ON THE FINITE PERIMETER SETS OF SUB-LORENTZIAN METRIC SPACES

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

GÜNSELİ ÇETİN

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
PHYSICS

AUGUST 2022

Approval of the thesis:

ON THE FINITE PERIMETER SETS OF SUB-LORENTZIAN METRIC SPACES

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ABSTRACT

ON THE FINITE PERIMETER SETS OF SUB-LORENTZIAN METRIC SPACES

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August 2022, 56 pages

In this thesis, inspired by the recent progress on sub-Riemannian geometry, a method of determining the finite perimeter sets of sub-Lorentzian manifolds independent of the original metric structure is suggested and a possible version of Riesz Representation theorem is discussed.

Keywords: BV Spaces, sub-Lorentzian metric, Carnot-Carathéodory distance, finite perimeter set

ÖZ

ALT LORENTZ METRİK UZAYLARINDA SONLU ÇEVREYE SAHİP KÜMELER ÜZERİNE

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Ağustos 2022 , 56 sayfa

Bu tezde, alt Riemann metrik uzayları üzerindeki çalışmalardan esinlenilerek, alt Lorentz metrik uzaylarında sonlu çevresi olan kümelerin metrikten bağımsız olarak belirlenmesi üzerine bir yöntem önerildi ve Riesz temsiliyet teoreminin olası bir alt Lorentz eşleniğinin varlığı tartışıldı.

Anahtar Kelimeler: BV Uzayları, sub-Lorentzian metrik, Carnot-Carathéodory norm, ölçülebilir çevre

ACKNOWLEDGMENTS

I would like to thank my supervisor Prof. Dr. Bahtiyar Özgür Sarıoğlu for his support and guidance throughout my studies. I also would like to thank my good friend and fellow mathematician M.Sc. Ahmed Yekta Ökten for his help and genuine interest in my work.

The author of this thesis is supported by TÜBİTAK BİDEB 2210 Scholarship Program throughout her Master's studies.

TABLE OF CONTENTS

LIST OF ABBREVIATIONS

$\overline{A}$	Closure of the set A.
C-C	Carnot-Carathéodory.
$C_C^n(\Omega;\mathbb{R}^n)$	n-differentiable real vector valued function, continuous over the set Ω that vanishes outside its domain.
$C_C^\infty(\Omega;\mathbb{R}^n)$	infinitely differentiable real vector valued function, continuous over the set Ω that vanishes outside its domain.
L^n	Lebesgue Space of dimension n.
$W^{1,1}$	Sobolev Space of dimension $(1,1)$.
$\lvert . \rvert$	Variation of a measure.
$\lVert v \rVert$	Riemannian norm of a vector v.
$\Subset$	$\Omega' \Subset \Omega$ implies that there exists a compact set K such that $\Omega' \subset K \subset \Omega$

CHAPTER 1

INTRODUCTION

In this thesis we investigate the measure theoretic methods one might employ to determine if a given set in a metric space is of finite perimeter, i.e. finitely measurable by the given measure. The motivation of this thesis is to eventually determine certain properties of boundaries in space-times in order to better understand the conditions for Gauss-Green formulas to exist. We improve the ideas given by Battista et al [1].

A physics student is usually introduced to the concept of Gauss-Green theorems while exploring the simple concepts that are precursors to field theories. Usually the integration of a function in a given closed volume is assumed to be related to another integral formulation relating to the boundary of said volume as:

$$\int_{\Omega} f dV = \oint_{\partial\Omega} \vec{\nabla} f \cdot d\vec{a}. \tag{1.1}$$

The fundamental question we examine is "what are the conditions on the domains of integrations and their boundaries for Gauss-Green formulas to hold?". Federer [2] explores the properties of such domains in a measure theoretic setup.

From a physicist's stand point, the most ambitious result desired is to make sure that an analogue of Stokes' Theorem holds for every bounded volume of spacetime. Moreover, we want to make sure that the result extends to infinity as we enlarge the volume or surface of integration. At first glance, this result not only seems desirable, it is crucial to prove the global conservation laws and determine the continuity equations.

We know that the proof of Stokes' theorem does not rely on the notion of metric.

Therefore it is reasonable to expect analogous formulas in other metric spaces other than strictly Euclidean or Riemannian. The higher dimensional Gauss-Green formulas are well understood in the Euclidean setting. It is safe to state that n-dimensional Stokes' Theorem holds in any Euclidean space of dimension n. According to the research initiated by Federer, it is discovered that under certain conditions on the domain of integration, Riemannian analogues certainly exist. Recent papers [3] [1] also extend this result into sub-Riemannian spaces, i.e. Carnot-Carathéodory (C-C) distance analogue of the Riemannian topological spaces.

In this thesis we investigate whether or not such results could be obtained in physical spacetimes. The method we used is to expand the methods proposed in recent literature in sub-Riemannian metric to sub-Lorentzian case.

The results obtained heavily rely on local Poincaré inequality and a local doubling assumption. It must be noted that it is not trivial to show that these conditions hold physical spacetimes. We may refer to the works of Erlich and Gromov [4] [5] [6] to ensure that these assumptions hold at least in some regions of Lorentzian manifolds.

Exploring local doubling property, one immediately realizes that it implies the existence of a volume form such that the volume of balls in the given space must have an ordering between each other. It is to say that the volume measure must allow us to state that the volume of balls of radii r_0 and r_1, where $r_0 < r_1$, might be ordered such that we may find a real number C for $\mathrm{Vol}(B_{r_1}) \leqslant C(r_1)\,\mathrm{Vol}(B_{r_0})$. It is assumed throughout the paper [3] that we may assume this property for the balls of sub-Riemannian metric spaces.

In order to write an analogous statement in Lorentzian case we must find regions for which the above condition holds. One immediately realizes that finding such volumes in Lorentzian manifolds is tricky. The volume of bounded sets of a Lorentz manifold might not be finite just by the virtue of being bounded. This poses a challenge to overcome. Hence, we resort to the corresponding CC distance function to restrict our volumes. Sub-Lorentzian distance is a great candidate to overcome this issue. It restricts the arcs on which a physical particle can travel to be the timelike future directed geodesics. All the other possible arcs are said to be of infinite length, hence non-rectifiable.

In the generalized Gauss-Green setup, if the integrand obeys a local Lipschitz condition and if the domain of integration is bounded by a rectifiable curve, Gauss-Green formulas are known to hold. In the case that the volume cannot be covered by a rectifiable curve (loop), we might resort to the theory of integration over non-rectifiable curves. To exemplify, these volumes might consist of regions bounded by fractals. It is evident that the volume of such regions are bounded in the Euclidian case and converge to some finite number. Recent literature suggests that Stokes' theorem also holds in such regions even for Riemannian metric spaces [7].

In Lorentzian (or pseudo-Riemannian) spaces, we may not compare the volumes of such bounded regions. In this thesis we restrict our attention to relatively simple regions of the manifold, namely balls and boxes. Then we may ask if they are contained in one another.

Another point of interest is the point at infinity in Lorentzian manifolds. In order to extend Stokes' theorem to the infinite volume containing the whole spacetime, one needs to determine if the space can be covered with balls of finite perimeter that are bounded by rectifiable sets. That is indeed not the case as we know that a curve with an end point outside the causal future is not rectifiable. Therefore we must carefully dissect our spacetime into the largest possible volumes in which Stokes' theorem holds.

Consider the generalized Gauss-Green formula, in terms of distributional derivatives:

$$\int_E D_j f(x)dx = \int_{\partial E} f(x)v_j(E,x)d\phi(x), \tag{1.2}$$

where E is the open subset of M and $v_j(E,x)$ is the jth component of the exterior normal v. Federer [8] states that (1.2) to hold, there must be a certain kind of regularity between the set and its boundary. If a set $E \subset M$ is of finite perimeter, then (1.2) holds [2].

Let us assume that we have a certain volume in spacetime that is measurable and bounded by the set ∂E. ∂E might contain uncountably many tentacles or spikes of sorts. We might determine whether this boundary satisfies Gauss-Green formulas by checking whether or not the set bounded by ∂E is of finite perimeter. Or as a physicist, we might be observing some part of the universe that has a rather tame boundary and we might wish to eliminate all those not so well-behaved chunks.

Then we can define the reduced boundary $\mathcal{F}E \subset \partial E$ where ∂E is the topological boundary of E in M that satisfies the following conditions

$$\liminf_{r\to 0} \frac{\min\left\{\mathbf{m}\left(B_r(x)\cap E\right), \mathbf{m}\left(B_r(x)\setminus E\right)\right\}}{\mathbf{m}\left(B_r(x)\right)} > 0, \tag{1.3}$$

$$\limsup_{r\to 0} \frac{\|D\chi_E\|\left(B_r(x)\right)}{\mathbf{m}\left(B_r(x)\right)/r} < \infty, \quad \forall x \in \Omega, \tag{1.4}$$

and

$$\lim_{r\to 0} \frac{1}{\|D\chi_E\|(B_r(x))} \int_{B_r(x)} |\nu_E^*(y) - \nu_E^*(x)|^2 d\|D\chi_E\|(y) = 0. \tag{1.5}$$

This new boundary does not contain such spikes or tentacles, yet still covers the volume of integration. Equivalently $\mathcal{F}E$ also satisfies the generalized Gauss-Green formula. The properties of such boundaries are well understood in Riemannian manifolds and they have remarkable properties, such as being rectifiable.

In this thesis we propose that some of this machinery can also be worked out in Lorentzian setting. We claim that given a Lorentzian spacetime satisfying certain properties, the convenience of reduced boundary and volumes of finite perimeter may be exploited.

In the first chapter, we introduce the basic notions in physical spacetimes and give some measure theoretic definitions along with simple theorems that are stated here without proof. We also introduce the C-C distance d of Riemannian and d_η of Lorentzian manifolds, and using these we construct the corresponding sub-Riemannian and sub-Lorentzian manifolds. We finally conclude that the topological structure of a Lorentzian and sub-Lorentzian manifold is different (due to the latter having positive definite metric) and introduce an Alexandrov basis for the latter case to deduce that our topology is indeed Hausdorff if our manifold is strongly causal.

In the second chapter we define the total variation of a function on Riemannian and sub-Riemannian manifolds and use the same definition to explore sub-Lorentzian ones. We also define the spaces of functions of bounded variation (BV) on a given open set of M with respect to the original metric g or η and the CC distance d or d_η, respectively. Also, in this chapter we give a theorem that was originally stated in [3] proving equivalence of BV spaces in Riemannian and sub-Riemannian manifolds. We also write a Lorentzian equivalent of the theorem and prove it, showing that BV spaces of Lorentzian and sub-Lorentzian manifolds are indeed equivalent. We also discuss the possibility of obtaining a Lorentzian version of the Riesz Representation Theorem and investigate the regions in which such result can be achieved.

Eventually we define balls and boxes in sub-Riemannian manifold and state a Ball-Box theorem given in [6]. We also discuss whether this result may be extended to spacetimes. At the end of this thesis, we give some warnings for infinitely large domains of integration and discuss the reason why Gauss-Green formula might fail for volumes covering the entirety of the spacetime.

CHAPTER 2

PRELIMINARIES

In this chapter some definitions and fundamental theorems exploited in the following chapters are introduced. We first start by exploring the very fabric of spacetime in a topological point of view. For ease of implementation to physical concepts, some well known notations are altered appropriately.

2.1 Structure of Spacetimes

We first explore the topological fabric of physical spacetimes. The definitions and theorems used in this first section are mainly based on the works of Hawking and Ellis [9] and Penrose [10].

Throughout this section, M is an n-dimensional, smooth, simply connected manifold. g or η is the intrinsic metric defined on M, depending on the Riemannian or the Lorentzian structure, respectively.

M is said to be a spacetime if it is real, four-dimensional, connected and Hausdorff and if for all $x \in M$, there exists a basis in T_xM such that the metric at the point x is $\eta_x = \mathrm{diag}(-1, 1, 1, 1)$ [10]. M is said to be time orientable if one can find a non vanishing vector field T such that $\eta(T, T) < 0$ everywhere, this vector is the so called time orientation. Given a coordinate basis $\{x^\mu\}$ at a point $x \in M$ where $\mu = 0, \ldots, n-1$, usually the vector $x^0 \equiv \partial/\partial t$ undertakes the role of time orientation. In fact the signature of time orientation is usually irrelevant as we can always find the vector with the opposite sign. Moreover time orientation is not unique, but one may always normalize it to find one that has unit length.

A metric η on a manifold M is an symmetric tensor field of type $(0, 2)$. A metric is an inner product on T_xM for all $x \in M$. Given a coordinate basis, η may be expanded in terms of its components as [11]

$$\eta = \sum_{\mu,\nu} \eta_{\mu\nu} dx^{\mu} \otimes dx^{\nu}.$$

Equivalently ds^2 may represent the metric tensor in the following manner,

$$ds^2 = \sum_{\mu,\nu} g_{\mu\nu} dx^{\mu} dx^{\nu}.$$

For example metric of Minkowski spacetime η, whose local coordinates are given as $\{t, x, y, z\}$, is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$

As in [11] assuming that our manifold M is time orientable we now introduce the concept of causality.

2.1.1 Causality Theory

A curve $\gamma : \mathbb{R} \to M$ is a function that assigns points of interval $I \subset \mathbb{R}$ to points of M. For all practical purposes we will restrict the interval I to the subinterval $[0, \tau]$. The starting point of the curve shall be denoted $\gamma(0) = p \in M$, and the endpoint will be $\gamma(\tau) = q \in M$. The curve is said to be C^k if it is k-differentiable. If γ is C^{∞} differentiable, then it is a smooth curve.

Let p and q be the starting and ending points of a piecewise smooth map γ. Assume that $\gamma(\epsilon_{i-1}, \epsilon_i)$ is a smooth line segment, where $\epsilon_i \in (0, \tau)$, if there exists such $\lim_{i \to n} \gamma(\epsilon_i) = q$ such that $n < \infty$ and the velocity of the curve $\dot{\gamma}$ is timelike (i.e. $g(\dot{\gamma}, \dot{\gamma}) < 0$), then γ is a trip from event p to event q [10].

