AN INTERPRETIVE EXAMINATION OF ANALOGIES USED IN NINTH-GRADE MATHEMATICS CLASSROOMS ON THE UNIT OF FUNCTIONS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

EMEL ÜNVER SEZER

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
SECONDARY SCIENCE AND MATHEMATICS EDUCATION

SEPTEMBER 2022

Approval of the thesis:

AN INTERPRETIVE EXAMINATION OF ANALOGIES USED IN NINTH-GRADE MATHEMATICS CLASSROOMS ON THE UNIT OF FUNCTIONS

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ABSTRACT

AN INTERPRETIVE EXAMINATION OF ANALOGIES USED IN NINTH-GRADE MATHEMATICS CLASSROOMS ON THE UNIT OF FUNCTIONS

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September 2022, 298 pages

The primary purpose of the current study is to examine how two mathematics teachers and their 121 ninth-grade students from five different classes employ analogies in the unit of functions in natural classroom settings and the features of used analogies. This study also portrays how the features of analogies vary from teacher to teacher and from class to class.

The corpus of the materials collected for the present study includes video recordings and transcriptions of lessons, field notes, informal conversations with teachers, and all teaching materials. Identified 215 analogies are analyzed according to the Analogy Features Framework (AFF) developed based on the criteria of Curtis and Reigeluth (1984) and of Thiele and Treagust (1994a).

The findings of the study revealed that teachers mainly chose analogies from their own experiences or knowledge bases, revolved around a few favorite analogies by repeating or extending, and used almost the same analogies with the same features in all their classes with a few differences that they were not aware. On the other hand, findings disclosed that the students generally generated analogies upon the request of their teachers, and their analogies had similar features to those of teachers.

The research suggests that teachers need to have a repertoire of sound analogies, including correctly identified analog and target attributes and epistemologically valid mappings, which will increase their ability to create effective analogies during classroom practices. It also recommends the systematic presentation of analogies through a theoretical model to maximize their potential benefits and minimize their potential limitations.

Keywords: Functions, Analogy, Interpretive Examination

ÖZ

DOKUZUNCU SINIF MATEMATİK SINIFLARINDA FONKSİYON ÜNİTESİNDE KULLANILAN ANALOJİLERİN YORUMLAYICI İNCELEMESİ

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Eylül 2022, 298 sayfa

Bu çalışmanın temel amacı, iki matematik öğretmeni ve onların beş farklı sınıftan 121 dokuzuncu sınıf öğrencisinin doğal sınıf ortamlarında fonksiyonlar ünitesinde analojileri nasıl kullandıklarını ve kullanılan analojilerin özelliklerini incelemektir. Bu çalışma aynı zamanda analojilerin özelliklerinin öğretmenden öğretmene ve sınıftan sınıfa nasıl değiştiğini de ortaya koymaktadır.

Bu çalışma için toplanan materyallerin bütünü, derslerin video kayıtları ve transkripsiyonlarını, alan notlarını, öğretmenlerle yapılan resmi olmayan sohbetleri ve tüm öğretim materyallerini içerir. Belirlenen 215 analoji, Curtis ve Reigeluth (1984) ve Thiele ve Treagust (1994a) kriterlerine dayalı olarak geliştirilen Analoji Özellikleri Çerçevesine (AFF)'ye göre analiz edilmektedir.

Araştırmanın bulguları, öğretmenlerin analojileri çoğunlukla kendi deneyimlerinden veya bilgi temellerinden seçtiklerini, tekrar ederek veya genişleterek birkaç favori analoji etrafında döndüklerini ve neredeyse aynı özelliklere sahip analojileri kendilerinin de farkında olmadıkları birkaç farklılıkla tüm sınıflarında kullandıklarını ortaya koymuştur. Öte yandan, bulgular öğrencilerin analojilerini

genellikle öğretmenlerinin istekleri üzerine oluşturduklarını ve analojilerinin öğretmenlerininkilere benzer özelliklere sahip olduklarını açığa çıkarmıştır.

Araştırma öğretmenlerin, sınıf uygulamaları sırasında etkili analojiler oluşturma yeteneklerini arttıracak, doğru tanımlanmış analog ve hedef nitelikleri ve epistemolojik olarak geçerli haritalanmalar içeren sağlam analojiler repertuarına sahip olmaları gerektiğini göstermektedir. Ayrıca, potansiyel faydalarını çıkarmak ve potansiyel sınırlamalarını en aza indirmek için analojilerin teorik bir model aracılığıyla sistematik olarak sunulmasını önermektedir.

Anahtar Kelimeler: Fonksiyonlar, Analoji, Yorumlayıcı İnceleme

To my husband, Burak, my little daughter, Elis Mila, and my mother, Elif

ACKNOWLEDGMENTS

Foremost, I would like to acknowledge and give my warmest thanks to my supervisor Prof. Dr. Behiye Ubuz, who made this work possible. Her guidance and advice carried me through all the stages of writing my dissertation. I would like to thank the rest of my thesis committee members: Prof Dr. Didem Akyüz, Prof Dr. Yezdan Boz, Prof. Dr. Lindsey Engle Richland, and Prof. Dr. Şenol Dost, for their enlightening comments and suggestions.

I would like to thank Tuta Sonbay Özgen, Füsun Ersoy, and İsmail Aydın for allowing me to collect data and for supporting me. I would also like to thank the teachers and their students for participating in this study.

Besides, I am deeply grateful to my parents, Elif and Gültekin; my sisters, Tülay and Derya; and my brother, Tanju, for their continuous support and understanding during this long journey and throughout my life.

Last but not least, I would like to give special thanks to my husband, Burak, and my daughter, Elis Mila; without their support and encouragement, none of this would exist.

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CHAPTER 1

INTRODUCTION

Is there anyone who learned functions without an analogy or teaches them without using an analogy? Or has anyone seen that the function is defined or explained without an analogy in various sources, from the books we read in high school and university years to the current mathematics curriculum to the textbooks we employ in our classrooms? Very few people engaged in education will answer these questions as “that’s me!” reflects how much function and analogy are riveted together.

Fundamental to the mathematical understanding of science, technology, engineering, and mathematics (STEM) disciplines (Evangelidou et al., 2004; McCulloch et al., 2019) and a springboard for college-level mathematical contents (Nyikahadzoyi, 2015), function concept is abstract and notoriously difficult for most students (Ayalon & Wilkie, 2017; Borke, 2021; Elia, Panaoura, Eracleous, & Gagatsis, 2007; Elia, Panaoura, Gagatsis, Gravvani, & Spyrou, 2008; Panaoura et al., 2017; Tabach & Nachlieli, 2015). Numerous research on secondary school and undergraduate students’ conceptions of function (e.g., Borke, 2021; L.L. Clement, 2001; Tabach & Nachlieli, 2015) pointed to many common misconceptions and difficulties.

Many (e.g., Hatisaru & Erbas, 2017; Panaoura, Michael-Chrysanthou, Gagatsis, Elia, & Philippou, 2017; Panaoura, Michael-Chrysanthou, & Philippou, 2015) found that students have trouble holding the Dirichlet-Bourbaki definition of a function, which defines a function as “a correspondence between two nonempty sets that assigns to every element in the first set (domain) exactly one element in the second set (co-domain)” (Vinner & Dreyfus, 1989, p. 357). As anticipated, many (e.g., Ayalon & Wilkie, 2017; Hatisaru & Erbas, 2017) also found that students have limited knowledge about essential features of functions – univalence, which refers to the requirement of mapping each element in the domain to exactly one element in the co-domain (Hatisaru, 2022; L. L. Clement, 2001; Vinner & Dreyfus, 1989), and arbitrariness, which refers to the relationship between the two sets and the sets themselves do not have to exhibit some regularity (Even, 1993). Apart from these, many disclosed that students are also unaware of multiple representations of functions such as tables, formulas, graphs, arrow diagrams, and verbal expressions and have difficulty handling the flexibility of these different representations (Akkoc & Tall, 2003; Elia & Spyrou, 2006; Hatisaru & Erbas, 2010, 2017; Markovits et al. 1986; Panaoura et al., 2015).

