# DEMONSTRATION OF QUANTUM CONTEXTUALITY VIA HARDY PARADOX

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**DEMONSTRATION OF QUANTUM CONTEXTUALITY VIA HARDY PARADOX**

submitted by **YANKI YOLSEVER** in partial fulfillment of the requirements for the degree of **Master of Science in Physics Department, Middle East Technical University** by,

Prof. Dr. Halil Kalıpçılar
Dean, Graduate School of **Natural and Applied Sciences** _______________

Prof. Dr. Seçkin Kürkçüoğlu
Head of Department, **Physics** _______________

Prof. Dr. Sadi Turgut
Supervisor, **Physics, METU** _______________

**Examining Committee Members:**

Assoc. Prof. Dr. Yusuf İpekoğlu
Physics, METU _______________

Prof. Dr. Sadi Turgut
Physics, METU _______________

Assist. Prof. Dr. Kıvanç Uyanık
Physics, Gazi University _______________

Date:

**I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.**

Name, Surname: Yankı Yolsever

Signature :

## ABSTRACT

## DEMONSTRATION OF QUANTUM CONTEXTUALITY VIA HARDY PARADOX

Yolsever, Yankı
M.S., Department of Physics
Supervisor: Prof. Dr. Sadi Turgut

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Quantum contextuality is one of the most fundamental foundations of Quantum Mechanics. In this thesis, we have examined the Hardy-type proof of quantum contextuality which is first developed by Lucien Hardy for quantum non-locality and later adapted to Quantum contextuality by Adan Cabello and his colleagues to 3-dimensional systems with a minimum of 5 observables. We have reformulated their method in terms of logical implications and adapted it to 4-dimensional and 5-dimensional systems with using fewer observables.

Keywords: Quantum contextuality, Bell Kochen Specker theorem, Hardy paradox

# ÖZ

## HARDY PARADOKSU YOLUYLA KUANTUM BAĞLAMSALLIĞININ GÖSTERİMİ

Yolsever, Yankı
Yüksek Lisans, Fizik Bölümü
Tez Yöneticisi: Prof. Dr. Sadi Turgut

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Kuantum bağlamsallığı, kuantum mekaniğinin en temel özelliklerinden biridir. Bu tezde, kuantum bağlamsallığının, ilk olarak Lucien Hardy tarafından kuantum yerelsizliği için geliştirilen ve daha sonra Adan Cabello ve meslektaşları tarafından 3 boyutlu ve en az 5 gözlemlenebilirli sistemlerde kuantum bağlamsallığına uyarlanan, Hardy tipi kanıtını inceledik. Onların yöntemini mantıksal çıkarımlar cinsinden ifade edip, bunu 4 boyutlu ve 5 boyutlu ve daha az gözlemlenebilir içeren sistemlere uyarladık.

Anahtar Kelimeler: Kuantum bağlamsallığı, Bell Kochen Specker teoremi, Hardy paradoksu

To my grandfather Yılmaz Düzakın who was curious, resilient and had a good sense of humor until the very last moment

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## LIST OF ABBREVIATIONS

| | |
|---|---|
| 3D | 3 Dimensional |
| 4D | 4 Dimensional |
| 5D | 5 Dimensional |
| NCR | Non-Contextual Realist |
| QM | Quantum Mechanics |
| S | Single Degenerate |
| D | Double Degenerate |

# CHAPTER 1

# INTRODUCTION

## 1.1 Realism and Contextuality

Realism is the assumption that observables have well-defined values prior to observation and the process of measurement reveals these pre-existing values. Under the realism assumption, two observables may not be co-measurable but yet can simultaneously have values as they exist before measurement. In contrast, observables do not have values prior to measurement in quantum mechanics. They gain values after they are measured and only co-measurable observables can simultaneously have values.

In theories with realism assumption ("realist theories" or "hidden variable theories"), value of an observable is defined to depend on some hidden parameter, which may not be accessible. As a result, full description of a physical system includes not only the wavefunction as in quantum mechanics, but also the hidden parameters. Moreover, aim of possible Realist Theories is not to replace quantum mechanics but to complete it by introducing the hidden parameters.

Context is defined as the set of observables which can be jointly measured [1] and non-contextuality is the assumption that the value of an observable is being independent of in which context it is measured. Context can be interpreted physically as the setup of the measurement apparatus [2]. As we change the context, we would need to change the setup of the measurement.

Let us consider the case that we have three observables $\{A, B, C\}$ such that $A$ is co-measurable with both $B$ and $C$ but $B$ and $C$ are not co-measurable with each other. In this case, there are three contexts that $A$ can be measured: $\{A, B\}$,$\{A, C\}$ and $A$

alone. If a theory with realism also has non-contextuality assumption, $A$ is considered to have the same value whether it is measured along with $B$ or with $C$ or on its own. [1]

The theories where the realism and non-contextuality are both assumed are referred as non-contextual realist theories (NCR) or non-contextual hidden variable theories (NCHV). We will not be dealing with a particular NCR theory and use the term to refer the general class of theories with the realism and non-contextuality assumptions.

For an NCR theory to be valid, there must be a way to assign pre-existing and context-independent values for observables in accordance with the constraints of the physical system at hand while being self-consistent and compatible with the predictions of quantum mechanics.

Quantum contextuality refers to the impossibility of reproducing quantum mechanical results with realism and non-contextuality assumptions and it can be proven by introducing a system where an NCR theory and quantum mechanics make different predictions.

Quantum contextuality is first proven by John Bell in 1966 [2] and Simon Kochen and Ernst Specker in 1967 [3] and named as Bell-Kochen-Specker Theorem or Kochen-Specker Theorem

## 1.2 Hardy-Type Proof of Contextuality

In his papers in 1992 [4] and 1993 [5], Lucien Hardy showed a proof for quantum contextuality by designing a system where a local realist theory makes the prediction that a certain outcome is impossible while that outcome is possible in quantum mechanics.

Adan Cabello and his colleagues adapted this method to the case of contextuality [6]. They have designed a system where a certain assignment of values to observables is impossible in an NCR theory while it is possible in quantum mechanics.

The system they have designed is subjected to constraints where the value of an observable implies that some other observable has a certain outcome. The source of discrepancy between an NCR theory and quantum mechanics is that these constraints are combined in the first but not in the latter.

They have worked in 3-dimensional Hilbert Space and showed that minimum 5 observables required to demonstrate quantum contextuality in 3-dimensions. We will examine their method and apply it to higher dimensions in order to make the demonstration with fewer number of observables.

As there is a set of implication constraints and the constraints are 'linked' in an NCR theory in the mentioned systems, we will call them Hardy-type Chains. Unlike the original authors, we will make a distinction based on the strictness of the constraints and name one type of systems as "Complete Hardy-type Chains" and the others as "Incomplete Hardy-type Chains".

## 1.3 Complete Hardy-Type Chain

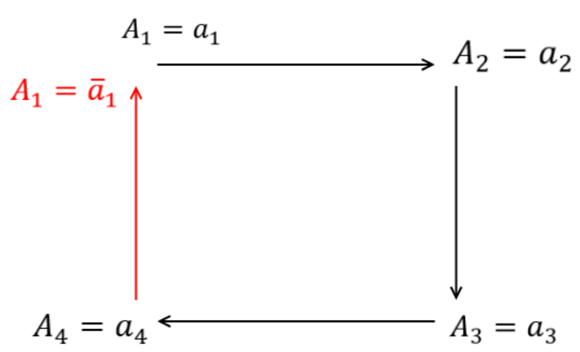


Figure 1.1: Representation of Complete Hardy-Type Chain with 4 observables

We consider $n$ dichotomic observables, $\{A_1, A_2, ...A_n\}$ where only two succeeding observables are co-measurable. In other words, $A_i$ and $A_j$ are co-measurable if and only if $j = i \pm 1 \mod n$. We show one possible value of $A_i$ as $a_i$ and use $\overline{a}_i$ to represent the other eigenvalue.

We construct a system which is subjected to implication constraints:

$$\{I(1), I(2), ..., I(n)\}$$

Implications from $1$ to $n-1$ are:

$$\begin{aligned}
I(1) &: \big(A_1 = a_1 \Longrightarrow A_2 = a_2\big) \\
I(2) &: \big(A_2 = a_2 \Longrightarrow A_3 = a_3\big) \\
&... \\
&... \\
I(n-1) &: \big(A_{n-1} = a_{n-1} \Longrightarrow A_n = a_n\big)
\end{aligned}$$

Where the general term is,

$$I(i) : \big(A_i = a_i \Longrightarrow A_{i+1} = a_{i+1}\big) \text{ for } 1 \leq i \leq n-1 \tag{1.3.1}$$

The $n$th implication is:

$$I(n) : \big(A_n = a_n \Longrightarrow A_1 = \overline{a}_1\big) \tag{1.3.2}$$

These implications are equivalent to following statements in terms of probabilities:

$$I(i) \equiv \big(p(A_i = a_i, A_{i+1} = \overline{a}_{i+1}) = 0\big) \tag{1.3.3}$$

$$I(n) \equiv \big(p(A_n = a_n, A_1 = a_1) = 0\big) \tag{1.3.4}$$

These implications are combined in an NCR theory as it is assumed that each observable has a pre-existing and context-independent value according to the realism and non-contextuality assumptions. But they are not combined in quantum mechanics as two implications refers to the values of non-subsequent observables and they cannot simultaneously have values according to quantum mechanics.

When the implications from $1$ to $n-1$ are combined, we end up with the following implication constraint:

$$I(1, n-1) : \big(A_1 = a_1 \Longrightarrow A_2 = a_2 \Longrightarrow A_3 = a_3 \Longrightarrow ... \Longrightarrow A_n = a_n\big) \tag{1.3.5}$$

$$: \big(A_1 = a_1 \Longrightarrow A_n = a_n\big) \tag{1.3.6}$$

When this combined implication is combined with $I(n)$, it gives a self-contradictory implication:

$$I(1,n) : \big(A_1 = a_1 \Longrightarrow A_1 = \overline{a}_1\big) \tag{1.3.7}$$

It means that, in a system where the constraints $\{I(1), I(2), ..., I(n)\}$ and $A_1 = a_1$, an NCR theory would lead to a self-contradictory statement $I(1,n)$. Therefore, for the theory to be logically consistent, value of $A_1$ should never be $a_1$. In other words, we must have $p(A_1 = a_1) = 0$ according to an NCR theory in such a system.

To demonstrate quantum contextuality, we need construct a system where the constraints $\{I(1), I(2), ..., I(n)\}$ hold but $p(A_1 = a_1) > 0$ using quantum mechanics. We will refer systems with these properties as Complete Hardy-type Chain systems.

As an NCR theory and quantum mechanics differ over $p(A_1 = a_1)$, we will refer this probability as the critical probability.

Experimental demonstration of quantum contextuality becomes easier as the number of observables (therefore constraints) gets fewer and the value of critical probability gets higher.

## 1.4 Incomplete Hardy-Type Chain

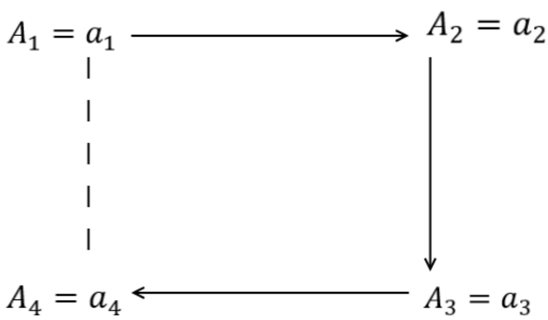


Figure 1.2: Representation of Incomplete Hardy-Type Chain with 4 observables

Quantum contextuality can also be demonstrated with systems where the implication constraints are relaxed. Considering a similar system as in the previous case with only difference is that $I(n)$ is now removed from the set of implications, an NCR theory leads to the following implication again:

$$I(1, n-1) : \big(A_1 = a_1 \Longrightarrow A_n = a_n\big) \tag{1.4.1}$$

It means that, in a system where the constraints $\{I(1), I(2), ..., I(n-1)\}$ hold, an NCR theory makes the prediction that if $A_1 = a_1$, then we must have $A_n = a_n$. In other words, $p(A_1 = a_1, A_n = \overline{a}_n) = 0$ according to an NCR theory in such systems.

Quantum contextuality can be demonstrated by constructing a system where $\{I(1), I(2), ..., I(n-1)\}$ hold but $p(A_1 = a_1, A_n = \overline{a}_n) > 0$ using quantum mechanics. We will refer this kind of systems as Incomplete Hardy-type Chain systems.

Similar to the previous case, an NCR theory and quantum mechanics differ on $p(A_1 = a_1, A_n = \overline{a}_n)$ and an experimental demonstration would be easier as its value gets higher. We will again refer to this probability as critical probability.

### 1.5 Relation Between Complete and Incomplete Chains

Complete Hardy-type Chain systems can be considered as a sub class of Incomplete Hardy-type Chains systems as the former has a one more implication constraint.

The critical probability in this case is identical to the critical probability of the Incomplete case. First we note that,

$$p(A_1 = a_1) = p(A_1 = a_1, A_n = a_n) + p(A_1 = a_1, A_n = \overline{a}_n) \tag{1.5.1}$$

holds irrespective of any implication. The complete chain has the implication $A_n = a_n \Longrightarrow A_1 = \overline{a}_1$ which is equivalent to $p(A_1 = a_1, A_n = a_n) = 0$. Therefore, the the critical probability of the Complete case becomes

$$p(A_1 = a_1) = p(A_1 = a_1, A_n = \overline{a}_n) \tag{1.5.2}$$

which is the critical probability for the Incomplete case.

The above identity holds under the implications of a Complete case. In an Incomplete Hardy-type Chain system, we would not have $p(A_1 = a_1, A_n = a_n) = 0$ as there is no $A_n = a_n \implies A_1 = \overline{a}_1$ implication. Therefore,

$$p(A_1 = a_1, A_n = \overline{a}_n) = p(A_1 = a_1) - p(A_1 = a_1, A_n = a_n) \tag{1.5.3}$$

As $p(A_1 = a_1) - p(A_1 = a_1, A_n = a_n)$ is a non-negative quantity,

$$p(A_1 = a_1, A_n = \overline{a}_n) \leq p(A_1 = a_1) \tag{1.5.4}$$

Predictions of an NCR theory and quantum mechanics are summarized in the table below:

| | Complete | Incomplete |
|---|---|---|
| NCR | $p(A_1 = a_1) = 0$ | $p(A_1 = a_1, A_n = \overline{a}_n) = 0$ |
| QM | $p(A_1 = a_1) > 0$ | $p(A_1 = a_1, A_n = \overline{a}_n) > 0$ |

Table 1.1: Critical probabilities in Complete and Incomplete cases

### 1.6 Probabilities

For the system to satisfy an implication, probability of an event which falsifies this implication should be zero. The event falsifies the implication $I(i) : \big(A_i = a_i \implies A_{i+1} = a_{i+1}\big)$ can be deduced from the truth table (we denote "true" with "T" and "false" with "F"):

| $A_i = a_i$ | $A_{i+1} = a_{i+1}$ | $A_i = a_i \implies A_{i+1} = a_{i+1}$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ |

Table 1.2: Truth table for an implication

As the only event falsifies the implication is the value of $A_i$ being $a_i$ and $A_{i+1}$ being

$\overline{a}_{i+1}$, it can be concluded that,

$$\big(A_i = a_i \Longrightarrow A_{i+1} = a_{i+1}\big) \leftrightarrow p(A_i = a_i, A_{i+1} = \overline{a}_{i+1}) = 0 \tag{1.6.1}$$

Similarly, for $I(n) : \big(A_n = a_n \Longrightarrow A_1 = \overline{a}_1\big)$,

$$\big(A_n = a_n \Longrightarrow A_1 = \overline{a}_1\big) \leftrightarrow p(A_i = a_i, A_1 = a_1) = 0 \tag{1.6.2}$$

Probabilities of having $A_i = a_i$ and $A_{i+1} = a_{i+1}$ can be expanded as,

$$p(A_i = a_i) = p(A_i = a_i, A_{i+1} = a_{i+1}) + \underbrace{p(A_i = a_i, A_{i+1} = \overline{a}_{i+1})}_{0} \tag{1.6.3}$$

$$p(A_{i+1} = a_{i+1}) = p(A_i = a_i, A_{i+1} = a_{i+1}) + p(A_i = \overline{a}_i, A_{i+1} = a_{i+1}) \tag{1.6.4}$$

$p(A_i = a_i, A_{i+1} = \overline{a}_{i+1})$ is 0 due to implication $I(i)$ (where $1 \leq i \leq n-1$). These implication hold both for Complete and Incomplete cases.