In M there might exist such points that does not have a trip between them, therefore we must restrict the regions we use in this study away from those. Consider the following definitions:

Definition 2.1.1 *Let M be an n-dimensional manifold. The vector $v \in TM$ is said to be timelike if and only if $g(v, v) < 0$. Also it is said to be null if and only if $g(v, v) = 0$ and spacelike if and only if $g(v, v) > 0$.*

Let γ be a curve such that $\gamma[0,\tau] \subset M$. If there exists a vector $\dot{\gamma}(t)$ such that $g(\dot{\gamma}(t),\dot{\gamma}(t)) < 0$ for almost all $t \in [0,\tau]$ then γ is also said to be timelike. A null or spacelike curve admits velocities $g(\dot{\gamma}(t),\dot{\gamma}(t)) = 0$ or $g(\dot{\gamma}(t),\dot{\gamma}(t)) > 0$, respectively.

Definition 2.1.2 *[12] Let $T \in TM$ be a time orientation (i.e. $g(T(x),T(x)) < 0$, $\forall x \in M$). A curve is said to be future directed if there exists a vector $\dot{\gamma}(t)$ for all $t \in [0,\tau]$, such that $g(T(x),\dot{\gamma}(t)) < 0$. Equivalently it is said to be past directed if $g(T(x),\dot{\gamma}(t)) > 0$, for all $t \in [0,\tau]$.*

Definition 2.1.3 *[10] It is customary to write $p \ll q$ and say that p chronologically preceeds q if there exists a smooth future directed timelike curve with endpoint q, starting at p. The set of such $q \in M$ is denoted $I^+(p,g) = \{q \in M \mid p \ll q\}$.*

It is denoted $p \prec q$ if there exists a smooth future directed non-spacelike curve joining p and q, and said that p causally preceeds q. The set of such q is $J^+(p,q) = \{q \in M \mid p \prec q\}$.

The sets $I^+(p)$ and $J^+(p)$ are called the timelike and causal future of the given point, respectively.

Penrose [10] shows that the relations $\ll$ and $\prec$ are transitive on a time orientable M.

Proposition 2.1.1: [10]
(i) $p < q \Rightarrow p \prec q$,

(ii) $p \ll q, q \ll y \Rightarrow p \ll y$,

(iii) $p \prec q, q \prec y \Rightarrow p \prec y$.

Example: Let M be the Minkowski spacetime. Let $\{x^\mu\}$ be the coordinate system around the origin. Then the timelike future of the origin O is

$$I^+(O) = \left\{ (x^\mu) : x^0 < \left[\sum_{i=1}^{3} \left(x^i\right)^2 \right]^{1/2} \right\}.$$

Whereas the causal future of the origin is

$$J^+(O) = \left\{ (x^\mu) : x^0 \leqslant \left[\sum_{i=1}^{3} \left(x^i\right)^2 \right]^{1/2} \right\}.$$

Notice that in this example $I^+(O)$ is open and $J^+(O)$ is closed with respect to the manifold topology. In fact timelike future of any point $p \in M$ is open by definition, but it must be noted that not every causal future is closed.

Now we define the timelike future of a set $U \in M$.

Definition 2.1.4 *[10] Timelike future of a set U is*

$$I^+(U) = \bigcup_{p \in U} I^+(p). \tag{2.1}$$

Causal future, timelike past and causal past are also defined similarly.

Let's tackle the following proposition on the structure of open and closed sets of M.

Proposition:[10] $I^+(U)$ is open in M with respect to manifold topology for any $U \subset M$.

Proof: Since the finite union of open sets is open, $\bigcup_i I^+(U_i)$ is open. Consider equation (2.1). Then:

$$\Rightarrow \bigcup_i I^+(U_i) = \bigcup_i \left(\bigcup_{p \in U_i} I^+(p) \right) = \bigcup_{p \in \bigcup_i U_i} (I^+(p)) = I^+ \left(\bigcup_i U_i \right) \tag{2.2}$$

is open. Thus we conclude that $I^+(U)$ is open regardless of U being open or closed.

Therefore we have a method of connecting the points with piecewise smooth future directing curves, provided that $U_i \cap U_j \neq \varnothing$.

2.1.2 Alexandrov Topology

Now we define the Alexandrov topology.

Definition 2.1.5 ***(Alexandrov topology)*** *Given a chronologically open M, the basis of Alexandrov topology is*

$$\mathcal{B} = \left\{ I^+(p) \cap I^-(q) \mid p, q \in M \right\}.$$

Let $p \in M$, take the curve $\gamma : [0,\tau] \to M$ such that $\gamma(\tau/2) = p, \gamma(0) = p_-$, $\gamma(\tau) = p_+$. This implies

$$p \in I^+(p_-) \cap I^-(p_+) \equiv O$$

Since $p \in \mathcal{B}, \exists O$ such that $p \in O \in \mathcal{B}, \forall p \in M$.

Now assume $p \in O_1 \cap O_2$ for two open sets $O_i \in \mathcal{B}$. That is $p \in I^+(p_-) \cap I^-(p_+) \cap I^+(q_-) \cap I^-(q_+) \Rightarrow \exists \gamma_1, \gamma_2 : [0,\tau] \to M$ such that $\gamma_1(0) = p_-$, $\gamma_1(\tau) = p_+$ and $\gamma_1(r) = p$, $\gamma_2(0) = q_-$, $\gamma_2(\tau) = q_+$ and $\gamma_2(r') = p$ for some $r, r' < \tau$. Without loss of generality assume that

$$p_- \ll p \ll q_+$$

$$\Rightarrow \exists \gamma_3 \text{ such that } \gamma_3(0) = p_-, \gamma_3(\tau) = q_+$$

$\Rightarrow \exists t \in (0,\tau), \gamma(t) = p \Rightarrow \exists \varepsilon < \tau$ such that $, p_- \ll \gamma_3(t-\varepsilon) \ll p \ll \gamma_3(t+\varepsilon) \ll q_+$

Then we conclude that $\exists O_3 \subset O_1 \cap O_2$. Then $\mathcal{B} = \{I^+(p) \cap I^-(q) \mid q, p \in M\}$ is a basis of topology $\mathcal{A}$. Usually $\mathcal{A}$ is coarser than the manifold topology for chronologically open spacetimes M [10]. Consider the following theorem:

Theorem 2.1.1 *[10] Following statements are equivalent for a space-time M.*
(i) M is strongly causal.
(ii) Alexandrov topology agrees with the manifold topology.
(iii) Alexandrov topology is Hausdorff.

2.2 Measure Theoretic Definitions

This section is dedicated to cite some basic definitions and elementary theorems that will be exploited in the proceeding chapters. The definitions and theorems cited are largely based on the classical textbook of H. Federer [2].

We start by defining the measure:

Definition 2.2.1 *[2]* ***(Measure)*** *A measure is a set function $\phi : \mathbf{2^X} \to \mathbb{R} \cup \{\infty\}$ such* *that*

$$\phi(A) \le \sum_{B \in \Omega} \phi(B), \tag{2.3}$$

where the countable set $\Omega \subset \mathbf{2^X}$ *and* $A \subset \bigcup \Omega$*. Then, it is said that* ϕ *is a measure on* X*.*

An immediate conclusion of this definition is that the measures of sets may be ordered by the innate property $\phi(A) \le \phi(B)$ whenever $A \subset B \subset X$. Naturally a measure is also additive and one may actually determine the measurability of a set exploiting this property as follows:

Definition 2.2.2 *[2] **(Measurable set)** A is a measurable set (with respect to* ϕ*) if and only if* $A \subset X$ *and for any subset* $K \subset X$*,* $\phi(K) = \phi(K \cap A) + \phi(K \setminus A)$ *holds.*

As suggested by Federer [2], this fact can be exploited to determine whether or not a set A is measurable, by showing that the inequality $\phi(K) \ge \phi(K \cap A) + \phi(K \setminus A)$ holds for any finitely measurable set K, i.e. $\phi(K) < \infty$.

Definition 2.2.3 *[2] **(Regular Measure)*** ϕ *is said to be a regular measure if and only if for each* $A \subset X$*, there exists a measurable set* B *s.t.* $A \subset B$ *implies* $\phi(A) = \phi(B)$*.*

Definition 2.2.4 *[2] **(Hull)** Let* X *be a* ϕ *measurable set. We say that* B *is a hull of* A *if and only if* $A \subset B \subset X$*,* B *is measurable and* $\phi(T \cap A) = \phi(T \cap B)$ *for every* ϕ *measurable set* T*.*

Federer [2] deduces that for every regular ϕ, if the measure of the set A, $\phi(A)$ is finite, the set A has a hull. Another property of a regular measure is that the sum of finite measures of sets equals the measure of the union of the sets. This fact can be summed up as shown in the following remark:

Remark 2.2.1 *Let* ϕ *be a regular measure, and A be a finitely* ϕ *measurable set such that* $A = \bigcup_{i \in I} A_i$*. If* $\phi(A_i) + \phi(A_j) = \phi(A_i \cup A_j) < \infty$ *then* A_i *and* A_j *are finitely* ϕ *measurable for* $\forall i, j \in I$*.*

For all practical purposes, all the measures discussed in this thesis are regular, yet it is beneficial to note that for any given measure, it is possible to construct a regular measure.

Remark 2.2.2 *Let ϕ be any measure and A be a measurable set. Then one can construct the regular measure $\mu(A) = \inf_{B \supset A} \{\phi(B)|A \subset B\}$ where infimum is taken over ϕ measurable sets B.*

Example: Let ϕ be the counting measure and $A = \{i \in I \mid a_i\} \subset \mathbb{R}$ for a finite index set $i \in I$. Observe that $\phi(A) = n$. Then A is definitely ϕ measurable. Let $\mu(A) = \inf_{B \supset A}\{\phi(B) \mid A \subset B\} = n$, since $A \subset A$, and A is indeed ϕ measurable.

We define the Borel sets:

Definition 2.2.5 *[2] **(Borel family)** A Borel family $\mathcal{B}$ of a given set X is a family of subsets in $\mathbf{2}^X$ satisfying the following conditions:*
(i) If $A \in \mathcal{B}$, then $X \setminus A \in \mathcal{B}$.
(ii) If $\mathcal{B}' \subset \mathcal{B}$ and $\mathcal{B}'$ is countable, then $\bigcup \mathcal{B}' \in \mathcal{B}$.
(iii)If $\emptyset \neq \mathcal{B}' \subset \mathcal{B}$ and $\mathcal{B}'$ is countable, then $\bigcap \mathcal{B}' \in \mathcal{B}$.

For every $S \subset \mathbf{2}^X$, there exists a smallest Borel family containing S, called the Borel family generated by S. In the case that X is a topological space and $\mathcal{T}$ is the topology, the members of the Borel family generated by $\mathcal{T}$ are the Borel sets associated with $\mathcal{T}$ [2].

Definition 2.2.6 *[2] **(Borel regular measure)** The measure ϕ is said to be Borel regular on the topological space X if and only if all open sets of X are ϕ measurable and $\forall A \subset X$, there exists a Borel set B such that $A \subset B \subset X$ for which $\phi(A) = \phi(B)$.*

Definition 2.2.7 ***(Support)** [2] Let ϕ measure the topological space X. Let $\mathcal{T}$ be the topology. Support of ϕ, denoted $\operatorname{spt}\phi$, is the set*

$$\operatorname{spt}\phi = X \setminus \bigcup \{A|A \in \mathcal{T}, \phi(A) = 0\}. \tag{2.4}$$

Another definition we will use throughout is the Radon measure which is defined over a locally compact Haussdorf space.

Definition 2.2.8 ***Radon Measure:*** *[2] A Radon measure ϕ over a locally compact Hausdorff space X is a measure that satisfies the following:*
(i) If K is a compact subset of X, $\phi(K) < \infty$,
(ii) If O is an open subset of X, then O is ϕ measurable and

$$\phi(O) = \sup\{\phi(K)|K \textit{ is compact, } K \subset O\},$$

(iii) For all $A \subset X$,

$$\phi(A) = \inf\{\phi(O)|O \textit{ is open, } A \subset O\}.$$

Definition 2.2.9 ***(Lipschitz function)****[2] The function $f : X \to Y$ is said to be Lipschitzian if and only if there exists a number $M \in \mathbb{R}^+$ such that $d_Y(f(x), f(y)) \leq M d_X(x, y)$, where d_X and d_Y are the metric on the metric spaces X and Y, respectively, and M is said to be the Lipschitz constant.*

This definition can be narrowed down to a locality condition, given that the vector space X is a convex subspace of a $V \subset TM$ with the inherited norm of the manifold M. That is, in $X \subset V \subset TM$, any line segment γ: $[0, \tau] \to X$ joining any points $x, y \in X$ is also contained in X, i.e. $\gamma = \{\gamma[0, \tau] \in V|\gamma(0) = x, \gamma(\tau) = y\} \subset X$. The merit of this definition will be appreciated later in the upcoming discussions as we concentrate our interest into Lorentzian vector spaces.

Definition 2.2.10 ***(Locally Lipschitz function)****[2] Assume that V is the vector space with the inherited norm. Let X be a convex subspace in V. Then, a function $f : X \to Y$ is said to be locally Lipschitz if and only if*

$$\limsup_{y \to x} \frac{d_Y(f(x), f(y))}{|d_X(x, y)|} \leq M(x), \qquad \forall x \in X, \tag{2.5}$$

where $M(x)$ is a constant depending on the point $x \in X$. The infimum of such constants is denoted by $|\nabla f|(x)$ for a given $x \in X$ and called the slope of f.

Consider the following theorems.

Theorem 2.2.1 *[2] If X is ϕ measurable metric space such that all open subsets $O \subset X$ are also ϕ measurable, the following statements hold:*

(i) Given a Borel set O such that $\phi(O) < \infty$ and a real number $\epsilon > 0$, there exists a closed set C, satisfying $C \subset O$ and $\phi(O \setminus C) < \epsilon$.

(ii) Let the Borel set O be a subset of the countable union $\bigcup_{i \in F} O_i$, $\phi(O_i) < \infty$ and $\epsilon > 0$. Then there exists an open set $\tilde{O}$ such that $\phi(\tilde{O} \setminus O) < \epsilon$.

Definition 2.2.11 ***(Characteristic function)*** *Characteristic function of a set $A \in X$ is defined to be the step function $\chi_A : X \to \{0, 1\}$ such that $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise.*

Note that we changed the widely used notation $\mathbf{1}_A$ for characteristic function to χ_A so that the former notation is reserved for the identity map over the set A.

