis of symmetry:
Vertex:
If exists, the maximum or the minimum:

Question 10: For each of the two graphs given below, please state the quadratic function.





Question 11: The graph of the function $f(x) = ax^2 + bx + c$ is shown in the figure below. State whether a, b, and c are negative, positive, or zero. Explain your reasoning.



Question 12: One of your students claims that it is possible to generate the graph of a quadratic function by applying some transformations on the graph of $f(x) = x^2$. What would be your reaction to this claim? Please explain why.

Question 13: Which function represents the parabola with the widest graph? Explain your answer.

A) $2(x+3)^2$ B) $x^2 - 5$ C) $0.5(x-1)^2 + 1$ D) $-x^2 + 6$

Question 14: Describe the similarities and differences between the parabolas generated from these two functions without drawing them: $f(x) = x^2 - 5$ and $g(x) = (x-5)^2$.

Question 15: The *Mathematical Magazine* is a popular magazine published once in three months is sold approximately 25,000 per issue for a 5.5 TL price. Due to the increasing cost of paper and production, a price increase has become inevitable. A survey was conducted with the readers in order to understand how a rise in the price of the magazine would affect the sales. Results of this survey revealed that each 50 Kr rise would result in a drop of 1,250 people buying the magazine. If you were the editor of the magazine, what would you suggest as the new selling price?

Question 16: The parabola with the vertex $(-1/4, 11/4)$ passes through the point $(-1, 5)$. What is the equation of that parabola?

Question 17: Find the equation of the quadratic function whose graph contains the points $(1, 9)$, $(-2, 27)$, and $(4, 3)$.

Question 18: Think about the parabola $f(x) = ax^2 + bx + c$ and the line $y = mx + n$. Under what conditions the parabola and the line intersect or not?

Follow-up: Decide whether the line $y = 11x - 13$ and the parabola $y = 2x^2 + 3x - 5$ intersect or not.

Question 19: State the quadratic formula and explain how it is derived, both geometrically and algebraically.

Question 20: Solve the quadratic equation $x^2 - x + 1 = 0$ without using the quadratic formula. Explain your method. Is there a special name for the method you have just used?

Question 21: Can you provide an example of a real-life problem you share with or ask to your students that can be modeled and solved by a quadratic function?

Question 22: Comment on the student's solution to the problem given below. State whether the result is correct or not. Explain your reason.

Problem: Find the minimum value of the function $f(x) = -3x^2 + 5x + 7$ in $[-2, 2]$.

Student's response: In order to find the minimum, we need to find the vertex of the function. It can be obtained by using the formula $-b/2a$.
$a = -3, b = 5$. Therefore $T = (-5)/(-6) = 5/6$ is the minimum of the function.

Question 23: Imagine that, in your exam, one of your students solved the quadratic equation $2x^2 + 5x - 7 = 0$ as presented below. What do you think about this solution? Is the solution correct? How would you describe the student's approach?

$$2x^2 + 5x = 7$$

$$x^2 + \frac{5}{2}x = \frac{7}{2}$$

$$x^2 + \frac{5}{2}x + \frac{25}{16} = \frac{7}{2} + \frac{25}{16}$$

$$\left(x + \frac{5}{4}\right)^2 = \frac{81}{16}$$

$$x + \frac{5}{4} = \frac{9}{4}, \qquad x + \frac{5}{4} = -\frac{9}{4}$$

$$x_1 = 1, \quad x_2 = -\frac{7}{2}.$$

Question 24: Please examine a student's solution to the below problem and decide whether the result is true or false. Describe the student's approach.

Problem: Find the (unique) quadratic polynomial such that all three of the following are true:

- All the coefficients are integers.
- The leading coefficient is 4.
- $7 + \sqrt{6}$ is one of the roots.

A student's Solution:

$x = 7 + \sqrt{6}$
$(x-7)^2 = 6$
$4(x-7)^2 = 24$
$4(x-7)^2 - 24 = 0$
$4(x^2 - 14x + 49) - 24 = 0$
$4x^2 - 56x + 196 - 24 = 0$
$p(x) = 4x^2 - 56x + 172.$

Question 25: In your class, while you are teaching the quadratic function $f(x) = ax^2 + bx + c$, one of the students asked that "While we are translating a parabola vertically, only c changes. However, translating a parabola horizontally changes both b and c. Why does it happen?"

How would you respond to your student's question? Please explain.

Question 26: Assumed that one of your students, Ali, provided the following solution for the equation $3x^2 = 15x$.

$3x^2 = 15x$

$x^2 = 5x$

$x = 5$

Then, the following conversation was made between Ali and another student of yours, Ayşe.

> Ayşe: You cannot divide both sides by *x*.
> Ali: If I can divide both sides by 3, why can't I divide by *x*?

At this moment, what could be the most proper explanation for your students?

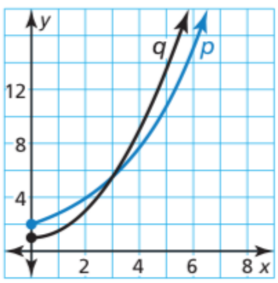
Question 27: In your class, you want to emphasize the symmetrical property of a parabola. In order for your students to understand the symmetrical property, what kind of an example might you use in the class?

Question 28: Task: The graph of the function $f(x) = x^2 - mx + m + 3$ intersects the x-axis at two different points, A and B. Given that the distance between A and B is 3 units, find the sum of the values that m might take?

a. Examine the task given above. Think about your students. Do you think that they could find the solution?
b. If you think that this is a difficult task for the students, how you can modify it to be an easier one.

Question 29: Think about the graph of the function $f(x) = ax^2 + bx + c$. You already know that if $a > 0$, the graph of the function is concave up, and if $a < 0$ the graph of the function is concave down. How can you provide a plausible explanation for this statement?

Question 30: Function p is an exponential function and function q is a quadratic function. One of your students says that after about $x = 3$, q will always have greater y-values than p. Is your student correct? Please explain your answer.



Question 31: Does the vertex of a quadratic function relate to the derivative in any way? If so, how?

Question 32: One of your students wonders if any concept from their physics course is related to quadratic equations and functions. What kind of examples/situations would you provide to him/her?

Question 33: One of your students asked whether and how the golden ratio is related to quadratic equations. How would you respond to this student?

Question 34: One of your students asked whether the graph of $y = x^4$ is a parabola or not. How would you respond? Please explain your answer.

Question 35: One of your students told you:

> "I heard the term 'reflection property of a parabola' while I was watching a documentary on TV last night. It was told that it has many practical uses in real-life. But, I missed the rest of the documentary after this introduction, because of a power cut in my area, and didn't understand the property. Could you explain what this property is and how/why it is useful in real-life?"