When the difference of (1.6.3) and (1.6.4) is taken:

$$p(A_{i+1} = a_{i+1}) - p(A_i = a_i) = p(A_i = \overline{a}_i, A_{i+1} = a_{i+1}) \geq 0 \tag{1.6.5}$$

which implies that

$$p(A_i = a_i) \leq p(A_{i+1} = a_{i+1}), \tag{1.6.6}$$

in other words, the probability $p(A_i = a_i)$ is non-decreasing along the chain for $1 \leq i \leq n-1$. This relation is same for both Complete and Incomplete cases. In the open form:

$$p(A_1 = a_1) \leq p(A_2 = a_2) \leq ... \leq p(A_n = a_n) \tag{1.6.7}$$

The probabilities for $A_n = a_n$ and $A_1 = \overline{a}_1$ can be expanded in a Complete Hardy-type Chain system [where $I(n)$ requires $p(A_1 = a_1, A_n = a_n) = 0$]:

$$p(A_n = a_n) = \underbrace{p(A_1 = a_1, A_n = a_n)}_{0} + p(A_1 = \overline{a}_1, A_n = a_n) \tag{1.6.8}$$

$$p(A_1 = \overline{a}_1) = p(A_1 = \overline{a}_1, A_n = a_n) + p(A_1 = \overline{a}_1, A_n = \overline{a}_n) \tag{1.6.9}$$

Again taking the difference of two equations,

$$p(A_1 = \overline{a}_1) - p(A_n = a_n) = p(A_1 = \overline{a}_1, A_n = \overline{a}_n) \geq 0 \tag{1.6.10}$$

$$\rightarrow p(A_n = a_n) \leq p(A_1 = \overline{a}_1) \tag{1.6.11}$$

Since $p(A_1 = \overline{a}_1) = 1 - p(A_1 = a_1)$,

$$p(A_n = a_n) \leq 1 - p(A_1 = a_1) \tag{1.6.12}$$

Using the recursive relation (1.6.7) and (1.6.7), probability of having $A_i = a_i$ for successive observables has the inequality:

$$p(A_1 = a_1) \leq p(A_2 = a_2) \leq ... \leq p(A_n = a_n) \leq 1 - p(A_1 = a_1) \tag{1.6.13}$$

In Complete Hardy-type chain systems, critical probability $p(A_1 = a_1)$ must be greater than $0$ to demonstrate quantum contextuality. Therefore the above inequality is bounded by $0$ and $1$ from left-hand and right-hand sides.

Hence, for Complete Hardy-type chain systems:

$$0 < p(A_1 = a_1) \leq p(A_2 = a_2) \leq ... \leq p(A_n = a_n) \leq 1 - p(A_1 = a_1) < 1 \tag{1.6.14}$$

## 1.7 Conditions

To construct a Hardy-type Chain System with quantum mechanics, we demand the following conditions to hold. We first list them and then explaing the motivations behind them in the following subsections:

- Compatibility condition: Two observables must be compatible if and only if they are subsequent.

$$[A_i, A_j] = 0 \leftrightarrow j = i \pm 1 \tag{1.7.1}$$

- Eigenstate condition: State vector of the system must not be an eigenstate of any of the observables.
- Implication conditions:

  We denote the projection to the joint eigen-subspace of observable $A_i$ and $A_{i+1}$ corresponding to their eigenvalues $a_i$ and $a_{i+1}$ as $\mathbb{P}_{a_i,a_{i+1}}$.

  For the implication $A_i = a_i \implies A_{i+1} = a_{i+1}$ to hold, the system must satisfy:

$$\mathbb{P}_{a_i,\overline{a}_{i+1}} |\Psi\rangle = 0 \tag{1.7.2}$$

These conditions are identical for both Complete and Incomplete Hardy-type Chains systems. Only difference between them is that, there are one less implication condition which the system requires to satisfy compared to the former case.

### 1.7.1 Compatibility Conditions

Each implication puts a constraint over the values of two subsequent observables. For them to be realizable according to quantum mechanics, subsequent observables must be compatible with each other:

$$[A_i, A_{i+1}] = 0 \tag{1.7.3}$$

All of the observables would simultaneously have values if all observables are compatible with each other. In that case, all the implications are combined also in quantum mechanics as in NCR theory and two theories cannot give different predictions for critical probabilities.

Also, some of the observables becomes uncessary to be defined if some of the non-subsequent observables are compatible.

Let us consider three observables: $A_i, A_{i+1}$ and $A_{i+2}$. All of them becomes mutually compatible if non-subsequent observables $A_i$ and $A_{i+2}$ are compatible and the following two implication can be replaced by the one on the right-hand side:

$$\left.\begin{array}{l} A_i = a_i \Longrightarrow A_{i+1} = a_{i+1} \\ A_{i+1} = a_{i+1} \Longrightarrow A_{i+2} = a_{i+2} \end{array}\right\} A_i = a_i \Longrightarrow A_{i+2} = a_{i+2} \tag{1.7.4}$$

Hence, the observable $A_{i+1}$ becomes obsolete.

We demand none of the non-subsequent observables to be compatible in order for a minimum number of observables to be defined.

### 1.7.2 Eigenstate Condition

Repeating the result derived in the Probabilities section (1.6.14),

$$0 < p(A_1 = a_1) \leq p(A_2 = a_2) \leq ... \leq p(A_n = a_n) \leq 1 - p(A_1 = a_1) < 1 \tag{1.7.5}$$

This inequality requires that,

$$0 < p(A_i = a_i) < 1 \tag{1.7.6}$$

This requirement cannot be satisfied if the state vector $|\Psi\rangle$ is an eigenstate of any of the observables $A_i$ with either eigenvalue $a_i$ or $\overline{a}_i$:

$$A_i |\Psi\rangle = a_i |\Psi\rangle \to p(A_i = a_i) = 1 \tag{1.7.7}$$

$$A_i |\Psi\rangle = \overline{a}_i |\Psi\rangle \to p(A_i = a_i) = 0 \tag{1.7.8}$$

### 1.7.3 Implication Conditions

We will denote the eigen-subspace of observable $A_i$ with the eigenvalue $a_i$ as $V_{a_i}$ and the projection onto that subspaces as $\mathbb{P}_{a_i}$. Also, we will denote the joint eigen-subspace of observables $A_i$ and $A_{i+1}$ corresponding to their $a_i$ and $a_{i+1}$ eigenvalues as $V_{a_i,a_{i+1}}$ and the projections onto that joint subspace as $\mathbb{P}_{a_i,a_{i+1}}$.

Considering two subsequent observables $A_i$ and $A_{i+1}$, the full Hilbert Space can be decomposed in terms of the eigen-subspaces corresponding to the eigenvalues of the two observable:

$$V_{full} = V_{a_i,a_{i+1}} + V_{a_i,\overline{a}_{i+1}} + V_{\overline{a}_i,a_{i+1}} + V_{\overline{a}_i,\overline{a}_{i+1}} \tag{1.7.9}$$

Sum of the projections onto these eigen-subspaces is the identity:

$$\mathbb{P}_{a_i,a_{i+1}} + \mathbb{P}_{a_i,\overline{a}_{i+1}} + \mathbb{P}_{\overline{a}_i,a_{i+1}} + \mathbb{P}_{\overline{a}_i,\overline{a}_{i+1}} = \mathbb{1} \tag{1.7.10}$$

Equivalence of implication $I(i) : \big(A_i = a_i \implies A_{i+1} = a_{i+1}\big)$ is:

$$p(A_i = a_i, A_{i+1} = \overline{a}_{i+1}) = \left|\mathbb{P}_{V_{a_i,\overline{a}_{i+1}}} |\Psi\rangle\right|^2 = 0 \tag{1.7.11}$$

The implication is satisfied either when the eigen-subspace being empty (i.e., $V_{a_i,\overline{a}_{i+1}} = \{0\}$, which is equivalent to $\mathbb{P}_{V_{a_i,\overline{a}_{i+1}}} = 0$) or the state vector $|\Psi\rangle$ being orthogonal to that subspace.

Once that implication holds, there will be limitations on other kind of implications to be defined:

- $A_i = a_i \rightarrow A_{i+1} = \overline{a}_{i+1}$:

  This implication requires that $\mathbb{P}_{V_{a_i,a_{i+1}}} |\Psi\rangle = 0$. Combining it with the requirement of the original implication (1.7.11),

$$|\Psi\rangle = \big( \underbrace{\mathbb{P}_{a_i,a_{i+1}}}_{0} + \underbrace{\mathbb{P}_{a_i,\overline{a}_{i+1}}}_{0} + \mathbb{P}_{\overline{a}_i,a_{i+1}} + \mathbb{P}_{\overline{a}_i,\overline{a}_{i+1}} \big) |\Psi\rangle \tag{1.7.12}$$

  Since the both projections on the last row are projections to the eigen-subspace corresponding to the $a_{i+1}$ eigenvalue of $A_{i+1}$, it can be re-stated as:

$$|\Psi\rangle = \mathbb{P}_{\overline{a}} |\Psi\rangle \tag{1.7.13}$$

  Thus, the state vector is an eigenstate of $A_i$ with eigenvalue $\overline{a}$ and it violates the eigenstate condition. Hence, the implication cannot hold.

- $A_i = \overline{a}_i \rightarrow A_{i+1} = a_{i+1}$ implication requires that $\mathbb{P}_{V_{\overline{a}_i,\overline{a}_{i+1}}} |\Psi\rangle = 0$. Similar to what we have done for the previous implication, combining it with the requirement of the original implication (1.7.11),

$$|\Psi\rangle = \mathbb{1} |\Psi\rangle = \big( \mathbb{P}_{V_{a_i,a_{i+1}}} + \underbrace{\mathbb{P}_{V_{a_i,\overline{a}_{i+1}}}}_{0} + \mathbb{P}_{V_{\overline{a}_i,a_{i+1}}} + \underbrace{\mathbb{P}_{V_{\overline{a}_i,\overline{a}_{i+1}}}}_{0} \big) |\Psi\rangle \tag{1.7.14}$$

$$\rightarrow |\Psi\rangle = \big( \mathbb{P}_{V_{a_i,a_{i+1}}} + \mathbb{P}_{V_{\overline{a}_i,a_{i+1}}} \big) |\Psi\rangle \tag{1.7.15}$$

  As both projections on the last row are projections to the eigen-subspace corresponding to the $a_{i+1}$ eigenvalue of $A_{i+1}$,

$$|\Psi\rangle = \mathbb{P}_{a_{i+1}} |\Psi\rangle \tag{1.7.16}$$

  Thus, the state vector is an eigenstate of $A_{i+1}$ with eigenvalue $a_{i+1}$ and it too violates the eigenstate condition. Hence, this implication also cannot hold.

- $A_i = \overline{a}_i \implies A_{i+1} = \overline{a}_{i+1}$ implication remains as the only other possible implication

To sum up, if a system satifies the implication

$$A_i = a_i \implies A_{i+1} = a_{i+1} \tag{1.7.17}$$

Then, only other possible implication that system can satisfy is:

$$A_i = \overline{a}_i \Longrightarrow A_{i+1} = \overline{a}_{i+1} \tag{1.7.18}$$

It means that there is a two-sided implication:

$$A_i = a_i \Longleftrightarrow A_i = a_{i+1} \tag{1.7.19}$$

## 1.8 Limitations

### 1.8.1 Critical Probability

In Complete Hardy-type Chain, due to the inequality (1.6.14),

$$p(A_1 = a_1) \leq 1 - p(A_1 = a_1) \tag{1.8.1}$$

$$\rightarrow 2\, p(A_1 = a_1) \leq 1 \tag{1.8.2}$$

From that, we can conclude that the critical probability in a Complete Hardy-type Chain cannot exceed $\frac{1}{2}$:

$$p(A_1 = a_1) \leq \frac{1}{2} \tag{1.8.3}$$

For the Incomplete Hardy-type Chain, using the inequality (1.6.7) which is common for both the Complete and Incomplete cases and the inequality (1.5.4) which holds in the Incomplete case:

$$p(A_1 = a_1, A_n = \overline{a}_n) \leq p(A_1 = a_1) = p(A_n = a_n) = 1 - p(A_n = \overline{a}_n) \tag{1.8.4}$$

Expanding $p(A_n = \overline{a}_n)$ as $\left[p(A_1 = a_1, A_n = \overline{a}_n) + p(A_1 = \overline{a}_1, A_n = \overline{a}_n)\right]$,

$$p(A_1 = a_1, A_n = \overline{a}_n) \leq 1 - p(A_1 = a_1, A_n = \overline{a}_n) - p(A_1 = \overline{a}_1, A_n = \overline{a}_n) \tag{1.8.5}$$

$$\rightarrow 2\, p(A_1 = a_1, A_n = \overline{a}_n) \leq 1 - p(A_1 = \overline{a}_1, A_n = \overline{a}_n) \tag{1.8.6}$$

$$\rightarrow p(A_1 = a_1, A_n = \overline{a}_n) \leq \frac{1}{2} - \frac{1}{2}\, p(A_1 = \overline{a}_1, A_n = \overline{a}_n) \tag{1.8.7}$$

Therefore, the critical probability in an Incomplete Hardy-type Chain system also cannot exceed $\frac{1}{2}$:

$$p(A_1 = a_1, A_n = \overline{a}_n) \leq \frac{1}{2} \tag{1.8.8}$$

### 1.8.2 Number of Observables

For an NCR theory and quantum mechanics to give different predictions, observables should not be all compatible. It is trivial that a Hardy-type Chain system cannot be constructed with 1 or 2 observables. It also cannot be constructed with 3 observables as compatibility condition demands the subsequent observables to be compatible. But all observables become mutually compatible as all of them are subsequent to each other.

Therefore, minimum number of observables required to be used is 4 to construct a Hardy-type chain system.

# CHAPTER 2

# 3-DIMENSIONS

In this chapter, we will restate the method of Cabello and his colleagues [6] for 3-Dimensional systems in terms of Hardy-Type Chains.

