SECONDARY MATHEMATICS TEACHERS' SUBJECT MATTER KNOWLEDGE OF QUADRATIC FUNCTIONS AND ITS CONTRIBUTION TO STUDENT LEARNING OUTCOMES

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

GÜLZADE KARACI YAŞA

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
SECONDARY SCIENCE AND MATHEMATICS EDUCATION

AUGUST 2022

Approval of the thesis:

SECONDARY MATHEMATICS TEACHERS' SUBJECT MATTER KNOWLEDGE OF QUADRATIC FUNCTIONS AND ITS CONTRIBUTION TO STUDENT LEARNING OUTCOMES

submitted by **GÜLZADE KARACI YAŞA** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Secondary Science and Mathematics Education Department, Middle East Technical University** by,

Prof. Dr. Halil Kalıpçılar
Dean, Graduate School of **Natural and Applied Sciences** _________________

Prof. Dr. Erdinç Çakıroğlu
Head of the Department, **Mathematics and Science Education** _________________

Prof. Dr. Ayhan Kürşat Erbaş
Supervisor, **Mathematics and Science Education, METU** _________________

Examining Committee Members:

Assoc. Prof. Dr. Bülent Çetinkaya
Mathematics and Science Education, METU _________________

Prof. Dr. Ayhan Kürşat Erbaş
Mathematics and Science Education, METU _________________

Assist. Prof. Dr. Şerife Sevinç
Mathematics and Science Education, METU _________________

Assist. Prof. Dr. Zeynep Sonay Ay
Mathematics and Science Education, Hacettepe Uni. _________________

Assist. Prof. Dr. Emine Gül Çelebi İlhan
Mathematics and Science Education, TED Uni. _________________

Date: 31.08.2022

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name Last Name: Gülzade Karacı Yaşa

Signature:

ABSTRACT

SECONDARY MATHEMATICS TEACHERS' SUBJECT MATTER KNOWLEDGE OF QUADRATIC FUNCTIONS AND ITS CONTRIBUTION TO STUDENT LEARNING OUTCOMES

Karacı Yaşa, Gülzade
Doctor of Philosophy, Department of Secondary Science and Mathematics Education
Supervisor: Prof. Dr. Ayhan Kürşat Erbaş

August 2022, 242 pages

The purpose of the present study was to investigate secondary mathematics teachers' subject matter knowledge (SMK) of quadratic functions and its contribution to teachers' instructional practice and student learning outcomes. The study was carried out in two stages. In the first stage, a questionnaire was administered to 18 secondary mathematics teachers to identify their SMK of quadratic functions, which has three sub-components: common content knowledge (CCK), specialized content knowledge (SCK), and horizon content knowledge (HCK). Two case studies were conducted to investigate the contribution of teachers' SMK to student learning outcomes regarding quadratic functions. Two teachers who were voluntary for further investigation were selected among 18 teachers based on their questionnaire results. Both teachers were interviewed before they started their instructional practice on the topic of quadratic functions. Observations were also made during their instructional practice on quadratic functions. Finally, the students of both teachers were administered a test to evaluate their performance on quadratic functions.

The result showed that the majority of teachers' CCK was stronger than their SCK and HCK regarding quadratic functions. The findings also indicated that teachers'

HCK regarding quadratic functions was fairly limited. Teachers have limited knowledge of associating quadratic functions with other content in the high school curriculum and any other concepts from advanced mathematics. Moreover, the data suggested evidence that teachers' SMK of quadratic functions contributed to student learning outcomes. Teachers' SMK affected their instructional practices, and their instructional practices contribute to student performance on quadratic functions.

Keywords: Secondary Mathematics Teachers, Subject Matter Knowledge, Quadratic Functions, Student Learning Outcomes.

ÖZ

LİSE MATEMATİK ÖĞRETMENLERİNİN İKİNCİ DERECEDEN FONKSİYONLARA YÖNELİK KONU ALAN BİLGİSİ VE BUNUN ÖĞRENCİLERİN ÖĞRENME ÇIKTILARINA KATKISI

Karacı Yaşa, Gülzade
Doktora, Orta Öğretim Fen ve Matematik Alanları Eğitimi Bölümü
Tez Yöneticisi: Prof. Dr. Ayhan Kürşat Erbaş

Ağustos 2022, 242 sayfa

Bu çalışmanın amacı lise matematik öğretmenlerinin ikinci dereceden fonksiyonlara yönelik konu alan bilgisini ve bunun öğrencilerin öğrenme çıktılarına olan katkısını incelemektir. Araştırma iki aşamada yürütülmüştür. İlk aşamada, öğretmenlerin ikinci dereceden fonksiyonlar kavramına ilişkin konu alan bilgisini, genel alan bilgisi, uzmanlık alan bilgisi ve ufuk alan bilgisi alt boyutları kapsamında değerlendirmek için 18 lise matematik öğretmenine açık uçlu bir ölçek uygulanmıştır. Öğretmenlerin ikinci dereceden fonksiyonlara yönelik konu alan bilgisi ile bunun öğrencilerin öğrenme çıktılarına olan katkısını incelemek amacıyla, iki ayrı durum çalışması yürütülmüştür. Bu amaçla, araştırmanın ilk aşamasına katılan 18 öğretmen arasından anket sonuçlarına göre iki öğretmen seçilmiştir. İkinci dereceden fonksiyonlar konusundaki öğretimlerine başlamadan önce bu iki öğretmenle görüşme yapılmıştır. Ayrıca, öğretmenler ikinci dereceden fonksiyonlar konusunu öğretirken araştırmacı tarafından sınıflarında gözlem yapılmıştır. Konunun öğretimi bittiğinde, iki öğretmenin sınıfındaki öğrencilere ikinci dereceler fonksiyonlarla ilgili açık uçlu bir test uygulanmıştır.

Araştırma sonucunda, öğretmenlerin büyük bir çoğunluğunun ikinci dereceden fonksiyonlara yönelik genel alan bilgilerinin uzmanlık alan bilgilerinden ve ufuk alan bilgilerinden daha iyi olduğu görülmüştür. Ayrıca, öğretmenlerin ikinci dereceden fonksiyonlara yönelik ufuk alan bilgilerinin oldukça sınırlı olduğu görülmüştür. Öğretmenler ikinci dereceden fonksiyon kavramının lise matematik öğretim programında ve ileri matematikteki diğer kavramlarla ilişkisini anlama konusunda sınırlı bilgiye sahiptir. Bunun yanında, araştırma sonuçları öğretmenlerin ikinci dereceden fonksiyonlara yönelik konu alan bilgilerinin öğrencilerin bu konudaki öğrenme çıktılarına katkısı olduğunu göstermiştir. Öğretmenlerin ikinci dereceden fonksiyonlara yönelik alan bilgileri öğretmenlerin sınıftaki öğretimlerini etkilemiştir. Öğretmenlerin sınıftaki öğretimleri de öğrencilerin öğrenme çıktılarını etkilemiştir.

Anahtar Kelimeler: Lise Matematik Öğretmenleri, Konu Alan Bilgisi, İkinci Dereceden Fonksiyonlar, Öğrencilerin Öğrenme Çıktıları.

To my lovely daughter

ACKNOWLEDGMENTS

Firstly, I would like to present my deepest gratitude to my supervisor, Prof. Dr. Ayhan Kürşat Erbaş, for his guidance, boundless patience, encouragement, and valuable advice throughout my dissertation study.

I would like to express my thanks to the jury members of my dissertation, Assist. Prof. Dr. Şerife Sevinç, Assist. Prof. Dr. Zeynep Sonay Ay, Assoc. Prof. Dr. Bülent Çetinkaya, and Assist. Prof. Dr. Emine Gül Çelebi İlhan for their time, and valuable comments which helped strengthen my dissertation.

I am also grateful to all participants in my dissertation study, especially two teachers, Can and Ahmet, and their students.

Last but not least, I would like to express my special and grateful thanks to my precious parents Sunay Karacı and Nevzat Karacı, my husband Muhammed Yaşa, and my little daughter Duru, for their endless love and lifelong support.

TABLE OF CONTENTS

LIST OF TABLES

TABLES

LIST OF FIGURES

FIGURES

CHAPTER 1

INTRODUCTION

Functions are one of the most important concepts in the field of mathematics (Cooney & Wilson, 1993; Zaslavsky, 1997). Understanding higher-level mathematical concepts is impossible without grasping the function concept (Dreyfuss, 1991). Burns-Childers and Vidakovic (2018) emphasized the importance of functions as a pre-requisite to learning calculus. While learning functions, students first encounter linear functions in middle school and then quadratic functions in later grades. Quadratic functions are seemed to be one of the most conceptually challenging contents in the secondary mathematics curriculum (Zaslavsky, 1997) and play a crucial role in the transition from linear functions to higher-degree functions (Movshovitzs-Hadar, 1993; Parent, 2015). Students must develop a deep conceptual understanding of all functions to perform well in the mathematics course (Cooney et al., 2010; Thompson & Carlson, 2017). There exists a body of research that investigated students' understanding of quadratic functions (Duarte, 2010; Eraslan, 2005; Metcalf, 2007; Parent, 2015). Despite quadratic functions being one of the most critical topics in the mathematics curriculum, learning quadratic functions might be challenging for many secondary students (Kotsopoulos, 2007; Metcalf, 2007).

Pre-service and practicing teachers also have some difficulties understanding quadratic functions and thus teaching the concept to their students (Bansilal et al., 2014; Even, 1990; Sibuyi, 2012). It is essential for teachers to gain a complete understanding of the mathematical concepts before they started to teach (Mutambara et al., 2019). An extensive body of research confirmed that teachers have an essential role in students' learning (Gençtürk, 2012; Hatisaru, 2013; Hatisaru, & Erbaş, 2017; Ibeawuchi, 2010; Mewborn, 2001, 2003; Shechtman et al., 2010). That is, teachers' knowledge is a significant predictor of students' achievement (Baumert et al., 2010; Hill et al., 2004). Since there is an important relationship between teacher knowledge and student achievement, many researchers investigated teacher knowledge (Shulman, 1986, Grossman, 1990, Ball et al., 2008). Educators have been searching for an answer to the question "what do teachers need to know and be able to do to teach effectively, or, what is required for effective teaching in terms of content understanding?" to increase the quality of teaching and learning. (Ball et al. 2008, p. 394).

Researchers have developed several frameworks to discuss the content of teachers' knowledge (Ball et al., 2008; Fennema & Franke, 1992; Shulman, 1986). One of the most critical works on teacher knowledge was presented by Shulman (1986). In his work, Shulman (1986) specified three domains of teachers' content knowledge: pedagogical content knowledge (PCK), curricular knowledge (CK), and subject matter knowledge (SMK). PCK was described as "the particular form of content knowledge that embodies the aspects of content most germane to its teachability" (p.9). According to Shulman (1987), PCK is a component of teachers' knowledge that distinguishes teachers from other professionals. CK was defined as "the understanding of curricular alternatives available for instruction" (p.10). On the other hand, SMK was defined as "the amount of an organization of knowledge per se in the mind of teacher" (p. 9).

In the past decades, teachers' SMK was evaluated quantitatively and described as the number of courses taken in university or the scores taken from standardized tests (Wilson et al., 1987). However, there are some problems with this quantitative description since it does not give adequate information about the quality of teachers' SMK. As Monk (1994) stated, the quantity of advanced mathematics training may not enhance the quality of teaching; the mathematical understanding that anyone possesses does not guarantee an ability to improve others' mathematical understanding. Furthermore, Even (1990) noted that defining SMK in terms of what it means to know mathematics can enhance teachers' quality of subject matter preparation and, as a result, the quality of instruction. As Krauss et al. (2008) noted, SMK is essential; but it is insufficient for effective mathematics teaching. Thus, other models were developed to identify what a mathematics teacher must know for effective teaching (Ball et al., 2008).

Ball et al. (2008) extended Shulman's (1986) categorization of teacher knowledge and proposed a new framework called mathematical knowledge for teaching (MKT). Ball et al. (2008) defined MKT as "the mathematical knowledge needed to carry out the work of teaching mathematics" (p. 395). According to the new framework, unlike Shulman's (1986), PCK was separated into three sub-components which are knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of the curriculum. SMK was also separated into three sub-components which are common content knowledge (CCK), specialized content knowledge (SCK), and horizon content knowledge (HCK). CCK is "the mathematical knowledge and skill used in settings other than teaching" (Ball et al., 2008, p. 399). In contrast, SCK is defined as "the mathematical knowledge and skill unique to teaching" (Ball et al., 2008, p. 400). Lastly, HCK is "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p. 403).

1.1. Statement of the Problem

Research confirms that teachers' SMK is critical for student learning (Tchoshanov et al., 2017). Teachers should have a grasp of the content that they would teach and learn how to teach it (Ma'rufi, 2016). "Teachers who have not mastered their subject well, of course, does not have the knowledge needed to help students learn the content." (Ma'rufi, 2016, p. 399). However, having high-quality content knowledge did not assure the quality of teaching (Ball, 1991; Ma, 1999). Rich SMK together with the knowledge of learners' conceptions, preconceptions and misconceptions, help to prevent the continued underachievement in learners' performance in mathematics. Teachers should thoroughly understand the subject they are teaching to analyze and evaluate students' ideas, help students develop and formalize intuitive understandings, and discover and correct students' mistakes (Ball, 1990).

Research also confirmed that SMK considerably affects the models and approaches teachers use while teaching mathematical concepts (Ball, 1990; Even, 1990, 1993), and thus student learning (Tchoshanov et al., 2017). A variety of research has attempted to identify the relationships among teachers' mathematics knowledge, their instructional practices, and student learning outcomes (Mewborn, 2003). However, there is a lack of understanding of how teachers' knowledge influences student learning outcomes and how teachers' instructional practice mediates the effects of teacher knowledge on student learning (Silverman & Thompson, 2008; Graeber & Tirosh, 2008). The present study investigated secondary mathematics teachers' SMK of quadratic functions and its contribution to teachers' instructional practice and thus, student learning outcomes. This study used Ball et al.'s (2008) model, which divided SMK into three sub-components: CCK, SCK, and HCK. In this context, the study was guided by two main research questions:

1. As regarding to quadratic functions, what SMK do secondary mathematics teachers have?
   a) As regarding to quadratic functions, what CCK do secondary mathematics teachers have?
   b) As regarding to quadratic functions, what SCK do secondary mathematics teachers have?
   c) As regarding to quadratic functions, what HCK do secondary mathematics teachers have?
2. How do CCK, SCK, and HCK contribute to the instructional practice and, thus student learning outcomes regarding quadratic functions?

1.2. Significance of the Study

The present study contributes to the field in several ways. First, there seems to be an agreement that strong SMK is a key element of teacher competence (NCTM, 2000). To prepare a productive learning environment for the students, teachers should have strong SMK that helps enhance students' mathematical understandings (Even, 1993). Moreover, a teacher with such knowledge can give details to students, direct students to several questions, and associate the subject with other areas (NCTM, 2000). Despite its importance, many prospective and practicing mathematics teachers lack confidence in their mathematical content knowledge (Askew, 2008). Mosvold and Fauskanger (2014) reported that teachers are more interested in the content they taught rather than the larger discipline and have an incomplete understanding of the association between the concepts they taught and advanced mathematics. "To make more effective decisions in both policy and practice, we need to understand better how teachers effectively draw on knowledge in teaching, what kinds of knowledge seem most important for teachers to use, and how to assess this knowledge" (Ferrini-Mundy et al., p.11). This study focused on the SMK of Ball et al.'s (2008) model, which has three sub-components: CCK, SCK, and HCK. Teachers' CCK, SCK, and HCK of quadratic functions were attempted to be measured and evaluated. Thus, the findings of this study help to clarify these three domains of teacher knowledge in the context of quadratic functions.

Secondly, this study examined teachers' SMK of quadratic functions. Quadratic functions are one of the most important subjects for students to understand higher mathematical ideas (Parent, 2015). However, several studies reported various difficulties or misconceptions that learners have regarding quadratic functions and equations (Ellis & Grinstead, 2008; Ibeawuchi & Ngoepe, 2012; Kotsopoulos, 2007; Makonye & Nhlanhla, 2014). Despite the importance of quadratic functions and equations in the history of mathematics and worldwide secondary mathematics curricula, there is limited research about these concepts (Vaiyavutjamai et al., 2005). Thus, investigating teachers' knowledge of quadratic functions might contribute to the literature to fill this gap. Moreover, using a subject-specific framework for identifying teacher knowledge will “give researchers and teachers a window into the ways in which teacher knowledge influences the work that they do with students” (Steele & Rogers, 2012, p. 178). Furthermore, content-specific studies, similar to the studies of those focusing on functions or geometry (Hatisaru & Erbas, 2017; Steele, 2013; Steele et al., 2013; Taşdan & Koyunkaya, 2017) help to provide an elaborate description of the relationship between knowledge and practice. The MKT framework, that “links knowledge, teaching practice, and students’ learning” (Speer et al., 2015, p. 120), may be more appropriate for the secondary level as a guide if a specific mathematical content is investigated to understand MKT.

Lastly, this study investigated the contribution of teachers’ SMK of quadratic functions to student learning outcomes. Research has confirmed that the mathematical knowledge of teachers “close” to teaching has a positive effect on student achievement (Baumert et al., 2010; Hill et al., 2005). “Does this mean that teachers should not learn mathematics on a more advanced level than what they teach?” (Jakobsen et al., 2012, p. 4636). As Jakobsen et al. (2012) stated, the answer to this question is obviously no. Advanced mathematics should be related to teaching at school (Jakobsen et al., 2012). However, there is no clear agreement on what and how much knowledge teachers need to facilitate student learning (Frome et al., 2005; Hatisaru & Erbaş, 2017). The empirical evidence regarding this relationship is inconsistent (Campbell et al., 2014). Thus, the present study’s findings would provide evidence for the contribution of teacher knowledge to student learning outcomes.

1.3. Definition of Important Terms

The following terms are commonly used in the present study.

Mathematical knowledge for teaching (MKT): MKT is a multidimensional construct that represents the professional knowledge of mathematics needed by teachers. (Ball & Bass, 2000). It was defined as "the mathematical knowledge needed to carry out the work of teaching mathematics" (Ball et al., 2008, p. 395).

Subject matter knowledge (SMK): Shulman (1986) defined SMK as "the amount and organization of knowledge per se in the mind of the teacher" (p. 9). Teachers should possess the knowledge of facts and procedures of concepts with the reasoning underlying them (Shulman, 1986). In this study, SMK includes common content knowledge (CCK), specialized content knowledge (SCK), and horizon content knowledge (HCK) (Ball et al., 2008).

Common content knowledge (CCK): CCK is "the mathematical knowledge and skill used in settings other than teaching" (Ball et al., 2008, p. 399).

Specialized content knowledge (SCK): SCK is "the mathematical knowledge and skill unique to teaching" (Ball et al., 2008, p. 400).

Horizon content knowledge (HCK): HCK is "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p. 403).

Pedagogical content knowledge (PCK): PCK is defined as "a special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding" (p. 8). It includes "an understanding of what makes specific topics easy or difficult for a certain group of learners" (Shulman, 1986, p. 9).

Student learning outcomes: Learning outcomes include a variety of student behavior and attitudes as well as cognitive indices (Guskey, 2002). In addition to students' scores on standardized tests, quizzes, and achievement tests, they include "students' attendance, their involvement in class sessions, their classroom behavior, their motivation for learning, and their attitudes toward school, the class, and themselves" (Guskey, 2002, p. 384). In this study, cognitive learning outcomes are focused on what students will know, be able to do, or demonstrate after completing instructional units on quadratic functions. In Turkey, at the end of the instructional unit on quadratic functions in the 11th grade, the students should be able to: find the vertex, *x*-intercept(s) and the *y*-intercept, and axis of symmetry; associate the vertex with the maximum or minimum value of the function; comment on the effect of the change in the coefficients of the function on the graph of the function; find the quadratic function whose two points such that one of them is the vertex, or three points such that one of them is on the *y*-axis are given; investigate the intersection of a line and a parabola; solve the problems which can be modeled by quadratic functions (Milli Eğitim Bakanlığı [MEB], 2018).

Procedural description: An explanation of processes or actions rather than about objects (Sfard, 1987). For example, defining the concept of "symmetry" as a transformation can be referred to as a procedural description (Sfard, 1987).

Structural description: An explanation of mathematical objects or products of some processes (Sfard, 1987). For example, defining the concept of "symmetry" as a static property of geometry can be referred to as a structural description (Sfard, 1987).

Instructional Practice: There are various dimensions of secondary mathematics teachers' instructional practices as suggested in the literature such as tasks (content, difficulty level, etc.), learning environment (social/intellectual climate, instructional routines, etc.) and discourse (teacher-student interaction, questioning, etc.) (e.g., see Artzt & Armour-Thomas, 1996). In the current study, teachers' instructional practice is examined based on the key components of CCK, SCK, and HCK as the focus of this study: the examples that teachers use, the use of multiple representations, making connections among mathematical concepts, making justifications of formulas, posing real-life problems, responding to students' why questions, analyzing the students' solutions, modifying the tasks while teaching quadratic functions.

CHAPTER 2

REVIEW OF LITERATURE

The present study investigates secondary mathematics teachers' SMK of quadratic functions and its contribution to student learning outcomes regarding this content. This section discusses a review of research about teacher knowledge, students' understanding of quadratic functions, teachers' knowledge of quadratic functions, and the interrelation between teacher knowledge and student learning.

2.1. Teacher Knowledge

Throughout history, researchers have developed several frameworks to define and distinguish the components of teacher knowledge (Carter, 1990; Elbaz, 1983; Shulman, 1986). In the early decades, Carter (1990) defined teachers' knowledge as the total knowledge which underlies their actions. However, this does not mean that all the knowledge teachers possess play a role in their actions. (Verloop et al., 2001).

Researchers attempted to identify domains of teacher knowledge (Cochran et al., 1993; Elbaz, 1983; Grossman, 1990; Shulman, 1986). Elbaz (1983) introduced five components of teacher knowledge as knowledge of curriculum, knowledge of students, knowledge of instruction, knowledge of the subject matter, and knowledge of self. Shulman (1986) brought a new perspective on teacher knowledge, emphasizing the importance of teachers' ability to integrate the understanding of subject matter and pedagogical skills. He proposed three domains for teacher knowledge: subject matter knowledge (SMK), pedagogical content knowledge (PCK), and curricular knowledge (CK). Shulman (1986) defined SMK as the "amount or organization of knowledge per se in the mind of the teacher" (p. 9).

Shulman (1986) noted that knowing facts and procedures about a subject would not be enough; teachers should also grasp why it is so. Teachers should provide evidence for the facts, the importance of the facts for learning, and the connections to other disciplines based on theory and practice (Shulman, 1986). The second component of teacher knowledge, PCK, was defined by Shulman (1987) as “a special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding” (p. 8). PCK allows teachers to effectively teach learners using various analogies, representations, illustrations, explanations, examples, and demonstrations. PCK also includes the knowledge of students’ preconceptions and misconceptions depending on their ages and backgrounds (Shulman, 1986). Curricular knowledge was defined as “a particular grasp of the materials and programs that serve as "tools of the trade" for teachers” (Shulman, 1987, p. 8).

After Shulman’s (1986) categorization, educational researchers have identified several components of teachers’ knowledge (Cochran et al., 1993; Grossman, 1990; Magnusson et al., 1999). Grossman (1990) divided teacher knowledge into four components—subject matter knowledge, general pedagogical knowledge, pedagogical content knowledge, and knowledge of context. Grossman’s (1990) SMK is related to the content teachers know and present in their classrooms. The general pedagogical knowledge is about learners and learning, curriculum and instruction, classroom management, and some other pedagogical concerns about teaching and learning. Another category, the knowledge of context, comprises knowledge about the school, such as its culture, the characteristics of the district in which it is located, and the structure of students’ families. Grossman’s (1990) categorization of PCK includes four core components: knowledge of students’ understanding, purposes for teaching, curriculum, and instructional strategies. Unlike Shulman (1986), who proposed curricular knowledge as a separate component, Grossman (1990) included curricular knowledge as a part of PCK.

Cochran et al. (1993) modified Shulman's (1986) categorization of teacher knowledge from a constructivist point of view. They proposed using the word "knowing" rather than the word "knowledge" when referring to PCK because "knowing" denoted the development of a process associated with a constructive approach. Cochran et al. (1993) defined pedagogical content knowing (PCKg), which was at the center of their model, as “a teacher’s integrated understanding of four components of pedagogy, subject matter content, student characteristics, and the environmental context of learning” (p. 266). Compared to Shulman’s (1986) framework, Cochran et al. (1993) emphasized teachers’ knowledge of students and the environmental context in which learning occurs. Cochran et al. (1993) noted that the contribution or impact of four components of PCKg might change over time. Thus, they recommended that teacher education programs should offer opportunities for pre-service teachers to introduce these components and improve their knowledge.

Based on the frameworks for teacher knowledge outlined above (Cochran et al., 1993; Grossman, 1990; Shulman, 1986), researchers in mathematics education proposed models for teacher knowledge, particularly in mathematics. Fennema and Franke (1992) built on Shulman’s work and suggested that the knowledge required for teaching should be interactive and dynamic. They proposed a model which indicates the dimensions of teacher knowledge as knowledge of mathematics, pedagogical knowledge, knowledge of learners’ cognition, and beliefs of teachers (see Figure 2.1). Mathematics content knowledge is understanding concepts, techniques, and problem-solving procedures (Fennema & Franke, 1992). This content knowledge necessitates conceptual comprehension, awareness of their relationships, and understanding of how to apply concepts and methods in mathematical contexts. Similar to Shulman’s (1986) definition of PCK, pedagogical knowledge is the knowledge of pedagogical concerns for teaching, such as classroom management, classroom organization, and methods and procedures for planning. Knowledge of learners’ cognition can be defined as the knowledge of learners’ thinking, learning, potential difficulties, and successes (Fennema & Franke, 1992).

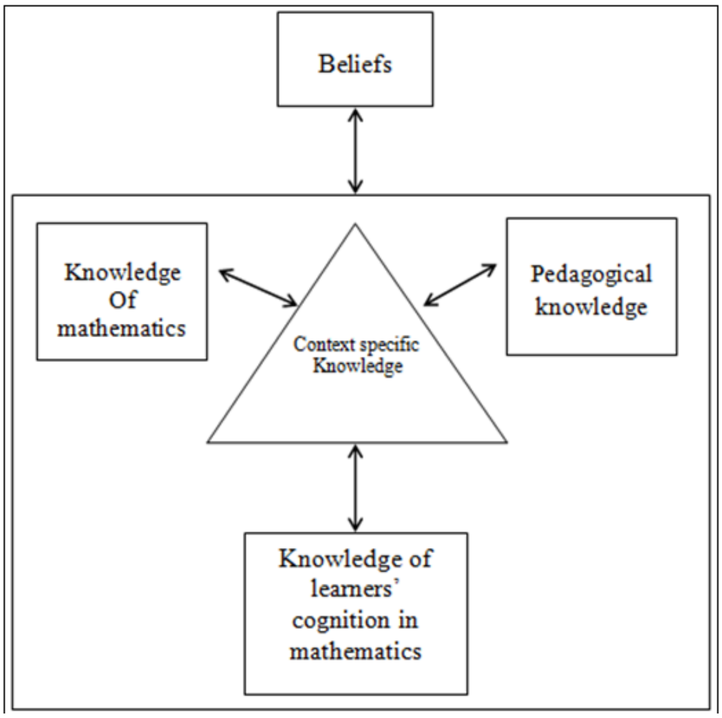


Figure 2.1. Teachers' knowledge: Developing in context (Fennema & Franke, 1992, p.162)

Another model for teachers' mathematics knowledge is named "Knowledge Quartet," which was suggested by Rowland et al. (2005). The researchers observed pre-service primary teachers' lessons based on their lesson plans and conducted research with them. Their goal was to uncover mathematics knowledge in pre-service teachers and mathematics knowledge in teaching. The model consists of the following four categories: foundation, transformation, connection, and contingency. Foundation includes mathematical knowledge for teaching, mathematical theoretical knowledge, and beliefs about this knowledge. The transformation includes teachers' transformation of content knowledge into teaching based on Shulman's (1987) definition. Connection is about teachers' decisions for planning and carrying out lessons. Contingency includes teachers' knowledge of how to deal with unexpected scenarios that arise in the classroom. This category covers teachers' feedback on students' questions or comments, readjusting the lessons, including unforeseen situations to the lesson plan, and using student-emergent ideas to enhance students' learning. The inclusion of the last category, contingency, differentiates this model from other models mentioned previously.

More recently, Ball et al. (2008) proposed a framework that identified mathematical knowledge needed by teachers for mathematics teaching and called it Mathematical Knowledge for Teaching (MKT), which is briefly discussed in the next section.

2.1.1. Mathematical Knowledge for Teaching

By refining the ideas of Shulman (1987), Ball and her colleagues (2008) created a new domain called Mathematical Knowledge for Teaching. They defined MKT as "the mathematical knowledge needed to carry out the work of teaching mathematics" (Ball et al., 2008, p.395). Although Ball and her colleagues (2008) developed the MKT framework for conceptualizing the knowledge needed by elementary mathematics teachers and never recommended generalizing this framework to other grade levels or content areas, some researchers have used this framework for understanding secondary mathematics teachers' knowledge and its relation to student learning at the secondary level (Hatisaru & Erbaş, 2017; Herbst & Kosko, 2014; Howell et al., 2016; Steele, 2013; Steele et al., 2013; Steele & Rogers, 2012).

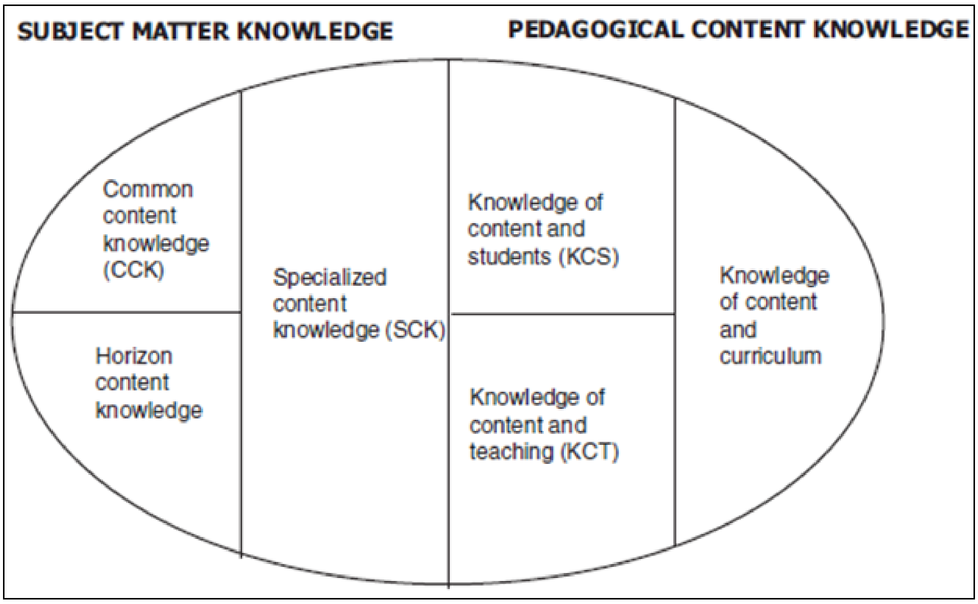


Figure 2.2. Domain map for mathematical knowledge for teaching (Ball et al., 2008, p. 403)

As shown in Figure 2.2, MKT was firstly divided into two components: SMK and PCK. PCK has three components: Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT), and Knowledge of Content and Curriculum (KCC). KCS is "the knowledge that combines knowing about students and knowing about mathematics" (Ball et al., 2008, p. 401). Ball et al. (2008) explain KCS with giving several examples. To illustrate, "when choosing an example, teachers need to predict what students will find interesting and motivating, or when assigning a task, teachers need to anticipate what students are likely to do with it and whether they will find it easy or hard" (p. 401). KCS is subject knowledge linked with knowledge about how students think about, know, or learn this specific content (Hill et al., 2008). The second component, KCT, was defined as "the knowledge that combines knowing about teaching and knowing about mathematics" (Ball et al., 2008, p. 401). As Hill et al. (2008) stated, KCT is about teaching moves considering "how to build on students' thinking or how to address and remedy student errors effectively" (p. 378). Finally, KCC is the curricular knowledge needed for teaching.

SMK was divided into three components as common content knowledge (CCK), specialized content knowledge (SCK), and horizon content knowledge (HCK). CCK is what Shulman likely meant by his original SMK (Hill et al., 2008). It was defined as "the knowledge that is used in the work of teaching in ways *in common with* how it is used in many other professions or occupations that also use mathematics" (Hill et al., 2008, p. 377). Hill and Ball (2004) exemplified CCK as: "…being able to compute $35x25$ accurately, identifying what power of 10 is equal to 1, solving word problems satisfactorily, and so forth" (p. 333). Ball et al. (2008) also illustrated a mathematics teacher's CCK as: "simply calculating an answer or, more generally, correctly solving mathematics problems, using terms and notation correctly writing on the board" (p. 399). In addition to these definitions, Sosa (2010, as cited in Carreño et al., 2013) identified key components of CCK:

- using definitions, rules, properties, and theorems regarding a specific topic
- using the mathematical notation
- understanding the importance of an item
- knowing how to apply mathematics
- doing demonstrations.

Based on the above components proposed by Sosa (2011), Girit (2016) also identified three components of CCK as the knowledge of:

- definitions, rules, properties, and theorems about a specific content
- using terms and notation correctly
- simply calculating an answer or solving mathematical problems correctly.

To sum up, teachers require this type of knowledge, and while other professions may benefit from it as well, it is an essential component of the knowledge of mathematics teachers as specialists (Carreño et al., 2013).

While CCK corresponds to Shulman's original SMK, SCK is a new concept (Hill et al., 2008). It was defined as "the mathematical knowledge that allows a teacher to engage in particular teaching tasks, including how to represent mathematical ideas accurately, provide explanations for common rules and procedures, and examine and understand unusual solution methods to problems" (Hill et al., 2008, pp. 377-378). Ball and Bass (2009) distinguished SCK from pure knowledge of the content by examining a multiplication task presented in Figure 2.3. In the multiplication tasks, it is easy to identify that the results are wrong and the result is 1225. However, a teacher should understand the reason for these errors by anticipating the students' thoughts in solving the task.

(a)	(b)	(c)
49	49	49
× 25	× 25	× 25
405	225	1250
108	100	25
1485	325	1275

Figure 2.3. Illustration of SCK in two-digit multiplication task (Ball & Bass, 2009)

Ball and Bass (2009) stated that many competent teachers are able to identify the source of the error by looking over the numerical solution. However, for some mathematically trained professionals (including mathematicians), it could be challenging. This kind of knowledge is "specialized in particular for the work of teaching" (Ball and Bass, 2009, p.3). This is why they called this *specialized* content knowledge. While searching for patterns in student errors, teachers must do some kind of mathematical work that others do not (Ball et al. 2008). Ball et al. (2008) presented a list of teaching tasks unique to this special work (see Table 2.1).

Table 2.1. Mathematical tasks of teaching (Ball et al., 2008, p. 400)

Mathematical Tasks of Teaching
Presenting mathematical ideas
Responding to students' "why" questions
Finding an example to make a specific mathematical point
Recognizing what is involved in using a particular representation
Linking representations to underlying ideas and other representations
Connecting a topic being taught to topics from prior or future years
Explaining mathematical goals and purposes to parents
Appraising and adapting the mathematical content of textbooks
Modifying tasks to be either easier or harder
Evaluating the plausibility of students' claims (often quickly)
Giving or evaluating mathematical explanations
Choosing and developing useable definitions
Using mathematical notation and language and critiquing its use
Asking productive mathematical questions
Selecting representations for particular purposes
Inspecting equivalencies

Researchers worked on the components of SCK to clarify this domain special to mathematics teaching. Sosa (2010, as cited in Carreño et al., 2013) identified some key components of SCK as follows:

- understanding the importance of concepts
- comprehending the unseen stages behind procedures
- intuiting the cause of students' mathematical errors.

Researchers have continued defining and distinguishing SCK from other teacher knowledge domains. Bair and Rich (2011) stated that the teachers themselves must have a rich and connected understanding of the mathematical concepts and connections among them to make their students grasp mathematical concepts. They proposed a framework for the development of SCK in algebraic reasoning and number sense. Bair and Rich (2011) determined four components as central to the development of SCK, related to the teacher's ability to:

- solve problems and justify their reasoning.
- use multiple representations.
- recognize, use, and generalize conceptually similar tasks.
- pose problems.

According to Harel (2008), teachers' mathematical way of thinking should also be included as an essential component of their SCK. Tallman and Frank (2018) also supported this idea of Harel (2008) in their research in which they examined the role of quantitative reasoning on the quality of a secondary teacher's teaching. More recently, Girit (2016) identified the main components of SCK as the knowledge of:

- relating a current topic to previous or future years' topics
- connecting representations to underlying concepts and other representations
- selecting/providing applicable definitions or explanations
- explaining/justifying one's mathematical ideas
- using mathematical language
- providing mathematical explanations for commonly used procedures and rules
- effectively choosing, creating, and using mathematical representations.

HCK is "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (p. 403). There are various interpretations of HCK by many researchers. Ball and Bass (2009) defined HCK as "an awareness- more as an experienced and appreciative tourist than as a tour guide- of the large mathematical landscape in which the present experience and instruction are situated" (p. 6). It involves those areas of mathematics that may not be covered in the curriculum, but are still beneficial to students' current learning (Ball & Bass, 2009). Ball and Bass (2009) determined four elements of teachers' HCK:

- a sense of the mathematical environment surrounding the current location in the instruction
- major disciplinary ideas and structures
- key mathematical practices
- core mathematical values and sensibilities.

Sosa (2010, as cited in Carreño et al., 2013) identified key components of HCK as the knowledge of:

- relationships between general and specific content
- interdisciplinary applications.

Zazkis and Mamolo (2011) stated that identifying HCK as a distinct category requires a more detailed description of it. They proposed a broader description of HCK as "advanced mathematical knowledge applied to ideas in the elementary or secondary curriculum, i.e., mathematical terms and their interrelations, structures, and key ideas of the discipline implemented in the curricula of elementary or secondary school" (Zazkis & Mamolo, 2011, p.4). Zazkis and Mamolo (2011) also differentiated learners' horizons from teachers' horizons and stated that Ball and Bass (2009) focused on the learners' horizons by emphasizing the learners' mathematical futures. They identified teachers' horizons related to their knowledge of advanced mathematics.

Researchers have worked to clarify the scope of teachers' HCK. Jakobsen et al. (2012) suggested a more detailed definition for HCK as "an orientation to and familiarity with the discipline that contributes to the teaching of the school subject at hand, providing teachers with a sense of how the content being taught is situated in and connected to the broader disciplinary territory" (p. 4642). Guberman and Gorev (2015) identified three components of HCK. These components are mathematical insight, mathematical connections, and understanding of meta-mathematics. Since the description of HCK might cause some confusion, Wasserman and Stockton

(2013) divided HCK into two parts: *a curricular mathematical horizon* that includes the knowledge of what mathematics is to come in the higher grades and an *advanced mathematical horizon* that includes the knowledge of connections with higher-level mathematical concepts.

Nyikahadzoi (2015) reported that in the context of functions, HCK includes the knowledge of how functions are related to other contents such as sequences, mappings, transformations, and determinants of matrices. More recently, Cho and Tee (2018) explained and described the construct of HCK. They stated that HCK not only includes a form of elementary perspective on advanced mathematics but also a higher perspective on elementary mathematics. This means that teachers with rich mathematical knowledge should transform this knowledge into pedagogically practical forms. Moreover, Cho and Tee (2018) reported that HCK can be viewed as a reciprocal bridge connecting the advanced and elementary levels of mathematical knowledge. As Ball et al. (2008) reported, a teacher who owns HCK has “peripheral vision” and they are aware of the questions to enhance understanding of mathematical proofs, when to support learning, to be patient, allowing the student to study on the problem individually. HCK is the least developed and understood domain among the other domains of the MKT framework (Jakobsen, 2014). Although empirical evidence for the existence of the six sub-domains of MKT was provided by Ball et al. (2008), and further studies defined or clarified most of these sub-domains, the explanations of what HCK embedded in MKT meant were limited.

2.2. Studies Using the MKT Model to Measure Secondary Teachers’ Knowledge

Steele (2013) measured teachers’ MKT for teaching geometry and measurement. For this purpose, Steele (2013) designed some tasks which directly measured teachers’ CCK and SCK. According to the study, teachers’ CCK was evaluated based on their ability to: calculate the perimeter and area of shapes; understand the association among lengths, perimeter, and area of shapes. Teachers’ SCK was measured by the tasks which evaluate their ability to: be aware of some constraints and affordances of different formulas of length, perimeter, and area; have representational fluency (moving between different representations such as symbolic, graphical/pictorial, tabular) while describing the interrelation between area, perimeter, and length; identify important aspects of the interrelation among length, perimeter, and area for students' learning; specify mathematical tasks for enhancing students' understandings of area, length, and perimeter. The results indicated that teachers with strong SCK are more likely to develop observable pedagogical practices that help improve students' understanding of the interrelations between length, perimeter, and area. The study also found some connections between teachers' CCK and SCK that previous research has not completely established. For example, teachers who clearly described the relationships between perimeter, length, and area on a given task (an indicator of CCK) were more likely to utilize multiple representations (an indicator of SCK) in their response to the same task. The result also suggested some evidence of the interrelation between teachers' SCK, CCK, and their ability to unpack mathematical goals for a mathematics lesson. That is to say, teachers who had better mathematical performances on a given task (minimizing the perimeter) could write more specific goals for students to use that task.

Steele et al. (2013) investigated the development of pre-service and in-service mathematics teachers' MKT for teaching functions through a methods course. They designed a pre-test and a post-test to evaluate their MKT. Their instrument included items that assessed teachers' CCK and SCK. They identified the indicators of CCK for functions to be able to state the definition of a function and write examples and non-examples of functions. The indicators of SCK for teaching functions were specified as their ability to evaluate different definitions of a function, consider their usefulness for teaching, and move between different representations of functions. The findings indicated that the content-focused method course improved teachers' CCK and SCK for teaching functions. Studies of this tradition reveal that courses can be designed to influence teachers' MKT and that pre-service and in-service teachers can both have opportunities to improve their CCK and SCK.

Howell et al. (2016) investigated the utility of the MKT framework at the secondary level. To extend the MKT framework to the secondary level, Howell et al. (2016) designed assessment items that search for evidence of MKT at the secondary level. Data were collected through think-aloud interviews from 23 pre-service and in-service teachers. Results showed that the items measured aspects of MKT such as SCK that extend beyond conventional mathematics knowledge. This validation study indicated that MKT can be extended to the secondary level.

Similarly, Herbst and Kosko (2014) reported a successful pilot of 34 tasks measuring secondary teachers' MKT for geometry. Their focus was on four of the six MKT subdomains, KCT, KCS, CCK, and SCK. The findings of the pilot study indicated a relationship between the specified MKT subdomains and teachers' years of experience. However, Herbst and Kosko (2014) noted that the initial findings are promising, but more testing is needed to understand the differences among teachers and how teachers possibly struggle with different domains of MKT.

Taşdan and Koyunkaya (2017) evaluated teacher knowledge, four of the six MKT subdomains. They focused on the MKT for a specific secondary content area, functions. Taşdan and Koyunkaya (2017) identified the CCK, SCK, KCS, and KCT subdomains regarding functions. HCK and KCC were not included in the study because the pre-service teachers lack real classroom experiences. Evaluating the pre-service teachers' design and implementation of a lesson plan regarding teaching functions, the result indicated that these pre-service teachers have limited knowledge of teaching functions, and they need more experience to develop all of the MKT subdomains.

Miheso-O'Connor Khakasa and Berger (2016) examined teachers' interpretations of students' unusual problem solving strategies to identify their MKT proficiency status. 117 practicing secondary mathematics teachers with different years of experience participated in the study. Data were collected via the MKT task questionnaire and an opinion questionnaire. Then, 14 teachers who participated in the lesson studies were interviewed. The result showed that mathematics teachers had partially fluent MKT proficiency in the secondary school level. The study also found that teachers had difficulty with concepts that were not included in the curriculum. The findings revealed that teachers were uncomfortable with engaging in responses that require KCS, HCK or SCK. The study also reported that teachers have limited knowledge of when and how to use HCK, which is related to the advanced mathematical knowledge.

Zembat (2013) examined the gap between the mathematics teachers' knowledge of mathematics and the ideal mathematical understanding for teaching, focusing on the SCK of mathematics teachers. The participants were 142 mathematics teachers from grade 1-12. A questionnaire including open-ended and multiple-choice items was administered to 142 mathematics teachers from grade 1-12. The result showed that teachers have quite limited understanding of the core mathematical ideas, analyzing the students' work, in the assessment of understanding mathematical ideas, and making curricular decisions. The study suggested that teachers should improve their SCK to fill the gap between where they are and where they need to be.

2.3. Quadratic Equations and Quadratic Functions in School Mathematics

In school mathematics, function is one of the most complicated and important topics (Pihlap, 2017). The quadratic function is a special and important case of the function. A complete understanding of the quadratic function is essential for students since it is later used in advanced mathematics, particularly while learning polynomial functions (Parent, 2015). Quadratic functions form a bridge between mathematical concepts, including linear functions, functions, and polynomials, as they have an important role in the transition from linear functions to higher degree functions (Movshovitzs-

Hadar, 1993; Sağlam & Alacacı, 2012). Quadratic functions are most commonly defined in their standard forms as $f(x) = ax^2 + bx + c$ where a, b, c are real numbers with $a \neq 0$ (Nielsen, 2015). They can also be written in the intercept form as $f(x) = a(x - x_1).(x - x_2)$ where x_1 and x_2 are the apsis of the x-intercepts; and the vertex form $f(x) = a(x - r)^2 + k$ where (r, k) is the vertex (Parent, 2015). Learners also engage in quadratic equations that result from setting a quadratic expression equal to a constant (often zero) while working with quadratic functions (Nielsen, 2015).

Zaslavsky (1997) identified components that are indicatives of understanding quadratic functions as: common algebraic forms of quadratic functions, connections between the x-intercepts of a parabola, the condition for determining the location of the x-intercepts of a parabola, the condition for determining the number of x-intercepts, the condition for determining the location of the y-intercept of a parabola, the condition for determining the type of concavity of a parabola, symmetrical properties of a parabola, extreme values of a quadratic function, connection to a linear function, and special cases of pairs of quadratic functions. Similarly, Parent (2015) identified the core content which would be focused on when dealing with quadratic functions. These are (1) axis of symmetry, (2) vertex, (3) graph orientation, (4) y-intercept, (5) graph transformations, (6) maximum/minimum point, and (7) location of roots (Parent, 2015). Metcalf (2007) also identified the objectives of the algebra curriculum regarding quadratic functions and equations for high school students in more detail. According to Metcalf (2007), the students will be able to:

- recognize $y = ax^2 + bx + c$ as the standard form of quadratic functions
- use several methods for solving quadratic equations and finding the x-intercepts of a quadratic function, such as the quadratic formula, completing the square, and factoring
- notice that the quadratic formula originated from completing the square
- graph a quadratic function by making a list of values

- notice that the *y*-intercept is the point at which the graph touches the *y*-axis and where $x = 0$
- notice that the *x*-intercepts are the places where the graph touches the *x*-axis, as well as where $y = 0$, and are the roots of quadratic equations
- be aware that the terms "*x*-intercept," "real root," "solution," and "zero of the polynomial" are equivalent
- notice that the graph of a quadratic function is a parabola
- know that the coefficient "a" in $y = ax^2 + bx + c$ determines the graph orientation (up or down)
- notice that the values in a table belonging to a quadratic function are ordered pairs on a graph as well as solutions to $f(x) = ax^2 + bx + c$
- notice that the graphical representations of solutions to quadratic equations are the *x*-intercepts.

In Turkey, students first recognize quadratic equations in 10th grade and quadratic functions in 11th grade. In Grade 10, at the end of the instructional unit on quadratic equations with one unknown, students will be able to:

- explain the concept of a quadratic equation with one unknown
- solve quadratic equations with one unknown
- explain that a complex number is denoted by $a + ib$ such that $a, b \in \mathbb{R}$
- make operations using the relationship between the roots and coefficients of a quadratic equation (Milli Eğitim Bakanlığı, 2018).

In grade 11, at the end of instructional units on quadratic functions, students will be able to:

(1) draw and interpret the graph of a quadratic function with one variable:

- find the vertex, *x* and *y*-intercepts, and axis of symmetry
- associate the vertex with the maximum or minimum value of the function

- comment on the effect of the change in the coefficients of the function on the graph of the function by using technology
- find the quadratic function whose two points such that one of them is the vertex or three points such that one of them is on the y-axis are given
- investigate the intersection of a line and a parabola.