How would you respond to this student?

Question 36: One of your students asked if/how a parabola and a hyperbola are related. How would you respond to this question?

Question 37: One of your students asked that she heard something called *the fundamental theorem of algebra*. She wonders what it is and if and how it applies to quadratic polynomials. What would you say to her?

Question 38: Which of the following students is most correct? Why do you think so?

Student 1: The graph of a quadratic function is a parabola.

Student 2: The graph of a quadratic function is called a parabola.

Question 39: What would be a proper definition of a *parabola*? Can you provide alternative definitions, if you think there are more?

Question 40: One of your students thinks that the shape of a uniform flexible chain or rope whose ends are suspended from the same height and sagging under the force of gravity resembles a parabola. How would you respond to this student?

APPENDIX B. SOURCES FOR THE QUESTIONS USED IN THE PRESENT STUDY

The Quadratic Function Concept Questionnaire	
Item Number	**Source**
1, 2, 3, 5, 6, 7, 8, 16, 17, 18, 19, 20, 21, 22, 25, 27, 28, 29, 36, 39.	Prepared by the researcher.
10, 11, 23, 26, 31, 32, 33, 35, 37, 38, 39, 40.	A. K. Erbaş (Personal Communication, October 1, 2019).
15.	Erbaş et al., 2016.
9, 13, 14.	Adapted from Parent, J. S. S., 2015.
12, 24, 34.	Adapted from Bremigan et al., 2011.
4, 30.	Adapted from Larson & Boswell, 2019.
The Quadratic Function Concept Test	
Item Number	**Source**
1, 2, 3, 4, 6, 7, 8.	Prepared by the researcher
5.	Adapted from Larson & Boswell, 2019.
9.	Adapted from Wu, 2016.
10.	Adapted from Bremigan et al., 2011.

APPENDIX C. FOLLOW-UP INTERVIEW

<u>Part I:</u> A review of Can's responses to the quadratic function concept questionnaire:

#1: Could you please find the axis of symmetry of the function $g(x) = -6x^2 + 12x + 5$?

#2: Could you please find the vertex of the function $f(x) = 3x^2 + 9x + 6$?

#3: In the 4th question, you have stated that the values in the first table belong to a linear function, the values in the second table belong to a quadratic function. Could you explain how did you decide it?

#4: In the 10th question, you have written the first function in the vertex form, the second function in the intercept form. Do you use these different algebraic demonstrations in your classroom instruction?

#5: In the 11th question, you wrote that $a > 0$, $b > 0$, and $c < 0$. How did you determine the signs?

#6: In the questionnaire, you did not solve the question 15. Could you examine the question again and think about how it can be solved?

#7: In question 19, you said that you would justify the quadratic formula by drawing the graph. I could not get what you meant. How do you justify the quadratic formula on the graph?

#8: In question 21, you were asked to provide an example of a real-life problem you share with or ask to your students that can be modeled and solved by a quadratic functions. You did not answer. Do you use this kind of problems during your instruction?

#9: In question 24, you have written that the student is right. Could you explain why did you think so?

#10: Could you explain your response to question 25? You said while translating a parabola upwards and downwards, only the ordinate value changes and the ordinate value only affects c.

#11: You said you would use Geogebra to emphasize the symmetrical property of a parabola. How would you do this? What kind of examples can you use?

#12: How can you explain why the graph of a quadratic function is concave down if $a < 0$, and concave up if $a > 0$?

#13: In question 30, you have stated that the quadratic function q will always take greater values than the exponential function p. Could you explain why did you think so?

#14: Is there a relationship between the vertex and derivative?

#15: In question 35, you were asked about reflection property of a parabola. Could you explain what this property is and where it is used in daily life?

#16: In question 33, you were asked to explain (if any) the relationship between the golden ratio and quadratic equations. You did not write anything. Do you have an idea about their relationship?

#17: In question 32, you were asked whether quadratic functions are related to any concept from physics course, you did not answer. Could you give some examples from physics course which might be related to quadratic functions?

#18: In the questionnaire, you were asked to state the fundamental theorem of algebra and its application to quadratic polynomials. You wrote that you have never heard this theorem.

#19: In question 38, you selected the second statement as the most correct? Could you explain why?

#20: You defined a parabola as the graph of a quadratic function. Do you know any alternative definitions?

#21: You did not respond to the last question. Have you ever heard the term catenary?

<u>Part II:</u> A review of Ahmet's responses to the quadratic function concept questionnaire:

#1: Could you please find the axis of symmetry of the function $g(x) = -6x^2 + 12x + 5$?

#2: Could you please find the vertex of the function $f(x) = 3x^2 + 9x + 6$?

#3: In the 10^{th} question, you have written the first function in the vertex form, the second function in the intercept form. Do you use these different algebraic demonstrations in your classroom instruction?

#4: In the eleventh question, you wrote that $a > 0$, $b > 0$, and $c < 0$. How did you determine?

#5: You made an algebraic justification of the quadratic formula. Do you know any geometrical justification?

#6: In question 21, you have written a profit-loss problem as example of real life problems regarding quadratic functions. Do you use this kind of problems as a part of your instruction? If yes, how often do you use?

#7: You stated that completing the square method is your favorite approach to solve quadratic equations and you gave importance on your students' use of this approach. Why do you care so much about completing the square, rather than using the quadratic formula, which might generally be more practical?

#8: In the questionnaire, you explained the change in the coefficients while translating parabolas. Do you tell your students about these changes?

#9: In question 28, you have changed the question by giving the apsis of the midpoint of A and B. Could you explain why?

#10: You wrote that we can use the second derivative to explain the relationship between the leading coefficient and the concavity of the quadratic function. Could you explain how?

#11: You wrote the definitions of a parabola and a hyperbola in your response to the questionnaire. What would you say about these two concepts?

#12: You stated that the reflection property is used in the construction of headlights and satellite dishes. Could you explain what this property is?

#13: You stated that the reflection property is used in the construction of headlights and satellite dishes. Could you explain what this property is?

#14: In question 38, you selected the second statement as the most correct? Could you explain why?

#15: In question 40, you have stated that this shape is a catenary. Could you explain why?

#16: In question 27, you have stated that you define a as the half of the sum of the roots of a quadratic equation. Could you explain how you do this?