## 2.1 Conditions in 3-Dimensional Hilbert Space

In terms of degeneracy, there is only one type of dichotomic observables in 3 - Dimensional Hilbert Space: single-degenerate observables where one eigenvalue has no degeneracy and the other has 2-fold degeneracy. We will take the non-degenerate eigenvalue as $+1$ (or simply as $+$) and the 2-fold degenerate eigenvalue as $-1$ (or simply as $-$) as a convention

We will denote the eigenvector corresponding to the $+$ eigenvalue of observable $A_i$ as $|i\rangle$. These vectors should be distinct in order for observables to be distinct. We will also denote the projection to the $\pm$ eigen-subspace of $A_i$ as $\mathbb{P}^i_\pm$.

Projections to $+$ eigen-subspace of $A_i$ is $|i\rangle\langle i|$ and the observable can be expressed as:

$$A_i = 2\mathbb{P}^i_+ - \mathbb{1} = 2\,|i\rangle\langle i| - \mathbb{1} \tag{2.1.1}$$

With all observables being singly-degenerate, conditions for constructing Hardy-Type Chains can be manifested as the following:

- Compatibility Condition:

  Compatibility of two observables is equivalent to the orthogonality of their eigen-vectors:

$$[A_i, A_j] = 0 \leftrightarrow \langle i|j\rangle = 0 \tag{2.1.2}$$

Then, compatbility condition of observables can be re-stated in terms of their eigen-vectors: Two eigen-vectors must be orthogonal if and only if they are subsequent:

$$\langle i|j\rangle = 0 \leftrightarrow j = i \pm 1 \tag{2.1.3}$$

- Eigenstate Condition:

  For the state vector not being an eigenstate of an observable $A_i$ with eigenvalue $+$ and $-$, it should not be equal or orthogonal to $|i\rangle$:

$$|\Psi\rangle \neq |i\rangle \tag{2.1.4}$$

$$|\Psi\rangle \not\perp |i\rangle \tag{2.1.5}$$

  More compactly:

$$|\langle\Psi|i\rangle| \neq 0, 1 \tag{2.1.6}$$

- Implication Conditions:

  We denote the the joint eigen-subspace of $A_i$ and $A_{i+1}$ corresponding to $+$ eigenvalues for both observables as $\mathbb{P}^{i,i+1}_{++}$. This projection can be expressed as the product of the individual projections to the eigen-subspaces of $A_i$ and $A_{i+1}$:

$$\mathbb{P}^{i,i+1}_{++} = \mathbb{P}^{i}_{+}\mathbb{P}^{i+1}_{+} = |i\rangle \underbrace{\langle i|i+1\rangle}_{0} \langle i+1| = 0 \tag{2.1.7}$$

  The projection is zero due to the orthogonality condition and the implication condition for implication $A_i = + \implies A_{i+1} = -$ is automatically satisfied for all observables independently of the state vector (It can be deduced by replacing $a_i$ with $+$ and $a_{i+1}$ with $-$ in (1.7.17) and (1.7.18) ):

$$\mathbb{P}^{i,i+1}_{++}|\Psi\rangle = 0 \tag{2.1.8}$$

$$\rightarrow p(A_i = +, A_{i+1} = +) = 0 \tag{2.1.9}$$

$$\rightarrow \big(A_i = + \rightarrow A_{i+1} = -\big) \tag{2.1.10}$$

Using the result derived previously (1.7.18), only other possible implication that can be defined without violating the Eigenstate Condition is:

$$A_i = - \implies A_{i+1} = + \tag{2.1.11}$$

With the requirement of $\mathbb{P}^{i,i+1}_{--} |\Psi\rangle = 0$

Unlike the previous one, it cannot be satisfied independently of the state vector as the projection in this case cannot vanish:

$$\mathbb{P}^{i,i+1}_{--} = (\mathbb{1} - \mathbb{P}^{i}_{+})(\mathbb{1} - \mathbb{P}^{i+1}_{+}) \tag{2.1.12}$$

$$= \mathbb{1} - \mathbb{P}^{i}_{+} - \mathbb{P}^{i+1}_{+} + \underbrace{\mathbb{P}^{i}_{+}\mathbb{P}^{i+1}_{+}}_{0} \neq 0 \tag{2.1.13}$$

As $\mathbb{P}^{i}_{+} + \mathbb{P}^{i+1}_{+} = |i\rangle \langle i| + |i+1\rangle \langle i+1| \neq \mathbb{1}$. Therefore, the condition must be satisfied state-dependently:

$$\mathbb{P}^{i,i+1}_{--} |\Psi\rangle = \left(\mathbb{1} - \mathbb{P}^{i}_{+} - \mathbb{P}^{i+1}_{+}\right) |\Psi\rangle = 0 \tag{2.1.14}$$

$$\to |\Psi\rangle = \left(\mathbb{P}^{i}_{+} + \mathbb{P}^{i+1}_{+}\right) |\Psi\rangle \tag{2.1.15}$$

$$\to |\Psi\rangle = \Big( |i\rangle \langle i| + |i+1\rangle \langle i+1| \Big) |\Psi\rangle \tag{2.1.16}$$

The condition is equivalent of $|\Psi\rangle$ lying within the 2D subspace spanned by $|i\rangle$ and $|i+1\rangle$:

$$|\Psi\rangle \in \text{ span}(|i\rangle, |i+1\rangle) \tag{2.1.17}$$

## 2.2 4-Observable System

As we have concluded previously, the Hardy-Type Chains require at least 4 observables. In this section, we will show that they also cannot be constructed with 4 observables in 3-dimensions due to the compatibility conditions.

When we have four observables

$$\{A_1, A_2, A_3, A_4\}$$

with their $+$ eigenvectors

$$\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$$

Compatibility conditions requires subsequent vectors to be orthogonal. Hence, $|1\rangle$ must be orthogonal to both $|2\rangle$ and $|4\rangle$:

$$\langle 1|2\rangle = \langle 1|4\rangle = 0 \tag{2.2.1}$$

Also, $|3\rangle$ must be orthogonal to both $|2\rangle$ and $|4\rangle$:

$$\langle 3|2\rangle = \langle 3|4\rangle = 0 \tag{2.2.2}$$

But in 3-dimensions, a vector which is orthogonal to two vectors is unique (up to a phase constant) and $|1\rangle$ and $|3\rangle$ cannot be chosen as independent vectors (as the vectors are 3-dimensional, we can treat them analogous to geometric vectors and use cross products to express them):

$$|1\rangle = e^{i\theta_1} |3\rangle = c_1 |2\rangle \times |4\rangle \quad \text{where } \theta_1 \text{ and } c_1 \text{ are constants} \tag{2.2.3}$$

Similarly, $|2\rangle$ and $|4\rangle$ also would be identical up to a phase constant:

$$|2\rangle = e^{i\theta_2} |4\rangle = c_2 |1\rangle \times |3\rangle \quad \text{where } \theta_2 \text{ and } c_2 \text{ are constants} \tag{2.2.4}$$

Therefore, it is not possible to define 4 different observables in 3-dimensions in accordance with the compatibility conditions.

## 2.3 5-Observable System

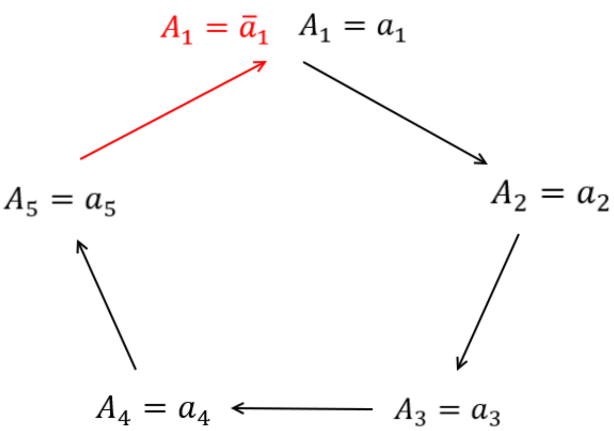


Figure 2.1: Representation of Complete Hardy Type Chain with 5 observables in 3D

As shown by Cabello and his colleagues, it is possible to construct Hardy-Type Chain systems in 3-dimensional Hilbert Space with a minimum number of 5 observables. We will quote the system they have designed [6] (and also experimentally verified [7]). Although they have not made this categorization, the system they have designed falls into the Complete Hardy-Type Chain category that we have proposed.

The state-vector they have defined is:

$$|\Psi\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{2.3.1}$$

And the $+$ eigenvectors of the observables are:

$$|1\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \ |2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \ |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ |4\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ |5\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \tag{2.3.2}$$

As all the subsequent vectors are orthogonal, $A_i = + \rightarrow A_{i+1} = -$ implication holds for each observables. Also, $A_2 - A_3$ and $A_4 - A_5$ pairs satisfies the implication conditions:

$$|\Psi\rangle \in \text{ span}(|2\rangle \, , |3\rangle) \tag{2.3.3}$$

$$|\Psi\rangle \in \text{ span}(|4\rangle \, , |5\rangle) \tag{2.3.4}$$

Therefore, we have the following implication constraints for the system:

$$A_1 = + \Longrightarrow A_2 = - \tag{2.3.5}$$

$$A_2 = - \Longrightarrow A_3 = + \tag{2.3.6}$$

$$A_3 = + \Longrightarrow A_4 = - \tag{2.3.7}$$

$$A_4 = - \Longrightarrow A_5 = + \tag{2.3.8}$$

$$A_5 = + \Longrightarrow A_1 = - \tag{2.3.9}$$

The critical probability is:

$$p(A_1 = +) = |\langle \Psi | \phi_1 \rangle|^2 = \frac{1}{9} \tag{2.3.10}$$

## 2.4 n-Observable Complete Hardy-Type Chain System

In this section, we will generalize this method and construct a Complete Hardy-Type Chain system with $n$ number of observables in order to maximize the critical probability. We again follow the method of Cabello and his colleagues [6] with a slightly different approach.

As concluded previously, there are two types of possible type of implications and they must follow each other by alternating:

$$A_i = + \Longrightarrow A_{i+1} = - \quad \text{which is automatically satisfied due to compatibility} \tag{2.4.1}$$

$$A_i = - \Longrightarrow A_{i+1} = + \quad \text{which requires state-dependent condition (2.1.17)} \tag{2.4.2}$$

An odd number of implications are needed in order for the last implication to imply the opposite of the initial value of $A_1$. We set the total number of observables as $n = 2m + 1$ where $m$ is an integer ($m \geq 2$). Also, we start with the first type of the implications above. By that, there will be $m + 1$ number of implications of the first type and $m$ type of the latter:

$$I(1) : \big(A_1 = + \Longrightarrow A_2 = -\big) \tag{2.4.3}$$

$$I(2) : \big(A_2 = - \Longrightarrow A_3 = +\big) \tag{2.4.4}$$

$$\dots$$

$$I(2m) : \big(A_{2m} = - \Longrightarrow A_{2m+1} = +\big) \tag{2.4.5}$$

$$I(2m+1) : \big(A_{2m+1} = + \Longrightarrow A_1 = -\big) \tag{2.4.6}$$

For even-numbered implications to hold, even-numbered and the subsequent odd-numbered vectors must satisfy (2.1.17):

$$|\Psi\rangle \in \ \text{span}(|2i\rangle \,, |2i+1\rangle) \tag{2.4.7}$$

We set the state-vector as:

$$|\Psi\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{2.4.8}$$

Then, we set the first vector using the general expression for 3-D geometrical vectors using spherical polar coordinates:

$$|1\rangle = \begin{pmatrix} \sin\theta_0 \cos\phi_0 \\ \sin\theta_0 \sin\phi_0 \\ \cos\theta_0 \end{pmatrix} \tag{2.4.9}$$

Then, we set the even-numbered vectors as:

$$|2i\rangle = \begin{pmatrix} \sin\theta_i \cos\phi_i \\ \sin\theta_i \sin\phi_i \\ \cos\theta_i \end{pmatrix} \tag{2.4.10}$$

Subsequent odd-numbered vector must be orthogonal to it. For (2.1.17) to be satisfied, it must be chosen such that the state vector lies in the plane spanned by the even-numbered vector and the subsequent odd-numbered vector. Two conditions simultaneously can hold only if the odd-vector is generated from the even-vector by transforming the coordinates as $\theta_i \to \frac{\pi}{2} - \theta_i$ and $\phi_i \to \pi + \phi_i$

$$|2i+1\rangle = \begin{pmatrix} \sin(\frac{\pi}{2} - \theta_i) \cos(\pi + \phi_i) \\ \sin(\frac{\pi}{2} - \theta_i) \sin(\pi + \phi_i) \\ \cos(\frac{\pi}{2} - \theta_i) \end{pmatrix} = \begin{pmatrix} -\cos\theta_i \sin\phi_i \\ -\cos\theta_i \sin\phi_i \\ \sin\theta_i \end{pmatrix} \tag{2.4.11}$$

Parameters must be defined in a way to make the rest of the subsequent vectors orthogonal. First, even-vectors should be orthogonal to the previous odd-vectors:

$$\langle 2i+1|2(i+1)\rangle = \begin{pmatrix} -\cos\theta_i \sin\phi_i & -\cos\theta_i \sin\phi_i & \sin\theta_i \end{pmatrix} \begin{pmatrix} \sin(\theta_{i+1}) \cos(\phi_{i+1}) \\ \sin(\theta_{i+1}) \sin(\phi_{i+1}) \\ \cos(\theta_{i+1}) \end{pmatrix} \tag{2.4.12}$$

$$= -\cos\theta_i \sin\theta_{i+1} \underbrace{(\cos\phi_i \cos\phi_{i+1} + \sin\phi_i \sin\phi_{i+1})}_{\cos(\phi_{i+1} - \phi_i)} + \sin\theta_i \cos\theta_{i+1} \tag{2.4.13}$$

$$\to \tan\theta_{i+1} \cos(\phi_{i+1} - \phi_i) = \tan\theta_i \quad \text{for } i > 1 \tag{2.4.14}$$

As $|1\rangle$ has the same form of even-vectors, (2.4.14) also provides that $\langle 2m+1|1\rangle = 0$, therefore hold for $i = m$. But providing the orthogonality of $|1\rangle$ and $|2\rangle$ requires a different relation as the both vectors has the form of even-vectors:

$$\langle 1|2\rangle = \begin{pmatrix}\sin\theta_0\cos\phi_0 & \sin\theta_0\sin\phi_0 & \cos\theta_0\end{pmatrix}\begin{pmatrix}\sin\theta_1\cos\phi_1\\ \sin\theta_1\sin\phi_1\\ \cos\theta_1\end{pmatrix} \tag{2.4.15}$$

$$= \sin\theta_1\sin\theta_0\underbrace{(\cos\phi_1\cos\phi_0 + \sin\phi_1\sin\phi_0)}_{\cos(\phi_1-\phi_0)} + \cos\theta_1\cos\theta_0 = 0 \tag{2.4.16}$$

$$\rightarrow \tan\theta_0\tan\theta_1\cos(\phi_1-\phi_0) = -1 \tag{2.4.17}$$

### 2.4.1 Maximizing the Critical Probability

We will try to determine the parameters $\theta_i$ and $\phi_i$ such that the critical probability is maximum while the relations (2.4.14) and (2.4.17) are satisfied.