(2) solve problems which can be modeled by quadratic functions (Milli Eğitim Bakanlığı, 2018).

According to the Common Core State Standards for Mathematics (CCSSM), which is a document including two sections as standards for mathematical practice and standards for mathematical content, the content on quadratic functions includes: solving quadratic equations with various methods, using multiple representations (graphical, tabular, and symbolic), comparing the properties of quadratic functions with other types of functions; and modeling real or natural patterns with quadratic functions as well as solving realistic problems by using quadratic functions (Common Core State Standards Initiative [CCSSI], 2010).

2.4. CCK, SCK, and HCK for Teaching Quadratic Functions

Categorizing a particular knowledge as unique to the mathematics domain or as unique to mathematics teaching might be difficult (Flores et al., 2013). This view coincides with Ball et al. (2008)'s argument that "it can be difficult to discern common from specialized knowledge in particular cases" (p. 403). Furthermore, the boundaries between CCK and SCK are not completely clear; what constitutes SMK may differ across teachers at different levels (Speer & Wagner, 2009) and across different countries (Delaney et al. 2008). However, these domains of knowledge are more easily distinguished when they are described clearly in terms of what having this knowledge enables a teacher to do. After an in-depth review of the literature, what CCK, SCK, and HCK mean regarding quadratic functions and equations is described in the following paragraphs of this section.

Ball et al. (2008) described a mathematics teacher's CCK as: "simply calculating an answer or, more generally, correctly solving mathematics problems, using terms and notation correctly writing on the board" (p. 399). Several researchers identified key components of CCK for a specific content (Girit, 2016; Steele, 2013). Furthermore, the key components of the concept of the quadratic function is identified by several researchers (Metcalf, 2007; Parent, 2015, Zaslavsky, 1997). Also, Turkish mathematics curriculum clearly defined the objectives that the students should understand at the end of the unit on quadratic functions (MEB, 2018). Based on the aforementioned definitions and descriptions of CCK, and the examples from the previous research together with an analysis of national and international contexts that identify key components of quadratic functions, these seven codes are framed for CCK for teaching quadratic functions in the scope of the present study:

- CCK1: Conception of quadratic equations/functions
- CCK2: Knowledge of solving quadratic equations with one unknown
- CCK3: Knowledge of sketching and interpreting the graphs of quadratic functions
- CCK4: Knowledge of graphing quadratic functions using transformations
- CCK5: Knowledge of solving real-life problems regarding quadratic functions
- CCK6: Knowledge of finding the quadratic function with given points
- CCK7: Knowledge of finding the intersection of a parabola and a line.

Since "conception of quadratic functions and equations" is a broad term, it might be useful to describe what it means in the current study. CCK1 includes the knowledge of defining a quadratic function and a quadratic equation, explaining the relationship among quadratic function, quadratic equation and quadratic polynomial, and distinguishing between linear functions and quadratic functions.

The second sub-component of teachers' SMK is teachers' SCK of quadratic functions. SCK is the knowledge that allows the teacher to engage in tasks specialized to teaching (Ball et al., 2004). More precisely, professions outside teaching do not need SCK, which is unique to teaching (Ball et al. 2008). Researchers have proposed several indicators of a teacher's SCK, i.e., understanding the importance of concepts, comprehending the unseen stages behind procedures, and intuiting the cause of students' mathematical errors (Bair & Rich, 2011; Ball & Bass, 2009; Ball et al., 2008; Girit, 2016; Hill et al., 2008). Based on the aforementioned definitions and descriptions of SCK, examples from the previous research, and the work on quadratic functions, these seven codes are framed for SCK for teaching quadratic functions in the current study:

- SCK1: Knowledge of explaining and justifying basic formulas of quadratic functions
- SCK2: Knowledge of posing problems regarding quadratic functions
- SCK3: Knowledge of recognizing students' incorrect solutions regarding quadratic functions
- SCK4: Knowledge of understanding students' unusual solutions regarding quadratic functions
- SCK5: Knowledge of responding to students' why questions about quadratic functions
- SCK6: Knowledge of finding an example to make a specific mathematical point about quadratic functions
- SCK7: Knowledge of modifying tasks regarding quadratic functions.

Finally, HCK is defined as the knowledge of how mathematical topics are related across the span of mathematics; it also involves the insight to see relations to higher mathematical ideas (Ball et al., 2008). Several researchers identified main components of HCK (Guberman & Gorev, 2015; Nyikahadzoi, 2015; Wasserman & Stockton, 2013). Based on the aforementioned definitions and descriptions of HCK, examples from the previous research, and the work on quadratic functions, these two codes are framed for HCK for teaching quadratic functions in the scope of the present study:

- HCK1: Knowledge of how quadratic functions are related to other contents in the high school curriculum
- HCK2: Knowledge of how quadratic functions are related to advanced mathematics.

Although there might be a relationship between HCK and KCC, HCK is independent from the curriculum (Fernández & Figueiras, 2014). HCK is not only an awareness of how mathematical topics are related over the span of mathematics included in the curriculum but it also refers to the "global knowledge of the evolution of the mathematical content and the relationship among its different areas needed for the teaching practice" (Fernández & Figueiras, 2014, p. 12). Sosa (2010, as cited in Carreno et al., 2013) defined KCC as "content in textbooks and the relation of previous and forthcoming mathematical topics". KCC is mostly about "an understanding of school mathematics and particular approaches to organizing the school curriculum" (Jakobsen et al., 2013, p. 1). In Ball et al.'s (2008) definition of HCK as "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum", the term "related" does not mean the curricular development of the content; it means some other kinds of connections that might exist between concepts (Jakobsen et al., 2013). It is important to note that there is a distinction between the sub-component HCK1 defined in this study and KCC. In this study, HCK1 includes the knowledge of the connections between mathematical concepts rather than the curricular order of the topics.

2.5. Students' Understanding of Quadratic Functions and Quadratic Equations

Despite their importance in mathematics education, students have a variety of difficulties or misconceptions in learning both quadratic functions and quadratic equations (Ellis & Grinstead, 2008; Ibeawuchi & Ngoepe, 2012; Kotsopoulos, 2007). Especially, moving between algebraic and graphical representations of a function is a struggle for many students (Baki & Güveli, 2008).

Some studies have focused on students' understanding and difficulties in solving quadratic equations. Vaiyavutjamai and Clements (2006) investigated high school students' understanding of quadratic equations. In particular, they investigated the effect of the instruction on students' performance in solving quadratic equations. The data were collected via students' written responses before and after the instruction. The result showed that students' performance in solving quadratic equations was improved. However, most students still had difficulty understanding the concept of a variable and a solution to a quadratic equation.

Kotsopoulos (2007) reported that factoring a quadratic equation is challenging for many high school students. Kotsopoulos (2007) also noted that when the students encounter a quadratic equation in a non-standard form, i.e., $x^2 + 3x + 1 = x + 4$, they struggle with related tasks. Likewise, Didiş et al. (2011) examined high school students' solution approaches for solving quadratic equations with one unknown. The data were collected from 113 tenth-grade students via an open-ended test. The findings indicated that factoring quadratic equations was difficult for them, especially when they encountered quadratic equations in a different structure than they were used to. Furthermore, despite knowing some procedures for solving quadratic equations, students used these rules without considering why they did so or whether what they were doing was mathematically accurate. The result also showed that students' understanding of solving quadratic equations was instrumental (or procedural) rather than relational (or conceptual).

Didiş and Erbaş (2015) examined 10th-grade students' performance in solving quadratic equations with one unknown, which were presented in two forms as symbolic equations or word problem representations. The data were collected from 217 tenth-grade students through an open-ended questionnaire, including symbolic equations and word problems. The findings indicated that students struggled to solve word problems and symbolic equations. However, their performance in solving symbolic equations was better than in solving word problems. The study also reported that students' failure in solving symbolic problems mainly stemmed from algebraic manipulation and arithmetical errors. Regarding the word problems, students had difficulty understanding the context; thus, they could not establish the quadratic equation needed to find the solution. These findings showed that the structure of the problems as a symbolic equation or a word problem had a considerable effect on students' performance in solving quadratic equations.

Makonye and Matuku (2016) explored 11th-grade students' misconceptions and errors while solving quadratic equations. The data were collected via quadratic equation tasks, including solving quadratic equations by completing the square, factorization, or using the quadratic formula. After that, six participants were interviewed for further investigation. The result showed that the students had algebraic incompetence, leading them to make errors while solving quadratic equations. Another study that focused on learners' misconceptions and difficulties regarding quadratic equations was conducted by Mazhindu (2016). The participants were 249 high school students from different schools. Data were collected via a questionnaire and interviews with selected participants. The main misconceptions and errors obtained from this study were: overgeneralizing the property of the zero product, linearizing quadratic equations, utilizing methods they have learned from linear equations for solving quadratic equations, and drawing quadratic graphs as straight lines.

Another body of research has focused on high school students' understanding of quadratic functions and their misconceptions regarding this content. Ellis and Grinstead (2008) investigated high school students' generalizations regarding the association between graphical and algebraic representations of quadratic functions. In particular, they focused on the roles of the coefficients a, b, and c in the standard form of a quadratic function, $f(x) = ax^2 + bx + c$. Data were collected through classroom observations and interviews with eight participants. The findings revealed that most of the students considered a coefficient as the slope of the parabola. The study reported that secondary students tend to overgeneralize the properties of linear functions to quadratic functions.

Parent (2015) examined how high school students approached quadratic functions. In particular, they investigated the students' thinking, their mathematical strategies, and the kinds of knowledge (conceptual or procedural) they have used while engaging in two types of tasks: traditional and multiple representation tasks. They also examined the effect of the kind of tasks on their understanding. The result indicated that students frequently focused on individual parts of the problem when attempting to solve quadratic problems. Students also often confused the y-intercept of the standard form with the ordinate of the vertex when the function was given in the vertex form. The result also indicated that the students preferred the standard form to the vertex form when solving quadratic equations, and preferred algebraic approaches to tabular or pictorial ones when solving a problem.

Eraslan (2005) analyzed the issue of learning quadratics from a different perspective. He identified the cognitive obstacles that students may face while learning quadratic functions. For this purpose, a multiple case study was conducted with two high school students in an honor algebra class. The analyses were done based on students' cognitive processes while working on the tasks regarding quadratic functions during the interviews. The result indicated four cognitive obstacles that arise from: the lack of making mathematical connections between algebraic and graphical representations of the concepts, the necessity to familiarize an unfamiliar idea, the image of the quadratic formula or absolute value function, and the disequilibrium between graphical and algebraic thinking,

Another research study that investigated the nature of the students' understanding of quadratic functions was Metcalf's (2007) case study, which focused on students' understanding of graphical and algebraic demonstrations of quadratic functions, their understanding of the relationship between the solutions of quadratics, and these two different representations, and how much their prior sources of knowledge regarding quadratics affect their understanding. For this purpose, three participants were interviewed and administered a variety of open-ended tasks. The result showed that one participant was good at procedural solutions, although s/he had just a limited relational knowledge of the ideas. The other two participants lacked procedural expertise and had isolated, unrelated parts of relational understanding. All participants had communication issues and exhibited little flexibility when switching between the representations.

In another study, Hoon et al. (2018) investigated the interrelation between students' understanding of functions and quadratic functions. Survey research was conducted with 103 students. The findings indicated a strong, positive, and significant relationship between students' understanding two concepts. They suggested that the techniques for teaching and learning both concepts should be considered simultaneously.

Some researchers have investigated students' understanding or difficulties of both quadratic functions and quadratic equations. Memnun et al. (2015) identified 11th grade students' inabilities and failures regarding quadratic functions and equations. They administered an open-ended achievement test to 182 students in the 11th grade. Results showed that many secondary students have misconceptions regarding quadratic functions and equations. Regarding quadratic equations, many of them had certain difficulties in factoring a quadratic equation, calculating the discriminant, and writing the quadratic equation by using the product and the sum of the roots. The study also reported various difficulties that students had regarding quadratic functions. Most of them failed in graphing quadratic functions, writing the quadratic function with its graph or x-intercepts given, and finding the vertex of a parabola.

Another example of research in this tradition is Tonui et al.'s (2021), who identified secondary students' difficulties in solving word and graphical problems about quadratic functions and equations. The result reported a variety of difficulties that secondary students have faced. For example, the study reported that students were confused with coordinates and intercepts, and they did not know the meaning of the root of a quadratic function. The authors also reported that students have difficulty drawing the correct graph and using roots to find the quadratic function. Similarly, Nielsen (2015) examined students' thinking about quadratic equations and functions. Data were collected via cognitive interviews with 27 students who had finished an Algebra 2 or pre-calculus course about quadratics in grades 9 to 11. The result showed that students could see the symmetry of the parent function, $f(x) = x^2$, but they frequently struggled to explain what causes that symmetry. The result also indicated that students use their previous knowledge about linear functions when solving quadratic equations and graphing quadratic functions.

Burns-Childers and Vidakovic (2018) investigated students' understanding of the relationship between the vertex of a quadratic function and the derivative. The participants were 30 first-year calculus students. The data were collected through students' written artifacts and group interviews. In analyzing the data, APOS (action-process-object-schema) theory was used. Students' interpretations and misconceptions of the concept of the vertex were identified together with their understanding of solving problems regarding the derivative of a quadratic function. The result showed that students have a weak schema of the vertex and limited connection between different types of problems. The study reported some misconceptions of students about the vertex: vertex as an intercept, vertex as the origin, vertex as an inflection point, and vertex related to symmetry. The authors noted that these misconceptions could be associated with a weak graphical schema of quadratic functions and emphasized the importance of comprehending functions and quadratic functions to understand calculus.

2.6. Teachers' Knowledge of Quadratic Functions and Quadratic Equations

Research showed that teachers also have difficulty understanding quadratic functions (Bansilal et al., 2014; Even, 1990; Sibuyi, 2012). Even (1990) suggested a theoretical framework of teachers' SMK for teaching functions. He used the data from research conducted with pre-service secondary teachers. The anecdotes included pre-service teachers' knowledge and understanding of quadratic functions. Even (1990) stated that many pre-service teachers had difficulties because of the lack of connectedness between different representations of functions. Most participants dealt with just symbolic representations of functions other than the graphic representation, which might be more appropriate to solve the given problem. Another example from the same study was that teachers had difficulty explaining why some basic rules hold for quadratic functions. Although they knew the basic rule that if the leading coefficient, a, is negative in the quadratic function $y = ax^2 + bx + c$, then the graph of the function is downwards, many of them failed to explain why this relation holds. Even (1990) emphasized the interrelation between the role of a coefficient in the algebraic representation and in the graphical demonstration of a quadratic function. Even (1990) noted that memorizing a rule does not help the learner make a generalization.

Li (2011) investigated an individual teacher's mathematical practices while teaching to solve quadratic equations using the quadratic formula. Li (2011) adapted the MKT framework and identified his own framework for MKT: knowledge of the mathematical subject-matter, knowledge of pedagogical representations, and knowledge of learners' conceptions. The result indicated that a teacher's SMK was reflected more in teaching rather than the other two knowledge domains.

Bansilal et al. (2014) investigated secondary mathematics teachers' CCK. They assessed teachers' CCK using an instrument that included questions related to quadratic equations and inequalities, patterns, hyperbolic functions, quadratic functions, derivatives, optimization, and linear programming. Related to quadratic functions, the teachers were given the graph of $f(x)$, then asked to find the maximum value of $1 - f(x)$. The teachers were expected to make some transformations on the vertex of the given parabola. The result indicated that most secondary teachers failed to make these transformations. The findings also showed that teachers did not have problems finding the x-intercepts of a parabola. However, teachers had difficulty transforming quadratic functions.

Sibuyi (2012) examined teachers' PCK in teaching quadratic functions. The study investigated three dimensions of PCK: SMK, knowledge of teaching strategies, and knowledge of learners' conceptions. In their study, teachers' SMK included the proper application of mathematical concepts, facts, and procedures as well as the reasons behind them and the connections between them during instruction on quadratic functions. The result showed that teachers have sufficient SMK of quadratic functions. However, teachers' knowledge of teaching strategies on quadratic functions and knowledge of students' conceptions and misconceptions of quadratic functions were limited.

Ubah and Bansilal (2018) investigated pre-service mathematics teachers' understanding of quadratic functions. More specifically, they explored how pre-service teachers found the algebraic demonstration of a quadratic function given in the graphical form. The result showed that although some participants could find the equation using one method, most failed to find the equation using two methods. The most common method for finding the equation of a quadratic function was using the vertex form. The results also indicated that many pre-service teachers even could not write the standard form of a quadratic function correctly. Thus, the study recommended that teacher training programs should provide more structural opportunities for students to improve their pedagogical content knowledge.

Aziz et al. (2018) investigated pre-service secondary mathematics teachers' views on distinguishing quadratic functions and quadratic equations. The participants were 55 pre-service secondary mathematics teachers. The analysis indicated that they encountered various obstacles in characterizing the differences, including improper constraints and erroneous interpretations. Moreover, participants' written responses emphasized differences between quadratic equations and functions based on their standard forms, main features, and geometrical aspects. The study also reported that factorization and the quadratic formula were two methods that were commonly used by the teachers to solve quadratic equations.

More recently, Sumartini (2021) examined the development of SMK of pre-service mathematics teachers through Problem Based Learning (PBL) model. A quasi-experimental study was conducted with two groups of pre-service teachers (each includes 40 students). One group included undergraduate students of the Mathematics Education Study Program from the PBL class and the other from the Conventional Learning (CL) class. In the PBL model, students are faced with mathematical applications that might improve their SMK. Data were collected via a written questionnaire about quadratic functions based on the SMK indicators identified by the researcher and interviews. They used Ball et al. (2008)'s model and examined SMK in three dimensions: CCK, SCK, and HCK. As an indicator of CCK, they examined their ability to present operational definitions of the quadratic function. As indicators of SCK, they examined their ability to use the quadratic function correctly to solve problems and provide more than one ways to solve mathematical problems related to quadratic functions. As indicators of HCK, they examined their ability to associate quadratic functions with other concepts and with daily life. The result suggested that the PBL model improved pre-service teachers' SMK. Moreover, among the five indicators, four of them indicated a high increase whereas one of them - the ability to connect quadratic functions with everyday life - indicated a moderate increase. The study suggested further investigation to understand the relationship between teachers' SMK and pedagogical abilities. Likewise, Duarte (2010) investigated the effects of an undergraduate course on pre-service teachers' understanding of quadratic functions (Duarte, 2010). The data were collected from 52 pre-service middle school teachers through questionnaires, interviews, artifacts, and observations. The result indicated that the course on quadratic functions improved the pre-service teachers' understanding, knowledge, and skills as well as developed their confidence.

Mbewe and Nkhata (2019) examined teachers' MKT for quadratic equations. The study had three distinct goals: to assess teachers' SMK of quadratic equations; to evaluate their instructional strategies, and to investigate how teachers respond to students' misconceptions and errors regarding quadratic equations. The data was collected from three participants via interviews, questionnaires, and classroom observation. The findings revealed that teachers have adequate SMK regarding quadratic equations; however, their knowledge was restricted to procedural knowledge.

Mutambara et al. (2019) investigated pre-service teachers' understanding of quadratic functions. Data were collected via students' written assessments and follow-up interviews. The study utilized the APOS (action-process-object-schema) theory and modified it to the quadratic functions to examine pre-service teachers' conceptual understanding of quadratic functions. According to the study, action level understanding includes: graphing using a table of values, stating the vertex, and minimum/maximum, stating the vertex using the formula, and describing a parabola. Process level understanding includes: defining the quadratic function, finding the vertex form, graphing without a table of values, and understanding the vertex. Object level includes: understanding word problems regarding quadratic functions and explaining transformations depending on the coefficients a, b, and c in $f(x) = ax^2 + bx + c$. The result revealed that the majority of the pre-service teachers operated at the action level of understanding, and very few teachers could reach the object level of understanding of quadratic functions.

2.7. Relationship Between Teacher Knowledge and Student Learning

Over the years, the factors that affect student learning have become one of the most important issues among educational researchers. An extensive body of research reported that teacher knowledge is among these factors (Baumert et al., 2010; Hatisaru, 2013; Hill et al., 2004; Mewborn, 2003; Smith & Esch, 2012; Thames, 2009).

Earlier studies investigated the relationship between teacher knowledge and student achievement quantitatively. They used variables including teachers' years of experience, educational level, and the number of undergraduate courses regarding mathematics or mathematics education to show the effect of teachers' knowledge on student learning. In general, researchers found no significant relationship between student performance and these variables (Begle, 1979; Monk, 1994). Begle (1979) reported that there is no indication of a substantial positive relationship between teacher knowledge and their students' mathematics achievement. Likewise, Monk (1994) reported that there is no linear relationship between a teacher's number of mathematics courses and student achievement, i.e., the impact of mathematics courses on student achievement decreased after five or more courses. Research also has shown that teachers with better content knowledge benefited students at secondary levels, but there was no discernible influence on student achievement at elementary levels (Monk, 1994; Monk & King, 1994; National Mathematics Advisory Panel, 2008).

In another body of research that focused on the interrelation between teacher knowledge and student learning, direct measurements of teacher knowledge were used. Hill et al. (2005) explored how the teachers' MKT contributed to student achievement in mathematics. The participants were 1190 first and 1773 third-grade students and 334 first and 365 third-grade teachers. Student data were collected from two sources: parent interviews and student assessments. A questionnaire was utilized to assess teachers' mathematical knowledge, focusing on the specialized knowledge and skills used in mathematics teaching. They found a significant relationship between teachers' mathematical knowledge and the mathematics test results of their students. In other words, teachers' mathematical knowledge positively affected the learning of the first and third-grade students. Hill et al. (2005) also reported that this study did not make any empirical or theoretical distinctions between content knowledge, and the impact on student achievement may differ depending on the type of knowledge (CCK, SCK, etc.).

Likewise, Baumert et al. (2010) investigated the interrelation between teachers' knowledge and student learning. Unlike the work of Hill et al. (2005), they empirically differentiated mathematical content knowledge (CK) and PCK. A total of 181 secondary mathematics teachers and 4353 students in the 9th grade participated in the longitudinal study. Teachers' CK was measured via a written test including items related to arithmetic, algebra, functions, probability, and geometry. To measure teachers' PCK, three aspects of PCK were considered: tasks, students, and instruction. The task component was about their ability to identify various approaches to the problem. The student component evaluated their ability to identify their students' solution ways, misconceptions, and difficulties. Lastly, the instruction component measured their ability to present different explanations or use multiple representations for standard mathematics problems. The analyses were done by using statistical models. The result indicated a significant and positive relationship between student achievement and teachers' PCK. The result also suggested that although CK is highly correlated with PCK, it has less predictive power for student progress. However, the authors underlined that CK should not be considered to be unimportant.

Thames (2009) proposed a model to understand how teacher knowledge affects student learning (see Figure 2.4). According to the model, "teacher content knowledge is key to improving teaching, which is key to improving instruction, which is key to improving student learning" (Thames, 2009, p. 7). Likewise, Girit (2016) investigated middle school mathematics teachers MKT for teaching the contents operations with algebraic expressions and generalization of patterns. Girit (2016) reported that teachers' CCK and SCK positively affect their PCK. When the teachers had strong knowledge of the subject matter, they were more likely to understand students' thinking (KCS) and utilize more effective teaching methods (KCT). Thus, teachers' strong SMK positively influences their instructional practice.

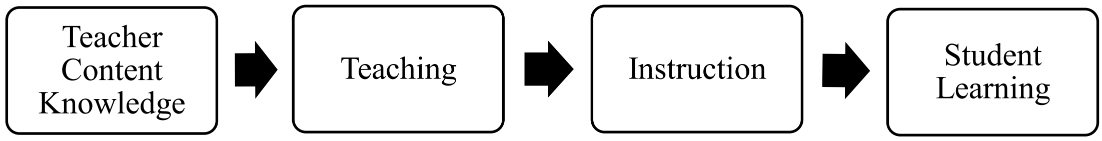


Figure 2.4. Model for effects of teacher content knowledge on student learning (Thames, 2009, p.7)

More recently, Tchoshanov, Quinones et al. (2017) investigated the connection between teacher knowledge and student content-specific knowledge regarding the division of fractions. An initial sample of 90 lower secondary (i.e., grades 5-9) mathematics teachers was administered a survey comprising items measuring their knowledge of facts and procedures, mathematical concepts and their connections, and models and generalizations. Two teachers were selected by purposive sampling based on their test results. Thus, two contrasting cases were examined to identify the interrelation between teacher knowledge and student performance. Interviews with these teachers were also conducted, asking questions about their content knowledge and PCK regarding fraction divisions. After teaching the topic of fraction division, the sixth-grade students of both teachers (n = 55) were administered a similar test that assessed their knowledge of facts and procedures, mathematical concepts and their connections, and models and generalizations. The results showed that teachers' topic-specific knowledge regarding the division of fractions contributes to student learning at lower secondary schools.

Tchoshanov, Cruz et al. (2017) investigated the interrelation between cognitive types of teachers' content knowledge and students' learning. Data were collected from 90 lower secondary teachers via the Teacher Content Knowledge Survey (TCKS), which consisted of 33 multiple choice items regarding probability, arithmetic, algebra and functions, and geometry. For evaluating student performance, teacher-reported student performance data was used. The first cognitive type measured participants' knowledge of basic facts and procedures. The second cognitive type assessed teachers' understanding of connections and concepts, whereas the third cognitive type assessed teachers' knowledge of generalizations and mathematical models. The result showed that there was a statistically significant relationship between the first and the second cognitive types of teacher content knowledge and student performance ($p < .05$). The study found no statistically significant correlation between the third cognitive type of teacher knowledge and student performance ($p = .0678$). These findings provided evidence for the association between teachers' content knowledge and student performance at the lower secondary level.

Several other studies have identified a relationship between teachers' PCK and student learning outcomes. Callingham et al. (2016) examined the relationship between teachers' PCK for teaching statistics and student learning outcomes. They conducted a survey measuring teachers' PCK for teaching statistics. The result indicated that student learning outcomes were positively affected by teachers' PCK. Likewise, Ibeawuchi (2010) investigated the interrelation between teachers' PCK for teaching quadratic functions and students' achievement. The study investigated PCK under three components: mathematical content knowledge, knowledge of students' conceptions and misconceptions, and knowledge of strategies. Seventeen mathematics teachers and 10 students from each teacher's classroom participated in the study. The analysis was carried out through descriptive statistics. The result showed that students of teachers who had a strong PCK performed better than the students of teachers who had weak PCK in the questionnaire.

Hatisaru (2013) identified teachers' KCS and examined the relationship between teachers' KCS of functions and student learning outcomes. The study included two parts. In the first part, 42 secondary mathematics teachers were administered a questionnaire that assessed their KCS regarding functions. Based on the questionnaire results, two teachers were selected for further investigation in the second part, where they were interviewed, and their classes were observed while teaching the function concept. After the lessons on functions ended, the students of both teachers were administered a test regarding the function concept. The result suggested evidence for the links between teachers' KCS and student learning outcomes. Interactions occurred between teachers' KCS and student learning outcomes in terms of the teachers' conceptions of the function and their understanding of the univalence feature of the functions. There was no interaction in terms of identifying two equal functions, associating a domain and a range with its graph, and locating images and pre-images on the graphs. The study also reported that teachers' KCS of the function concept affected their instruction, and their instructional practice influenced student learning outcomes.

A body of research also reported some mediating factors such as teachers' perceptions and beliefs for the links between teacher knowledge and student learning. Campbell et al. (2014) investigated the interrelation among teachers' CK and PCK, teachers' perceptions, and student achievement. In the study, teachers' perceptions were considered as teachers' beliefs about teaching and learning of mathematics and teachers' awareness of the mathematical disposition of the students. The participants of the study were 266 upper-elementary (i.e., grades 4-5) and 193 middle grade mathematics teachers (grades 6-8). The instrument that assessed teachers' CK included items regarding number and operations, geometry, measurement, probability, data analysis, patterns, functions, and algebra. They found a significant relationship between both upper-elementary and middle-grade teachers' CK of mathematics and their students' mathematics achievement. Furthermore, the study suggested some evidence for the interrelation between teachers' perceptions (beliefs and awareness) and their CK.

Hill, Blunk et al. (2008) investigated the links between teachers' MKT and the mathematical quality of instruction (MQI) which includes "several dimensions that characterize the rigor and richness of the mathematics of the lesson, including the presence or absence of mathematical errors, mathematical explanation and justification, mathematical representation, and related observables" (Hill, Blunk et al., 2008). For this purpose, they conducted five case studies. The result indicated a positive, strong and significant relationship between MKT and MQI. However, the study also reported that there are many critical mediating factors that either support or hinder practical use of teachers' knowledge. As the study reported, these factors might be teachers’ beliefs about mathematics teaching and learning, beliefs about curriculum materials, and the accessibility of these materials. However, the study also reported that these mediating factors are mostly shaped by teacher knowledge.

Shechtman et al. (2010) investigated the interrelation between teachers’ mathematical knowledge, their classroom decision-making, and student learning outcomes on the concepts of linear function, rate, and proportionality. The study included a part of a research project called “Scaling Up SimCalc” that was conducted in 56 eighth-grade and 125 seventh-grade classrooms. The overall research consisted of two experimental designs. The result indicated that MKT might have a non-linear interrelation with student learning and that interaction may be mostly mediated by other aspects of instruction. The study also reported that teachers' mathematical knowledge, the curriculum and other learning resources such as technology and student-student interactions are undeniably crucial for student learning.

Similarly, Gençtürk (2012) also examined the interrelation between teachers’ mathematical knowledge, their instructional practices, and student performance. Multiple instruments such as interviews, classroom observations, written assessments, and surveys were used to collect data from 21 teachers and 873 students. In the quantitative part of the study, three-level growth models were utilized to identify the impact of teachers’ knowledge and teaching practices on students’ achievement. Teachers’ beliefs about mathematics teaching and learning were also involved in some analyses. The result showed that there was a significant relationship between only student engagement and students’ gain scores. Teachers’ mathematical knowledge and instructional practices, such as the quality of the tasks they used, inquiry-based teaching, and the classroom climate, did not interact with students’ gain scores. The study also reported that teachers’ beliefs had a mediating role in the interrelation between teachers’ mathematical knowledge and their teaching practices.

Steele and Rogers (2012) used the MKT model to examine the relationship between teacher knowledge and their instructional practice. Their framework called MKT-P focused mainly on CCK and SCK components of Ball et al.'s (2008) model. The MKT-P includes the knowledge of defining proof, creating proofs, distinguishing proofs and non-proofs, and comprehending the roles of proofs in mathematics. Among the participants of a larger study with 25 teachers, they selected two contrasting cases of an expert and a novice teacher in different districts (suburban and rural) and compared their classroom practice. Data were collected through written assessments, semi-structured interviews, and classroom observations. Results showed that the use of MKT that is evident in clinical settings (i.e., interviews and written assessment) was different in the classroom of both teachers. While both teachers exhibited a wealth of MKT in clinical settings, this knowledge was utilized more in practice in the expert's classroom. The tasks they preferred to use in the classroom and the means by which they performed the task influenced the ways in which students were positioned according to the task and provided opportunities for particular aspects of this knowledge of proof to be made available to students. The result suggested that student positioning might be a mediating factor between teacher knowledge, instructional practice, and student learning.

In another study of this tradition, Hatisaru and Erbaş (2017) examined the relationship between teachers' MKT for teaching functions and student learning outcomes regarding this concept. A function concept test for teachers and students, follow-up interviews with teachers, and classroom observations were used to gather data from two teachers teaching in a vocational high school and their students. Teachers' MKT and students' learning outcomes were found to be somewhat associated, but the relationship was not linear. The findings indicated that teachers' MKT for teaching functions influenced the quality of their instructional practices, and instructional practices had a mediating role in student learning on functions. However, the study reported some mediating factors for the relationship between teacher knowledge and student learning outcomes. These are; inherent complexities of the function concept, the students' difficulties in arithmetic, the students' academic background.

2.8. Summary

Several studies in the literature provided evidence for the links between teachers' mathematical knowledge and student learning of a specific content (Gençtürk, 2012; Hatisaru, 2013; Hatisaru, & Erbaş, 2017; Ibeawuchi, 2010; Shechtman et al., 2010). On the other hand, it may not be so simple to determine how teacher knowledge, instructional strategies, and student learning relate to one another. The impact of teachers' knowledge on instructional strategies and student learning may be moderated by teacher beliefs in addition to some other variables (Campbell et al., 2014; Gençtürk, 2012; Hatisaru & Erbaş, 2017; Hill & Blunk et al., 2008; Shechtman et al., 2010).

The above literature confirms that teachers should have an extensive and organized body of knowledge (Shulman, 1986). The studies presented above showed that secondary prospective and practicing mathematics teachers have insufficient understanding of quadratic functions (Mutambara et al., 2019; Sibuyi, 2012; Ubah & Bansilal, 2018). Moreover, investigating particularly teachers' SMK of particularly quadratic functions and how it contributes to student learning seems to be undervalued. These results indicate a need for further studies to examine practicing teachers' SMK of quadratic functions and its contribution student learning. The current study is built on the results of the former research in two ways: (a) it examined practicing secondary mathematics teachers' SMK of quadratic functions, based on their CCK, SCK, and HCK, in more detail than it was previously done, and (b) it identified the contribution of teachers' SMK of quadratic functions to teachers' instructional practice, and thus student learning outcomes of this concept.

CHAPTER 3

METHODOLOGY

This chapter describes the research methodology used to investigate secondary mathematics teachers’ SMK of quadratic functions and its contribution to student learning outcomes. The chapter starts with the overview and rationale for the research design conducted in the present study. Then, it describes the participants of the study, including a detailed explanation of the criteria for selecting them. The chapter continues with a description of procedures for data collection and data analysis. It ends with a discussion of the trustworthiness and ethical issues of the study.

3.1. Design of the Study

Yin (2009) suggested that the type of research question(s) being investigated through the study is the best way to decide which research method is appropriate. Yin (2009) categorized the main types of questions as “who,” “what,” “where,” “how,” and “why.”

The first research question of the present study explores what SMK secondary teachers have for teaching quadratic functions with three sub-questions, each focusing on a different sub-component of teachers’ SMK: CCK, SCK, and HCK. To answer this question, a survey research was conducted (Fraenkel, et al., 2012). In survey research, information is gathered from a sample to describe some characteristics of the population, such as beliefs, knowledge, abilities, or opinions, by asking questions (Fraenkel et al., 2012). The present study investigated secondary mathematics teachers' SMK of quadratic functions via an open-ended questionnaire.

The second research question investigates how CCK, SCK, and HCK contribute to the instructional practice and thus student learning regarding quadratic functions. Since this is a "how" question, a qualitative research design was used to answer the second research question as qualitative research allows for a more in-depth examination of people's perspectives, emotions, beliefs, and mental structures as they arise from their experiences (Hogan et al., 2009).

Several researchers differentiated various approaches to qualitative research (Creswell, 2008; Denzin & Lincoln, 2005; Merriam, 2009). Creswell (2008) identified five approaches to qualitative research: ethnography, grounded theory, case study, phenomenological research, and critical research. In the second stage of the present study, the case study research methodology was carried out to examine the contribution of teachers' SMK of quadratic functions to instructional practice, and thus student learning outcomes. Creswell (2007) defined the case study as "a qualitative approach in which the investigator explores a bounded system (a case) or multiple bounded systems (cases) over time, through detailed, in-depth data collection involving multiple sources of information" (p. 73). In addition, Sanders (1981) noted that "case studies help us to understand processes of events, projects, and programs and to discover context characteristics that will shed light on an issue or object" (p. 44). A case study involves "multiple sources of evidence, with data needing to converge in a triangulating fashion" (Yin, 2009, p. 18).

Researchers utilize the term "multiple case study" when more than one case is conducted to investigate the same concern (Creswell, 2007; Yin, 2009). In the current study, two case studies with two teachers and their students were carried out to find out the contribution of SMK to student learning outcomes. Thus, the second stage of the present study employed a multiple case study research design.

3.2. Context of the Study

Secondary education is the third stage of the Turkish educational system, offering a minimum four-year education from the 9th grade to the 12th grade. Secondary education schools include vocational and technical education and general education schools comprising five school types: Anatolian High School, Science High School, Social Sciences High School, Fine Arts High School, and Sports High School.

In both general and vocational high schools, all teachers—including math teachers—receive the same university training. During the data collection of this study, the teachers were teaching based on the mathematics curriculum, which had been updated in 2018 (Milli Eğitim Bakanlığı, 2018). The mathematics curriculum comprises three learning domains: Numbers and Algebra; Data, Counting and Probability; and Geometry. The topic that was the concern of the present study is titled “Quadratic Functions and their Graphs,” which is a part of the sub-domain “Applications of Functions” under the learning domain Numbers and Algebra.

3.3. Participants

This study was conducted in two stages. In the first stage, the participants were determined by convenience sampling, where the researcher selected a group of teachers who were available for the study (Fraenkel et al., 2012). The participants were 18 secondary mathematics teachers (9 male, 9 female) from different types of high schools in Zonguldak including Science High Schools, Anatolian High Schools, and Vocational and Technical High Schools. The teachers’ average years of teaching experience was 15 years, varying from 3 to 22 years.

The purpose of the second stage of the study was to identify the contribution of teachers’ SMK of quadratic functions to students learning outcomes. In the second stage of the study, two case studies were conducted. To select two teachers among the 18 teachers, the teachers' responses to the quadratic function concept questionnaire were analyzed. The bar graph in Figure 4.1 (on page XX) shows each teachers' overall score as well as their scores on CCK, SCK, and HCK items separately. It can easily be observed from the graph that T16 has the highest overall score, whereas T3 has the lowest overall score among the participating teachers. The graph also shows the contribution of each teachers' CCK, SCK, and HCK on their overall SMK scores. Some teachers (T1, T4, T15, and T16) have relatively balanced distribution that means their scores on CCK, SCK, and SCK items were close to each other (high CCK, SCK, and HCK scores or low CCK, HCK, SCK scores). However, some of them (T7, T8, T17) have unbalanced distribution which means that teachers' scores on each sub-component are remarkably different (i.e., high CCK score with very low HCK score). Teachers generally performed better on CCK items, when compared to SCK and HCK items. Two cases were planned to be selected so that one of them has a balanced distribution and performed well on CCK, SCK and HCK items, whereas one of them has an unbalanced distribution and performed well on CCK items but relatively lower on SCK and HCK items. By this way, it would be easier to distinguish the contribution of teachers' CCK, SCK and HCK to student learning outcomes. T16 was the teacher who had the highest overall score as well as the highest CCK, SCK, and HCK scores. Thus, the researcher asked him wanted to be a participant in the second stage of the study and he accepted to be a participant. The second teacher was selected among the teachers who had unbalanced distributions of the three sub-components. There were 9 teachers with high/medium CCK, lower SCK, and the lowest HCK score. Of the 9 participants, only one of them (T17) was willing to participant in the second stage. Thus, T16 and T17 were the participants of the second stage of the study. The teachers are given pseudonyms as Can (T17) and Ahmet (T16) to ensure confidentiality.

These two teachers were teaching at different high schools, at the time of data collection for the study. Two schools were Anatolian high schools that select students according to the results of a high school entrance exam conducted at the national level by the Ministry of National Education. In Can's school, there were a total of 964 students and 61 teachers. In Ahmet's school, there were a total of 968 students 63 teachers. Both teachers held a bachelor's degree in mathematics education and were registered to a master's degree program in mathematics education.

A total of 51 eleventh grade students (28 from Ahmet's classroom and 23 from Can's classroom) participated in the second stage of the study. The students in both groups were coming from the middle-income families. They were seventeen years old. There were 23 students (12 female, 11 male) enrolled in Can's classroom. There were 28 students (15 male, 13 female) enrolled in Ahmet's classroom.

3.4. Instruments

Three instruments are used to collect data on teachers' SMK of quadratic functions and investigate the contribution of SMK to student learning outcomes: the quadratic function concept questionnaire, follow-up interview, and quadratic function concept test. These instruments are described in the following sections.

3.4.1. The Quadratic Function Concept Questionnaire

The first stage of the study used the quadratic function concept questionnaire to assess teachers' SMK of quadratic functions. The questionnaire included 40 open ended-items. The researcher developed the questionnaire by reviewing the literature, personal communication with the dissertation supervisor, the national mathematics curriculum (MEB, 2018), and taking expert opinions (see Appendix A). The first consideration for preparing the questionnaire items was the mathematics curriculum for grades 9-12. Firstly, the mathematics curriculum for high schools and science high schools were reviewed. The second and the most important consideration for preparing the questionnaire items was the literature review regarding the frameworks of teachers' SMK.

Three experts in mathematics education were consulted regularly to establish the validity of the questionnaire. Before the study began, the researcher made some revisions on the questionnaire items several times. For example, at the beginning, 45 items were prepared. Then, some items were excluded based on the consensus between the researcher and the experts. Furthermore, the sub-dimensions of some questions were changed. For example, the researcher prepared the question about the definition of a parabola (question 39) to measure teachers' CCK. At the end of discussion with the experts, it was concluded that question 39 measures teachers' HCK rather than CCK as it evaluates whether the teachers know the geometrical definition of a parabola, which is not included in the curriculum. The source of each question in the questionnaire is given in Appendix B. The questionnaire was piloted with two secondary mathematics teachers who were not the participants of this study to anticipate the time needed for completing the questionnaire and to check the language of the questions. Table 3.1. shows important sub-components of SMK and the corresponding items in the questionnaire.

Table 3.1. The quadratic function concept questionnaire items and important sub-components of SMK of quadratic functions

Three Components of SMK	Important sub-components	Items
CCK	CCK1: Conception of quadratic equations and functions	1, 2, 3, 4
	CCK2: Knowledge of solving quadratic equations with one unknown	5
	CCK3: Knowledge of sketching and interpreting the graphs of quadratic functions	6, 7, 8, 9, 10, 11
	CCK4: Knowledge of graphing quadratic functions using transformations	12, 13, 14
	CCK5: Knowledge of solving real-life problems regarding quadratic functions	15
	CCK6: Knowledge of finding the quadratic with given points	16,17
	CCK7: Knowledge of finding the intersection of a parabola and a line	18
SCK	SCK1: Knowledge of explaining and justifying basic formulas of quadratic functions	19, 20
	SCK2: Knowledge of posing problems regarding quadratic functions	21
	SCK3: Knowledge of recognizing students' incorrect solutions regarding quadratic functions	22
	SCK4: Knowledge of understanding students' unusual solutions regarding quadratic functions	23, 24
	SCK5: Knowledge of responding to students' why questions about quadratic functions	25, 26
	SCK6: Knowledge of finding an example to make a specific mathematical point about quadratic functions	27
	SCK7: Knowledge of modifying tasks regarding quadratic functions	28
HCK	HCK1: Knowledge of how quadratic functions are related to other contents in middle school or high school curriculum	29, 30, 31, 32, 33, 34.
	HCK2: Knowledge of how quadratic functions are related to advanced mathematics	35, 36, 37, 38, 39, 40.

As shown in Table 3.1, each item in the questionnaire corresponds to a sub-component of SMK of quadratic functions identified in the current study. The number of items in each sub-component are not equal since their content are different from each other. For example, HCK1 and HCK2 include broader concepts; thus, they include more numbers of questions. Similarly, of the CCK items, CCK3 has the most numbers of items since the graph of quadratic functions have many elements such as the vertex, the axis of symmetry, etc. A sample set of questions for each component of SMK is given in Table 3.2.

Table 3.2. A sample set of questions in the quadratic function concept questionnaire

Components	Sample Question
CCK	**#18:** Think about the parabola $f(x) = ax^2 + bx + c$ and the line $y = mx + n$. Under what conditions the parabola and the line intersect or not?
SCK	**#19:** State the quadratic formula and explain how it is derived, both geometrically and algebraically.
HCK	**#37:** One of your students asked that she heard something called the fundamental theorem of algebra. She wonders what it is and if and how it applies to quadratic polynomials. What would you say to her?

3.4.2. The Follow-up Interview

In the second stage of the study, the data obtained from the questionnaire would be sufficient to get a general picture of the two teachers' SMK of quadratic functions. However, getting a detailed and clear picture of their SMK would be limited. Thus, a follow-up interview was conducted with these two teachers (Can and Ahmet). The aim of the interview was to clarify the data obtained from the questionnaire. During the interview, the two teachers were asked questions related to their answers on the questionnaire, but this time requiring responses in more detail (see Appendix C). Sample questions from each interview were given in Table 3.3.

Table 3.3. Sample questions from the interviews

Teacher	Sample Question
Can	In the 4th question, you have stated that the values in the first table belong to a linear function, the values in the second table belong to a quadratic function. Could you explain how you decide it?
Ahmet	In question 21, you have written a profit-loss problem as an example of real-life problems regarding quadratic functions. Do you use this kind of problems as a part of your instruction? If yes, how often do you use?

3.4.3. The Quadratic Function Concept Test

The second research question of the study investigated the contribution of teachers' SMK of quadratic functions to student learning outcomes of this concept. For this purpose, the quadratic function concept test is administered to the students of both teachers (see Appendix D). The test was prepared by the researcher on the basis of the content of the secondary mathematics curriculum. Each item represents an objective regarding quadratic functions as shown in Table 3.2. Three experts in mathematics education were consulted several times to identify the difficulty level of the items and whether the test really measures the objectives. The source of each question in the quadratic function concept test is given in Appendix B.

Table 3.4. The quadratic function concept test items

Objective	Items
Find the vertex, x- and y- intercepts, and axis of symmetry.	1, 2
Associate the vertex with the maximum or minimum value of the function	3, 4
Comment on the effect of the change in the coefficients of the function on the graph of the function by using technology	5
Find the quadratic function whose two points such that one of them is the vertex or three points such that one of them is on the y-axis are given.	6, 7
Investigate the intersection of a line and a parabola.	8
Solve the problems which can be modeled by quadratic functions.	9, 10

3.4.4. Observations

Creswell (2012) defined observation as "the process of gathering open-ended, firsthand information by observing people and places at a research site" (p. 213). As Yin (2003) stated, observation is one of the data sources in a case study. Since the interactions between teaching and learning occur in the classroom environment, classroom observation is a critical part of the research to find an answer to the second research question. For each of the two cases, the classroom observation started on the day the quadratic function was introduced and continued until the end of the concept. The purpose of the observations was to understand how teachers used their SMK in their instructional practice, that is, to find out the reflections of teachers' written responses to the questionnaire and to the interview on their instructional practice.

The researcher observed a total of 24 classes (12 for each case) that included the instructional unit on quadratic functions. During the observations, the researcher was a non-participant observer. She sat at the backside of the classroom, watched the teachers' instructional practice, and took observation notes. Observation field notes are about participants, routines, interactions, and interpretations (Denzin, 1989). The researcher wrote her interpretations of how the instructional practice was mediating to the contribution of teachers' SMK to student learning.

At the end of the instruction, for each classes, the researcher also took one students' (who participated in all the lessons on quadratic functions and wrote all the definitions, illustrations, examples, and problems the teacher used) notebook. The student's notebook included some information about the examples, problems, definitions, and representations the teacher used during instructional practice. Finally, two teachers' responses to the quadratic function concept questionnaire and the interviews, students' responses to the quadratic function concept test, analysis of the notebooks, and the observation notes helped to examine the contribution of teachers' SMK to instructional practice and thus student learning outcomes regarding quadratic functions.

3.5. Procedure and Data Collection

Data were collected in two stages from October 2019 to March 2020. The administration of the first stage lasted from October 2019 until the end of November 2019. During this stage, the questionnaire was used to collect data from 18 secondary mathematics teachers. Firstly, 15 high schools of different types (Anatolian High school, Science High School, etc.) in Zonguldak, Turkey were determined. The researcher visited the schools and contacted school administrators to meet mathematics teachers. This process was difficult for the researcher since most teachers were reluctant to be a participant in the study for several reasons. Their excuses were like: "I don't have much time to respond to these questions," "These questions are not a part of the curriculum," "I have got bored of filling the questionnaires that came from your faculty," "Will you assess our knowledge?" and so on. At the end of this process, 18 secondary mathematics teachers accepted to be a participant in this study. Then, appointments were made with each teacher to complete the questionnaire.

For the second stage of the study, two teachers willing for further investigation were selected based on their responses to the questionnaire. The second stage of the study started with the administration of interviews with two selected teachers on December 2019. Both teachers were interviewed approximately two weeks before they started to teach quadratic functions. The interviews were conducted in their schools during their leisure times and lasted about two hours for each teacher. The questionnaire and interview data provided a detailed description of teachers' SMK of quadratic functions.