This being the case, the pedagogical questions are how to overcome these troubles in the classroom (Sierpinska, 1992, p. 25) and how to improve students' understanding of functions (Hatisaru, 2022, p.2). As the people who plan and carry out the teaching process, teachers try to meet these questions in many ways. Although it is not known how each teacher finds a pedagogical solution to these questions, a large body of works demonstrates that in-service teachers regularly use analogies and pre-service teachers tend to use analogies during their teaching functions in mathematics classrooms around the world (for example, see Akkoc, 2006; Ayalon & Wilkie, 2017; Bayazit & Aksoy, 2011; Bayazit & Gray, 2006; Bayazit & Ubuz, 2008; Elia & Spyrou, 2006; Espinoza-Vásquez et al., 2017; Evangelidou et al., 2004; Hatisaru, 2020, 2021, 2022; Hatisaru & Erbas, 2010, 2017; Nachlieli & Tabach, 2012; Tasdan & Koyunkaya, 2017; Ubuz, Eryilmaz, Aydin, & Bayazit, 2009; Ubuz, Ozdil, & Cevirgen, 2013; Unver, 2009; Unver-Sezer, 2021; Unver-Sezer & Ubuz, 2008). For instance, Espinoza-Vásquez et al. (2017) observed an experienced mathematics teacher (Arturo) and noted that Arturo used the *washing machine* analogy to support his students' understanding of functions. Similarly, Hatisaru & Erbas (2017) observed and interviewed two experienced mathematics teachers (Ali and Fatma) and disclosed that both teachers frequently resorted to analogies during their function concept teaching. They reported that Ali, who defined function as a transformation, exemplified it with analogies such as *a function is like the grinding of wheat at the mill to make flour* or *a function is like a machine* *transforming coffee beans into powder form*. Moreover, they notified that both teachers used a textbook analogy (*restaurant analogy*) to reinforce function definition, especially its univalence feature. More recently, Hatisaru (2020, 2021, 2022) analyzed teacher responses referencing analogies for the function in a questionnaire administered to a group of teachers that included Ali and Fatma. They noted that teachers' definitions, examples, and questions had analogies such as *a function is like a mother-child relationship* or *a function is like a machine transforming inputs into outputs*, which were likely used to reduce the abstractness of the formal definition of the function. In another instance, Bayazit and Ubuz (2008) observed another experienced mathematics teacher (Burak) during his function concept teaching. They reported that Burak offered analogies as an advanced organizer to prepare his students for the concept of function or as an activator to arouse their knowledge when solving problems related to functions.

Shulman (1986) suggested that teachers must know ways to represent and formulate the subject to make it understandable to others, and one way to support this is through analogies. Analogies, which are the comparisons of similarities between two concepts (Glynn, 2015), provide a bridge between what students already know (analog) and what they are about to learn (target) (Dagher, 1995; Thiele & Treagust, 1992). In other words, they make the unfamiliar familiar (Treagust et al., 1992). For instance, comparing *functions* (target) with *the children-mother relationship* (analog) helps to infer that domain and co-domain sets of a function consist of any objects other than numbers, such as children in the domain and mothers in the co-domain in this analogy (arbitrariness requirement). The fact that each child has only one mother helps infer that for each element in the domain, there must be only one element in the co-domain (univalence requirement).

Analogies used in mathematics are not proofs themselves (N. Adams & Elliot, 2013; Zwicky, 2010); however, they are "illustrations" (Pimm, 1981, p. 47), "vehicles of insight" (Zwicky, 2010, p. 11), or "alternative representations of a situation" (Fast, 1997, p.10). They help simplification of new concepts through the medium of connection to already understood concepts (Fu, 2019). For instance, when finding the domain of the function $f(x) = \frac{x^3+1}{x^2-x-6}$ by analogy, it is required to find a simpler one (for example, $f(x) = \frac{x+1}{x-2}$) to compare with the original function. Since simplification is a type of conversion, which is the basic idea of problem-solving (ibid), analogies are used to find solutions to mathematical problems by comparison (English, 1993; Gick & Holyoak, 1983; Novick & Holyoak, 1991; Polya, 1954).

Analogies capture parallels across different concepts, contexts, or domains (Gentner, 1998; Gick & Holyoak, 1983; D. P. Newton, 2012, Richland, 2011; Vamvakoussi, 2017; Vendetti et al., 2015). Thus, analogies allow teachers to demonstrate common structures among concepts, representations, problems, or procedures and transfer knowledge and inferences (Gentner, 1998). Such analogies may compare two mathematics concepts, representations, problems, or procedures (e.g., the addition of functions is like the addition of algebraic expressions). They also may compare a mathematical concept to a non-mathematical entity (Richland & McDonough, 2010) (e.g., functions are like a machine).

Analogies may not be beneficial for all kinds of target concepts (Curtis & Reigeluth, 1983). However, it is known that the target concepts must be abstract, sufficiently novel, and challenging for analogies to be potentially helpful for students (Duit, 1991; Gick & Holyoak, 1983; Gray & Holyoak, 2021; Harrison & Treagust, 2006; Hayes & Tierney, 1980; Treagust et al., 1992; Venville & Treagust, 1997). At this juncture, the abstract nature of the function and the potential advantages of analogies make analogical reasoning highly relevant to the function concept. However, despite all these indications that analogies may be an essential component of teaching and learning the concept of function, surprisingly, little is known about how analogies are employed for functions in regular classroom settings and the features of analogies used.

Despite nearly seventy years of research generated on analogies in mathematics education, limited research focused on or addressed the use of analogies in functions. One group of studies (e.g., Akkoc, 2006; Aksu & Kul, 2016; Ayalon et al., 2017;

Ayalon & Wilkie, 2017; Bayazit & Gray, 2006; Bardini et al., 2014; Elia & Spyrou, 2006; Evangelidou et al., 2004; Gülbagci Dede et al., 2022; Hatisaru, 2020; Hatisaru & Erbas, 2010, 2017; Nachlieli & Tabach, 2012; Tasdan & Koyunkaya, 2017; Vinner, 1983) addressed analogies in functions generated by pre-service or in-service teachers or students; however, it was not their primary focus. Another group of studies (e.g., DeMarois, 1997, 1998; DeMarois & Tall, 1999; Davis & McGowen, 2002; Kabael, 2011; McGowen et al., 2000; Tall et al., 2000) focused on particularly the use of the function machine (or input-output box) and recently the use of vending machine applet (e.g., Bailey et al., 2019; McCulloch, Lovett, Dick, Sherman, Edgington, & Meagher, 2020; McCulloch, Lovett, & Edgington, 2017, 2019; M. Meagher et al., 2019; Sherman, Lovett, McCulloch, Edgington, Dick, & Casey, 2018; Sherman, Meagher, Lovett, & McCulloch, 2019), characterizing them as a kind of metaphor. On the other hand, remaining studies centered around pre-service or in-service teachers' knowledge and/or beliefs about both function and analogy use and their association with teachers' teaching practices (e.g., Bayazit & Aksoy, 2011; Espinoza-Vásquez et al., 2017; Ubuz et al., 2013), analysis of directly pre-service or in-service teacher-generated analogies in functions (e.g., Bayazit & Aksoy, 2011; Bayazit & Ubuz, 2008; Font et al., 2010; Hatisaru, 2021, 2022; Ubuz, Eryilmaz, Aydin, & Bayazit, 2009; Ubuz, Ozdil, & Cevirgen, 2013; Unver, 2009), analysis of textbook analogies in functions (e.g., Unver, 2009), and students' perceptions about teacher-generated analogies during teaching and learning functions (e.g., Unver-Sezer, 2021; Unver-Sezer & Ubuz, 2018).

Although a few (e.g., Ayalon et al., 2017; Ayalon & Wilkie, 2017; Bayazit & Ubuz, 2008; Elia & Spyrou, 2006; Evangelidou et al., 2004; Hatisaru & Erbas, 2017) directly worked with students, it is clear from the existing literature that almost all prior studies directly or indirectly examining analogies in functions tended to involve working with either pre-service or in-service teachers. Accordingly, they mostly concentrated on teacher-generated analogies, and student-generated analogies only appeared when quoting teacher-generated ones. Differently, only one study (Ubuz et al., 2013) was interested in analogies generated by both teachers and students.

Still, it did not go beyond reporting how many of the whole classroom analogies were student-generated. However, to more fully understand the mental processes that students employ when using analogies, the focus needs to be on the use of analogies not only by teachers but also by both teachers and students (Thiele & Treagust, 1991, 1994a).