$$p(A_1=+) = |\langle 1|\Psi\rangle|^2 = \left|\begin{pmatrix}\sin\theta_0\cos\phi_0 & \sin\theta_0\sin\phi_0 & \cos\theta_0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\right|^2$$

$$= \cos^2\theta_0 = \frac{1}{1+\tan^2\theta_0} \tag{2.4.18}$$

Using the relation (2.4.14) repeatedly, $\tan\theta_0$ can be expressed as:

$$\begin{aligned}\tan\theta_0 &= \frac{\tan\theta_m}{\cos(\phi_0-\phi_m)} = \frac{\tan\theta_{m-1}}{\cos(\phi_0-\phi_m)\cos(\phi_m-\phi_{m-1})}\\ &= \frac{\tan\theta_1}{\cos(\phi_0-\phi_m)\cos(\phi_m-\phi_{m-1})...\cos(\phi_2-\phi_1)}\end{aligned} \tag{2.4.19}$$

Using also (2.4.17),

$$\tan\theta_0 = \frac{-1}{\cos(\phi_0-\phi_m)\cos(\phi_m-\phi_{m-1})...\cos(\phi_2-\phi_1)\cos(\phi_1-\phi_0)\tan\theta_0} \tag{2.4.20}$$

$$\rightarrow \tan^2\theta_0 = \frac{-1}{\prod_i^m\cos(\phi_{i+1}-\phi_i)} \tag{2.4.21}$$

When this expression for $\tan\theta_0$ is inserted in the expression for the critical probability:

$$p(A_1=+) = |\langle 1|\Psi\rangle|^2 = \frac{1}{1+\left(\frac{-1}{\prod_i^m\cos(\phi_{i+1}-\phi_i)}\right)} \tag{2.4.22}$$

To shorten the notation, we will define:

$$\alpha_i = \phi_{i+1} - \phi_i \tag{2.4.23}$$

In this notation,

$$p(A_1 = +) = |\langle 1|\Psi\rangle|^2 = \frac{1}{1 + \left(\frac{-1}{\prod_{i=1}^{m} \cos\alpha_i}\right)} \tag{2.4.24}$$

There are two properties regarding the $\alpha$'s:

- As $\tan^2\theta_0$ is a positive quantity, the product in (2.4.21) must be negative

$$\prod_{i=0}^{m} \cos\alpha_i < 0 \tag{2.4.25}$$

- Since the each $\phi$ term appears once with $+$ sign and once with $-$ sign, sum of the arguments of the cosines is 0 in modulo $2\pi$

$$\sum_{i=0}^{m} (\phi_{i+1} - \phi_i) = \sum_{i=0}^{m} \alpha_i = 2\pi k, \quad \text{where } k \text{ is an integer} \tag{2.4.26}$$

The critical probability has the form:

$$y = \frac{1}{1-x} \tag{2.4.27}$$

Since the absolute value of product of cosines cannot be greater than 1 and it must be negative due to the first property above, the range of $x$ is:

$$-1 \leq x \leq 0 \tag{2.4.28}$$

In this domain, $x$ must get close to $-1$ as much as possible to maximize $y$. Then, we need to minimize $\prod_{i=1}^{m} \cos\alpha_i$. Using the method of Lagrange multiplier to find its minima:

$$\nabla f = \lambda \nabla g \rightarrow \frac{\partial f}{\partial \alpha_j} = \lambda \frac{\partial g}{\partial \alpha_j} \tag{2.4.29}$$

Where $f = \prod_i^n \cos\alpha_i$ and $g = \sum_{i=0}^m \alpha_i - 2\pi k = 0$

Lagrange multiplier method gives:

$$\frac{\partial}{\partial\alpha_j}\prod_{i=1}^{n}\cos\alpha_i = \lambda\frac{\partial}{\partial\alpha_j}\left(\sum_{i=0}^{m}\alpha_i - 2\pi k\right) \tag{2.4.30}$$

$$\rightarrow \underbrace{\sin\alpha_j}_{\tan\alpha_j\cos\alpha_j}\prod_{i=1,i\neq j}^{n}\cos\alpha_i = \lambda \tag{2.4.31}$$

$$\rightarrow \prod_{i}^{n}\cos\alpha_i = \frac{\lambda}{\tan\alpha_j} \tag{2.4.32}$$

It means that $\tan\alpha_i$ is same for all $i$, which means that $\alpha_i$ is identical for all $i$ with $\pm\pi$ difference:

$$\alpha_i = \alpha \pm \pi \tag{2.4.33}$$

Also, product of the cosines must be minimized while keeping it negative (due to (2.4.25)) and sum of the $\alpha_i$'s must be $0$ mod $2\pi$ (due to (2.4.26)).

The solution for $m$ even is:

$$\alpha_i = \alpha = \frac{m\pi}{m+1} \tag{2.4.34}$$

Maximum critical probability for $n = 2m+1$ obversables becomes:

$$p(A_1 = +)_{max} = \frac{1}{1 + \left[\frac{-1}{\cos^{(n+1)}\left(\frac{n\pi}{n+1}\right)}\right]} \tag{2.4.35}$$

Its value approaches to $\frac{1}{2}$, the value we have concluded as the maximum possible value, as $m$ (therefore $n = 2m+1$) approaches infinity:

$$\lim_{m\to\infty} p(A_1 = +)_{max} = \frac{1}{2} \tag{2.4.36}$$

# CHAPTER 3

# 4-DIMENSIONS

In this chapter, we will apply the method of Cabello and his colleagues [6] to 4-dimensional Hilbert Space and try to construct Hardy-Type Chain systems.

Unlike the case of 3-dimensional Hilbert Space, there are two kind of dichotomic observables in 4-dimensions, in terms of degeneracy: single-degenerate and double-degenerate.

In single-degenerate observables (we will denote them as $S$), one eigenvalue is non-degenerate and the other is 3-fold degenerate. We again set the non-degenerate eigenvalue as $+$ and show the corresponding eigenvector as $|s\rangle$.

In double-degenerate observables (we will denote them as $D$), each eigenvalue has 2-fold degeneracy. We will denote the corresponding eigen-subspaces for these eigenvalues as $V_{D_+}$ and $V_{D_-}$. We will take $+$ as the generic eigenvalue in discussions whenever there is no loss of generality.

## 3.1 Conditions

### 3.1.1 Compatibility

- Two single-degenerate observables:

  Compatibility conditions are same as in the case of 3-dimensions. Let $S$ and $S'$ be two singly-degenerate observables with $+$ eigenvectors $|s\rangle$ and $|s'\rangle$. $S$ and $S'$ are compatible if $|s\rangle$ and $|s'\rangle$ are orthogonal. Therefore the compatibility condition is:

$$\langle s|s'\rangle = 0 \text{ if and only if } S \text{ and } S' \text{ are consecutive} \tag{3.1.1}$$

- A single-degenerate and a double-degenerate observables:

  Let $S$ be the single-degenerate and $D$ be the double-degenerate observables. Two observables are compatible if and only if:

$$|s\rangle \in V_{D_+} \text{ or } |s\rangle \in V_{D_-} \tag{3.1.2}$$

  In terms of projections,

$$\mathbb{P}_{D_+} |s\rangle = |s\rangle \text{ or } \mathbb{P}_{D_-} |s\rangle = |s\rangle \tag{3.1.3}$$

  The two eigen-subspaces are complete and orthogonal:

$$\mathbb{P}_{D_+} + \mathbb{P}_{D_+} = \mathbb{1} \text{ and } \mathbb{P}_{D_+}\mathbb{P}_{D_-} = 0 \tag{3.1.4}$$

  Therefore, if $|s\rangle$ is orthogonal to one of the eigen-subspaces, then it means it is inside the other eigen-subspace:

$$\mathbb{P}_{D_\pm} |s\rangle = 0 \rightarrow \mathbb{P}_{D_\mp} |s\rangle = |s\rangle \tag{3.1.5}$$

  When $S$ and $D$ are compatible, it can also be thought as the two observables have a common orthonormal eigen-basis. If we call this basis $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$, eigenvalue correspondances for these basis would be:

| | $S$ | $D$ |
|---|---|---|
| $\|1\rangle$ | $+$ | $+$ |
| $\|2\rangle$ | $-$ | $+$ |
| $\|3\rangle$ | $-$ | $-$ |
| $\|4\rangle$ | $-$ | $-$ |

Table 3.1: Common eigen-basis of a single-degenerate and a double-degenerate observable

- Two double-degenerate observables:

  Let $D$ and $D'$ be two double-degenerate observables. For them to be compatible, there must exist a common orthonormal eigen-basis for them. But basis vectors must correspond to different eigenvalues in each observables if the two observables are distinct. Let $\{|d_1\rangle, |d_2\rangle, |d_3\rangle, |d_4\rangle\}$ be such a basis. We show the eigenvalue correspondance with a table:

| | $D$ | $D'$ |
|---|---|---|
| $\|d_1\rangle$ | $+$ | $+$ |
| $\|d_2\rangle$ | $+$ | $-$ |
| $\|d_3\rangle$ | $-$ | $+$ |
| $\|d_4\rangle$ | $-$ | $-$ |

Table 3.2: Common eigen-basis of two double-degenerate observables

$D$ and $D'$ would be compatible if and only if such a common basis exists.

### 3.1.2 Eigenstate

- Single-Degenerate Observables:

  Similar to the case of 3-dimensions, for the state-vector $|\Psi\rangle$ to not be an eigenstate of a single-degenerate observable $S$, it must not be same or ortogonal to $|s\rangle$:

$$\langle\Psi|s\rangle \neq \{0, 1\} \tag{3.1.6}$$

- Double-Degenerate Observables:

  For the state-vector $|\Psi\rangle$ to not be an eigen-state of a double-degenerate observable $D$:

$$\mathbb{P}_{D_\pm} |\Psi\rangle \neq |\Psi\rangle \tag{3.1.7}$$

  The condition can be re-stated in terms of one of the projections:

$$\mathbb{P}_{D_+} |\Psi\rangle \neq \{0, 1\} \tag{3.1.8}$$

### 3.1.3 Implications

The possible implications and required conditions are:

- $\big(S = + \implies S' = -\big)$: Similar to the 3-dimensional case, this implication is automatically satisfied due to compatibility of the observables.
- $\big(S = - \implies S' = +\big)$: Similar to the 3-dimensional case, it is the only other possible implication once two consecutive single-degenerate observables are compatible and satisfies the above implication. It requires that,

$$\Psi \in \text{span}(|s\rangle\,, |s'\rangle) \tag{3.1.9}$$

- $\big(S = + \implies D = +\big)$: As $|s\rangle \in V_{D_+}$ due to compatability condition, this condition is automatically satisfied.
- $\big(S = - \implies D = +\big)$ and $\big(D = \pm \implies S = \pm\big)$: Considering the common eigen-basis as in table (3.1), these conditions require that the state-vector being orthogonal to the vector corresponding to $-$ eigenvalue of $S$ and $+$ eigenvalue of $D$:

$$\langle \Psi | 2 \rangle = 0 \tag{3.1.10}$$

- $\big(D = \alpha \implies D' = \beta\big)$: Considering a common eigen-basis for $D$ and $D'$ as in table (3.2),

$$\langle \Psi | d_1 \rangle = 0 \leftrightarrow \big(D = + \implies D' = -\big) \tag{3.1.11}$$

$$\langle \Psi | d_2 \rangle = 0 \leftrightarrow \big(D = + \implies D' = +\big) \tag{3.1.12}$$

$$\langle \Psi | d_3 \rangle = 0 \leftrightarrow \big(D = - \implies D' = -\big) \tag{3.1.13}$$

$$\langle \Psi | d_4 \rangle = 0 \leftrightarrow \big(D = - \implies D' = +\big) \tag{3.1.14}$$

## 3.2 Configurations

In this section, we will investigate the possibility of constructing Hardy-Type Chain systems with 4 observables in different combinations of single and double degenerate observables and show that only eligible configuration is the one where all the observables are double degenerate.

### 3.2.1 SSSS

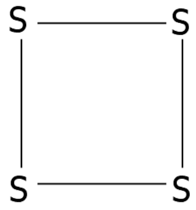


Figure 3.1: Hardy-Type Chain with 4 single-degenerate observables

Let us call the single-degenerate observables $\{A_1, A_2, A_3, A_4\}$ and their eigen-vectors with $+$ eigen-values as $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$. Similar to the 3-dimensional case, following implication holds due to the compatibility conditions regardless of what the state vector is:

$$A_i = + \Longrightarrow A_{i+1} = - \tag{3.2.1}$$

Using the result derived earlier (1.7.17) (1.7.18), only other possible implication between the subsequent observables is:

$$A_i = - \Longrightarrow A_{i+1} = + \tag{3.2.2}$$

Again similar to the 3-dimensional case, it is not possible to construct a Complete Hardy-Type Chain system with 4 single-degenerate observables as the value of the observables alternates in each implication and odd number of observables needed in order for the last implication to imply the negative of the inital value of $A_1$.

In the case of Incomplete Hardy-Type Chain case, implication chain can start with either $A_1 = +$ or $A_1 = -$.

When it is started with $A_1 = +$, the implication will be:

$$A_1 = + \Longrightarrow A_2 = - \tag{3.2.3}$$

$$A_2 = - \Longrightarrow A_3 = + \tag{3.2.4}$$

$$A_3 = + \Longrightarrow A_4 = - \tag{3.2.5}$$

When these are combined, NCR Theory leads to the prediction:

$$\big(A_1 = + \Longrightarrow A_4 = -\big) \leftrightarrow p(A_1 = +, A_4 = +) = 0 \tag{3.2.6}$$

But it is not possible to design a system with quantum mechanics which $p(A_1 = +, A_4 = +) > 0$. Thus, an Incomplete Hardy-Type Chain cannot be constructed in this case.

When the chain starts with $A_1 = -$, the implications will be:

$$A_1 = - \implies A_2 = + \tag{3.2.7}$$

$$A_2 = + \implies A_3 = - \tag{3.2.8}$$

$$A_3 = - \implies A_4 = + \tag{3.2.9}$$

The first and the third implication requires a condition regarding the state vector (2.1.17):

$$|\Psi\rangle \in \text{span}(|1\rangle, |2\rangle) \tag{3.2.10}$$

$$|\Psi\rangle \in \text{span}(|3\rangle, |4\rangle) \tag{3.2.11}$$

It requires that the intersection of the two span spaces not being empty:

$$\text{span}(|1\rangle, |2\rangle) \cap \text{span}(|3\rangle, |4\rangle) \neq 0 \tag{3.2.12}$$

It means that, for arbitrary coefficients $x, y, z$ and $w$, there should be a relation as:

$$x|1\rangle + y|2\rangle = z|3\rangle + w|4\rangle \tag{3.2.13}$$

As in the section where 4-Observable systems in 3-dimensions discussed, compatibility conditions require subsequent vectors being orthogonal and inner products of the non-subsequent vectors can be taken as real.

When both sides of (3.2.13) multiplied with $\langle 1|$:

$$x\underbrace{\langle 1|1\rangle}_{1} + y\underbrace{\langle 1|2\rangle}_{0} = z\langle 1|3\rangle + w\underbrace{\langle 1|4\rangle}_{0} \tag{3.2.14}$$

$$\rightarrow x = z\langle 1|3\rangle \tag{3.2.15}$$

And when both sides multiplied with $\langle 3|$:

$$x\langle 3|1\rangle + y\underbrace{\langle 3|2\rangle}_{0} = z\underbrace{\langle 3|3\rangle}_{1} + w\underbrace{\langle 3|4\rangle}_{0} \tag{3.2.16}$$

$$\rightarrow x \langle 3|1 \rangle = z \tag{3.2.17}$$

Combining (3.2.15) and (3.2.17),

$$\langle 1|3 \rangle = \langle 3|1 \rangle = \frac{x}{z} = \frac{z}{x} \tag{3.2.18}$$

Above equation hold only when $\langle 1|3 \rangle = \pm 1$. But it it would mean that $A_1 = \pm A_3$ (Similarly, it can be shown that $\langle 2|4 \rangle = \pm 1$ by multiplying (3.2.13) with $\langle 2|$ and $\langle 4|$ combining their result).