Data collection for the second phase lasted until the middle of March 2020. At the beginning of February 2020, the teachers started the instructional unit on quadratic functions. Then, the classroom observation started and lasted approximately three weeks for each case. Twelve lesson hours were observed in both classrooms. Can's classroom was observed on Monday mornings and Wednesday afternoons, each time for 80 minutes. Similarly, Ahmet's class was observed on Tuesday and Thursday mornings, each time for 80 minutes. During classroom observations, no video or audio recording was done. During the observations in both classrooms, the researcher took observation notes. After the instructional unit on quadratic functions were completed in each classroom, the students of both teachers were administered the quadratic function concept test on March 2020.

3.6. Data Analysis

Data analysis includes "organizing, accounting for and explaining the data; in short, making sense of data in terms of the participants' definitions of the situation, noting patterns, themes, categories and regularities" (Cohen et al., 2007, p.461). The major purpose of this study was to identify secondary teachers' SMK of quadratic functions and to investigate the contribution of teachers' SMK to their instructional practice and student learning outcomes. Multiple sources of data were used to increase the trustworthiness of the results: written assessments, interviews, and observations. Both quantitative and qualitative analyses of data were used to find out answers to the research questions.

The first research question of the study examined the teachers' SMK quadratic function based on their CCK, SCK, and HCK. To answer this, the responses of 18 teachers to the questionnaire were analyzed to describe their SMK of quadratic functions. For this purpose, a qualitative analysis of the 40 items in the questionnaire was done through content analysis that is "a technique that enables researchers to study human behavior in an indirect way through an analysis of their communications" (Fraenkel et al., 2012, p. 478). The questionnaire was designed to reveal teachers' SMK based on their CCK, SCK, and HCK regarding quadratic functions. In the analyzing process, three components were individually analyzed and integrated to get a clear description of teachers' SMK of quadratic functions.

For each question, the teachers' responses were coded into the categories that emerged throughout the analysis. The coding process was more than identifying the incorrect or correct answer. The researcher focused on the nuances in the teachers' written responses. For example, when the teachers were asked to define a quadratic function, the responses that include algebraic definitions like $f(x) = ax^2 + bx + c$ were coded as *structural description*. They were coded as *procedural description* if there was some reference to the parabolas generated from quadratic functions. In another question, if the teachers solved the real-life problem using a mathematical model, their responses were coded as *using an algebraic model*. If their responses included numerical computations without using an algebraic model, they were coded as *using a numerical approach*. If teachers had no answer to the question, it was coded as *no answer*. Incorrect responses or the responses that were not directly related to the focus of the question were coded as *incorrect*. These categories were elaborately discussed with three experts in mathematics education. In this process, some categories were eliminated, and similar categories were combined. When the researcher was faced with ambiguous cases, the experts were consulted. Finally, a coding scheme and a scoring rubric were generated to analyze the quadratic function concept test (see Appendix E).

Each item in the questionnaire (except one item) was scored. Question 28 was not scored since it included a yes/no answer. Any answer that included a correct explanation/result was assigned 2 points. Responses that included partially correct explanations/results were assigned 1 point. Incorrect/no answers were assigned 0 point. The maximum score of CCK items were 36 points, the maximum score of SCK items were 18 points, and the maximum score of HCK items were 24 points. The main purpose of scoring the questionnaire was to select the participants for the second stage of the study and to obtain a general picture of teachers' SMK based on their scores on CCK, SCK, and HCK items in the questionnaire. To enhance the reliability of the analysis, the researcher gave the coding scheme to a mathematics education doctoral student and asked to code all the responses of two participants who were randomly selected. There was agreement on 63 out of 72 statements (87.5 %). The items that were not agreed were discussed again and the two raters reached a consensus for all the items.

To answer the second research question, the data gathered from the interviews, the classroom observations, and the quadratic function concept test results of 48 students were analyzed and described. The aim of the interviews was to clarify two teachers' responses to the questionnaire and to obtain a more detailed picture of their SMK of quadratic functions. The analysis of the interviews started with listening to audio records, then writing the transcripts of them and editing the transcripts. After that, the researcher read each transcript and analyzed the interview based on the key components of CCK, SCK, and HCK (see Table 3.1). Any piece of data that is related to these sub-components was recorded and then used to identify the patterns of the contribution of teachers' SMK to instructional practice.

The analysis has continued with examining the observation notes and two students' notebooks. The key components identified by the researcher (see Table 3.1) formed the basis for analyzing these pieces of data. More specifically, instructional episodes that indicated teachers' CCK, SCK, or HCK regarding quadratic functions were identified. Any indicator of the key components was recorded and then triangulated with the questionnaire results and the interviews. For each case, the sections from one student's notebook were also used to either support the data obtained from the other instruments or obtain a new data. For example, in the interview, one of the teachers said that he likes to use completing the square method for solving quadratic functions during his instruction. If the student's notebook contained an exercise or a problem that was solved by the teacher with this method, the section about this content was directly presented in the results section of the study.

Lastly, a total of 48 students' responses to the quadratic function concept test were analyzed to evaluate their performance on the test. The analyses were made based on the objectives regarding the concept of the quadratic function identified by the mathematics curriculum (see page 24). A total of 10 items were analyzed using content analysis. Like the analysis of teachers' responses, no preliminary categories were used. For each question, the students' responses were coded into the categories that emerged throughout the analysis. The coding process was more than identifying the incorrect or correct answer. The researcher focused on the nuances in the students' written responses. For example, the students were asked to find the x-intercepts of a quadratic function. If the students find the roots of the quadratic equation by completing the square, their answers were coded as *using completing the square method*. The responses of the students who solved the quadratic equations by factorization were coded as *using factorization*. In another question, the students were asked to find the quadratic function with its vertex and one point given. The responses of students who used the vertex form were coded as *using the vertex form*. The responses that included the standard form were coded as *using the standard form*. If the students did not respond, it was coded as "no answer". Incorrect responses or the responses that were not directly related to the focus of the question were coded as *incorrect*. These categories were also discussed in detail with three experts in mathematics education. In this process, some categories were eliminated, and similar categories were combined. When the researcher was faced with ambiguous cases, the experts were consulted. Finally, a coding scheme for analyzing the students' responses to the quadratic function concept test was created by the researcher. The scoring of the test was also made based on the answers being correct or incorrect. The total score of the test is 100 points, 10 points for each item (see Appendix F for the coding scheme and the scoring rubric).

After the main analyses were finished, the researcher wrote summary cases for each teacher. Each case consisted of the teacher's CCK, SCK, and HCK of quadratic functions based on their responses to the questionnaire and interviews, their instructional practice, and their students' learning outcomes of quadratic functions. During the data analyses, two teachers were contacted face-to-face and also given a summary of the research findings to justify their answers to the questionnaire, interviews.

3.7. Trustworthiness

Validity and reliability are two critical issues that any researcher must consider while designing a research study, analyzing the results, and making judgments about the quality of the study (Patton, 2002). "Validity refers to the appropriateness, meaningfulness, correctness, and usefulness of the inferences a researcher makes. Reliability refers to the consistency of scores or answers from one administration of an instrument to another, and from one set of items to another" (Fraenkel et al., 2012, p. 147). Since the nature and main purposes of qualitative and quantitative research traditions differ, the terms validity and reliability in quantitative research are replaced with trustworthiness in qualitative research (Gay et al., 2006; Krefting, 1990; Merriam, 2009). Lincoln and Guba (1985) stated four criteria to establish the trustworthiness of a qualitative study: credibility (for internal validity), transferability (for external validity), dependability/consistency (for reliability), and confirmability (for objectivity).

Credibility refers to whether or not the research results represent an honest and authentic interpretation of the participants' original views through depth and rich information derived from the data collected in the study (Anney, 2014; Lincoln & Guba, 1985). In the present study, some common strategies suggested by Lincoln and Guba (1985) were taken into consideration to increase the credibility of the study. These were; prolonged engagement in the field or research site, triangulation, peer examination, and member checking. Firstly, as Lundy (2008) stated, the researcher should establish a trusting relationship with the participants to effectively engage them on a prolonged basis. Before the study began, the researcher had done several preliminary visits to the high schools to understand their culture and recognize the context. During these visits, the participants were informed about the aim of the research, their role in this study, and the steps to be followed during the research. Accordingly, they also had the chance to know the researcher more closely. The researcher spent approximately five months in the field. So, she had the opportunity to establish good relationships with the teachers, students, and school administrators. This process allowed the researcher to access suitable data sources and develop more sense of its nature. The second strategy for increasing the credibility of a research study is triangulation. Triangulation is "using a variety of instruments to collect data" (Fraenkel et al., 2012, p. 458). In the current study, different data collection tools such as questionnaires, interviews, observation, and written artifacts were used. Another way to improve credibility is peer examination which is an external review or examination of the study process (Creswell, 1998). The researcher shared several stages of the present study with other colleagues, and some modifications were made based on their feedback. For example, the items in the quadratic function concept questionnaire were modified based on their feedback. Furthermore, three experts in mathematics education regularly evaluated the research methodology, instruments, and data analysis to strengthen the study. Another crucial process to enhance the credibility of the qualitative data was member checking that is "the single most important way of ruling out the possibility of misinterpreting the meaning of what participants say and do…, as well as being an important way of identifying your own biases and misunderstanding of what you observed" (Maxwell, 2005; p. 111). In this study, two teachers of the second phase of the study were asked to review the interview transcripts and the researcher's interpretations of these transcripts to give them an opportunity to examine whether what they said in the interviews was the same as what they actually intended to say. They were also asked to modify, clarify, or expand their responses if necessary. They expressed no disagreements with their transcripts and the researcher's interpretations of them. Such checks improved the verification of the results and helped ensure that the results of the study represent an accurate reflection of teachers' knowledge. It also helped the researcher to prevent the researcher's possible bias in analyzing and interpreting the study results (Merriam, 2002).

Transferability deals with the issue that the results of the qualitative research can be applied to another contexts with different participants (Lincoln & Guba, 1985). In the present study, thick description and purposive sampling techniques were used to enhance the transferability (Creswell, 1998; Erlandson et al., 1993; Guba, 1981). Accordingly, all the research processes, from the data collection to the preparation of the final report, were written in detail. Thus, other researchers could replicate the study in a similar context with different participants. Moreover, purposive sampling was used to increase the range of the information obtained from the data. A total of 18 teachers from different high schools were the participants of the first part of the study. For the second part of the study, two teachers who were volunteers for further investigation were selected based on the questionnaire results. Thus, the examination of different cases that yielded maximum variation sampling contributed to enhancing the transferability of the study (Merriam, 2009; Miles & Huberman, 1994).

3.8. Researcher Role

The researchers play a vital role in data collection and analysis processes (Merriam, 1998). As Patton (2002) stated, the researchers should describe their roles within the study to avoid discrediting it. During classroom observations, researchers may take several roles, such as “complete participant, observer as a participant, participant as an observer, and complete observer” (Creswell, 2002, p. 213). During classroom observations, the researcher observed but did not take part in the instruction or any classroom discussion. However, the participants were informed about the researcher’s identity. Creswell (2012) noted that a non-participant observer takes notes without participating in the activities. Thus, the researcher was a non-participant observer in the current study.

3.9. Ethical Consideration

For ethical issues, firstly, necessary permission was obtained from the ethical committee of a public university (see Appendix G). After taking the permission, the researcher appealed to the National Education Directorate of the district and took permission from all the relevant authorities and schools to administer questionnaires, conduct interviews and observations before the study began.

Another important issue to be addressed before conducting any research is the informed consent taken from individuals to be allowed to agree or refuse to participate in the study, in the light of comprehensive information about the purpose and nature of the study (Cohen et al., 2007). Accordingly, the school administrators were given information about the study in general before starting the study. Then, the teachers were informed about the purpose of the study, and the questionnaire and informed consent was orally taken from each participant. Personal information of the participants and the names of their schools were kept private. The two teachers who participated in the second part of the study were informed about the process, especially the interviews and classroom observations. The teachers were asked for their permission to allow the audio recording of the interviews. During classroom observation, no audio or video recording was done. Moreover, none of the participants' names were used anywhere; pseudonyms were used to present the data. All the questionnaire results and interviews were kept confidential. This means that anyone except the researcher could access the research data. Moreover, the instruments were not used for grading mathematics teachers. Lastly, during the study, the participants were not harmed either physically or psychologically.

CHAPTER 4

RESULTS

This chapter includes three sections. The first section presents the results of the quadratic function concept questionnaire administered to 18 secondary mathematics teachers. This data reveals secondary mathematics teachers' SMK of quadratic functions based on their CCK, SCK, and HCK. The second section presents the results of the quadratic function concept questionnaire, the observation of the teacher's instructional practice in Case 1, and the follow-up interview with him. This section provides a detailed description of Can's SMK of quadratic functions, his instructional practice and his students' performance on the quadratic function concept test. The third section is presented in parallel with the previous section, and summarizes the results for Ahmet in Case 2.

4.1. Teachers' Subject Matter Knowledge of Quadratic Functions

Teachers' SMK of quadratic functions was identified based on their responses to the quadratic function concept questionnaire (see Appendix A). For an elaborate discussion of teachers' SMK of quadratic functions, the results are presented under three sections: common content knowledge, specialized content knowledge, and horizon content knowledge.

4.1.1. Teachers' Common Content Knowledge of Quadratic Functions

The first 18 items of the questionnaire evaluated the teachers' CCK regarding quadratic functions. Figure 4.1 shows a general review of teacher's CCK based on their scores taken from the CCK items of the questionnaire. The maximum score that can be taken from CCK items is 36 points. The participants' scores range from 9 points (T5) to 34 points (T16), out of 36 points.

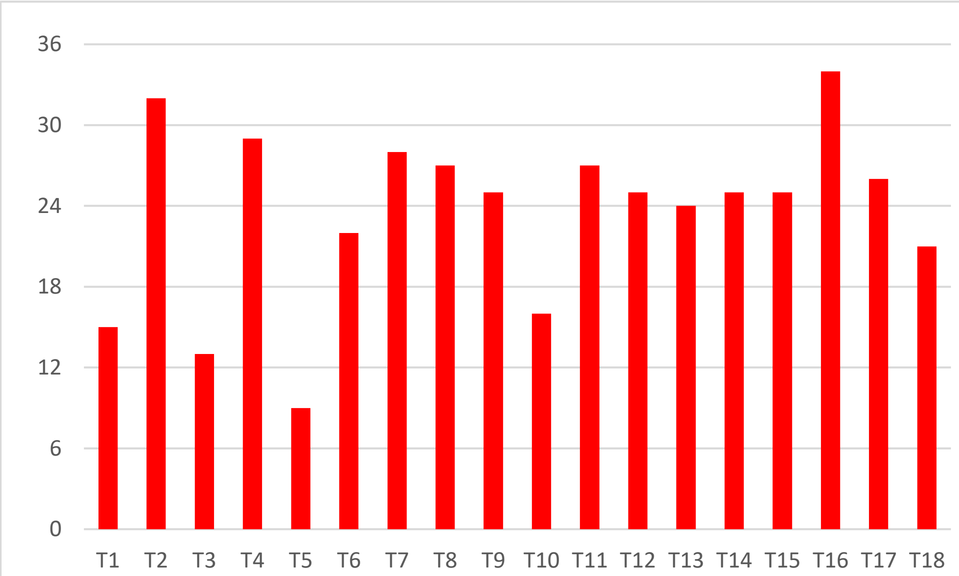


Figure 4.1. The teachers' scores on CCK items on the quadratic function concept questionnaire

For an elaborate discussion of teachers' CCK, the results are presented under seven headings in the following sections.

4.1.1.1. Teachers' Conceptions of Quadratic Functions and Equations

There were 4 items for assessing and evaluating the teachers' conceptions of quadratic functions and equations. Table 4.1 summarizes the teachers' responses to these questions.

Table 4.1. Teachers' (n=18) responses to the questions related to CCK1—Teachers' conception of quadratic equations and functions

Category of Responses	f	Teachers
Question 1: Defining a quadratic equation		
Structural description (2 points)	13	T2, T6, T8, T9, T11, T12, T13, T14, T15, T1, T17, T10, T3.
Structural and procedural descriptions (2 points)	1	T16.
Writing main characteristics of quadratic equations (1 point)	1	T18.
No answer (0 point)	3	T4, T5, T7.
Question 2: Defining a quadratic function		
Structural description (2 points)	14	T1, T2, T4, T5, T6, T7, T8, T9, T11, T12, T13, T14, T15, T17.
Structural and procedural descriptions (2 points)	1	T16.
Writing main characteristics of quadratic functions (1 point)	1	T18.
No answer (0 point)	2	T3, T10.
Question 3: Explaining the relationship between quadratic equations, quadratic functions, and quadratic polynomials		
Based on their standard forms (2 points)	7	T12, T7, T1, T4, T15, T13, T3.
Based on their geometrical aspects (2 points)	2	T16, T10.
Based on their characteristics (2 points)	4	T18, T2, T11, T17.
Incorrect (0 point)	3	T9, T8, T14.
No answer (0 point)	2	T6, T5.
Question 4: Deciding whether given x and y values belong to a quadratic function or a linear function		
Examining the first differences (1 point)	11	T5, T4, T3, T11, T15, T18, T17, T16, T8, T7, T1.
Using their algebraic forms (1 point)	2	T12, T2.
Examining their graphs (1 point)	3	T9, T6, T10.
No answer (0 point)	2	T14, T13.

In the first question, the teachers were asked to define a quadratic equation. Most of the participants (n = 13) made the algebraic definition of a quadratic equation using its standard form. Their answers were mostly like: "A quadratic equation is of the form $ax^2 + bx + c = 0$; $a, b, c \in \mathbb{R}$; $a \neq 0$." (T2). One teacher (T16) defined a quadratic equation by emphasizing its meaning as a process. He wrote: "A quadratic equation $(ax^2 + bx + c = 0)$ is a tool for finding the x-intercepts of a quadratic function." Another teacher (T18) wrote some main characteristics of quadratic equations, rather than writing a clear definition. He stated that: "The highest degree of x is 2. It is not linear."

In the second question, the teachers were asked to define a quadratic function. Most teachers (n=14) defined a quadratic function using its standard form as in the previous question. Their definitions were like: "A quadratic function is $f(x) = ax^2 + bx + c$; $a, b, c \in \mathbb{R}$; $a \neq 0$" (T1). Unlike the others, one teacher (T16) considered geometrical aspects of a quadratic function. He stated: "Quadratic functions ($f(x) = ax^2 + bx + c$) generate parabolas." There was also one teacher (T18) who emphasized some properties of functions in general, not in particular to quadratic functions. He wrote: "There are one domain and one range; a function is a relation between two sets." This kind of answer does not contain any specific information about quadratic functions.

In the third question, the teachers were asked to identify the differences or similarities between quadratic equations, quadratic functions, and quadratic polynomials. Seven teachers wrote some similarities and differences based on their standard forms. For example, one of them (T3) stated: "Quadratic equation: $ax^2 + bx + c = 0$, quadratic function: $f(x) = ax^2 + bx + c$, quadratic polynomial: $p(x) = ax^2 + bx + c$." However, four teachers explained the differences based on the main characteristics of them. For example, T11 stated: "The quadratic functions involve some relation between two sets; however, quadratic equations involve equality to a constant." Two teachers mentioned some geometrical aspects of these concepts. One of them stated, "As a difference, the graph of a quadratic function is a parabola" (T10). Some explanations were either incorrect or not directly related to the interrelation between the three concepts. To illustrate, T8 wrote that "$p(x, y) = 2xy + x^2 + y^2$ is a quadratic polynomial." Likewise, T14 stated incorrectly that "The graph of a quadratic equation is a quadratic function."

In the fourth question, the teachers were given two tables, each including some numerical x and y values. Then, they were asked to decide whether those values belong to a linear function or a quadratic function. Some participants (*n*=11) calculated the first differences of the two functions, then decided whether they were linear or quadratic. For example, T7 wrote: "The first difference is constant, so the first one is linear. In the second one, the first difference is not constant. It is not linear." Two teachers wrote standard forms of linear and quadratic functions, as $y = ax + b$ and $y = ax^2 + bx + c$, respectively. Then, they calculated a, b, and c coefficients and found the equations of both functions. Three participants sketched the graphs of both functions roughly to decide whether the given values belong to linear or quadratic functions.

4.1.1.2. Teachers' Knowledge of Solving Quadratic Equations with One Unknown

There was one question that assessed and evaluated teachers' knowledge of solving quadratic equations with one unknown. The teachers were asked to solve three quadratic equations and state some alternative methods for solving quadratic equations. Table 4.2 summarizes the teachers' responses to this question.

Table 4.2. Teachers' (n=18) responses to the question related to CCK2— Teachers' knowledge of solving quadratic equations with one unknown

Answers	***f***	**Teachers**
Question 5: Solving quadratic equations with one unknown		
Using the quadratic formula (2 points)	17	T18, T17, T15, T14, T13, T12, T11, T9, T8, T7, T6, T5, T4, T2, T1, T3, T10.
Completing the square (2 points)	1	T16.
Follow-up question: Alternative ways for solving quadratic equations		
Algebra tiles	2	T9, T11.
Factorization	6	T9, T8, T13, T14, T12, T16, T17.
Change of variables	2	T9, T12.
Completing the square	3	T8, T4, T7.
The quadratic formula	1	T16.
None	7	T18, T6, T15, T2, T3, T10.

As seen in Table 4.2, almost all teachers (n=17) solved the equations using the quadratic formula. They first calculated the discriminant of the quadratic equation, and then used the quadratic formula to find the roots of the quadratic equation. Only one teacher (T16) solved the quadratic equations without using the quadratic formula. He used the method named "completing the square." A follow-up question also asked for alternative ways to solve quadratic equations other than those they had just used. The teachers stated various methods for solving quadratic equations such as algebra tiles, factorization, change of variables, and completing the square. Some teachers (n=7) did not suggest any alternative method. One teacher (T16), who had solved the equations by completing the square, suggested using the quadratic formula as an alternative method for solving quadratic equations.

4.1.1.3. Teachers' Knowledge of Sketching and Interpreting the Graphs of Quadratic Functions

Six items in the questionnaire assessed and evaluated the teachers' knowledge of sketching and interpreting the graphs of quadratic functions. Table 4.3 summarizes the teachers' responses to these items.

Table 4.3. Teachers' (n=18) responses to the questions related to CCK3—Teachers' knowledge of sketching and interpreting the graphs of quadratic functions

Answers	*f*	Teachers
Question 6: Defining the axis of symmetry of a quadratic function		
Structural description (2 points)	3	T16, T2, T7.
Procedural description (1 point)	11	T15, T10, T4, T12, T6, T14, T13, T11, T1, T9, T8.
Incorrect (0 point)	1	T17.
No answer (0 point)	3	T3, T5, T18.
Question 7: Defining the vertex of a quadratic function		
Structural description (2 points)	9	T2, T4, T7, T9, T11, T13, T14, T16, T18.
Procedural description (1 point)	5	T15, T1, T6, T8, T12, T17.
Incorrect (0 point)	1	T17
No answer (0 point)	3	T3, T5, T10.
Question 8: Defining the concavity of a quadratic function		
Structural description (2 points)	3	T4, T7, T16.
Procedural description (1 point)	7	T8, T9, T11, T15, T12, T13, T14.
Incorrect (0 point)	1	T2.
No answer (0 point)	7	T6, T5, T3, T17, T18, T10, T1.
Question 9: Finding some properties of a function and sketching the graph of it		
Drawing the correct graph (2 points)	18	All.
Question 10: Writing the quadratic function whose graph is given		
Finding the correct quadratic functions (2 points)	15	T15, T13, T6, T12, T4, T7, T9, T14, T18, T17, T16, T11, T8, T2, T10.
No answer (0 point)	3	T1, T3, T5.
Question 11: Determining the signs of the coefficients of a quadratic function by examining its graph		
Finding the signs of all the coefficients correctly (2 points)	17	All.
Finding one or two of the coefficients wrongly (1 point)	1	T18.

In the first one, the teachers were asked to explain what the axis of symmetry means for a quadratic function. Although the teachers were asked to explain the meaning of the axis of symmetry, most of them (n=11) described how to find the axis of symmetry. Their responses included: "It is the line $x = -b/2a$, passing through the vertex." (T6). Three teachers wrote a structural definition of the axis of symmetry.

For example, one of them stated: "It is the line separating the parabola into two symmetrical parts." (T16). One teacher (T17) incorrectly defined the axis of symmetry as "the apsis of the vertex".

Another question asked the teachers to explain what the vertex of a quadratic function means. As in the previous item, some participants (*n*=5) defined how to find the vertex, rather than explaining what it is. Their responses were like: "The vertex of a quadratic function is $(\frac{-b}{2a}, f(\frac{-b}{2a})$." (T12). This statement was considered as a procedural description of the vertex, rather than a structural one since it is about how to find the vertex on a parabola, rather than explaining what it means for a quadratic function. One teacher (T17) wrote an incorrect definition of the vertex: "It is the ordinate of the maximum or the minimum point of a parabola." Half of the teachers (*n*=9) wrote a structural definition of the vertex. To illustrate, one of them stated: "The vertex is the maximum or minimum point of the quadratic function." (T13). This statement described what the vertex means for a quadratic function, rather than how to find it, or where it is located on a parabola.

Another question asked the teachers to explain what concavity means for quadratic functions. Few teachers (*n*=3) made structural descriptions of the concavity of a parabola by referring to its shape. For example, one of them wrote: "A parabola is concave down if it is ∩-shaped; concave up if it is ∪-shaped." (T4). Some teachers (*n*=7) made procedural descriptions of concavity by referring to its relationship with the leading coefficient of a quadratic function. For example, one of them, T8, wrote: "If $a > 0$, f is concave up; if $a < 0$, f is concave down." Another teacher, T12, wrote: "If f'' is positive, f is concave up; if f'' is negative, f is concave down."

In the next question, the teachers were asked to find some properties of a given quadratic function such as the vertex, the axis of symmetry, and *x*-intercept(s); then graph it. All of the teachers (*n*=18) found all the properties of the function and sketched its graph appropriately. Then, the teachers were given two parabolas and asked to find the corresponding quadratic functions. Most teachers (n=15) found the quadratic functions for both graphs correctly. They used different algebraic representations of quadratic functions. When the vertex was given in the graph, the teachers used the vertex form to obtain the quadratic function. When the x-intercepts were given, they used the intercept form to find the quadratic function.

Lastly, the teachers were given a graph and asked to comment on the signs of a, b, and c coefficients of the corresponding quadratic function. Almost all teachers (n=17) found the signs of the three coefficients correctly. While finding the sign of a, all the teachers used the same pattern. They checked the concavity of the graph and stated that $a > 0$ since the parabola is upwards." To determine the sign of b, two different approaches were observed. Most of the teachers (n=11) examined the sign of the apsis of the vertex, used the information that a is positive, and concluded that b is positive. For example, T3 wrote: "We know a is positive, $-b/2a$ is negative; so, b should be positive." (T3). However, some teachers (n=6) examined the sign of the sum of the roots to decide the sign of b. Since the roots were given in the graph, the teachers could easily comment on the sign of the sum of the roots. For example, T6 stated: "a is positive, and the sum of the roots $-b/2a$ is negative, so $b > 0$." There was also an incorrect response: "Since there are two roots, b should be positive." (T18). While determining the sign of c, most participants (n=15) examined the y-intercept. Since the ordinate of the y-intercept was on the lower side of the y-axis, they found that c should be negative. There were also a few teachers (n=3) who found the sign of c by checking the sign of the multiplication of the roots. For example, T9 stated: "a is positive. The multiplication of roots (c/a) is negative, so $c < 0$."

4.1.1.4. Teachers' Knowledge of Graphing Quadratic Functions Using Transformations

There were three items that assessed and evaluated teachers' knowledge of graphing quadratic functions using transformations. Table 4.4 summarizes the teachers' responses to these items.

Table 4.4. Teachers' (*n*=18) responses to questions related to CCK4— Teachers' knowledge of graphing quadratic functions using transformations

Answers	***f***	**Teachers**
Question 12: Explaining how to generate any quadratic function from the graph of $f(x) = x^2$		
Describing some of the transformations (1 point)	7	T17, T13, T7, T8, T6, T4, T12.
Describing all the transformations (2 points)	2	T2, T16.
No answer (0 point)	9	T1, T3, T5, T9, T10, T11, T14, T15, T18
Question 13: Comparing the width of the graphs of quadratic functions		
Examining the leading coefficients of quadratic functions (2 points)	7	T2, T4, T8, T14, T15, T17, T16.
Incorrect (0 point)	2	T9, T18.
No answer (0 point)	9	T7, T11, T13, T1, T3, T10, T5, T6, T12.
Question 14: Comparing the graphs of the quadratic functions $f(x) = x^2 - 5$ and $g(x) = (x-5)^2$		
Comparing the transformations made onto $f(x) = x^2$ to obtain the two functions (2 points)	7	T17, T9, T7, T8, T6, T4, T2.
Comparing some characteristic of the quadratic functions (1 point)	6	T16, T18, T13, T11, T15, T14.
Incorrect (0 point)	1	T12.
No answer (0 point)	4	T1, T3, T5, T10.

Firstly, the teachers were asked to write their responses to a student's claim that it is possible to generate the graph of any quadratic function by applying some transformations on the graph of $f(x) = x^2$. Half of the teachers (*n*=9) stated that the student is right. However, their explanations were different from each other. For example, seven teachers stated that it is possible by making some translations on the graph of $f(x) = x^2$. One of them, T17, elaborately explained vertical and horizontal translations, and he wrote: “$y = f(x - a)$ is the translation along the x-axis a unit right; $y = f(x + a)$ is the translation along the x-axis a unit left. $y = f(x) - a$ and $y = f(x) + a$ are translations along the y-axis a unit below and above.” Two teachers mentioned reflection, translations, and stretching as graph transformations. One of them, T2, stated: “ The student is right. $f(x) = a(x - r)^2 + k$. We can first make horizontal and vertical translations. Then, reflect the graph according to the sign of a, then shrink or stretch it.”

The second question asked the teachers to find the quadratic function generating the widest parabola, among the given four ones. Some teachers (n=7) correctly found the quadratic function that generated the widest parabola. For example, T17 wrote: “The smaller the $|a|$ becomes, the wider the parabola becomes. So, the answer is C.” Two teachers suggested some incorrect strategies to decide the widest parabola. One of them, T9, calculated the difference of the roots and wrote: “$x_1 - x_2$=$\sqrt{\Delta}/|a|$. The answer is D, because $x_1 - x_2 = \sqrt{24}$, the biggest difference.*”* Another teacher, T18, established a relationship between the b coefficient and the width of the parabola, and stated: “The answer is A, because b is the biggest.”

The last question about graph transformations is about comparing the graphs of the two functions $f(x) = x^2 - 5$ and $g(x) = (x - 5)^2$. Some teachers (n=7) made comparisons based on the transformations made on the quadratic function $y = x^2$. For example, T8 wrote: “Both of the functions can be obtained by applying some translations on $y = x^2$. $f(x) = x^2 - 5$ is obtained by translating $y = x^2$, 5 units below along the y-axis; whereas $g(x) = (x - 5)^2$ is obtained by translating $y = x^2$, 5 units right along the x-axis.” There were also six teachers who compared the two functions based on their some characteristics without referring to any transformations. For example, T14 stated: “$f(x) = x^2 - 5$ intersects the x-axis at two different points, whereas $g(x) = (x - 5)^2$ is tangent to the x-axis." There was also an incorrect response that T12 wrote: "$g(x) = (x - 5)^2$ is parallel to the x-axis."

4.1.1.5. Teachers' Knowledge of Solving Real-Life Problems regarding Quadratic Functions

To assess and evaluate teachers' knowledge of solving real-life problems regarding quadratic functions, they were given a real-life problem and asked to solve it. Table 4.5 shows the teachers' responses to this question.

Table 4.5. Teachers' responses to questions related to CCK5— Teachers' knowledge of solving real-life problems regarding quadratic functions

Question 15: Solving a real-life problem that can be modeled by a quadratic function		
Answers	***f***	**Teachers**
Using an algebraic model (2 points)	1	T16.
Using a numerical approach (1 point)	2	T11, T2.
No answer (0 point)	15	T1, T3, T5, T10, T12, T4, T7, T9, T13, T14, T15, T18, T17, T8, T6.

In the question, there was a mathematical magazine whose price should be increased due to an increase in paper and production costs. The problem also stated that an increase in the selling price would cause a decrease in sales. The teachers were asked to suggest the new price that would yield the maximum profit. Most of the teachers (n=15) did not respond to this question. Only one teacher (T16) used an algebraic model to solve the problem. He defined a quadratic function that represented the income and calculated its vertex to find the maximum income. His solution was (as reproduced for readability):

Income: $(5,5).25000$

Income after the increase in the price: $g(x)$: $(5,5+\frac{1}{2}x).(25000-1250x)$

For $r=4,5$ the function has the maximum. $5,5+\frac{1}{2}.4,5=7,75$. So, the selling price should be 7,75 TL.

There were also some numerical approaches used by two teachers, without using a quadratic function. For example, the solution of T2 is presented below (as reproduced for readability):

$25000.5,5=137500$
$23750.6=142500$
$22500.6,5=146500$
$21250.7=148750$
$\underline{20000.7,5=150000}$
$\underline{18750.8=150000}$
$17500.8,5=148750$. So, I could suggest the selling price as 7,5 TL.

4.1.1.6. Teachers' Knowledge of Finding the Quadratic Functions with Given Points

There were two items in the questionnaire that assessed and evaluated teachers' knowledge of finding the quadratic functions passing through specific points. Table 4.6 summarizes the teachers' answers to these two items.

Table 4.6. Teachers' responses to questions related to CCK6— Teachers' knowledge of finding the quadratic function with given points

Answers	***f***	**Teachers**
Question 16: Finding the quadratic equation with its vertex and one point given		
Finding the correct quadratic function (2 points)	14	T6, T2, T8, T11, T16, T17, T18, T15, T14, T13, T9, T7, T4, T12.
No answer (0 point)	4	T1, T3, T5, T10.
Question 17: Finding the quadratic function with three points given		
Finding the correct quadratic function (2 points)	12	T12, T3, T7, T9, T14, T17, T16, T11, T8, T2, T4, T18.
No answer (0 point)	6	T1, T5, T10, T6, T13, T15.

In the first one, the teachers were asked to find the quadratic function given the vertex and one point on it. Most of the teachers (*n*=14) correctly found the quadratic function in the vertex form $y = a(x - r)^2 + k$. In the second one, the teachers were asked to determine the quadratic function whose arbitrary three points were given. This time, most teachers (*n*=12) used the standard form $y = ax^2 + bx + c$ to find the quadratic function. Teachers used different algebraic demonstrations of the quadratic functions.

4.1.1.7. Teachers' Knowledge of Finding the Intersection of Parabolas and Lines

There were one question and a follow-up question assessing and evaluating teachers' knowledge of finding the intersection of a line and a parabola. The teachers' responses to these items were summarized in Table 4.7.

Table 4.7. Teachers' answers to questions related to CCK7—Teachers' knowledge of finding the intersection of a parabola and a line

Answers	*f*	Teachers
Question 18: Explaining the conditions for the intersection of a parabola and a line		
Stating the three conditions for the intersection of a line and a parabola (2 points)	15	T15, T13, T6, T12, T4, T7, T9, T14, T18, T17, T16, T11, T8, T2, T10.
No answer (0 point)	3	T1, T3, T5.
Follow-up: Finding the intersection of a line and a parabola		
Correctly finding the point of intersection	15	T15, T13, T6, T12, T4, T7, T9, T14, T18, T17, T16, T11, T8, T2, T10.
No answer	3	T1, T3, T5.

Firstly, the teachers were asked to explain the conditions for a parabola $y = ax^2 + bx + c$ and a line $y = mx + n$ to intersect. Almost all teachers (*n*=15) correctly stated the conditions for the intersection of a line and a parabola. They equated the y-values of the parabola and the line; and obtained a quadratic equation. Then, they wrote similar statements to this one: "If $\Delta < 0$, they do not intersect. If $\Delta = 0$, the parabola is tangent to the line. If $\Delta > 0$, they intersect at two different points." (T14). In the follow-up of this question, the teachers were asked to find the points of intersection of a parabola and a line. Most of them (*n*=15) found the solution by using the same strategy they had explained previously. First, they obtained a new quadratic equation by equating the parabola and the line. Then, they calculated the discriminant of this new quadratic equation and stated similar statements like "the parabola and the line intersect at one point", or, "the parabola is tangent to the line."

4.1.2. Teachers' Specialized Content Knowledge of Quadratic Functions

For a general review of each teacher's SCK, the following graph is presented (Figure 4.2). The graph shows the teachers' scores from the SCK items of the questionnaire. There were 10 items (questions 19-28) in the questionnaire that assessed and evaluated teachers' SCK. The maximum score that can be taken from SCK items is 18 points. The teachers' scores range between 2 points (T6) and 17 points (T16).

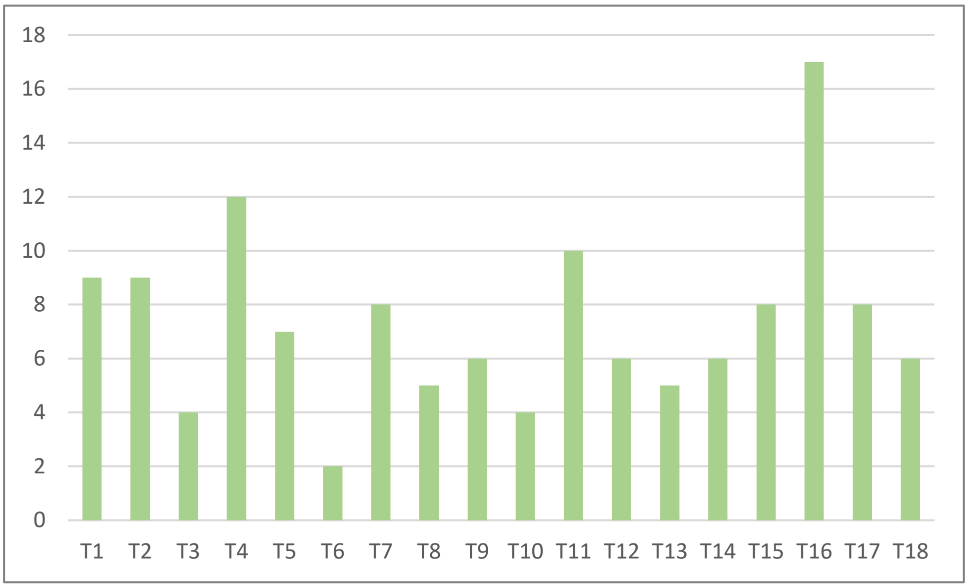


Figure 4.2. The teachers' scores on SCK items on the quadratic function concept questionnaire

For a detailed description of teachers' SCK, the results are presented under seven headings in the following sections.

4.1.2.1. Teachers' Knowledge of Explaining and Justifying Basic Formulas of Quadratic Functions

There were two items assessing and evaluating teachers' knowledge of explaining and justifying basic formulas of quadratic functions. The teachers' responses to these items were summarized in Table 4.8.

Table 4.8. Teachers' (n=18) responses to questions related to SCK1— teachers' knowledge of explaining and justifying basic formulas of quadratic functions

Answers	*f*	Teachers
Question 19: Stating and justifying the quadratic formula		
Algebraic justification only (1 point)	6	T18, T16, T4, T3, T1, T11.
No justification (0 point)	9	T14, T13, T12, T10, T9, T8, T2, T17, T15.
No answer (0 point)	3	T7, T6, T5.
Question 20: Solving a quadratic equation without using the quadratic formula		
Solving by completing the square (2 points)	11	T17, T1, T2, T3, T4, T5, T8, T11, T12, T13, T16.
Incorrect (0 point)	2	T10, T15.
No answer (0 point)	5	T6, T7, T9, T14, T18.

Firstly, the teachers were asked to state the quadratic formula, and explain how it is derived, both geometrically and algebraically. Some teachers (n=9) just wrote the quadratic formula as "$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$" without any justification. None of the teachers made a geometrical justification of the quadratic formula. However, six teachers made an algebraic justification. One of them, T11, wrote (as reproduced for readability):

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right) = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Then, the teachers were given a quadratic equation $x^2 - x + 1 = 0$ and they were asked to solve it without using the quadratic formula. Many teachers (n=11) solved the equation by completing the square method. To illustrate, the solution of T13 is presented below (as reproduced for readability):

$$x^2 - x + \frac{1}{4} - \frac{1}{4} + 1 = 0$$

$$\left(x - \frac{1}{2}\right)^2 = -\frac{3}{4}.$$

$x_1 = \frac{1+\sqrt{3}i}{2}$ and $x_2 = \frac{1-\sqrt{3}i}{2}$."

There was one teacher (T15) who did not solve the equation but stated: "The equation can be solved by factorization." This claim is not correct since the given equation cannot be factorized. There was also one teacher (T10) who attempted to solve the quadratic equation by using a third-order equation (as reproduced for readability):

$$x^3 + 1 = (x + 1)(x^2 - x + 1) = 0$$

$$x^3 = -1$$

$$x = -1,\ x = i,\ x = -i.$$

The above solution is incorrect since the numbers -1, i and $-i$ are not the roots of the given quadratic equation. Moreover, the use of a third-order equation is not one of the strategies for solving quadratic equations.

4.1.2.2. Teachers' Knowledge of Posing Real-Life Problems Regarding Quadratic Functions

For assessing and evaluating the teachers' knowledge of posing real-life problems regarding quadratic functions, they were asked to provide an example of a real-life problem that can be modeled and solved by a quadratic function. The teachers' responses were summarized in Table 4.9.

Table 4.9. Teachers' responses to questions related to SCK2— Teachers' knowledge of posing real-life problems regarding quadratic functions

Question 21: Stating a real-life problem about quadratic functions		
Answers	***f***	**Teachers**
Writing a problem statement (2 points)	1	T16
Writing a problem context (1 point)	10	T18, T15, T12, T11, T9, T8, T7, T4, T2, T1, T10.
No answer (0 point)	7	T14, T13, T6, T5, T3, T17.

Many teachers (*n*=10) wrote a problem context rather than the full statement of a real-life problem. These contexts included projectile motion, velocity-acceleration problems, and calculation of cost and profit-loss in economics. Only one teacher (T16) stated a problem regarding the maximum/minimum of quadratic functions: "Let the cost of a product be x TL. If the product is sold $\mathrm{x}^2 - 5\mathrm{x} + 14$ TL, what would be the minimum profit?"

4.1.2.3. Teachers' Knowledge of Recognizing Students' Incorrect Solutions Regarding Quadratic Functions

To assess and evaluate the teachers' knowledge of recognizing students' incorrect solutions regarding quadratic functions, they were given a problem related to quadratic functions with an incorrect student solution. They were asked to examine the solution and state whether it is correct or not, by explaining their reasons. The teachers' responses to this item were summarized in Table 4.10.

Table 4.10. Teachers' responses to questions related to SCK3— Teachers' knowledge of recognizing students' incorrect solutions regarding quadratic functions

Question 22: Examining a student's incorrect solution to a given problem regarding quadratic functions		
Answers	***f***	**Teachers**
Explaining all the incorrect steps (2 points)	6	T16, T14, T11, T4, T2, T17.
Explaining some of the incorrect steps (1 point)	9	T15, T13, T12, T10, T9, T7, T5, T3, T1.
No answer (0 point)	3	T6, T18, T8.

Most teachers (*n*=15) noticed that the student's solution is incorrect. Some of them (*n*=6) identified all the incorrect steps in the student's solution and explained them in detail. For example, T17 stated: "The solution is incorrect. The parabola is downwards, the vertex gives the maximum, not the minimum. Also, the ordinate of the vertex gives the max/min value, not the apsis. The endpoints should be checked." (T17). However, half of the teachers (*n*=9) detected some of the student's errors and ignored some others. For example, one teacher (T1) stated: "The student is wrong because the parabola is downwards, the vertex gives the maximum." Another teacher (T10) wrote: "The student did not check the endpoints of the function." These teachers did not make an elaborate description of the student's errors.

4.1.2.4. Teachers' Knowledge of Understanding Students' Unusual Solutions Regarding Quadratic Functions

There were two items in the questionnaire assessing the teachers' knowledge of understanding students' unusual solutions regarding quadratic functions. In both of them, the teachers were given a question and a student's response to this question; then they were asked to comment on the student's solution. Table 4.11 summarizes the teachers' responses to these items.

Table 4.11. Teachers' responses to questions related to SCK4— Teachers' knowledge of understanding students' unusual solutions regarding quadratic equations and functions

Answers	*f*	Teachers
Question 23: Examining a student's solution to a task regarding quadratic equations		
Explaining the student's solution (2 points)	9	T16, T15, T11, T9, T7, T5, T4, T2, T17.
Only stating the solution is correct (0 point)	7	T18, T14, T13, T12, T10, T8, T1.
No answer (0 point)	2	T6, T3.
Question 24: Examining a student's solution to a task regarding quadratic functions		
Explaining the student's solution (2 points)	3	T16, T7, T15.
Only stating the solution is correct (0 point)	6	T17, T18, T12, T8, T5, T4.
Solving the problem using another approach (0 point)	6	T14, T11, T10, T9, T2, T1.
No answer (0 point)	3	T6, T7, T9, T14, T18.

In the first one, a quadratic equation and a student's solution to this equation were given. The student solved the quadratic equation by completing the square, without using the quadratic formula. The teachers were asked to examine the student's solution and decide whether it is correct or not, by explaining the reason for their answers. Some participants (*n*=7) stated that the solution is correct, without writing any explanation. Half of the teachers (*n*=9) stated that the solution is correct, and wrote the name of the student's approach as "completing the square". One teacher (T16) also wrote: "The student solved the equation by completing the square, without using the quadratic formula. This approach is my favorite while teaching quadratic equations. I care about my students understanding the origin of the formula."

In another question, the teachers were asked to examine a student's solution to a problem regarding quadratic polynomials, and decide whether the result is correct or incorrect by explaining their reason. In the problem, some information about the coefficients of a quadratic polynomial was given. Also, one of the roots of the polynomial was given. The question was to find the (unique) quadratic polynomial that satisfies the given conditions. The student found the quadratic polynomial by following some steps. Some teachers (n=6) stated that the solution is correct, without explaining why they thought so. Some other teachers (n=6) also stated that the solution is correct, and they justified the student's solution by finding the quadratic polynomial using a different approach. For example, the solution of one teacher (T10) is: "If one root is $7 + \sqrt{6}$, another is $7 - \sqrt{6}$. The sum of the roots is $\frac{-b}{a} = 14$, and the multiplication of the roots is $\frac{c}{a} = 43$. We know $a = 4$, hence $b = -56$ and $c = 172$. The students' solution is correct." Even though these teachers noticed that the student's solution is correct, they did not really engage in the student's approach. However, their focus was directly on the result, rather than the student's approach or what the student has thought while solving the problem. The teachers obtained the quadratic polynomial using their own approach and compared their results to the student's result. Only three teachers examined the student's approach. For example, one of them (T7) wrote: "The solution is correct. The student made some inverse operations. First, he wrote one of the roots as equal to *x*. Then, he squared the equation and found the result."

4.1.2.5. Teachers' Knowledge of Responding to Students' Why Questions About Quadratic Functions

Two items in the questionnaire assessed and evaluated teachers' knowledge of responding to students' why questions about quadratic functions and equations. The teachers' responses to these items were summarized below (Table 4.12).

Table 4.12. Teachers' responses to questions related to SCK5— Teachers' knowledge of responding to students' why questions about quadratic functions

Answers	*f*	Teachers
Question 25: Responding to a student's question about the effects of the translations on the coefficients of quadratic functions		
Making a correct explanation (2 points)	3	T16, T14, T1.
Incorrect (0 point)	13	T18, T17, T15, T12, T11, T10, T9, T8, T7, T5, T4, T3, T2.
No answer (0 point)	2	T13, T6.
Question 26: Responding to a student's question about dividing both sides of a quadratic equation by a variable		
Making a correct explanation (2 points)	17	T2, T11, T15, T16, T18, T1, T4, T5, T6, T7, T8, T9, T10, T12, T14, T13, T17.
No answer (0 point)	1	T3.

One of the questions was about the transformations made on the parabolas. The teachers were asked to respond to a student's question that why translating a parabola upwards and downwards changes only c while translating a parabola to the left and right changes both b and c in the quadratic function $y = ax^2 + bx + c$. Most of the teachers (n=13) failed to explain the reason for these interrelations. For example, one teacher, T3, stated: "While translating upwards and downwards, the roots do not change. So, only c changes. While translating it to the left and right, roots change. So, everything changes." This statement is not correct; because while translating a parabola upwards and downwards, the roots change. Another incorrect explanation was T18's, who stated: "While moving the parabola upwards and downwards, only c changes because the x value stays constant." Similarly, T12 wrote: "While moving up and down, only c changes because $x = 0$. While moving left and right, the roots change, then the sum and the multiplication of the roots change. So, b and c change." Another teacher (T9) stated: "The reason for this is that the vertical translation does not affect the roots. While translating left and right, the roots change; so both of the values change." Three teachers suggested a plausible explanation for the effects of the translations on the coefficients of the parabola. Their common idea was based on the location of the vertex. For example, one of them (T16) wrote:

> While translating upwards and downwards, the apsis of the vertex does not change. So, the sum of the roots stays constant but the roots change. So, the multiplication of the roots changes. Thus, b stays constant, and c changes. While translating left and right, both the sum of the roots and the multiplication of the roots change. Hence, b and c change.