Definition 2.2.12 ***(Measurable functions)*** *Let X be a ϕ measurable space and Y be a topological space. Let $f : D \to Y$ where $D \subset X$. f is said to be ϕ measurable if and only if $\phi(X \setminus D) = 0$ (i.e. domain of f contains ϕ-almost all of X), $f(D) \subset Y$ and $f^{-1}(B)$ is ϕ-measurable for all open $B \subset Y$.*

Now we must note an important result for metric spaces called the "Carathéodory's Criterion". Consider the identity mapping over X, ($\mathbf{1}_A : X \to X$). Assume that $\mathbf{1}_A$ is ϕ-measurable. Then, by the definition $f^{-1}(B) = B$ is ϕ-measurable, requiring that all the open subsets $B \subset X$ are ϕ-measurable. If our measurable space is inherently a metric space, one obtains another fundamental result:

Remark 2.2.3 ***(Carathéodory's Criterion:)****[2] Given any open subset $O, \tilde{O} \subset X$ is ϕ measurable if and only if $\phi(O \cup \tilde{O}) \geq \phi(O) + \phi(\tilde{O})$ whenever $\inf\{d(x, y) | x \in O, y \in \tilde{O}\} > 0$ which is the distance between sets O and $\tilde{O}$, i.e. $\operatorname{dist}(O, \tilde{O})$.*

2.3 Distributions and Geodesics

Let M be an n dimensional manifold. A distribution is a linear sub-bundle of the tangent bundle TM of the manifold. The distribution is referred to as the horizontal space or a horizontal distribution. Curves tangent to this space is said to be horizontal curves [13].

A k dimensional distribution where $k \leqslant n$ can be visualized as a function D such that for any $p \in M$, $D(p) \subset T_pM$. A distribution D is of class C^∞ at $p \in M$, if there exists a set of vector fields $\{X_1, \ldots, X_k\}$ of class C^∞ defined on a neighborhood U of p, $p \in U \subset M$, such that for all $q \in U$, $\{X_1(q), \ldots, X_k(q)\}$ spans $D(q)$ [14].

A vector field X is in the distribution D if for every p in the domain of X, $X(p) \in D(p)$. Then we define the involutive distribution.

If two vector fields belong to the involutive distribution D, then the Lie bracket $[X, Y]$ belongs to D as well. A distribution having this property could also be considered a bracket generating distribution. Let us define it formally below.

2.3.1 Bracket Generating Distributions

Let D be a subbundle of TM such that [3]

$$D(p) = \{v \in T_pM \mid g_p(v, v) < \infty\}. \tag{2.6}$$

Assume that there exists a sequence $\{D^i\}$ of vector fields in D, which is closed under Lie bracket, i.e. $[D^i, D^j] \in \{D^i\}$.

Define the elements of $\{D^j\}$ using the following setup: [14]

$$D^2 = D + [D, D], \ldots, D^i = D^{i-1} + \left[D, D^{i-1}\right]$$

Then

$$\left\{D^j\right\} = \left\{D, D + [D, D], \ldots \cdot D^{i-1} + \left[D, D^{i-1}\right], \ldots, TM\right\}$$

$$D \subset D^2 \subset \ldots \subset D^i \subset \cdots \subset TM$$

This implies that every horizontal vector may be expressed in terms of iterated Lie brackets $[D, [D^2, [D^3 \ldots]]]$ [15].

Definition 2.3.1 ***(Bracket generating distribution):*** *[15] [13] Let $D \subset TM$. If there exists o sequence $\{D^j\}$, such that $D^j \subset TM$ where $j \in \mathbb{Z}^+$, $D^1 = D$ and, $D^j = D^{j-1} + [D, D^{j-1}]$. The last term indicates Lie bracket and for some $k \in \{j\}, D^k = TM$, then D is said to be bracket generating (or equivalently, completely non -integrable).*

Here are some examples of how to construct bracket generating distributions.

Example: (Heisenberg Distribution)[14]

Let $M = \mathbb{R}^3$, and let $D = \text{span}\{D^1, D^2\} = \text{span}\{\frac{\partial}{\partial x^1} - \frac{1}{2}x^2\frac{\partial}{\partial t}, \frac{\partial}{\partial x^2} + \frac{1}{2}x^1\frac{\partial}{\partial t}\}$. One can easily see that $\left[\frac{\partial}{\partial x^1} - \frac{1}{2}x^2\frac{\partial}{\partial t}, \frac{\partial}{\partial x^2} + \frac{1}{2}x^1\frac{\partial}{\partial t}\right] = D^3 = \frac{\partial}{\partial t}$. Notice that $\text{span}\{D^1, D^2, D^3\} = \mathbb{R}^3$. Then Heisenberg distribution is bracket generating.

Example: (Lie Group)[3] Let G be a connected Lie group and $\mathfrak{g}$ be the associated Lie algebra. Let $\mathfrak{g}_1 = \text{span}\{X_\mu\}$, where $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$. To exemplify, we let $X_1 = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}$ and $X_2 = \frac{\partial}{\partial\theta}$. Then, $\left[\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}, \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right] = \partial_x^2 + \partial_y^2 = \partial_\theta = X_2$. The Lie group associated is $\mathbb{R}^2 \times S^1$.

2.3.2 Geodesics

Consider the following theorem:

Theorem 2.3.1 ***(Ball-Box Theorem)****[13][6] Let M be a Riemannian manifold with coordinates $\{x^\mu\}$ at the point p and $B_\rho(p)$ be the ball of radius ρ around $p \in M$. Define the box $\text{Box}_\rho(p) = \{y \in M| \mid y^\mu - x^\mu \mid \leq \rho\}$. Then there exists constants $c, C \in \mathbb{R}^+$, $\rho_0 > 0$ such that for all $\rho < \rho_0$, the following holds*

$$\text{Box}_{c\rho}(p) \subset B_\rho(p) \subset \text{Box}_{C\rho}(p). \tag{2.7}$$

A direct consequence of this theorem is the existence of finite length curves connecting the points enclosed by $B_\rho(p)$.

For a Lorentzian manifold Ball-Box theorem does not necessarily hold. Under the assumption of the existence of bracket generating distributions, we may state a weaker result than the original ball box theorem, called Chow-Rashevskii theorem.

Theorem 2.3.2 ***(Chow - Rashevskii)****[13] A bracket generating distribution $D \subset TM$ of a connected manifold M implies that a set of points in M can be connected by a horizontal curve.*

Theorem (2.3.2) holds for Riemannian manifolds. It also has local and global implications, namely a local doubling assumption and a local Poincaré inequality. If theorem (2.3.2) holds, we say that these assumptions are satisfied automatically. In the Lorentzian case, we may not be able to have such a strong result. Therefore throughout this thesis the assumptions on the metric space is generalized to somewhat weaker conditions that will later be discussed (see (A1) (4.25) and (A2) (4.26)).

CHAPTER 3

SUB-METRIC MANIFOLDS

In this chapter we define the sub-Riemannian and sub-Lorentzian manifolds. These manifolds are endowed with a Riemannian or a Lorentzian metric, respectively.

3.1 Sub-Riemannian Manifolds

A sub-Riemannian manifold, i.e the manifold endowed with the distance d, is the generalization of a Riemannian manifold via Carnot-Carathéodory metric. In this setting, extraordinarily, the distance function is defined by the help of so-called length minimizer. This property is especially worth investigating from a physicist's point of view as this condition may simplify a plethora of problems concerning how to define trajectories of some particles moving under constraints.

A sub-Riemannian manifold is a a manifold M of dimension n, endowed with a distribution defined in the preceding chapter and an accompanying fiber inner product [13]. On M we might define the distance between the two points $p, q \in M$ as the usual Riemannian distance if they are connected by a horizontal curve previously defined. If such a curve does not exist, we say that points p and q have infinite distance between them.

Consider the Riemannian manifold M. The metric $g : T_pM \times T_pM \to \mathbb{R}$ on the tangent bundle TM is positive definite. The definition of Riemannian norm $\|.\| : T_pM \to \mathbb{R}$ is $\|v\| = \sqrt{g_p(v, v)}$. We might realize this norm as a measure over the metric space.

Consider the points $p, q \in M$. Let $\gamma : [0, \tau] \to M$ be the curve connecting these two.

Therefore we have $\gamma(0) = p$, $\gamma(\tau) = q$. A vector $v \in T_z M$, tangent to the curve γ at the point $\gamma(t) \in M$ will be denoted $\dot{\gamma}_t \in T_{\gamma(t)} M$ as usual. The associated Riemannian norm $\|\dot{\gamma}_t\| = \int_0^\tau \sqrt{g_{\gamma(s)}(\dot{\gamma}(s), \dot{\gamma}(s))} ds$ is nonnegative by definition. This translates to

$$\inf_\gamma \int_0^\tau \sqrt{g_{\gamma(s)}(\dot{\gamma}(s), \dot{\gamma}(s))} ds > 0.$$

Now, restrict the curves γ to be the horizontal curves, i.e. curves whose tangent vectors belong to a bracket generating distribution D. The resulting distance function $\inf_{\{\gamma : \dot{\gamma} \in D\}} \int_0^\tau \sqrt{g_{\gamma(s)}(\dot{\gamma}(s), \dot{\gamma}(s))} ds$, is said to be the Carnot-Carathéodory distance. This distance can be realized as a metric on a Riemannian manifold, and historically is called as the *C-C metric* [6].

3.1.1 Definition and Construction

Now we are ready to give the formal definition of sub-Riemannian norm.

Let M be an n-dimensional Riemannian manifold. Choose any Euclidean bundle $\mathbf{U}$ over TM, and let the inclusion map $u : \mathbf{U} \to TM$ be smooth. Assume that under such map u, the fiber $\mathbf{U}_p$ is mapped linearly to $T_p M$, $u(\mathbf{U}_p) \subset T_p M$. Then we construct the so called horizontal, bracket generating distribution

$$D = \{u \circ \sigma \mid \sigma \in C^\infty(M, U), \sigma(p) \in \mathbf{U}_p\}. \tag{3.1}$$

Here $C^\infty(M, \mathbf{U})$ denotes the infinitely differentiable functions from M to $\mathbf{U} \subset TM$. A great candidate for σ is the exponential function over M. A fiber $D(p)$ is given as:

$$D(p) = u(\mathbf{U}_p) \subset T_p M. \tag{3.2}$$

Sub-Riemannian distance at the point $p \in M$ is the following function $d_p(v) : T_p M \to M$,

$$d_p(v) = \begin{cases} \min \left\{ \|\nu\|_p : \nu \in U_p, u(\nu) = v, & v \in D(p) \\ \infty, & v \notin D(p) \end{cases} \tag{3.3}$$

where $\|.\|_p$ is the usual Riemann norm at the point p.

As an example, let $\{\sigma_i\}$ where $i = 1, \ldots, n$ be the orthonormal frame at some $\mathbf{U}_\Omega$ where $\Omega \subset M$ is open and $\sigma_i \equiv \sigma_i(x^\mu)$ is a function of $\{x^\mu\}$, i.e. the usual coordinate basis. Let $X_i = u \circ \sigma_i$. We observe that $\operatorname{span}\{X_i\}$ is a local frame of $\mathbf{U}_\Omega \subset TM$. We may write any vector $v \in \mathbf{U}_\Omega$ in component form $v = \sum_{i=1}^n c_i X_i(p)$, then the sub-Riemannian norm is given as

$$d_p(v) = \min_{\{c_i : v = c_i X_i(p)\}} \left[\sum_{i=1}^n c_i^2 \right]^{1/2}. \tag{3.4}$$

We established the sub-Riemannian norm of a vector. Now, we need to write down the sub-Riemannian distance between two points. Let $\{X_i\}$ be the orthonormal basis with respect to the metric g, and $D = \operatorname{span}\{X_1, \ldots, X_n\}$ be the distribution. Let $\gamma(t) \in M$. Then $u \circ \sigma(\gamma) = \sum_{i=1}^n c_i(t) X_i(\gamma)$. Therefore the sub-Riemannian length of a curve γ is

$$d(p,q) = \min \left\{ \int_0^\tau \|\dot{\gamma}(t)\| dt \mid \gamma(0) = p, \gamma(\tau) = q, \dot{\gamma}(t) \in D(\gamma(t)) \right\}. \tag{3.5}$$

Recall theorem (2.3.2) that definitely holds for Riemannian manifolds, thus justifying the choice of sub-Riemannian distance via the existence of geodesics.

3.1.2 Sub-Riemannian Distance as a Solution of the Optimal Control Problem

Definition 3.1.1 ***(Admissible Pair)*** *[15] Let M be an n-dimensional, smooth, sub-Riemannian manifold, and V be a subspace in M. Let $\pi : V \to M$ be a submersion. Notice that π induces a fiber bundle. We denote $U = \pi^{-1}(q)$ for a chosen point $q \in M$. $(u, \gamma_u) : [0, \tau] \to V$ is said to be an admissible pair where γ_u is a Lipschitzian curve on M and $\dot{\gamma}_u = f(u, \gamma_u)$ where $f : V \to TM$ is a morphism of vector bundles.*

A given $\tau \in \mathbb{R}$ is a parameter that determines the domain of admissible pairs. As in [15], we denote $\mathcal{V}_\tau$ as the space of such admissible pairs. Moreover, $\mathcal{V}_\tau$ is a smooth Banach submanifold of $L^\infty([0, \tau], V)$ [15]. Note that if any points $p, q \in V \subset M$ admits at least one admissible pair connecting them, then (V, f) is said to be a controllable system [15].

As given in [13], a direct corrolary for theorem (2.3.2) is the following:

Theorem 3.1.1 *[13] Let M be a connected manifold, and $D \subset TM$ be a bracket-qenerating distribution, and $V \subset M$ be a subset.*

(i) Then $\forall p \in V$, $\exists U \in V$ such that U is a neighborhood of p, every $q \in U$ admits a pair $(u, \gamma_u) \in \mathcal{V}_\tau$ such that $\gamma_u(0) = p, \gamma_u(\tau) = q$, hence a minimizing geodesic.
(ii) Let M be a complete manifold relative to the Carnot - Carathédory distance. Then any $p, q, \in M$ can be connected by a minimizing geodesic.