Therefore, it is not possible to define 4 distinct observables which satisfies both the compatibility and the implication conditions.

### 3.2.2 SSSD

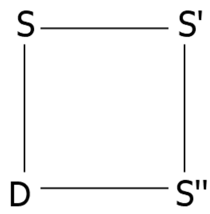


Figure 3.2: Hardy-Type Chain with 3 single-degenerate and a double-degenerate observables

Let us call the single-degenerate observables as $\{S, S', S''\}$ and their $+$ eigen-vectors as $\{|s\rangle, |s'\rangle, |s''\rangle\}$ and the double-degenerate observables as $D$.

For $S$ and to $D$ to be compatible:

$$|s\rangle \in V_{D_+} \tag{3.2.19}$$

As $S$ and $S''$ must not be compatible,

$$\langle s|s'' \rangle \neq 0 \tag{3.2.20}$$

For $S''$ and to $D$ to be compatible, $|s''\rangle$ should be inside one of the eigen-subspaces of $D$. But it cannot be inside $V_{D_-}$ as it is orthogonal to $V_{D_+}$ and being inside $V_{D_-}$ makes

$|s''\rangle$ orthogonal to $|s\rangle$. So the two vectors must be inside the same eigen-subspace:

$$|s\rangle \in V_{D_+} \tag{3.2.21}$$

As $|s\rangle$ and $|s''\rangle$ are two distinct vectors inside the subspace $V_{D_+}$, they span the subspace:

$$V_{D_+} = \text{span}(|s\rangle, |s''\rangle) \tag{3.2.22}$$

For $S'$ to be compatible with both $S$ and $S''$,

$$\langle s'|s\rangle = \langle s'|s''\rangle = 0 \tag{3.2.23}$$

But since $|s'\rangle$ is orthogonal to both $|s\rangle$ and $|s''\rangle$, it is also orthogonal to the subspace $V_{D_+}$. Hence, it means it is inside $V_{D_-}$

$$|s'\rangle \perp \text{span}(|s\rangle, |s''\rangle) = V_{D_+} \rightarrow |s'\rangle \in V_{D_-} \tag{3.2.24}$$

It makes $S'$ compatible to the non-subsquent $D$ and violates the compatibility condition.

### 3.2.3 SDSD

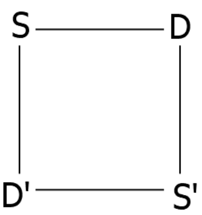


Figure 3.3: Hardy-Type Chain with 2 single-degenerate and 2 double-degenerate observables distributed evenly

Similar to the previous case, $S$ and $S'$ must be inside the same eigen-subspace of $D$ in order for not being compatible:

$$|s\rangle \in V_{D_+} \text{ and } |s'\rangle \in V_{D_+} \tag{3.2.25}$$

Again, like the previous case,

$$V_{D_+} = \text{span}(|s\rangle, |s'\rangle) \tag{3.2.26}$$

Similarly, $|s\rangle$ and $|s'\rangle$ must also be inside the same eigen-subspace of $D'$:

$$|s\rangle \in V_{D'_\gamma} \text{ and } |s'\rangle \in V_{D_\gamma} \tag{3.2.27}$$

$$\rightarrow V_{D'_\gamma} = \text{span}(|s\rangle, |s'\rangle) \tag{3.2.28}$$

It makes the eigen-subspaces of $D$ and $D'$ either identical or orthogonal:

$$\gamma = + \rightarrow V_{D_+} = V_{D'_+} \rightarrow D = D' \tag{3.2.29}$$

$$\gamma = - \rightarrow V_{D_+} = V_{D'_-} \rightarrow D = -D' \tag{3.2.30}$$

In both cases, observables $D$ and $D'$ would be identical (up to multiplicative constant) and two distinct double-degenerate observables cannot be defined while keeping the compatibility conditions.

### 3.2.4 SSDD

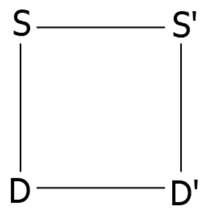


Figure 3.4: Hardy-Type Chain with 2 single-degenerate and 2 double-degenerate observables distributed unevenly

For $D$ and $D'$ to be compatible, there must exists a common orthonormal eigen-basis with the corresponding eigenvalues as:

| | $D$ | $D'$ |
|---|---|---|
| $\|1\rangle$ | $+$ | $+$ |
| $\|2\rangle$ | $+$ | $-$ |
| $\|3\rangle$ | $-$ | $+$ |
| $\|4\rangle$ | $-$ | $-$ |

Table 3.3: Common eigen-basis of two double-degenerate observables

For $S$ to be compatible with $D$ and for $S'$ to be compatible with $D'$ (eigenvalue of the both eigen-subspaces can be taken as $+$ without loss of generality):

$$|s\rangle \in V_{D_+} = \text{span}(|1\rangle, |2\rangle) \tag{3.2.31}$$

$$|s'\rangle \in V_{D'_+} = \text{span}(|1\rangle, |3\rangle) \tag{3.2.32}$$

$|s\rangle$ and $|s'\rangle$ can be expressed as a linear combination of the vectors which spans the subspaces they are in where $a$, $b$, $c$ and $d$ are constants:

$$|s\rangle = a|1\rangle + b|2\rangle \tag{3.2.33}$$

$$|s\rangle = c|1\rangle + d|3\rangle \tag{3.2.34}$$

For $S$ and $S'$ to be compatible,

$$\langle s|s'\rangle = a^*c\langle 1|1\rangle + a^*d\langle 1|3\rangle + b^*c\langle 2|1\rangle + b^*d\langle 2|3\rangle = 0 \tag{3.2.35}$$

Due to orthonormality of the basis vectors:

$$\langle s|s'\rangle = a^*c = 0 \tag{3.2.36}$$

In order for the equality to hold, either $a$ or $c$ should be 0.

$$a = 0 \rightarrow |s\rangle = |2\rangle \rightarrow |s\rangle \in V_{D'_+} \tag{3.2.37}$$

$$c = 0 \rightarrow |s'\rangle = |3\rangle \rightarrow |s'\rangle \in V_{D_+} \tag{3.2.38}$$

In both cases, one of the single-degenerate observables becomes compatible with the non-subsequent double-degenerate observables and hence, the compatibility condition is violated.

### 3.2.5 SDDD

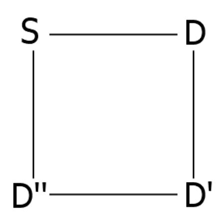


Figure 3.5: Hardy-Type Chain with 3 double-degenerate observables and a single-degenerate observable

For $D$ to be compatible with $D'$ and for $D'$ to be compatible $D''$, there must exists a two common orthonormal eigen-basis for the two pairs:

| | $D$ | $D'$ | | $D'$ | $D''$ |
|---|---|---|---|---|---|
| $\|1\rangle$ | $+$ | $+$ | $\|1'\rangle$ | $+$ | $+$ |
| $\|2\rangle$ | $+$ | $-$ | $\|2'\rangle$ | $+$ | $-$ |
| $\|3\rangle$ | $-$ | $+$ | $\|3'\rangle$ | $-$ | $+$ |
| $\|4\rangle$ | $-$ | $-$ | $\|4'\rangle$ | $-$ | $-$ |

Table 3.4: Common eigen-basis sets for two pairs of double-degenerate observables

Since the $|1\rangle$ and $|3\rangle$ pair and the $|1'\rangle$ and $|2'\rangle$ pair both span the $+$ eigen-space of D', their span spaces must be equal to each other:

$$\text{span}(|1\rangle, |3\rangle) = \text{span}(|1'\rangle, |2'\rangle) \tag{3.2.39}$$

Due to this span relation, $|1'\rangle$ and $|2'\rangle$ can be expressed in terms of $|1\rangle$ and $|3\rangle$. But we can also express $|3\rangle$ and $|2'\rangle$ in terms of $|1\rangle$ and $|1'\rangle$. If we define $\langle 1|1'\rangle = a$ and considering the orthogonality of $|3\rangle$ to $|1\rangle$ and $|2'\rangle$ to $|1'\rangle$

$$|3\rangle = a\,|1\rangle - |1'\rangle \tag{3.2.40}$$

$$|2'\rangle = |1\rangle - a\,|1'\rangle \tag{3.2.41}$$

With these values inserted, common basis sets become:

| | $D$ | $D'$ | | $D'$ | $D''$ |
|---|---|---|---|---|---|
| $\|1\rangle$ | $+$ | $+$ | $\|1'\rangle$ | $+$ | $+$ |
| $\|2\rangle$ | $+$ | $-$ | $\|1\rangle - a\,\|1'\rangle$ | $+$ | $-$ |
| $a\,\|1\rangle - \|1'\rangle$ | $-$ | $+$ | $\|3'\rangle$ | $-$ | $+$ |
| $\|4\rangle$ | $-$ | $-$ | $\|4'\rangle$ | $-$ | $-$ |

Table 3.5: Common eigen-basis sets for two pairs of double-degenerate observables

Due to the orthogonality relations of the remaining vectors in each basis set:

$$\langle 2|3\rangle = 0 = \langle 2|\big(a\,|1\rangle - |1'\rangle\big) = a\underbrace{\langle 2|1\rangle}_{0} + \langle 2|1'\rangle \tag{3.2.42}$$

$$\rightarrow \langle 2|1'\rangle = 0 \tag{3.2.43}$$

$$\langle 3'|2'\rangle = 0 = \langle 3'|\big(|1\rangle - a\,|1'\rangle\big) = \langle 3'|1\rangle + a\underbrace{\langle 3'|1'\rangle}_{0} \tag{3.2.44}$$

$$\rightarrow \langle 3'|1\rangle = 0 \tag{3.2.45}$$

For $S$ to be compatible with $D$ and $D''$, it must be inside one of the eigen-subspaces of $D$ and $D''$ (They can be set as the $+$ eigenvalued subspaces without loss of generality):

$$|s\rangle \in V_{D_+} = \text{span}(|1\rangle\,,|2\rangle) \tag{3.2.46}$$

$$|s\rangle \in V_{D'_+} = \text{span}(|1'\rangle\,,|3'\rangle) \tag{3.2.47}$$

$|s\rangle$ can be expressed as the linear combination of the vectors in the spans:

$$|s\rangle = x\,|1\rangle + y\,|2\rangle \tag{3.2.48}$$

$$|s\rangle = z\,|1'\rangle + w\,|3'\rangle \tag{3.2.49}$$

$$\rightarrow x\,|1\rangle + y\,|2\rangle = z\,|1'\rangle + w\,|3'\rangle \tag{3.2.50}$$

When each side of (3.2.50) is multiplied by $\langle 1|$,

$$x\underbrace{\langle 1|1\rangle}_{1} + y\underbrace{\langle 1|2\rangle}_{0} = z\underbrace{\langle 1|1'\rangle}_{a} + w\underbrace{\langle 1|3'\rangle}_{0} \rightarrow x = za \tag{3.2.51}$$

And when each side is multiplied by $\langle 1'|$,

$$x\underbrace{\langle 1'|1\rangle}_{a} + y\underbrace{\langle 1'|2\rangle}_{0} = z\underbrace{\langle 1'|1'\rangle}_{1} + w\underbrace{\langle 1'|3'\rangle}_{0} \rightarrow xa = z \tag{3.2.52}$$

Combining (3.2.51) and (3.2.52):

$$a = \frac{z}{x} = \frac{x}{z} \rightarrow a = \pm 1 \tag{3.2.53}$$

The value of $a = \langle 1|1'\rangle$ being $\pm 1$ makes it impossible for two distinct eigen-basis and makes it impossible for defining 3 distinct double-degenerate observables satisfying compatibility conditions.

## 3.3 System with 4 Double-Degenerate Observables

We will denote the observables as $\{A_1, A_2, A_3, A_4\}$. For consecutive observables to be compatible, there must exist four sets of common orthonormal eigen-basis for the pair of consecutive observables as:

| | $A_1$ | $A_2$ | | $A_2$ | $A_3$ | | $A_3$ | $A_4$ | | $A_4$ | $A_1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\|a_1\rangle$ | $+$ | $+$ | $\|b_1\rangle$ | $+$ | $+$ | $\|c_1\rangle$ | $+$ | $+$ | $\|d_1\rangle$ | $+$ | $+$ |
| $\|a_2\rangle$ | $+$ | $-$ | $\|b_2\rangle$ | $+$ | $-$ | $\|c_2\rangle$ | $+$ | $-$ | $\|d_2\rangle$ | $+$ | $-$ |
| $\|a_3\rangle$ | $-$ | $+$ | $\|b_3\rangle$ | $-$ | $+$ | $\|c_3\rangle$ | $-$ | $+$ | $\|d_3\rangle$ | $-$ | $+$ |
| $\|a_4\rangle$ | $-$ | $-$ | $\|b_4\rangle$ | $-$ | $-$ | $\|c_4\rangle$ | $-$ | $-$ | $\|d_4\rangle$ | $-$ | $-$ |

Table 3.6: Common eigen-basis sets for 4 pairs of double-degenerate observables

Following set of vectors span the same eigen-subspaces. So, their spans should be equal:

$$V_{(A_2=+)} = \text{span}(|a_1\rangle, |a_3\rangle) = \text{span}(|b_1\rangle, |b_2\rangle) \tag{3.3.1}$$

$$V_{(A_3=+)} = \text{span}(|b_1\rangle, |b_3\rangle) = \text{span}(|c_1\rangle, |c_2\rangle) \tag{3.3.2}$$

$$V_{(A_4=+)} = \text{span}(|c_1\rangle, |c_3\rangle) = \text{span}(|d_1\rangle, |d_2\rangle) \tag{3.3.3}$$

$$V_{(A_1=+)} = \text{span}(|d_1\rangle, |d_3\rangle) = \text{span}(|a_1\rangle, |a_2\rangle) \tag{3.3.4}$$

Using this span and ortogonality relations, all vectors can be defined in terms of the first vectors in each set and their inner products. We change the notation of these vectors to simplfy the notation:

$$|a_1\rangle \equiv |1\rangle \tag{3.3.5}$$

$$|b_1\rangle \equiv |2\rangle \tag{3.3.6}$$

$$|c_1\rangle \equiv |3\rangle \tag{3.3.7}$$

$$|d_1\rangle \equiv |4\rangle \tag{3.3.8}$$

We define their inner products as:

$$\langle 1|2\rangle = a \tag{3.3.9}$$

$$\langle 2|3\rangle = b \tag{3.3.10}$$

$$\langle 3|4\rangle = c \tag{3.3.11}$$

$$\langle 4|1\rangle = d \tag{3.3.12}$$

Values of $a$, $b$ and $c$ can be taken as real and positive as $|1\rangle$, $|2\rangle$ and $|3\rangle$ can be multiplied by a phase constant to make $a$, $b$ and $c$ real and positive. $d$ cannot be taken as real-positive by multiplying $|4\rangle$ by a phase constant since it could make $a$ negative or complex. But it will turn out that $d$ is also real as once the other inner products taken as real due to the orthogonality relations.