In another question, the teachers were given an imaginary conservation between two students about the division of a quadratic equation by a variable, x. In the conservation, one student claimed that both sides of the equation cannot be divided by x. Another student responded, "If we can divide both sides by 3, why can't we divide by x?" The teachers were asked to state the most proper explanation for their students. Almost all teachers (n=17) provided plausible explanations. For example, T2 wrote: "I would say that an equation cannot be divided by x, because we can eliminate one of the roots, which is equal to 0." Another similar response was: "I would say that an equation cannot be divided by x, because we don't know the value of x. It might be equal to 0, and 0 cannot divide any number." (T17). As seen in these two responses, some teachers (n=7) provided an explanation based on the elimination of one root, while some (n=10) mentioned the division rule that 0 cannot divide any number.

4.1.2.6. Teachers' Knowledge of Finding an Example to Make a Specific Mathematical Point About Quadratic Functions

To assess and evaluate the teachers' knowledge of finding an example to make a specific mathematical point about quadratic functions, they were asked to state what kind of examples they would use in the classroom to emphasize the symmetrical property of a parabola. The summary of their responses is presented in Table 4.13.

Table 4.13. Teachers' responses to questions related to SCK6— Teachers' knowledge of finding an example to make a specific mathematical point about quadratic functions

Question 27: Stating examples to emphasize the symmetrical property of a parabola		
Answers	***f***	**Teachers**
Writing relevant examples (2 points)	3	T16, T18, T4.
Writing irrelevant examples (0 point)	12	T15, T14, T12, T11, T9, T8, T7, T5, T2 T17, T1, T10.
No answer (0 point)	3	T13, T6, T3.

Some teachers (n=12) made explanations that were not directly related to the symmetrical property of parabolas. One of them, T15, responded: "I would use geometrical examples." Another response was: "I show my students some symmetrical shapes such as a heart shape." Similarly, T11 stated: "I would show them butterfly shape as an example of symmetrical shape and make them understand what symmetrical means." Another teacher (T5) stated: "I would use a mirror." Some of them (n=3) suggested the use of mathematical software to emphasize the symmetrical property of a parabola. For example, T10 stated: "I would draw some parabolas using mathematical software and demonstrate the symmetrical property of parabolas on them." Three teachers proposed a different way of emphasizing the symmetrical property of a parabola. In his response to the questionnaire, one of them, T16, wrote: "I define r (the apsis of the vertex) as the half of the sum of the roots. I tell my students that the x-values that add up to $2r$ are symmetrical. For example, if $r = 5$, $f(1) = f(9)$ or $f(-5) = f(15)$. I want my students to notice this property."

4.1.2.7. Teachers' Knowledge of Modifying Tasks of Quadratic Functions

The teachers were asked one question to assess and evaluate their ability to modify tasks of quadratic functions. The summary of the teachers' responses to this question is shown in Table 4.14.

Table 4.14. Teachers' responses to questions related to SCK7— Teachers' knowledge of modifying tasks of quadratic functions

Question 28: Examining a given task about quadratic functions and modifying the task for their students		
Answers	***f***	**Teachers**
Making some reasonable modifications	7	T16, T14, T12, T11, T9, T8, T4.
Making no modification	4	T17, T15, T10, T1.
Making some unnecessary/irrelevant modifications	2	T18, T7.
No answer	5	T13, T6, T5, T3, T2.

The teachers were given a task about finding an unknown coefficient in a quadratic function and were asked two questions. Firstly, they were asked to examine the task and state whether their students could solve this task or not. Secondly, if they thought that this was an easy/difficult task for their students, they were asked to explain how they could modify it to be harder or easier. Four teachers stated that their students could solve the task; so they made no modifications. Half of the teachers (n=9) stated that the task could be hard for their students. So, they made some modifications to the task. For example, T18 wrote: "I would give extra information about the sign of the sum of the roots or the multiplication of the roots." Another teacher, T7, responded: "I would delete the statement "the distance between A and B is 3 units" and write "one of the roots is 3 more than the other"." However, these modifications do not seem to contribute to making the task easier, so they might be unnecessary. On the other hand, some teachers (n=7) made some plausible modifications. These included:

> "I would give the numerical value of the b coefficient." (T8).
>
> "I would give the sum of the roots as extra information." (T9).
>
> "I would ask a very simple question like $f(x) = x^2 - mx + m + 3$ intersects the x-axis $at\ x = 3$, what is the value of m?" (T11).
>
> "I would give the apsis of the vertex as extra information." (T12).

"I would change the problem as "$f(x) = x^2 - 5x + m - 1$ intersects the x-axis at two different points, A and B. If $|AB| = 3$ units, what is m?"" (T14).

4.1.3. Teachers' Horizon Content Knowledge of Quadratic Functions

For a general review of each teacher's HCK, the following graph is presented (Figure 4.3).

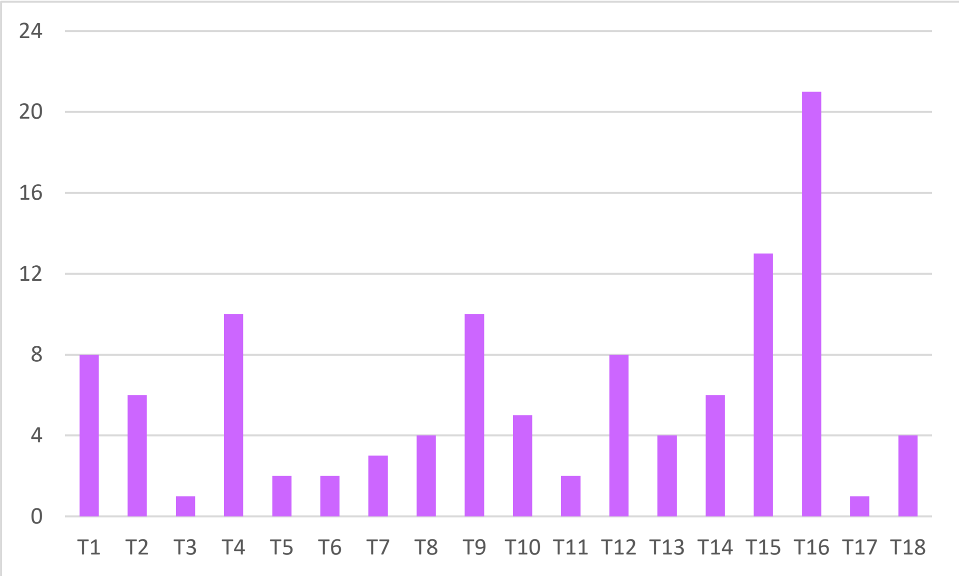


Figure 4.3. The teachers' scores on HCK items on the quadratic function concept questionnaire

The graph shows the teachers' scores from the HCK items on the questionnaire. The items from 29 to 40 evaluated the teachers' HCK. The maximum score that can be taken from the HCK items is 24 points. The teachers' scores range between 1 point (T3, T17) and 21 points (T16). For a detailed description of teachers' HCK, the results are presented under two headings in the following sections.

4.1.3.1. Teachers' Knowledge of How Quadratic Functions Are Related to Other Contents in the High School Curriculum

There were 6 items that assessed and evaluated the teachers' knowledge of how quadratic functions are related to other contents in the high school curriculum. Table 4.15 summarizes the teachers' responses to these items.

Table 4.15. Teachers' responses to questions related to HCK1—Teachers' knowledge of how quadratic functions are related to other contents in the high school curriculum

Answers	*f*	Teachers
Question 29: Explaining the relationship between the concavity of a parabola and the derivative		
Relating the concavity of the graph with the second derivative of the quadratic function (2 points)	1	T16
Irrelevant explanations (0 point)	11	T15, T12, T11, T9, T4, T2, T18, T6, T7, T1, T10.
No answer (0 point)	6	T17, T14, T13, T10, T5, T3.
Question 30: Comparing the graphs of quadratic functions and exponential functions		
Comparing the graphs of quadratic functions and exponential functions correctly(2 points)	4	T10, T12, T15, T16.
Incorrect (0 point)	7	T2, T4, T8, T13, T17, T3, T5.
No answer (0 point)	7	T1, T6, T7, T9, T11, T14, T18.
Question 31: Explaining the relationship between the vertex of a parabola and the derivative		
Explaining the relationship between the vertex and the derivative partially (1 point)	10	T18, T16, T15, T14, T12, T9, T13, T8, T7, T4.
Incorrect (0 point)	1	T2.
No answer (0 point)	7	T6, T5, T1, T17, T11, T10, T3.
Question 32: Relating quadratic functions to any concept from the physics course		
Writing any concept from the physics course related to quadratic functions (2 points)	7	T1, T4, T15, T16, T2, T12, T6.
No answer (0 point)	11	T3, T7, T8, T10, T11, T13, T17, T18, T9, T14, T5.
Question 33: Explaining the relationship between the golden ratio and quadratic equations		
Relating quadratic equations with the golden ratio (2 points)	5	T2, T8, T9, T15, T16.
Only stating the golden ratio (0 point)	4	T7, T14, T3, T5.
No answer (0 point)	9	T6, T10, T11, T13, T17, T18, T12, T4, T1.
Question 34: Determining whether the graph of $y = x^4$ is a parabola or not		
Stating that the graph of $y = x^4$ is not a parabola (2 points)	12	T1, T4, T7, T9, T10, T12, T13, T14, T15, T16, T18, T5.
Incorrect (0 point)	3	T8, T2, T17.
No answer (0 point)	3	T3, T6, T11.

In question 29, the teachers were asked to provide a plausible explanation for why the graph of the quadratic function $f(x) = ax^2 + bx + c$ is concave up if $a > 0,$ concave down if $a < 0$. Only one teacher (T16) made a plausible explanation for this well-known fact of quadratic functions. He explained this fact referring to the second derivative of quadratic functions. Most teachers (n=11) wrote some irrelevant explanations that do not explain the reason for the aforementioned fact of quadratic functions. Their answers were like: "Sketching the graphs of several quadratic functions helps students understand the relationship between the sign of a, and the concavity of the graph." (T7). Similarly, T15 wrote: "I draw different parabolas and demonstrate to my students the change in the concavity of them, according to the sign of a." (T15).

The next question evaluated whether the teachers are able to compare the graphs of quadratic functions and the exponential functions. The teachers were asked to comment on a student's claim related to the patterns of the graphs of functions p and q, where p is an exponential function and q is a quadratic function. In the question, the student stated that after about $x = 3,$ the quadratic function will always take greater values than the exponential function. Some teachers correctly (n=4) stated that the student's claim is false and made plausible explanations. One of them, T12, wrote: "The student is wrong because an exponential function eventually will get bigger than a quadratic function." (T12). There were also some teachers (n=7) who incorrectly stated that the student is right. For example, T3 stated: "When we examine the graphs of the two functions, we see that y-values of the function q is always greater than of the function p."

In another question, the teachers were asked to explain the relationship (if any) between the vertex of a quadratic function and the derivative of the function. One participant (T2) wrote an incorrect answer: "The first derivative of a function can be found by drawing tangents from the vertex of the function." Some teachers (n=10) explained the relationship between the vertex and the first derivative partially. They reported that the first derivative of the function is 0 at the vertex. Although this statement is correct, it is insufficient to explain the relationship between the vertex and the first derivative since it does not include information about the rate of change or the maximum-minimum points.

Then, the teachers were asked to state whether any concept from the physics course is related to quadratic functions. Seven teachers wrote some concepts from the physics course which might be related to quadratic functions. These are; free fall, projectile motion, and velocity-acceleration problems. Most teachers (n=11) did not respond to this question.

In the next question, the teachers were asked to explain how the golden ratio and quadratic equations are related. Half of the teachers (n=9) did not answer. Some teachers (n=4) only stated the numerical value of the golden ratio as $1+\frac{\sqrt{5}}{2}$, without explaining its relation to quadratic equations. Five teachers explained the relationship between the golden ratio and quadratic equations. Their answers included: "The golden ratio is the positive root of the quadratic equation $\mathrm{x}^2-\mathrm{x}-1=0$." (T2).

Another question asked the teachers whether the graph of $y = x^4$ is a parabola or not. Most teachers correctly (n=12) stated that $y = x^4$ is not a parabola, but they did not state a reason for their response. Three teachers gave incorrect answers. Two of them stated that the graph of the function $y = x^4$ is a parabola. For example, T17 wrote: "It is a parabola because it is U-shaped." On the other hand, another teacher, T8, stated: "It is not a parabola because it is so wide. The arms of the parabolas are narrower."

4.1.3.2. Teachers' Knowledge of How Quadratic Functions Are Related to Advanced Mathematics

There were 6 items evaluating the teachers' knowledge of how quadratic functions are related to advanced mathematics. The summary of the teachers' responses is presented in Table 4.16.

Table 4.16. Teachers' responses to questions related to HCK2—Teachers' knowledge of how quadratic functions are related to advanced mathematics

Answers	*f*	Teachers
Question 35: Stating the reflection property of a parabola and its daily use		
Explaining the reflection property and its daily use correctly (2 points)	3	T16, T9, T4.
Incorrect (0 point)	6	T2, T18, T12, T7, T8, T14.
No answer (0 point)	9	T17, T15, T13, T11, T10, T6, T5, T3, T1.
Question 36: Explaining the relationship between a parabola and a hyperbola		
Explaining some differences between parabolas and hyperbolas (2 points)	3	T16, T2, T9.
Incorrect (0 point)	5	T8, T4, T12, T15, T18.
No answer (0 point)	10	T1, T3, T5, T6, T7, T10, T11, T13, T14, T17.
Question 37: Stating the fundamental theorem of algebra and its application to quadratic polynomials		
Applying the fundamental theorem of algebra to quadratic polynomials (2 points)	5	T1, T14, T15, T16, T4.
Incorrect (0 point)	1	T2.
No answer (0 point)	12	T3, T5, T6, T7, T8, T9, T10, T11, T12, T13, T17, T18.
Question 38: Choosing the most proper statement about a parabola		
Statement 1 is correct (0 point)	2	T1, T2.
Statement 2 is correct (0 point)	13	T12, T16, T7, T8, T9, T11, T14, T15, T17, T18, T3, T5, T6.
None (0 point)	2	T4, T13.
No answer (0 point)	1	T10.
Question 39: Defining a parabola and stating alternative definitions		
Describing the parabola as the graph of a quadratic function and stating the geometrical definition of the parabola (2 points)	1	T16.
Describing the parabola as the graph of a quadratic function only (1 point)	11	T1, T4, T8, T10, T11, T13, T14, T15, T17, T18, T3.
Incorrect (0 point)	4	T2, T7, T9, T12.
No answer (0 point)	2	T5, T6.
Question 40: Deciding whether a given shape is a parabola or not		
Distinguishing between a parabola and a catenary (2 points)	1	T16.
Stating that it is not a parabola without explanation (1 point)	5	T15, T9, T1, T12, T11.
Incorrect (0 point)	7	T18, T14, T13, T8, T4, T2, T3.
No answer (0 point)	5	T17, T10, T7, T6, T5.

Firstly, the teachers were asked to explain the reflection property of a parabola and its daily use. Half of the teachers (*n*=9) did not respond. Some teachers (*n*=6) made some explanations that were not related to the reflection property. One of them, T14, stated: "Arch bridges have a parabolic shape." Another teacher, T12, stated: "I would tell my students that the vertex of a parabola is the axis of symmetry." On the other hand, three teachers correctly stated the reflection property and its daily use. For example, one of them, T16, wrote: "A ray that is parallel to the axis of symmetry of the parabola is reflected and passes through the focus. It is used in the real-life in the construction of headlights and satellite dishes."

The teachers were also asked to explain the relationship (if exists) between a parabola and a hyperbola. Most of the teachers (*n*=10) did not respond to this question. Few teachers (*n*=3) described the properties of a parabola and a hyperbola. One of them, T16, wrote: "They both are conic sections. A parabola is the set of points which are equidistant from a straight line and focus whereas a hyperbola is the set of points whose distances to two fixed points have a constant difference." Some teachers (*n*=5) made some incorrect explanations as illustrated below:

> "A hyperbola is the symmetry of a parabola." (T4)
>
> "A parabola is of the form $y = ax^2$, whereas a hyperbola is of the form $x = ay^2$." (T15)
>
> "They both are the graphs of quadratic functions." (T18)

In question 37, the teachers were asked to state the Fundamental Theorem of Algebra and how it applies to quadratic polynomials. Most teachers (*n*=12) did not respond to this question. One teacher (T2) gave an answer that was not directly related to this theorem and its application to quadratic polynomials. He drew some squares and represented a quadratic function by completing the square. Five teachers stated the theorem and its application to quadratic polynomials. For example, one of them, T14, stated: "A polynomial with degree n has n roots. Quadratic polynomials have two roots". Another teacher wrote: "Quadratic equations have 2 roots. If the discriminant is less than 0, it has no roots.” In the next question, the teachers were given two statements that were written by two students. These are:

Student 1: The graph of a quadratic function is a parabola.

Student 2: The graph of a quadratic function is called a parabola.

Then, they were asked to select the most correct statement with a justification for their answer. A few teachers (n=2) stated that the statement of Student 1 is the most correct. One of them, T1, who selected the first statement wrote: “The second statement is a definition, but a parabola cannot be defined. So Student 1 is correct.” (T1). Another teacher, T2, wrote: “Student 1 is right because the other name for the parabolas is quadratic functions.” Both of these explanations are incorrect. On the other hand, most of the teachers (n=13) thought that Student 2 is correct. Most of them (n=11) did not state any reason for their answer. Only one of them, T12, made an explanation: “The second statement is correct because it is a definition.” There were also two teachers who stated that none of the statements is correct. One of them, T4, wrote that the correct statement should be: “The graph of a quadratic polynomial function is called a parabola.” Another teacher, T13, stated that the correct statement should be: “The graph of a polynomial function $f(x) = ax^2 + bx + c$, $(a \neq 0,\ a, b, c \in \mathbb{R})$ is a parabola.” The common idea in the previous two responses is based on adding the term *polynomial* before the word *function*.

In the next question, the teachers were asked to define a parabola and state some alternative definitions for it. The purpose of this question was to examine whether the teachers have any idea about the geometrical definition of a parabola as the question that asked about the parabola-hyperbola relationship. Some teachers (n=4) wrote some incorrect statements like:

“A parabola is a quadratic function.” (T2)

“A parabola is the graph of a quadratic equation.” (T9)

“A parabola is a quadratic equation. (T12)”

Most of the teachers (*n*=11) defined a parabola as the graph of a quadratic function and did not suggest any alternative definitions. Only one teacher, T16, presented an alternative definition. He wrote: "A parabola is the graph of a quadratic function. Alternative definition: A parabola is the set of points that are equidistant from both the directrix and the focus."

The next question evaluated whether the teachers could distinguish a parabola from a catenary. For this purpose, the teachers were shown a figure (the shape of a uniform flexible chain) and asked to state whether that shape is a parabola or not. Some participants (*n*=7) stated that it is a parabola, without further explanation, while some (*n*=5) stated that it was not a parabola. For example, T15 wrote: "I would say that it resembles a parabola, but it is not." Only one teacher, T16, mentioned a *catenary*, which is a curve formed by a wire, rope, or chain hanging freely from two points that are not in the same vertical line. He wrote: "I would say that it is a catenary."

4.1.4. Summary of Teachers' Subject Matter Knowledge of Quadratic Functions

Teachers' SMK of quadratic functions was discussed in the previous sections, on the basis of their CCK, SCK, and HCK. In analyzing the results of the questionnaire, the teachers' responses to CCK, SCK, and HCK items were discussed separately. The graph in the Figure 4.4 summarizes the teachers' overall performance on the quadratic function concept questionnaire. It also enables the reader to compare an individual teacher's scores on each dimension of the questionnaire. The maximum scores of each dimension in the questionnaire were not equal (36 points for CCK items, 18 points for SCK items, and 24 points for HCK items) as the number of items included in these dimensions were not equal. Thus, the teachers' scores on each dimension were modified to be out of 100 points to make a more meaningful comparison between teachers' performances on each dimension. As shown in Figure 4.4, the total score of the questionnaire is 300 points.

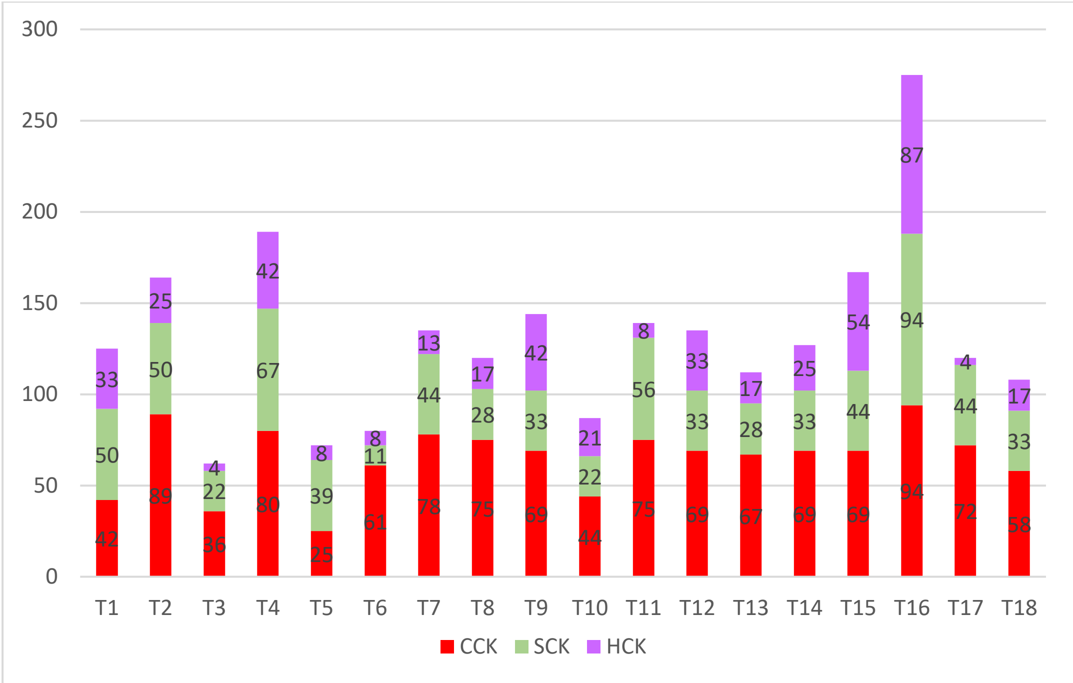


Figure 4.4. The teachers' scores on CCK, SCK, and HCK items

In the graph, each dimension is represented by a different color. The purple area represents the teachers' HCK scores, the green area represents the teachers' SCK scores, and the red area represents the teachers' CCK scores. However, it is not intended to say that these dimensions are disjoint. Although this study attempted to differentiate and measure each dimension, it is an undeniable fact that these sub-dimensions of teacher knowledge interact with each other. Thus, it might be useful to note that the purpose of this graph is to present a general picture of teachers' SMK and compare their scores on CCK, SCK, and HCK items in the questionnaire. As it can be seen in Figure 4.4, the majority of teachers' performance on the CCK items are remarkably better than their performances on the SCK and HCK items. Moreover, teachers' performances on the HCK items are the lowest among the three dimensions for most of the participants.

4.2. Contribution of Subject Matter Knowledge to Student Learning Outcomes: The Case of Can

In this section, the case of Can was presented. The data obtained from the questionnaire, the interview, and the classroom observation were combined and triangulated. Can is the teacher who was referred to as “T17” in the first phase of the study. He has 19 years of teaching experience, and he was teaching at an Anatolian High School when this study was conducted.

His overall performance on the quadratic function concept questionnaire was moderate. He performed better in the questions related to CCK, however, he failed to solve questions that require a deeper and more connected understanding of the mathematical concepts. His scores from the HCK items were extremely lower than his scores from the CCK items. His performance on SCK items was moderate. These will elaborately be discussed in the following sections.

4.2.1. Can’s Subject Matter Knowledge of Quadratic Functions

The descriptions of Can’s subject matter knowledge were developed from his responses to the questionnaire (see Appendix A), the follow-up interview (see Appendix C), and classroom observations. These descriptions are presented under three headings in the following sections.

4.2.1.1. Can’s Common Content Knowledge of Quadratic Functions

For an elaborate discussion of Can’s CCK, the results are presented under seven headings that indicate the sub-dimensions of teachers’ CCK.

Can's conception of quadratic equations and functions

Can defined both quadratic functions and quadratic equations referring to their algebraic representations. When he was asked to define a quadratic equation, he wrote: "A quadratic equation is $ax^2 + bx + c = 0$ where a, b, c are real numbers and $a \neq 0$." Similarly, he defined a quadratic function: "A quadratic function is $f(x) = ax^2 + bx + c$ where a, b, c are real numbers and $a \neq 0.$" When he was asked to distinguish quadratic functions, quadratic equations, and quadratic polynomials, he stated: "An equation involves an equality. The exponents of quadratic polynomials must be natural numbers."

In the questionnaire, when the teachers were asked to decide whether two given tables containing x and y values indicate a linear or a quadratic function, he calculated the first differences and stated: "The first one is linear since first differences are constant. The second one is quadratic since the first differences are not constant". In the interview, the researcher asked him:

> **Researcher:** In the 4th question, you have stated that the values in the first table belong to a linear function, the values in the second table belong to a quadratic function. Could you explain how did you decide it?
>
> **Can:** I examined the differences. Four, four, four, four. The first one is linear. In the second table, there is no linear increase or decrease. Here is 3, here is 2, here is 5. Since the differences were different, I said that this was a quadratic function.
>
> **Researcher:** Okay. You examined the differences. Does a non-constant difference always indicate a quadratic function?
>
> **Can:** Hmm… I did not think about that. I don't know.

As it can be seen in his words, Can did not have any idea about the constant second differences of quadratic functions. He examined only the first differences of the two functions and thought that if the first difference is constant, the function is linear, and if it is non-constant, the function is quadratic.

Can's knowledge of solving quadratic equations with one unknown

In the questionnaire, he solved the quadratic equations by using the quadratic formula. In the follow-up, the teachers were required to state alternative methods to solve quadratic equations. He suggested factorization as an alternative way of solving quadratic equations. During his instruction he used the quadratic formula and the factorization for solving quadratic equations. He never used completing the square method to solve a quadratic equation.

Can's knowledge of sketching and interpreting the graphs of quadratic functions

He made some incorrect or structural descriptions of the concepts like the vertex, the axis of symmetry, and the concavity of quadratic functions, in his responses to the questionnaire. For example, he defined the axis of symmetry as "the apsis of the vertex". This is not a correct definition of the axis of symmetry since it is a line that separates the parabola into two symmetrical parts, not a single point. In the follow-up interview, when he was asked to find the axis of symmetry of the function $g(x) = -6x^2+12x + 5$, he calculated the r-value (i.e., the apsis of the vertex) and wrote "$r = 1$" as the axis of the symmetry of the function $g(x)$.

He defined the vertex as "the ordinate of the maximum or the minimum point of a parabola". This is also an incorrect definition since the vertex is the point (r, k) in the coordinate plane. In the follow-up interview, when he was asked to find the vertex of the function $f(x) = 3x^2 + 9x + 6$, he found the ordinate of the vertex and wrote "37/4" as the vertex of the function. This response is consistent with his definition of the vertex in the questionnaire. He did not write a definition for concavity in the questionnaire.

The teachers were also asked to find some properties of a quadratic function such as the axis of symmetry, the vertex, the *x*-intercepts, and the *y*-intercept, then sketch the graph of it. Like all the other teachers, he found all the properties and then drew the correct graph. While finding the axis of symmetry, he wrote "$r = -1$" as the axis of symmetry, rather than writing "$x = -1$". In the next question, two graphs were given and the teachers were asked to find the quadratic functions. He correctly found the quadratic functions. When the vertex was given in the graph, he used the vertex form of quadratic functions. When the *x*-intercepts were given, he used the intercept form of quadratic functions. He used different algebraic forms of quadratic functions according to the nature of the task. In connection with this, in the interview the researcher asked him:

> **Researcher:** In the 10th question, you have written the first function in the vertex form, the second function in the intercept form. Do you use these different algebraic demonstrations in your classroom instruction?
>
> **Can:** We use the vertex form in the questions about translations because we tell the translations as adding a constant to the inside or outside of the function. We say that if we add inside, the function moves along the *x*-axis; if we add outside, the function moves along the *y*-axis.
>
> **Researcher:** In what cases do you use the standard form?
>
> **Can:** We use the standard form if the graph is not given, and three arbitrary points are given. We move based on the types of questions. Maybe, we might do wrong since we give importance to the types of questions, not the concept of the parabola.

During the classroom instruction, as he told in the interview, he used different algebraic forms of quadratic functions. To illustrate this, an example from his instruction is given below (Figure 4.5).

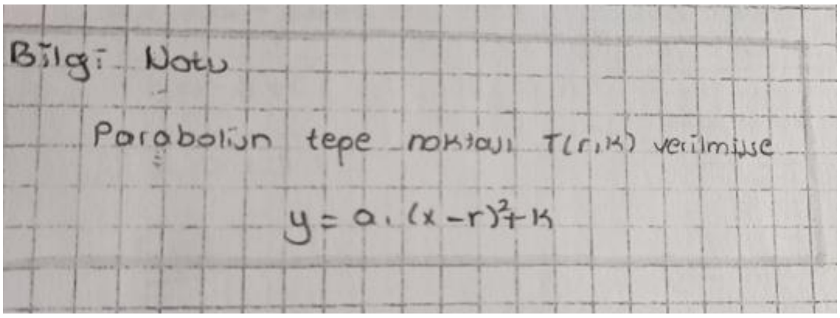


Figure 4.5. A section from Can's instruction

Lastly, the teachers were shown a graph. They were asked to comment on the signs of the coefficients of the corresponding quadratic function. The response of Can is presented below (Figure 4.6).

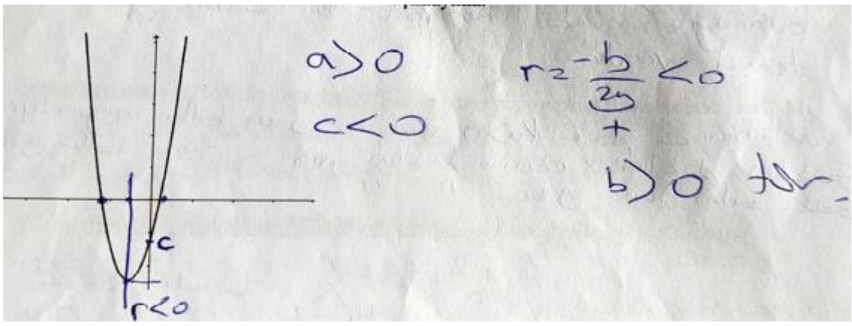


Figure 4.6. Can's response to question 11

As seen in Figure 4.6, he first determined the sign of a. He determined the sign of b by using the information that r is negative. He determined the sign of c by writing the ordinate of the y-intercept as c in the graph. In the interview, the following conservation also reveals the pattern that he used to determine the signs of the coefficients.

> **Researcher:** In the 11th question, you wrote that $a > 0$, $b > 0$, and $c < 0$. How did you determine the signs?
>
> **Can:** The parabola is upwards, so $a > 0$. We know a is positive, the vertex $(-b/2a)$ is negative; so b must be greater than 0. Since the ordinate of the y-intercept is negative, c must be negative.

Can's knowledge of graphing quadratic functions using transformations

When he was asked to explain how to generate the graph of any quadratic function from the graph of $f(x) = x^2$, he stated that it can be done by vertical and horizontal translations (question 12). In question 13, he properly found the quadratic function that generates the widest parabola among the given four quadratic functions. He stated: "The smaller the $|a|$ gets, the wider the parabola becomes. So, the answer is C." During classroom instruction, he did not tell his students about this property of the parabolas. When he was asked to compare the graphs of the functions $f(x) = x^2 - 5$ and $g(x) = (x-5)^2$ without drawing their graphs, he compared the functions by explaining the transformations made on $y = x^2$ (question 14). He stated: "$f(x)$ is the translation of $y = x^2$, 5 units below along the y-axis and $g(x)$ is the translation of x^2, 5 units right along the *x*-axis. Their shape is the same." As it can be seen in his words, he knows the shape is conserved during vertical or horizontal translations.

Can's knowledge of solving real-life problems regarding quadratic functions

Can did not solve the real-life problem of quadratic functions in the questionnaire (question 15). In the interview, the researcher asked him to read the question again and think about it.

> **Researcher:** In the questionnaire, you did not solve the question 15. Could you examine the question again and think about how it can be solved?
>
> **Can:** Hmm… Let me look at the question (examines the question and his answer). Well, I have tried to calculate the loss for each 0,5 cent. I multiplied 1250 and 0,5 and found that each 0,5 cent increase in the price causes 625 dollars loss.
>
> **Researcher:** Well, you did not continue.
>
> **Can:** Yes, I don't know how to move on. I am stuck here.

As it can be seen in his above words, he could not write a quadratic function for calculating the new income. He just made some numerical operations that were not enough to find solution for the problem.

Can's knowledge of finding the quadratic function with given points

Can correctly found the quadratic functions when some specific points on them were given. In the first one, the vertex and an arbitrary point on the function were given. In this case, he used the vertex form and found the quadratic function correctly. In the second one, three arbitrary points on the function were given. This time, he used the standard form to find the quadratic function. His solution is presented below (Figure 4.6). As shown in Figure 4.7, he used the standard form and calculated the coefficients a, b, and c correctly.

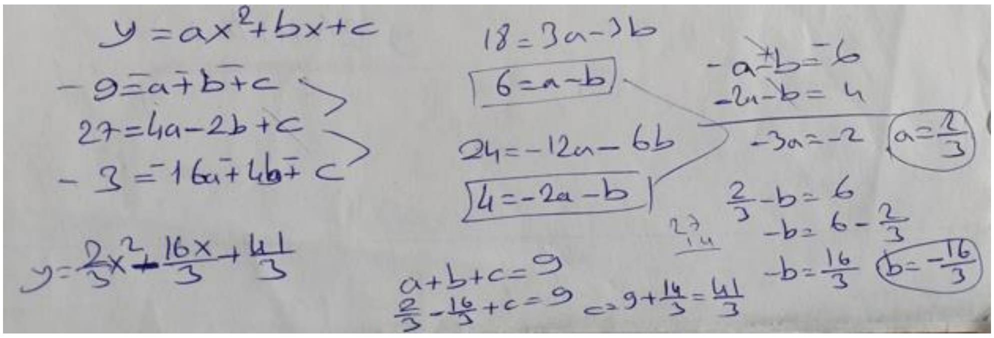


Figure 4.7. Can's response to question 17

Can's knowledge of finding the intersection of a parabola and a line

Can correctly explained the intersection of parabolas and lines. In question 18, he identified three conditions for the intersection of a parabola and a line. He stated: "

$ax^2 + bx + c = mx + n$. We examine the discriminant of this new equation. If Δ<0, they do not intersect; if Δ>0, they intersect at two points; if Δ=0, the parabola is tangent to the line." Then, he found the intersection of the line $y = 11x - 13$ and the parabola $y = 2x^2 + 3x - 5$ as shown in Figure 4.8.

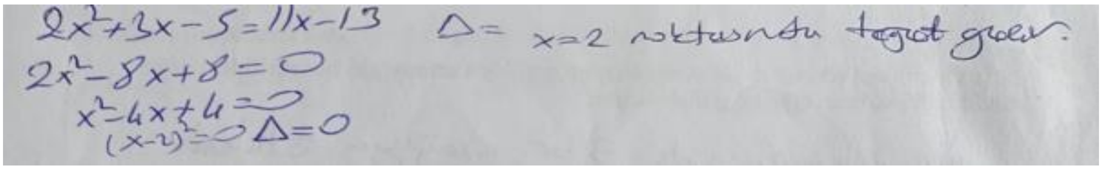


Figure 4.8. Can's response to the follow-up of question 18

During his instruction, Can told his students this content and solved several examples of finding the intersection of a parabola and a line. One of them is presented below (Figure 4.9).

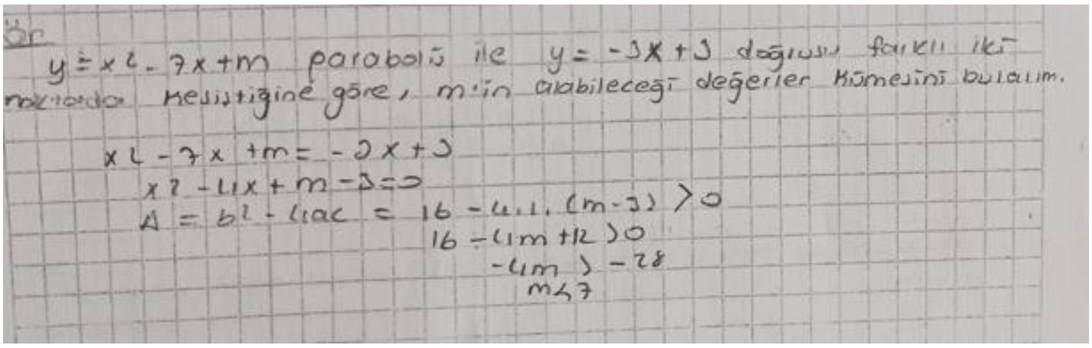


Figure 4.9. A section from Can's instruction

4.2.1.2. Can's Specialized Content Knowledge of Quadratic Functions

For a detailed description of Can's SCK, the results are presented under seven headings that indicate the sub-dimensions of teachers' SCK in the present study.

Can's knowledge of explaining and justifying basic formulas of quadratic functions

In the questionnaire, the teachers were asked to state and justify the quadratic formula both geometrically and algebraically (question 19). He did not make any algebraic or geometrical justification; he just stated the quadratic formula. Then, the teachers were asked to solve the quadratic equation $x^2 - x + 1 = 0$ without using the quadratic formula (question 20). He correctly solved the equation by completing the square. He did not use this method in the classroom to make an algebraic justification of the quadratic formula or to solve quadratic equations. In the interview, the researcher asked him:

> **Researcher:** In question 19, you said that you would justify the quadratic formula by drawing the graph. I could not get what you meant. How do you justify the quadratic formula on the graph?
>
> **Can:** I mean if a parabola has two x-intercepts, the discriminants must be greater than zero.
>
> **Researcher:** Do you think that this is the justification of the quadratic formula?
>
> **Can:** Actually no. I explain why the discriminant is zero for perfect square functions.
>
> **Researcher:** Do your students ask about where the quadratic formula has come from?
>
> **Can:** Usually no. It is easier for them to memorize the formula rather than to prove it. We have time restrictions and so we cannot engage in proofs. In the book which was given by the government, there exist some proofs but we skip them.

As seen in the above conservation, he stated that he does not have time for the justifications or proofs of formulas. During the classroom instruction, as he also stated in the interview, he did not tell his students the justification of the quadratic formula. He introduced completing the square method shortly and found the vertices of two functions by his method. Then, he moved on without using the completing the square method in any part of his instruction.

Can's knowledge of posing real-life problems regarding quadratic functions

Can did not answer the question in the questionnaire which asked them to write a real-life problem they might use during classroom instruction (question 21). In the interview, the researcher asked him whether he uses real-life problems in his instruction:

> **Researcher:** In question 21, you were asked to provide an example of a real-life problem you share with or ask to your students that can be modeled and solved by a quadratic functions. You did not answer. Do you use this kind of problems during your instruction?
>
> **Can:** I generally do not use, I cannot. In the textbooks, there are real-life problems but we do not use them. While beginning unit on parabola, there are some examples of the Eifel Tower, the Bosphorus Bridge, and the satellite dishes as daily examples of parabolic curves. We tell students these examples, but we fail to solve real-life problems. This is our biggest weakness.
>
> **Researcher:** Why do you think so?
>
> **Can:** Because we have arithmetical thinking rather than algebraic thinking. We only make quantitative operations; we do not follow the new education system which is based on non-routine problems. As secondary mathematics teachers, I think we do not completely know what a non-routine problem is.

The classroom observation is consistent with his above words. While introducing quadratic functions, he shortly mentioned arch bridges and satellite dishes as daily examples of parabolas. Then, he did not solve any real-life problems about quadratic functions during his instruction.

Can's knowledge of recognizing students' incorrect solutions regarding quadratic functions

When the teachers were asked to describe the incorrect steps in a student's solution to a problem about the vertex and maximum/minimum points of a quadratic function (question 22), he wrote: "The student is wrong because the parabola is downwards, the vertex does not give the minimum. Also, the ordinate of the vertex gives the max/min value, not the apsis. The endpoints should be checked." As seen in his response, he recognized all the mistakes in the student's solution and explained them clearly.

Can's knowledge of understanding students' unusual solutions regarding quadratic functions

The teachers were given two problems, each together with a student's solution about quadratic functions and equations. In the first one (question 23), there was a quadratic equation, which can be factorized. However, the student solved that equation by completing the square method. He responded: "The student solved the equation by completing the square. The result is correct." As he stated, he knew the method of completing the square for solving quadratic equations, as a secondary mathematics teacher. However, he does not prefer to use this method in his instructional practice.

In the second one (question 24), the question was to find the (unique) quadratic polynomial, with some information about the coefficients and one of the roots were given. He stated that the solution is correct without explaining why he thought so. Thus, in the interview, the researcher asked him to express why he thought so. This is illustrated below:

> **Researcher:** In question 24, you have written that the student is right. Could you explain why did you think so?

Can: If one of the roots is $7 + \sqrt{6}$, another one must be $7 - \sqrt{6}$.

Researcher: Yes.

Can: We can find the solution since two roots are known. We can use the sum and multiplication of the roots to find the coefficients.

In fact, the aim of that question was to lead teachers to analyze the student's approach. However, he focused on finding the result by his own approach rather than analyzing the student's solution. He examined the question and explained how to find the solution without paying attention to the student's solution strategy.

Can's knowledge of responding to students' why questions about quadratic functions

In the questionnaire, two questions were asked to understand teachers' ability to respond to students' why questions. In the first one (question 25), they were asked to respond to a student's question that asked why translating a parabola upwards and downwards changes only *c*, while translating a parabola left and right changes both b and c in $f(x) = ax^2 + bx + c$. In the questionnaire, he did not present a plausible explanation for this question. He said: "While translating upwards and downwards, only the ordinate value changes. The ordinate value only affects *c*." In the interview, the researcher asked him:

Researcher: Could you explain your response to question 25? You said while translating a parabola upwards and downwards, only the ordinate value changes and the ordinate value only affects c.

Can: Hmm…I use the vertex form. When we add values to k, the parabola goes up. When we subtract values from k, the parabola goes down. The width of the parabola does not change. So, only c changes.

Researcher: How do you explain why both b and c change while translating the parabola horizontally?

Can: *c* changes because of *r*. In fact, *r* depends on *b*. When I write

$(x - r)^2 + k$, for each number I add to r, the parabola moves right or left, in opposite direction with the sign of the number which was added.

The above conversation indicated that Can did not present a plausible explanation to explain the relationship between the coefficients and the translations made on the parabolas.

Can's knowledge of finding an example to make a specific mathematical point about quadratic functions

In the questionnaire, when he was asked what kind of examples, he would use in the classroom to emphasize the symmetrical property of parabolas (question 27), he wrote: "I would draw parabolas on Geogebra." In the interview, the researcher asked him to explain his response in detail, as illustrated below:

Researcher: You said you would use Geogebra to emphasize the symmetrical property of a parabola. How would you do this? What kind of examples can you use?

Can: As I said, I would draw some parabolas and find their vertices. To be honest, I had never had an extra effort to emphasize the symmetrical property. Of course, I say that parabolas are symmetrical shapes; but I mean I did not think about a specific example to highlight the importance of the symmetrical property.

During the classroom observation, Can used some examples which might help to emphasize symmetrical property, as presented below. (Figure 4.10). He did not use a mathematical software during his instruction. As seen in Figure 4.10, he wrote that $f(r + m) = f(r - m)$ and solved some examples about this property. However, while teaching this content, he did not underline the symmetrical property.

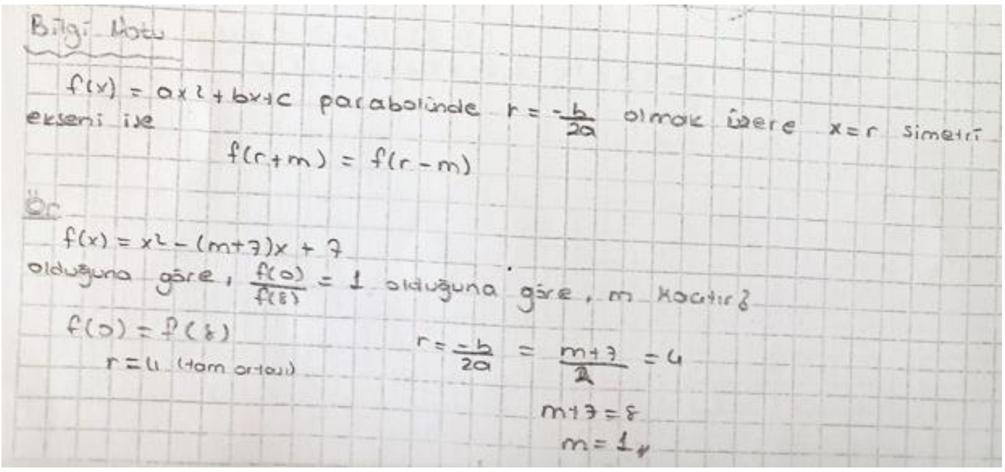


Figure 4.10. A section from Can's instruction

Can's knowledge of modifying tasks regarding quadratic functions

In the questionnaire, when the teachers were asked to modify a task considering their students, he did not make any change in the question. He told that his students could easily solve the task and find the correct result.

4.2.1.3. Can's Horizon Content Knowledge of Quadratic Functions

For an elaborate discussion of Can's HCK regarding quadratic functions and equations, the results are presented under two headings that include Can's knowledge of: how quadratic functions are related to other contents in the high school curriculum and how quadratic functions are related to advanced mathematics.

Can's knowledge of how quadratic functions are related to other contents in the high school curriculum

Based on the questionnaire results and the interview, Can's knowledge of the relationship between quadratic functions and other contents in the high school curriculum was extremely weak. Can presented no correct answer to the items in the questionnaire that evaluated his knowledge of how quadratic functions are related to other contents in the high school curriculum (questions 29-34). He gave either incorrect or no answer to all of those items. In question 29, he did not explain the relationship between the derivative and the concavity of a parabola, as illustrated below:

> **Researcher:** How can you explain why the graph of a quadratic function is concave down if $a < 0$, and concave up if $a > 0$?
>
> **Can:** I give this to my students as a rule. I say if $a > 0$ the arms of the parabola opens up, if $a < 0$ the arms of the parabola opens down.
>
> **Researcher:** Well, did you think why this is so?
>
> **Can:** No. This is a well-known rule. When we draw the graph, we can see it easily.

In the next question (question 30), the teachers were asked to compare the graphs of exponential and quadratic functions. He stated: "According to the graph, q is always greater than p." In the interview, the researcher asked him the same question again and required him to explain his answer.

> **Researcher:** In question 30, you have stated that the quadratic function q will always take greater values than the exponential function p. Could you explain why did you think so?
>
> **Can:** I said that q is always greater because exponential function grows faster.
>
> **Researcher:** But q is not exponential, q is a quadratic function.
>
> **Can:** I supposed that q is exponential. I meant to say that exponential function grows faster. For example, think about 2^x. When x gets bigger, it grows faster, and the arms of the graph approach the y-axis faster.
>
> **Researcher:** Well, I get it. You misunderstood the notations (p and q) of the functions. You say that exponential function eventually gets bigger.
>
> **Can:** Yes.

The above conservation indicates that Can knows that the exponential function grows faster than the quadratic function. In the next question, he did not explain the relationship between the vertex of a quadratic function and the derivative. In the interview, the researcher asked him the same question again (question 31).

> **Researcher:** Is there a relationship between the vertex and derivative?
>
> **Can:** Yes, but in grade 12.
>
> **Researcher:** How?
>
> **Can:** In grade 12, a third-order equation is given. Its derivative becomes a second-order equation. When the minimum value is to be found, we use r.
>
> Also, the fact that the first order derivative is the slope, is told in grade 12.
>
> **Researcher:** Well, I do not mean their curricular relationship. I am asking how do yourself associate them conceptually?
>
> **Can:** I cannot give a certain answer to this question. They are always related.