Furthermore, studies directly collected data on teacher-generated analogies in functions used research methods including pencil and paper questionnaires (e.g., Hatisaru, 2021, 2022; Ubuz et al., 2013) and knowledge tests (e.g., Bayazit & Aksoy, 2011; Ubuz et al., 2013), interviews (e.g., Bayazit & Aksoy, 2011; Bayazit & Ubuz, 2008; Ubuz et al., 2013; Unver-Sezer, 2021; Unver-Sezer & Ubuz, 2018), and observations (e.g., Bayazit & Ubuz, 2008; Espinoza-Vasquez et al., 2017; Ubuz, Eryilmaz, Aydin, & Bayazit, 2009, Ubuz, Ozdil, & Cevirgen, 2013; Unver, 2009). While all were carried out to understand analogy use in function teaching and learning, a limited number of studies made observations. Some of these observations were carried out in fictitious mathematics classroom settings, where one pre-service teacher acted as a teacher and others acted as students in a university classroom (e.g., in Ubuz, Eryilmaz, Aydin, & Bayazit, 2009, Ubuz, Ozdil, & Cevirgen, 2013). Although observations in fictitious classroom settings provide some insight into analogies created in natural classroom environments, they are inadequate to fully elucidate how these analogies are employed in everyday interactions to teach and learn (Richland et al., 2004) and the features of analogies. Although a few studies (e.g., Bayazit & Ubuz, 2008; Unver, 2009) made observations in a natural classroom environment, they are still insufficient to thoroughly explain how classroom analogies are produced during teaching and learning function concepts. However, classroom setting, function-related concepts themselves, interactions between teachers and students, and many uncontrollable factors mentioned in many studies (e.g., Curtis & Reigeluth, 1983; Donnelly, 1990; Duit, 1991; Harrison & Jong, 2005a, 2005b; Iding, 1997; Mastrilli, 1997; Newby et al., 1995; Newby & Stepich, 1991; Nottis & McFarland, 2001; Orgill & Bodner, 2005; Schenke & Richland, 2017; Thiele, 1995; Thiele & Treagust, 1992; Treagust, 1993; Venville & Treagust,

1997) affect analogy use (Thiele & Treagust, 1994b). As Guerra-Ramos (2011) said, how the analogies are presented, the contexts in which they are offered, and how much students are involved in mapping the analogical relations are more determinant than the analogies themselves. Therefore, if analogies are such widely used pedagogical and learning tools for functions in mathematics classrooms, they merit being explored in situ, that is, in natural classroom settings that will provide more prosperous and more reliable data.

In addition, very few studies concentrated on analyzing the features of analogies. For instance, Bayazit & Ubuz (2008) focused on whether analogies were generated to explain function-related ideas or stress procedures and whether analogs had intrinsic power to represent function-related ideas. Similarly, Ubuz et al. (2009) concentrated on whether analogies were appropriate to illustrate the essence and the properties functions and whether there was a structural relation between analog and target concepts. In addition, Bayazit & Aksoy (2011) concentrated on whatever Bayazit and Ubuz (2008) focused on. Additionally, if analogies were developed to explain function-related concepts, they focused more on whether analogies addressed the function as a relation matching between elements of two sets or as a process transforming every input into a unique output (p. 123). More recently, Hatisaru (2021, 2022) was concerned with understanding the structure of analogy mappings of identified analogies according to Gentner's (1983) structure-mapping theory and the sorts of function conceptions addressed by these analogies. She investigated whether comparisons were an "analogy" (based on only common relational attributes) or an "anomaly" (based on just common object attributes) and whether they addressed functions as "an input-output machine" or "a mapping between two sets."

As far as it is known, only two studies (Ubuz et al., 2013; Unver, 2009) concentrated on more analogy features on the concept of functions. These two studies analyzed the features of teacher-generated analogies according to a framework proposed by Thiele and Treagust (1994a). Based on the original criteria presented by Curtis and Reigeluth (1984), this framework was developed to evaluate the characteristics of chemistry textbook analogies, including (1) the content of the target concept, (2) the location of the analogy through the textbook, (3) analogical relationship between analog-target pairs, (4) their format, (5) the position of the analog relevant to the target, (6) their position in the instruction, (7) the level of enrichment, (8) the presence of analog explanation and strategy identification, and (9) the presence and amount of limitations. Ubuz et al. (2013) classified analogies as appropriate or inappropriate, depending on whether the basic properties of functions (targets) were correctly mapped to the analogs and the characteristics included in this framework. While Ubuz et al. (2013) used the framework with a few modifications, Unver (2009) used almost the original version with only minor adaptations needed to classify instructional analogies. However, the fact that mathematics has some differences from science and examining classroom analogies instead of textbook analogies will cause differences arising from the context; the compatibility of the analogies to be determined during function teaching and learning with the aforementioned framework requires a more detailed examination. Besides, many years ago, Curtis and Reigeluth (1984) and their successor framework developers recommended further exploration and analysis of analogy characteristics.

Although the use of analogies while teaching and learning functions in mathematics classrooms is fairly common, the specific ways mathematics teachers use analogies differ from teacher to teacher and class to class. However, studies that analyzed the characteristics of teacher-generated analogies mentioned so far gave information about how their properties changed, albeit partially, from teacher to teacher but did not provide any information about how their properties changed from class to class. Namely, Bayazit and Ubuz (2008) and Espinoza-Vasquez et al. (2017) did not have a chance to compare the analogies created by other teachers and created in different classes since they examined analogies created by only one teacher (Burak and Arturo, respectively) in his one class. On the other hand, Ubuz, Eryilmaz, Aydin, and Bayazit (2009) inspected analogies generated by five pre-service teachers (PTs), Bayazit and Aksoy (2011) 22 PTs, Ubuz, Ozdil, and Cevirgen (2013) 7 PTs, Unver (2009) 2 mathematics teachers, and Hatisaru (2021, 2022) 26 mathematics teachers.

However, except for Unver (2009), no study specified the characteristics of the analogies established by each teacher, and no investigation did not focus mainly on how the characteristics change from teacher to teacher. Instead, while only discussing the general features of teacher-generated analogies, they gave examples from analogies created by teachers from time to time in relevant places. However, instead of considering all teacher-generated analogies as a whole and evaluating their features, a separate evaluation for each teacher may make it easier to see the connections between analogy features more clearly. Moreover, showing how analogy features change or do not change from teacher to teacher provides a more coherent picture of analogy features, allowing for explicit consideration of the teacher variable and making more grounded recommendations. In addition, as expected, none of these studies examined how the characteristics of the analogies employed varied from class to class. However, studying and comparing the features of analogies used in different teachers' classrooms with the features of analogies used in different classrooms of the same teachers will provide a more consistent picture of analogy use and allow precise handling of teacher, student, and context variables. Besides, such a study will heed the calls of Thiele and Treagust (1991, 1994a), who emphasized the necessity of designing studies that report not only the end results of analogy use but also the processes that occur.

A review of available literature combining both functions and analogy suggested focusing on both student- and teacher-generated analogies during function concepts teaching and learning in regular mathematics classrooms and further exploration and the analysis of their features proposed by Curtis and Reigeluth (1984) and Thiele and Treagust (1994a), and searching for how the features of analogies vary from teacher to teacher and from class to class to contribute to a growing body of research in this field. From this point of view, the current case study aimed to conduct an interpretive examination focusing on how two mathematics teachers and their students from different classes employed analogies while function unit teaching and learning in the natural classroom settings and the features of their generated analogies. More specifically, the present study aimed to answer the subsequent research questions:

(1) how are the features of teacher- and student-generated analogies in the ninth-grade functions unit? (2) how do the features of analogies employed in the ninth-grade functions unit differ from teacher to teacher and from class to class?

Analysis of analogies used in this study opens new directions to research on teaching and learning function concepts with analogies and alternative representations of functions. The new framework developed in this study for the classification of both teacher-and student-generated analogies will assist future researchers in building and testing the analogy features proposed here or examine the use of analogies in other mathematics contents other than functions or non-math contents.

In addition, the current study will provide a source on how to structure and present analogies for pre-service and in-service mathematics teachers, textbook writers, and policymakers planning and searching for appropriate and efficient instruction at the high school level. Namely, assertions and interpretations regarding the features of analogies reported in this study and suggestions for analogy implementation will assist pre-service and in-service teachers in improving or refining their analogies, deepen their understanding of function concepts and help them foresee possible analogy-caused student misconceptions.