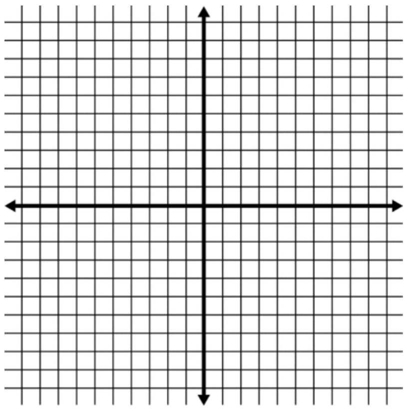
#17: In the 4^{th} question, you have stated that the values in the first table belong to a linear function, the values in the second table belong to a quadratic function. Could you explain how did you decide it?

#18: You solved the quadratic equations by completing the square. Do you use this method in the classroom?

APPENDIX D. THE QUADRATIC FUNCTION CONCEPT TEST

Question #1: For $f(x) = x^2 + 2x - 8$, find the following properties of the function and sketch the graph.

i) x- intercept(s) (ii) y- intercept (iii) vertex (iv) axis of symmetry



2. Given that the vertex of the function $f(x) = x^2 + bx + c$ is T (2,6), what is c?

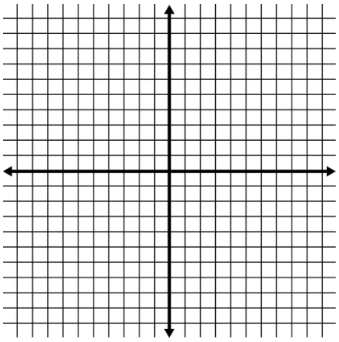
3. Find the minimum value of the function $f(x) = 3x^2 + 6x + 5$.

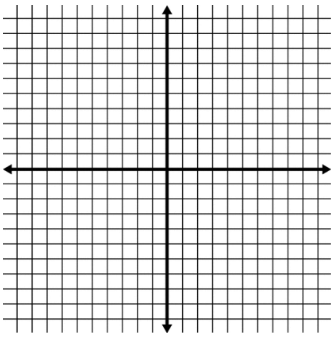
4. Find the maximum value of the function $h(x) = -x^2 + 4x + 6$.

5. Given that $f(x) = 2x^2$, interpret how the graph of f changes when the coefficient of x^2 changes as in the two cases given below:

<u>Case 1:</u> If $f(x) = 2x^2$ becomes x^2

<u>Case 2:</u> If $f(x) = 2x^2$ becomes $3x^2$





6. Find the quadratic function whose vertex is T (-2,2) and passing through A (1,11).

7. Find the quadratic function passing through $A\ (-1,-3)$ and $B\ (2,6)$, with the *y*-intercept $C(0,-4)$.

8. Which of the following is correct for the function $f(x) = x^2 + 5x + 2$ and the line
$y = 3x + 1$? Explain your reason.

a. They are tangent to each other.

b. They intersect at two different points.

c. They never intersects.

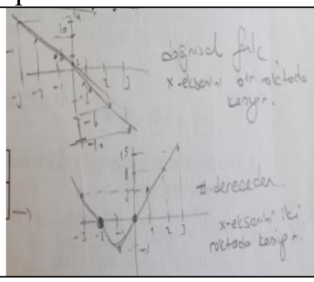
9. The figure represents a rectangular garden whose perimeter is 36 meters. What is the maximum area of this garden?

10. Ali and his friends are playing football. $f(x) = -t^2 + 4t$ represents the height of the ball, h (meters), time, t (seconds), when he hits the ball. So, at what time the reaches 3 meters above the ground?

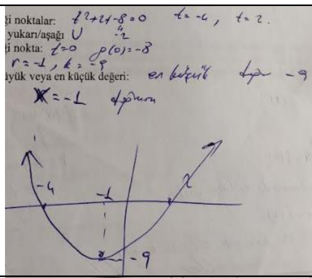
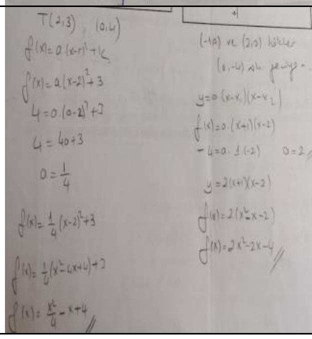
APPENDIX E. CODING SCHEME AND SCORING RUBRIC: THE QUADRATIC FUNCTION CONCEPT QUESTIONNAIRE

	Code	Meaning	Examples
#1	Structural description **(2 points)**	Defining the quadratic equation in its standard form.	$ax^2 + bx + c = 0$, $a, b, c \in \mathbb{R}$ and $a \neq 0$.
	Structural and procedural descriptions **(2 points)**	Defining quadratic equations in its standard form and also referring to quadratic functions.	$ax^2 + bx + c = 0$, $a, b, c \in \mathbb{R}$ and $a \neq 0$. The quadratic equation is a tool for finding the *x*-intercepts of a quadratic function.
	Writing main characteristics of quadratic equations **(1 point)**	Stating some properties of quadratic equations.	-The highest degree of *x* is 2. It is not linear.
	No answer **(0 point)**	-	-
#2	Structural description **(2 points)**	Defining the quadratic function in its standard form.	$f(x) = ax^2 + bx + c, a, b, c \in \mathbb{R}$ and $a \neq 0$.
	Procedural and structural descriptions **(2 points)**	Defining quadratic functions in its standard form and also referring to parabolas.	Quadratic functions are the functions whose graphs generate parabolas.
	Writing main characteristics of quadratic functions **(1 point)**	Defining functions referring to some properties of them.	Function is a relation between two sets. There are one domain and one range.
	No answer **(0 point)**	-	-

#3	Based on their standard forms **(2 points)**	Expressing the differences or similarities based on their standard forms.	Quadratic equation: $ax^2 + bx + c = 0$ Quadratic function: $f(x) = ax^2 + bx + c$ Quadratic polynomial: $p(x) = ax^2 + bx + c$ is also a function.
	Based on their some characteristic **(2 points)**	Expressing the differences or similarities based on their characteristics.	The quadratic function involves some relation between two sets, however, quadratic equations involve equality to a constant.
	Based on their geometrical aspects **(2 points)**	Expressing the differences or similarities based on their geometrical aspects.	As a difference, the graph of a quadratic function is a parabola.
	Incorrect **(0 point)**	Making incorrect explanations.	The graph of a quadratic equation is a quadratic function.
	No answer **(0 point)**	-	-
#4	Examining the second differences **(2 points)**	Examining the first differences and then examining the second differences to be sure that the function is quadratic.	-
	Examining only the first differences **(1 point)**	Calculating the first differences for both cases.	The rate of change is constant, so the first one is linear. In the second one, the rate of change is not constant, thus it is quadratic.