Using the span relations and the orthogonality between the first vector and the other vectors in each set, the other vectors can be expressed in terms of $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$ and their inner products (Last vectors are omitted as they can be identified uniquely once the other 3 vectors are determined. Normalization factors are also omitted to keep the tables simple):

| | $A_1$ | $A_2$ | | $A_2$ | $A_3$ | | $A_3$ | $A_4$ | | $A_4$ | $A_1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\|1\rangle$ | + | + | $\|2\rangle$ | + | + | $\|3\rangle$ | + | + | $\|4\rangle$ | + | + |
| $\|4\rangle - d^* \|1\rangle$ | + | − | $\|1\rangle - a\|2\rangle$ | + | − | $\|2\rangle - b\|3\rangle$ | + | − | $\|3\rangle - c\|4\rangle$ | + | − |
| $a\|1\rangle - \|2\rangle$ | − | + | $b\|2\rangle - \|3\rangle$ | − | + | $c\|3\rangle - \|4\rangle$ | − | + | $d\|4\rangle - \|1\rangle$ | − | + |

Table 3.7: Common eigen-basis sets for 4 pairs of double-degenerate observables

From the orthogonality between the second and the third vectors of each set:

- 

$$\big(\langle 4| - d^* \langle 1|\big)\big|\big(a|1\rangle - |2\rangle\big) = 0 \tag{3.3.13}$$

$$\rightarrow a \underbrace{\langle 4|1\rangle}_{d} - \langle 4|2\rangle - ad^* \underbrace{\langle 1|1\rangle}_{1} + d^* \underbrace{\langle 1|2\rangle}_{a} = 0 \tag{3.3.14}$$

$$\rightarrow \langle 4|2\rangle = ad \rightarrow \langle 2|4\rangle = ad^* \tag{3.3.15}$$

- $$\big(\langle 1| - a\langle 2|\big)|\big(b|2\rangle - |3\rangle\big) = 0 \tag{3.3.16}$$
  $$\rightarrow b\underbrace{\langle 1|2\rangle}_{a} - \langle 1|3\rangle - ab\underbrace{\langle 2|2\rangle}_{1} + a\underbrace{\langle 2|3\rangle}_{b} = 0 \tag{3.3.17}$$
  $$\rightarrow \langle 1|3\rangle = ab \tag{3.3.18}$$

- $$\big(\langle 2| - b\langle 3|\big)|\big(c|3\rangle - |4\rangle\big) = 0 \tag{3.3.19}$$
  $$\rightarrow c\underbrace{\langle 2|3\rangle}_{b} - \langle 2|4\rangle - bc\underbrace{\langle 3|3\rangle}_{1} + b\underbrace{\langle 3|4\rangle}_{c} = 0 \tag{3.3.20}$$
  $$\rightarrow \langle 2|4\rangle = cb \tag{3.3.21}$$

- $$\big(\langle 3| - c\langle 4|\big)|\big(d|4\rangle - |1\rangle\big) = 0 \tag{3.3.22}$$
  $$\rightarrow d\underbrace{\langle 3|4\rangle}_{c} - \langle 3|1\rangle - cd\underbrace{\langle 4|4\rangle}_{1} + c\underbrace{\langle 4|1\rangle}_{d} = 0 \tag{3.3.23}$$
  $$\rightarrow \langle 3|1\rangle = bd \rightarrow \langle 1|3\rangle = bd^* \tag{3.3.24}$$

From (3.3.18) and (3.3.24),

$$\langle 1|3\rangle = ab = cd^* \rightarrow \frac{a}{c} = \frac{d^*}{b} \tag{3.3.25}$$

And from (3.3.15) and (3.3.21),

$$\langle 2|4\rangle = ad^* = cb \rightarrow \frac{a}{c} = \frac{b}{d^*} \tag{3.3.26}$$

Only solution to satisfy the two equations is $a = c$ and $b = d$ (Also it means that $d$ is real since it is equal to $b$). Plugging these values in and adding the normalization factors to the table (3.7), the final form of the set of basis vectors are (again the fourth vectors are omitted) :

| | $A_1$ | $A_2$ | | $A_2$ | $A_3$ | | $A_3$ | $A_4$ | | $A_4$ | $A_1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\|1\rangle$ | $+$ | $+$ | $\|2\rangle$ | $+$ | $+$ | $\|3\rangle$ | $+$ | $+$ | $\|4\rangle$ | $+$ | $+$ |
| $\frac{\|4\rangle - b\|1\rangle}{\sqrt{1-b^2}}$ | $+$ | $-$ | $\frac{\|1\rangle - a\|2\rangle}{\sqrt{1-a^2}}$ | $+$ | $-$ | $\frac{\|2\rangle - b\|3\rangle}{\sqrt{1-b^2}}$ | $+$ | $-$ | $\frac{\|3\rangle - a\|4\rangle}{\sqrt{1-a^2}}$ | $+$ | $-$ |
| $\frac{a\|1\rangle - \|2\rangle}{\sqrt{1-a^2}}$ | $-$ | $+$ | $\frac{b\|2\rangle - \|3\rangle}{\sqrt{1-b^2}}$ | $-$ | $+$ | $\frac{a\|3\rangle - \|4\rangle}{\sqrt{1-a^2}}$ | $-$ | $+$ | $\frac{b\|4\rangle - \|1\rangle}{\sqrt{1-b^2}}$ | $-$ | $+$ |

Table 3.8: Common eigen-basis sets for 4 pairs of double-degenerate observables

In summary, given any two real parameters $a$ and $b$ such that $0 < a < 1$ and $0 < b < 1$; we can find four double-degenerate dichotomic observables having common eigen-basis sets described on table (3.8)

### 3.3.1 Complete Hardy-Type Chain

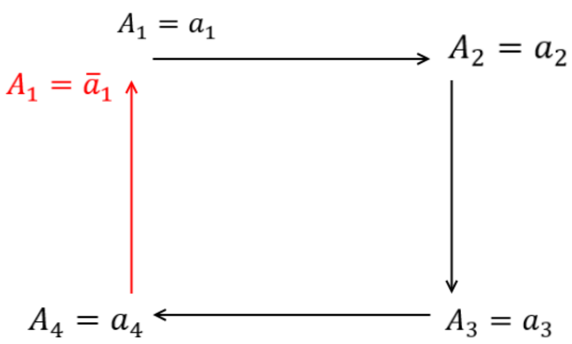


Figure 3.6: Complete Hardy-Type Chain with 4 observables

In this subsection, we will show that a system with 4 double-degenerate observables are not eligible for Complete Hardy Chain case.

In Complete Hardy-Type Chain system, the implications and the condition for them are:

$$\big(A_1=+\Longrightarrow A_2=+\big)\leftrightarrow \langle\Psi|\,\big(\,|4\rangle-b\,|1\rangle\,\big)=0 \tag{3.3.27}$$

$$\big(A_2=+\Longrightarrow A_3=+\big)\leftrightarrow \langle\Psi|\,\big(\,|1\rangle-a\,|2\rangle\,\big)=0 \tag{3.3.28}$$

$$\big(A_3=+\Longrightarrow A_4=+\big)\leftrightarrow \langle\Psi|\,\big(\,|2\rangle-b\,|3\rangle\,\big)=0 \tag{3.3.29}$$

$$\big(A_4=+\Longrightarrow A_1=-\big)\leftrightarrow \langle\Psi|4\rangle=0 \tag{3.3.30}$$

In order for the state-vector being orthogonal to all of these 4 vectors, these 4 vectors should be linearly dependent and not span the whole 4-dimensional Hilbert Space. To investigate this, we will calculate their Gram matrix. We re-define the vectors for ease of notation:

$$|1'\rangle=|4\rangle-b\,|1\rangle \tag{3.3.31}$$

$$|2'\rangle=|1\rangle-a\,|2\rangle \tag{3.3.32}$$

$$|3'\rangle=|2\rangle-b\,|3\rangle \tag{3.3.33}$$

$$|4'\rangle=|4\rangle \tag{3.3.34}$$

Elements of the Gram matrix are:

$$G_{1'1'}=\big(\,\langle 4|-b\,\langle 1|\,\big)\big(\,|4\rangle-b\,|1\rangle\,\big)=\underbrace{\langle 4|4\rangle}_{1}-b\underbrace{\langle 4|1\rangle}_{b}-b\underbrace{\langle 1|4\rangle}_{b}+b^2\underbrace{\langle 1|1\rangle}_{1}=1-b^2 \tag{3.3.35}$$

$$G_{2'2'}=\big(\,\langle 1|-a\,\langle 2|\,\big)\big(\,|1\rangle-a\,|2\rangle\,\big)=\underbrace{\langle 1|1\rangle}_{1}-a\underbrace{\langle 1|2\rangle}_{a}-a\underbrace{\langle 2|1\rangle}_{a}+a^2\underbrace{\langle 2|2\rangle}_{1}=1-a^2 \tag{3.3.36}$$

$$G_{3'3'}=\big(\,\langle 2|-b\,\langle 3|\,\big)\big(\,|2\rangle-b\,|3\rangle\,\big)=\underbrace{\langle 2|2\rangle}_{1}-b\underbrace{\langle 2|3\rangle}_{b}-b\underbrace{\langle 3|2\rangle}_{b}+b^2\underbrace{\langle 3|3\rangle}_{1}=1-b^2 \tag{3.3.37}$$

$$G_{4'4'}=\langle 4|4\rangle=1 \tag{3.3.38}$$

$$G_{1'2'}=G_{2'1'}=\big(\,\langle 4|-b\,\langle 1|\,\big)\big(\,|1\rangle-a\,|2\rangle\,\big) \tag{3.3.39}$$

$$=\underbrace{\langle 4|1\rangle}_{b}-a\underbrace{\langle 4|2\rangle}_{ab}-b\underbrace{\langle 1|1\rangle}_{1}+ab\underbrace{\langle 1|2\rangle}_{a}=0 \tag{3.3.40}$$

$$G_{1'3'} = G_{3'1'} = \big( \langle 4| - b\,\langle 1| \big)\big( |2\rangle - b\,|3\rangle \big) \tag{3.3.41}$$

$$= \underbrace{\langle 4|2\rangle}_{ab} - b\,\underbrace{\langle 4|3\rangle}_{a} - b\,\underbrace{\langle 1|2\rangle}_{a} + b^2\,\underbrace{\langle 1|3\rangle}_{ab} = -ab(1-b^2) \tag{3.3.42}$$

$$G_{1'4'} = G_{4'1'} = \big( \langle 4| - b\,\langle 1| \big)\,|4\rangle = \underbrace{\langle 4|4\rangle}_{1} - b\,\underbrace{\langle 1|4\rangle}_{b} = 1 - b^2 \tag{3.3.43}$$

$$G_{2'3'} = G_{3'2'} = \big( \langle 1| - a\,\langle 2| \big)\big( |2\rangle - b\,|3\rangle \big) \tag{3.3.44}$$

$$= \underbrace{\langle 1|2\rangle}_{a} - b\,\underbrace{\langle 1|3\rangle}_{ab} - a\,\underbrace{\langle 2|2\rangle}_{1} + ab\,\underbrace{\langle 2|3\rangle}_{b} = 0 \tag{3.3.45}$$

$$G_{2'4'} = G_{4'2'} = \big( \langle 1| - a\,\langle 2| \big)\,|4\rangle = \underbrace{\langle 1|4\rangle}_{b} - a\,\underbrace{\langle 2|4\rangle}_{ab} = b(1-a^2) \tag{3.3.46}$$

$$G_{3'4'} = G_{4'3'} = \big( \langle 2| - b\,\langle 3| \big)\,|4\rangle = \underbrace{\langle 2|4\rangle}_{ab} - b\,\underbrace{\langle 3|4\rangle}_{a} = 0 \tag{3.3.47}$$

With these elements,

$$G = \begin{pmatrix} 1-b^2 & 0 & -ab(1-b^2) & 1-b^2 \\ 0 & 1-a^2 & 0 & b(1-a^2) \\ -ab(1-b^2) & 0 & 1-b^2 & 0 \\ 1-b^2 & b(1-a^2) & 0 & 1 \end{pmatrix} \tag{3.3.48}$$

For the vectors $\{|1'\rangle, |2'\rangle, |3'\rangle, |4'\rangle\}$ being linearly dependent, determinant of the Gram matrix should be 0 [8]:

$$\det(G) = a^2(1-a^2)^2\, b^4(1-b^2)^2 = 0 \tag{3.3.49}$$

As $a$ and $b$ are inner products of two normalized vectors, their value is less than 1. For the determinant to vanish, $a$ or $b$ should be 0. Each case leads to the basis vectors to have the form (now we also include the fourth vector in each set):

- When a=0:

| | $A_1$ | $A_2$ | | $A_2$ | $A_3$ | | $A_3$ | $A_4$ | | $A_4$ | $A_1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\|1\rangle$ | $+$ | $+$ | $\|2\rangle$ | $+$ | $+$ | $\|3\rangle$ | $+$ | $+$ | $\|4\rangle$ | $+$ | $+$ |
| $\|4\rangle - b\,\|1\rangle$ | $+$ | $-$ | $\|1\rangle$ | $+$ | $-$ | $\|2\rangle - b\,\|3\rangle$ | $+$ | $-$ | $\|3\rangle$ | $+$ | $-$ |
| $\|2\rangle$ | $-$ | $+$ | $b\,\|2\rangle - \|3\rangle$ | $-$ | $+$ | $\|4\rangle$ | $-$ | $+$ | $b\,\|4\rangle - \|1\rangle$ | $-$ | $+$ |
| $b\,\|2\rangle - \|3\rangle$ | $-$ | $-$ | $\|4\rangle - b\,\|1\rangle$ | $-$ | $-$ | $\|4\rangle - b\,\|1\rangle$ | $-$ | $-$ | $\|2\rangle - b\,\|3\rangle$ | $-$ | $-$ |

Table 3.9: Common eigen-basis sets for 4 pairs of double-degenerate observables when $a = 0$

- When b=0:

| | $A_1$ | $A_2$ | | $A_2$ | $A_3$ | | $A_3$ | $A_4$ | | $A_4$ | $A_1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\|1\rangle$ | $+$ | $+$ | $\|2\rangle$ | $+$ | $+$ | $\|3\rangle$ | $+$ | $+$ | $\|4\rangle$ | $+$ | $+$ |
| $\|4\rangle$ | $+$ | $-$ | $\|1\rangle - a\,\|2\rangle$ | $+$ | $-$ | $\|2\rangle$ | $+$ | $-$ | $\|3\rangle - a\,\|4\rangle$ | $+$ | $-$ |
| $a\,\|1\rangle - \|2\rangle$ | $-$ | $+$ | $\|3\rangle$ | $-$ | $+$ | $a\,\|3\rangle - \|4\rangle$ | $-$ | $+$ | $\|1\rangle$ | $-$ | $+$ |
| $\|3\rangle - a\,\|4\rangle$ | $-$ | $-$ | $a\,\|3\rangle - \|4\rangle$ | $-$ | $-$ | $\|1\rangle - a\,\|2\rangle$ | $-$ | $-$ | $a\,\|1\rangle - \|2\rangle$ | $-$ | $-$ |

Table 3.10: Common eigen-basis sets for 4 pairs of double-degenerate observables when $b = 0$

In each case, vectors in two sets are identical with the vectors in the two other sets. It means that there exists a common eigen-basis for non-subsequent observables. Hence, a Complete Hardy-Type Chain cannot be constructed due to compatibility conditions.