On the other side, interpretations and assertions related to student- and teacher-generated analogies and suggestions for their implementation will provide a source for textbook authors on how to integrate analogies into their textbooks while guiding on how to present them in textbooks. Notably, by helping textbook authors recognize the importance of textbook analogies for teachers and students, they may encourage them to define and articulate the limitations of analogies.

Similarly, interpretations and suggestions on using analogies in this study will help policymakers realize the importance of analogies for teaching and learning mathematics and that their use should not be left to chance. Parallel to this, curriculum designers may design mathematics curricula that allow teachers to implement analogies properly.

CHAPTER 2

LITERATURE REVIEW

In this chapter, the literature review relevant to current research is presented as follows: (1) the concept of function in school mathematics, (2) the nature of analogy, (3) features of analogies and their classifications, (4) analogies and metaphors in mathematics education, and (5) factors limiting the use of analogies in a classroom context.

2.1 The Concept of Function in School Mathematics

2.1.1 Definition of Function and Its Place in School Mathematics

The concept of function, which has a fundamental and unifying role in modern mathematics (Ayalon & Wilkie, 2017, 2019; Carlson & Oehrtman, 2005; L. L. Clement, 2001; Eisenberg, 2002; Jones, 2006; Mesa, 2004; Tabach & Nachlieli, 2015; P. W. Thompson & Carlson, 2017), reached its accepted definition and current value in school mathematics at the end of a gradual evolution dating back to 4000 years (Cooney & Wilson, 1993; Kleiner, 1989, 1993). Function evolved from being an implicit tool with no name and definition in mathematics and science to the soul of mathematics. Briefly, it progressed from a relation or a dependence between varying quantities to a geometric object, an algebraic equation, an analytic expression, arbitrary correspondence between (real) numbers (Dirichlet), and lastly, a correspondence (or mapping) between arbitrary sets (Bourbaki). This progress can also be defined as a move from a dynamic covariation approach, emphasizing how changes in one variable (dependent variable) are related to changes in another variable (independent variable) or how two change together (Ayalon et al., 2017; P. W. Thompson & Carlson, 2017) to a static correspondence approach. The correspondence definition (also called the contemporary or modern definition) of the function, first introduced by Dirichlet and later refined by Bourbaki in the 1930s, is often referred to as the Dirichlet-Bourbaki definition, Dirichlet-Bourbaki, or simply Bourbaki approach. This definition states that a function is "a correspondence between two nonempty sets that assigns to every element in the first set (domain) exactly one element in the second set (co-domain)" (Vinner & Dreyfus, 1989, p. 357). Besides, a function is also referred to as a set of ordered pairs with the Dirichlet-Bourbaki definition since Bourbaki later formulated an equivalent function definition as a particular subset of the Cartesian product $E \times F$. (For an extended discussion of this issue, see Kleiner, 1989). In this definition, the relation between dependent and independent variables is tacitly stated.

According to the Dirichlet-Bourbaki definition, a function has two essential features: arbitrariness and univalence (or one-valuedness). The former, implicitly specified in the definition, refers to the arbitrary nature of the domain and co-domain and their relationship (Even, 1993). The arbitrary nature of the domain and co-domain means that these two sets can be any set of objects (not especially sets of numbers). On the other hand, the arbitrary nature of the relationship means that functions do not require any regularity. That is, they can be defined by arbitrary expressions or graphs with any particular shapes (Even, 1993; Even & Tirosh, 1995) and arbitrarily do matching or transformation (Bayazit, 2011; Bayazit & Aksoy, 2011). The latter, explicitly specified in the definition, refers to that (1) for every element in the domain, (2) there must be a unique element (image) in the co-domain. This feature allows one-to-one or many-to-one mappings, but not one-to-many mappings (Ayalon & Wilkie, 2019; Bayazit & Ubuz, 2008; Hatisaru, 2021, 2022; Steele et al., 2013).

Although the evolution regarding the definition of function is still maintained (Cooney & Wilson, 1993; Gok, et al., 2019; P. W. Thompson & Carlson, 2017), the Dirichlet-Bourbaki definition is the most commonly used one in mathematics today. The gradual evolution in the accepted meaning of function naturally influenced the pedagogical philosophy of function. Felix Klein, a well-known German mathematician, in 1908, in his “Meran programme,” emphasized the importance of functions and proposed incorporating functions into school mathematics (Cooney & Wilson, 1993; Mesa, 2004; Sears et al., 2016; Sierpinska, 1992). This movement first influenced mathematics education in countries such as France, Austria, Hungary, America, and England and then spread worldwide quickly (Hamley, 1934). Calls for emphasizing functions as relations between quantities in all areas of secondary mathematics and suggestions for using graphs to teach functions and relating functions to real-world situations were central themes of the pedagogical movement (Cooney & Wilson, 1993). With the “New Math” era in the late 1950s, the function was rigidly integrated into the school curriculum. Function as an ordered pair was accepted as a compulsory part of school mathematics. Although there is still no complete agreement, the Dirichlet-Bourbaki definition, or its some versions, is still the most generally accepted and used textbook definition in the school curricula in many countries today (Akkoc & Tall, 2005; Ayalon et al., 2017). With this definition, teaching methods such as mapping diagrams (also called arrow diagrams or set correspondence diagrams), input-output models, and cartesian graphs are emphasized.

After the New Math reform, the themes of discussions, calls, and recommendations did not change over time. Discussions about the definition of a function, its placement in the school curriculum, and related teaching techniques of function in school mathematics have been going on for decades. Calls for greater emphasis on function in the school curricula and focus on functional thinking in every area of secondary mathematics, other sciences, problems in living, and the outside world and related recommendations have remained unchanged.

Decades of recommendations are evident in today’s curriculum standards. Most notably, functions are one of five categories in 8^{th}-grade mathematics and one of six in high school mathematics in The Common Core State Standards for Mathematics (2010). Moreover, Principles and Standards for School Mathematics (NCTM, 2000, p.296) recommend that teaching programs from prekindergarten through Grade 12 should support all students to understand patterns, relations, and functions. In short, functions currently have a fundamental, unifying role (Akkoc, 2006; Akkoc & Tall, 2005; L. L. Clement, 2001; Cooney & Wilson, 1993; McCulloch et al., 2019; Steele et al., 2013) and have a regular place throughout the school-mathematics curriculum in many countries despite varying applications.

In Israel, functions are introduced with the word "function" in the context of numerical functions as a relationship between two depending quantities as early as Grade 7 (Ayalon et al., 2016, 2017; Ayalon & Wilkie, 2019). Likewise, the function concept is formally introduced to USA students in Grade 7 and Chinese students in Grade 8 (Son & Hu, 2016). On the contrary, in Sweden, functions are presented as a dependency relation between two variables by rarely using the term function in the compulsory school curriculum. Later, functions are explicitly defined in the upper-secondary grades (Hansson, 2006). In the UK, the idea of a function machine is introduced in Grade 7 (Ayalon & Wilkie, 2019; Tall & Bakar, 1992), but functions are presented with the name function in Grade 12 for those who continue to advance study (Ayalon et al., 2015; Ayalon & Wilkie, 2019). In Australia, although the idea of function is introduced without using the input-output machine (or function machine) analogy, the word function is introduced by mapping at Grade 10 (ibid). Similarly, in the Turkish curriculum, at the time of this study, although the function concept is covered at a more fundamental level starting with pattern exploration in the early grades, it is first introduced in Grade 9 (ages 15-16) under the title of functions. However, since the latest 2017 mathematics curriculum was published, the introduction of the function has been moved to Grade 10. In the curriculum, a function is defined as a relation that maps each element of one set (domain) to one and only one element of another set (co-domain). Instead of an abstract approach, it is advised to introduce functions in the context of table-graphic analysis and the relationship between dependent-independent variables using one-to-one and non-one-to-one function cases and modellable real-life situations. It is also suggested to explain function with an analogy of a machine generating output ($f(x)$) values for some input (x) values within the framework of a certain rule.