	Using their algebraic forms **(1 point)**	Using algebraic forms of linear functions and finding the coefficients that satisfy the given points.	For the first table: $y = ax + b$ $14 = -3a + b$ $10 = -2a + b$ $a = -4, b = -2.$ $6 = -4.(-1) + 2$ $2 = -4.0 + 2$ $-2 = -4.1 + 2$ $-6 = -4.2 + 2$ $-10 = -4.3 + 2$ For the second table: $3 = -3a + b$ $0 = -2a + b$ $a = -3, b = -6.$ $0 \neq (-3).0 - 6$ Thus it is not linear. It is quadratic.
	Examining their graphs **(1 point)**	Drawing graphs roughly and deciding according to the shape of the graph.	
	No answer **(0 point)**	-	-
#5	Using the quadratic formula **(2 points)**	Solving the quadratic equations by using the quadratic formula.	$3x^2 - 5x + 1 = 0$ $\Delta = b^2 - 4ac$ $= 25 - 4.3.1 = 13.$ $\Delta > 0.$ $x_{1,2} = \frac{-b \mp \sqrt{\Delta}}{2a}$ $x_1 = \frac{5 + \sqrt{13}}{6}$ $x_2 = \frac{5 - \sqrt{13}}{6}$

	Completing the square **(2 points)**	Solving the quadratic equations by completing the square method.	$3x^2 - 5x + 1 = 0$ $x^2 - \frac{5}{3}x + \frac{1}{3} = 0$ $x^2 - \frac{5}{3}x + \left(\frac{5}{6}\right)^2 = \frac{25}{36} - \frac{1}{3}$ $\left(x - \frac{5}{6}\right)^2 = \frac{13}{36}$ $\left\|x - \frac{5}{6}\right\| = \frac{\sqrt{13}}{6}$ $x_1 = \frac{5 + \sqrt{13}}{6}$ $x_2 = \frac{5 - \sqrt{13}}{6}$
#6	Structural description **(2 points)**	Defining what the axis of symmetry is.	It is the line that separates the parabola into two symmetrical parts.
	Procedural description **(1 point)**	Describing how to find the axis of symmetry.	$x = -b/2a$.
	Incorrect **(0 point)**	Writing an incorrect definition.	The apsis of the vertex.
	No answer **(0 point)**	-	-
#7	Structural description **(2 points)**	Defining what the vertex means for a quadratic function.	The vertex is the maximum or the minimum point of a quadratic function.
	Procedural description **(1 point)**	Describing how to find the vertex of a quadratic function.	The apsis of the vertex is $-b/2a$.
	Incorrect **(0 point)**	Writing an incorrect definition.	It is the ordinate of the maximum or the minimum point of a parabola
	No answer **(0 point)**	-	-
#8	Structural description **(2 points)**	Defining the concavity related to the curvature.	A parabola is concave down if it is ∩-shaped; concave up if it is ∪-shaped.

	Procedural description **(1 point)**	Defining concavity related to the sign of the leading coefficient or the second derivative.	-If $a > 0$, f is concave up; if $a < 0$, f is concave down. -If f'' is positive, f is concave up; if f'' is negative, f is concave down.
	Incorrect **(0 point)**	Making incorrect explanations.	It represents the vertex.
	No answer **(0 point)**	-	-
#9	Drawing the correct graph **(2 points)**	Calculating the properties of the quadratic function and drawing its graph correctly.	
#10	Finding the correct quadratic functions **(2 points)**	Finding the correct quadratic functions using different algebraic forms of quadratic functions.	
	No answer **(0 point)**	-	-
#11	Finding the sign of all the coefficients correctly **(2 points)**	Determining the sign of a by examining the concavity of the parabola.	The parabola is upwards, so $a > 0$. a is positive, and the apsis of the vertex $(-b/2a)$ is negative. $b > 0$. The ordinate of the y-intercept is negative. $c < 0$.
	Finding one or two of the coefficients wrongly **(1 point)**	Determining the sign of b by examining the sign of the apsis of the vertex.	$b < 0$ since there are two different roots.
	Incorrect or no answer **(0 point)**	-	-

#12	Describing some of the transformations **(1 point)**	Describing vertical and horizontal translations.	$y = f(x - a)$ and $y = f(x + a)$ are the translations along the x-axis a unit right and left. $y = f(x) - a$ and $y = f(x) + a$ are translations along the y-axis a unit below and above.
	Describing all the transformations **(2 points)**	Describing how to translate, reflect, and stretch or shrink the graphs.	The student is right. $f(x) = a(x - r)^2 + k$ We can first make horizontal and vertical translations. Then, reflect the graph according to the sign of a, then shrink or stretch it.
	No answer **(0 point)**	-	-
#13	Examining the leading coefficients of quadratic functions **(2 points)**	Comparing the absolute values of the leading coefficients of the functions.	The smaller the $\|a\|$ gets, the wider the parabola becomes. The answer is A.
	Incorrect **(0 point)**	Comparing the difference of roots.	-The answer is D because the difference of the roots is the biggest, $x_1 - x_2 = \sqrt{24}$. -The answer is A because b value is the biggest.
	No answer **(0 point)**	-	-
#14	Comparing the transformations made onto $f(x) = x^2$ to obtain the two functions **(2 points)**	Describing translations made onto the parent quadratic function.	$f(x)$ can be obtained by vertical translation of x^2, 5 units below. $g(x)$ can be obtained by horizontal the translation of x^2, 5 units right.
	Comparing some characteristic of the quadratic functions **(1 point)**	Describing their characteristics such as the number of x-intercepts, the vertex, etc.	$f(x)$ intersects the x-axis at two different points. $g(x)$ is tangent to the x-axis.
	Incorrect **(0 point)**	Making incorrect explanations.	$g(x)$ is parallel to the x-axis.
	No answer **(0 point)**	-	-