When he was asked to determine whether $y = x^4$ is a parabola or not (question 34), he wrote: "It is a parabola, because it is U-shaped." Another question was about the relationship between the golden ratio and quadratic equations (question 33). In the questionnaire, Can did not answer this question. So, in the interview, the researcher asked the same question again, as illustrated below:

> **Researcher:** In question 33, you were asked to explain (if any) the relationship between the golden ratio and quadratic equations. You did not write anything. Do you have an idea about their relationship?
>
> **Can:** Golden ratio is not a parabolic curve. I have no idea about their relation. Maybe, there is, but I don't know.

The teachers were also asked to tell what kind of examples they would provide their students to emphasize the relationship between any concept from the physics course and quadratic functions (question 32). Can did not respond to this question. In the interview, the researcher asked him:

> **Researcher:** In question 32, you were asked whether quadratic functions are related to any concept from the physics course, you did not answer. Could you give some examples from physics course which might be related to quadratic functions?
>
> **Can:** Sometimes, students say that this is similar to projectile motion; but they do not make a connection between this concept and parabolas.
>
> **Researcher:** Do you associate them during your classroom instruction?
>
> **Can:** No, I don't.

During his instruction, he did not emphasize the connection between quadratic functions and any other content in the high school curriculum. Based on the questionnaire results, the classroom observation, and the interview, Can has poor knowledge of how quadratic functions are related to other contents in the high school curriculum.

Can's knowledge of how quadratic functions are related to advanced mathematics

In the questionnaire, Can did not answer the question that asked the reflection property and its daily use (question 35). In the interview, the researcher asked him:

> **Researcher:** In question 35, you were asked to explain reflection property of a parabola. Could you explain what this property is and where it is used in daily life?
>
> **Can:** I have no idea about the reflection property.

In another question, Can did not explain the relationship between a hyperbola and a parabola (question 36). He also stated that he has never heard about the fundamental theorem of algebra, in his response to question 37. So, the researcher asked him the same question in the interview. This is illustrated below:

Researcher: In the questionnaire, you were asked to state the fundamental theorem of algebra and its application to quadratic polynomials. You wrote that you have never heard this theorem.

Can: I have no idea about this theorem.

When the teachers were asked to define a parabola and give alternative definitions (question 39), he wrote: "The graph of a quadratic function is called a parabola." In the interview, the researcher asked him to state any alternative definitions:

Researcher: You defined a parabola as the graph of a quadratic function. Do you know any alternative definitions?

Can: I don't know. Maybe there is, we can investigate. I only know that definition. According to me, a parabola is the graph of a quadratic function.

As he confirmed in the questionnaire, he considers a parabola as the graph of a quadratic function. He is not aware of the geometrical definition of a parabola, which is related to a point (focus) and a line (directrix). In connection with the previous question, the teachers were asked to select the most correct statement among the given two ones, which are presented below (question 38).

Statement 1: The graph of a quadratic function is a parabola.

Statement 2: The graph of a quadratic function is called a parabola.

In the questionnaire, Can wrote that the second statement is correct, without further explanation. In the interview, the researcher asked him:

Researcher: In question 38, you selected the second statement as the most correct? Could you explain why?

Can: I selected the second statement because it is a definition.

In the last question of the questionnaire, which asked the teachers whether a given curve is a parabola or not, he had no answer (question 40). So, the researcher asked him:

Researcher: You did not respond to the last question. Have you ever heard the term catenary?

Can: No.

Researcher: Do you think that this shape is a parabola? What is required to be a parabola?

Can: Parabolas have two symmetrical roots. They are symmetrical shapes.

Researcher: Isn't this shape (the shape in question 40) symmetrical?

Can: It looks symmetrical. It can be a parabola.

Researcher: If a shape is symmetrical, is it enough criteria to become a parabola?

Can: I think yes.

On the basis of the questionnaire results, the classroom observation, and his responses to the interview, Can's knowledge of how quadratic functions and equations are related to advanced mathematics is fairly limited.

4.2.2. The contribution of Can's Subject Matter Knowledge of Quadratic Functions to Student Learning Outcomes

A total of 23 students were in Can's class. Three of them were absent on the day the questionnaire was administered. So, 20 students were administered the quadratic function concept test (see Appendix C), which provided the data for interpreting his students' learning outcomes of quadratic functions. The responses of 3 students were excluded from the analysis, since they did not respond any questions in the test. Thus, the responses of 17 students were analyzed to evaluate students' learning outcomes regarding quadratic functions. A summary of the results of Can's students' performance on the test is presented below (Figure 4.11).

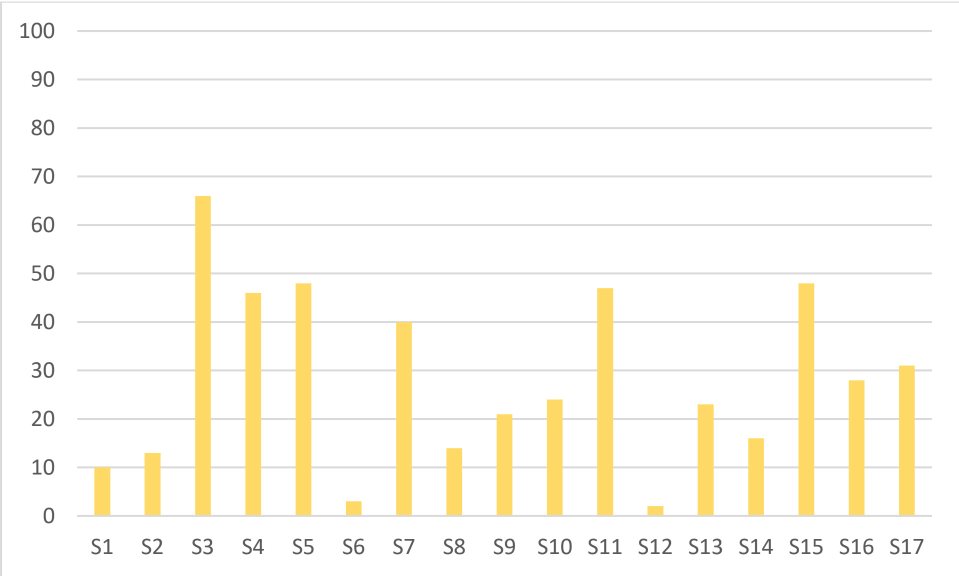


Figure 4.11. Can's students' scores on the quadratic function concept test

As seen in Figure 4.11, Can's students' test scores range between a minimum of 2 and a maximum of 66 out of 100 points. The average score of the students on the test is 28.2. In general, his students' performance on the test is very limited. The students' performance is discussed in detail in the next sections based on the objectives of the mathematics curriculum regarding quadratic functions.

4.2.2.1. Finding the Vertex, x-Intercepts, the y-Intercept, and the Axis of Symmetry

In the first question of the quadratic function concept test, the students were asked to find *x*-intercepts, the *y*-intercept, vertex, and axis of symmetry of the graph of the function $f(x) = x^2 + 2x - 8$ and then graph it. While they were finding *x*-intercepts, some students (*n*=7) factorized the quadratic equation and correctly found *x*-intercepts as -4 and 2. Some students (*n*=4) used the quadratic formula to find the *x*-intercepts of the quadratic function. Six students did not answer this part of the question. None of the students used completing the square method to solve the quadratic equation. When the students were asked to find the *y*-intercept, two of them found it by calculating $f(0)$ and correctly wrote $(0, -8)$ as the *y*-intercept. Many students (*n*=10) wrote only "-8" as the *y*-intercept, without using the proper notation as (x, y). Four students had no answer. There was only one student who found the *y*-intercept incorrectly as 0. She wrote -4 and 2 (the roots of the quadratic equation) for x and obtained $y = 0$.

When the students were asked to find the vertex, some students (*n*=8) found it correctly as the point (r, k). They firstly found r by using the formula. Then, six of them found k by calculating $f(r)$, while one of them used the formula "k=$\frac{4\mathrm{ac}-b^2}{4\mathrm{a}}$". Two students wrote the apsis of the vertex (r) as the vertex, whereas seven students wrote the ordinate of the vertex (k) as the vertex. When the students were asked to find the axis of symmetry, none of them correctly wrote it as a line equation. There were five students who were aware of the interrelation between the apsis of the vertex and the axis of symmetry. They wrote the axis of symmetry as "$r = -1.$" Two participants wrote the vertex $(-1, -9)$ as the axis of symmetry. Two participants wrote some irrelevant numbers without explaining how they found these numbers. Eight participants did not respond. When they were asked to graph the function, only four students correctly sketched the graph. Six students did not sketch any graphs whereas seven of them sketched incorrect graphs.

In another question, the students were asked to find c in the function $f(x) = x^2 + bx + c$ with its vertex given. Most students (*n*=13) correctly found c. Firstly, they found b by using the formula for r. Twelve of them found c by using that $f(r) = k$. Their solutions were like: "$-b/2a = 2,\ b = -4,\quad f(r) = k,\quad f(2) = 6,$ $2^2 - 8 + c = 6,\ c = 10.$" Unlike the others, one student used the formula for finding the ordinate of the vertex: "$-b/2a\ = 2, b = -4;\quad k = \frac{4ac-b^2}{4a} = 6;$ $4c = 40, c = 10.$" Three students did not respond to this question, whereas one student found an incorrect result.

The result suggested some evidence that teachers' content knowledge of quadratic functions interact with student learning outcomes. First of all, Can solved the the quadratic equations in the questionnaire by using the quadratic formula and he suggested "factorization" as an alternative strategy for solving quadratic equations. During his instruction, he used factorization for solving the quadratic equations which can be factorized; he used the quadratic formula for those which cannot be factorized. He shortly mentioned completing the square method, but he did not use this method for finding the roots of a quadratic equation. As so Can, his students used the quadratic formula and factorization to find the x-intercepts of a quadratic function. None of his students used completing the square method for finding the roots of a quadratic equation. This finding provides evidence of the relationship between teachers' subject matter knowledge and students' learning outcomes regarding solving quadratic equations or finding the x-intercepts of a quadratic function.

Secondly, half of the students wrote the ordinate of the vertex, as the vertex, as Can did in his response to the questionnaire. Can defined the vertex as "it is the ordinate of the maximum or the minimum point of a parabola" and found the vertex of the function $f(x) = 3x^2 + 9x + 6$ as $\frac{37}{4}$, which represents the ordinate of the vertex. This finding also provides evidence of the relationship between teachers' content knowledge and students' learning outcomes regarding finding the vertex of a quadratic function.

Thirdly, Can's students failed to find the axis of symmetry of a parabola. Some of them (n=5) wrote the axis of symmetry as "$r = -1$", as Can did. In the questionnaire, he defined the axis of symmetry as "the apsis of the vertex" and found the axis of symmetry of the function $g(x) = -6x^2 + 12x + 5$ as "$r = 1$." As so Can, his students perceive the axis of symmetry as the apsis of the vertex, rather than a line passing through the vertex. Thus, an interaction could be made between teacher knowledge and students' learning outcomes regarding finding the axis of symmetry of a parabola.

When compared to the first question, Can's students performed better on the second question, which was about finding an unknown coefficient of a quadratic function whose vertex is given. During his instruction, Can solved similar kinds of questions. One of them is illustrated below (Figure 4.12).

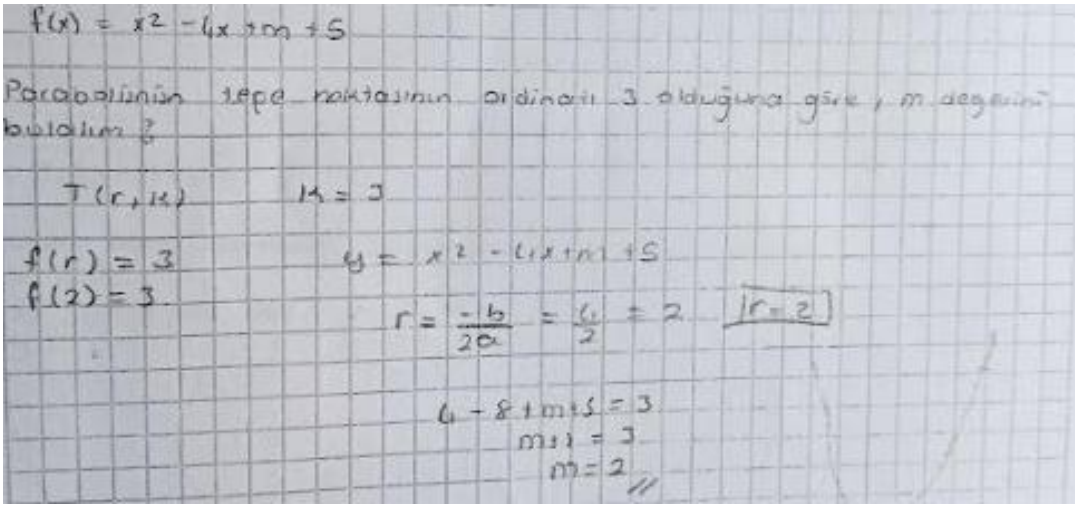


Figure 4.12. A section from Can's instruction

4.2.2.2. Associating the Vertex with the Maximum or the Minimum of a Quadratic Function

In the third question, the students were asked to find the minimum of a quadratic function. Seven students correctly associated the minimum of the function with the ordinate of the vertex. Six of them calculated r firstly; then found $f(r)$ as the minimum. Their responses were like: "$T(r, k)$, $r = -b/2a = -1$, $f(-1) = k = 2$." One participant directly used the formula for k, without calculating r, and made a calculation error while applying the formula $k = \frac{4ac-b^2}{4a}$. Five students gave some incorrect answers. Three of them calculated $f(0)$ as the minimum of the function.

One student tried to find the x-intercepts to find the minimum. Another student found the discriminant (Δ) of the quadratic equation and stated that "Δ is the minimum". Five participants did not answer this question. Similarly, in the fourth question, the students were asked to find the maximum value of a given function. Seven students correctly associated the maximum of the function with the ordinate of the vertex, k. Six of them calculated r firstly; then found $f(r)$ as the minimum: "$r = -b/2a = 2$, $k = f(2) = -4 + 8 + 6 = 10$". As in the previous question, one student directly used the formula for k, without calculating r, and made a calculation error while applying the formula. Two students found "$f(1) = 9$" as the maximum, which is incorrect. Eight students did not respond to this question.

Most of Can's students performed poorly in associating the vertex of a parabola with the minimum or the maximum of the function. During his instruction, although he solved several problems about finding the vertex of a quadratic function, he did not solve problems about the minimum or the maximum of quadratic functions and did not associate the vertex with the maximum or the minimum of a quadratic function. He solved several questions about finding the vertex of a parabola, but he did not solve problems that asked to find the minimum or the maximum of a quadratic function. Thus, his students' poor performance in finding the maximum or the minimum of a quadratic function might be associated with his instructional practice.

4.2.2.3. Commenting on the Effect of the Change in the Coefficients on the Graph of the Function

In the fifth question of the quadratic function concept test, none of Can's students provided a correct answer. In the question, the students were asked to comment on two cases about how the graph of the quadratic function $y = 2x^2$ changes depending on the leading coefficient. Four students did not respond to this question, whereas the remaining (n=16) made incorrect explanations. For example, nine of them made some vertical or horizontal translations on the graph. The response of a student who made a vertical translation on the graph is illustrated in Figure 4.13.

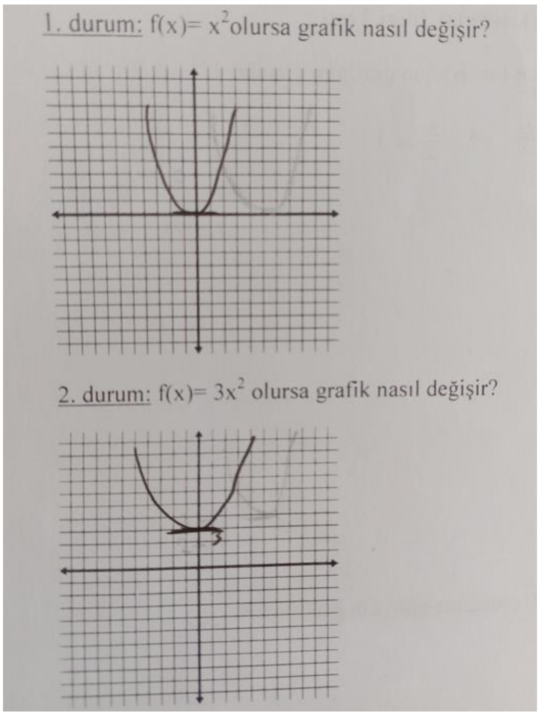


Figure 4.13. An example from Can's students' responses to question 5

A few students (n=3) tried to explain the relationship between the leading coefficient and the width of the graph, but they expressed it wrongly. Their explanations were like: "the arms of the parabola become larger, if |a| gets larger." Accordingly, one of them sketched the following graph shown in Figure 4.14.

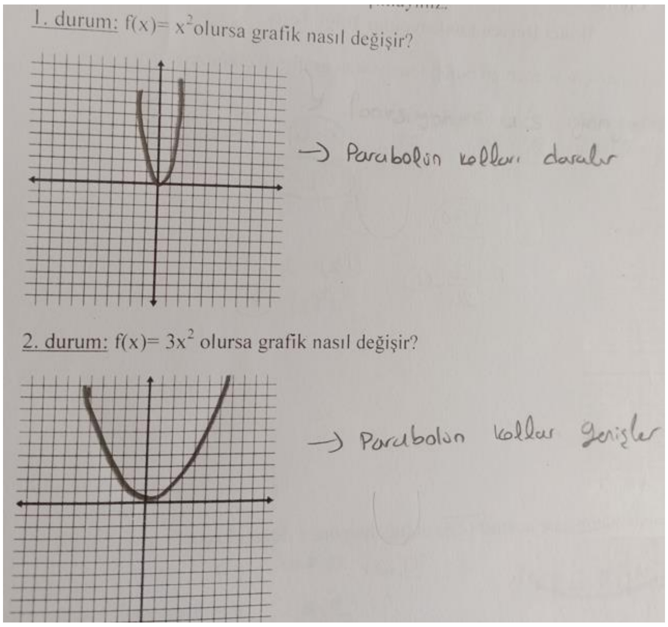


Figure 4.14. An example from Can's students' responses to question 5

Based on their performance on the quadratic function concept test, Can's students' performance in associating the coefficients of a quadratic function with its graph was fairly limited. On the other hand, the teachers were asked to find the widest parabola among the given ones. Can stated: "The smaller the |a| becomes, the wider the parabola becomes." However, during his instruction, he did not solve exercises about this property of the parabolas. He introduced the translation and symmetry of functions in general and solved some questions about these contents. Although he solved several questions about function transformations, he did not tell the stretching of parabolas (multiplying a quadratic function by a constant). He wrote on the board: "If the function $y = f(x)$ is multiplied by a constant k as $y = k.f(x)$, all the y-values in the range of the function are multiplied by k." Then, he applied this property to the linear functions as illustrated in Figure 4.15.

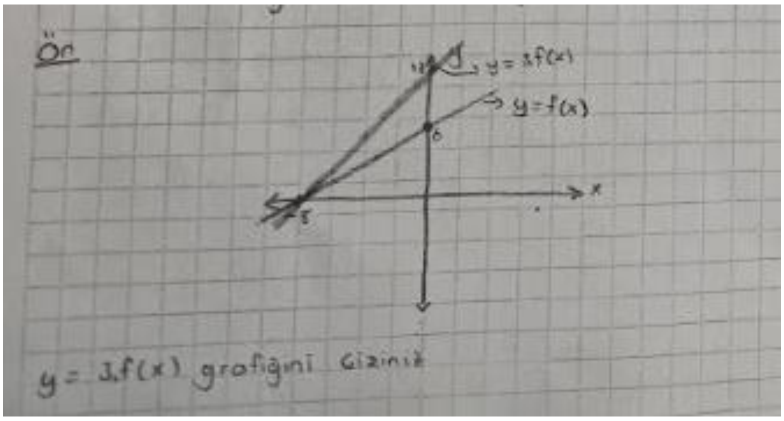


Figure 4.15. A section from Can's instruction

During his instruction, he did not solve any example related to the stretching of parabolas. Thus, students' low performance in commenting on the effect of the change in the leading coefficient on the shape of the parabola might be associated with his classroom instruction.

4.2.2.4. Finding the Quadratic Function Given Three Points or Two Points that One of Them is the Vertex

The students were asked two questions regarding this objective of the mathematics curriculum. In the first one (question 6), they were asked to find the quadratic function with given two points such that one of them is the vertex. Only three students could find the quadratic function. They used the vertex form of quadratic functions. For example, one of them responded: "$f(x) = a(x-r)^2 + k$, $f(x) = a(x+2)^2 + 2$, $f(1) = 11$, $9a + 2 = 11$, $a = 1$, $f(x) = (x+2)^2 + 2$." However, most of them (n=14) failed to find the quadratic function. Eleven students had no answer. Two of them tried to find the quadratic function; one of them used the intercept form whereas another one used the standard form. The solution of the student who used the intercept form was: "$y = a(x-x_1).(x-x_2)$, $y = a(x-1).(x+2)$, $y = -2a$, $y = -2(x-1).(x+2)$." Another student who tried to find the quadratic function by using the standard form wrote: "$f(x) = ax^2 + bx + c$,

$c = 11,\ f(-2) = 2,\ 4a - 2b + 11 = 2,\ 4a - 2b = -9,\ r = -\frac{b}{2a},\ b = 4a,\ 4a - 8a = -9,\ a = 9/4$."

In the second one (question 7), the students were asked to find the quadratic function whose three points such that one of them is on the y-axis were given. None of the students could find the quadratic function. Most of them (n=12) did not answer, whereas five students found an incorrect result. Three of them used the intercept form and considered the given two points as the x-intercepts. For example, one of them wrote: "$y = a.(x - x_1).(x - x_2),\ y = 4.(x + 1).(x - 2),\quad y = 4x^2 - 4x - 8$." One of them used the vertex form; and thought that one of the given points is the vertex: "$f(x) = a(x - r)^2 + k, y = a(x - 2)^2 + 6; y = x^2 - 4x + 10$."

When the above responses are examined, Can's students had some difficulty in finding the quadratic function whose some points were given. In the questionnaire that was administered to the teachers, there were similar questions that asked to find the quadratic functions with some points given (questions 16&17). Can found the quadratic function by using the intercept form, when the vertex was given; he used the standard form when three arbitrary points were given, as most of the teachers did. However, his students failed to solve similar questions. During his classroom instruction, he also solved several questions about this content as illustrated in Figure 4.16.

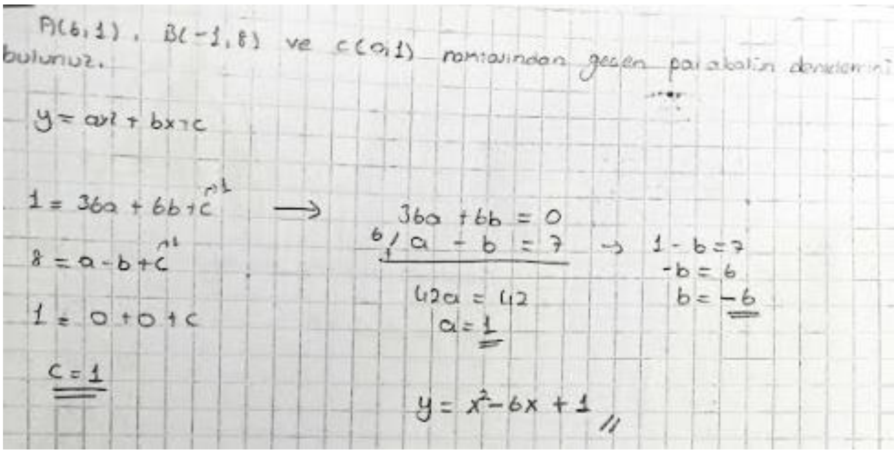


Figure 4.16. A section from Can's instruction

4.2.2.5. Investigating the Intersection of a Line and a Parabola

When the students were asked to examine the intersection of a line and a parabola (question 8), approximately half of the students (*n*=8) gave the correct answer. They equated the line equation and quadratic function, and obtained a new quadratic equation. Then, they investigated the discriminant of this new quadratic equation. Their solutions were like: "$x^2 + 5x + 2 = 3x + 1$, $x^2 + 2x + 1 = 0$, $\Delta = b^2 - 4ac$=0. They are tangent." Two students did not respond whereas seven students gave incorrect answers. Some of them (*n*=3) investigated the discriminant of the given quadratic equation and wrote: "$f(x) = x^2 + 5x + 2, \Delta = b^2 - 4ac = 17 > 0$, they have two points of intersection." Four students made some incorrect explanations. For example, one student responded: "They do not intersect because the line $y = 3x + 1$ is not a quadratic line."

When compared to other questions in the test, the students performed better on this question regarding investigating the intersection of a line and a parabola. In the questionnaire, the teachers were also asked to find the intersection of a line and a parabola (question 18). As most of the teachers did, Can investigated the discriminant of the new quadratic equation and easily found the intersection of the line and the parabola. During his instruction, he solved several problems about this content, as illustrated in Figure 4.17.

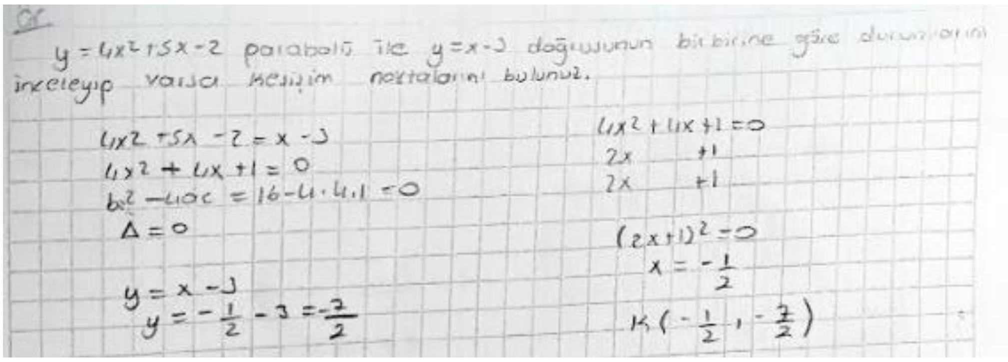


Figure 4.17. A section from Can's instruction

4.2.2.6. Solving Problems that can be Modeled by Quadratic Functions

In question 9, the students were asked to find the maximum area of a rectangle with its perimeter given. The students were expected to solve this question by using a quadratic model. However; none of Can's students constructed a quadratic model. Some of them (*n*=3) found the correct answer by trying some numbers, which adds up to 18, that is the half of the perimeter of the rectangle. Their solutions were like:

$a + b = 18$
1.17=17
2.18=36
.
.
$8.10 = 80$
$9.9 = 81$ (maximum area)

Six participants also followed similar steps and replaced numerical values to find the maximum area. However, they found the result as 80 since they thought that "if the dimensions would be $9\,x\,9$, it would be a square, not a rectangle". Thus, they selected the dimensions that would yield the maximum area as $8\,x\,10$. There was also one student who tried to find the solution by determining a ratio between the dimensions of the rectangle. He determined a $3{:}\,2$ ratio between the long side and the short side of the rectangle and made some operations that yielded an incorrect result. Seven participants did not respond to this question.

In question 10, the students were given a quadratic function that represented the height of a ball that was hit by someone playing football. The question was to find at what time the ball reaches 3 meters above the ground. Only two participants correctly solved the problem. To illustrate, one of them wrote: "$h(t) = -t^2 + 4t = 3$, $-t^2 + 4t - 3 = 0$, $t_1 = 1$, and $t_2 = 3$." Seven students found incorrect results. Some of them (n=5) calculated $h(3)$ whereas some of them (n=2) calculated the maximum of the function. Eight students did not respond to this question.

As it can be seen in their responses, Can's student's ability to solve real-life problems about quadratic functions was fairly limited. During classroom instruction, Can never solved real-life problems regarding quadratic functions, as he stated in the interview. Furthermore, Can could not solve the real-life problem in the questionnaire that required a mathematical model of quadratic functions. Thus, Can's students' inability to solve real-life problems about quadratic functions might be related to his inadequate subject matter knowledge. Moreover, in the interview, when the researcher asked him whether he emphasized the relationship between quadratic functions and some other concepts, he said that he did not tell these kinds of interrelations between the concepts. Thus, his students' low performance in solving real-life problems about quadratic functions might also be related to his low HCK.

4.3. Contribution of Subject Matter Knowledge to Student Learning Outcomes: The Case of Ahmet

In this section, the case of Ahmet was presented. The data obtained from the questionnaire, interview, and classroom observation were combined and triangulated.

Ahmet is the teacher who was called “T16” in the previous sections. He had 22 years of teaching experience and he was teaching at an Anatolian high school when this study was conducted.

He had a very good performance on the quadratic function concept questionnaire. His scores on the CCK, SCK, and HCK items were the highest among all the participating teachers. These will elaborately be discussed in the following sections.

4.3.1. Ahmet’s Subject Matter Knowledge of Quadratic Functions

The descriptions of Ahmet’s content knowledge were developed from his responses to the questionnaire (see Appendix A), the follow-up interview (see Appendix C), and classroom observations. These descriptions are presented under three headings in the following sections.

4.3.1.1. Ahmet’s Common Content Knowledge of Quadratic Functions

For an elaborate discussion of Ahmet’s CCK, the results are presented under seven headings that indicate the sub-dimensions of teachers’ CCK identified in the present study.

Ahmet’s conception of quadratic equations and functions

When he was asked to define a quadratic equation (question 1), he stated: “A quadratic equation ($ax^2 + bx + c = 0$) is a tool for finding the x-intercepts of a quadratic function.” Similarly, when he was asked to define a quadratic function (question 2), he wrote: “A quadratic function ($ax^2 + bx + c = 0$) generates a parabola.” When he was asked to distinguish quadratic functions, quadratic equations and quadratic polynomials (question 3), he focused on their geometrical aspects. He stated: “As I stated above, quadratic functions generate parabolas and the intersection of the parabola with the *x*-axis can be determined by a quadratic equation.”

Unlike Can, Ahmet underlined the geometrical aspects of quadratic functions and equations, rather than stating their demonstrations. When he was asked to determine whether the *x* and *y* values that were given in the tables belong to a linear or a quadratic function (question 4), he checked the first differences, as Can did. In the interview, the researcher asked him:

> **Researcher:** In the 4th question, you have stated that the values in the first table belong to a linear function, the values in the second table belong to a quadratic function. Could you explain how did you decide it?
>
> **Ahmet:** I examined the first differences.
>
> **Researcher:** How do you guarantee that it is quadratic if the first differences are non-constant?
>
> **Ahmet:** Hmm… According to the table, there are two roots. So we can generate a quadratic equation from those values.

Ahmet’s knowledge of solving quadratic equations with one unknown

In the questionnaire, when the teachers were asked to solve some quadratic equations (question 5), Ahmet was the only teacher who solved all of them by completing the square. His solutions are presented in Figure 4.18.

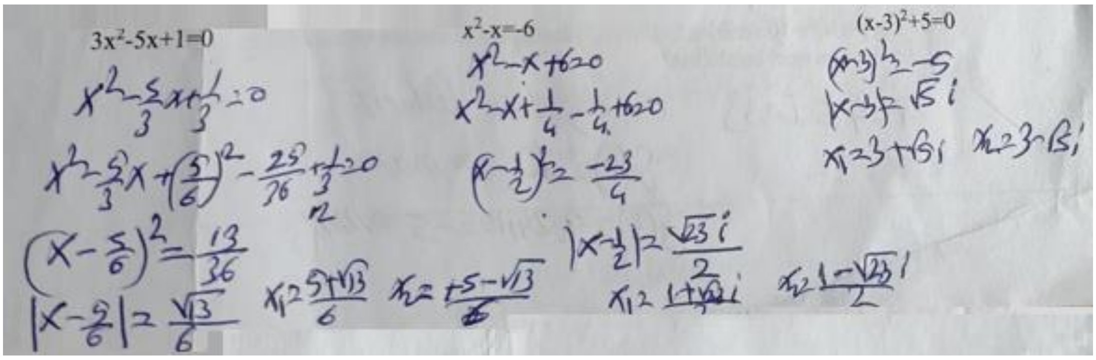


Figure 4.18. Ahmet's solution to question 5

In the follow-up, he stated the quadratic formula as an alternative solution method. In the interview, the researcher asked him:

> **Researcher:** You solved the quadratic equations by completing the square. Do you use this method in the classroom?
>
> **Ahmet:** Yes, of course. I frequently use it. This is my favorite solution method.

As he stated in the interview, he used completing the square method for solving quadratic equations several times.

Ahmet's knowledge of sketching and interpreting the graphs of quadratic functions

Unlike Can, he made structural descriptions of the vertex and the axis of symmetry. For example, he defined the axis of symmetry (question 6) as "the line according to which the parabola is symmetrical". Similarly, he defined the vertex of a parabola (question 7) as "the maximum or the minimum point of a quadratic function depending on the sign of the leading coefficient". In his response to question 8 that asked to define the concavity, he wrote: "It tells us about the direction of the parabola." In question 9, he properly found the vertex, the x-intercepts, the y-intercept, graph orientation, and the minimum value of the given quadratic function and sketched the graph of it accurately. He also correctly found the equations of two functions whose graphs were given (question 10). As so Can, he used multiple algebraic demonstrations of quadratic functions. He wrote the first function in the vertex form and the second function in the intercept form. In the interview, the researcher asked:

> **Researcher:** In the 10th question, you have written the first function in the vertex form, the second function in the intercept form. Do you use these different algebraic demonstrations in your classroom instruction?
>
> **Ahmet:** Yes, of course. I show my students some graphs and explain how to write the equations. For example, I say that if we know the x-intercepts, we use this form $f(x) = a.(x - x_1).(x - x_2)$.

His above words were supported by classroom observation. In the classroom, he emphasized the use of multiple algebraic demonstrations as shown in Figure 4.19.

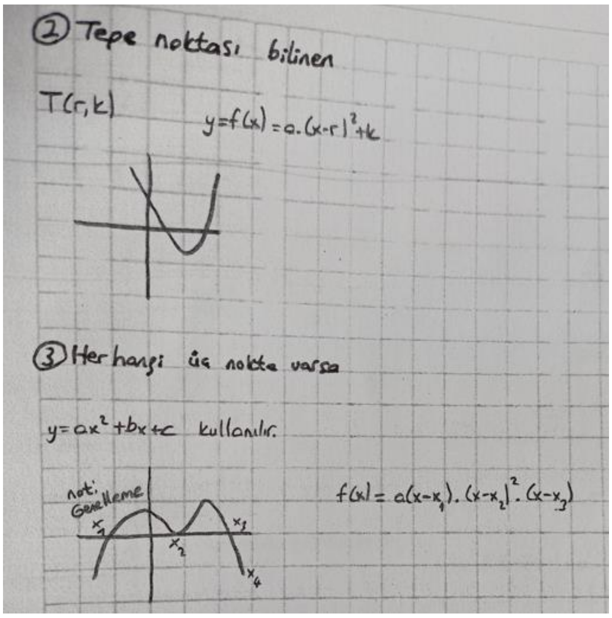


Figure 4.19. A section from Ahmet's instruction

Lastly, the teachers were given a parabola and asked to comment on the signs of the coefficients of the corresponding quadratic function (question 11). Ahmet's response is presented below (Figure 4.20).

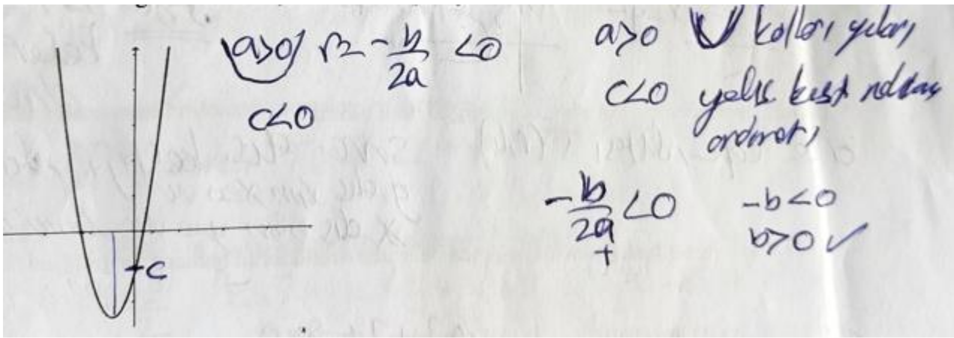


Figure 4.20. Ahmet's response to question 11

When the Figure 4.20 is examined, he firstly determined the sign of the leading coefficient, a. Then, as so Can, he determined the sign of b by examining the sign of the apsis of the vertex, r. He determined the sign of c by noting that it is the ordinate of the point where the parabola cuts the y-axis. In the interview, the researcher asked him to explain how he determined the signs of the coefficients:

> **Researcher:** In the eleventh question, you wrote that $a > 0$, $b > 0$, and $c < 0$. How did you determine?
>
> **Ahmet:** The parabola is upwards, so $a > 0$. The vertex $(-b/2a)$ is negative; so $b > 0$. Since the ordinate of the y-intercept is negative, *c* must be negative.

Ahmet's knowledge of graphing quadratic functions using transformations

When he was asked to explain how to generate the graph of any quadratic function from the graph of $f(x) = x^2$ (question 12), he wrote: "We can do it by reflections and translations." In question 13, he could find the widest parabola among the given four parabolas. He wrote: "For $f(x) = ax^2 + bx + c$, the larger the $|a|$ becomes, the arms of the parabola is getting closer to the *y*-axis." As so Can, during his instructional practice, he did not tell this property. When he was asked to compare the graphs of the functions $f(x) = x^2 - 5$ and $g(x) = (x-5)^2$ (question 14), he wrote some properties of them: "Both of them are concave up. $g(x)$ is a perfect square and thus tangent to the *x*-axis. $f(x)$ intersects the *x*-axis at two different points." Unlike Can, he compared the functions in terms of their some characteristics. He did not describe the transformations made on the function $f(x) = x^2$.

Ahmet's knowledge of solving real life problems regarding quadratic functions

Ahmet was the only participant who properly constructed a quadratic model and solved the problem stated in question 15. His original solution is presented below (Figure 4.21). During his instruction, he solved several real-life problems.

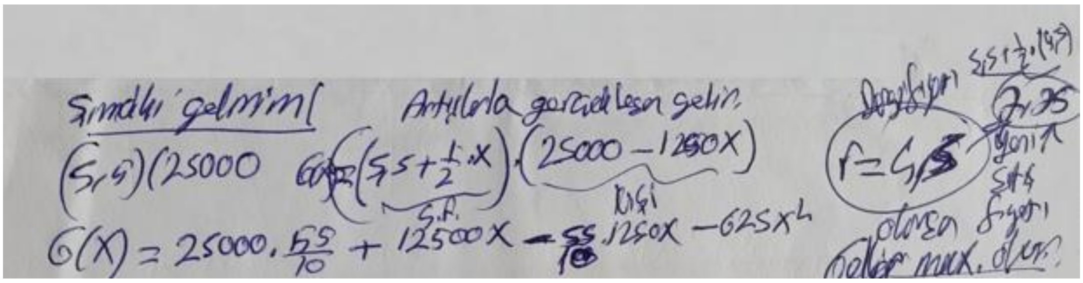


Figure 4.21. Ahmet's solution to question 15

Ahmet's knowledge of finding the quadratic functions with given points

As so Can, Ahmet correctly found the quadratic functions with some points given. In the first one, when the vertex and another point on the parabola were given; he used the vertex form to find the quadratic function (question 16). In the second one, when three arbitrary points on the parabola were given, he used the standard form to find the quadratic function. His solution to question 16 is presented below (as reproduced for readability):

$$
\begin{aligned}
&y = a(x-r)^2 + k\\
&y = a(x+\frac{1}{4})^2 + \frac{11}{4}\\
&5 = a(-1+\frac{1}{4})^2 + \frac{11}{4}\\
&a = 4.\\
&y = 4(x+\frac{1}{4})^2 + \frac{11}{4}
\end{aligned}
\tag{4.4}
$$

During his instruction, he also emphasized the use of different algebraic demonstrations. He told his students that if three arbitrary points are given, it is better to use the standard form. On the other hand, he stated that if the vertex is known, it is much practical to use the vertex form to find the quadratic function.

Ahmet's knowledge of finding the intersection of a parabola and a line

In question his response to question 18, he identified three conditions for the intersection of a line and a parabola. His solution was (as reproduced for readability):

$$
\begin{aligned}
&ax^2 + bx + c = mx + n.\\
&ax^2 + (b-m)x + c - n = 0\\
&\Delta = (b-m)^2 - 4a(c-n)\\
&\Delta > 0 \text{ (two points of intersection)}\\
&\Delta = 0 \text{ (one point of intersection)}\\
&\Delta < 0 \text{ (no point of intersection)}
\end{aligned}
\tag{4.5}
$$

In the follow-up, the teachers were asked to find the point(s) of the intersection of a given line and a parabola. He correctly found their point of intersection as seen in his original solution presented in Figure 4.22.

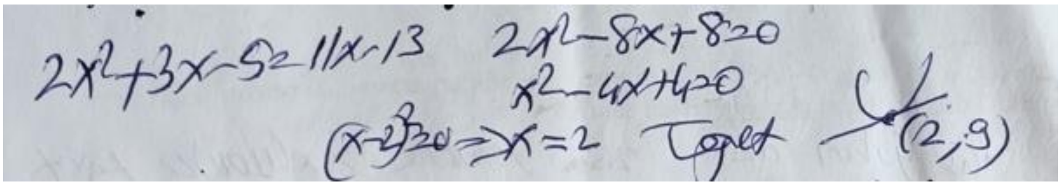


Figure 4.22. Ahmet's response to the follow-up of question 18

During the classroom instruction, as so Can, he told the intersection of a parabola and a line. This is illustrated in Figure 4.23.

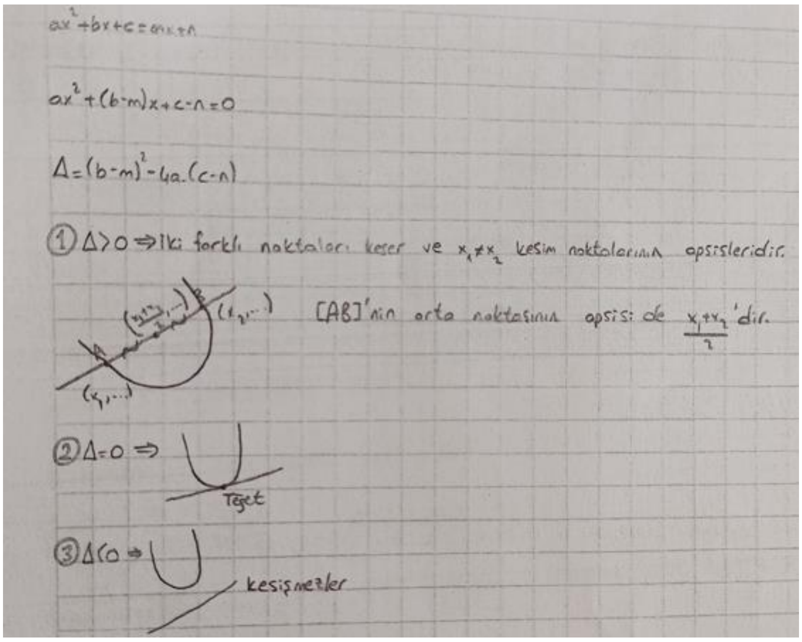


Figure 4.23. A section from Ahmet's instruction

4.3.1.2. Ahmet's Specialized Content Knowledge of Quadratic Functions

For a detailed description of Can's SCK, the results are presented under seven headings that indicate the sub-dimensions of teachers' SCK in the present study.

Ahmet's knowledge of explaining and justifying basic formulas of quadratic functions

In the questionnaire, the teachers were asked to state and derive the quadratic formula both geometrically and algebraically (question 19). Ahmet made only an algebraic justification of the formula. Then, he correctly solved the quadratic equation $x^2 - x + 1 = 0$ by completing the square (question 20). In the interview, the researcher asked:

> **Researcher:** You made an algebraic justification of the quadratic formula. Do you know any geometrical justification?
>
> **Ahmet:** Yes, we can use rectangles and squares but I don't show it geometrically in the classroom. I use geometrical demonstrations while teaching mathematical identities. However, for quadratic functions, I want my students to notice that they could solve quadratic equations by completing the square.

As he said in the interview, he told his students completing the square method and solved several exercises about this method as illustrated in Figure 4.24. He gives much importance to the use of completing the square method while solving quadratic equations.

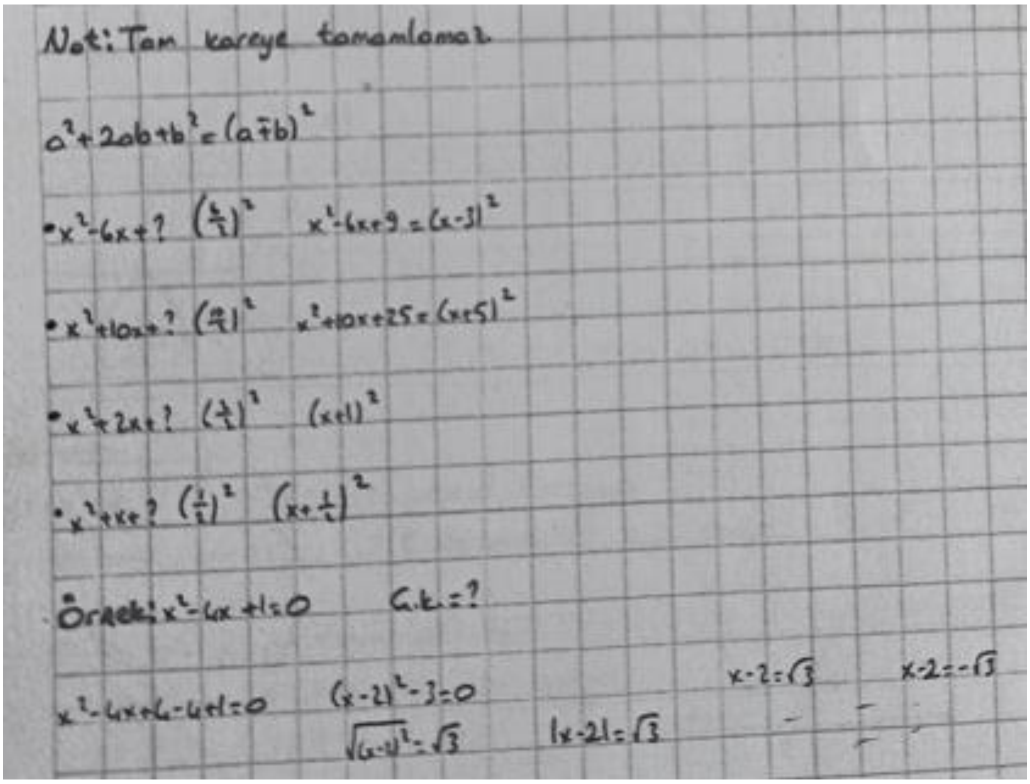


Figure 4.24. A section from Ahmet's instruction

Ahmet's knowledge of posing real-life problems regarding quadratic functions

In the questionnaire, when the participants were asked what kind of real-life examples they would use while teaching quadratic functions and equations (question 21), he stated a problem: "Let the cost of a product be *x* TL. If the product is sold $x^2 - 5x + 14$ TL, what would be the minimum profit?" In the interview, the researcher asked:

> **Researcher:** In question 21, you have written a profit-loss problem as example of real life problems regarding quadratic functions. Do you use this kind of problems as a part of your instruction? If yes, how often do you use?
>
> **Ahmet:** At the beginning, I say that parabolas are related to the construction of arch bridges. At the end, I solve maximum-minimum problems that are mostly related to profit-loss or the maximum areas of a rectangle.

His above words were confirmed by the classroom observation. As he stated, he mentioned arch bridges while introducing quadratic functions. In the final, he solved several problems about quadratic functions, as presented in Figure 4.25.