2.1.2 Conceptual Difficulties of Students and Teachers About Function

The long historical evolution of function suggests that "there is no single way of looking at functions" (Ponte, 1992, p.8). The notion of function evokes various definitions, representations, and interpretations (Ayalon et al., 2017; Ayalon & Wilkie, 2017; Jones, 2006; Selden & Selden, 1992; P. W. Thompson & Carlson, 2017), each of which suggests the other(s) and any of which can be convertible (translatable) to any other(s) at will (Selden & Selden, 1992). Therefore, it seems not to be easy for students and even teachers to grasp the whole meaning and range of relevant applications of functions with the one-sentence Bourbaki definition (Christou et al., 2005; Cuoco, 1995; Evangelidou et al., 2004; Jones, 2006; Selden & Selden, 1992; Vinner, 1983; 1992; Vinner & Dreyfus, 1989).

Numerous research on secondary school and undergraduate students' conceptions of function pointed to many common misconceptions and difficulties (Borke, 2021; L. L. Clement, 2001; Marbán & Sintema, 2020; Tabach & Nachlieli, 2015). Some studies (see, for instance, Akkoc & Tall, 2005; L. L. Clement, 2001; Bardini et al., 2014; Elia, Panaoura, Eracleous, & Gagatsis, 2007; Elia, Panaoura, Gagatsis, Gravvani, & Spyrou, 2008; Gagatsis et al., 2006; Hatisaru & Erbas, 2017; Panaoura, Michael-Chrysanthou, Gagatsis, Elia, & Philippou, 2017; Panaoura, Michael-Chrysanthou, & Philippou, 2015) found that students have problems presenting an accurate contemporary definition of the function. As anticipated, it was revealed that students mostly have limited knowledge about the essential features of functions (Ayalon & Wilkie, 2017; Hatisaru & Erbas, 2017). On the other hand, some studies (e.g., Akkoc & Tall, 2005; L. L. Clement, 2001; Hatisaru & Erbas, 2017; Vinner, 1983) showed that even when students provide accurate definitions, they often do not hold a modern conception of function. It was identified that students tend to view functions such as a formula (Breidenbach et al., 1992; Hatisaru & Erbas, 2017; Vinner, 1983), a rule (Bardini et al., 2014; Hatisaru & Erbas, 2017), an algebraic term (Vinner, 1983) or calculation (Ayalon et al.,2017), an equation to be solved

(Elia et al., 2007; Elia & Spyrou, 2006; Hatisaru & Erbas, 2010, 2017; Tall & Bakar, 1992; Vinner, 1983), a mathematical operation to do (Hatisaru & Erbas, 2017; Vinner, 1983), an analogy (particularly machine analogy) focusing on input-output process (Ayalon et al., 2017; Ayalon & Wilkie, 2017; L. L. Clement, 2001; Hatisaru & Erbas, 2017; Tall & Bakar, 1992), or covariation (Ayalon et al., 2017; Ayalon & Wilkie, 2017; L. L. Clement, 2001; Evangelidou et al., 2004). Apart from these, some studies disclosed that students are unaware of multiple representations of functions and have difficulty handling the flexibility of different representations of functions (Akkoc & Tall, 2003; Elia & Spyrou, 2006; Hatisaru & Erbas, 2010, 2017; Markovits et al. 1986; Panaoura et al., 2015). Moreover, investigations revealed that students have significant misconceptions about functions, such as graphs of functions must be continuous (L. L. Clement, 2001; Elia et al., 2008; Elia & Spyrou, 2006; Gagatsis et al., 2006; Hatisaru & Erbas, 2017; Tall & Bakar, 1992), a constant function is not a function (Tall & Bakar, 1992), and a circle is a function (Tall & Bakar, 1992), a function must contain two variables or unknowns (Elia & Spyrou, 2006; Hatisaru & Erbas, 2010), a function is one-to-one (Evangelidou et al., 2004), functionality can only be decided using vertical line test (L. L. Clement, 2001), and function should be in the form of a list or set (Hatisaru & Erbas, 2017).

Although teachers are expected to have a solid understanding of functions (Sherman et al., 2019); however, unfortunately, teachers' knowledge of function at the content level is quite similar to that of students (Zazkis & Marmur, 2018). A vast amount of research on pre-service and in-service teachers' conceptions of function pointed to many common misconceptions and difficulties ascribed to students. This comprises: trouble presenting an accurate contemporary definition of function (Gulbagci Dede et al., 2022; Hatisaru, 2020; Hatisaru & Erbas, 2017; Meel, 2003; Sherman et al., 2019; Ubuz et al., 2013); limited knowledge about the essential features (particularly univalence feature) of functions and the underlying reasons behind these features (Gulbagci Dede et al., 2022; Even, 1990,1993; Hatisaru, 2020; Hatisaru & Erbas, 2017; M. Meagher et al., 2019; Meel, 2003; Sherman et al., 2019; Vinner & Dreyfus, 1989); difficulty in identifying core concepts of functions such as identification of pre-image-image pairs or difference between image and range of a function (Hatisaru, 2020); tend to view functions as a rule (Gulbagci Dede et al., 2022; Hatisaru, 2020; Hatisaru & Erbas, 2017; M. Meagher et al., 2019), an identity or an equation (Sherman et al., 2019), or a formula (Meel, 2003); difficulty in deciding what is a function and what is not (Hatisaru & Erbas, 2017; McCulloch et al., 2019); incorrect assumptions such that a one-to-one function is not a function (Borke, 2021) or that functions are continuous (M. Meagher et al., 2019); overemphasis on the vertical line test (Even, 1993; Hatisaru & Erbas, 2017; Hitt, 1998; Sherman et al., 2019) or incorrectly performing the vertical line test (Hatisaru & Erbas, 2017); using a limited repertoire of representations to help students understand functions (Even, 1993; Hatisaru, 2020; Hatisaru & Erbas, 2017); privileging algebraic presentations of function (Gulbagci Dede et al., 2022; Even 1990, 1993; Hatisaru, 2020; Sherman et al., 2019); difficulty in passing from one representation to another (Hitt, 1998); and failure to address essential features of a function when using different modes of representations (especially when using analogies) (Bayazit & Gray, 2006; Bayazit & Ubuz, 2008; Espinoza-Vásquez et al., 2017; Ubuz, Eryilmaz, Aydin, & Bayazit, 2009; Ubuz, Ozdil, & Cevirgen, 2013).

Since definition plays a vital role in the construction of a mathematical concept (Panaoura, Michael-Chrysanthou, Gagatsis, Elia, & Philippou, 2017; Panaoura, Michael-Chrysanthou, & Philippou, 2015; Tabach & Nachlieli, 2015), some researchers (e.g., Dreyfus et al., 1990; Jones, 2006; Markovits et al. 1986; McCulloch et al., 2019; Oliveira et al., 2021; Sfard, 1992; Vinner & Dreyfus, 1989) suggested that many (perhaps not all) of these misconceptions and difficulties stem from the “modern” function definition itself.

Although it was accepted as a compulsory part of school mathematics, many researchers (e.g., Malik, 1980; Markovits et al., 1986; Meel, 2003; Sfard, 1992; Tall & Bakar, 1992; Vinner & Dreyfus, 1989, to name but a few) found the Dirichlet-Bourbaki definition is quite abstract and not accessible for students to grasp. Jones (2006) argued that since a full grasp of the definition requires a fairly solid understanding of set theory, without this understanding, it is inevitable that words like “set and subset” and the idea of infinity and the notion of the infinite set will be confusing. In addition, Markovits et al. (1986) indicated that the fundamental difficulty in the modern definition is that it comprises many components (domain, range, rule of correspondence, pre-image, and image). Similarly, Tabach and Nachlieli (2005) pointed out that the determiners “unique” and “for every” in the modern definition are the primary source of the difficulty. Apart from these, Biehler (2005) and P. W. Thompson and Carlson (2017) emphasized that covariation conception lost value with the modern definition of the function. P. W. Thompson and Carlson (2017) stated that even though this definition solved problems that mathematicians encountered; however, introducing it in school mathematics made it almost impossible for school students to have any intellectual need for it (p. 423). Moreover, some researchers (e.g., Malik, 1980; Thorpe, 2018; Vinner, 1983) went further and urged to avoid this formalistic Dirichlet-Bourbaki definition, at least at the school level.