#15	Using an algebraic model **(2 points)**	Forming a mathematical model with a quadratic function.	The income function *g(x)*= $(5.5 + 0.5x) . (25000 - 1250x)$ x: the number of each 50-cent rise in the price. $r = 4{,}5$ $f(r) = k = 5{,}5 + \frac{1}{2}.4{,}5 = 7{,}75$ TL.
	Using a numerical approach **(1 point)**	Trying numerical values without forming a quadratic model.	$25000.5{,}5 = 137500$ $23750.6 = 142500$ $22500.6{,}5 = 146500$ $21250.7 = 148750$ $\underline{20000.7{,}5 = 150000}$ $\underline{18750.8 = 150000}$ $17500.8{,}5 = 148750$ So, I could suggest the selling price as 7,5 TL.
	No answer **(0 point)**	-	-
#16	Finding the correct quadratic function **(2 points)**	Find the correct quadratic function using the vertex form.	$y = a(x - r)^2 + k$ $5 = a(-1 + 1/4)^2 + 11/4$ $a = 4$.
	No answer **(0 point)**	-	-
#17	Finding the correct quadratic function **(2 points)**	Finding the correct quadratic function using the standard form.	$y = ax^2 + bx + c$ $a + b + c = 9$ $4a - 2b + c = 27$ $16a + 4b + c = 3$.
	No answer **(0 point)**	-	-
#18	Correctly explaining the three conditions for the intersection of a line and a parabola **(2 points)**	Equating two functions, obtaining a quadratic equation and examining its discriminant, and stating three conditions.	$ax^2 + bx + c = mx + n$ $ax^2 + (b - m)x + c - n = 0$. If $\Delta < 0$, they do not intersect. If $\Delta = 0$, the parabola is tangent to the line. If $\Delta > 0$, they intersect at two different points.
	No answer **(0 point)**	-	-
#19	Algebraic and geometrical justifications **(2 points)**	-	-

	Algebraic justification only **(1 point)**	Deriving the quadratic formula by completing the square.	$ax^2 + bx + c = 0$ $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$ $= \frac{b^2}{4a^2} - \frac{c}{a}$ $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$ $\left(x + \frac{b}{2a}\right) = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
	No justification **(0 point)**	Stating the quadratic formula without any justification.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
	No answer **(0 point)**	-	-
#20	Solving by completing the square **(2 points)**	Finding the roots by completing the square method.	$x^2 - x + \frac{1}{4} - \frac{1}{4} + 1 = 0$ $\left(x - \frac{1}{2}\right)^2 = -\frac{3}{4}$ $x_{1,2} = \frac{1 \pm \sqrt{3}i}{2}$.
	Incorrect **(0 point)**	Making incorrect explanations.	The equation can be solved by factorization.
	No answer **(0 point)**	-	-
#21	Writing a problem statement **(2 points)**	Writing a well-defined and solvable real-life problem.	Let the cost of a product be x TL. If the product is sold $x^2 - 5x + 14$ TL, what would be the minimum profit?
	Writing a problem context **(1 point)**	Giving some examples about the use of quadratics in real-life.	The maximum height of a ball thrown by projectile motion can be calculated by using quadratic functions.
	No answer **(0 point)**	-	-

#22	Explaining all the incorrect steps **(2 points)**	Stating that the solution is incorrect and explaining all the errors.	The student is wrong. Since the parabola is downwards, the vertex does not give the minimum. Also, the ordinate of the vertex gives the max/min values, not the apsis of it. The endpoints should also be checked.
	Explaining some of the incorrect steps **(1 point)**	Stating that the solution is incorrect and explaining some of the errors.	The student is wrong. Since the parabola is downwards, the vertex gives the maximum.
	No answer **(0 point)**	-	-
#23	Explaining the student's solution **(2 points)**	Stating that the solution is correct and writing the name of the approach.	The student solved the equation by completing the square. This approach is my favorite while teaching quadratic equations.
	Only stating that the solution is correct **(0 point)**	Stating that the solution is correct without giving the name of the approach.	It is correct.
	No answer **(0 point)**	-	-
#24	Explaining the student's solution **(2 points)**	Stating that the solution is correct explaining the student's method.	The solution is correct. The student went from the final to the beginning. He/she wrote one of the roots equal to *x*, squared the equation, and obtained the quadratic polynomial.
	Only stating that the solution is correct **(0 point)**	Stating that the solution is correct without further information.	The solution is correct.
	Solving the problem using another approach **(0 point)**	Finding the quadratic polynomial using another approach, comparing the solution with the student's.	If one root is $7 + \sqrt{6}$, another is $7 - \sqrt{6}$. The sum of roots: $\frac{-b}{a} = 14$. The multiplication of roots: $\frac{c}{a} = 43$. $a = 4, b = -56, c = 172$. The student is correct.

	No answer **(0 point)**	-	-
#25	Making a correct explanation **(2 points)**	Presenting a plausible response to student's question.	While translating upwards and downwards, the apsis of the vertex does not change. So, the sum of the roots stay constant but the roots change. So, the multiplication of the roots changes. Hence, b stays constant and c changes. While translating left and right, both the sum of the roots and multiplication of the roots change. Hence, b and c change.
	Incorrect **(0 point)**	Making some incorrect explanations	While translating upwards and downwards, the roots do not change. While translating upwards and downwards, roots change. So, everything changes.
	No answer **(0 point)**	-	-
#26	Making a correct explanation **(2 points)**	Presenting a plausible response to student's question.	I would say that an equation cannot be divided by *x*, because we can eliminate one of the roots, which is 0. I would say that an equation cannot be divided by x, because we don't know the value of x. It might be 0. Zero cannot divide any number.
	No answer **(0 point)**	-	-
#27	Writing relevant examples **(2 points)**	Defining *r* (the apsis of the vertex) as the half of the sum of the roots; and emphasizing that the x values sum up to 2r have the same ordinate, meanly they are symmetrical.	I am defining *r* as half of the sum of the roots. I tell my students that the *x*-values sum up to *2r* are symmetrical. For example if $r = 5$, $f(1) = f(9)$ or $f(-5) = f(15)$. I want my students to notice this property.

	Writing irrelevant examples **(0 point)**	Making some irrelevant explanations.	-I demonstrate symmetrical shapes to my students like the shape of a heart.
	No answer **(0 point)**	-	-
#28	Making some reasonable modification	Stating that the question is hard and making some reasonable modifications.	I would change the problem: $f(x) = x^2 - 5x + m - 1$ intersects the x-axis at two different points, A and B. If $\lvert AB \rvert = 3\ br$, what is m?
	Making no modification	Stating that the question is easy to solve and making no modification.	I think it is an easy question for my students. I would not change.
	Making unnecessary/irrelevant modification	Stating that the question is hard and making some irrelevant/unnecessary modifications.	I would give extra information about the sign of the sum of the roots.
	No answer	-	-
#29	Relating the concavity of the graph with the second derivative of quadratic functions **(2 points)**	Explaining the relationship between the second derivative of the quadratic function and the concavity of its graph.	$f''(x)$ is always constant and there is no inflection point.
	Irrelevant explanations **(0 point)**	Making some irrelevant explanations.	I draw different parabolas and show the change in their shapes depending on the sign of a.
	No answer **(0 point)**	-	-
#30	Comparing the graphs of a quadratic function and an exponential function correctly **(2 points)**	Stating that an exponential function grows faster than any quadratic function after some point.	The student is wrong. After some point, an exponential function will increase faster than a quadratic function.
	Incorrect **(0 point)**	Examining their graphs and stating that the quadratic function has always greater y-values than the exponential.	When we look at the graphs, we notice that y-values of the q function is always greater than of the p function after $x = 3$.