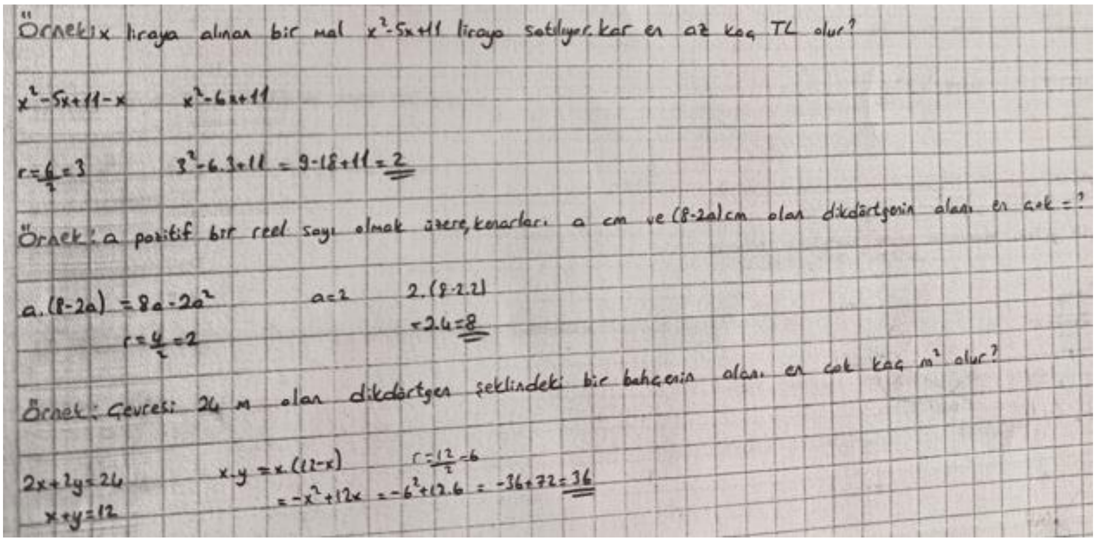


Figure 4.25. A section from Ahmet's instruction

Ahmet's knowledge of recognizing students' incorrect solutions regarding quadratic functions

As so Can, Ahmet identified all the errors in a given student solution (question 22). He wrote: "It is wrong. $k = f(r)$ gives maximum if $a > 0$, and gives minimum if $a > 0$. Here $a = -3$, so the vertex gives maximum. The student did not notice this. Also, the student did not check the values of the function at the endpoints."

Ahmet's knowledge of understanding students' unusual solutions regarding quadratic functions

The teachers were given two problems, each together with a student's solution. In the first one (question 23), there was a quadratic equation that can be factorized and the student solved the equation by completing the square method. Ahmet's comment on this solution was: “The student solved the equation by completing the square, without using the quadratic formula. This approach is my favorite while teaching quadratic equations. I give importance to my students to comprehend the origin of the formula.” In the questionnaire, the researcher asked him

> **Researcher:** You stated that completing the square method is your favorite approach to solve quadratic equations and you gave importance to your students understanding this approach. Why do you care so much about completing the square, rather than using the quadratic formula, which might generally be more practical?
>
> **Ahmet:** Because it is necessary in analytic geometry to understand the equation of a circle. I care about understanding the origins of the formulas. I do not prefer my students to memorize all the formulas. I advise them to avoid overloading the mass of information into their brains.
>
> **Researcher:** How is it related to analytic geometry? Could you explain?
>
> **Ahmet:** For example, if the equation of a circle is not given, they could find it by completing the square by using the analytics of the circle. They can find the radius and the center of the circle by completing the square. Mathematics is like a loop. Students should understand the connection between topics. They should have mathematical literacy. They should make sense of what they have learned. As I said before, for example, knowing that the roots of a quadratic equation are the *x*-intercepts of a parabola has vital importance.

In another item (question 24), the question was to find the (unique) quadratic polynomial with given some information about the coefficients and one of the roots. Unlike Can, Ahmet focused on the student’s solution process rather than the result. He wrote: “The solution is correct. The student went from the result (the root) to the beginning. He/she wrote the root of the polynomial equal to *x* and took the square to get rid of the radical. Then, the student multiplied the expression by 4 since the leading coefficient is 4.”

Ahmet's knowledge of responding to students' why questions about quadratic functions

In question 25, the teachers were asked to provide a plausible reason for why translating a parabola upwards and downwards changes only c while translating a parabola left and right changes both b and c. Ahmet responded:

> While translating upwards and downwards, the apsis of the vertex does not change. So, the sum of the roots stays constant, and thus b stays constant. As the roots change, the multiplication of the roots also changes. So, c changes. While translating left and right, both the sum of the roots and the multiplication of the roots change. Thus, both b and c change.

In the interview, the researcher asked:

> **Researcher:** In the questionnaire, you explained the change in the coefficients while translating parabolas. Do you tell your students about these changes?
>
> **Ahmet:** Yes. I introduce the parabola by drawing the graph of $y = ax^2$. Then, I draw $y = ax^2$+k, and say that if k is positive we will move the parabola k units above the y-axis, if k is negative we will move it k units below the y-axis. Translation of the parabola is the basis for translating all the functions. We can apply this on all the functions. This tells us how to draw $f(x) - r$ when $f(x)$ is given. As I said before, since we will encounter this in the next sections, I show them these translations.
>
> **Researcher:** You move on by considering the next concepts.
>
> **Ahmet:** Yes. This is valid for all function translations.

As he said in the interview, he introduced the graphs of quadratic functions step by step. This is illustrated below in Figures 4.26 and 4.27.

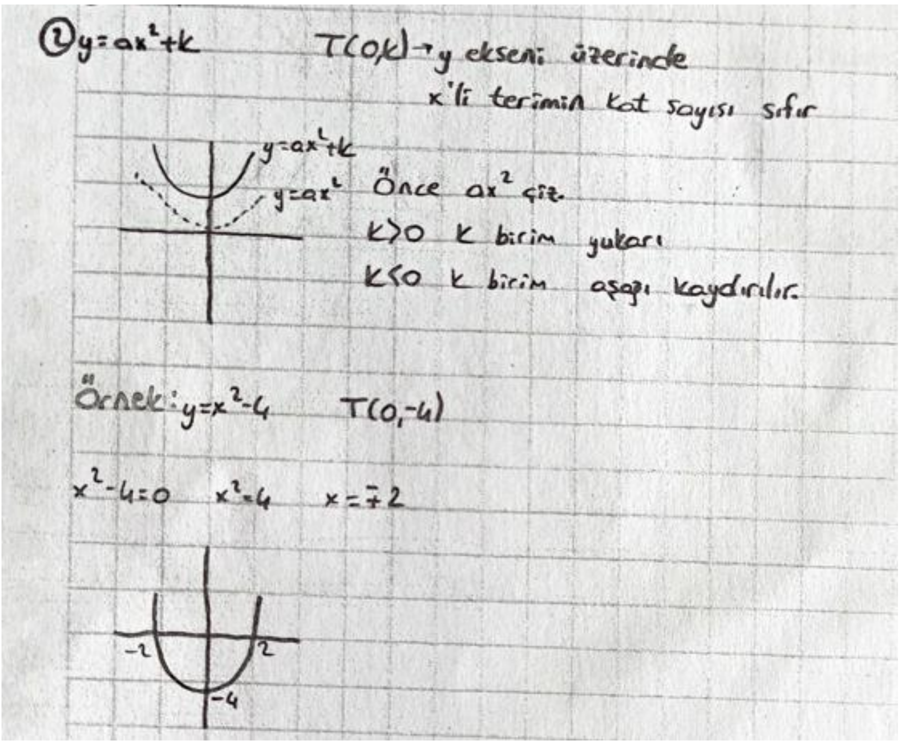


Figure 4.26. A section from Ahmet's instruction

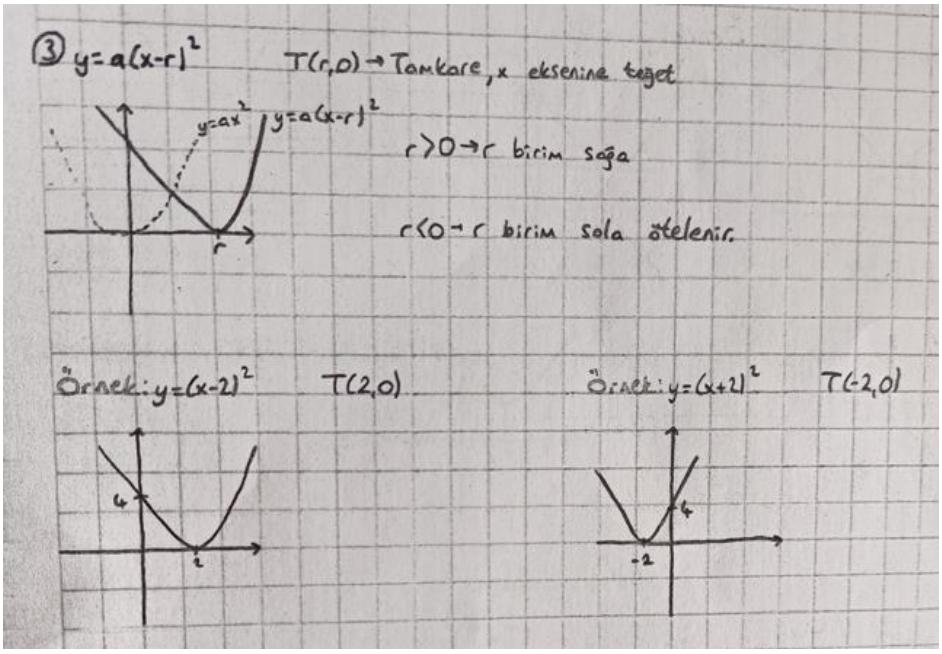


Figure 4.27. A section from Ahmet's instruction

Ahmet's knowledge of finding an example to make a specific mathematical point about quadratic functions

When he was asked what kind of examples he would use in the classroom to emphasize the symmetrical property of parabolas (question 27), he wrote: "I am defining r (the apsis of the vertex) as the half of the sum of the roots. I tell my students that the x-values that sum up to $2r$ are symmetrical. For example if $r = 5$, $f(1) = f(9)$ or $f(-5) = f(15)$. I want my students to notice this property." In the interview, the researcher asked him:

> **Researcher:** In question 27, you have stated that you define a as the half of the sum of the roots of a quadratic equation. Could you explain how you do this?
>
> **Ahmet:** (draws a parabola with $r = 5, f(0) = 4$) For example, I give my students this graph and I want my students to find $f(10)$. Students find the quadratic function and then calculate $f(10)$ as 4 but I want them to notice this short way without finding the equation. In short, I use symmetrical property here. I always ask this kind of questions. I want my students to notice $f(1) = f(9)$ or $f(100) = f(-90)$.

As he stated in the questionnaire and the interview, during his instruction, he defined r as the half of the sum of the roots. Then, he defined the line $x = r$ as the axis of symmetry. This is illustrated in Figure 4.28.

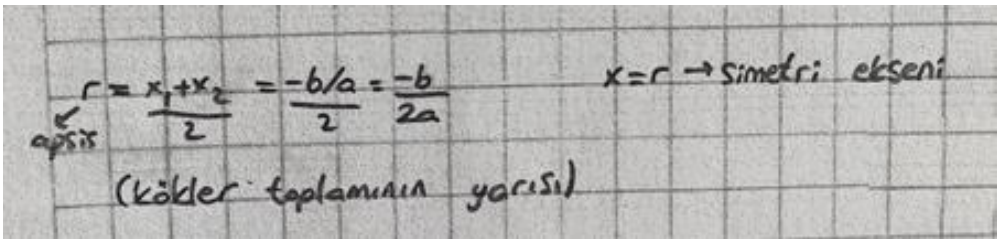


Figure 4.28. Section from Ahmet's instruction

Ahmet's knowledge of modifying tasks regarding quadratic functions

In question 28, the teachers were asked to determine whether their students could solve a task regarding quadratic functions. They were also asked to make some modifications to the task if they think that their students could not solve it. Ahmet made some modifications to the task. He replaced one statement in the task "…given that the distance between A and B is 3 units…" with the statement "…given that the apsis of the midpoint of A and B is 2…". In the interview, the researcher asked him:

> **Researcher:** In question 28, you have changed the question by giving the apsis of the midpoint of A and B. Could you explain why?
>
> **Ahmet:** I think, some of my students can find the solution by using the roots difference formula. However, if the apsis of the vertex was given, the task would be easier and most of my students can solve it.

As it can be seen in his above words, he thought that some students could solve the task, and changed one statement in the task to make the task easier for most of his students.

4.3.1.3. Ahmet's Horizon Content Knowledge of Quadratic Functions

For an elaborate discussion of Ahmet's HCK regarding quadratic functions, the results are presented under two headings that include Ahmet's knowledge of: how quadratic functions are related to other contents in the high school curriculum and how quadratic functions are related to advanced mathematics.

Ahmet's knowledge of how quadratic functions are related to other contents in the high school curriculum

Based on the questionnaire results and the interview, Ahmet's knowledge of the relationship between quadratic functions and other contents in the high school curriculum was strong. Unlike Can, he responded to all the questions regarding the connection between quadratic functions and other contents such as the derivative, exponential functions, etc. (questions 29-34). In question 29, he wrote that the second derivative can be used to explain the relationship between the sign of the leading coefficient and the concavity of the parabola. In the interview, the researcher asked him:

> **Researcher:** You wrote that we can use the second derivative to explain the relationship between the leading coefficient and the concavity of the parabola. Could you explain how?
>
> **Ahmet:** For the function $\mathrm{f(x)} = \mathrm{ax}^2 + \mathrm{bx} + \mathrm{c}$, $\mathrm{f}''(\mathrm{x}) = 2\mathrm{a}$ is always constant. If $a > 0$, then $\mathrm{f}''(\mathrm{x}) > 0$, so the graph is concave up. If $a < 0$, then $\mathrm{f}''(\mathrm{x}) < 0$, so the graph is concave down. There is no inflection point to change the concavity of the graph.

On the basis of his response to the questionnaire (question 30), he is also aware that exponential functions (with base greater than 1) grow faster than quadratic functions. Then, he explained the association between the vertex and the first derivative (question 31) as: "The vertex is a local extremum point. At the vertex, the first derivative is zero (the slope of the tangent line is zero)." As regarding to the relationship between the quadratic functions and the physics course (question 32), he stated: "I would give the example of projectile motion." Ahmet has the knowledge of the relationships between the quadratic function and some other mathematical concepts as well as its relationship with interdisciplinary areas such as the physics.

In the next item (question 33), he explained the relationship between the golden ratio and quadratic equations. He stated: "It is $1 + \frac{\sqrt{5}}{2}$, which is the positive root of the quadratic equation $\mathrm{x}^2 - \mathrm{x} - 1 = 0$." Lastly, when the teachers were asked to decide whether the graph of $y = x^4$ is a parabola or not (question 34), he responded: "It is not a parabola because it is not of the second order." This explanation is correct since all the parabolas can be modeled by a quadratic function.

Ahmet's knowledge of how quadratic functions are related to advanced mathematics

Ahmet's knowledge of the relationships between quadratic functions and advanced mathematics was strong, on the basis of his responses to the questionnaire and the interview. For example, when the participants were asked to describe (if exists) the relationship between a parabola and a hyperbola (question 36), he wrote: "A parabola is the set of points which are equidistant from a straight line and a focus whereas a hyperbola is the set of points whose distances to two fixed points have a constant difference." Then, in the interview the researcher asked him:

> **Researcher:** You wrote the definitions of a parabola and a hyperbola in your response to the questionnaire. What would you say about these two concepts?
>
> **Ahmet:** They are both conic sections. I can say this. Their graphs are of course different. The definition of a hyperbola does not take place in the current high school mathematics curriculum.

He also gave some examples of the daily use of the reflection property of parabolas (question 35). He stated: "It is used in real-life, for example, in the construction of headlights and satellite dishes." In the interview, the researcher asked him:

> **Researcher:** You stated that the reflection property is used in the construction of headlights and satellite dishes. Could you explain what this property is?
>
> **Ahmet:** Reflection property says that any ray parallel to the axis of the parabola will reflect and pass through the focus of the parabola. The logic behind the headlights and satellite dishes is this property. I do not know exactly, the engineers must know and use it better.

In question 37, he explained the fundamental theorem of algebra and its application to quadratic equations. He stated: "Any polynomial of degree n has n roots. So, a quadratic equation has 2 roots." Then, the participants were also required to select the most correct statement among the given two ones, which are presented below (question 38).

Statement 1: The graph of a quadratic function is a parabola.

Statement 2: The graph of a quadratic function is called a parabola.

In the questionnaire, Ahmet selected the second statement without further explanation. In the interview, the researcher asked him:

> **Researcher:** In question 38, you selected the second statement as the most correct? Could you explain why?
>
> **Ahmet:** The graph of a quadratic function defines a parabola like a first-degree function defines a line. In this case, we call them parabolas. We do not use the conic definition; it was removed from the curriculum.

In question 39, he defined a parabola as "the graph of a quadratic function" and stated an alternative definition as "the set of points that are equidistant from both the directrix and the focus". He was the only teacher who stated the geometrical definition of a parabola. Even though he does not use this definition in his instruction because it is not included in the curriculum, he knows the geometrical definition of a parabola and its interrelation with hyperbolas.

The last question was about distinguishing a parabola from a catenary (question 40). In the question, there was a figure which had the shape of a uniform flexible chain. Of the 18 teachers, only Ahmet stated that the shape is a catenary. In the interview, the researcher asked him:

> **Researcher:** In question 40, you have stated that this shape is a catenary. Could you explain why?
>
> **Ahmet:** Yes. I think it resembles a parabola at first sight, but I think that is different. Who discovered the catenaries? Hmmm… Was he Leibniz? I am not sure.
>
> **Researcher:** Do you know the equations of catenaries?
>
> **Ahmet:** Catenaries have different equations, but I don't know exactly. I will investigate it.

4.3.2. The contribution of Ahmet's Subject Matter Knowledge of Quadratic Functions to Student Learning Outcomes

A total of 28 students were enrolled in Ahmet's course. All the students in his class were administered the quadratic function concept test which provided the data for interpreting Ahmet's students' learning outcomes of quadratic functions (see Appendix C). A summary of the results of Ahmet's students' performance on the test is presented below (Figure 4.29).

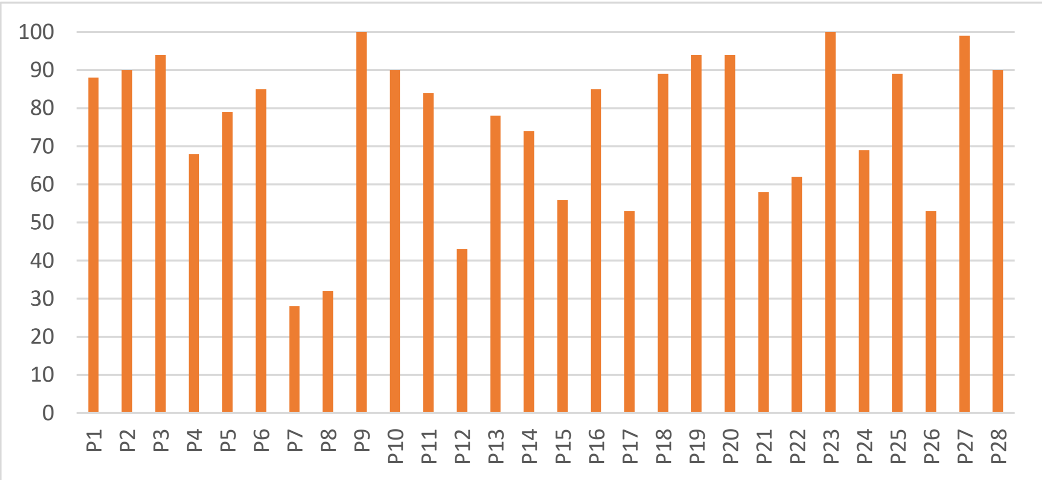


Figure 4.29. Ahmet's students' scores on the quadratic function concept test

Figure 4.29 shows that Ahmet's students' test scores range between a minimum of 28 and a maximum of 100 points. The average test score of the students is 75,8. In general, his students' performance on the test is good. The students' performance is discussed in detail in the next sections based on the objectives of the mathematics curriculum regarding quadratic functions.

4.3.2.1. Finding the Vertex, x-Intercepts, the y-Intercept, and Axis of Symmetry

In the first question of the quadratic function concept test, the students were asked to find the x-intercept(s), the y-intercept, the vertex, and the axis of symmetry of the parabola generated by $f(x) = x^2 + 2x - 8$, and graph it. While finding x-intercepts, unlike Can's students, some of them (n=9) used completing the square method. For example, one student wrote: "$f(x) = (x+1)^2 - 9 = 0,\ x+1 = 3,\ x+1 = -3,$ $x_1 = 2, x_2 = -4$." Fourteen students used factorization, whereas five students did not answer. None of Ahmet's students used the quadratic formula for finding the x-intercepts of the quadratic function. During his instruction, unlike Can, Ahmet emphasized the importance of completing the square method and used this method for solving quadratic equations and finding the x-intercepts of quadratic functions.

When they were asked to find the y-intercept, almost all the students (n=27) correctly found it. Fifteen of them properly that the y-intercept is $(0, -8)$, whereas twelve of them wrote only the number "-8" as the y-intercept.

While finding the vertex, most students (n=25) found it correctly as the point $(-1, -9)$. They all firstly found r by using the formula; then, they found k, by finding $f(r)$. Only two students found an incorrect point as the vertex. They calculated $f(2)$ as the vertex.

When the students were asked to find the axis of symmetry, most of them (n=26) were aware of the interrelation between the apsis of the vertex and the axis of symmetry. However, they had a problem with the use of mathematical terminology. Most of them (n=14) wrote "$r = -1$" as the axis of symmetry. Twelve students correctly stated that the axis of symmetry is the line $x = -1$. Two students wrote an irrelevant number as the axis of symmetry. When they were asked to graph the function, most students (n=26) sketched the graph correctly. Only two students sketched incorrect graphs.

There is some evidence for the contribution of Ahmet's SMK to student learning outcomes. To illustrate, in the questionnaire, he was the only participant who solved the quadratic equations by completing the square method. During his instruction, he frequently used completing the square method, for finding the x-intercepts of a quadratic function. Unlike Can's students, some of his students (n=9) used completing the square method for finding the x-intercepts of a quadratic function. None of his students used the quadratic formula, whereas half of them used the factorization method. When his students were asked to find the vertex of a quadratic function, almost all of them (n=26) correctly found the vertex as the point (r, k). Most of them (n=17) calculated firstly r, then found $k = f(r)$. Unlike Can's students, Ahmet's students noticed that the vertex is the point (r, k), not the "k" value. Moreover, approximately half of Ahmet's students were aware that the axis of symmetry is a line, not a point as they wrote "$x = -1$" as the axis of symmetry of the given function. In the questionnaire, Ahmet defined the axis of symmetry as "a line separating the parabola into two symmetrical parts", and found the axis of symmetry of the function $g(x) = -6x^2 + 12x + 5$ as "$x = 1$". However, as so Can's students, half of Ahmet's students wrote "$r = -1$" as the axis of symmetry. In the second question, which asked to find a missing coefficient in a quadratic function whose vertex is given, most students found the correct result. As so Can, during his instruction, Ahmet solved similar kinds of questions.

4.3.2.2. Associating the Vertex with the Maximum or the Minimum of a Function

When the students were asked to find the minimum of a given function (question 3) and the maximum of a given function (question 4), all of Ahmet's students correctly associated the minimum and the maximum of a quadratic function with the ordinate of the vertex, k. In both questions, they calculated firstly r, and then found k as $f(r)$. For example, one of his student's response to question 4 is: "$r = -\frac{b}{2a} = 2,\ k = h(2) = -4 + 8 + 6 = 10$."

Ahmet's students' performance on finding the minimum or the maximum of a parabola was good. During his instruction, Ahmet solved several problems about the maximum and minimum of quadratic functions, as illustrated in Figure 4.30.

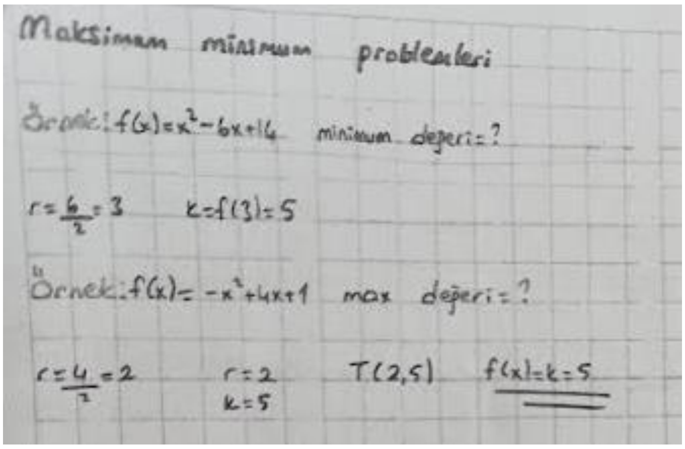


Figure 4.30. A section from Ahmet's instruction

4.3.2.3. Commenting on the Effect of the Change in the Coefficients on the Graph of the Function

In the fifth question, the students were asked to comment on two cases about how the graph of a quadratic function changes when the leading coefficient a gets smaller or larger. Most students (n=20) provided a reasonable explanation for the change in the graph as the leading coefficient changes. One of the students' responses is presented in Figure 4.31. Three students did not answer this question whereas five students wrote incorrect statements and drew incorrect graphs. For example, in the first case, one of them stated: "x values increase, y values do not change, and the vertex increases."

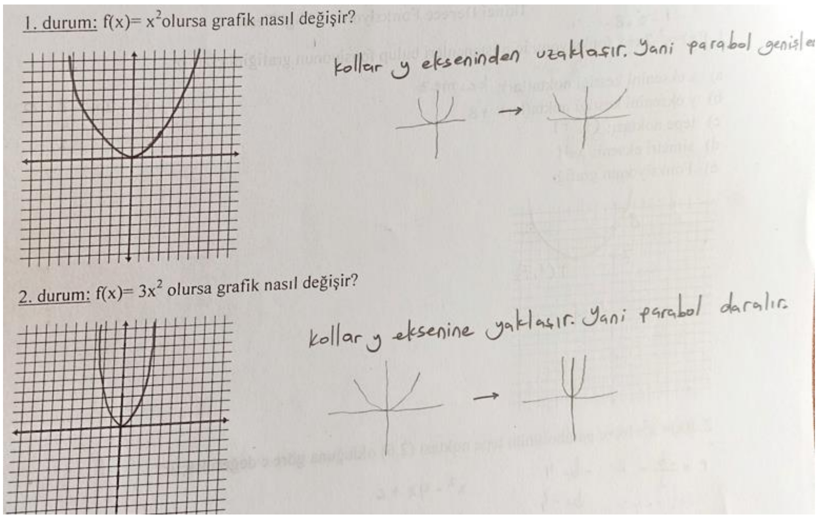


Figure 4.31. An example from Ahmet's students' responses to question 5

Most of Ahmet's students performed well in explaining the relationship between the leading coefficient of a quadratic function and its graph. Based on the questionnaire results, it can also be said that Ahmet also has knowledge of the relationship between the change in the coefficients and the graph of a quadratic function. During the classroom instruction, he emphasized the interrelation between algebraic and graphical representations of quadratic functions. Thus, Ahmet's content knowledge might contribute to his instruction positively, and his instruction might positively have affected his students' learning outcomes about interpreting the graphs of quadratic functions.

4.3.2.4. Finding the Quadratic Function Given Three Points or Two Points That One of Them is the Vertex

In question 6, the students were asked to find the quadratic function whose vertex and one point were given. Most of them (*n*=22) were able to find the quadratic function by using the vertex form. For example, one of them responded: "$f(x) = a(x-r)^2 + k,\ \ f(x) = a(x+2)^2 + 2,\ \ f(1) = 11,\ \ a = 1,\ \ f(x) = a(x+2)^2$." Three students did not respond. A few students (*n*=3) used the vertex form and correctly found a, too. However; they also wrote the quadratic equation in the standard form $y = ax^2 + bx + c$, and tried to calculate the other coefficients b and c. They made calculation errors and came up with an incorrect equation.

In question 7, the students were asked to find the quadratic function whose three points such that one of them is the y-intercept was given. Thirteen students found the quadratic function by using the standard form. Since the y-intercept was given, they easily found c in the quadratic function $y = ax^2 + bx + c$. Then, they substituted the two given points in the function, and obtained two equations. Finally, they found a and b coefficients by solving both equations. The students who found an incorrect answer (n=10) tried to find the quadratic function by using the intercept form. They considered the given two points as the x-intercepts.

Most of Ahmet's students (n=22) were able to find the quadratic function when the vertex is given. However, when they were given three points, not the vertex, approximately half of them (n=15) had difficulty in finding the quadratic function. All of those who found incorrect results used the intercept form, as so Can's students. In the questionnaire, as Can and most teachers did, Ahmet used the intercept form when the vertex is given, and he used the standard form when three points were given. During his classroom instruction, as so Can, Ahmet solved several questions about finding the quadratic function whose some points were given, as shown in Figure 4.32. However, the students in two cases performed better when the vertex is given. When three points were given, the students of both teachers had some difficulties in writing the quadratic function.

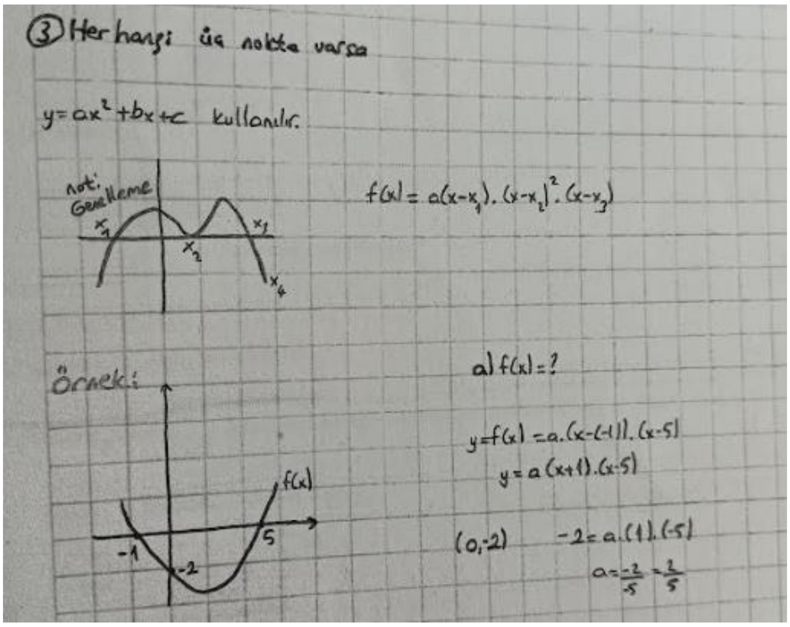


Figure 4.32. A section from Ahmet's instruction

4.3.2.5. Investigating the Intersection of a Line and a Parabola

Most of his students (n=24) correctly examined the intersection of a given line and a parabola. The majority of them (n=18) investigated the discriminant of the new quadratic equation that they obtained by equating the y- values of two functions. They wrote that they are tangent to each other since the discriminant equals to 0. Some students (n=6) did not calculate the discriminant of the new quadratic equation. They stated that the parabola and the line are tangent to each other since the quadratic equation is a perfect square. These students established a relationship between the discriminant of a quadratic equation and its form (being a perfect square or not). Three students did not respond to this question whereas one student had an incorrect answer. He calculated the discriminant of the given quadratic equation.

In the questionnaire, the teachers were also asked to find the intersection of a line and a parabola. As most teachers and Can did, Ahmet obtained a new quadratic equation and investigated its discriminant. Furthermore, as so Can, he solved several problems about investigating the intersection of a parabola and a line in the classroom, as illustrated in Figure 4.33.

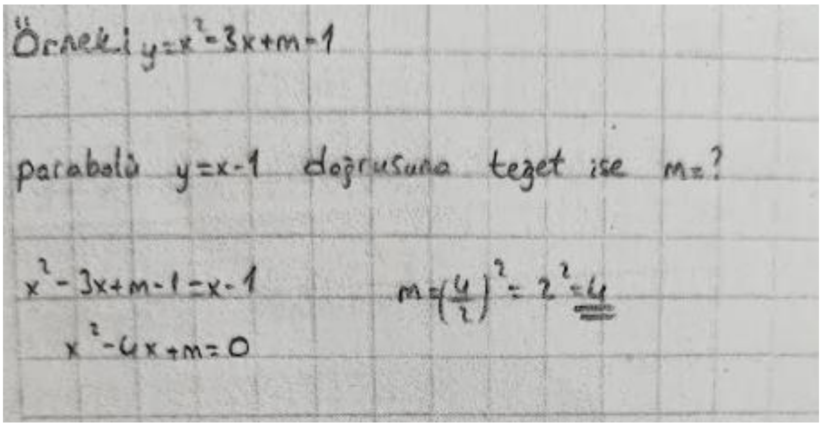


Figure 4.33. Section from Ahmet's instruction

4.3.2.6. Solving Problems That Can Be Modeled by Quadratic Functions

In question 9, the students were asked to find the maximum area of a rectangle with its perimeter given. Twelve students correctly found the result by constructing a quadratic model. One of the student's solution is presented below (Figure 4.34). There were also some students (n=7) who found the correct result without constructing a quadratic model, by trying some numbers for the dimensions. Similarly, three students replaced several numerical values for the dimensions of the rectangle. However, they eliminated the case $9x9$ since they thought that it would be a square, not a rectangle.

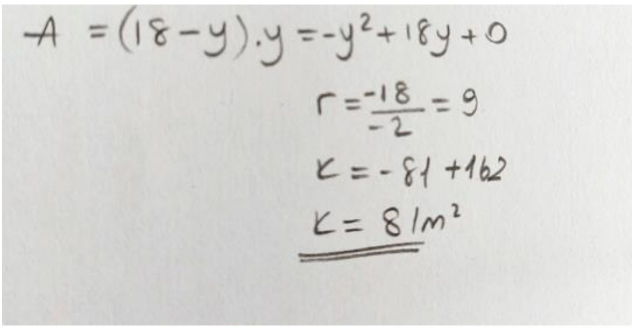


Figure 4.34. An example from Ahmet's students' responses to question 9

In question 10, the students were given a quadratic function, which represents the height of a ball that was hit by someone playing football. The question was to find at what time the ball reaches 3 meters above the ground. Most students (n=15) correctly found the solution. For example, one of them wrote: $h(t) = -t^2 + 4t = 3$, $-t^2 + 4t - 3 = 0$, $t_1 = 1$, $t_2 = 3$. Four participants found only one of the roots $t = 3$ or $t = 1$. Some students (n=6) found incorrect results. Two of them calculated the value of $h(3)$ whereas four of them found the maximum of the function. Three students did not respond.

CHAPTER 5

CONCLUSION AND DISCUSSION

The primary purposes of this study were to identify high school mathematics teachers' SMK in three dimensions, CCK, SCK, and HCK, and to examine its contribution to student learning outcomes on quadratic functions. There were two research questions:

1. As regarding to quadratic functions, what SMK do secondary mathematics teachers have?
    d) As regarding to quadratic functions, what CCK do secondary mathematics teachers have?
    e) As regarding to quadratic functions, what SCK do secondary mathematics teachers have?
    f) As regarding to quadratic functions, what HCK do secondary mathematics teachers have?
2. How do CCK, SCK, and HCK contribute to the instructional practice and thus, student learning outcomes regarding quadratic functions?

To answer the first research question about teachers' SMK of quadratic functions, a questionnaire was administered to 18 high school mathematics teachers. Two case studies were carried out to address the second research question that investigates the contribution of teachers' SMK of quadratic functions to instructional practice, and thus student learning outcomes. Two teachers from the participants of the first part of the study were selected for the second part of the study, which included interviews and classroom observations. In this chapter, the main findings of the study were concluded and discussed. This chapter also included implications of the study, recommendations for future studies, and limitations of the study.

5.1. Teachers' Subject Matter Knowledge of Quadratic Functions

The teachers' SMK of quadratic functions was discussed under three headings: teachers' CCK, SCK, and HCK.

5.1.1. Teachers' Common Content Knowledge of Quadratic Functions

In this study, teachers' CCK of quadratic functions includes seven components teachers' conceptions of quadratic equations and functions, teachers' knowledge of solving quadratic equations with one unknown, teachers' knowledge of sketching and interpreting the graphs of quadratic functions, knowledge of graphing quadratic functions using transformations, teachers' knowledge of solving real-life problems regarding quadratic functions, teachers' knowledge of finding the quadratic function with given points, and teachers' knowledge of finding the intersection of a parabola and a line.

The result showed that although teachers defined quadratic functions, quadratic equations, and quadratic polynomials in their algebraic forms, most of them have limited knowledge of the interrelation between these concepts. Most of the teachers examined the first differences for the given (x, y) ordered pairs to decide whether the function is linear or quadratic; however, none of the teachers mentioned the constant second differences of quadratic functions. The findings also indicated that the majority of teachers used the quadratic formula for solving a quadratic equation. When they were asked to define some basic elements of a parabola, like the axis of symmetry, vertex, or the concavity of a parabola, the majority made procedural descriptions rather than structural descriptions. For example, most of them described the axis of symmetry as the line $x = -b/2a$, passing through the vertex. Likewise, some teachers defined the vertex as the point $(\frac{-b}{2a}, f(\frac{-b}{2a})$. Some teachers described how to find the type of the concavity of a parabola based on the leading coefficient, whereas some associated the concavity with the second derivative of the function. All the teachers correctly sketched the graph of a given quadratic function.

When the participants were given a parabola and asked to find the corresponding quadratic function, they used multiple algebraic forms of quadratic functions. That is, when the vertex was given in the graph, they used the vertex form; and when the x-intercepts were given, they used the intercept form. When the participants were given a graph and asked to comment on the signs of a, b, and c coefficients of the corresponding quadratic function, all of them examined the shape of the parabola and stated that $a > 0$ since the parabola is upwards. To determine the sign of b, two different approaches were observed. Most teachers examined the sign of the apsis of the vertex to determine the sign of b. However, some participants examined the sign of the sum of the roots to find the sign of b. While determining the sign of c, most participants examined the y-intercept whereas some participants found the sign of c by examining the multiplication of the roots.

The result also indicated that teachers mostly stated translation as a graph transformation. They did not mention other types of transformations such as reflection, stretching or shrinking of parabolas. Only a few teachers stated reflection and stretching as graph transformations. The majority of teachers did not explain how the graph of any quadratic function can be obtained from the graph of the function $y = x^2$. The teachers were also asked to compare two functions without graphing them; many participants noticed that their shape was the same. However, some teachers compared the graphs in terms of their x-intercepts, the y-intercept, and the vertices without referring to the transformations and the conservation of the shape. Moreover, many participants did not explain the effect of the coefficients on the shape of a parabola. When the teachers were asked to determine the widest parabola among the given ones, few of them correctly stated the relationship between the absolute value of the leading coefficient and the width of the parabola.

The findings also revealed that teachers have limited knowledge of solving real-life problems about quadratic functions. That is, most of the teachers could not construct a mathematical model to solve the real-life problem in the questionnaire. Some teachers attempted to solve the problem without forming a quadratic model, trying some numerical values to obtain the result, but they were unsuccessful. Teachers' performance in finding the quadratic function with some points given was good. When the vertex was known, they used the vertex form; and when the three points were known, they used the standard form to obtain the quadratic function. Lastly, the majority of teachers found the intersection of a parabola and a line correctly.

Although there were some differences between their solution strategies or approaches to solving problems, secondary mathematics teachers' performance on CCK items was good. They were able to explain basic facts or procedures about quadratic functions, use correct mathematical notations, and correctly solve simple questions about quadratic functions.

The results of this study coincide with the study of Bansilal et al. (2014) that investigated secondary mathematics teachers' CCK. In their study, the teachers were given the graph of a function $f(x)$, and they were asked to find the maximum of $1 - f(x)$. They were expected to make some transformations on the vertex of the given parabola. Bansilal et al. (2014) reported that most secondary teachers failed to make these transformations. However, they did not have problems finding the x-intercepts of a parabola. Likewise, in the present study, most teachers did not have a problem finding the x-intercepts of a parabola; however, they have limited knowledge of explaining graph transformations. In another study, Ubah and Bansilal (2018) explored how pre-service teachers found the algebraic equation for a quadratic function given in the graphical form. Ubah and Bansilal (2018) reported that although some of the participants could find the equation using one method; most failed to find the equation using two methods and the most common method function was using the intercept form of quadratic functions. On the contrary, in the current study, teachers used more than one method (using the intercept form and using the standard form). In the study of Ubah and Bansilal (2018), many pre-service teachers could not even write the correct algebraic form of a quadratic function (Ubah, & Bansilal, 2018). On the contrary, in the present study, secondary teachers did not have difficulty writing the algebraic form of a quadratic function given in the graphical form.

This study also confirms the findings of the study by Aziz et al. (2018) that investigated pre-service secondary mathematics teachers' views on distinguishing quadratic functions and quadratic equations. Like the participants in Aziz et al.'s (2018) study, the teachers in the current study wrote differences between quadratic equations and quadratic functions mostly based on their standard forms. The findings of the current study also revealed that some teachers compared quadratic functions and quadratic equations based on their main characteristics and based on geometrical aspects. Aziz et al. (2018) also reported that factorization and the quadratic formula were the two methods the participants commonly used to solve quadratic equations; and a few participants used completing the square method for solving a quadratic equation. Similarly, in this study, the quadratic formula and factorization was the most common method for solving a quadratic equation; only one teacher used completing the square method to determine the roots of a quadratic equation.

5.1.2. Teachers' Specialized Content Knowledge of Quadratic Functions

In this study, teachers' SCK is discussed under seven components including teachers' knowledge of: explaining and justifying basic formulas of quadratic functions, posing problems regarding quadratic functions, recognizing students' incorrect solutions regarding quadratic functions, understanding students' unusual solutions regarding quadratic functions, responding to students' why questions about quadratic functions, finding an example to make a specific mathematical point about quadratic functions, and modifying tasks regarding quadratic functions.

The result showed that the majority of teachers' performances on SCK items were lower than their performances on CCK items. Although teachers used the quadratic formula, the majority of them did not justify it. Some teachers made an algebraic justification of the formula by completing the square; none of the participants made a geometrical justification. When the teachers were asked to state a real-life problem about quadratic functions, the majority of them did not state a problem. Most teachers wrote a problem context rather than the full statement of the problem, including projectile motion and velocity-acceleration problems from the physics course. The teachers were also asked to recognize a student's incorrect solution. Some participants noticed all the incorrect steps, whereas half of them recognized some of the incorrect steps. The comments that the teachers made about the student's solutions were incomplete.

The findings also indicated that the majority of teachers could not explain the effect of the translations on the coefficients of the quadratic functions. That is, most teachers failed to identify the association between the shape of the parabola, the coefficients, and the location of the vertex. When they were asked to explain why a quadratic function $ax^2 + bx + c$ cannot be divided by the variable x, almost all of them explained the reason. The teachers were also asked to state examples to underline the symmetrical property of a parabola. The majority of teachers made some incorrect or irrelevant explanations. Only a few teachers provided a plausible example to emphasize the symmetrical property during instructional practice. Lastly, the teachers were given a task of quadratic functions and asked to explain whether their students could solve it or not. Many teachers made some reasonable changes to the task since they thought it might be hard for their students.

In this study, teacher have limited SCK for teaching quadratic functions. Similarly, Zembat (2013) reported that teachers have quite limited understanding of the core mathematical ideas, analyzing the students' work, in the assessment of understanding mathematical ideas, and making curricular decisions. Zembat (2013) suggested that teachers should improve their SCK to fill the gap between where they are and where they need to be.

5.1.3. Teachers' Horizon Content Knowledge of Quadratic Functions

In this study, teachers' HCK was analyzed in two dimensions: the knowledge of how quadratic functions are related to other contents in the high school curriculum and the knowledge of how quadratic functions are related to advanced mathematics.

The result indicated that although teachers know basic facts, operations or procedures about quadratic functions, the majority of them were unable to connect quadratic functions with other content in the high school curriculum. Most teachers did not explain the relationship between the concavity of a parabola and the second derivative of the quadratic function. Furthermore, the majority of teachers could not compare the graphs of parabolas with the graphs of exponential functions. Although more than half of the teachers were able to explain the relationship between the vertex of a parabola and the first derivative partially, none of them mentioned the rate of change or the maximum-minimum points of the quadratic function. In addition, few teachers explained the connection between the golden ratio and quadratic equations. Lastly, the teachers were asked to relate the concept of the quadratic function with any concept from the physics course. Some participants stated that the quadratic function is related to free fall, projectile motion, and velocity-acceleration problems from the physics course, whereas most of them did not give any examples.

The results also revealed that the teachers' knowledge of how quadratic functions relate to advanced mathematics is limited. When the teachers were asked to compare a parabola and a hyperbola, only a few defined a hyperbola and explained their relationship. Most of them did not state the fundamental theorem of algebra and its application to quadratic polynomials. Furthermore, almost all the teachers did not state the geometrical definition of a parabola as a conic section. In addition, most of the participants did not explain the term *catenary*, which has a shape that looks like a parabola but is somehow different. This finding coincides with the study of Miheso-O'Connor Khakasa and Berger (2016) who reported that teachers were uncomfortable with engaging in responses that require HCK. Miheso-O'Connor Khakasa and Berger (2016) also reported that teachers have limited knowledge of when and how to use the advanced mathematical knowledge.

5.2. The Contribution of Teachers' Subject Matter Knowledge of Quadratic Functions to Student Learning Outcomes

The second phase of the study revealed the contribution of teachers' SMK of the quadratic function concept and student learning outcomes. The data suggested evidence of that teachers' SMK of quadratic functions contributed to student learning outcomes. The study also revealed that teachers' SMK of quadratic functions affected their instructional practices, and their instructional practices interacted with students' performance.

The study showed that the teachers' SMK of quadratic functions influenced their instructional practices. Can's instruction was mostly based on the procedural aspects of the quadratic functions rather than the conceptual aspects. He solved the quadratic equations by the quadratic formula or factorization. He did not use the completing the square method. He did not give much importance to justifying basic formulas of quadratic functions, such as the quadratic formula. He sometimes used incorrect notation for mathematical equations or incorrect explanations for some properties. For example, he defined the vertex as the ordinate of a parabola's maximum or minimum point. Also, he defined the axis of symmetry as the apsis of the vertex and found the axis of symmetry of a function as $r = 1$ rather than a line equation that $x = 1$. The exercises he solved in the classroom mostly required procedural knowledge. He showed many examples of finding the vertex of a quadratic function during his instruction. However, he did not solve problems about the maximum or minimum of a parabola. He failed to explain the effect of the translations of parabolas on the coefficients of the functions. During his instructional practice, he did not emphasize the transformations of quadratic functions. Also, he failed to solve the real-life problem in the questionnaire. Accordingly, he did not solve any real-life problems about quadratic functions in the classroom. However, he was good at using multiple representations of functions. In his instructional practice, he solved several questions about finding the equation of a quadratic function. He emphasized using the intercept form if the vertex is given, and he told his students to use the standard form if the y-intercept and two other points are given. He also solved several problems about finding the intersection of a parabola and a line.

On the other hand, unlike Can's instructional practice, Ahmet's instruction involved more detailed explanations, justifications of the mathematical rules, and connections between mathematical concepts. In contrast to Can, Ahmet focused primarily on the conceptual aspects of quadratic functions as well as the procedural aspects during his instruction. His performance on the quadratic function concept questionnaire was clearly better than the other participating teachers. He had the highest scores on the CCK, SCK, and HCK items of the questionnaire. His SMK was stronger than Can and also the other teachers in the first phase of the study. He was the only teacher who solved the quadratic equations by completing the square. He emphasized justifying basic formulas or properties of quadratic functions and explaining the connections to higher mathematical ideas. He made structural definitions of the concepts such as the vertex and the axis of symmetry that are crucial to understanding quadratic functions. Unlike Can, he solved several real-life problems about quadratic functions, including maximum-minimum problems. He was also the only participant who solved the real-life problem in the questionnaire by forming a quadratic model. Like Can, he solved several questions about finding the equation of a quadratic function and the intersection of a parabola and a line. As Can did, he emphasized multiple representations of quadratic functions during his instruction. He could also explain the effect of the translations of parabolas on the coefficients of the functions. In his instructional practice, he introduced the graphs of quadratic functions step by step, first introducing the quadratic function $y = ax^2$, and obtaining other quadratic functions by making translations over this function. Thus, his instruction seems more planned and connected since he cares about conceptual knowledge rather than procedural.

The influence of teacher knowledge on the quality of the instructional practice is consistent with those reported in the literature (e.g., Hatisaru, 2013; Sánchez & Llinares, 2003). Hatisaru's (2013) study found that teachers' KCS influenced the quality of their instruction regarding the function concept. Similarly, Sánchez and Llinares (2003) reported that pre-service teachers' ways of knowing the subject matter affected their pedagogical reasoning, i.e., what they considered important for students and which representations they use in the classroom.