On the other hand, the disparity between concept definition (a verbal definition that accurately explains the concept) and concept image (a set of all mental pictures created in a person’s mind) that discussed in several papers (Hershkowitz & Vinner, 1980; Tall & Vinner, 1981; Vinner, 1983) can be seen as the cause of many of the above misconceptions and difficulties. Results of many of the studies mentioned above investigating students’ and teachers’ conceptions of functions supported this discrepancy. Most of them (e.g., Akkoc & Tall, 2005; L. L. Clement, 2001; Even, 1990, 1993; Gulbagci Dede et al., 2022; Hatisaru & Erbas, 2017; Sherman et al., 2019; Vinner, 1983, to name but a few) showed that even though both students and teachers know by heart an accurate modern definition, they mostly do not hold a modern conception of a function. For example, Hatisaru and Erbas (2017) found that most students of an experienced mathematics teacher (Ali) defined functions as mathematical operations and viewed functions as formulas, although they encountered correspondence definitions of function. Similarly, Elia and Spyrou (2006) found that although they were introduced to the function concept with its modern meaning, most students did not give a correct definition. In addition, they narrowed the idea into a one-to-one function when deciding whether a given correspondence is a function or not.

To give another example, Even (1993) found that albeit prospective secondary teachers who used the contemporary definition of a function, referring to its arbitrary nature, did not do so when approaching a student with difficulties. In another example, Espinoza-Vásquez et al. (2017) found that although an experienced mathematics teacher (Arturo) provided the correct correspondence definition of a function, his generated "washing machine" analogy was lacking in highlighting the arbitrariness property of the function. Likewise, Bayazit and Ubuz (2008) also showed that although another experienced mathematics teacher (Burak) correctly defined the identity function, he suggested that his students think of the identity function like identity numbers with his generated analogy.

Apart from the inherent difficulties of the function definition itself, all work done so far strongly supported that the disparity between concept definition and concept image is a product of various function definitions and numerous treatments of functions. Moreover, all these potential difficulties experienced by students and even by teachers emphasize the importance of teachers' content knowledge about functions and their teaching practices to close this discrepancy between concept definition and students' concept images.

2.2 The Nature of Analogy

2.2.1 The Meaning of Analogy and Related Theoretical Approaches

Various definitions and descriptive terms are used in the literature to describe analogy and its features. The word "analogy" originally comes from the Greek mathematical term "αναλογία (analogia)," which describes the equivalence of two ratios or a proportion. The original conceptualization of the term involved the determination of the "mean or middle term (b)" between two numbers (a and c) to construct a series (a : b : : b : c), that is, the relation in the first pairs (a, b) is the same as that in the two numbers in the second pair (b and c) (Dawis & Siojo, 1972). This conception was later extended to the Aristotelian analogy (also known as proportional, classical, four-term, or four-place analogy in the literature), which describes a mathematical construction comparing four numbers or terms (a : b : : c : d) such that "the second is related to the first as the fourth is to the third" (b is to a as d is to c). An example of this type of analogy from biology is "spine is to fish as bone is to animal" (spine : fish : : bone : animal) (Aristotle, Poetics, cited in Hesse, 1965, p. 330).

This mathematical construction was later reflected in psychology, and analogies in this format (often referred to as test or item analogies) were widely used as a test tool in many standardized tests. It was Burt (1911) who added the term "analogy" to the glossary of mental testing by using the familiar proportional format a: b : : c : d items (Dawis & Siojo, 1972). An example of Burt's (1911) test item was "Paris : France : : London : …?" (p. 9). Later, tests consisting only of analogies were developed, and the most known of these is the Miller Analogies Test (MAT), which is still used as a graduate school admission test today. An example test item is: "*homophone* : (a. articulation, b. principle, c. significance, d. synonym) : : *pronunciation* : *meaning*" (D. Meagher, 2006, p. 3). In this item, a relationship similar to that between the second pair of words (*pronunciation* and *meaning*) must be sought between the first pair (*homophone* and the missing word). Examining the four options for the fourth term, the expected answer is d. *synonym*. Because, homophones each have the same *pronunciation*, and synonym words each have the same or close *meaning.*

In 1966, Mary Hesse pioneered modern views of analogy by criticizing the adequacy of the proportional format. She argued that analogies should not be reduced to proportionality, stating that they cannot find a single fourth missing term and do not have a transitive property like mathematical proportions (Haglund, 2012). Similarly, finding the proportional format (i.e., a : b : : c :?) limited, Gick and Holyoak (1980) developed another type of analogy called story (or problem) analogy. They put forward that the main limitation of such a format is that it does not allow subjects to recognize the analogy spontaneously, which is often a prerequisite for problem-solving (Gick & Holyoak, 1983). As an alternative format, Gick and Holyoak (1980, 1983) offered a story describing a problem and its solution, which guides to generate analogous potential solution to a target problem. For example, Gick and Holyoak (1980) presented a story (attack-dispersion) to subjects about a military problem, then asked them to solve Dunker's (1945) radiation problem, which has the same relational structure as the story. The story was about capturing a fortress in the center of the country without detonating mines. On the other hand, the radiation problem was about finding a way for a doctor to attack a malignant tumor using sufficiently high-intensity rays without destroying healthy tissue. The solution required sending low-intensity rays from different directions whose procedure was structurally similar to the small groups of army attacks on the fortress.

Holyoak (1984) declared that although it is possible to define a story analogy between a base and a target problem in a proportional format such that Problem_B : Solution_B : : Problem_T : Solution_T, they are structurally and functionally different. They are structurally different because proportional analogies have arbitrary relations, whereas story analogies have casual relations. On the other hand, they are functionally different because the purpose of proportional analogies is to solve the analogy, while the purpose of story analogies is to solve the problem.

In 1954, Polya defined that “two systems are analogous if they agree in clearly definable relations of their respective parts” (p. 13). This definition based on “proportionality or agreement in the ratios of corresponding parts” formed the basis for the structure-mapping theory of analogy (Gentner, 1980, 1982, 1983, 1986, 1988; Gentner & Gentner, 1983), which is the cornerstone of many current analogy models. Noting that the analogy is a kind of similarity, Polya (1954) emphasized that the main difference of analogy between other similarities lies in the thinker’s intention. Besides, he implied that an analogy is valuable in its defined context; for example, referring to “a triangle in a plane is analogous to a tetrahedron in space” is valuable in its place (pp. 14-15). These ideas later led to the development of another important theory called multiconstraint theory (Holyoak & Thagard, 1989, 1995).

Most of the analogy definitions commonly used today are derived from Gentner’s formalization (English 1993; Richland, 2003). According to the structure-mapping, an analogy is a mapping from B to T, in the form “a T is like a B” (Gentner, 1983, p. 157). B is the base domain that serves as a source of knowledge, and T is the target domain being explained. For instance, in the Rutherford analogy used by Gentner (1983), “the *atom* is like *solar system* (p. 159)”, the base domain is the *solar system*, and the target domain is the *atom*. Either both domains are equally well known, or the base is better known than the target (Gentner, 1989; Richland, 2011).

In this theory, domains are accepted as systems of objects and predicates (object-attributes and relations). Objects are entities that function as wholes in the given organization. Using the examples presented in Gentner (1983), they can be net entities (e.g., “rabbit”), parts of a larger object (e.g., “rabbit’s ear”), or a combination of smaller ones (e.g., “herd of rabbits”) (p.156, footnote # 2). Object-attributes are predicates that describe one object, and relations are predicates of more than one. From the example given above, “the sun is *yellow*”, YELLOW (sun) is an object-attribute, and “the planets *revolve around* the sun”, REVOLVE AROUND (planet, sun) is a relation between objects.

According to the theory, mapping is the process of aligning objects and predicates of base and target domains, deriving commonalities, and making inferences from one to another (Gentner, 1998). In mapping, the base domain provides a kind of model for making inferences about the target domain when little or nothing is known about the target domain (Gentner, 1989, 1998). Gentner's theory allows for mapping relational predicates but few or no attributes. For instance, "the electron *revolves around* the nucleus, just as the planets *revolve around* the sun, but not the nucleus is yellow, hot, massive like the sun" (Gentner, 1983, p. 159). Higher-order relations (relations between relations) such as causal, mathematical, or functional relations play an essential role in analogy (Gentner, 1983; Holyoak et al., 2001). For example, "the fact that the nucleus *attracts* the electron *causes* the electron to *revolve around* the nucleus" likes "the fact that the sun *attracts* the planets *causes* the planets to *revolve around* the sun" (Gentner, 1983, p. 159).