	No answer **(0 point)**	-	-
#31	Explaining the relationship between the vertex and the derivative partially **(1 point)**	Stating that the vertex is the point where the first derivative of the function is zero.	The vertex is the point where the first derivative of the function is 0.
	Incorrect **(0 point)**	Making incorrect explanations.	The first derivative of a function can be found by drawing tangent lines passing through the vertex.
	No answer **(0 point)**	-	-
#32	Writing any concept from the physics course related to quadratic functions **(2 points)**	Mentioning some topics from the physics course that are related to quadratic functions.	Free fall, projectile motion.
	No answer **(0 point)**		
#33	Relating golden ratio with quadratic equations **(2 points)**	Stating the golden ratio and its relation to quadratic equations.	The golden ratio is $1 + \frac{\sqrt{5}}{2}$. It is the positive root of the quadratic equation $x^2 - x - 1 = 0$.
	Only stating the golden ratio **(0 point)**	Stating the golden ratio without explaining its relation to quadratics.	$1 + \frac{\sqrt{5}}{2}$.
	No answer **(0 point)**	-	-

#34	Stating that the graph of $y = x^4$ is not a parabola **(2 points)**	Stating that the graph of $y = x^4$ is not a parabola by emphasizing that it is not a quadratic function.	It is not a parabola; because parabolas are the graphs of quadratic functions.
	Incorrect **(0 point)**	Making some incorrect explanations.	-It is a parabola because it is U-shaped. -It is not a parabola because it is so wide. -The arms of the parabolas are narrower.
	No answer **(0 point)**	-	-
#35	Explaining the reflective property and its daily use correctly **(2 points)**	Describing what reflective property is and its daily use.	It is used in real life in the construction of headlights and satellite dishes.
	Incorrect **(0 point)**	Making irrelevant/incorrect explanations.	I would say that the vertex of a parabola is the axis of symmetry.
	No answer **(0 point)**	-	-
#36	Explaining some differences between parabolas and hyperbolas **(2 points)**	Stating that they both are conic sections and explaining their differences.	They are both conic sections. A parabola consists of one curve, a hyperbola consists of two curves. A parabola is a set of points that are equidistant from a straight line and focus. A hyperbola is a set of points whose distances to two fixed points have a constant difference.
	Incorrect **(0 point)**	Making incorrect explanations.	Hyperbola is the symmetry of a parabola. Parabola is $y = ax^2$, hyperbola is $x = ay^2$.
	No answer **(0 point)**	-	-

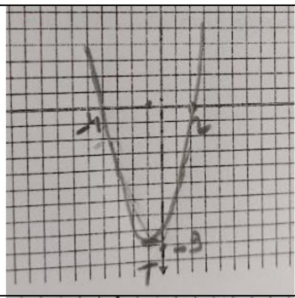
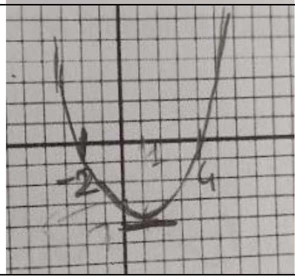
#37	Applying the fundamental theorem of algebra to quadratic polynomials **(2 points)**	Writing the Fundamental Theorem of Algebra with some and explaining its application to quadratic polynomials.	-A polynomial with degree n has n roots. Quadratic polynomials have two roots. -Quadratic equations have 2 roots. If the discriminant<0, it has no roots.
	Incorrect **(0 point)**	Writing irrelevant information.	Demonstrating a quadratic equation by drawing a square.
	No answer **(0 point)**	-	-
#38	Stating that Student 1 is correct with a correct justification **(2 points)**	Choosing the first statement as the most correct and defining a parabola as a conic section.	-
	Stating that Student 1 is correct with incorrect explanation **(0 point)**	Stating that Student 1 is correct by making some incorrect explanations.	Second statement is a definition, but parabola is undefined. So, Student 1 is right.
	Stating that Student 2 is correct with incorrect explanation **(0 point)**	Stating that Student 2 is correct by making some incorrect explanations.	Because it is a definition.
	None **(0 point)**	Stating that both of the arguments are incorrect.	-The graph of a quadratic polynomial function is called a parabola. -The graph of a polynomial function $f(x) = ax^2 + bx + c$ ($a \neq 0$, a, b, c € $\mathbb{R}$) is a parabola.
	No answer **(0 point)**		

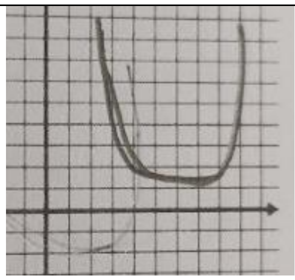
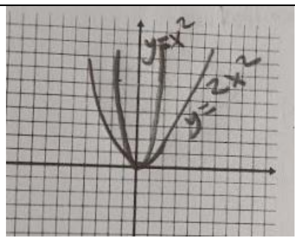
#39	Describing parabola as the graph of a quadratic function and stating geometrical definition of a parabola **(2 points)**	Defining parabola as the graph of a quadratic function and also stating geometrical definition of a parabola.	A parabola is the graph of a quadratic function. Alternative definition: A parabola is defined as the set of points that are equidistant from both the directrix and the focus.
	Describing parabola as the graph of a quadratic function only **(1 point)**	Defining parabola as graph of a quadratic function; not giving any alternative definition.	A parabola is the graph of a quadratic function.
	Incorrect **(0 point)**	Making some incorrect definitions for a parabola.	-A parabola is a quadratic function. -A parabola is the graph of a quadratic equation. -A parabola is a quadratic equation.
	No answer **(0 point)**	-	-
#40	Distinguishing between a parabola and a catenary **(2 points)**	Stating that the shape is not a parabola, it is a catenary.	I would say that it is a catenary.
	Stating that it is not a parabola without explanation **(1 point)**	Stating that the shape is not a parabola but looks like a parabola.	-I would say that it resembles a parabola, but it is not. -I would say that is not a parabola.
	Incorrect **(0 point)**	Stating that the shape is a parabola.	I would say that it is a parabola.
	No answer **(0 point)**	-	-