Now, we not only know that in a Riemannian manifold, a pair of points can be connected by a curve admitting sub-Riemannian distance, but we also know that such a curve is indeed a geodesic of the initial Riemannian manifold. Therefore we give the definition below:

Definition 3.1.2 ***C-C distance on sub-Riemannian manifolds***

The corresponding Carnot - Carathéodory distance between two points $p, q \in M$ is the following:

$$d\,(p, q) = \inf_{\gamma} l(\gamma),$$

where the infimum is over horizontal curves such that $\gamma(0) = q_0, \gamma(\tau) = q_1$. More precisely

$$d\,(p, q) = \begin{cases} \inf_\gamma l(\gamma) & \textit{if } \exists \gamma \textit{ s.t. } \gamma(0) = p, \gamma(\tau) = q \\ \infty & \textit{otherwise.} \end{cases}$$

Let $I : \mathcal{V}_\tau \to \mathbb{R}$ be the functional minimizing the length $l(\gamma)$. Then for an admissible pair (u, γ_u), we need to minimize the functional

$$\frac{1}{2}\int_0^\tau \sum_{j=1}^{n} u_j^2 dt, \qquad \text{where } u \in \{u \in \mathbb{R}^n \mid \|u\| < 1\}, \tag{3.6}$$

in order to obtain C-C distance between two points connected by γ, where $\|.\|$ denotes the Riemannian norm as usual.

3.1.3 Existence of Geodesics (Ball-Box Theorem)

We established that minimizing geodesics exist for Riemannian manifolds. We need to discuss their existence in sub-Riemannian setting as well. It is somewhat easy to conclude that the existence in Riemannian case translates into the sub-Riemannian counterpart.

We need to find the sub-Riemannian equivalent of the Ball-Box theorem. In [13], the following theorem is stated:

Theorem 3.1.2 *[13] If D is a bracket generating distribution on a connected manifold M then any $p, q \in M$ can be connected by a horizontal curve.*

A horizontal curve, as defined before, is a curve whose tangent vectors is in the horizontal (bracket generating) distribution. If $D \subset TM$ is a bracket generating distribution, then we know that for any point $p, q \in M$, there exists a curve γ such that $\dot{\gamma}_t \in D$, $\gamma(0) = p$, $\gamma(\tau) = q$. Therefore we are justified in choosing the distance between two points as the infimum of length of the curves connecting those given that we may write a bracket generating distribution for the base Riemannian manifold M. In general sub-Riemannian distance is the metric distance between the points of the manifold.

3.2 Sub-Lorentzian Manifolds

A sub-Lorentzian manifold is also a topological manifold endowed with a distance metric. Structurally it resembles its sub-Riemannian counterpart.

3.2.1 Definition

We choose a horizontal distribution $D \subset TM$ of the Lorentzian manifold M endowed with the Lorentzian metric η.

Let M be an n-dimensional Lorentzian manifold, with the metric signature $(-1, 1, \ldots, 1)$. Consider the vector $v \in D \subset TM$. Vectors are classified in the usual sense as timelike, spacelike or null with respect to η.

Timelike and causal pasts of the point $x \in M$ is also determined similarly with the condition $\eta(T, \dot{\gamma}) > 0$. Finally the Carnot-Carathéodory distance is given as

$$d_\eta (p, q) = \begin{cases} \sup_\gamma (l(\gamma)) & \text{if } q \in J^+ (p, \eta) \\ 0 & \text{otherwise }, \end{cases}$$

where $l(\gamma) = \int_0^\tau \mid \eta(\dot{\gamma}(t), \dot{\gamma}(t)) \mid^{1/2} dt$. As opposed to the sub-Riemannian case, the distance between two points of M is determined by a maximizing curve, rather than a minimizing one. Similar to the previous case

$$d_\eta(p, q) = \max \left\{ \int_0^\tau \mid \eta(\dot{\gamma}(t), \dot{\gamma}(t)) \mid^{1/2} dt \middle| \gamma(0) = p, \gamma(\tau) = q, \dot{\gamma}(t) \in D(\gamma(t)) \right\}. \tag{3.7}$$

Consider the following theorem.

Theorem 3.2.1 *[16] Following statements are equivalent for a sub-Lorentzian M.*
(i) M is strongly causal, i.e. given an open neighborhood $U = I^+(p) \cap I^-(q)$, the timelike future directed curve which leaves U never returns to U.
(ii) Open topology $\mathcal{A}$ is equal to the manifold topology $\mathcal{T}$, $\mathcal{T} = \mathcal{A}$.
(iii) $\mathcal{A}$ is Hausdorff.

Rephrasing the above theorem, we state that the curves γ are restricted to be the timelike curves whose velocity vector $\dot{\gamma}$ are included in a bracket generating distribution. This prevents γ to constitute a loop. Therefore the curve connecting $p \in M$ to $p \in M$ has length 0; i.e. $d_\eta(p, p) = 0$ as usual. Whether or not d_η constitutes a metric over M is an open problem and not necessarily trivial for most η. However, we do not require M to be a metric space under d_η in our thesis since we exploit this function as a measure rather than a metric.

Let's consider a horizontal smooth distribution $D \subset TM$. Consider any point $p \in M$, and a neighborhood U_p around it. $\forall q \in U_p$, there exists a set of vector fields $\{X_i\}, i \in \{0, 1, \ldots, k\}$ such that $D_q = \operatorname{span}\{X_0(q), \ldots, X_k(q)\}$.

As given in the preceding section, an admissible pair (u, γ_u) is the control u, and an admissible curve $\gamma : [0, \tau] \to M$ with $\dot{\gamma}(t) \in D_{\gamma(t)}$ almost everywhere. Existence of admissible pairs is determined via the bracket generating property of D.

One must realize that admissible pairs might not exist for a sub-Lorentzian M for all $p, q \in M$. Therefore, we may only define admissible pairs if $q \in M$ is reachable by a timelike curve.

In the case that D is bracket generating, and $T \subset D$ is a time orientation the distance between $p, q \in M$ is given by

$$d_\eta(p,q) = \sup_\gamma l(\gamma) \text{ for } q \in J^+(p,\eta).$$

To find the admissible pair (u^*, γ_{u^*}) whose length equals to the distance, one needs to maximize the following functional J, for the set U [15]

$$J = \int \left[u_0^2 - \sum_{j=1}^{n-1} u_j^2 \right]^{1/2} dt, \quad U = \left\{ u \in \mathbb{R}^n \mid u_0 = \left(1 + \sum_{j=1}^{n-1} u_j^2 \right)^{1/2} \right\}$$

Notice that the integrand of J appears as if we changed the signature of the metric, but remember that the u the so called control function where $u(p) \in \mathbb{R}^n$ for a point p on the curve γ, not a Lorentzian vector.

Compared to the sub-Riemannian case, determining the maximum of J is usually more cumbersome since J is a non-convex functional.

3.2.2 Existence of Geodesics

If M is a spacetime, the curves γ whose length is equal to $d_\eta(p,q) = \sup_\gamma l(\gamma)$ are said to be the geodesics of the Lorentzian manifold. We may state that in sub-Lorentzian manifold the distance between two points is the length of the geodesic curve connecting them. Given a sub-Lorentzian M, the particles moving towards the point $q \in M$ may only travel along the geodesics. Also the particles are not allowed to loop, therefore they may never travel back in time. This automatically satisfies the causality condition imposed on a spacetime. Therefore we may carry out our physical calculations in the corresponding sub-Lorentzian manifold instead of working in the original manifold where we cannot get rid of spacelike regions and timelike loops. This allows us to carry out calculations without possibly violating causality or considering spacelike regions of the spacetime.

CHAPTER 4

SETS OF FINITE PERIMETER

Throughout this section it is assumed that M is an n dimensional sub-Riemannian manifold and $\omega \in \Lambda^n(M)$ is a top form. Hence, we define the volume measure over a Borel set $E \subset M$ as:

$$\mathbf{m}(E) = \int_E \omega. \tag{4.1}$$

As discussed in the preliminaries, the characteristic function of the set $E \subset M$, $\chi_E(x)$, is the following:

$$\chi_E = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}.$$

The volume form in the local coordinates is

$$\omega = \rho dx^1 \wedge \cdots \wedge dx^n \tag{4.2}$$

for some $\rho(x) > 0$ [17]. Let us consider the Lie derivative of the volume form with respect to the vector field X, $L_X\omega$. By Cartan's formula:

$$L_X\omega = \iota_X(d\omega) + d(\iota_X\omega). \tag{4.3}$$

Since ω is a top form $d\omega = 0$. Therefore we have

$$L_X\omega = d(\iota_X\omega). \tag{4.4}$$

Writing explicitly

$$
\begin{aligned}
L_X\omega &= d(\iota_X\omega) \\
&= d\sum_k (-1)^{k-1}\rho dx^1\wedge\cdots\wedge\iota_X dx^k\wedge\cdots\wedge dx^n \\
&= d\sum_k (-1)^{k-1}(\rho X^k)\rho dx^1\wedge\cdots\wedge\widehat{dx^k}\wedge\cdots\wedge dx^n \\
&= \sum_k (-1)^{k-1}\left\{\frac{\partial}{\partial x^k}(\rho X^k)dx^k\right\}\wedge dx^1\wedge\cdots\wedge\widehat{dx^k}\wedge\cdots\wedge dx^n \qquad (4.5)\\
&= \sum_k \left\{\frac{\partial}{\partial x^k}(\rho X^k)\right\}\wedge dx^1\wedge\cdots\wedge dx^k\wedge\cdots\wedge dx^n \\
&= \left\{\frac{1}{\rho}\sum_k \frac{\partial}{\partial x^k}(\rho X^k)\right\}\rho dx^1\wedge\cdots\wedge dx^n.
\end{aligned}
$$

As in [17]

$$\operatorname{div} X = \frac{1}{\rho}\sum_{k=1}^{n}\frac{\partial}{\partial x^k}(\rho X^k). \tag{4.6}$$

Therefore we write

$$L_X\omega = (\operatorname{div} X)\omega. \tag{4.7}$$

Take a test function $\varphi\in C_c^\infty(M)$. We will integrate $\varphi L_X\omega$ over the manifold M. Consider the identity given in [3]

$$\int_M \varphi L_X\omega = \int_M \varphi(\operatorname{div} X)\omega = -\int_M (X\varphi)\omega. \tag{4.8}$$

Now consider a function $f\in C^1(M)$. Applying the above identity to $f\varphi$, we obtain

$$\int_M f\varphi(\operatorname{div} X)\omega = -\int_M (X(f\varphi))\omega. \tag{4.9}$$

Applying Leibniz rule

$$\int_M f\varphi(\operatorname{div} X)\omega = -\int_M f(X\varphi)\omega - \int_M \varphi(Xf)\omega. \tag{4.10}$$

Then

$$-\int_M \varphi(Xf)\omega = \int_M f\varphi(\operatorname{div} X)\omega + \int_M f(X\varphi)\omega. \tag{4.11}$$

Assume that X is a subset of the horizontal distribution D defined in the preceding chapters. Then for an open subset $\Omega\subset M$, (Xf) may also be realized as a distribution [3]. Consider the following definition:

Definition 4.0.1 *[3] Given an open set $\Omega \subset M$, a measure with finite total variation in Ω is the derivative of f in the sense of distributions, along X in Ω. We denote this measure by $D_X f$. If $f \in C^1(M)$, then $D_X f = (Xf)\mathbf{m}$.*

Therefore we write

$$(Xf)\mathbf{m} = \int_\Omega (Xf)\omega = \int_\Omega d(D_X f) \tag{4.12}$$

Rewriting (4.11) in Ω, we obtain

$$-\int_\Omega \varphi d(D_X f) = \int_\Omega f\varphi(\operatorname{div} X)\omega + \int_\Omega f(X\varphi)\omega. \tag{4.13}$$

Total variation of the measure $D_X f$ over Ω is denoted by $|D_X f|(\Omega)$, or simply $|D_X f|$. We use the following proposition to determine and define the variation of a measure $D_X f$.

Proposition 4.0.2 *[3] Given $\Omega \subset M, f \in L^1_{loc}(\Omega, \mathbf{m})$, $D_X f$ has finite total variation in Ω if and only if*

$$\sup_\varphi \left\{ \int_\Omega f \operatorname{div}(\varphi X)(\omega) \mid \varphi \in C_c^\infty(\Omega), |\, \varphi \,| \leqslant 1 \right\} < \infty \tag{4.14}$$

In this case we denote $|D_X f|$ as the total variation of the measure $D_X f$.

Therefore we write the following definition:

Definition 4.0.2 *The total variation of the measure $D_X f$ over $\Omega \subset M$ is*

$$\sup_\varphi \left\{ \int_\Omega f \operatorname{div}(\varphi X)(\omega) \mid \varphi \in C_c^\infty(\Omega), |\, \varphi \,| \leqslant 1 \right\} \equiv |D_X f|, \tag{4.15}$$

where $f \in L^1_{\text{loc}}(\Omega, \mathbf{m})$.

We have established how to determine whether f has finite total variation. Now we can define the space of functions of bounded variation.

4.1 Space of Functions of Bounded Variation

In this thesis, we define two spaces of functions of bounded variation, namely $BV(\Omega, g, \omega)$ and $BV(\Omega, d, \mathbf{m})$ where Ω is an open subset of the manifold M.

The space $BV(\Omega, g, \omega)$ is defined with respect to the metric g of M and the volume form ω whereas $BV(\Omega, d, \mathbf{m})$ is defined with respect to the **C-C** distance d and the volume measure $\mathbf{m}$.

4.1.1 $BV(\Omega, g, \omega)$ of Riemannian Manifolds

Remembering the definition given in the previous chapters, we pick a smooth section $\Gamma^g(\Omega, D)$ of the distribution D, where $\Gamma(\Omega, D)$ is the set of smooth vector fields $X \in TM$ such that $X(p) \in D(p)$ for all $p \in \Omega$:

$$\Gamma^g(\Omega, D) \equiv \{X \in \Gamma(\Omega, D) \mid \; g_p(X(p), X(p)) \leqslant 1, \forall p \in \Omega\}\,. \tag{4.16}$$

Consider the following definition:

Definition 4.1.1 *[3] **The space** $BV(\Omega, g, \omega)$**:** Let $\Omega \subset M$ be an open set and $f \in L^1(\Omega, \mathbf{m})$. f is said to have bounded variation in Ω if $D_X f$ exists for all $X \in \Gamma^g(\Omega, D)$ and*

$$\sup_{X \in \Gamma^g(\Omega, D)} \{|D_X f|(\Omega) : X \in \Gamma^g(\Omega, D)\} < \infty.$$

Then it is said that f belongs to the space of functions of bounded variation, $f \in BV(\Omega, g, \omega)$.