The basic idea of this theory is that analogy requires finding a relational alignment of corresponding elements in the domains (Holyoak et al., 2001). This alignment is characterized by two constraints: parallelism and systematicity. Parallelism entails structurally consistent and one-to-one correspondences between mapped predicates in the base and target (Gentner, 1998; Gentner & Markman, 1997; Holyoak et al., 2001). Systematicity refers to a tacit preference to map interconnected systems of relations (Gentner & Markman, 1997; Holyoak et al., 2001). It supports a hierarchy between predicates; namely, coherent, higher-order predicates have priority over isolated, lower-order predicates (e.g., attributes) (Gentner, 1983). In the contrast model, Tversky (1977) suggested that enlarging the measure of the common features increases similarity and decreases the difference; on the contrary, enlarging the measure of the distinctive features decreases similarity and increases difference. Unlike Tversky, Gentner (1983) argued that an analogy's power does not depend on the degree-of-overlap because not all matching relations are equally relevant.

Another valuable analogy approach is Holyoak and Thagard's multiconstraint theory or pragmatic approach (Holyoak 1984, 1985; Holyoak & Thagard, 1989, 1995). Basically, according to this approach, analogical thinking is governed by three constraints – structure, similarity, and purpose (Holyoak & Thagard, 1997; Hummel & Holyoak, 1997; Spellman & Holyoak, 1996). Based on Gentner's structure mapping, the multiconstraint theory is similar to it in stressing structural constraint while differing in highlighting similarity and purpose constraints. The first constraint refers to (1) one-to-one and (2) structurally consistent mappings between elements of source and target domains, as proposed in Gentner's structure-mapping theory. On the other side, the second constraint refers to predicate similarity, which allows direct similarity between objects in an analogy. According to this constraint, although the analogy depends on the similarity between relational predicates, attribute similarity between objects in the source and target domains guides to some extent analogy mapping, particularly at the first access of an analogy (Holyoak et al., 2001). The third constraint concentrates on pragmatics' role and explains that context and reasoners' current goals guide analogical thinking (ibid). According to this constraint, predicates to be mapped are chosen concerning the reasoners' current goals (C. A. Clement & Gentner, 1991).

In Wilbers and Duit's (2006) words, these two "predominant and the most prominent" theoretical approaches soon caught the education research community's attention. Gentner and Gentner (1983) had advocated the use of analogies in education with their words, "analogical comparisons are commonly used in the discussion and teaching of scientific topics" (p. 1). Some educational researchers (e.g., Donnelly, 1990; Duit et al., 2001; Wilbers & Duit, 2006; Vosniadou, 1988, to name but a few) have criticized the feasibility of some of their notions for education and have tried ways to improve them. However, most have widely accepted the overall views of these two theoretical frameworks.

The view that an analogy involves the process of transferring selected relational structure from the better-understood *source* domain to the lesser-known *target* domain, which both theories stand for, has been globally accepted (Holyoak & Koh, 1987). In the literature, various terms such as "mapping, matching, aligning, relating, linking, corresponding, or comparing" are used to define the process of *transferring* knowledge from one entity to another. For instance, Glynn et al.

(1989) favored using the word “mapping”; Duit (1991), Glynn (2015), and Orgill and Bodner (2004) preferred to use the word “comparison” while others used other terms. Similarly, there are also differences in the terms used to describe compared entities. Some researchers (e.g., Gentner, 1998; Gentner & Maravilla, 2018; Vosniadou, 1988) described both entities as analogs, while others (e.g., Duit, 1991, to mention but a few) described both entities as domains. Further, some of them continued to use the terms “source” and “target” for familiar and unfamiliar entities, while some preferred to use words such as “base, analogue, vehicle, anchor, or analog” and “topic” respectively, instead of these terms. Although there are differences in the terminology used, they are still alternatives to describe the process of comparing base and target in analogy (Aubusson et al., 2006).

The role of pragmatics in analogy construction suggested by the multiconstraint theory has received the same attention in the educational community. After object similarity in analogy was voiced with multiconstraint theory, analogy theories expanded the analogy boundaries to overall similarity, which is considered a rich analogy that shares relational and object commonalities (Gentner, 1998). With this change, although object similarity has been evaluated in different ways by education researchers, the relational similarity in the analogy defended by both approaches remained indispensable for analogy.

The literature abounds with various definitions of analogy; however, the definition adopted in this study is that of Glynn’s (2015), namely, “an analogy is a comparison of the similarities of two concepts” (p. 45). In line with the use of terms in the work of Glynn (2015), an analogy consists of three main components: (1) an analog concept (or analog), (2) a target concept (or target), and (3) mapping between them. An abstract representation of an analogy with its components is given in Figure 2.1.

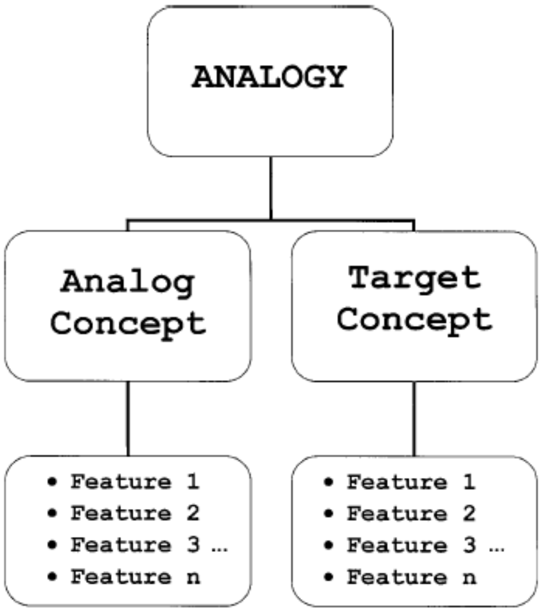


Figure: 2.1: An abstract representation of an analogy with its components (Glynn, 2015, p.45).

An example of a classic analogy is "function is like a machine". Analogies make the unfamiliar familiar (Treagust et al., 1992). The more familiar concept (machine, from the example) and the less familiar or unfamiliar concept (function, from the example) are analog and target concepts, respectively. Both analog and target have some features (also called attributes), and mapping is a systematic comparison between the features of analog and target. No analogs are alike, and no analog perfectly matches the target since analog and target have both similar and dissimilar (or shared and unshared) features (Glynn, 1989; 1994; Gray & Holyoak, 2021). So, a good mapping also indicates unshared features (Thiele & Treagust, 1991, 1992). On the other hand, not every shared feature is equally relevant to the goal of the analogy (Gentner, 1983). Some features even one feature may be more related to the analogy goal than others (Glynn, 1994).

Analogical reasoning can exist between two concepts, which belong to different conceptual domains (e.g., between domains of mathematics and physics), or within a conceptual domain (e.g., within the domain of mathematics) (Glynn, 2015; Vosniadou, 1988). As stated in structure-mapping theory, analogy derives its power from inter-object casual and logical connections. So, an analogy can be formed between two concepts that share no object attributes, or those that share both object attributes and relational similarity (Richland, 2011). However, contrary to what Gentner (1983) emphasized, there is no constraint that analogy should have little or no object attribute. In line with multiconstraint theory, analogical reasoning can be employed between any two concepts with the same relational similarity and varying degrees of object similarity. On the other hand, whether a comparison can be accepted as an analogy depends on the reasoning behind it (Vosniadou, 1988). Moreover, when making this decision, the presenter's current goals and the presented context should be taken into account. For instance, using the example presented in Vosniadou (1988), if the comparison “Earth is like the moon” emphasizes object attributes such that earth is solid and spherical like the moon, it cannot be considered an analogy. On the contrary, if the same comparison helps to conclude that there must be a day/night circle on the moon like the earth by evaluating other commonalities between the moon and the earth, it can undoubtedly be considered an analogy.

2.2.2 Analogies and Metaphors

The terms analogy and metaphor, which are frequently encountered in the educational literature and used in various ways, are often seen as close relatives (Duit, 1991) and are even used interchangeably (Aubusson et al., 2006; Glynn et al., 1989). Moreover, analogy relation is often used to understand a metaphor (Bailer-Jones, 2002). Therefore, to better understand what an analogy is, it is crucial to understand its relationship with metaphor.