### 3.3.2 Incomplete Hardy-Type Chain

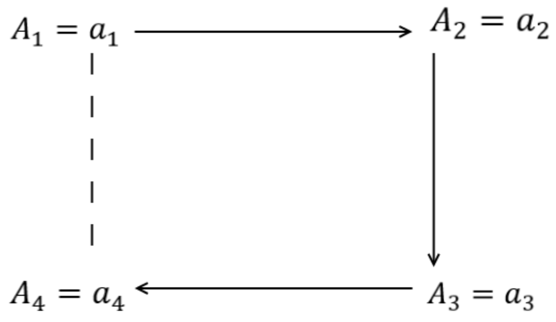


Figure 3.7: Incomplete Hardy-Type Chain with 4 observables

When the last implication is removed, the remaining implications and their requirements are:

$$
\begin{aligned}
\big(A_1 = + \Longrightarrow A_2 = +\big) &\leftrightarrow \langle\Psi| \left( |4\rangle - b\,|1\rangle \right) = 0 \\
\big(A_2 = + \Longrightarrow A_3 = +\big) &\leftrightarrow \langle\Psi| \left( |1\rangle - a\,|2\rangle \right) = 0 \\
\big(A_3 = + \Longrightarrow A_4 = +\big) &\leftrightarrow \langle\Psi| \left( |2\rangle - b\,|3\rangle \right) = 0
\end{aligned} \tag{3.3.50}
$$

An NCR Theory would lead to the prediction:

$$
\big(A_1 = + \Longrightarrow A_4 = +\big) \leftrightarrow p(A_1 = +, A_4 = -) = 0 \tag{3.3.51}
$$

The critical probability, according to quantum mechanics:

$$
p(A_1 = +, A_4 = -) = \left| \langle\Psi| \left( \frac{b\,|4\rangle - |1\rangle}{\sqrt{1-b^2}} \right) \right|^2 \tag{3.3.52}
$$

#### 3.3.2.1 Expression of the Basis Vectors

To find the expressions of the vectors $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$ in terms of standard basis, we calculate the Gram matrix, whose elements are $G_{ij} = \langle i|j\rangle$:

$$G = \begin{pmatrix} 1 & a & ab & b \\ a & 1 & b & ab \\ ab & b & 1 & a \\ b & ab & a & 1 \end{pmatrix} \tag{3.3.53}$$

It can be diagonalized in the form of $G = U^{-1}\Lambda U$ where,

$$U = U^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \tag{3.3.54}$$

$$\Lambda = \begin{pmatrix} (1-a)(1+b) & 0 & 0 & 0 \\ 0 & (1+a)(1-b) & 0 & 0 \\ 0 & 0 & (1-a)(1-b) & 0 \\ 0 & 0 & 0 & (1+a)(1+b) \end{pmatrix} \tag{3.3.55}$$

Its square root can be taken in the form of $\sqrt{G} = U^{-1}\sqrt{\Lambda}U$ where,

$$\sqrt{\Lambda} = \begin{pmatrix} \sqrt{(1-a)(1+b)} & 0 & 0 & 0 \\ 0 & \sqrt{(1+a)(1-b)} & 0 & 0 \\ 0 & 0 & \sqrt{(1-a)(1-b)} & 0 \\ 0 & 0 & 0 & \sqrt{(1+a)(1+b)} \end{pmatrix} \tag{3.3.56}$$

From the columns of $U^{-1}\sqrt{\Lambda}$, expressions of the vectors are obtained as:

$$|1\rangle = \frac{1}{2}\begin{pmatrix} \sqrt{(1-a)(1+b)} \\ -\sqrt{(1+a)(1-b)} \\ -\sqrt{(1-a)(1-b)} \\ \sqrt{(1+a)(1+b)} \end{pmatrix}, \quad |2\rangle = \frac{1}{2}\begin{pmatrix} -\sqrt{(1-a)(1+b)} \\ -\sqrt{(1+a)(1-b)} \\ \sqrt{(1-a)(1-b)} \\ \sqrt{(1+a)(1+b)} \end{pmatrix}$$

$$|3\rangle = \frac{1}{2}\begin{pmatrix} -\sqrt{(1-a)(1+b)} \\ \sqrt{(1+a)(1-b)} \\ -\sqrt{(1-a)(1-b)} \\ \sqrt{(1+a)(1+b)} \end{pmatrix}, \quad |4\rangle = \frac{1}{2}\begin{pmatrix} \sqrt{(1-a)(1+b)} \\ \sqrt{(1+a)(1-b)} \\ \sqrt{(1-a)(1-b)} \\ \sqrt{(1+a)(1+b)} \end{pmatrix} \tag{3.3.57}$$

#### 3.3.2.2 Expression of the State Vector

Using the orthogonality relations (3.3.50), inner products between the state vector and the basis vectors $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$ can be computed and they can be used to decompose the state vector in terms of these vectors. But as they do not constitute an orthogonal basis, we first define reciprocal basis (where $G_{ij} = \langle i|j\rangle$):

$$|i'\rangle = \sum_j G^{-1}_{ij} |j\rangle \tag{3.3.58}$$

When its inner product taken with the basis $\{|i\rangle\}$:

$$\langle k|i'\rangle = \sum_j G^{-1}_{ji} \underbrace{\langle k|j\rangle}_{G_{kj}} = \sum_j G_{kj} G^{-1}_{ji} = (\underbrace{GG^{-1}}_{\mathbb{1}})_{ki} = \delta_{ik} \tag{3.3.59}$$

Inner product of the state vector and $\{|i\rangle\}$ basis:

$$\langle k|\Psi\rangle = \langle k| \left( \sum_i c_i |i'\rangle \right) = \langle k| \left( \sum_{ij} c_i G^{-1}_{ji} |j\rangle \right) \tag{3.3.60}$$

$$= \sum_{ij} c_i G^{-1}_{ji} \underbrace{\langle k|j\rangle}_{G_{kj}} = \sum_i c_i \underbrace{\sum_j G_{kj} G_{ji}}_{\delta_{ik}} \tag{3.3.61}$$

$$\rightarrow \langle k|\Psi\rangle = c_k \tag{3.3.62}$$

Inner product between the vectors of $|i'\rangle$ basis:

$$\langle i'|j'\rangle = \sum_k (G^{-1}_{ki})^T \sum_l G^{-1}_{lj} |l\rangle \tag{3.3.63}$$

$$= \sum_{kl} \underbrace{(G^{-1}_{ki})^T}_{G^{-1}_{ik}} G^{-1}_{lj} \underbrace{\langle k|l\rangle}_{G_{kl}} = \sum_k G^{-1}_{ki} \underbrace{\sum_l G_{kl} G^{-1}_{lj}}_{\delta_{kj}} \tag{3.3.64}$$

$$\rightarrow \langle i'|j'\rangle = G^{-1}_{ij} \tag{3.3.65}$$

Where inverse of the Gram matrix is:

$$G^{-1} = \frac{1}{(1-a^2)(1-b^2)} \begin{pmatrix} 1 & -a & ab & -b \\ -a & 1 & -b & ab \\ ab & -b & 1 & -a \\ -b & ab & -a & 1 \end{pmatrix} \tag{3.3.66}$$

Using (3.3.62) and the orthogonality relations (3.3.50),

$$\langle \Psi| \big( |4\rangle - b\,|1\rangle \big) = \langle \Psi|4\rangle - b\,\langle \Psi|1\rangle = 0 \rightarrow c_4 - bc_1 = 0 \tag{3.3.67}$$

$$\langle \Psi| \big( |1\rangle - a\,|2\rangle \big) = \langle \Psi|1\rangle - a\,\langle \Psi|2\rangle = 0 \rightarrow c_1 - ac_2 = 0 \tag{3.3.68}$$

$$\langle \Psi| \big( |2\rangle - b\,|3\rangle \big) = \langle \Psi|2\rangle - b\,\langle \Psi|3\rangle = 0 \rightarrow c_2 - bc_3 = 0 \tag{3.3.69}$$

When the other coefficients are expressed in terms of $c_4$:

$$c_1 = \frac{1}{b}c_4,\ c_2 = \frac{1}{ab}c_4,\ c_3 = \frac{1}{ab^2}c_4 \tag{3.3.70}$$

Then, the state vector becomes:

$$|\Psi\rangle = c_4\left(\frac{1}{b}\,|1'\rangle + \frac{1}{ab}\,|2'\rangle + \frac{1}{ab^2}\,|3'\rangle + |4'\rangle\right) \tag{3.3.71}$$

Using the relations (3.3.65) and the fact that the $|\Psi\rangle$ is normalized to find $c_4$:

$$\langle\Psi|\Psi\rangle = c_4^2\Bigg(\frac{1}{b^2}\langle 1'|1'\rangle + \frac{1}{a^2b^2}\langle 2'|2'\rangle + \frac{1}{a^2b^4}\langle 3'|3'\rangle + \langle 4'|4'\rangle$$
$$+\frac{2}{ab^2}\langle 1'|2'\rangle + \frac{2}{ab^3}\langle 1'|3'\rangle + \frac{2}{b}\langle 1'|4'\rangle$$
$$+\frac{2}{a^2b^3}\langle 2'|3'\rangle + +\frac{2}{ab}\langle 2'|4'\rangle + +\frac{2}{ab^2}\langle 3'|4'\rangle\Bigg) \tag{3.3.72}$$
$$= \frac{c_4^2}{(1-a^2)(1-b^2)}\Bigg[\frac{1}{b^2} + \frac{1}{a^2b^2} + \frac{1}{a^2b^4} + 1$$
$$\frac{2}{ab^2}(-a) + \frac{2}{ab^3}ab + \frac{2}{b}(-b)$$
$$+\frac{2}{a^2b^3}(-b) + \frac{2}{ab}ab + \frac{2}{ab^2}(-a)\Bigg] \tag{3.3.73}$$
$$= \frac{c_4^2}{(1-a^2)(1-b^2)}\left(1 + \frac{1}{a^2b^4} - \frac{1}{b^2} - \frac{1}{a^2b^2}\right) \tag{3.3.74}$$
$$= \frac{c_4^2}{(1-a^2)(1-b^2)}\frac{(1-b^2)(1-a^2b^2)}{a^2b^4} = \frac{c_4^2(1-a^2b^2)}{(1-a^2)a^2b^4} = 1 \tag{3.3.75}$$
$$\rightarrow c_4 = ab^2\sqrt{\frac{1-a^2}{1-a^2b^2}} \tag{3.3.76}$$

With that value inserted, state vector expressed in the reciprocal basis:

$$|\Psi\rangle = ab^2\sqrt{\frac{1-a^2}{1-a^2b^2}}\left(\frac{1}{b}|1'\rangle + \frac{1}{ab}|2'\rangle + \frac{1}{ab^2}|3'\rangle + |4'\rangle\right) \tag{3.3.77}$$

To express it in the standard basis, we use its Gram matrix. Its elements are: $G'_{ij} = \langle i'|j'\rangle = G^{-1}_{ij}$ (3.3.66). It can be diagonalized in the form of $G = U^{-1}\Lambda U$ where,

$$U = U^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \tag{3.3.78}$$

$$\sqrt{\Lambda} = \begin{pmatrix} \frac{1}{(1-a)(1+b)} & 0 & 0 & 0 \\ 0 & \frac{1}{(1+a)(1-b)} & 0 & 0 \\ 0 & 0 & \frac{1}{(1-a)(1-b)} & 0 \\ 0 & 0 & 0 & \frac{1}{(1+a)(1+b)} \end{pmatrix} \tag{3.3.79}$$

Its square root can be taken in the form of $\sqrt{G} = U^{-1}\sqrt{\Lambda}U$ where,

$$\sqrt{\Lambda} = \begin{pmatrix} \frac{1}{\sqrt{(1-a)(1+b)}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{(1+a)(1-b)}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{(1-a)(1-b)}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{(1+a)(1+b)}} \end{pmatrix} \tag{3.3.80}$$

From the columns of $U^{-1}\sqrt{\Lambda}$, reciprocal basis in terms of standard basis can be obtained as:

$$|1'\rangle = \frac{1}{2}\begin{pmatrix} \frac{1}{\sqrt{(1-a)(1+b)}} \\ \frac{-1}{\sqrt{(1+a)(1-b)}} \\ \frac{-1}{\sqrt{(1-a)(1-b)}} \\ \frac{1}{\sqrt{(1+a)(1+b)}} \end{pmatrix}, \quad |2'\rangle = \frac{1}{2}\begin{pmatrix} \frac{-1}{\sqrt{(1-a)(1+b)}} \\ \frac{-1}{\sqrt{(1+a)(1-b)}} \\ \frac{1}{\sqrt{(1-a)(1-b)}} \\ \frac{1}{\sqrt{(1+a)(1+b)}} \end{pmatrix}$$

$$|3'\rangle = \frac{1}{2}\begin{pmatrix} \frac{-1}{\sqrt{(1-a)(1+b)}} \\ \frac{1}{\sqrt{(1+a)(1-b)}} \\ \frac{-1}{\sqrt{(1-a)(1-b)}} \\ \frac{1}{\sqrt{(1+a)(1+b)}} \end{pmatrix}, \quad |4'\rangle = \frac{1}{2}\begin{pmatrix} \frac{1}{\sqrt{(1-a)(1+b)}} \\ \frac{1}{\sqrt{(1+a)(1-b)}} \\ \frac{1}{\sqrt{(1-a)(1-b)}} \\ \frac{1}{\sqrt{(1+a)(1+b)}} \end{pmatrix} \tag{3.3.81}$$

When these expressions inserted in (3.3.77),

$$|\Psi\rangle = \frac{1}{2\sqrt{1-a^2b^2}}\begin{pmatrix} -(1-ab)\sqrt{(1+a)(1+b)} \\ (1-ab)\sqrt{(1-a)(1-b)} \\ -(1+ab)\sqrt{(1+a)(1-b)} \\ (1+ab)\sqrt{(1-a)(1+b)} \end{pmatrix} \tag{3.3.82}$$

#### 3.3.2.3 Maximizing the Critical Probability

Considering the expression of the state vector in reciprocal basis again, the critical probability is:

$$p(A_1 = +, A_4 = -) = \left| \langle\Psi| \left( \frac{b\,|4\rangle - |1\rangle}{\sqrt{1-b^2}} \right) \right|^2 = |bc_4 - c_1|^2 \tag{3.3.83}$$

$$= \frac{1}{1-b^2} \left| bc_4 - \frac{1}{b} c_4 \right|^2 = \frac{1}{1-b^2} \left( \frac{b^2-1}{b} \right)^2 c_4^2 \tag{3.3.84}$$

$$= \frac{1-b^2}{b^2} a^2 b^4 \frac{1-a^2}{1-a^2b^2} \tag{3.3.85}$$

$$\rightarrow p(A_1 = +, A_4 = -) = \frac{a^2b^2(1-a^2)(1-b^2)}{1-a^2b^2} \tag{3.3.86}$$

Numerical solution (obtained using Mathematica) which maximizes the value of the critical probability is:

$$p(A_1 = +, A_4 = -) = \frac{5\sqrt{5}-11}{2} \cong 9\% \tag{3.3.87}$$

For the values of $a$ and $b$,

$$a = b = \sqrt{\frac{\sqrt{5}-1}{2}} \cong 0.79 \tag{3.3.88}$$

# CHAPTER 4

# 5-DIMENSIONS

As it is shown that Complete Hardy-Type Chain system cannot be constructed in 4-Dimensional Hilbert Space with 4 observables, we attempt to achieve it in higher dimensions.