Observe that

$$\begin{aligned}
&\sup_{X \in \Gamma^g(\Omega, D)} \{|D_X f|(\Omega) \mid X \in \Gamma^g(\Omega, D)\} \\
&= \sup_{X \in \Gamma^g(\Omega, D)} \left\{ \sup_{\varphi \in C_c^\infty(\Omega)} \left\{ \int_\Omega f \operatorname{div}(\varphi X)\omega \mid \varphi \in C_c^\infty(\Omega), |\varphi| \leqslant 1 \right\} \Big| X \in \Gamma^g(\Omega, D) \right\} \\
&= \sup_{X \in \Gamma^g(\Omega, D)} \left\{ \sup_{\varphi \in C_c^\infty(\Omega)} \left\{ \int_\Omega f L_{(\varphi X)}\omega \mid \varphi \in C_c^\infty(\Omega), |\varphi| \leqslant 1, X \in \Gamma^g(\Omega, D) \right\} \right\}
\end{aligned}$$

Since φX can also be realized as an element of the distribution D, we write

$$= \sup_{\varphi X \in \Gamma^g(\Omega, D)} \left\{ \int_\Omega f L_{(\varphi X)} \omega \mid \varphi \in C_c^\infty(\Omega), |\varphi| \leqslant 1, X \in \Gamma^g(\Omega, D) \right\}.$$

Therefore we define

$$|Df| = \sup_{\varphi X \in \Gamma^g(\Omega, D)} \left\{ \int_\Omega f L_{(\varphi X)} \omega \mid \varphi \in C_c^\infty(\Omega), |\varphi| \leqslant 1, X \in \Gamma^g(\Omega, D) \right\}. \quad (4.17)$$

Since φ is only a test function we might as well consider $\varphi X \equiv X \in \Gamma^g(\Omega, D)$. Then we say that following holds if $f \in BV(\Omega, g, \omega)$:

$$|Df| = \sup_{X \in \Gamma^g(\Omega, D)} \left\{ \int_\Omega f L_{(\varphi X)} \omega \mid \varphi \in C_c^\infty(\Omega), |\varphi| \leqslant 1, X \in \Gamma^g(\Omega, D) \right\} < \infty. \quad (4.18)$$

Example: Let $M = \mathbb{R}^n$. In this case a function f is said to be of bounded variation, i.e. $f \in BV(\Omega, g, \omega)$, where $\Omega \subset \mathbb{R}^n$ is an open set with the metric g and the volume form $\omega \in \Lambda^n(M)$ if its distributional gradient is a finite $\mathbb{R}^n$ vector valued Radon measure over Ω [18]. Notice that the supremum given in (4.18) may be written equivalently for this case as

$$|Df|(\Omega) \equiv \sup_{\varphi} \left\{ \int_\Omega f \operatorname{div} \varphi dV \left| \varphi \in C_c^1 (\Omega; \mathbb{R}^n), \right| \varphi \,|\leqslant 1 \right\} < \infty. \quad (4.19)$$

Divergence of the "test function" φ is given by:

$$\operatorname{div} \varphi = \sum_{j=1}^n X_j \varphi_j, \quad (4.20)$$

where $X_j = X_j^\mu \frac{\partial}{\partial x^\mu}$. Equivalently if $\varphi \in C_c^\infty(\Omega)$

$$\int_\Omega f \frac{\partial \varphi}{\partial x_i} dV = - \int_\Omega \varphi d(D_i f). \quad (4.21)$$

Here we state a theorem given in [3] that will be of use in the proceeding chapters.

Theorem 4.1.1 *Let $\Omega \subset M$ be open and $f \in L^1(\Omega, \mathbf{m})$. Let X be a smooth vector field in $\Gamma(\Omega, D)$ and ϕ_t^X be the flow generated by X on M. If $D_X f$ is a signed measure with finite total variation, then in Ω, the following holds*

$$|D_X f|(\Omega) = \sup_{\Omega' \Subset \Omega} \left\{ \liminf_{t \to 0} \int_{\Omega'} \frac{f(\phi_t^X) - f}{|t|} \omega \mid \Omega' \Subset \Omega \right\}. \quad (4.22)$$

4.1.2 $BV(\Omega, d, \mathbf{m})$ of sub-Riemannian Manifolds

We have defined the space $BV(\Omega, g, \omega)$ above. An equivalent space of functions of bounded variation can be defined as:

Definition 4.1.2 *[18] Let $\Omega \subset M$ be open and $f \in L^1(\Omega, \mathbf{m})$. $f \in BV(\Omega, d, \mathbf{m})$ if there exists a set $\{f_n\} \subset L^1(\Omega, \mathbf{m})$ such that*

$$\limsup_{n\to\infty} \int_\Omega |\nabla f_n| d\mathbf{m} < \infty. \tag{4.23}$$

Then we define

$$\|Df\|(\Omega) \equiv \inf_{f_n} \left\{ \liminf_{n\to\infty} \int_\Omega |\nabla f_n| \, d\mathbf{m} : f_n \in \textit{Lip}_{loc}(\Omega), \lim_{n\to\infty} \int_\Omega |f_n - f| \, d\mathbf{m} = 0 \right\}, \tag{4.24}$$

where infimum is taken over f_n that satisfies $\lim_{n\to\infty} \int_\Omega |f_n - f| \, d\mathbf{m} = 0$.

$\|Df\|(\Omega)$ is the total variation of f in $\Omega \subset M$ where M is a sub-Riemannian manifold. In case $\|Df\|(\Omega)$ is finite, f is a function of bounded variation, i.e. $f \in BV(\Omega, d, \mathbf{m})$. This is understood as f is a function of bounded variation with respect to the distance measure d instead of the metric g.

Hence we have defined a space $BV(\Omega, d, \mathbf{m})$ which is the analogue of the space $BV(\Omega, g, \omega)$.

4.1.3 Equivalence of $BV(\Omega, g, \omega)$ and $BV(\Omega, d, \mathbf{m})$

In this section we give a theorem proven in [3] that shows the equivalence of the spaces $BV(\Omega, g, \omega)$ and $BV(\Omega, d, \mathbf{m})$. In order for the following theorem to hold, we must impose two conditions on the metric space and the metric measure structure. These are stated as the following.

(A1) **(Local doubling assumption)**
Let M be a sub-Riemannian manifold and U be an open set, $U \subset M$. $\forall K \subset U$ compact, $\exists l > 0$, $C \geqslant 0$ such that $\forall p \in K$, $r \in (0, l)$

$$\mathbf{m}\left(B_{2r}(p)\right) \leqslant C\mathbf{m}\left(B_r(p)\right). \tag{4.25}$$

(A2) **(Local Poincaré inequality)**

for all compact $K \subset U$, $\exists l, c, \lambda > 0$ such that

$$\int_{B_r(p)} |f - f_{p,r}| \, d\mathbf{m} \leqslant cr \int_{B_{\lambda r}(p)} |\nabla f| d\mathbf{m}, \tag{4.26}$$

where f is a locally Lipschitz function, $p \in K$ and $r \in (0, l)$ and $f_{p,r}$ is the mean value of f on the ball $B_r(p)$. Here ∇f is the slope of f as in (2.2.10).

Remark 4.1.1 *Let M be a metric space, endowed with the distance function d. Given a local Lipschitz function $f : M \to \mathbb{R}$, $|\nabla f|$ at the point $p \in M$ is defined as*

$$|\nabla f|(p) = \limsup_{q \to p} \frac{d(f(q) - f(p))}{d(q, p)}.$$

Although skipped in [3], we may show that (A1) implies (A2).

Proposition 4.1.2 *Let U be the sub-Riemannian manifold and $U \subset M$ be open. Then the condition (A1) (4.25) stated above implies (A2) (4.26) for sufficiently small $r > 0$.*

Proof

(i) (A1) holds in sub-Riemannian M.

The proof of this follows from the Ball-Box theorem (2.3.1).

(ii) (A1) implies (A2) for sufficiently small r.

Let $K \subset X$ be a compact subset of X. If f is locally Lipschitz, then $\forall y \in B_r(p)$, $|f(x) - f(p)| \leqslant kd(p, x)$ for some $k \geqslant 0, k \in \mathbb{R}$. Then

$$\int_{B_r(p)} d(f(x), f(p)) d\mathbf{m} \leqslant k \int_{B_r(p)} d(x, p) d\mathbf{m}. \tag{4.27}$$

Now use the mean value inequality

$$d(f(x), f(p)) \leqslant d(x, p) \sup_{y \in B_{d(x,p)}(p)} |f'(y)| \, . \tag{4.28}$$

Since $f_{p,r}$ is the mean value of the function in some domain, without loss of generality, we write $f_{p,r} = f(\tilde{x})$ for some fixed $\tilde{x} \in B_r(p)$. p is the center of the ball $B_r(p)$. Then there exists some $x \in B_r(p)$ that satisfies $|f(x) - f(\tilde{x})| \leqslant |f(x) - f(p)|$.

$$d(f, f_{p,r}) \equiv d(f(x) - f(\tilde{x})) \leqslant \sup_{x \in B_r(p)} d(f(x), f(p)). \tag{4.29}$$

$$\Rightarrow \int_{B_r(x)} (f, f_{p,r}) d\mathbf{m} \leqslant \int_{B_r(x)} \sup_{x \in B_r(p)} d(f(x), f(p)) d\mathbf{m}$$
$$\leqslant \int_{B_r(p)} \sup_{x \in B_r(p)} \left\{ d(x,p) \sup_{y \in B_{d(x,p)}(p)} |f'(y)| \right\} d\mathbf{m}.$$

Note that $\sup_{y \in B_r(x)} \{d(y,x)\} \leqslant r$. Then we write:

$$\Rightarrow \int_{B_r(x)} \sup_{x \in B_r(p)} \left\{ d(x,p) \sup_{y \in B_{d(x,p)}(p)} |f'(y)| \right\} d\mathbf{m} \leqslant \int_{B_r(p)} r \sup_{x \in B_r(p)} |f'(x)| \, d\mathbf{m}$$
$$= r \int_{B_r(p)} \sup_{x \in B_r(p)} |f'(x)| \, d\mathbf{m}. \tag{4.30}$$

Using the definition of $f'(z)$ and $|\nabla f|(z)$

$$r \int_{B_r(p)} \sup_{x \in B_r(p)} |f'(x)| \, d\mathbf{m} = r \int_{B_r(p)} \sup_{x \in B_r(p)} \left\{ \lim_{y \to x} \frac{d(f(y) - f(x))}{d(y,x)} \right\} d\mathbf{m}$$
$$= r \int_{B_r(p)} \limsup_{y \to x \in B_r(p)} \frac{d(f(y) - f(x))}{d(y,x)} d\mathbf{m} \tag{4.31}$$
$$= r \int_{B_r(p)} |\nabla f|(x) d\mathbf{m}.$$

Therefore we obtain

$$\int_{B_r(p)} d(f - f_{p,r}) d\mathbf{m} \leqslant r \int_{B_r(p)} |\nabla f|(x) d\mathbf{m}. \tag{4.32}$$

This resembles Poincaré inequality somehow. We will manipulate the radius of the ball to satisfy the local Poincaré inequality.

Now we use equation (4.27)

$$\int_{B_r(p)} d(f(x), f(p)) d\mathbf{m} \leqslant \int_{B_r(p)} \sup_{x \in B_r(p)} d(f(x), f(p)) d\mathbf{m}$$
$$\leqslant k \int_{B_r(p)} \sup_{x \in B_r(p)} d(x,p) d\mathbf{m}$$
$$\leqslant kr \int_{B_r(p)} d\mathbf{m}.$$

Here $k = k(p)$ is a real constant determined by $p \in M$. Now, let us use the local doubling property (A1) (4.25)

$$\int_{B_r(p)} d\mathbf{m} \leqslant C \int_{B_{\frac{r}{2}}(p)} d\mathbf{m}, \tag{4.33}$$

which can also be extended in the following manner for some finite $c \in \mathbb{R}$:

$$\int_{B_r(p)} d\mathbf{m} \leqslant C \int_{B_{\frac{r}{2}}(p)} d\mathbf{m} \leqslant C' \int_{B_{\frac{r}{4}}(p)} d\mathbf{m} \leqslant \cdots \leqslant c \int_{B_{\lambda r}(p)} d\mathbf{m}. \tag{4.34}$$

Then we have

$$kr \int_{B_r(p)} d\mathbf{m} \leqslant \tilde{c}r \int_{B_{\lambda r}(p)} d\mathbf{m}, \tag{4.35}$$

for some $\tilde{c} \in \mathbb{R}$.

If $r > 0$ is sufficiently small region of the domain such that $\sup_{x \in B_{\lambda r}(p)} |f'(x)| = \sup_{x \in B_r(p)} |f'(x)|$ is constant, we have

$$kr \int_{B_r(p)} \sup_x |f'(x)| d\mathbf{m} \leqslant \tilde{c}r \int_{B_{\lambda r}(p)} \sup_x |f'(x)| d\mathbf{m}, \tag{4.36}$$

for some finite constant $\tilde{c} > 0$. Then we obtain

$$r \int_{B_r(p)} \sup_x |f'(x)| \, d\mathbf{m} \leqslant \tilde{c}r \int_{B_{\lambda r}(p)} \sup_x |f'(x)| d\mathbf{m}. \tag{4.37}$$

Working out the equation (4.31) for this case, and combining with the equation (4.30), we obtain

$$\int_{B_r(x)} d(f - f_{x,r}) d\mathbf{m} \leqslant \tilde{c}r \int_{B_{\lambda r}(p)} |\nabla f|(x) d\mathbf{m}. \tag{4.38}$$

■

Hence we conclude that sufficiently small compact neighborhoods of M satisfies local Poincaré inequality. Therefore we may state that the results given in this thesis hold for small neighborhoods automatically.