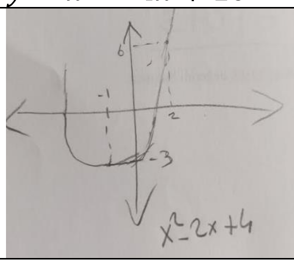
APPENDIX F. CODING SCHEME AND SCORING RUBRIC: THE QUADRATIC FUNCTION CONCEPT TEST

1-a		Code	Meaning	Examples
	Correct	Using completing the square method **(2 points)**	Solving the quadratic equation by completing the square.	$f(x) = (x+1)^2 - 9 = 0$ $x + 1 = 3$ $x + 1 = -3$ $x_1 = 2,\ x_2 = -4.$
	Correct	Using the quadratic formula **(2 points)**	Solving the quadratic equation by using the quadratic formula.	$\Delta = 4 - 4.1.(-8) = 36$ $x_1 = \frac{-2+\sqrt{36}}{2} = 2$ $x_2 = \frac{-2-\sqrt{36}}{2} = -4.$
	Correct	Using factorization **(2 points)**	Solving the quadratic equation by factorization.	$f(x) = x^2 + 2x - 8.$ $f(x) = (x-2)(x+4)$ $x_1 = 2,\ x_2 = -4.$
		No answer **(0 point)**	-	-
1-b	Correct	Writing $(0, y)$ as the y-intercept **(2 points)**	Finding $y = -8$ for $x = 0$, and writing correctly $(0, -8)$ as the y-intercept.	For $x = 0, y = -8$ $(0, -8).$
	Partially correct	Writing y as the y-intercept **(1 point)**	Writing only the ordinate of the y-intercept without any explanation.	-8.
	Incorrect	Writing $y = 0$ as the y-intercept **(0 point)**	Calculating the y value for the apsis of the x-intercepts.	$y = x^2 + 2x - 8$ for $x = -4, y = 0$ for $x = 2, y = 0.$
		No answer **(0 point)**	-	-

1-c	**Correct**	Writing (r, k) by using the vertex form **(2 points)**	Finding r and k by turning the function into vertex form.	$f(x) = (x+1)^2 - 9$ $r = -1, \quad k = -9.$ $(-1, -9).$
		Writing (r, k) by using the formula for finding the vertex **(2 points)**	Calculating r and k values, and writing vertex as (r, k).	$r = -2/2 = -1$ $k = f(r)$=$f(-1) = -9$ $(-1, -9)$
	Incorrect	Writing only the apsis of the vertex (r) as vertex **(0 point)**	Calculating only the apsis of the vertex (r) and writing vertex as r.	$r = -b/2a = -2/2$ $r = -1.$ $-1.$
		Writing the y-intercept as the vertex **(0 point)**	Writing vertex as -8 without further explanation.	-8.
		$f(2)$ as the vertex **(0 point)**	Writing $f(2)$ as the vertex.	$f(2) = 4 + 4 - 8 = 0.$
1-d	**Correct**	$x = r$ as the axis of symmetry **(2 points)**	Writing axis of symmetry as the line $x = r$.	$x = -1.$ $x + 1 = 0.$
	Partially correct	r value as the axis of symmetry **(1 point)**	Writing axis of symmetry as $r = -1$ or -1.	$r = -1.$
	Incorrect	The vertex as the axis of symmetry **(0 point)**	Writing axis of symmetry as the vertex.	$(-1, -9).$
		Other incorrect answers **(0 point)**	Writing some irrelevant numbers as the axis of symmetry without any explanation.	3.
		No answer	-	-

#1-e	**Correct**	Sketching the accurate graph **(2 points)**	Replacing all the elements of the function correctly on the graph.	
	Incorrect	Sketching inaccurate graphs **(0 point)**	Sketching the graph inaccurately.	
		No answer **(0 point)**	-	-
#2	**Correct**	Using the formula for *r* and the equation $f(r) = k$. **(10 points)**	Calculating firstly *r* by using $-b/2a$, then using the $f(r) = k$, finding *c*.	$-\frac{b}{2a} = -\frac{b}{2} = 2$ $b = -4.$ $f(r) = k, f(2) = 6$ $22 - 8 + c = 6\ c = 10.$
		Using the vertex form **(10 points)**	Rewriting the quadratic function in the vertex form and finding the coefficients.	$f(x) = a(x - r)^2 + k$ $f(x) = (x - 2)2 + 6$ $= x^2 - 4x + 10, c = 10.$
	Incorrect	Other incorrect answers **(0 point)**	Skipping some steps and finding an incorrect result	$-b/2a = 2$ $-b = 4a$ $x^2 - 4ax + c$ $c = 6.$
		No answer **(0 point)**	-	-
#3	**Correct**	k as the minimum **(10 points)**	Calculating *r* by using the formula $-b/2a$ and then finding *k* by using $f(r) = k$.	$r = -b/2a = -6/6 = -1.$ $k = f(-1) = 3 - 6 + 5 = 2$
	Incorrect	The *y*-intercept as the minimum **(0 point)**	Finding the *y*-intercept as minimum	$f(0) = 3.0 + 6.0 + 5$ $= 5.$

		Δ as the minimum **(0 point)**	Calculating the discriminant as the minimum	$\Delta = b^2 - 4ac$ $= 36 - 60 = -24.$
		Other incorrect answers **(0 point)**	Trying to find the *x*-intercepts as the minimum	$(3x + 3).(x + 2) = 0$ $x = 3, \quad x = 2.$
#4	**Correct**	*k* as the maximum **(10 points)**	Calculating *r* by –*b/2a* and then finding k by using the equation $h(r) = k$.	$r = -\frac{b}{2a} = 2$ $h(2) = -2^2 + 4.2 + 6$ $k = -4 + 8 + 6 = 10.$
	Incorrect	$h(1)$ as the maximum **(0 point)**	Replacing *x* with 1 in the function and calculating $h(1)$ as the maximum.	$-1^2 + 4 + 6 = 9.$
		No answer **(0 point)**	-	-
#5	**Correct**	Stating that the parabola becomes wider, if $\|a\|$ gets smaller. **(10 points)**	Writing that the parabola becomes larger, if $\|a\|$ gets smaller and sketching proper graphs for two cases.	(for case 1): The arms of the parabola would move away from the *y*-axis. (for case 2): The arms would get closer to the *y*-axis.
	Incorrect	Making translations **(0 point)**	Changing the graph by making horizontal or vertical translations.	
		Inverse application of the rule **(0 point)**	Sketching the graph of $y = x^2$ narrower than the graph of $y = 2x^2$ and sketching the graph of $y = 3x^2$ wider than the graph of $y = 2x^2$.	
		Other incorrect answers **(0 point)**	Writing some other irrelevant/incorrect statements.	(for case 1): the value of *x* increases. (for case 2) the value of *x* decreases.
		No answer **(0 point)**	-	-