## 4.1 General Structure

We define a state vector $|\Psi\rangle$ and 4 dichotomic observables with $\pm 1$ eigenvalues: $\{A_1, A_2, A_3, A_4\}$. Unlike the previous cases in 3 and 4 Dimensions, we do not specify their degeneracy.

With these observables, we would have the set of implications with their equivalances in terms of probabilities:

$$A_1 = + \Longrightarrow A_2 = + \tag{4.1.1}$$

$$A_2 = + \Longrightarrow A_3 = + \tag{4.1.2}$$

$$A_3 = + \Longrightarrow A_4 = + \tag{4.1.3}$$

$$A_4 = + \Longrightarrow A_1 = - \tag{4.1.4}$$

Due to compatibility conditions, subsequent observables would have a common eigen-subspaces. We will denote the projections onto the eigen-subspace corresponding to the $\alpha$ eigenvalue of the observable $A_i$ and the $\beta$ eigenvalue of the observable $A_j$ as $\mathbb{P}^{ij}_{\alpha\beta}$.

For an implication to hold, projection of the state-vector corresponding the eigen-subspace prohibited by that implication must be 0 (1.7.2). For example, for the implication to $\left(A_i = + \Longrightarrow A_j = +\right)$ to hold, the system must satisfy $\mathbb{P}^{12}_{+-}\,|\Psi\rangle = 0$.

Unlike what we have done in the 3-Dimensions and 4-Dimensions, we will not define eigenvectors or common eigen-basis for the observables. Instead, we decompose the state-vector in terms of its projections onto the eigen-subspaces of the observables (To simplfy the notation, we will rename each projected state vector):

$$|\Psi\rangle = \underbrace{\mathbb{P}^{12}_{++}|\Psi\rangle}_{|a_{++}\rangle} + \underbrace{\mathbb{P}^{12}_{+-}|\Psi\rangle}_{0} + \underbrace{\mathbb{P}^{12}_{-+}|\Psi\rangle}_{|a_{-+}\rangle} + \underbrace{\mathbb{P}^{12}_{--}|\Psi\rangle}_{|a_{--}\rangle} \tag{4.1.5}$$

$$|\Psi\rangle = \underbrace{\mathbb{P}^{23}_{++}|\Psi\rangle}_{|b_{++}\rangle} + \underbrace{\mathbb{P}^{23}_{+-}|\Psi\rangle}_{0} + \underbrace{\mathbb{P}^{23}_{-+}|\Psi\rangle}_{|b_{-+}\rangle} + \underbrace{\mathbb{P}^{23}_{--}|\Psi\rangle}_{|b_{--}\rangle} \tag{4.1.6}$$

$$|\Psi\rangle = \underbrace{\mathbb{P}^{32}_{++}|\Psi\rangle}_{|c_{++}\rangle} + \underbrace{\mathbb{P}^{32}_{+-}|\Psi\rangle}_{0} + \underbrace{\mathbb{P}^{32}_{-+}|\Psi\rangle}_{|c_{-+}\rangle} + \underbrace{\mathbb{P}^{32}_{--}|\Psi\rangle}_{|c_{--}\rangle} \tag{4.1.7}$$

$$|\Psi\rangle = \underbrace{\mathbb{P}^{41}_{++}|\Psi\rangle}_{0} + \underbrace{\mathbb{P}^{41}_{+-}|\Psi\rangle}_{|d_{+-}\rangle} + \underbrace{\mathbb{P}^{41}_{-+}|\Psi\rangle}_{|d_{-+}\rangle} + \underbrace{\mathbb{P}^{41}_{--}|\Psi\rangle}_{|d_{--}\rangle} \tag{4.1.8}$$

In the simplified notation:

$$\begin{aligned}
|\Psi\rangle &= |a_{++}\rangle + |a_{-+}\rangle + |a_{--}\rangle \\
&= |b_{++}\rangle + |b_{-+}\rangle + |b_{--}\rangle \\
&= |c_{++}\rangle + |c_{-+}\rangle + |c_{--}\rangle \\
&= |d_{+-}\rangle + |d_{-+}\rangle + |d_{--}\rangle
\end{aligned} \tag{4.1.9}$$

Projections to an eigen-subspace of observable $A_i$ can be decomposed in terms of the projections to the joint eigen-subspaces of $A_i$ and the previous and next observable:

$$\mathbb{P}^{i}_{+}|\Psi\rangle = \mathbb{P}^{i-1,i}_{-+}|\Psi\rangle + \mathbb{P}^{i-1,i}_{-+}|\Psi\rangle \tag{4.1.10}$$

$$\mathbb{P}^{i}_{+}|\Psi\rangle = \mathbb{P}^{i,i+1}_{++}|\Psi\rangle + \mathbb{P}^{i,i+1}_{+-}|\Psi\rangle \tag{4.1.11}$$

Using this relation and projections of the state-vector to some of the joint eigen-subspaces being 0, relations between the vectors in (4.1.9) become (we again rename some of the vectors for further simplify the notation):

$$\begin{aligned}
\underbrace{|a_{++}\rangle}_{|a\rangle} + |a_{-+}\rangle &= |b_{++}\rangle \\
|a_{--}\rangle &= |b_{-+}\rangle + |b_{--}\rangle \\
\underbrace{|b_{++}\rangle}_{|b\rangle} + |b_{-+}\rangle &= |c_{++}\rangle \\
|b_{--}\rangle &= |c_{-+}\rangle + |c_{--}\rangle \\
\underbrace{|c_{++}\rangle}_{|c\rangle} + |c_{-+}\rangle &= |d_{++}\rangle \\
|c_{--}\rangle &= |d_{-+}\rangle + |d_{--}\rangle \\
|d_{-+}\rangle &= |a_{++}\rangle \\
\underbrace{|d_{+-}\rangle}_{|d\rangle} + |d_{--}\rangle &= |a_{-+}\rangle + |a_{--}\rangle
\end{aligned} \tag{4.1.12}$$

Using the relations in (4.1.9) and in (4.1.12), all the vectors can be expressed in terms of the five vectors $\{|a\rangle, |b\rangle, |c\rangle, |d\rangle, |\Psi\rangle\}$:

$$|a_{++}\rangle = |a\rangle \tag{4.1.13}$$

$$|a_{-+}\rangle = |b\rangle - |a\rangle \tag{4.1.14}$$

$$|a_{--}\rangle = |\Psi\rangle - |b\rangle \tag{4.1.15}$$

$$|b_{++}\rangle = |b\rangle \tag{4.1.16}$$

$$|b_{-+}\rangle = |c\rangle - |b\rangle \tag{4.1.17}$$

$$|b_{--}\rangle = |\Psi\rangle - |c\rangle \tag{4.1.18}$$

$$|c_{++}\rangle = |c\rangle \tag{4.1.19}$$

$$|c_{-+}\rangle = |d\rangle - |c\rangle \tag{4.1.20}$$

$$|c_{--}\rangle = |\Psi\rangle - |d\rangle \tag{4.1.21}$$

$$|d_{+-}\rangle = |d\rangle \tag{4.1.22}$$

$$|d_{-+}\rangle = |a\rangle \tag{4.1.23}$$

$$|d_{--}\rangle = |\Psi\rangle - |a\rangle - |d\rangle \tag{4.1.24}$$

As all the projections of the state-vector to the eigen-subspaces can be represented in terms of 5 five vectors, a Complete Hardy-Type Chain system with 4 observables can be constructed in 5-Dimensions if these 5 vectors can be independently chosen.

We set the norm of the vectors as (we assume the state-vector is normalized):

$$\langle a|a\rangle = a \tag{4.1.25}$$

$$\langle b|b\rangle = b \tag{4.1.26}$$

$$\langle c|c\rangle = c \tag{4.1.27}$$

$$\langle d|d\rangle = d \tag{4.1.28}$$

$$\langle \Psi|\Psi\rangle = 1 \tag{4.1.29}$$

Using the orthogonality relations between the vectors, all the inner products between them except $\langle c|a\rangle$ and $\langle d|b\rangle$ can be expressed in terms of $a$, $b$, $c$ and $d$. We the value of $\langle c|a\rangle$ as $x$ and the value of $\langle d|b\rangle$ as $y$. With these inner products, the Gram matrix can be computed as:

$$G = \begin{pmatrix} \langle a|a\rangle & \langle b|a\rangle & \langle c|a\rangle & \langle d|a\rangle & \langle \Psi|a\rangle \\ \langle a|b\rangle & \langle b|b\rangle & \langle c|b\rangle & \langle d|b\rangle & \langle \Psi|b\rangle \\ \langle a|c\rangle & \langle b|c\rangle & \langle c|c\rangle & \langle d|c\rangle & \langle \Psi|c\rangle \\ \langle a|d\rangle & \langle b|d\rangle & \langle c|d\rangle & \langle d|d\rangle & \langle \Psi|d\rangle \\ \langle a|\Psi\rangle & \langle b|\Psi\rangle & \langle c|\Psi\rangle & \langle d|\Psi\rangle & \langle \Psi|\Psi\rangle \end{pmatrix} = \begin{pmatrix} a & a & x & 0 & a \\ a & b & b & y & b \\ x^* & b & c & c & c \\ 0 & y^* & c & d & d \\ a & b & c & d & 1 \end{pmatrix} \tag{4.1.30}$$

Linearly independent 5 vectors $\{|a\rangle, |b\rangle, |c\rangle, |d\rangle, |\Psi\rangle\}$ exist if the Gram matrix is positive definite.

According to Slyvester's criterion [8] , the matrix $G$ is positive definite if the determinant of the following matrices are greater than 0:

$$\begin{vmatrix} d & d \\ d & 1 \end{vmatrix} > 0 \tag{4.1.31}$$

$$\begin{vmatrix} c & c & c \\ c & d & d \\ c & d & 1 \end{vmatrix} > 0 \tag{4.1.32}$$

$$\begin{vmatrix} b & b & y & b \\ b & c & c & c \\ y^* & c & d & d \\ b & c & d & 1 \end{vmatrix} > 0 \tag{4.1.33}$$

$$\begin{vmatrix} a & a & x & 0 & a \\ a & b & b & y & b \\ x^* & b & c & c & c \\ 0 & y^* & c & d & d \\ a & b & c & d & 1 \end{vmatrix} > 0 \tag{4.1.34}$$

The critical probability is:

$$p(A_1 = +) = |\mathbb{P}^1_+ |\Psi\rangle |^2 = |(\mathbb{P}^{12}_{++} + \mathbb{P}^{12}_{+-}) |\Psi\rangle |^2 \tag{4.1.35}$$

$$= |\underbrace{\mathbb{P}^{12}_{++} |\Psi\rangle}_{|a\rangle} + \underbrace{\mathbb{P}^{12}_{+-} |\Psi\rangle}_{0} |^2 \tag{4.1.36}$$

$$= \langle a|a\rangle = a \tag{4.1.37}$$

## 4.2 Specific case: $x = a$ and $y = b$

We attempt to find solutions for $\{a, b, c, d\}$ which makes $G$ a positive-definite matrix. The Gram matrix in this case:

$$\begin{pmatrix} a & a & a & 0 & a \\ a & b & b & b & b \\ a & b & c & c & c \\ 0 & b & c & d & d \\ a & b & c & d & 1 \end{pmatrix} \tag{4.2.1}$$

The parameters should satisfy the following inequalities in order for the matrix to be positive definite:

$$0 < b < c < d < 1 \tag{4.2.2}$$

$$0 < a < \frac{b(1-d)(d-c)}{b(1-c) + (1-d)(d-c)} \tag{4.2.3}$$

Numerical solution (obtained using Mathematica) which maximizes the value of $a$ while the other variables approach to the limits are:

$$a_{\text{max}} < \frac{1}{9} \text{ as } b \to \frac{1}{3}, c \to \frac{1}{3}, d \to \frac{2}{3},$$

As the value of the critical probability is equal to $a$ (4.1.37), upper bound for the critical probability in this case is:

$$p(A_1 = +) = a < \frac{1}{9} \tag{4.2.4}$$

In summary, we have shown that, given four real parameters $a$, $b$, $c$ and $d$ and two complex parameters $x$ and $y$ such that satisfying the determinant inequalities (4.1.31), (4.1.32), (4.1.33) and (4.1.34); it is possible to define 4 dichotomic observables which can be used to construct Complete Hardy-Type Chain system in 5-Dimensional Hilbert Space. In the specific case of $x = a$ and $y = b$, the problem reduces to finding 4 real parameters $a$, $b$, $c$ and $d$ such that satisfies the inequalities (4.2.2) and (4.2.3). In that case, a Complete Hardy-Type Chain can be constructed with 4 observables which has a maximum critical probability $\frac{1}{9}$.

# CHAPTER 5

# CONCLUSION

In this thesis, we have reformulated Hardy-Type proof of quantum contextuality in terms of implication chains which we have called as Hardy-Type Chains. In the first chapter, we have derived the conditions that a system must satisfy in order for Hardy-Type Chains can be defined on it. We have derived the limitations for such systems: Minimum number of observables required is 4 and maximum critical probability that can be reached is $\frac{1}{2}$. In the following chapters, our aim was to construct systems with using 4 observables and making the critical probability high as possible.

We also defined two subtypes for Hardy-Type Chain systems based on the strictness of the conditions: Complete and Incomplete. Although we have not quantified and compared the 'strength' of the contextuality in these cases (for example, in the framework of Abramsky and Brandenburger [9]), we think that the Complete case represents a more effective demonstration of quantum contextuality and we have tried to construct such systems whenever possible.

In the second chapter, we have reformulated the work of Cabello and his colleagues [6] within our framework and reproduced their results in 3-Dimensional Hilbert Space for systems with 5 observables and $n$ observables.

In the third chapter, we have constructed Incomplete Hardy-Type Chain system with 4 observables by making use of double degenerate observables. We have defined common eigen-basis sets for each consecutive observable pairs and showed that such basis sets satisfying necessary conditions exist. The system we have designed can be compared to the system designed in the original paper of Hardy [4]. But in contrast, our system is not bipartite and its maximum critical probability is $\sim 9\%$ instead of

$\frac{1}{16} = 6.25\%$

We have also concluded that Complete Hardy-Type Chain system cannot be constructed with 4 observables in 4-Dimensions, hence we have attempted to achieve it in 5-Dimensions in the fourth chapter. Instead of defining common eigen-basis sets for observables as we have done in the previous chapters, we have decomposed the state vectors in terms of its components in the eigen-subspaces of consecutive observable pairs. In the end, we have showed that 4 observables can be defined for a specific case of variables.

To sum up, we have presented a demonstration of quantum contextuality with using logical implication chains in various dimensional Hilbert Spaces and by using the minimum number of observables and achieving a maximal critical probability for each case. The systems we have designed in 4-Dimensions and 5-Dimensions constitute simplest possible demonstration of quantum contextuality within this framework.

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