This result cannot be extended to an arbitrary r such that $B_r(p) \subset U$, where $p \in K \subset U$ lies in a compact K. Therefore local doubling (A1) (4.25) is not a sufficient condition for regions of arbitrary volume. Therefore our results are valid for volumes that satisfy (A1) (4.25) and (A2) (4.26) separately. We can state other properties of such regions. Consider the following theorem:

Theorem 4.1.3 ***((A1) and (A2) implies rectifiability:)*** *[19] Let M be locally compact and $\mathbf{m}$ be a locally doubling measure. Assume that in M a local Poincaré inequality holds. Then M is locally rectifiably path-connected. This means that $\forall p \in M$ there exists finite constants $r_0, k(p) > 0$ such that every $p, q \in B_{r_0}(p)$ can be connected by a curve γ whose length is $l(\gamma) \leqslant \sup_k\{k(p)d(p,q)\}$.*

It can be deduced from the above proof also that for sufficiently small r_0, local doubling and rectifiability also imply local Poincaré inequality. However, it must be noted that rectifiability itself does not necessarily imply (A1) and (A2) if larger neighborhoods are considered.

Also it must be noted that conditions (A1) and (A2) imply rectifiability if M is locally compact. Therefore even if we take a region Ω of the Lorentzian manifold M, which satisfies (A1) and (A2), it might fail to be locally path-connected. If we are integrating some function over Ω, we might need to consider the integration techniques on non-rectifiable domains.

We might visualize the problem at hand assuming that we integrate a Lipschitz function over some non-rectifiable Ω and we wish to make use of the divergence theorem. Tackling this problem without ensuring path connectivity and rectifiablity might result in failure of the divergence theorem. For the time being, our framework is resticted to Riemannian manifolds and we do not encounter this issue. Therefore we might consider the following theorem.

Theorem 4.1.4 *[3] Let $\Omega \subset M$ be an open set. Assume that $f \in L^1(\Omega)$. Let X be a local orthonormal frame such that $X_i = u \circ \sigma_i$, and $\{\sigma_i\}$ is a local orthonormal frame of $\mathbf{U} \in TM$ where $i = 1, \ldots, n$. Then the following conditions are equivalent:*

(i) $\sup\{|D_X f|(\Omega) \big| X = u \circ \sigma, \sigma \in \Gamma^g(\mathbf{U}_\Omega), |\sigma| \leqslant 1\} < \infty$;

(ii) $f \in BV(\Omega, d, \mathbf{m})$;

(iii) $f \in BV(\Omega, g, \omega)$.

A corollary of this theorem can be stated as below:

Corollary 1 *Let $\Omega \subset M$ be an open set. Assume that $f \in L^1(\Omega)$. If the conditions given in (4.1.4) hold, then we write*

$$\|Df\|(\Omega) = \sup_X \{|D_X f|(\Omega) : X = u \circ \sigma, \sigma \in \Gamma(\mathbf{U}|_{\boldsymbol{\Omega}}), |\sigma| \leqslant 1\} < \infty. \tag{4.39}$$

We state the proof of this corollary as given in [3] since we will modify this to the sub-Lorentzian case later on.

Proof: [3]

Define the following:

$$s(\Omega) = \sup_X \{|D_X f|(\Omega) : X = u \circ \sigma, \sigma \in \Gamma(U|_\Omega), |\sigma| \leqslant 1\}. \tag{4.40}$$

In order to show the equality in (4.39), we show $s(\Omega) \leqslant \|Df\|(\Omega)$ and $\|Df\|(\Omega) \leqslant s(\Omega)$.

In [3] $s(\Omega) \leqslant \|Df\|(\Omega)$ is cited without proof, given that the class of vector fields considered in the definition of $\|Df\|(\Omega)$ is larger. We may write this result more openly. Remember the definitions of $\|Df\|(\Omega)$ and $|D_X f|$.

$$\|Df\|(\Omega) \equiv \inf_{f_n} \left\{ \liminf_{n\to\infty} \int_\Omega |\nabla f_n| \, d\mathbf{m} : f_n \in \text{Lip}_{\text{loc}}(\Omega), \lim_{n\to\infty} \int_\Omega |f_n - f| \, d\mathbf{m} = 0 \right\}. \tag{4.41}$$

$$\begin{aligned} |D_X f| &= \sup_\varphi \left\{ \int_\Omega f \operatorname{div}(\varphi X)(\omega) \mid \varphi \in C_c^\infty(\Omega), |\, \varphi \,| \leqslant 1 \right\}, \\ &= \sup_\varphi \left\{ \int_\Omega f L_{\varphi X}(\omega) \mid \varphi \in C_c^\infty(\Omega), |\, \varphi \,| \leqslant 1 \right\}, \end{aligned} \tag{4.42}$$

where $X \in \Gamma(U|_\Omega, D)$. Let $\varphi X = Y \in \Gamma(U|_\Omega, D)$. Then Y induces curves γ with speed $g(\dot\gamma, \dot\gamma) < 1$ via the flow ϕ_t^Y, $t \in [0, \tau]$. Then $L_Y \omega$ is, by using the theorem

(4.1.1),

$$
\begin{aligned}
|D_Y f|(\Omega) &= \sup_{\Omega' \Subset \Omega} \left\{ \liminf_{t \to 0} \int_{\Omega'} \frac{|f(\phi_t^Y) - f|}{|t|} \omega \mid \Omega' \Subset \Omega \right\}, \\
&\leqslant \sup_{\Omega' \Subset \Omega} \left\{ \limsup_{t \to 0} \int_{\Omega'} \frac{|f(\phi_t^Y) - f|}{|t|} \omega \mid \Omega' \Subset \Omega \right\},
\end{aligned} \tag{4.43}
$$

where $|.|$ is realized as the distance in the image of f.

We know that f is a Lipschitz function, therefore $\frac{|f(\phi_t^Y)(x) - f(x)|}{|t|}$ is uniformly bounded on a compact $K \subset \Omega$, then $\limsup_{t \to 0} \frac{|f(\phi_t^Y)(x) - f(x)|}{|t|} \leqslant |\nabla f|(x)$ [3].

$$
\int_\Omega \limsup_{t \to 0} \frac{|f(\phi_t^Y)(x) - f(x)|}{|t|} d\mathbf{m} \leqslant \int_\Omega |\nabla f| d\mathbf{m}. \tag{4.44}
$$

Hence, it is concluded that $|D_X f|(\Omega) \leqslant \|Df\|(\Omega)$. So that $s(\Omega) \leqslant \|Df\|(\Omega)$. So now, we must prove that $\|Df\|(\Omega) \leqslant s(\Omega)$ holds.

Assume that in Ω, $X_i = u \circ \sigma_i$, where $X_i = \sum_{\mu=1}^n X_i^\mu \frac{\partial}{\partial x^\mu} \equiv X_i^\mu \partial_\mu$. Let $f \in C^1(\Omega)$. In this case, the measure $|\nabla f|$ is the slope of f and $D_X f = (Xf)\mathbf{m}$. Then it sufices to prove

$$
\int_\Omega |\nabla f| d\mathbf{m} \leqslant |Xf|(\Omega),
$$

where

$$
|Xf|(\Omega) = \sqrt{\sum_{i=1}^n (X_i f)^2 \mathbf{m}},
$$

since

$$
\sqrt{\sum_{i=1}^n (X_i f)^2}
$$

is the Riemannian norm of Xf.

f is a $C^1(\Omega)$ function. Assume further that there exists a curve γ such that $\dot{\gamma} = \sum_{i=1}^n c_i X_i(\gamma)$ where $c \in L^2([0, \tau]; \mathbb{R}^m)$ and $\gamma(0) = x$, $\gamma(\tau) = y$. In [3], $|.|$ denote the distance in Y, where $f(\Omega) \subset Y$. Let us write the distance $|f(x) - f(y)|$ as in [3] which is actually the distance between $f(x)$ and $f(y)$ in the sub-Riemannian manifold, i.e. $|f(x)-f(y)| = d(f(x), f(y))$.

$$
\begin{aligned}
|f(x)-f(y)| = d(f(x), f(y)) &= \left| \int_0^\tau d_{\gamma(t)} f(\dot{\gamma_t}) dt \right|, \\
&= \left| \int_0^\tau \sum_{i=1}^n c_i(t) X_i f(\gamma_t) dt \right|, \\
&\leqslant c^2 \sup_{t\in[0,\tau]} \sqrt{\sum_{i=1}^n (X_i f(\gamma_t))^2}.
\end{aligned}
\tag{4.45}
$$

Since X_i is arbitrary we might normalize c^2.

$$
d(f(x), f(y)) \leqslant \sup_{t\in[0,\tau]} \sqrt{\sum_{i=1}^n (X_i f(\gamma_t))^2}. \tag{4.46}
$$

By the definition of ∇f,

$$
|\nabla f| = \lim_{y\to x} \frac{d(f(x), f(y))}{d(x,y)} \leqslant \lim_{y\to x} \left\{ \sup_{z\in B_{d(x,y)}x} \left\{ \sqrt{\sum_{i=1}^n (X_i f(z))^2} \right\} \right\}. \tag{4.47}
$$

Integrating over the limit Ω we must consider every little ball $B_{d(x,y)}x$ in Ω, then

$$
\int_\Omega |\nabla f| d\mathbf{m} \leqslant \int_\Omega \sup_{z\in\Omega} \left\{ \sqrt{\sum_{i=1}^n (X_i f(z))^2} \right\} d\mathbf{m}. \tag{4.48}
$$

The following result is stated in [3] as the supremum becomes redundant for continuous Xf,

$$
\int_\Omega |\nabla f| d\mathbf{m} \leqslant \int_\Omega \left\{ \sqrt{\sum_{i=1}^n (X_i f(z))^2} \right\} d\mathbf{m}. \tag{4.49}
$$

Hence we conclude

$$
\|Df\|(\Omega) \leqslant |Xf|(\Omega). \tag{4.50}
$$

Therefore $s(\Omega) = \|Df\|$. ■

This implies that if a function f is determined to be of bounded variation in the manifold endowed by the sub-Riemannian distance, then it is certainly a function of bounded variation in the original Riemannian manifold.

4.1.4 $BV(\Omega, \eta, \omega)$ of Lorentzian Manifolds

As defined in the previous sections, we might as well define the set $BV(\Omega, g, \omega)$ for a Lorentzian manifold.

Let M be a Lorentzian manifold of dimension n. Here $X \in TM$ is a smooth vector field as usual. The definition of total variation does not necessarily require the metric to be positive definite. Remember the definition (4.0.2)

$$\sup_{\varphi} \left\{ \int_{\Omega} f \operatorname{div}(\varphi X)(\omega) \mid \varphi \in C_c^{\infty}(\Omega), \mid \varphi \mid \leqslant 1 \right\} \equiv |D_X f|_{\eta}.$$

If this supremum exists, we will use it as $|D_X f|_{\eta}$. If it does not exist, it is said that the total variation of $D_X f$ is not finite.

We repeat the given definitions of BV spaces for the Lorentzian and sub-Lorentzian cases.

Definition 4.1.3 ***The space*** $BV(\Omega, \eta, \omega)$***:*** *Let $\Omega \subset M$ be an open set and $f \in L^1(\Omega, \mathbf{m})$. If $D_X f$ exists for all $X \in \Gamma^{\eta}(\Omega, D)$ and*

$$\sup_{X \in \Gamma^{\eta}(\Omega, D)} \{|D_X f|_{\eta}(\Omega) : X \in \Gamma^{g}(\Omega, D)\} < \infty,$$

then $f \in BV(\Omega, \eta, \omega)$.

The Lorentzian nature of the metric does not pose any constraint over this definition.

4.1.5 $BV(\Omega, d_{\eta}, \mathbf{m})$ of sub-Lorentzian Manifolds

In order to define $BV(\Omega, d_{\eta}, \mathbf{m})$ for the sub-Lorentzian manifold, we use the definition (4.1.2). Remember for a function to be in $BV(\Omega, d, \mathbf{m})$

$$\inf_{f_n} \left\{ \liminf_{n \to \infty} \int_{\Omega} |\nabla f_n| \, d\mathbf{m} : f_n \in \mathrm{Lip}_{\mathrm{loc}}(\Omega), \lim_{n \to \infty} \int_{\Omega} |f_n - f| \, d\mathbf{m} = 0 \right\} \leqslant \infty$$

must hold for the sequence $\{f_n\} \subset L^1(\Omega, \mathbf{m})$ that satisfies $\limsup_{n \to \infty} \int_{\Omega} |\nabla f_n| d\mathbf{m} < \infty$ and $\lim_{n \to \infty} \int_{\Omega} |f_n - f| \, d\mathbf{m} = 0$.

Similar to the Lorentzian case, as sub-Lorentzian distance is already positive definite there is virtually no reason for this definition not to hold. Then we will denote the total variation of f in a sub-Lorentzian manifold with respect to C-C distance d_η as, $\|Df\|_\eta$.

4.1.6 Equivalence of $BV(\Omega, \eta, \omega)$ and $BV(\Omega, d_\eta, \mathbf{m})$

In this section we must show that spaces $BV(\Omega, \eta, \omega)$ and $BV(\Omega, d_\eta, \mathbf{m})$ are equivalent for the spacetime M. To achieve this result we prove a theorem similar to (4.1.4)

Theorem 4.1.5 *Let $\Omega \subset M$ be an open set. Assume that (A1) (4.25) and (A2) (4.26) hold in Ω. Also assume that $f \in L^1(\Omega)$. Let X be a local orthonormal frame such that $X^i = u \circ \sigma^i$ $(i = 0, \ldots, n-1)$ and $\{\sigma^i\}$ is a local orthonormal frame of $\mathbf{U} \in TM$ and $u : U \to TM$ is a smooth map, linear over vector bundles such that $f(U_x) \subset T_x M$, for all $x \in M$. Then the following conditions are equivalent for Minkowski metric η:*

(i) $f \in BV(\Omega, d_\eta, \mathbf{m})$;

(ii) $f \in BV(\Omega, \eta, \omega)$.

Moreover, if one of the above holds

$$\|Df\|_\eta(\Omega) = \sup_X \{|D_X f|_\eta(\Omega) : X = u \circ \sigma, \sigma \in \Gamma(\mathbf{U}|_{\mathbf{\Omega}}), |\sigma| \leqslant 1\} < \infty. \tag{4.51}$$

Proof: Define

$$s(\Omega) = \sup_X \{|D_X f|_\eta(\Omega) : X = u \circ \sigma, \sigma \in \Gamma(\mathbf{U}|_{\mathbf{\Omega}}), |\sigma| \leqslant 1\} < \infty. \tag{4.52}$$

We must show $\|Df\|_\eta(\Omega) \leqslant s(\Omega)$ and $s(\Omega) \leqslant \|Df\|_\eta(\Omega)$.