The findings also indicated a relationship between instructional practice and student learning outcomes. When the teachers finished the lessons on quadratic functions, the majority of Can's students did not find basic elements of the quadratic function like the x-intercepts, the vertex, and the axis of symmetry and did not sketch the graph. In addition, none of them used completing the square method for solving a quadratic equation. However, the majority of Ahmet's students correctly found the x-intercepts, the vertex, and the axis of symmetry of a given function and sketched its correct graph. Additionally, some used completing the square to find the x-intercepts of the function. Unlike Can's students, none of his students used the quadratic formula to find the roots of a quadratic equation. Moreover, when the students were asked to find the minimum or the maximum of a quadratic function, most of Can's students did not find it. In contrast, all of Ahmet's students could correctly find the maximum or the minimum of a quadratic function. Ahmet's students were able to associate the vertex with the minimum or the maximum of a quadratic function.

None of Can's students could comment on the effect of the change in the leading coefficient on the parabola. They did not explain the basic interrelation between the leading coefficient and the parabola. However, most of Ahmet's students provided an explanation for the interrelation between the leading coefficient and the parabola. Unlike Can's students, they were aware that when the absolute value of the leading coefficient gets larger, the parabola becomes narrower.

Can's students' performances in investigating the intersection of a line and a parabola were relatively better than when compared to the other questions in the test. Many students correctly commented on the intersection of a line and a parabola examining the discriminant of the new quadratic equation that they obtained by equating the *y*-values of their standard forms. Likewise, the majority of Ahmet's students examined the intersection of a parabola and a line. Some of them did not calculate the discriminant of the new quadratic equation, rather, they stated that the parabola is tangent to the line since the new quadratic equation is a perfect square. This shows that some of Ahmet's students understand the relationship between the discriminant of a quadratic equation and its algebraic form (being a perfect square). During his instructional practice, Ahmet mostly used completing the square method and emphasized the vertex form of quadratic functions. Furthermore, the majority of Can's students could not solve real-life problems about quadratic functions. Ahmet's students' performances were clearly better in solving real-life problems regarding quadratic functions than Can's students. These findings provide strong evidences for the contribution of instructional practice to the student learning outcomes.

When the students were asked to find the quadratic function given its vertex and one point, only a few of Can's students found it correctly, using the vertex form. In Ahmet's class, the majority of students could find the quadratic function by using the vertex form. The students were also asked to find the quadratic function given the *y*-intercept and two points. In this case, none of Can's students found the quadratic function; approximately half of Ahmet's students found the quadratic function using the standard form. Although the majority of Ahmet's students found the quadratic function when the vertex was given, some of them did not find the quadratic function when the *y*-intercept and two points were given. In both cases, the students' performance was better in finding the quadratic function when the vertex is given.

Educators seem to have a consensus that instructional practice affects students' performance (Gençtürk, 2012; Hatisaru, 2013; Hatisaru, & Erbaş, 2017; Ibeawuchi, 2010; Shechtman et al., 2010). The NCTM (2000) reported that "students learn mathematics through the experiences teachers provide. Thus, students' understanding of mathematics, their ability to use it to solve problems, and their confidence in, and disposition toward, mathematics are all shaped by the teaching they encounter in school" (p. 16-17). The data also revealed that the performance of students of the teacher with strong MKT was better than those of teachers who had relatively low MKT for teaching quadratic functions. This is consistent with the literature (Callingham et al., 2016, Tchoshanov et al., 2017). Callingham et al.'s (2016) study reported that students of teachers who had a strong PCK performed better than the students of teachers who had weak PCK in the questionnaire. Tchoshanov et al. (2017) also provided evidence for the relationship between teachers' mathematics content knowledge and student performance at the lower secondary level.

The result also indicated the existence of some mediating factors for the relationship among teacher knowledge, instructional practice, and student learning. For example, during his instructional practice, although Can solved several questions about finding the quadratic function with some points (the vertex, the y-intercept, or any point) given, his students had some difficulty in finding the quadratic function especially when the y-intercept and two points given. This finding might be explained by some mediating factors such as teacher beliefs, inherent complexities of the function concept, the students' academic background, and the students' difficulties in arithmetic (Hatisaru & Erbaş, 2017; Hill, Blunk et al., 2008; Shechtman et al., 2010).

5.3. Implications

This study highlights secondary mathematics teachers' SMK of quadratic functions and its contribution to instructional practice and student learning outcomes regarding quadratic functions. The results have both methodological and practical implications for mathematics educators, secondary mathematics teachers, and policymakers.

First of all, research confirms that teachers' content knowledge is critical for student learning (Tchoshanov et al., 2017). Although its critical role in student learning, the findings of this study revealed that some aspects of the teachers' SMK of quadratic functions are inadequate, which might adversely affect student learning. Teachers' performance on CCK items were better than their performance on SCK and HCK items. That is, secondary mathematics teachers in this study have adequate knowledge of basic facts or procedures of the quadratic function concept, which is expected for them to teach the content. However, their SMK is limited to basic facts or procedures. Most teachers did not explain the relationship between quadratic functions and other concepts in the secondary mathematics curriculum or in the advanced mathematics. Teacher education programs or professional development programs might be organized to enhance pre-service teachers' and in-service teachers' SCK and HCK regarding a specific content.

The mathematics educators in faculties of education may design some method courses regarding specific learning areas such as geometry, functions, numbers, or algebra. This might give the pre-service teachers the opportunity to develop their SMK of a specific content, understanding all the definitions, representations, teaching methods, the use of mathematical terminology, and the use of mathematical models. Likewise, professional development programs might also be designed to enhance practicing teachers' SMK of a specific content. The findings suggested evidence of the link between teachers' SMK and student learning outcomes regarding quadratic functions, mediated by instructional practice.

Secondly, the questions that were asked to teachers in this study were beyond basic facts or procedures about quadratic functions and they addressed many other aspects of knowledge that a teacher might possess to yield an effective teaching. Some questions were about the daily use of quadratic functions whereas some were about explaining basic formulas of quadratic functions. There were also some questions that addressed students' solutions. Engaging in these questions might give the teachers opportunity for self-assessment and allow them to evaluate the extent of their own SMK regarding a specific content. Thus, teachers might have an opportunity to reflect their own strengths and weaknesses about this particular content and make plan to improve themselves for effective teaching.

Another implication of this study is related to the specification of teacher knowledge for teaching quadratic functions. This study contributed to clarifying what a secondary mathematics teacher should know for teaching quadratic functions. For this purpose, the key components of teachers' CCK, SCK, and HCK were identified. This specification of teacher knowledge regarding a specific-content might contribute to the field of mathematics education. Mathematics educators should extend this literature examining teacher knowledge and specifying the key aspects of teacher knowledge for teaching any other content from other mathematical domains such as algebra, numbers, etc.

5.4. Recommendations for Future Studies

This study helps to understand the complex relationship between teachers' SMK, their instructional practices, and student learning outcomes regarding a specific content, the quadratic function. There is much more to be learned about this relationship. In this study, no student achievement data is used. Whether and what changes occur in students' learning outcomes with different academic backgrounds might be an important question to investigate. Thus, this study should be replicated with different teachers and students including the student achievement data. Data obtained from these studies will contribute to understanding how students' academic backgrounds mediate the influence of teachers' SMK on student learning outcomes of a specific content.

This study found that secondary teachers' SMK – especially SCK and HCK - regarding the quadratic function concept is limited. Although the participants of the current study included teachers in various high school types (i.e., science high schools, Anatolian high schools, etc.), this study did not examine teachers' SMK based on the school types. Further studies should compare teachers' SMK of quadratic functions based on the school types (i.e., one teacher from science high school and one teacher from a vocational high schools).

The present study was limited to quadratic functions. More aspects of the polynomial functions of higher degrees should be added to the results of this study. The results of these studies might contribute to getting a clear picture of teachers' SMK for polynomial functions.

The present study used Ball et al.'s (2008) model to investigate teachers' SMK and its contribution to student learning. Further studies should investigate the PCK of the model, which includes KCS, KCT, and the knowledge of curriculum. These studies should investigate which dimensions of teacher knowledge have the most influence on student learning outcomes of specific mathematical content. For example, they should examine whether SMK or PCK is more influential on student performance.

In the present study, two different cases were investigated. One of the teachers got the highest scores on CCK, SCK, and HCK dimensions of the questionnaire that indicates he has a balanced distribution of three dimensions of SMK. Another teacher got an intermediate score from the CCK items, relatively low score from the SCK items and the lowest score from the HCK items, that indicates he has an unbalanced distribution of three dimensions of SMK. Further studies should investigate different cases with different teachers. For example, T5 in this study might be a different case with low CCK score, relatively higher SCK score, and the lowest HCK score.

5.5. Limitations of the Study

This study has some limitations in terms of the participants, the instruments, and the procedure for data collection. First of all, this study distinguished and measured secondary mathematics teachers' CCK, SCK, and HCK of quadratic functions. Distinguishing between sub-domains of teacher knowledge has been a concern for many researchers (Hill et al., 2008; Howell, 2012). As Ball et al. (2008) stated "it is not always easy to discern where one of our categories divides from the next" (Ball et al., 2008, p. 403). Thus, trying to distinguish the sub-domains was a challenge throughout the study. To overcome this problem, the researcher created a table of specifications that identified the key components of CCK, SCK, HCK (see Table 3.1). Moreover, the researcher frequently consulted the experts in mathematics education to discuss ambiguous cases. However, there might be still some small ambiguities with matching the items in the questionnaire with a sub-domain of teacher knowledge. An item that aimed to evaluate teachers' SCK might also include any piece of knowledge from CCK or HCK. These three dimensions cannot be not strictly separated from each other. This is the biggest limitation of the present study.

Secondly, a total of 18 volunteer teachers accepted to participate in the first phase of this study. Also, the study was conducted in a particular setting with two teachers and their students, and included a detailed description of two cases. Even though the present study does not aim to generalize the results to other secondary mathematics teachers and their students, the number of participants might be a limitation since a limited number of teachers may not represent a variety of perspectives of teacher knowledge regarding quadratic functions. Thus, the results of the present study may not necessarily be generalizable to other teachers and students in different school settings.

In this study, no student achievement data is used. The contribution of teacher knowledge to student performance cannot be exactly known without student achievement data. However, it is evident from the previous research that teacher knowledge strongly and positively affects student performance. In this study, SMK dimension of teacher knowledge that has been linked to gains in student performance is investigated. Thus, the present study helps to identify those links to some extent.

Another limitation of the study is the absence of video recording during teachers' instructional practice. The notebook of one student and the observation notes of the researcher were used to analyze the instructional practice of teachers. In this study, some elements of teachers' instructional practice were analyzed: the examples that teachers use, the use of multiple representations, making connections among mathematical concepts, making justifications of formulas, and using real-life problems while teaching quadratic functions. Some elements of teachers' instructional practice were not included in the analysis, i.e., responding to students' why questions, and analyzing the students' solutions while teaching quadratic functions in the classroom.

Lastly, during classroom observations, the researcher's presence may have influenced the teachers' instructional practices. To minimize this, the researcher participated in two classes before the instruction on quadratic functions started and established a close relationship with the teacher and the students to make them feel more comfortable. Thus, the teacher and the students were more likely to accept the presence of the researcher as a part of their classroom environment.

REFERENCES

Akkoç, H. (2006). Fonksiyon kavramının çoklu temsillerinin çağrıştırdığı kavram görüntüleri. *Hacettepe Üniversitesi Eğitim Fakültesi Dergisi, 30,* 1-10.

Anney, V. N. (2014). Ensuring the quality of the findings of qualitative research: Looking at trustworthiness criteria. *Journal of Emerging Trends in Educational Research and Policy Studies, 5*(2), 272-281.

Artzt, A. F., & Armour-Thomas, E. (1996). *Evaluation of instructional practice in the secondary school mathematics classroom: A cognitive perspective* (ED397131). https://files.eric.ed.gov/fulltext/ED397131.pdf

Askew, M. (2008). Mathematical discipline knowledge requirements for prospective primary teachers, and the structure and teaching approaches of programs designed to develop that knowledge. *Knowledge and Beliefs in Mathematics Teaching and Teaching Development*, *1*, 11-35.

Aziz, T. A., Pramudiani, P., & Purnomo, Y. W. (2018). Differences between quadratic equations and functions: Indonesian pre-service secondary mathematics teachers' views. *Journal of Physics: Conference Series, 948*(1), 012043. https://doi.org/10.1088/1742-6596/948/1/012043

Bair, S. L. & Rich, B. S. (2011). Characterizing the development of specialized mathematical content knowledge for teaching in algebraic reasoning and number theory. *Mathematical Thinking and Learning*, *13(4),* 292-321.

Baki, A., & Güveli, E. (2008). Evaluation of a web based mathematics teaching material on the subject of functions. *Computers & Education*, *51*(2), 854-863.

Ball, D. L, & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on the teaching and learning of mathematics (*pp. 83-104). Ablex.

Ball, D. L. & Bass, H. (2009). With an eye on the mathematical horizon: Knowing mathematics for teaching to learners' mathematical futures. Paper based on keynote address at the *43rd Jahrestagung für Didaktik der Mathematik* held in Oldenburg, Germany.

Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *The Elementary School Journal*, *90*(4), 449-466.

Ball, D. L. (1991). Research on teaching mathematics: Making subject matter part of the equation. In J. Brophy (Ed.), *Advances in research in teaching* (pp. 1-48). CAI.

Ball, D. L., Bass, H. & Hill, C. H. (2004). Knowing and using mathematical knowledge in teaching: Learning what matters. In *Proceedings of 12th Annual Conference of the Southern African Association for Research in Mathematics, Science and Technology Education*. Cape Town, South Africa.

Ball, D. L., Thames, M.H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? J*ournal of Teacher Education*, *59*(5), 389–407.

Bansilal, S., Mkhwanazi, T., & Brijlall, D. (2014). An exploration of the common content knowledge of high school mathematics teachers. *Perspectives in Education, 32*(1), 34-50.

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., Klusmann, U., Krauss, S., Neubrand, M., & Tsai, Y. M. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal*, *47*(1), 133-180.

Begle, E. G. (1979). *Critical variables in mathematics education: Findings from a survey of empirical literature.* Washington, DC: Mathematical Association of America and National Council of Teachers of Mathematics.

Bremigan, E. G., Bremigan, R. J., & Lorch, J. D. (2011). *Mathematics for secondary school teachers* (Vol. 21). MAA.

Burns-Childers, A. & Vidakovic, D. (2018). Calculus students’ understanding of the vertex of the quadratic function in relation to the concept of derivative, *International Journal of Mathematical Education in Science and Technology, 49*(5), 660-679, DOI: 10.1080/0020739X.2017.1409367

Callingham, R., Carmichael, C., & Watson, J. M. (2016). Explaining student achievement: The influence of teachers’ pedagogical content knowledge in statistics. *International Journal of Science and Mathematics Education*, *14*(7), 1339-1357.

Campbell, P. F., Nishio, M., Smith, T. M., Clark, L. M., Conant, D. L., Rust, A. H., ... & Choi, Y. (2014). The relationship between teachers' mathematical content and pedagogical knowledge, teachers' perceptions, and student achievement. *Journal for Research in Mathematics Education*, *45*(4), 419-459.

Carreño, E., Rojas, N., Montes, M.A., & Flores, P. (2013). Mathematics teacher’s specialized knowledge. Reflections base on specific descriptors of knowledge. *Eighth Congress of European Research in Mathematics Education (CERME 8),* Manavgat-Side, Antalya, Turkey, 6-10 February 2013. http://cerme8.metu.edu.tr/wgpapers/WG17/WG17_Rojas.pdf

Carter, K. (1990). Teachers’ knowledge and learning to teach. In W. R. Houston (Ed.), *Handbook of Research on Teacher Education* (pp. 291-310). New York: MacMillan.

Cho, Y. and Tee, F. (2018). Complementing mathematics teachers’ horizon content knowledge with an elementary-on-advanced aspect. *Pedagogical Research*, *3*(1), 03. https://doi.org/10.20897/pr/85172

Cochran, K. F., DeRuiter, J. A., & King, R. A. (1993). Pedagogical content knowing: An integrative model for teacher preparation. *Journal of Teacher Education, 44*, 263-272.

Cohen, L., Manion, L., & Morrison, K. (2007). *Research methods in education (6th ed.).* London & New York: Routledge-Falmer.

Common Core State Standards Initiative (2010). Mission statement. Retrieved from http://www.corestandards.org/

Cooney, T. J., & Wilson, M. R. (1993). Teachers' thinking about fractions; Historical and research perspectives. In T. Romberg, E. Fennema, & T. Carpenter (Eds.), *Integrating research on the graphical representation of functions*, (pp. 131-151). Hillsdale, NJ: Lawrence Erlbaum Associates.

Cooney, T., Beckmann, S., & Lloyd, G. (2010). *Developing essential understanding of functions for teaching mathematics in grades 9-12*. Reston, VA: National Council of Teachers of Mathematics.

Creswell, J. W. (1998). *Qualitative inquiry and research design: Choosing among five traditions*. Thousand Oaks, CA: Sage Publications.

Creswell, J. W. (2002). *Educational research: Planning, conducting, and evaluating quantitative*. Upper Saddle River, NJ: Prentice Hall.

Creswell, J. W. (2007). *Qualitative inquiry and research design (2nd ed.)*. Thousand Oaks, CA: Sage Publications.

Creswell, J. W. (2008). *Research design: Qualitative, quantitative, and mixed methods approaches*. Thousand Oaks, CA: Sage Publications.

Creswell, John W. (2012). *Educational Research: Planning, conducting, and evaluating quantitative and qualitative research (4th ed.)*. Nebraska: Pearson Education.

Delaney, S., Ball, D. L., Hill, H. C., Schilling, S. G., & Zopf, D. (2008). Mathematical knowledge for teaching: Adapting U.S. measures for use in Ireland. *Journal of Mathematics Teacher Education, 11*(3), 171–197.

Denzin, N. K. (1989). *Interpretive Interactionism.* Newbury Park, CA: SAGE Publications.

Denzin, N.K., & Lincoln, Y.S. (2005). *The SAGE Handbook of Qualitative Research* (3rd. ed.). Thousand Oaks, CA: Sage Publications.

Didiş, M. G., & Erbaş, A. K. (2015). Performance and difficulties of students in formulating and solving quadratic equations with one unknown. *Educational Sciences: Theory & Practice*, *15*(4).

Didiş, M. G., Baş, S., & Erbaş, A. K. (2011). Students' reasoning in quadratic equations with one unknown. In *The Seventh Congress of the European Society for Research in Mathematics Education (CERME-7)* (pp. 479-489).

Dreyfus, T. (1991). Aspects of computerized learning environments which support problem solving. In J. P. Ponte, J. E Matos, J. M. Matos & D. Femandes (Eds.), *Mathematical problem solving and new information technologies* (pp. 255-266). Berlin, Germany: Springer.

Duarte, J. T. (2010). *The effects of an undergraduate algebra course on prospective middle school teachers' understanding of functions, especially quadratic functions*. (Unpublished doctoral dissertation). Illinois State University, USA.

Elbaz, F. (1983). *Teacher thinking. A study of practical knowledge: Croom helm curriculum policy and research series*. New York, NY: Nichols Publishing Company.

Ellis, A. B., & Grinstead, P. (2008). Hidden lessons: How a focus on slope-like properties of quadratic functions encouraged unexpected generalizations. *The Journal of Mathematical Behavior, 27*, 277-296.

Eraslan, A. (2005). *A qualitative study: Algebra honor students' cognitive obstacles as they explore concepts of quadratic functions* (Unpublished doctoral dissertation). Florida State University, Florida, USA.

Erbaş, A. K., Çetinkaya, B., Alacacı, C., Çakıroğlu, E., Aydoğan-Yenmez, A., Şen-Zeytun, A., Korkmaz, H., Kertil, M., Didiş, M. G., Baş, S. & Şahin, Z. (2016). *Lise matematik konuları için günlük hayattan modelleme soruları [Daily life modeling questions for high school math subjects]*. Turkish Academy of Sciences.

Erlandson, D. A. Harris, E. L., Skipper, B. L., & Allen, S. D. (1993). *Doing naturalistic inquiry*. Newbury Park, CA: Sage Publications.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions.*Educational Studies in Mathematics, 21,* 521-554.

Even, R. (1993). Subject matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. *Journal for Research in Mathematics education*, 94-116.

Fennema, E. and Franke, L. M (1992). Teachers' knowledge and its impact, in D.A. Grouws (Ed) *Handbook of research on Mathematics Teaching and Learning* (New York, Macnillan), 147-164.

Fernández, S., & Figueiras, L. (2014). Horizon Content Knowledge: Shaping MKT for a Continuous Mathematical Education. *Redimat, 3*(1), 7-29.

Ferrini-Mundy, J., Floden, R., McCrory, R., Burrill, G., Sandow, D. (2005). *Knowledge for teaching school algebra: Challenges in developing an analytic framework*. East Lansing, MI: Michigan State University, Knowledge for Teaching Algebra Project.

Flores, E., Escudero, D. I., & Carrillo, J. (2013). A theoretical review of specialised content knowledge. In *Proceedings of the Eighth Congress of European Research in Mathematics Education (CERME 8)*, 6–10 February 2013, Antalya, Turkey.

Fraenkel, J. R., Wallen, N. E., & Hyun, H. H. (2012). *How to design and evaluate research in education.* New York: McGraw-hill.

Frome, P., Lasater, B., & Cooney, S. (2005). Well-Qualified Teachers and High Quality Teaching: Are They the Same? Research Brief. *Southern Regional Education Board*.

Gay, L. R., Mills, G. E., & Airasian, P. (2006). *Educational research: Competencies for analysis and applications*. New Jersey, Pearson Merrill Prentice Hall.

Gençtürk, Y. C. (2012). *Teachers' mathematical knowledge for teaching, instructional practices, and student outcomes* (Doctoral dissertation). University of Illinois at Urbana-Champaign.

Girit, D. (2016). *Investigating middle school mathematics teachers' mathematical knowledge for teaching algebra: A multiple case study* (Doctoral dissertation). Middle East Technical University, Ankara.

Graeber, A., & Tirosh, D. (2008). Pedagogical content knowledge: Useful concept or elusive notion. In P. Sullivan & T. Wood (Eds.), *Knowledge and beliefs in mathematics teaching and teaching development. The international handbook of mathematics teacher education* (Vol. 1, pp. 117-132).

Grossman, P. (1990). *The making of a teacher: Teacher knowledge and teacher education*. New York: Teachers College Press.

Guba, E. G. (1981). Criteria for assessing the trustworthiness of naturalistic inquiries. *Educational Communication and Technology Journal, 29*(2), 75-91.

Guberman, R., & Gorev, D. (2015). Knowledge concerning the mathematical horizon: A close view. *Mathematics Education Research Journal*, *27*(2), 165-182.

Guskey, T. R. (2002). Professional Development and Teacher Change, *Teachers and Teaching, (8)*3, 381-391, DOI: 10.1080/135406002100000512.

Harel, G. (2008). A DNR perspective on mathematics curriculum and instruction. Part II: with reference to teacher's knowledge base. *ZDM*, *40*(5), 893-907.

Hatısaru, V. (2013). *Teachers' knowledge of content and students about the function concept and its interrelation with student learning outcomes in vocational high schools.* (Doctoral dissertation). Middle East Technical University, Ankara.

Hatisaru, V., & Erbaş, A. K. (2017). Mathematical knowledge for teaching the function concept and student learning outcomes. *International Journal of Science and Mathematics Education*, *15*(4), 703-722.

Herbst, P., & Kosko, K. (2014). Mathematical knowledge for teaching and its specificity to high school geometry instruction. In J-J Lo, K. R. Leatham, & L. R. Van Zoest (Eds.), *Research trends in mathematics teacher education* (pp. 23–45). Springer.

Hill, H. C., & Ball, D. L. (2004). Learning mathematics for teaching: Results from California's mathematics professional development institutes. *Journal for Research in Mathematics Education, 35*(5), 330-351.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' content specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372-400.

Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction, 26*(4), 430-511.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.

Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *The Elementary School Journal, 105*(1), 11-30.

Hogan, J., Dolan, P., & Donnelly, P. (2009). Introduction. In J. Hogan, P. Dolan, & P. Donnelly (Eds.), *Approaches to qualitative research: Theory and its practical application - A guide for dissertation students*, (pp. 1-18). Ireland: Oak Tree Press.

Hoon, T., Singh, P., & Halim, U. K. (2018). Understanding of function and quadratic function among secondary school students in Selangor. *Asian Journal of University Education (AJUE)*, *14*(1), 77-88.

Howell, H. (2012). *Characterizing mathematical knowledge for secondary teaching: A case from high school algebra* (Doctoral dissertation), New York University.

Howell, H., Lai, Y., Phelps, G., & Croft, A. (2016). Assessing mathematical knowledge for teaching beyond conventional mathematical knowledge: Do elementary models extend? https://doi.org/10.13140/RG.2.2.14058.31680

Ibeawuchi, E. & Ngoepe, M. G. (2012). Investigating grade 11 learners' misconceptions in understanding quadratic functions in some South African schools. In Mogari, D. & Ogbonnaya, U.I. (Eds). *ISTE International Conference on Mathematics, Science and Technology Education "Proceeding Towards Effective teaching and Meaningful Learning in Mathematics, Science and Technology"* 22-25 October 2012, Mopani camp, Kruger National Park, Limpopo, South Africa.

Ibeawuchi, E. O. (2010). *The role of pedagogical content knowledge in the learning of quadratic functions* (Doctoral dissertation), University of South Africa.

Jakobsen, A. (2014). Developing an understanding of horizon content knowledge: Experiences from a practice-based approach in Norway. 1-6.

Jakobsen, A., Thames, M. & Ribeiro, C., M. (2013). Delineating issues related to Horizon Content Knowledge for mathematics teaching. In B. Ubuz, Ç. Haser & M. A. Mariotti (Eds.*), Proceedings of CERME 8* (pp. 3125-3134). Antalya, Türkiye: ERME. ISBN: 978-975-429-315-9.

Jakobsen, A., Thames, M. H., Ribeiro, C. M., & Delaney, S. (2012, July). Using practice to define and distinguish horizon content knowledge. In *12th International Congress in Mathematics Education (12th ICME)*, 4635-4644.

Kotsopoulos, D. (2007). Unraveling student challenges with quadratics: A cognitive approach. *Australian Mathematics Teacher, 63*(2), 19-24.

Krauss, S., Brunner, M., Kunter, M., Baumert, J., Blum, W., Neubrand, M., & Jordan, A. (2008). Pedagogical content knowledge and content knowledge of secondary mathematics teachers. *Journal of Educational Psychology*, *100* (3), 716-725. doi: 10.1037/0022-0663.100.3.716

Krefting, L. (1990). Rigor in qualitative research: The assessment of trustworthiness. *The American Journal of Occupational Therapy, 45*(3), 214-222.

Larson, R., & Boswell, L. (2019). *Algebra 1: A common core curriculum.* Big Ideas Learning.

Li, X. (2011). Mathematical knowledge for teaching algebraic routines: a case study of solving quadratic equations. *Journal of Mathematics Education*, *4*(2), pp.1-16.

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry*. Thousand Oaks, CA: Sage.

Lundy, K. (2008). Prolonged engagement. In L. Given (Ed.), *The SAGE encyclopedia of qualitative research methods* (pp. 691-693). Thousand Oaks, CA: Sage.

Ma, Liping, 1999. *Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Fundamental Mathematics in China and the United States*. New York: Routledge.

Ma'rufi, I. (2016). Teacher's Pedagogical Content Knowledge Concerned to Students Knowledge on Quadratic Function. *Proceeding Of 3rd International Conference On Research, Implementation and Education Of Mathematics And Science*, Yogyakarta, 16 – 17 May, 2016.

Magnusson, S., Krajcik, L., & Borko, H. (1999). Nature, sources and development of pedagogical content knowledge. In J. Gess-Newsome, & N. G. Lederman (Ed.), *Examining pedagogical content knowledge* (pp. 95-132). Dordrecht, The Netherlands: Kluwer.

Makonye, J. & Nhlanhla, S. (2014). Exploring 'non-science' grade 11 Learners Errors in Solving Quadratic Equations. *Mediterranean Journal of Social Science, 5* (27).

Makonye, J. P., & Matuku, O. (2016). Exploring learner errors in solving quadratic equations. *International Journal of Educational Sciences*, *12*(1), 7-15.

Maxwell, J. A. (2005). *Qualitative research design: An interactive approach* (2nd ed.). Thousand Oaks, CA: Sage Publications.

Mazhindu, L. (2016). *Misconceptions learners encounter in solving quadratic equations: a case of 3 secondary schools in Zimbabwe (*Doctoral dissertation). BUSE.

Mbewe, T. L., & Nkhata, B. (2019). Secondary teachers' mathematics knowledge for teaching quadratic equations: A case of selected secondary schools in Katete district. *Zambia Journal of Teacher Professional Growth, 5*(1), 38 – 55.

Memnun, D. S., Aydin, B., Dinç, E., Çoban, M., & Sevindik, F. (2015). Failures and Inabilities of High School Students about Quadratic Equations and Functions. *Journal of Education and Training Studies*, *3*(6), 50-60.

Merriam, S. B. (1998). *Qualitative research and case study applications in education* (Rev. ed.). San Francisco: Jossey-Bass.

Merriam, S. B. (2002). *Qualitative research in practice examples for discussion and analysis*. San Francisco, Jossey-Bass.

Merriam, S. B. (2009). *Qualitative research: A guide to design and implementation*. San Francisco, CA: Jossey-Bass.

Metcalf, R. C. (2007). *The nature of students' understanding of quadratic functions*. State University of New York at Buffalo.

Mewborn, D. (2001). Teachers content knowledge, teacher education, and their effects on the preparation of elementary teachers in United States. *Mathematics Education Research Journal, 3*, 28-36.

Mewborn, D.S. (2003). Teaching, teachers' knowledge, and their Professional development. In J. Kilpatrick, W.G. Martin, & D. Schifter (Ed.), *A research companion to the principles and standards for school mathematics* (pp. 45-52). Reston, VA: NCTM.

Miheso-O'Connor Khakasa, M., & Berger, M. (2016). Status of teachers' proficiency in mathematical knowledge for teaching at secondary school level in Kenya. *International Journal of Science and Mathematics Education, 14*(2), 419-435.

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis (2nd ed.)*. Thousand Oaks, CA: Sage.

Milli Eğitim Bakanlığı (MEB). 2018. *Ortaöğretim Matematik Dersi (9, 10, 11 ve 12. Sınıflar) Öğretim Programı.*

Monk, D. H. & King, J. A. (1994). Multi-level teacher resource effects in pupil performance in secondary mathematics and science: The case of teacher subject matter preparation. In R. G. Ehrenberg (Ed.), *Choices and consequences: Contemporary policy issues in education*, (pp. 29-58). Ithaca, NY: ILR Press.

Monk, D.H. (1994). Subject area preparation of secondary mathematics and science teachers and student achievement. *Economics of Education Review, 13*(2), pp. 125-145.

Mosvold, R., & Fauskanger, J. (2014). Teachers' beliefs about mathematical horizon content knowledge. *International Journal for Mathematics Teaching and Learning*, *9*(3), 311-327.

Movshovitzs-Hadar, N. (1993). A constructive transition from linear to quadratic functions. *School Science and Mathematics, 93*(6), 288-298.

Mutambara, L. H. N., Tendere, J., & Chagwiza, C. J. (2019). Exploring the conceptual understanding of the quadratic function concept in teachers' colleges in Zimbabwe. *EURASIA Journal of Mathematics, Science and Technology Education, 16*(2), 1-17. https://doi.org/10.29333/ejmste/112617

National Council of Teachers of Mathematics [NCTM]. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

National Mathematics Advisory Panel (2008). *Foundations for success. The final report of the National Mathematics Advisory Panel.* Washington, DC: U.S. Department of Education.

Nielsen, L. E. J. (2015). *Understanding quadratic functions and solving quadratic equations: An analysis of student thinking and reasoning* (Doctoral dissertation). University of Washington.

Nyikahadzoyi, M. R. (2015). Teachers' knowledge of the concept of a function: a theoretical framework. *International Journal of Science and Mathematics Education*, *13*(2), 261-283.

Parent, J. S. S. (2015). *Students' understanding of quadratic functions: Learning from students' voices.* (Unpublished doctoral dissertation). The University of Vermont, USA.

Patton, M. (2002). *Qualitative research and evaluation methods* (3rd ed.) Thousand Oaks, CA: Sage.

Pihlap, S. (2017). The impact of computer use on learning of quadratic functions. International. *Journal for Technology in Mathematics Education, 24*(2), 59-66.

Rowland, T., Huckstep, P. and Thwaites, A. (2005) Elementary teachers' mathematics subject knowledge: the knowledge quartet and the case of Naomi, *Journal of Mathematics Teacher Education, 8*(3), 255-281.

Sağlam, R., & Alacacı, C. (2012). A comparative analysis of quadratics unit in Singaporean, Turkish and IBDP mathematics textbooks. *Turkish Journal of Computer and Mathematics Education (TURCOMAT)*, *3*(3).

Sánchez, V., & Llinares, S. (2003). Four student teachers' pedagogical reasoning on functions. *Journal of Mathematics Teacher Education, 6,* 5-25.

Sanders, J. R. (1981). Case Study Methodology: A Critique. In W.W. Welsh (ed.), Case Study Methodology in Educational Evaluation. *Proceedings of the Minnesota Evaluation Conference*. Minneapolis: Minnesota Research and Evaluation Center.

Shechtman, N., Roschelle, J., Haertel, G. & Knudsen, J. (2010). Investigating links from teacher knowledge, to classroom practice, to student learning in the instructional system of the middle-school mathematics classroom. *Cognition and Instruction, 28*(3), 317-359, DOI: 10.1080/07370008.2010.487961

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, *57*(1), 1-23.

Sibuyi, C. D. (2012). *Effective teachers' pedagogical content knowledge in teaching quadratic functions in mathematics* (Doctoral dissertation). University of Pretoria.

Silverman, J., & Thompson, P. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education, 11*(6), 499-511.

Smith, P. S., & Esch, R. K. (2012, April). Identifying and measuring factors related to student learning: The promise and pitfalls of teacher instructional logs. *Paper presented at the Annual Meeting of the American Educational Research Association*. Vancouver, British Columbia, Canada.

Sosa, L. (2010). *Conocimiento matemático para la enseñanza en bachillerato: un estudio de dos casos* (Doctoral dissertation), Universidad de Huelva.

Speer, N. M., & Wagner, J. F. (2009). Knowledge needed by a teacher to provide analytic scaffolding during undergraduate mathematics classroom discussions. *Journal for Research in Mathematics Education, 40*(5), 530–562.

Speer, N. M., King, K. D., & Howell, H. (2015). Definitions of mathematical knowledge for teaching: Using these constructs in research on secondary and college mathematics teachers. *Journal of Mathematics Teacher Education*, *18*(2), 105–122.

Steele, M. (2013). Exploring the mathematical knowledge for teaching geometry and measurement through the design and use of rich assessment tasks. *Journal of Mathematics Teacher Education, 16*(4), 245–268.

Steele, M., & Rogers, K. (2012). Relationships between mathematical knowledge for teaching and teaching practice: The case of proof. *Journal of Mathematics Teacher Education, 15*(2), 159–180.

Steele, M., Hillen, A., & Smith, M. (2013). Developing mathematical knowledge for teaching in a methods course: The case of function. *Journal of Mathematics Teacher Education, 16*(6), 451–482. https://doi.org/10.1007/s10857-013-9243-6

Sumartini, T. S. (2021). Subject matter knowledge of prospective mathematics teachers on quadratic functions using problem-based learning. *Jurnal Pendidikan dan Pengajaran*, *54*(1), 141-149.

Tallman, M. A., & Frank, K. M. (2018). Angle measure, quantitative reasoning, and instructional coherence: an examination of the role of mathematical ways of thinking as a component of teachers' knowledge base. *Journal of Mathematics Teacher Education*, *23*(1), 69-95.

Taşdan, B. T., & Koyunkaya, M. Y. (2017). Examination of pre-service mathematics teachers' knowledge of teaching function concept. *Acta Didactica Napocensia, 10*(3), 1–17.

Tchoshanov, M., Cruz, M. D., Huereca, K., Shakirova, K., Shakirova, L., & Ibragimova, E. N. (2017). Examination of lower secondary mathematics teachers' content knowledge and its connection to students' performance. *International Journal of Science and Mathematics Education, 15*(4), 683–702. https://doi.org/10.1007/s10763- 015-9703-9.

Tchoshanov, M., Quinones, M. C., Shakirova, K. B., Ibragimova, E. N., & Shakirova, L. R. (2017). Analyzing connections between teacher and student topic-specific knowledge of lower secondary mathematics. *The Journal of Mathematical Behavior, 47*, 54-69.

Thames, M. H. (2009). *Coordinating mathematical and pedagogical perspectives in practice-based and discipline-grounded approaches to studying mathematical knowledge for teaching (K-8)* (Doctoral dissertation), University of Michigan.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. *Compendium for Research in Mathematics Education*, 421–456.

Tonui, G.K., Ayiro, L., & Ongeti, K. (2021). Learner difficulties in solving word and graphical problems in quadratic equations. *International Journal of Recent Research in Physics and Chemical Sciences (IJRRPCS), 8*(2), 13-25.

Ubah, I. J. A., & Bansilal, S. (2018). Pre-service mathematics teachers' knowledge of mathematics for teaching: quadratic functions. *Problems of Education in the 21st Century, 76*(6), 847.

Vaiyavutjamai, P., & Clements, M. A. (2006). Effects of classroom instruction on students' understanding of quadratic equations. *Mathematics Education Research Journal, 18*(1), 47-77.

Vaiyavutjamai, P., Ellerton, N. F., & Clements, M. A. (2005). Students' attempts to solve two elementary quadratic equations: A study in three nations. In *Building connections: Theory, research and practice: Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia*.

Verloop, N., Driel, J. V., & Meijer, P. (2001). Teacher knowledge and the knowledge base of teaching. *International Journal of Educational Research, 35*, 441-461.

Wasserman, N., & Stockton, J. C. (2013). Horizon content knowledge in the work of teaching: A focus on planning. *For the Learning of Mathematics, 33*(3), 20-22.

Wilson, S. M., Shulman, L. S., & Richert, A. E. (1987). "150 different ways" of knowing: Representations of knowledge in teaching. In J. Calderhead (Ed.), *Exploring teachers' thinking* (pp. 104–124). London: Cassell.

Wu, H. (2016). *Teaching school mathematics: Algebra. Providence.* RI: American Mathematical Society

Yin, R. K. (2003). *Case study research: Design and methods* (3rd ed.). Thousand Oaks, CA: Sage.

Yin, R. K., (2009). *Case study research: Design and methods* (4th ed.). Thousand Oaks, CA: Sage.

Zaslavsky, O. (1997). Conceptual Obstacles in the Learning of Quadratic Functions. *Focus on Learning Problems in Mathematics*, *19*(1), 20-44.

Zazkis, R., & Mamolo, A. (2011). Reconceptualizing knowledge at the mathematical horizon. *For the Learning of Mathematics*, *31*(2), 8-13.

Zembat, I. O. (2013, February). Specialized content knowledge of mathematics teachers in UAE context. In *Proceedings of the Eighth Congress of European Research in Mathematics Education-CERME* (Vol. 8).

APPENDICES

APPENDIX A. THE QUADRATIC FUNCTION CONCEPT QUESTIONNAIRE

Question 1: What is a quadratic equation?

Question 2: What is a quadratic function?

Question 3: How are quadratic equations, quadratic functions, and quadratic polynomials interrelated? Could you explain the differences or similarities between them?

Question 4: Decide whether the following table of values belongs to either linear or quadratic functions:

x	-3	-2	-1	0	1	2	3
y	14	10	6	2	-2	-6	10

x	-3	-2	-1	0	1	2	3
y	3	0	-1	0	3	8	15

Question 5: Solve the quadratic equations below.

a. $3x^2 - 5x + 1 = 0$ **b.** $x^2 - x = -6$ **c.** $(x-3)^2 + 5 = 0$

Follow-up: What other ways (if any) are there to solve it?

Question 6: What does the *axis of symmetry* mean for a quadratic function?

Question 7: What does the *vertex* of a quadratic function mean?

Question 8: What does *concavity* of a quadratic function mean?

Question 9: Find the following properties of the function $g(t) = t^2 + 2t - 8$, and then graph it.

Graph orientation: up/down

x-intercept(s):

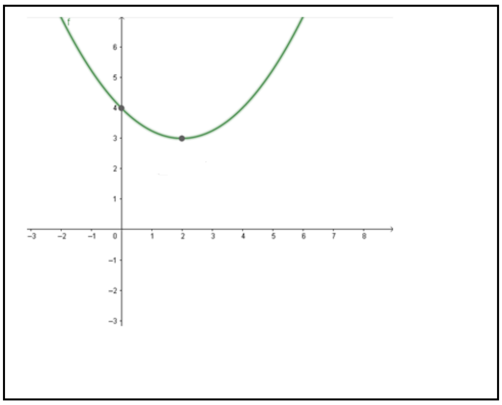
y-intercept:

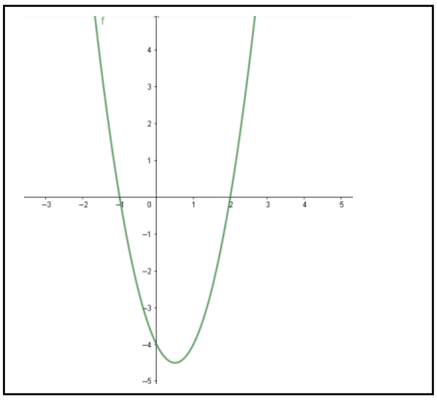
Axis of symmetry:

Vertex:

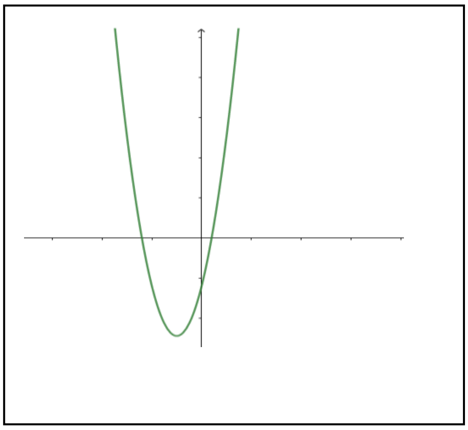
If exists, the maximum or the minimum:

Question 10: For each of the two graphs given below, please state the quadratic function.





Question 11: The graph of the function $f(x) = ax^2 + bx + c$ is shown in the figure below. State whether a, b, and c are negative, positive, or zero. Explain your reasoning.



Question 12: One of your students claims that it is possible to generate the graph of a quadratic function by applying some transformations on the graph of $f(x) = x^2$. What would be your reaction to this claim? Please explain why.

Question 13: Which function represents the parabola with the widest graph? Explain your answer.

A) $2(x+3)^2$ B) $x^2 - 5$ C) $0.5(x-1)^2 + 1$ D) $-x^2 + 6$

Question 14: Describe the similarities and differences between the parabolas generated from these two functions without drawing them: $f(x) = x^2 - 5$ and $g(x) = (x-5)^2$.

Question 15: The *Mathematical Magazine* is a popular magazine published once in three months is sold approximately 25,000 per issue for a 5.5 TL price. Due to the increasing cost of paper and production, a price increase has become inevitable. A survey was conducted with the readers in order to understand how a rise in the price of the magazine would affect the sales. Results of this survey revealed that each 50 Kr rise would result in a drop of $1{,}250$ people buying the magazine. If you were the editor of the magazine, what would you suggest as the new selling price?

Question 16: The parabola with the vertex $(-1/4, 11/4)$ passes through the point $(-1, 5)$. What is the equation of that parabola?

Question 17: Find the equation of the quadratic function whose graph contains the points $(1, 9)$, $(-2, 27)$, and $(4, 3)$.

Question 18: Think about the parabola $f(x) = ax^2 + bx + c$ and the line $y = mx + n$. Under what conditions the parabola and the line intersect or not?

Follow-up: Decide whether the line $y = 11x - 13$ and the parabola $y = 2x^2 + 3x - 5$ intersect or not.

Question 19: State the quadratic formula and explain how it is derived, both geometrically and algebraically.

Question 20: Solve the quadratic equation $x^2 - x + 1 = 0$ without using the quadratic formula. Explain your method. Is there a special name for the method you have just used?

Question 21: Can you provide an example of a real-life problem you share with or ask to your students that can be modeled and solved by a quadratic function?

Question 22: Comment on the student's solution to the problem given below. State whether the result is correct or not. Explain your reason.

Problem: Find the minimum value of the function $f(x) = -3x^2 + 5x + 7$ in $[-2, 2]$.

Student's response: In order to find the minimum, we need to find the vertex of the function. It can be obtained by using the formula $-b/2a$.
$a=-3, b=5$. Therefore $T=(-5)/(-6)=5/6$ is the minimum of the function.

Question 23: Imagine that, in your exam, one of your students solved the quadratic equation $2x^2+5x-7=0$ as presented below. What do you think about this solution? Is the solution correct? How would you describe the student's approach?

$$2x^2+5x=7$$

$$x^2+\frac{5}{2}x=\frac{7}{2}$$

$$x^2+\frac{5}{2}x+\frac{25}{16}=\frac{7}{2}+\frac{25}{16}$$

$$\left(x+\frac{5}{4}\right)^2=\frac{81}{16}$$

$$x+\frac{5}{4}=\frac{9}{4}, \qquad x+\frac{5}{4}=-\frac{9}{4}$$

$$x_1=1, \quad x_2=-\frac{7}{2}.$$

Question 24: Please examine a student's solution to the below problem and decide whether the result is true or false. Describe the student's approach.

Problem: Find the (unique) quadratic polynomial such that all three of the following are true:

- All the coefficients are integers.
- The leading coefficient is 4.
- $7+\sqrt{6}$ is one of the roots.

A student's Solution:

$x = 7 + \sqrt{6}$
$(x-7)^2 = 6$
$4(x-7)^2 = 24$
$4(x-7)^2 - 24 = 0$
$4(x^2 - 14x + 49) - 24 = 0$
$4x^2 - 56x + 196 - 24 = 0$
$p(x) = 4x^2 - 56x + 172.$

Question 25: In your class, while you are teaching the quadratic function $f(x) = ax^2 + bx + c$, one of the students asked that "While we are translating a parabola vertically, only c changes. However, translating a parabola horizontally changes both b and c. Why does it happen?"

How would you respond to your student's question? Please explain.

Question 26: Assumed that one of your students, Ali, provided the following solution for the equation $3x^2 = 15x$.

$3x^2 = 15x$

$x^2 = 5x$

$x = 5$

Then, the following conversation was made between Ali and another student of yours, Ayşe.

Ayşe: You cannot divide both sides by *x*.
Ali: If I can divide both sides by 3, why can't I divide by *x*?

At this moment, what could be the most proper explanation for your students?

Question 27: In your class, you want to emphasize the symmetrical property of a parabola. In order for your students to understand the symmetrical property, what kind of an example might you use in the class?

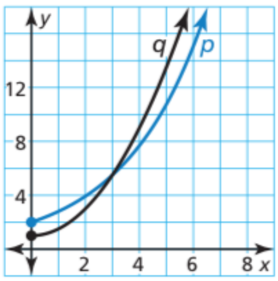
Question 28: <u>Task:</u> The graph of the function $f(x) = x^2 - mx + m + 3$ intersects the x-axis at two different points, A and B. Given that the distance between A and B is 3 units, find the sum of the values that m might take?

a. Examine the task given above. Think about your students. Do you think that they could find the solution?

b. If you think that this is a difficult task for the students, how you can modify it to be an easier one.

Question 29: Think about the graph of the function $f(x) = ax^2 + bx + c$. You already know that if $a > 0$, the graph of the function is concave up, and if $a < 0$ the graph of the function is concave down. How can you provide a plausible explanation for this statement?

Question 30: Function p is an exponential function and function q is a quadratic function. One of your students says that after about $x = 3$, q will always have greater y-values than p. Is your student correct? Please explain your answer.



Question 31: Does the vertex of a quadratic function relate to the derivative in any way? If so, how?

Question 32: One of your students wonders if any concept from their physics course is related to quadratic equations and functions. What kind of examples/situations would you provide to him/her?

Question 33: One of your students asked whether and how the golden ratio is related to quadratic equations. How would you respond to this student?

Question 34: One of your students asked whether the graph of $y = x^4$ is a parabola or not. How would you respond? Please explain your answer.

Question 35: One of your students told you:

> "I heard the term 'reflection property of a parabola' while I was watching a documentary on TV last night. It was told that it has many practical uses in real-life. But, I missed the rest of the documentary after this introduction, because of a power cut in my area, and didn't understand the property. Could you explain what this property is and how/why it is useful in real-life?"