#6	**Correct**	Using the vertex form **(10 points)**	Using the vertex form to find the quadratic function.	$f(x) = a(x-r)^2 + k$ $f(x) = a(x+2)^2 + 2$ $f(1) = 11, 9a + 2 = 11$ $a = 1$ $f(x) = (x+2)2 + 2.$
	Incorrect	Using the intercept form **(0 point)**	Using the intercept form and finding an incorrect function.	$y = a(x-x_1)(x-x_2)$ $y = a(x-1)(x+2)$ $y = -2a$ $y = -2(x-1)(x+2).$
		Using the standard form **(0 point)**	Using the standard form and finding an incorrect function.	$f(x) = ax^2 + bx + c.$ $c = 11$ $f(-2) = 2$ $4a - 2b + 11=2$ $4a - 2b = -9$ $r = -b/2a \quad b = 4a$ $4a - 8a = -9$ $a = 9/4.$
		No answer **(0 point)**	-	-
#7	**Correct**	Standard form **(10 points)**	Using the standard form and finding the quadratic function.	$f(x) = ax^2 + bx + c$ $f(x) = ax^2 + bx - 4.$
	Incorrect	Intercept form **(0 point)**	Using the intercept form and finding an incorrect function.	$y = a(x-x_1)(x-x_2)$ $y = 4(x+1)(x-2)$ $= 4x^2 - 4x - 8.$
		Vertex form **(0 point)**	Using the vertex form finding an incorrect function.	$y = a(x-r)^2 + k$ $y = a(x-2)^2 + 6$ $y = x^2 - 4x + 10$
		Others **(0 point)**	Drawing a graph and writing an incorrect quadratic function.	
		No answer **(0 point)**	-	-

#8	**Correct**	Investigating the discriminant of the quadratic equation **(10 points)**	Equating two functions and obtaining a quadratic equation; then investigating Δ of the common equation.	$x^2 + 5x + 2 = 3x + 1$ $x^2 + 2x + 1 = 0$ $\Delta = b^2 - 4ac$ $= 4 - 4.1.1 = 0$ Tangent.
		Noticing that the new equation is a perfect square **(10 points)**	Stating that the quadratic equation is a perfect square and has one root, so they are tangent to each other.	$x^2 + 5x + 2 = 3x + 1$ $x^2 + 2x + 1 = 0$ (perfect square) $x_1 = x_2 = 1$. One point of intersection, they are tangent.
	Incorrect	Investigating the discriminant of the quadratic equation $f(x)$=0 **(0 point)**	Calculating Δ of the given quadratic equation.	$f(x) = x^2 + 5x + 2 = 0$ $\Delta = b2 - 4ac$ $= 25 - 4.1.2 = 17$ $\Delta > 0$.
		Other incorrect answers **(0 point)**	Writing some other irrelevant/incorrect statements.	They do not intersect because the line $y = 3x + 1$ is not a quadratic line.
		No answer **(0 point)**	-	-
#9	**Correct**	Using a quadratic model **(10 points)**	Forming a quadratic function for the area of the rectangle and calculating its maximum.	$A = x(18 - x)$ $A = -x^2 + 18x$ $r = -18/-2 = 9$ $k = f(9) =$ $-9^2 + 18.9$ $= 81$.
	Partially correct	Using a numerical approach **(5 points)**	Finding the maximum area by trying some numerical values for dimensions of the rectangle.	$a + b = 18$ 1.17 2.18 . . $8.10 = 80$ $9.9 = 81$ (the maximum)

	Incorrect	Using a numerical approach **(0 point)**	Trying to find the maximum area by trying some numerical values for dimensions of the rectangle and finding an incorrect result.	$a + b = 18$ 1.17 2.18 . . 8.10 = 80 (the maximum) Dimensions cannot be 9x9, the shape would be a square then.
		Other incorrect answers **(0 point)**	Assigning 3k for long side, 2k for the short side and calculating k; then finding the dimensions and calculating the area.	$2(3k + 2k) = 36$ $10k = 36 \quad k = 3{,}6$. $a = 3k = 10{,}8$ $b = 2k = 7{,}2$ $10{,}8 \,.\, 7{,}2 = 77{,}76$.
		No answer **(0 point)**	-	-
#10	**Correct**	Finding two values of t at $h(t) = 3$ **(10 points)**	Equating the function $h(t)$ to 3, and finding two roots of the quadratic equation.	$h(t) = -t^2 + 4t$ $3 = -t^2 + 4t$ $0=-t^2+4t+3$ $t = 1$ and $t = 3$. 1st and 3rd seconds.
	Partially correct	Finding one of the values of t at $h(t) = 3$ **(5 points)**	Equating the function $h(t)$ to 3, and finding one of the roots of the quadratic equation.	$h(t) = -t^2 + 4t$ $3 = -t^2 + 4t$ $3 = t(-t + 4)$ $t = 3$. (3rd second).
	Incorrect	Finding $h(3)$ **(0 point)**	Finding the value of $h(t)$ at $t = 3$, $h(3)$.	$t = 3$ $h(3) = -3^2 + 4.3 = 3$ 3rd second.
		Finding the maximum of the function **(0 point)**	Finding the apsis of the vertex, *r*, and calculating h(r).	$-b/2a = -4/-2 = 2$ $h(2) = \ -22 + 4.2$ $= 4$.
		No answer **(0 point)**	-	-

APPENDIX G. ETHICAL PERMISSION

Kayıt Tarihi: 05.11.2019	Protokol No: 652

07/11/2019

T.C

BÜLENT ECEVİT ÜNİVERSİTESİ

İNSAN ARAŞTIRMALARI ETİK KURULU KARARI

ÇALIŞMANIN TÜRÜ:	Anket
BAŞLIK:	Lise Matematik Öğretmenlerinin İkinci Dereceden Fonksiyon Kavramına Yönelik Alan Bilgisi ve Bunun Öğrencilerin Öğrenme Çıktıları İle İlişkisi
SORUMLU ARAŞTIRMACI:	Ayhan Kürşat Erbaş
KARAR:	Uygun

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1. Karacı-Yaşa, G., & Karataş, İ. (2018). Effects of the instruction with mathematical modeling on pre-service mathematics teachers' mathematical modeling performance. *Australian Journal of Teacher Education, 43*(8), 1-14.

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