$s(\Omega) \leqslant \|Df\|_\eta(\Omega)$ follows from the proof given for (4.1.4) and the theorem (4.1.1). Notice that theorem (4.1.1) does not depend on the Riemannian nature of the metric but rather on the assumption that $\Omega' \subset K \subset \Omega$ such that

$$\begin{aligned} |D_X f|_\eta(\Omega) &= \sup_\varphi \left\{ \int_\Omega f \operatorname{div}(\varphi X)(\omega) \mid \varphi \in C_c^\infty(\Omega), | \varphi | \leqslant 1 \right\}, \\ &= \sup_{\Omega' \Subset \Omega} \left\{ \liminf_{t \to 0} \int_{\Omega'} \frac{|f(\phi_t^X) - f|}{|t|} \omega \mid \Omega' \Subset \Omega \right\}, \end{aligned} \tag{4.53}$$

which follows from Meyers-Serrin theorem as stated in [3]. So, we expect that functions of bounded variation of a sub-Lorentzian M with respect to d_η to be also of bounded variation of a Lorentzian metric space.

We concentrate on whether or not $\|Df\|_\eta(\Omega) \leqslant s(\Omega)$ holds.

Assume that for $X \in \Gamma(U|_\Omega)$, $X_i = u \circ \sigma_i$. Let γ be a timelike curve such that $\dot{\gamma}(t) = \sum_{i=0}^{n-1} c_i(t) X_i(\gamma(t))$ for some $c \in L^2([0,\tau]; \mathbf{R}^n)$ and $\gamma(0) = x$, $\gamma(\tau) = y$. Let $d_\eta(f(x), f(y))$ be denoted by $|f(x) - f(y)|$. Since γ is timelike, $\eta(\dot{\gamma}, \dot{\gamma}) < 0$. Observe that

$$d_\eta(f(x), f(y)) = \left| \int_0^\tau d_{\gamma(t)} f(\dot{\gamma}_t) dt \right|. \tag{4.54}$$

We might square this equation to get

$$\begin{aligned}(d_\eta(f(x), f(y)))^2 &= \left[\int_0^\tau d_{\gamma(t)} f(\dot{\gamma}_t) dt \right]^2, \\ &= \left[\int_0^\tau \sum_{i=0}^{n-1} c_i(t) X_i f(\gamma(t)) dt \right]^2, \\ &\leqslant \sup_{t \in [0,\tau]} \left| \left(-(c_0(t) X_0 f(\gamma_t))^2 + \sum_{i=1}^{n-1} (c_i(t) X_i f(\gamma_t))^2 \right) \Big|_{t=0}^{t} \right|\end{aligned} \tag{4.55}$$

If $d_\eta(f(x), f(y))$ and $\sup_{t \in [0,\tau]} c_i(t)$ is finite, then we can find a real number $\tilde{c}$ such that

$$\begin{aligned}(d_\eta(f(x), f(y)))^2 &\leqslant \sup_{t \in [0,\tau]} \left| \left(-(c_0(t) X_0 f(\gamma_t))^2 + \sum_{i=1}^{n-1} (c_i(t) X_i f(\gamma_t))^2 \right) \Big|_{t=0}^{t} \right| \\ &\leqslant \tilde{c}^2 \sup_{t \in [0,\tau]} \left| \left(-(X_0 f(\gamma_t))^2 + \sum_{i=1}^{n-1} (X_i f(\gamma_t))^2 \right) \Big|_{t=0}^{t} \right|\end{aligned} \tag{4.56}$$

Assume that there is another curve $\tilde{\gamma}$ such that $\dot{\tilde{\gamma}} = \sum_{i=0}^{n} X_i(f(\gamma_t))$. Then above equation becomes

$$\Rightarrow (d_\eta(f(x), f(y)))^2 \leqslant \tilde{c}^2 \left[\sup_{t \in [0,\tau]} \left| \int_0^\tau \sum_{i=0}^{n} X_i(\tilde{\gamma}(t)) dt \right| \right]^2.$$

Since X is arbitrary, we may choose X_i such that $\tilde{c}$ is set to 1. Therefore

$$\Rightarrow (d_\eta(f(x), f(y)))^2 \leqslant \sup_{t\in[0,\tau]} \left| \left(-(X_0\tilde{\gamma}(t))^2 + \sum_{i=1}^{n-1} (X_i\tilde{\gamma}(t))^2 \right) \Bigg|_{t=0}^{t} \right|.$$

By the definition of ∇f,

$$\begin{aligned} |\nabla f|^2 &= \left[\lim_{y\to x} \frac{d_\eta(f(x), f(y))}{d_\eta(x,y)} \right]^2 \\ &= \left[\lim_{y\to x} \frac{d_\eta(f(x), f(y))}{\sup_{\gamma\in\{\gamma|\gamma(0)=x,\gamma(\tau)=y\}} l(\gamma)} \right]^2 \\ &\leqslant \left[\lim_{t\to 0} \left\{ \frac{\sup_{t\in[0,\tau]} \left| \int_0^\tau \sum_{i=0}^n X_i(\tilde{\gamma}(t))dt \right|}{\sup_\gamma l(\gamma)} \right\} \right]^2 . \end{aligned} \tag{4.57}$$

The right hand side of the equation (4.57) can also be written equivalently as

$$\lim_{y\to x} \frac{\sup_{z\in B_{d(x,y)}x} \left\{ \left| -(X_0 f(z))^2 + \sum_{i=1}^{n-1} (X_i f(z))^2 \right| \right\}}{(d_\eta(x,y))^2}.$$

Integrating the square root of both sides over Ω we obtain

$$\int_\Omega |\nabla f(x)| d\mathbf{m} \leqslant \int_\Omega \left(\limsup_{z\in B_r(x)\subset\Omega} \frac{\left| -(X_0 f(z))^2 + \sum_{i=1}^{n-1} (X_i f(z))^2 \right|}{r^2} \right)^{1/2} \omega. \tag{4.58}$$

We exploit the fact that X is arbitrary to manipulate the above equation to give

$$\int_\Omega |\nabla f(x)| d\mathbf{m} \leqslant \int_\Omega \left(\limsup_{z\in B_r(x)\subset\Omega} \left| -(X_0 f(z))^2 + \sum_{i=1}^{n-1} (X_i f(z))^2 \right| \right)^{1/2} \omega. \tag{4.59}$$

Remember for $f \in C^1$ we have $D_X f = (Xf)\mathbf{m} = (\sum_{i=0}^{n-1} X_i^\mu \partial_\mu f)\mathbf{m}$. Moreover $s(\Omega) = \sup_X\{|D_X f|_\eta(\Omega)\} = |Df|_\eta(\Omega)$. Exploiting the definition of $|D_X f|_\eta$ we rewrite the above equation

$$\int_\Omega |\nabla f(x)| d\mathbf{m} \leqslant |Xf|(\Omega) \equiv s(\Omega). \tag{4.60}$$

The rest follows similarly to the proof of (4.1.4). Therefore we write

$$\|Df\|_\eta(\Omega) \leqslant |Xf|_\eta(\Omega). \tag{4.61}$$

Hence $s(\Omega) = \|Df\|$. Thus (ii) implies (i). ■

The validity of $BV(\Omega, g, \omega) \subset BV(\Omega, d, \mathbf{m})$ is usually an open problem for a given C-C distance d_η. In [3] by theorem (4.1.4), it is shown that for a sub-Riemannian theorem this inclusion is valid. Here in our thesis, we have shown that it is also valid for the sub-Minkowski case. Therefore we conclude that if a function is of bounded variation in sub-Lorentzian M where η is the Minkowski metric, it is also bounded variation in the original Minkowski spacetime. Moreover if we can find such finite $\tilde{c}$ that satisfies

$$(d_\eta(f(x), f(y)))^2 \leqslant \tilde{c}^2 \sup_{t\in[0,\tau]} \left| \sum_{i=0}^{n-1} X_i f(\gamma(t)) \right|^2, \tag{4.62}$$

for the given $\{X_i\}$, then the theorem holds independent of the metric η given.

4.2 Sets of Finite Perimeter

Let us give the following definition for a Riemannian manifold M.

Definition 4.2.1 *[3] A set $E \subset \Omega$ is said to be of finite perimeter if its characteristic function $\chi_E \in BV(\Omega, g, \omega)$, for a Riemannian manifold $M \supset \Omega$.*

Perimeter of E in a subset $U \subseteq \Omega$ is defined as the following

$$P(E, U) \equiv |D\chi_E|\,(U), \tag{4.63}$$

$$P(E, U) = \sup \left\{ \int_U \chi_E \operatorname{div} \varphi dx \,\middle|\, \varphi \in C_c^1\left(U, \mathbb{R}^n\right), |\, \varphi \,| \leqslant 1 \right\} < \infty. \tag{4.64}$$

Therefore if E is a finite perimeter set in Ω, $P(E, \Omega)$ is a finite Radon measure. Given this case, it is not hard to see that the Sobolev space $W^{1,1}(\Omega) \subset BV(\Omega)$ [18], therefore the Poincaré inequality holds.

The $|.|$ defines the variation of sets with respect to the measure ω, which is the volume measure in our study, hence $|\Omega|$ simply denotes the volume of the open set Ω.

Assume that we have such a region E in $\Omega \subset M$ that has $\chi_E \in BV(\Omega)$. Then $D\chi_E$ is an $\mathbb{R}^n$ valued vector measure with finite total variation $|D\chi_E|$.

Example: Let M be $\mathbb{R}^2$, and $\Omega = B_1(0) \subset M$, the unit ball. The basis $X = \{e_1, e_2\}$ is the usual $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$. Then the perimeter defined on the unit ball is:

$$P\left(B_1(0), \Omega\right) = \sup \int_{B_1(0)} \chi_E \operatorname{div} \phi \, dx = \sup \int_{B_1(0)} X_j^* \phi_j dx = \sup \int_{B_1(0)} \frac{\partial \phi_j}{\partial x} dx, \tag{4.65}$$

since characteristic function equals 1 on the unit ball.

Note that $\phi_j \in C_c^1\left(B_1(0), \mathbb{R}^2\right)$ and $\frac{\partial \phi_j}{\partial x^j}$ is continuous, and we have $|\phi| \leqslant 1$. Hence it is evident that $\sup \int \partial \phi_j / \partial x^j dx$ is bounded, therefore $P\left(B_1(0), \Omega\right) < \infty$. This shows that the unit ball of $\mathbb{R}^2$ is a finite perimeter set.

4.3 Finite Perimeter sets of sub-Lorentzian Manifolds

In this section we explore the finite perimeter sets of Lorentzian manifolds. Similar to the Riemannian case, a set $E \subset M$ is said to be of finite perimeter if $\chi_E \in BV(\Omega, \eta, \omega)$, where η is the original Lorentz (Minkowski for the extend of this thesis) metric and ω is the volume form on M.

Let $D \subset TM$ be a bracket generating horizontal distribution. Let $D = \operatorname{span}\{X^i\}$. Using the definition of the total variation of the measure $D\chi_E$ the perimeter of E in Ω is defined as [3]

$$\begin{aligned} P\left(E, \Omega\right) &= \sup_{\varphi X \in \Gamma(\Omega, D)} \int_\Omega \chi_E \operatorname{div}(\varphi X)(\omega) dx, \\ &= \sup_{\varphi X \in \Gamma(\Omega, D)} \int_E X_j(\varphi_j) dx, \\ &= \sup \int_E \sum_{i=0}^{n-1} \frac{\partial \varphi_j}{\partial x^i} dx^i. \end{aligned} \tag{4.66}$$

Then,

$$D\chi_E = \sup_{\varphi X \in \Gamma(\Omega, D)} \int_E \sum_{i=0}^{n-1} \frac{\partial \varphi_j}{\partial x^i} dx^i = \sup \int_E d(D_X \varphi). \tag{4.67}$$

If $\varphi \in C_C^\infty$ then φ is bounded and Lipschitzian on $\Omega \subset M$. Therefore the measure $D\chi_E$ is bounded if the directional derivative of φ along X does not blow up. In this case, the set $E \in M$ is a finite perimeter set of the Lorentzian manifold M. That is, if $D\chi_E$ is finite, then $\chi_E \in BV(\Omega, \eta, \omega)$.

Since we have shown the equivalence of the spaces $BV(\Omega, \eta, \omega)$ and $BV(\Omega, d_\eta, \mathbf{m})$, if $\chi_E \in BV(\Omega, \eta, \omega)$ then $\chi_E \in BV(\Omega, d_\eta, \mathbf{m})$ and vice versa. Therefore instead of determining a volume being finite perimeter in the Lorentzian manifold, we might as well use the sub-Lorentzian counterpart which is already endowed by a positive definite distance function. Using this method we might work out the perimeter measures of our regions of interest as if they were already Riemannian volumes.

CHAPTER 5

GAUSS-GREEN THEOREM

The motivation of this thesis is to discuss the validity of Gauss-Green theorem on arbitrary volumes of physical spacetimes. It seems reasonable that Gauss-Green theorem must hold for any given volume as long as the volume is finite perimeter.

5.1 The Gauss-Green Formula

The generalized Gauss-Green formula given in [8] is

$$\int_E D_j f(x)dx = \int_{\partial E} f(x)\nu_j(E,x)ds, \tag{5.1}$$

where $\nu_j(E,x)$ is the jth component of the exterior normal $\nu(E,x)$. Our main problem is to establish a condition between E and ∂E such that 5.1 holds.

We know that if ∂E is a Gaussian surface 5.1 indeed holds. Since a spacetime in general is not locally compact, we might not simply take ∂E to be a Gaussian surface regardless of the local properties of the spacetime.

5.2 Riezs Representation Theorem

Consider the following lemma:

Lemma 5.2.1 ***Urysohn's Lemma:*** *[21] A space X is normal if and only if for all closed $A, B \subset X$ such that $A \cap B = \emptyset$, there exists a continuous function $f : X \to [0,1]$ satisfying $f(A) = 0$, $f(B) = 1$.*

Remembering the fact that every metrizable or pseudo-metrizable space is normal [21], then locally compact subspaces of a Lorentzian manifold satisfies (5.2.1).