How would you respond to this student?

Question 36: One of your students asked if/how a parabola and a hyperbola are related. How would you respond to this question?

Question 37: One of your students asked that she heard something called *the fundamental theorem of algebra*. She wonders what it is and if and how it applies to quadratic polynomials. What would you say to her?

Question 38: Which of the following students is most correct? Why do you think so?

Student 1: The graph of a quadratic function is a parabola.

Student 2: The graph of a quadratic function is called a parabola.

Question 39: What would be a proper definition of a *parabola*? Can you provide alternative definitions, if you think there are more?

Question 40: One of your students thinks that the shape of a uniform flexible chain or rope whose ends are suspended from the same height and sagging under the force of gravity resembles a parabola. How would you respond to this student?

APPENDIX B. SOURCES FOR THE QUESTIONS USED IN THE PRESENT STUDY

The Quadratic Function Concept Questionnaire	
Item Number	**Source**
1, 2, 3, 5, 6, 7, 8, 16, 17, 18, 19, 20, 21, 22, 25, 27, 28, 29, 36, 39.	Prepared by the researcher.
10, 11, 23, 26, 31, 32, 33, 35, 37, 38, 39, 40.	A. K. Erbaş (Personal Communication, October 1, 2019).
15.	Erbaş et al., 2016.
9, 13, 14.	Adapted from Parent, J. S. S., 2015.
12, 24, 34.	Adapted from Bremigan et al., 2011.
4, 30.	Adapted from Larson & Boswell, 2019.
The Quadratic Function Concept Test	
Item Number	**Source**
1, 2, 3, 4, 6, 7, 8.	Prepared by the researcher
5.	Adapted from Larson & Boswell, 2019.
9.	Adapted from Wu, 2016.
10.	Adapted from Bremigan et al., 2011.

APPENDIX C. FOLLOW-UP INTERVIEW

<u>Part I:</u> A review of Can's responses to the quadratic function concept questionnaire:

#1: Could you please find the axis of symmetry of the function $g(x) = -6x^2 + 12x + 5$?

#2: Could you please find the vertex of the function $f(x) = 3x^2 + 9x + 6$?

#3: In the 4^{th} question, you have stated that the values in the first table belong to a linear function, the values in the second table belong to a quadratic function. Could you explain how did you decide it?

#4: In the 10^{th} question, you have written the first function in the vertex form, the second function in the intercept form. Do you use these different algebraic demonstrations in your classroom instruction?

#5: In the 11th question, you wrote that $a > 0$, $b > 0$, and $c < 0$. How did you determine the signs?

#6: In the questionnaire, you did not solve the question 15. Could you examine the question again and think about how it can be solved?

#7: In question 19, you said that you would justify the quadratic formula by drawing the graph. I could not get what you meant. How do you justify the quadratic formula on the graph?

#8: In question 21, you were asked to provide an example of a real-life problem you share with or ask to your students that can be modeled and solved by a quadratic functions. You did not answer. Do you use this kind of problems during your instruction?

#9: In question 24, you have written that the student is right. Could you explain why did you think so?

#10: Could you explain your response to question 25? You said while translating a parabola upwards and downwards, only the ordinate value changes and the ordinate value only affects c.

#11: You said you would use Geogebra to emphasize the symmetrical property of a parabola. How would you do this? What kind of examples can you use?

#12: How can you explain why the graph of a quadratic function is concave down if $a < 0$, and concave up if $a > 0$?

#13: In question 30, you have stated that the quadratic function q will always take greater values than the exponential function p. Could you explain why did you think so?

#14: Is there a relationship between the vertex and derivative?

#15: In question 35, you were asked about reflection property of a parabola. Could you explain what this property is and where it is used in daily life?

#16: In question 33, you were asked to explain (if any) the relationship between the golden ratio and quadratic equations. You did not write anything. Do you have an idea about their relationship?

#17: In question 32, you were asked whether quadratic functions are related to any concept from physics course, you did not answer. Could you give some examples from physics course which might be related to quadratic functions?

#18: In the questionnaire, you were asked to state the fundamental theorem of algebra and its application to quadratic polynomials. You wrote that you have never heard this theorem.

#19: In question 38, you selected the second statement as the most correct? Could you explain why?

#20: You defined a parabola as the graph of a quadratic function. Do you know any alternative definitions?

#21: You did not respond to the last question. Have you ever heard the term catenary?

<u>Part II:</u> A review of Ahmet's responses to the quadratic function concept questionnaire:

#1: Could you please find the axis of symmetry of the function $g(x) = -6x^2 + 12x + 5$?

#2: Could you please find the vertex of the function $f(x) = 3x^2 + 9x + 6$?

#3: In the 10^{th} question, you have written the first function in the vertex form, the second function in the intercept form. Do you use these different algebraic demonstrations in your classroom instruction?

#4: In the eleventh question, you wrote that $a > 0$, $b > 0$, and $c < 0$. How did you determine?

#5: You made an algebraic justification of the quadratic formula. Do you know any geometrical justification?

#6: In question 21, you have written a profit-loss problem as example of real life problems regarding quadratic functions. Do you use this kind of problems as a part of your instruction? If yes, how often do you use?

#7: You stated that completing the square method is your favorite approach to solve quadratic equations and you gave importance on your students' use of this approach. Why do you care so much about completing the square, rather than using the quadratic formula, which might generally be more practical?

#8: In the questionnaire, you explained the change in the coefficients while translating parabolas. Do you tell your students about these changes?

#9: In question 28, you have changed the question by giving the apsis of the midpoint of A and B. Could you explain why?

#10: You wrote that we can use the second derivative to explain the relationship between the leading coefficient and the concavity of the quadratic function. Could you explain how?

#11: You wrote the definitions of a parabola and a hyperbola in your response to the questionnaire. What would you say about these two concepts?

#12: You stated that the reflection property is used in the construction of headlights and satellite dishes. Could you explain what this property is?

#13: You stated that the reflection property is used in the construction of headlights and satellite dishes. Could you explain what this property is?

#14: In question 38, you selected the second statement as the most correct? Could you explain why?

#15: In question 40, you have stated that this shape is a catenary. Could you explain why?

#16: In question 27, you have stated that you define a as the half of the sum of the roots of a quadratic equation. Could you explain how you do this?

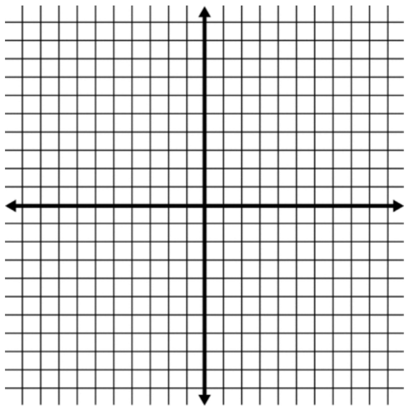
#17: In the 4^{th} question, you have stated that the values in the first table belong to a linear function, the values in the second table belong to a quadratic function. Could you explain how did you decide it?

#18: You solved the quadratic equations by completing the square. Do you use this method in the classroom?

APPENDIX D. THE QUADRATIC FUNCTION CONCEPT TEST

Question #1: For $f(x) = x^2 + 2x - 8$, find the following properties of the function and sketch the graph.

i) x- intercept(s) (ii) y- intercept (iii) vertex (iv) axis of symmetry



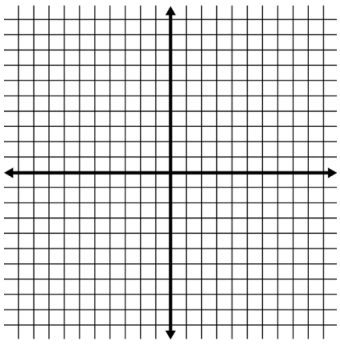
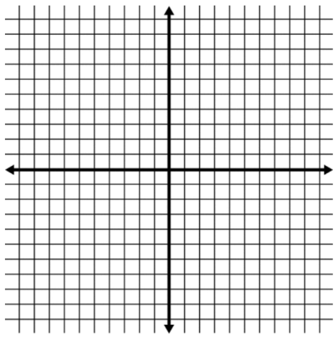
2. Given that the vertex of the function $f(x) = x^2 + bx + c$ is T (2,6), what is c?

3. Find the minimum value of the function $f(x) = 3x^2 + 6x + 5$.

4. Find the maximum value of the function $h(x) = -x^2 + 4x + 6$.

5. Given that $f(x) = 2x^2$, interpret how the graph of f changes when the coefficient of x^2 changes as in the two cases given below:

<u>Case 1:</u> If $f(x) = 2x^2$ becomes x^2 <u>Case 2:</u> If $f(x) = 2x^2$ becomes $3x^2$

6. Find the quadratic function whose vertex is T (-2,2) and passing through A (1,11).

7. Find the quadratic function passing through $A\ (-1, -3)$ and $B\ (2, 6)$, with the *y*-intercept $C(0, -4)$.

8. Which of the following is correct for the function $f(x) = x^2 + 5x + 2$ and the line

$y = \ 3x + 1$? Explain your reason.

a. They are tangent to each other.

b. They intersect at two different points.

c. They never intersects.

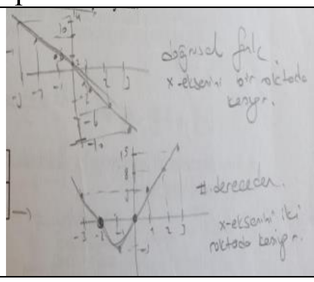
9. The figure represents a rectangular garden whose perimeter is 36 meters. What is the maximum area of this garden?

10. Ali and his friends are playing football. $f(x) = -t^2 + 4t$ represents the height of the ball, h (meters), time, t (seconds), when he hits the ball. So, at what time the reaches 3 meters above the ground?

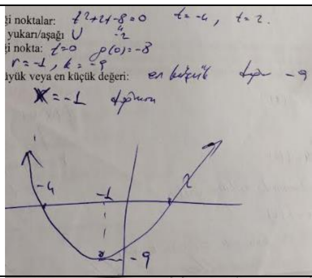
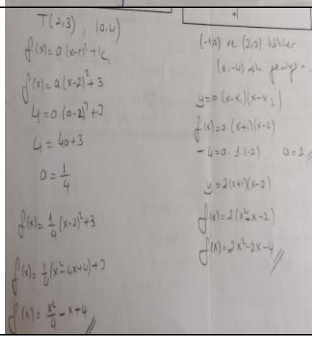
APPENDIX E. CODING SCHEME AND SCORING RUBRIC: THE QUADRATIC FUNCTION CONCEPT QUESTIONNAIRE

	Code	Meaning	Examples
#1	Structural description **(2 points)**	Defining the quadratic equation in its standard form.	$ax^2 + bx + c = 0$, $a, b, c \in \mathbb{R}$ and $a \neq 0$.
	Structural and procedural descriptions **(2 points)**	Defining quadratic equations in its standard form and also referring to quadratic functions.	$ax^2 + bx + c = 0$, $a, b, c \in \mathbb{R}$ and $a \neq 0$. The quadratic equation is a tool for finding the *x*-intercepts of a quadratic function.
	Writing main characteristics of quadratic equations **(1 point)**	Stating some properties of quadratic equations.	-The highest degree of x is 2. It is not linear.
	No answer **(0 point)**	-	-
#2	Structural description **(2 points)**	Defining the quadratic function in its standard form.	$f(x) = ax^2 + bx + c$, $a, b, c \in \mathbb{R}$ and $a \neq 0$.
	Procedural and structural descriptions **(2 points)**	Defining quadratic functions in its standard form and also referring to parabolas.	Quadratic functions are the functions whose graphs generate parabolas.
	Writing main characteristics of quadratic functions **(1 point)**	Defining functions referring to some properties of them.	Function is a relation between two sets. There are one domain and one range.
	No answer **(0 point)**	-	-

#3	Based on their standard forms **(2 points)**	Expressing the differences or similarities based on their standard forms.	Quadratic equation: $ax^2 + bx + c = 0$ Quadratic function: $f(x) = ax^2 + bx + c$ Quadratic polynomial: $p(x) = ax^2 + bx + c$ is also a function.
	Based on their some characteristic **(2 points)**	Expressing the differences or similarities based on their characteristics.	The quadratic function involves some relation between two sets, however, quadratic equations involve equality to a constant.
	Based on their geometrical aspects **(2 points)**	Expressing the differences or similarities based on their geometrical aspects.	As a difference, the graph of a quadratic function is a parabola.
	Incorrect **(0 point)**	Making incorrect explanations.	The graph of a quadratic equation is a quadratic function.
	No answer **(0 point)**	-	-
#4	Examining the second differences **(2 points)**	Examining the first differences and then examining the second differences to be sure that the function is quadratic.	-
	Examining only the first differences **(1 point)**	Calculating the first differences for both cases.	The rate of change is constant, so the first one is linear. In the second one, the rate of change is not constant, thus it is quadratic.

	Using their algebraic forms **(1 point)**	Using algebraic forms of linear functions and finding the coefficients that satisfy the given points.	For the first table: $y = ax + b$ $14 = -3a + b$ $10 = -2a + b$ $a = -4, b = -2.$ $6 = -4.(-1) + 2$ $2 = -4.0 + 2$ $-2 = -4.1 + 2$ $-6 = -4.2 + 2$ $-10 = -4.3 + 2$ For the second table: $3 = -3a + b$ $0 = -2a + b$ $a = -3, b = -6.$ $0 \neq (-3).0 - 6$ Thus it is not linear. It is quadratic.
	Examining their graphs **(1 point)**	Drawing graphs roughly and deciding according to the shape of the graph.	
	No answer **(0 point)**	-	-
#5	Using the quadratic formula **(2 points)**	Solving the quadratic equations by using the quadratic formula.	$3x^2 - 5x + 1 = 0$ $\Delta = b^2 - 4ac$ $= 25 - 4.3.1 = 13.$ $\Delta > 0.$ $x_{1,2} = \frac{-b \mp \sqrt{\Delta}}{2a}$ $x_1 = \frac{5 + \sqrt{13}}{6}$ $x_2 = \frac{5 - \sqrt{13}}{6}$

	Completing the square **(2 points)**	Solving the quadratic equations by completing the square method.	$3x^2 - 5x + 1 = 0$ $x^2 - \frac{5}{3}x + \frac{1}{3} = 0$ $x^2 - \frac{5}{3}x + \left(\frac{5}{6}\right)^2 = \frac{25}{36} - \frac{1}{3}$ $\left(x - \frac{5}{6}\right)^2 = \frac{13}{36}$ $\left\|x - \frac{5}{6}\right\| = \frac{\sqrt{13}}{6}$ $x_1 = \frac{5 + \sqrt{13}}{6}$ $x_2 = \frac{5 - \sqrt{13}}{6}$
#6	Structural description **(2 points)**	Defining what the axis of symmetry is.	It is the line that separates the parabola into two symmetrical parts.
	Procedural description **(1 point)**	Describing how to find the axis of symmetry.	$x = -b/2a$.
	Incorrect **(0 point)**	Writing an incorrect definition.	The apsis of the vertex.
	No answer **(0 point)**	-	-
#7	Structural description **(2 points)**	Defining what the vertex means for a quadratic function.	The vertex is the maximum or the minimum point of a quadratic function.
	Procedural description **(1 point)**	Describing how to find the vertex of a quadratic function.	The apsis of the vertex is $-b/2a$.
	Incorrect **(0 point)**	Writing an incorrect definition.	It is the ordinate of the maximum or the minimum point of a parabola
	No answer **(0 point)**	-	-
#8	Structural description **(2 points)**	Defining the concavity related to the curvature.	A parabola is concave down if it is ∩-shaped; concave up if it is ∪-shaped.

	Procedural description **(1 point)**	Defining concavity related to the sign of the leading coefficient or the second derivative.	-If $a > 0$, f is concave up; if $a < 0$, f is concave down. -If f'' is positive, f is concave up; if f'' is negative, f is concave down.
	Incorrect **(0 point)**	Making incorrect explanations.	It represents the vertex.
	No answer **(0 point)**	-	-
#9	Drawing the correct graph **(2 points)**	Calculating the properties of the quadratic function and drawing its graph correctly.	
#10	Finding the correct quadratic functions **(2 points)**	Finding the correct quadratic functions using different algebraic forms of quadratic functions.	
	No answer **(0 point)**	-	-
#11	Finding the sign of all the coefficients correctly **(2 points)**	Determining the sign of a by examining the concavity of the parabola.	The parabola is upwards, so $a > 0$. a is positive, and the apsis of the vertex $(-b/2a)$ is negative. $b > 0$. The ordinate of the y-intercept is negative. $c < 0$.
	Finding one or two of the coefficients wrongly **(1 point)**	Determining the sign of b by examining the sign of the apsis of the vertex.	$b < 0$ since there are two different roots.
	Incorrect or no answer **(0 point)**	-	-

#12	Describing some of the transformations **(1 point)**	Describing vertical and horizontal translations.	$y = f(x - a)$ and $y = f(x + a)$ are the translations along the *x*-axis a unit right and left. $y = f(x) - a$ and $y = f(x) + a$ are translations along the *y*-axis a unit below and above.
	Describing all the transformations **(2 points)**	Describing how to translate, reflect, and stretch or shrink the graphs.	The student is right. $f(x) = a(x - r)^2 + k$ We can first make horizontal and vertical translations. Then, reflect the graph according to the sign of a, then shrink or stretch it.
	No answer **(0 point)**	-	-
#13	Examining the leading coefficients of quadratic functions **(2 points)**	Comparing the absolute values of the leading coefficients of the functions.	The smaller the $\lvert a \rvert$ gets, the wider the parabola becomes. The answer is A.
	Incorrect **(0 point)**	Comparing the difference of roots.	-The answer is D because the difference of the roots is the biggest, $x_1 - x_2 = \sqrt{24}$. -The answer is A because *b* value is the biggest.
	No answer **(0 point)**	-	-
#14	Comparing the transformations made onto $f(x) = x^2$ to obtain the two functions **(2 points)**	Describing translations made onto the parent quadratic function.	$f(x)$ can be obtained by vertical translation of x^2, 5 units below. $g(x)$ can be obtained by horizontal the translation of x^2, 5 units right.
	Comparing some characteristic of the quadratic functions **(1 point)**	Describing their characteristics such as the number of *x*-intercepts, the vertex, etc.	$f(x)$ intersects the *x*-axis at two different points. $g(x)$ is tangent to the *x*-axis.
	Incorrect **(0 point)**	Making incorrect explanations.	$g(x)$ is parallel to the *x*-axis.
	No answer **(0 point)**	-	-

#15	Using an algebraic model **(2 points)**	Forming a mathematical model with a quadratic function.	The income function *g(x)*= $(5.5 + 0.5x) \,.\, (25000 - 1250x)$ x: the number of each 50-cent rise in the price. $r = 4{,}5$ $f(r) = k = 5{,}5 + \frac{1}{2}.4{,}5 =$ 7,75 TL.
	Using a numerical approach **(1 point)**	Trying numerical values without forming a quadratic model.	$25000.5{,}5 = 137500$ $23750.6 = 142500$ $22500.6{,}5 = 146500$ $21250.7 = 148750$ $\underline{20000.7{,}5 = 150000}$ $\underline{18750.8 = 150000}$ $17500.8{,}5 = 148750$ So, I could suggest the selling price as 7,5 TL.
	No answer **(0 point)**	-	-
#16	Finding the correct quadratic function **(2 points)**	Find the correct quadratic function using the vertex form.	$y = a(x - r)^2 + k$ $5 = a(-1 + 1/4)^2 + 11/4$ a =4.
	No answer **(0 point)**	-	-
#17	Finding the correct quadratic function **(2 points)**	Finding the correct quadratic function using the standard form.	$y = ax^2 + bx + c$ $a + b + c = 9$ $4a - 2b + c = 27$ $16a + 4b + c = 3.$
	No answer **(0 point)**	-	-
#18	Correctly explaining the three conditions for the intersection of a line and a parabola **(2 points)**	Equating two functions, obtaining a quadratic equation and examining its discriminant, and stating three conditions.	$ax^2 + bx + c = mx + n$ $ax^2 + (b - m)x + c - n = 0.$ If $\Delta < 0$, they do not intersect. If $\Delta = 0$, the parabola is tangent to the line. If $\Delta > 0$, they intersect at two different points.
	No answer **(0 point)**	-	-
#19	Algebraic and geometrical justifications **(2 points)**	-	-

	Algebraic justification only **(1 point)**	Deriving the quadratic formula by completing the square.	$ax^2 + bx + c = 0$ $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$ $= \frac{b^2}{4a^2} - \frac{c}{a}$ $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$ $\left(x + \frac{b}{2a}\right) = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
	No justification **(0 point)**	Stating the quadratic formula without any justification.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
	No answer **(0 point)**	-	-
#20	Solving by completing the square **(2 points)**	Finding the roots by completing the square method.	$x^2 - x + \frac{1}{4} - \frac{1}{4} + 1 = 0$ $\left(x - \frac{1}{2}\right)^2 = -\frac{3}{4}$ $x_{1,2} = \frac{1 \pm \sqrt{3}i}{2}$.
	Incorrect **(0 point)**	Making incorrect explanations.	The equation can be solved by factorization.
	No answer **(0 point)**	-	-
#21	Writing a problem statement **(2 points)**	Writing a well-defined and solvable real-life problem.	Let the cost of a product be x TL. If the product is sold $x^2 - 5x + 14$ TL, what would be the minimum profit?
	Writing a problem context **(1 point)**	Giving some examples about the use of quadratics in real-life.	The maximum height of a ball thrown by projectile motion can be calculated by using quadratic functions.
	No answer **(0 point)**	-	-

#22	Explaining all the incorrect steps **(2 points)**	Stating that the solution is incorrect and explaining all the errors.	The student is wrong. Since the parabola is downwards, the vertex does not give the minimum. Also, the ordinate of the vertex gives the max/min values, not the apsis of it. The endpoints should also be checked.
	Explaining some of the incorrect steps **(1 point)**	Stating that the solution is incorrect and explaining some of the errors.	The student is wrong. Since the parabola is downwards, the vertex gives the maximum.
	No answer **(0 point)**	-	-
#23	Explaining the student's solution **(2 points)**	Stating that the solution is correct and writing the name of the approach.	The student solved the equation by completing the square. This approach is my favorite while teaching quadratic equations.
	Only stating that the solution is correct **(0 point)**	Stating that the solution is correct without giving the name of the approach.	It is correct.
	No answer **(0 point)**	-	-
#24	Explaining the student's solution **(2 points)**	Stating that the solution is correct explaining the student's method.	The solution is correct. The student went from the final to the beginning. He/she wrote one of the roots equal to *x*, squared the equation, and obtained the quadratic polynomial.
	Only stating that the solution is correct **(0 point)**	Stating that the solution is correct without further information.	The solution is correct.
	Solving the problem using another approach **(0 point)**	Finding the quadratic polynomial using another approach, comparing the solution with the student's.	If one root is $7+\sqrt{6}$, another is $7-\sqrt{6}$. The sum of roots: $\frac{-b}{a}=14$. The multiplication of roots: $\frac{c}{a}=43$. $a=4, b=-56, c=172$. The student is correct.

	No answer **(0 point)**	-	-
#25	Making a correct explanation **(2 points)**	Presenting a plausible response to student's question.	While translating upwards and downwards, the apsis of the vertex does not change. So, the sum of the roots stay constant but the roots change. So, the multiplication of the roots changes. Hence, b stays constant and c changes. While translating left and right, both the sum of the roots and multiplication of the roots change. Hence, b and c change.
	Incorrect **(0 point)**	Making some incorrect explanations	While translating upwards and downwards, the roots do not change. While translating upwards and downwards, roots change. So, everything changes.
	No answer **(0 point)**	-	-
#26	Making a correct explanation **(2 points)**	Presenting a plausible response to student's question.	I would say that an equation cannot be divided by *x*, because we can eliminate one of the roots, which is 0. I would say that an equation cannot be divided by x, because we don't know the value of x. It might be 0. Zero cannot divide any number.
	No answer **(0 point)**	-	-
#27	Writing relevant examples **(2 points)**	Defining *r* (the apsis of the vertex) as the half of the sum of the roots; and emphasizing that the x values sum up to 2r have the same ordinate, meanly they are symmetrical.	I am defining *r* as half of the sum of the roots. I tell my students that the *x*-values sum up to *2r* are symmetrical. For example if $r = 5$, $f(1) = f(9)$ or $f(-5) = f(15)$. I want my students to notice this property.

	Writing irrelevant examples **(0 point)**	Making some irrelevant explanations.	-I demonstrate symmetrical shapes to my students like the shape of a heart.
	No answer **(0 point)**	-	-
#28	Making some reasonable modification	Stating that the question is hard and making some reasonable modifications.	I would change the problem: $f(x) = x^2 - 5x + m - 1$ intersects the x-axis at two different points, A and B. If $\lvert AB \rvert = 3\ br$, what is m?
	Making no modification	Stating that the question is easy to solve and making no modification.	I think it is an easy question for my students. I would not change.
	Making unnecessary/irrelevant modification	Stating that the question is hard and making some irrelevant/unnecessary modifications.	I would give extra information about the sign of the sum of the roots.
	No answer	-	-
#29	Relating the concavity of the graph with the second derivative of quadratic functions **(2 points)**	Explaining the relationship between the second derivative of the quadratic function and the concavity of its graph.	$f''(x)$ is always constant and there is no inflection point.
	Irrelevant explanations **(0 point)**	Making some irrelevant explanations.	I draw different parabolas and show the change in their shapes depending on the sign of a.
	No answer **(0 point)**	-	-
#30	Comparing the graphs of a quadratic function and an exponential function correctly **(2 points)**	Stating that an exponential function grows faster than any quadratic function after some point.	The student is wrong. After some point, an exponential function will increase faster than a quadratic function.
	Incorrect **(0 point)**	Examining their graphs and stating that the quadratic function has always greater y-values than the exponential.	When we look at the graphs, we notice that y-values of the q function is always greater than of the p function after $x = 3$.

	No answer **(0 point)**	-	-
#31	Explaining the relationship between the vertex and the derivative partially **(1 point)**	Stating that the vertex is the point where the first derivative of the function is zero.	The vertex is the point where the first derivative of the function is 0.
	Incorrect **(0 point)**	Making incorrect explanations.	The first derivative of a function can be found by drawing tangent lines passing through the vertex.
	No answer **(0 point)**	-	-
#32	Writing any concept from the physics course related to quadratic functions **(2 points)**	Mentioning some topics from the physics course that are related to quadratic functions.	Free fall, projectile motion.
	No answer **(0 point)**		
#33	Relating golden ratio with quadratic equations **(2 points)**	Stating the golden ratio and its relation to quadratic equations.	The golden ratio is $1 + \frac{\sqrt{5}}{2}$. It is the positive root of the quadratic equation $x^2 - x - 1 = 0$.
	Only stating the golden ratio **(0 point)**	Stating the golden ratio without explaining its relation to quadratics.	$1 + \frac{\sqrt{5}}{2}$.
	No answer **(0 point)**	-	-

#34	Stating that the graph of $y = x^4$ is not a parabola **(2 points)**	Stating that the graph of $y = x^4$ is not a parabola by emphasizing that it is not a quadratic function.	It is not a parabola; because parabolas are the graphs of quadratic functions.
	Incorrect **(0 point)**	Making some incorrect explanations.	-It is a parabola because it is U-shaped. -It is not a parabola because it is so wide. -The arms of the parabolas are narrower.
	No answer **(0 point)**	-	-
#35	Explaining the reflective property and its daily use correctly **(2 points)**	Describing what reflective property is and its daily use.	It is used in real life in the construction of headlights and satellite dishes.
	Incorrect **(0 point)**	Making irrelevant/incorrect explanations.	I would say that the vertex of a parabola is the axis of symmetry.
	No answer **(0 point)**	-	-
#36	Explaining some differences between parabolas and hyperbolas **(2 points)**	Stating that they both are conic sections and explaining their differences.	They are both conic sections. A parabola consists of one curve, a hyperbola consists of two curves. A parabola is a set of points that are equidistant from a straight line and focus. A hyperbola is a set of points whose distances to two fixed points have a constant difference.
	Incorrect **(0 point)**	Making incorrect explanations.	Hyperbola is the symmetry of a parabola. Parabola is $y = ax^2$, hyperbola is $x = ay^2$.
	No answer **(0 point)**	-	-

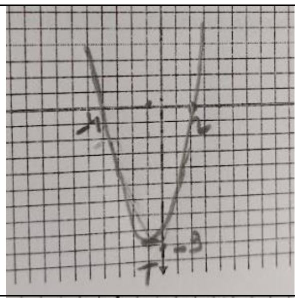
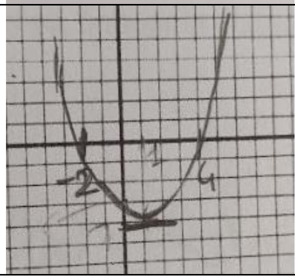
#37	Applying the fundamental theorem of algebra to quadratic polynomials **(2 points)**	Writing the Fundamental Theorem of Algebra with some and explaining its application to quadratic polynomials.	-A polynomial with degree n has n roots. Quadratic polynomials have two roots. -Quadratic equations have 2 roots. If the discriminant<0, it has no roots.
	Incorrect **(0 point)**	Writing irrelevant information.	Demonstrating a quadratic equation by drawing a square.
	No answer **(0 point)**	-	-
#38	Stating that Student 1 is correct with a correct justification **(2 points)**	Choosing the first statement as the most correct and defining a parabola as a conic section.	-
	Stating that Student 1 is correct with incorrect explanation **(0 point)**	Stating that Student 1 is correct by making some incorrect explanations.	Second statement is a definition, but parabola is undefined. So, Student 1 is right.
	Stating that Student 2 is correct with incorrect explanation **(0 point)**	Stating that Student 2 is correct by making some incorrect explanations.	Because it is a definition.
	None **(0 point)**	Stating that both of the arguments are incorrect.	-The graph of a quadratic polynomial function is called a parabola. -The graph of a polynomial function $f(x) = ax^2 + bx + c$ ($a \neq 0$, a, b, c € $\mathbb{R}$) is a parabola.
	No answer **(0 point)**		

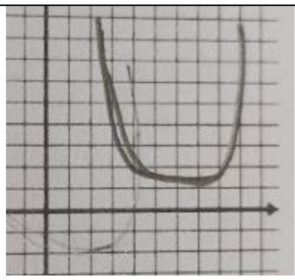
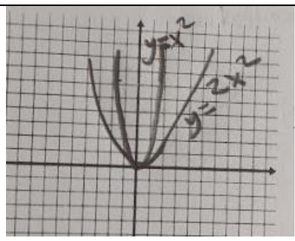
#39	Describing parabola as the graph of a quadratic function and stating geometrical definition of a parabola **(2 points)**	Defining parabola as the graph of a quadratic function and also stating geometrical definition of a parabola.	A parabola is the graph of a quadratic function. Alternative definition: A parabola is defined as the set of points that are equidistant from both the directrix and the focus.
	Describing parabola as the graph of a quadratic function only **(1 point)**	Defining parabola as graph of a quadratic function; not giving any alternative definition.	A parabola is the graph of a quadratic function.
	Incorrect **(0 point)**	Making some incorrect definitions for a parabola.	-A parabola is a quadratic function. -A parabola is the graph of a quadratic equation. -A parabola is a quadratic equation.
	No answer **(0 point)**	-	-
#40	Distinguishing between a parabola and a catenary **(2 points)**	Stating that the shape is not a parabola, it is a catenary.	I would say that it is a catenary.
	Stating that it is not a parabola without explanation **(1 point)**	Stating that the shape is not a parabola but looks like a parabola.	-I would say that it resembles a parabola, but it is not. -I would say that is not a parabola.
	Incorrect **(0 point)**	Stating that the shape is a parabola.	I would say that it is a parabola.
	No answer **(0 point)**	-	-

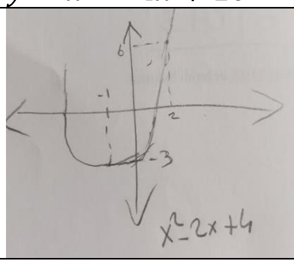
APPENDIX F. CODING SCHEME AND SCORING RUBRIC: THE QUADRATIC FUNCTION CONCEPT TEST

1-a		Code	Meaning	Examples
	Correct	Using completing the square method **(2 points)**	Solving the quadratic equation by completing the square.	$f(x) = (x+1)^2 - 9 = 0$ $x + 1 = 3$ $x + 1 = -3$ $x_1 = 2,\ x_2 = -4.$
	Correct	Using the quadratic formula **(2 points)**	Solving the quadratic equation by using the quadratic formula.	$\Delta = 4 - 4.1.(-8) = 36$ $x_1 = \dfrac{-2+\sqrt{36}}{2} = 2$ $x_2 = \dfrac{-2-\sqrt{36}}{2} = -4.$
	Correct	Using factorization **(2 points)**	Solving the quadratic equation by factorization.	$f(x) = x^2 + 2x - 8.$ $f(x) = (x-2)(x+4)$ $x_1 = 2,\ x_2 = -4.$
		No answer **(0 point)**	-	-
1-b	Correct	Writing $(0, y)$ as the y-intercept **(2 points)**	Finding $y = -8$ for $x = 0$, and writing correctly $(0, -8)$ as the y-intercept.	For $x = 0, y = -8$ $(0, -8).$
	Partially correct	Writing y as the y-intercept **(1 point)**	Writing only the ordinate of the y-intercept without any explanation.	-8.
	Incorrect	Writing $y =$ 0 as the y-intercept **(0 point)**	Calculating the y value for the apsis of the x-intercepts.	$y = x^2 + 2x - 8$ for $x = -4, y = 0$ for $x = 2, y = 0.$
		No answer **(0 point)**	-	-

1-c	**Correct**	Writing (r, k) by using the vertex form **(2 points)**	Finding r and k by turning the function into vertex form.	$f(x) = (x+1)^2 - 9$ $r = -1, \quad k = -9.$ $(-1, -9).$
		Writing (r, k) by using the formula for finding the vertex **(2 points)**	Calculating r and k values, and writing vertex as (r, k).	$r = -2/2 = -1$ $k = f(r)$=$f(-1) = -9$ $(-1, -9)$
	Incorrect	Writing only the apsis of the vertex (r) as vertex **(0 point)**	Calculating only the apsis of the vertex (r) and writing vertex as r.	$r = -b/2a = -2/2$ $r = -1.$ $-1.$
		Writing the y-intercept as the vertex **(0 point)**	Writing vertex as -8 without further explanation.	-8.
		$f(2)$ as the vertex **(0 point)**	Writing $f(2)$ as the vertex.	$f(2) = 4 + 4 - 8 = 0.$
1-d	**Correct**	$x = r$ as the axis of symmetry **(2 points)**	Writing axis of symmetry as the line $x = r$.	$x = -1.$ $x + 1 = 0.$
	Partially correct	r value as the axis of symmetry **(1 point)**	Writing axis of symmetry as $r = -1$ or -1.	$r = -1.$
	Incorrect	The vertex as the axis of symmetry **(0 point)**	Writing axis of symmetry as the vertex.	$(-1, -9).$
		Other incorrect answers **(0 point)**	Writing some irrelevant numbers as the axis of symmetry without any explanation.	3.
		No answer	-	-

#1-e	**Correct**	Sketching the accurate graph **(2 points)**	Replacing all the elements of the function correctly on the graph.	
	Incorrect	Sketching inaccurate graphs **(0 point)**	Sketching the graph inaccurately.	
		No answer **(0 point)**	-	-
#2	**Correct**	Using the formula for *r* and the equation $f(r) = k$. **(10 points)**	Calculating firstly *r* by using $-b/2a$, then using the $f(r) = k$, finding *c*.	$-\frac{b}{2a} = -\frac{b}{2} = 2$ $b = -4.$ $f(r) = k, f(2) = 6$ $22 - 8 + c = 6\ c = 10.$
		Using the vertex form **(10 points)**	Rewriting the quadratic function in the vertex form and finding the coefficients.	$f(x) = a(x - r)^2 + k$ $f(x) = (x - 2)2 + 6$ $= x^2 - 4x + 10, c = 10.$
	Incorrect	Other incorrect answers **(0 point)**	Skipping some steps and finding an incorrect result	$-b/2a = 2$ $-b = 4a$ $x^2 - 4ax + c$ $c = 6.$
		No answer **(0 point)**	-	-
#3	**Correct**	*k* as the minimum **(10 points)**	Calculating *r* by using the formula $-b/2a$ and then finding *k* by using $f(r) = k$.	$r = -b/2a = -6/6 = -1.$ $k = f(-1) = 3 - 6 + 5 = 2$
	Incorrect	The *y*-intercept as the minimum **(0 point)**	Finding the *y*-intercept as minimum	$f(0) = 3.0 + 6.0 + 5$ $= 5.$

		Δ as the minimum **(0 point)**	Calculating the discriminant as the minimum	$\Delta = b^2 - 4ac$ $= 36 - 60 = -24.$
		Other incorrect answers **(0 point)**	Trying to find the x-intercepts as the minimum	$(3x + 3).(x + 2) = 0$ $x = 3, \quad x = 2.$
#4	**Correct**	k as the maximum **(10 points)**	Calculating r by $-b/2a$ and then finding k by using the equation $h(r) = k$.	$r = -\frac{b}{2a} = 2$ $h(2) = -2^2 + 4.2 + 6$ $k = -4 + 8 + 6 = 10.$
	Incorrect	$h(1)$ as the maximum **(0 point)**	Replacing x with 1 in the function and calculating $h(1)$ as the maximum.	$-1^2 + 4 + 6 = 9.$
		No answer **(0 point)**	-	-
#5	**Correct**	Stating that the parabola becomes wider, if $\|a\|$ gets smaller. **(10 points)**	Writing that the parabola becomes larger, if $\|a\|$ gets smaller and sketching proper graphs for two cases.	(for case 1): The arms of the parabola would move away from the y-axis. (for case 2): The arms would get closer to the y-axis.
	Incorrect	Making translations **(0 point)**	Changing the graph by making horizontal or vertical translations.	
		Inverse application of the rule **(0 point)**	Sketching the graph of $y = x^2$ narrower than the graph of $y = 2x^2$ and sketching the graph of $y = 3x^2$ wider than the graph of $y = 2x^2$.	
		Other incorrect answers **(0 point)**	Writing some other irrelevant/incorrect statements.	(for case 1): the value of x increases. (for case 2) the value of x decreases.
		No answer **(0 point)**	-	-

#6	**Correct**	Using the vertex form **(10 points)**	Using the vertex form to find the quadratic function.	$f(x) = a(x-r)^2 + k$ $f(x) = a(x+2)^2 + 2$ $f(1) = 11, 9a + 2 = 11$ $a = 1$ $f(x) = (x+2)2 + 2.$
	Incorrect	Using the intercept form **(0 point)**	Using the intercept form and finding an incorrect function.	$y = a(x-x_1)(x-x_2)$ $y = a(x-1)(x+2)$ $y = -2a$ $y = -2(x-1)(x+2).$
		Using the standard form **(0 point)**	Using the standard form and finding an incorrect function.	$f(x) = ax^2 + bx + c.$ $c = 11$ $f(-2) = 2$ $4a - 2b + 11$=2 $4a - 2b = -9$ $r = -b/2a \quad b = 4a$ $4a - 8a = -9$ $a = 9/4.$
		No answer **(0 point)**	-	-
#7	**Correct**	Standard form **(10 points)**	Using the standard form and finding the quadratic function.	$f(x) = ax^2 + bx + c$ $f(x) = ax^2 + bx - 4.$
	Incorrect	Intercept form **(0 point)**	Using the intercept form and finding an incorrect function.	$y = a(x-x_1)(x-x_2)$ $y = 4(x+1)(x-2)$ $= 4x^2 - 4x - 8.$
		Vertex form **(0 point)**	Using the vertex form finding an incorrect function.	$y = a(x-r)^2 + k$ $y = a(x-2)^2 + 6$ $y = x^2 - 4x + 10$
		Others **(0 point)**	Drawing a graph and writing an incorrect quadratic function.	
		No answer **(0 point)**	-	-

#8	**Correct**	Investigating the discriminant of the quadratic equation **(10 points)**	Equating two functions and obtaining a quadratic equation; then investigating Δ of the common equation.	$x^2 + 5x + 2 = 3x + 1$ $x^2 + 2x + 1 = 0$ $\Delta = b^2 - 4ac$ $= 4 - 4.1.1 = 0$ Tangent.
		Noticing that the new equation is a perfect square **(10 points)**	Stating that the quadratic equation is a perfect square and has one root, so they are tangent to each other.	$x^2 + 5x + 2 = 3x + 1$ $x^2 + 2x + 1 = 0$ (perfect square) $x_1 = x_2 = 1$. One point of intersection, they are tangent.
	Incorrect	Investigating the discriminant of the quadratic equation $f(x)$=0 **(0 point)**	Calculating Δ of the given quadratic equation.	$f(x) = x^2 + 5x + 2 = 0$ $\Delta = b2 - 4ac$ $= 25 - 4.1.2 = 17$ $\Delta > 0$.
		Other incorrect answers **(0 point)**	Writing some other irrelevant/incorrect statements.	They do not intersect because the line $y = 3x + 1$ is not a quadratic line.
		No answer **(0 point)**	-	-
#9	**Correct**	Using a quadratic model **(10 points)**	Forming a quadratic function for the area of the rectangle and calculating its maximum.	$A = x(18 - x)$ $A = -x^2 + 18x$ $r = -18/-2 = 9$ $k = f(9) =$ $-9^2 + 18.9$ $= 81$.
	Partially correct	Using a numerical approach **(5 points)**	Finding the maximum area by trying some numerical values for dimensions of the rectangle.	$a + b = 18$ 1.17 2.18 . . $8.10 = 80$ $9.9 = 81$ (the maximum)

	Incorrect	Using a numerical approach **(0 point)**	Trying to find the maximum area by trying some numerical values for dimensions of the rectangle and finding an incorrect result.	$a + b = 18$ 1.17 2.18 . . 8.10 = 80 (the maximum) Dimensions cannot be 9x9, the shape would be a square then.
		Other incorrect answers **(0 point)**	Assigning 3k for long side, 2k for the short side and calculating k; then finding the dimensions and calculating the area.	$2(3k + 2k) = 36$ $10k = 36 \quad k = 3{,}6.$ $a = 3k = 10{,}8$ $b = 2k = 7{,}2$ $10{,}8 \,.\, 7{,}2 = 77{,}76.$
		No answer **(0 point)**	-	-
#10	**Correct**	Finding two values of t at $h(t) = 3$ **(10 points)**	Equating the function $h(t)$ to 3, and finding two roots of the quadratic equation.	$h(t) = -t^2 + 4t$ $3 = -t^2 + 4t$ $0=-t^2+4t+3$ $t = 1$ and $t = 3$. 1st and 3rd seconds.
	Partially correct	Finding one of the values of t at $h(t) = 3$ **(5 points)**	Equating the function $h(t)$ to 3, and finding one of the roots of the quadratic equation.	$h(t) = -t^2 + 4t$ $3 = -t^2 + 4t$ $3 = t(-t + 4)$ $t = 3$. (3rd second).
	Incorrect	Finding $h(3)$ **(0 point)**	Finding the value of $h(t)$ at $t = 3$, $h(3)$.	$t = 3$ $h(3) = -3^2 + 4.3 = 3$ 3rd second.
		Finding the maximum of the function **(0 point)**	Finding the apsis of the vertex, r, and calculating h(r).	$-b/2a = -4/-2 = 2$ $h(2) = \ -22 + 4.2$ $= 4.$
		No answer **(0 point)**	-	-

APPENDIX G. ETHICAL PERMISSION

Kayıt Tarihi: 05.11.2019	Protokol No: 652

07/11/2019

T.C

BÜLENT ECEVİT ÜNİVERSİTESİ

İNSAN ARAŞTIRMALARI ETİK KURULU KARARI

ÇALIŞMANIN TÜRÜ:	Anket
BAŞLIK:	Lise Matematik Öğretmenlerinin İkinci Dereceden Fonksiyon Kavramına Yönelik Alan Bilgisi ve Bunun Öğrencilerin Öğrenme Çıktıları İle İlişkisi
SORUMLU ARAŞTIRMACI:	Ayhan Kürşat Erbaş
KARAR:	Uygun

ETİK KURUL ÜYELERİ İMZA

1- Prof. Dr. Hamza ÇEŞTEPE (Başkan)

2- Doç. Dr. Ayça DEMİR (Başkan Yrd.)

3- Prof. Dr. Ali ARSLAN (Başkan Yrd.)

4- Prof. Dr. Mehmet Ali KURÇER

5- Prof. Dr. Ertuğrul YILDIRIM

6- Doç. Dr. Hasan MEYDAN

7- Dr. Öğr. Üyesi Elif KARAHAN

29.05.2014 tarih ve 2014/08-13 sayılı Senato Kararı ile kabul edilmiştir.

CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name : Karacı Yaşa, Gülzade
Phone : +90 372 323 38 70
email : gulzade.karaci@metu.edu.tr

EDUCATION

Degree	Institution	Year of Graduation
MS	Zonguldak Bülent Ecevit University Elementary Mathematics Education	2016
Integrated BS & MS	Boğaziçi University Secondary School Science and Mathematics Education	2013
High School	Zonguldak I.M.K.B. Anatolian Teacher High School	2008

WORK EXPERIENCE

Year	Place	Enrollment
2014- Present	Zonguldak Bülent Ecevit University	Research Assistant

FOREIGN LANGUAGES

Advanced English

PUBLICATIONS

1. Karacı-Yaşa, G., & Karataş, İ. (2018). Effects of the instruction with mathematical modeling on pre-service mathematics teachers' mathematical modeling performance. *Australian Journal of Teacher Education, 43*(8), 1-14.

2. Karataş, İ., Pişkin-Tunç, M., Yılmaz, N., & Karacı, G. (2017). An investigation of technological pedagogical content knowledge, self-confidence, and perception of pre-service mathematics teachers towards instructional technologies. *Educational Technology Society, 20*(3), 122-132.

3. Karacı, G., & Pişkin-Tunç, M. (2018). Sınıf öğretmeni adaylarının görsel matematik okuryazarlığı öz-yeterlik algılarının incelenmesi: Zonguldak ili örneği. *II. Uluslararası Sınırsız Egitim ve Arastırma Sempozyumu (USEAS 2018)*, Muğla.

4. Karacı, G., & Yildiz, A. (2018). Lisansüstü öğrenim yapmakta olan matematik ögretmenlerinin bilimsel arastırma yapmaya yönelik kaygıları ve bunlarla bas etme stratejileri. *II. Uluslararası Uluslararası Sınırsız Egitim ve Arastırma Sempozyumu (USEAS 2018)*, Muğla.

5. Karacı, G., & Pişkin-Tunç, M. (2018). Sınıf öğretmeni adaylarının orantısal durumları orantısal olmayan durumlardan ayırt edebilmeleri. *II. Uluslararası Sınırsız Egitim ve Arastırma Sempozyumu (USEAS 2018)*, Muğla.

6. Karataş, İ. & Karacı, G. (2017). Matematiksel modellemeye yönelik hazırlanan bir ögretim sürecinden yansımalar. *7th International Congress of Research in Education*, Çanakkale.

7. Karacı, G. & Erbaş, A.K. (2017). An investigation of eight grade students' misconceptions regarding the concept of variable. *7th International Conference on Research in Education*, Çanakkale.

8. Pişkin-Tunç, M., & Karacı, G. (2015). Sınıf öğretmeni adaylarının matematik kavramına ilişkin algılarının zihinsel imgeler yardımıyla incelenmesi. *II. Türk Bilgisayar ve Matematik Eğitimi Sempozyumu*, Adıyaman.

9. Karataş, İ., Yılmaz, N., Karacı, G., & Atasoy, E. (2015). İlkögretim matematik öğretmen adaylarının sayılarla ilgili alan egitimi bilgilerinin incelenmesi. *II. Türk Bilgisayar ve Matematik Egitimi Sempozyumu*, Adıyaman.

10. Karacı, G. (2015). İlköğretim matematik ögretmen adaylarının limit ve süreklilik konuları hakkındaki anlamaları. *II Türk Bilgisayar ve Matematik Egitimi Sempozyumu*, Adıyaman.

11. Pişkin-Tunç, M., Karataş, İ., Yılmaz, N., & Karacı, G. (2015). İlköğretim matematik öğretmen adaylarının teknolojik pedagojik alan bilgisi. *II. International Dynamic, Explorative and Active Learning (IDEAL) Conference*, Amasya.

12. Karataş, İ. & Karacı, G. (2016). İlköğretim matematik öğretmen adaylarının matematiksel modelleme becerilerinin değerlendirilmesi. *III. International Eurasian Education Research Congress,* Muğla.