Consider the sub-Lorentzian manifold M. The topology on M is Alexandrov topology. Since it is finer than metric topology we may also state that (5.2.1) holds for sub-Lorentzian case too.

Notice that this lemma suggests that we may approach the characteristic function χ_E by a continuous f. In [21] f is explicitly stated. For a sequence of nested open sets U_{r_i} such that

$$A \subset U_{r_1} \cdots \subset U_{r_k}, \overline{U_{r_k}} \cap B = \emptyset,$$

$$f(x) = \begin{cases} 1 & x \notin U_{r_i}, \forall i \in \{1, \ldots, k\}, \\ \inf_{U_{r_i} \in \{U_{r_i}\}} \{r_i | x \in U_{r_i}\} & \text{otherwise.} \end{cases} .$$

The continuity of this function is worked out in [21]. Realizing this function as a measure we may construct its compact support as

$$\operatorname{spt} f = X \setminus \bigcup \{O | O \in \mathcal{T}, f(O) = 0\} . \tag{5.2}$$

Therefore the compact support of f contains all the sets that are not contained in the set A, for which $f(A) = 0$.

Then we write the compact support of f as

$$\operatorname{spt} f = \{B | B \in \mathcal{T}, f(B) \neq 0\} = \{x \in X | f(x) \neq 0\} , \tag{5.3}$$

assuming that f continuously exists everywhere on X. Notice that it might not be trivial to show this for any given Lorentzian manifold.

Now, let us consider the Riesz's representation theorem as stated in [2]:

Theorem 5.2.2 ***Riesz's Representation Theorem*** *[2] Let X be a locally compact Hausdorff space. Let L be the set of all continuous maps $f : X \to \mathbb{R}$ having the following compact support in a Borel subset $E \subset X$*

$$\operatorname{spt} f = \overline{\{x | x \in E, f(x) \neq 0\}}. \tag{5.4}$$

(This means that we take such f whose support is closed in E.) Let I be a positive linear functional such that

$$\sup_{g\in L} I(L \cap \{g|0 \leqslant g \leqslant f\}) < \infty, \qquad \forall f \in L. \tag{5.5}$$

Then there exists a unique Radon measure μ such that

$$I(f) = \int_E f d\mu, \qquad \forall f \in L. \tag{5.6}$$

In [3], a version of 5.2.2 is written for the sub-Riemannian case. Let us examine this in the following section.

5.2.1 Riezs Representation Theorem for sub-Riemannian Manifold

Theorem 5.2.3 *[3]* ***Riesz's theorem in sub-Riemannian manifolds***
Let $f \in BV(\Omega, g, \omega)$. There exists a Borel vector field ν_f such that $g(\nu_f, \nu_f) = 1$ satisfying

$$D_X f = g(X, \nu_f)\|Df\|, \qquad \forall X \in \Gamma^g(\Omega, D) \subset TM \tag{5.7}$$

where X is a smooth vector field and $\|Df\|$ is the variation defined as before.

If $\chi_E \in BV(\Omega, g, \omega)$ for $E \subset \Omega$, we denote ν_{χ_E} by ν_E. Let $X_i = u \circ \sigma^i$ be given. In this case $\nu_E = u(\sum_i \nu^*_{E,i}\sigma^i)$ where $\nu^*_{E,i}$ is the ith component of the dual of the vector ν_E. ν_E and ν^*_E are called geometric normal and dual normal of E, respectively [3]. We will soon conclude that this normal ν_E is the vector $\nu(A, x)$ of (5.1).

Notice that in theorem (5.2.2), the local compactness of X is required. However as cited in [3], theorem (5.2.3) follows from the equivalence of BV sets. Therefore conditions required to obtain this result are (A1) (4.25) and (A2) (4.26) only. We stated that (A1) and (A2) combined implies rectifiability for locally compact M. Therefore we know by (4.1.3) that a subset Ω in which theorem (5.2.3) holds is rectifiable. The converse might not hold.

By Hopf-Rinow theorem, a closed and bounded subset C of the connected Riemannian manifold M is compact, therefore locally compact [6]. However, theorem (5.2.3) does not require the open set Ω to be contained in such C. This is a strong indication that one might deduce the same result for a Lorentzian manifold M.

5.2.2 The sub-Lorentzian Case

We observe that theorem (5.2.3) does not require manifold M to be locally compact. This is good news as a Lorentzian M is generally not locally compact. Given a locally doubling region $\Omega \subset M$ that satisfies a local Poincaré inequality, we may write an equivalent theorem for Ω. Although it is not stated and proven in this thesis, we established a strong premise towards the existence of ν_f as given in (5.7). The condition $g(\nu_f, \nu_f) = 1$ might be replaced by $\eta(\nu_f, \nu_f) = -1$ depending on the signature of given Lorentz metric.

5.3 Reduced Boundary

We know that for a finite perimeter set E (5.1) holds [1]. The boundary satisfying (5.1) need not be unique. To visualize, consider the case $B_r(p) \subset \mathbb{R}^n$. $\mathbb{R}^n$ is evidently a Riemannian manifold. The boundary $\partial B_r(p)$ satisfies (5.1) as well as $\partial B_r(p) \cup \gamma$ where γ is simply a line. Given such a boundary, we may eliminate this line as $\partial B_r(p) \cup \gamma$ overcompensates for $\partial B_r(p)$.

Assume that we have a Borel set $E \subset M$, we might as well eliminate objects such as hairs, tentacles, etc. attached to ∂E that does not contribute to the integral. The boundary we obtain is said to be the reduced boundary. It is not always trivial to determine this set. Let us give the formal definition below.

Let E be a finite perimeter set of M. As suggested in [3] the implications of (A1) (4.25) and (A2) (4.26) on E are given in the following proposition.

Proposition 5.3.1 *[3] : Under the assumption that (A1) and (A2) hold and $E \subset \Omega$ with characteristic function $\chi_E \in BV(\Omega, d, \mathbf{m})$, the following is true:*

$$\liminf_{r \to 0} \frac{\min \{\mathbf{m}\,(B_r(x) \cap E)\,, \mathbf{m}\,(B_r(x) \setminus E)\}}{\mathbf{m}\,(B_r(x))} > 0 \tag{5.8}$$

and

$$\limsup_{r \to 0} \frac{\|D\chi_E\|\,(B_r(x))}{\mathbf{m}\,(B_r(x))\,/r} < \infty, \quad \forall x \in \Omega. \tag{5.9}$$

Basically, as the ball shrinks at $x \in E$, the dual normals must approach each other. If we have a region that does not satisfy this, then the region does not rectify properly.

Definition 5.3.1 *[3][1]* ***The Reduced Boundary:*** *The reduced boundary $\mathcal{F}E$ of $E \subset M$ is the set of points in the topological boundary $x \in \partial E$ that satisfies the conditions (5.8), (5.9) and*

$$\lim_{r \to 0} \frac{1}{\|D\chi_E\|(B_r(x))} \int_{B_r(x)} |\nu_E^*(y) - \nu_E^*(x)|^2 d\|D\chi_E\|(y) = 0. \tag{5.10}$$

The reduced boundary is especially useful in the study of General Relativity as the reduced boundary is a rectifiable set that has a finite measure. This guarantees that on these sets we can exploit Gauss-Green formulas. Using the reduced boundaries of spacetime regions we can remarkably reduce our surfaces of integration from totally unknown shapes with possibly uncountably many blow-up points, to relatively tame and simple curves without imposing assumptions on the geometric properties of a spacetime region.

5.3.1 C-C Balls and Boxes and Some Volumes of Interest

Let M be a Riemannian manifold and $D \subset TM$ be a bracket generating distribution spanned by $\{X_i\}$, where $i = \{1, \ldots, n\}$. We define the number $\deg X_j$ as the number of X_is involved in the successive commutation equivalent to X_j. For example, if $[X_1, X_2] = X_3$, $\deg X_3 = 2$. As D is bracket generating, we must use all the vector fields X_i in successive commutation to define TM, therefore $\deg TM = n$ [6].

Let $\phi_t^{X_i} = \exp(tX_i)$ be the flow generated by X_i. Then $\exp(t_1, \ldots, t_n) = \exp(t_1 X_1 + \cdots + t_n X_n)$ as in [6].

The $\mathbb{R}^n$ box is defined as

$$\text{Box}(r) = \{|t_i| \leqslant r^{\deg X_i} | i = \{1, \ldots, n\}\}. \tag{5.11}$$

By Hölder equivalence of C-C and Euclidean metrics stated in [6], we consider the image of a box in $\mathbb{R}^n$ under exponential map as the C-C box. Therefore a version of Ball-Box theorem holds for sufficiently small balls.

Theorem 5.3.2 *The sufficiently small $B_r(p)$ satisfies the following condition for some continuous function $c(p) > 0$ and a constant $r_0(p)$*

$$\exp \mathrm{Box}(c^{-1}r) \subset B_r(p) \subset \exp \mathrm{Box}(Cr); \qquad \forall p \in M, r \leqslant r_0(p). \tag{5.12}$$

If M is not compact, exponential map need not be defined everywhere in its domain $\mathbb{R}^n$. This implies that the above Ball-Box theorem might not hold globally for a Lorentzian M. As discussed before, this implies that a ball covering the spacetime cannot be approximated by given C-C boxes. Therefore the integral of the volume form need not be equivalent for balls and boxes covering larger regions of spacetime. As a result, we might fail to have $\lim_{r\to\infty} \int_{B_r} f dV = \lim_{r\to\infty} \int_{\mathrm{Box}(r)} f dV$.

We have stated the conditions for finite perimeter sets of sub-Lorentzian and Lorentzian manifolds to be equivalent. Therefore we propose the following method to determine the largest domain on which Gauss-Green formulas hold and the integration is independent of the choice of local coordinates. Take a sequence of rectifiable, finite perimeter sets $E_1 \subset \ldots E_i \subset \cdots \subset M$, find its finite perimeter upper bound (if exists) E, then construct the possibly infinite union $\bigcup_i B_{r_i} \supset E$ such that $\chi_{B_{r_i}} \in BV(M, d, \mathbf{m})$. For each sufficiently small B_{r_i}, find $\exp^{-1}(B_{r_i}) \subset \mathbb{R}^n$ and use the union of such sets in $\mathbb{R}^n$ as the volume of integration.

Notice that $\exp$ is always defined in a sufficiently small neighborhood U of origin in $\mathbb{R}^n$. If M is a local doubling space we might cover a larger region with a possibly infinite union of small neighborhoods $\exp(U)$. Therefore we might find the largest volume of integration containing E that satisfies Gauss-Green theorem bypassing the condition of local compactness.

As an example, we might be interested in calculating the total flux across the universe by integrating the field over a surface covering the entirety of the physical spacetime. Evidently, we must assume the surface of integration is locally compact to ensure Gauss-Green formulas hold. However the local compactness of such an arbitrary surface is not guaranteed, therefore we might be wrong in our assumption to begin with. Therefore we propose to use sub-Lorentzian metric to define the volumes of integration and make use of reduced boundaries to use as surfaces of integration to eliminate such difficulties.

CHAPTER 6

CONCLUSIONS

This thesis shows us how to define finite perimeter sets of Lorentzian spacetimes. We have shown that if $E \subset M$ is a finite perimeter set, then $|\chi_E| \leqslant \infty$ and $\chi_E \in BV(\Omega, \eta, \omega)$ for the Minkowski metric η. Equivalently we have also shown that whenever $\chi_E \in BV(\Omega, \eta, \omega)$, also $\chi_E \in BV(\Omega, d_\eta, \mathbf{m})$. Therefore we have established how to define and measure the perimeter of a given set. We have also discussed the conditions for $BV(\Omega, \eta, \omega)$ and $BV(\Omega, d_\eta, \mathbf{m})$ to be equivalent spaces.

It must be noted that if we, as physicists, wish to calculate the surface integral over the whole spacetime, there are some “traps” we must be aware of. If we choose to integrate over concentric spheres that cover the whole spacetime, we might not find a box that will “fit” in between those spheres. Suppose we travel along a curve on the box that covers the entirety of the spacetime, as we pass through the corners of the box, velocity of the curve might fail to be a timelike vector. Hence we conclude that this cube fails to be a finite perimeter set in sub-Lorentzian setting. Therefore we cannot assume the boundary of the box to satisfy Gauss-Green formulas, whereas we might not encounter the same issue while traveling on the boundary of a ball.

Another point of discussion is the works of Battista and Esposito [1]. They suggest a method for determining the sets of finite perimeter of a Lorentzian manifold M by relating a Riemannian metric to the original Lorentzian one and checking whether or not the set is finite perimeter. Notice that this method fails to exclude the points that are not reachable in Lorentzian setting, therefore does not map the physically not-so-tame regions of Lorentzian manifold onto the light cone. It also fails to exclude the curves that might not be tangent to the horizontal distribution, or the ones that might pass through the regions that cannot be mapped onto regions of positive curvature.

In [1], it is suggested that, one may find a metric g equivalent to the metric η such that there exists a diffeomorphism between them. This is done by constructing the equivalence class between infinitely many Riemannian metrics that correspond to η and associate the set of points that form the reduced boundary of E. We have shown that the use of sub-Lorentzian structure is just as efficient in determining the perimeter of the given subset and it uses the natural choice of Carnot-Carathéodory metric. Moreover choosing to use the sub-Lorentzian machinery, we need not assume local compactness on the surfaces of integration. This result is important as most Lorentzian spacetimes fail to be locally compact.

Schoen and Yau [22] state that there is a connection between the geometry of minimal hypersurfaces and scalar curvature, exploited to prove the famous positive mass conjecture. This proof mainly relies on choosing the hypersurfaces in an asymptotically flat manifold that can be mapped onto a Riemann surface of positive curvature. We propose that this result can be generalized into the Lorentzian case with the use of finite perimeter sets and reduced boundary. Also it can be extended to show that the conjecture might fail in certain metric spaces due to the failure we might face in mapping the boundaries of certain volumes onto a Riemannian surface.

We know that by Hopf-Rinow Theorem [6] the closed and bounded subsets of a complete Riemannian manifold is compact, therefore locally compact. But for the spacetimes we do not have the luxury of assuming compactness. Hence the generality of positive mass theorem should be checked by choosing hypersurfaces in an asymptotically flat manifold that can be mapped into a sub-Lorentzian manifold.

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