In literature, metaphors are defined as a similarity category spanning a range from object to relational commonalities and containing analogies. Analogies are described as a particular type of metaphor (or relational metaphor) (for an extended discussion of this issue, see Gentner & Jeziorski, 1993; see also Gentner & Markman, 1997). The definitions of analogy and metaphor support the generalization made by Aubusson et al. (2006), "all analogies are metaphors, but not all metaphors are extended into analogies" (p. 3).

Although all analogies are metaphors simultaneously, they can be distinguished from other metaphors with small differences (Duit, 1991). First, they are different in their formula, namely, in metaphor, "an A is a B" (Gentner, 1988; Perrine, 1971) or "A as B" (Black, 1993); however, in analogy, "an A is like B" (Aubusson et al., 2006). Using examples from Aubusson and his colleagues' study, the metaphor "the student as tabula rasa" suggests that the student has no previous knowledge before entering the classroom. On the other side, the analogy "the student is like a sponge" implies that the student and a sponge have common features, but they also have some differences.

Another discrepancy is that a metaphor is mostly used with views of teaching (e.g., Ezra Pound's metaphor of education as sheepherding as cited in Black, 1993). On the other hand, an analogy is mostly used with explanations of scientific and technical contexts (e.g., birthdays are like functions) (Aubusson et al., 2006; Glynn et al., 1989).

One more distinction is that although both metaphor and analogy are comparisons, in a metaphor, comparisons are made hidden (Aubusson et al., 2006; Duit, 1991), whereas comparisons are made apparent in an analogy (Aubusson et al., 2006). According to Black (1993), metaphors "say one thing and mean another" (p. 21). In other words, "what the speaker means is not identical with what the sentence means" (Searle, 1993, p. 84). In this sense, metaphor concerns relations existing between what the speaker said (word or sentence meaning) and what s/he meant (conveyed meaning or utterance meaning) (Black, 1993; Ortony et al., 1978b; Searle, 1993).

Sentence meaning can be literal (the language of science, which is precise and unambiguous) or metaphorical. The words and sentences merely have one meaning they have; however, the main concern in a metaphor is the speakers' probable intentions (Duit, 1991; Searle, 1993). The main problem is to clarify how it is understood; otherwise, there is no matter in understanding metaphors (Grey, 2000).

Scholars thought that all metaphors are not the same, but there is a continuum. Live (or novel) and dead (or frozen) metaphors are endpoints of this continuum (Fraser, 1993; Ortony et al., 1978a). According to some scholars (e.g., Black, 1993; White, 1987), proverbs and idioms are classic examples of dead metaphors. On the other side, according to others (e.g., Cacciari, 1993; Gibbs, 1993, 1994; Ortony et al., 1978a), everyday occurring metaphors like proverbs and idioms might be in the middle of the continuum. They argued that although proverbs and idioms are not original, they are certainly not frozen because of their active metaphorical roots.

In this study, like scholars mentioned earlier, the concept of metaphor covers proverbial and idiomatic expressions. Although their slight differences, most researchers agree that both analogy and metaphor are based on the same analogical process (or mapping) to interpret newer and less well-structured situations (target) in terms of previously encountered and well-structured ones (analog) (White, 1987). Ultimately, the comparisons classified as analogies for this study involved metaphors, proverbs, idioms, and comparisons that could rigorously be considered analogies.

2.3 Features of Analogies and Their Classifications

Analogy-related literature abounds various analogies with different features and many criteria classifying these analogies according to their features. Generally, these criteria classify the nature of analogies and their use in different contexts.

The criteria that classify the nature of analogies concentrate specifically on the nature of the analog, target, and their relation. To this end, Curtis and Reigeluth (1984) initiated classifying the level of abstraction (or condition) of analog and target in their work. Later many other studies (e.g., L. D. Newton, 2003; Orgill & Bodner, 2006; Thiele et al., 1995; Thiele & Treagust, 1994a, to name only a few) used the same variable. Mary Hesse (1966) initially classified analogy relations into two broad categories: formal and material. Formal analogies are based on structural similarities (e.g., "a light wave likes a simple pendulum") and material analogies are based on material similarities (e.g., "gas particles are like billiard balls in all mechanical properties relevant to Newton's law") (Hesse, 2000, p. 299). Then, Gentner (1983) also made a similar classification of analog and target similarities: relational similarity (sharing relational properties of objects) and object similarity (sharing descriptive properties of objects). Afterward, some researchers preferred to use such as structural and surface similarities (e.g., Holyoak & Koh, 1987; Vosniadou, 1988), structural and superficial similarities (e.g., Blanchette & Dunbar, 2001), functional and structural relationships (e.g., Curtis & Reigeluth, 1984; Thiele & Treagust, 1994a) or functional and visual/geometric similarities (Else et al., 2002, 2003) as an alternative to Hesse's formal and material relations. Although there are some differences in the terminology and the way of their usage, the same rationale lies behind them.

Curtis and Reigeluth (1984) identified three contexts for analogies: (1) testing (e.g., standardized tests), (2) oral communication (e.g., oral instruction), and (2) text (e.g., written textbooks). Since analogy features vary depending on the contexts in which they are used, there are many criteria in the literature that focus on analogies used in these specified contexts. However, the most comprehensive classification systems for analogies have been developed for textbook analysis. Curtis and Reigeluth (1984), who devised a "classification system" to examine analogy use in physical science textbooks ranging from elementary to post-secondary level, pioneered the development of more extensive systems for analyzing textbook analogies. A detailed paraphrased summary of the "classification system" with original examples cited in

Curtis and Reigeluth (1984) is given below in Figure 2.2. Analogies are underlined, and relevant parts are indicated in italics to make the examples more understandable.

---

1. Analogical relationship - *Is the analogical relationship between analog and target structural, functional, or structural-functional?*

Structural – when the analog-target pair have the same general physical appearance or are similarly constructed. For example, "each cell in the onionskin is something like a room. It has a 'floor' and a "ceiling" as well as four 'walls'" (Smith & Lawrence, 1966, p.23)

Functional – when they share similar functions. For example, "feedback works like a building thermostat. The thermostat is set at a certain temperature, a signal turns the heat off. The signal to turn a gland on or off differs for different glands" (Ramsey et al., 1978, p.291).

Structural-functional – when they share both relationships at the same time. An example is the following:

> *The structure and functions of our cells could be compared to a factory.* The manufacturing processes may be compared to the life processes carried on in a cell. The finished products are the compounds that form the many parts of the cell … The main office and planning department of our factory cell is the nucleus. The nucleus is the control center of the cell. It controls everything that goes on inside the cell (Ramsey et al., 1978, p. 69).

2. Presentational format - Is the analogy presented in a *verbal (or written) format* or *pictorial-verbal format?*

Verbal – where the analogy is presented merely in words,

Pictorial – where a picture, drawing, or photograph of the analog supports the verbal analogy.

3. Condition - Does the analog-target pair have a *concrete-concrete*, *abstract-abstract*, or *concrete-abstract* nature?

Concrete-concrete - where both analog and target are concrete,

Abstract-abstract - where both analog and target are abstract,

Concrete-abstract - where analog is concrete and target is abstract.

4. Position - Is the analog domain presented as an *advance organizer*, an *embedded activator*, or a *post-synthesizer*?

Advance organizer - analog is presented before the target. An example is included below.

> *Two hundred years ago, our nation began to expand westward over the Appalachian Mountains and on to the Central Plains. Just staying alive was a full-time job. Each family unit that settled far from others ... As time went on, more and more families settled in the wilderness areas ... As these families settled closer together, a kind of exchange system developed ... So as time went on, people tended to become specialists. They became more and more dependent on each other for survival ...*
>
> <u>Single-cell organisms are similar to those early families of settlers</u>. Each must perform all of the life functions by itself … (Ramsey et al., 1978, pp. 81-2).

Embedded activator - analog is presented somewhere at a point where the target becoming complex. Here is one of them.

> *Energy is needed to move objects through the air. In aircraft, that energy comes ... In the early aircraft, the engines turned propellers.* <u>A propeller works something like a screw</u>. *When you turn a screw in a piece of wood, it moves into the wood. A fast-turning propeller "bites" into the air in much the same way a screw "bites" into wood. As the propeller turns, ...* (Mallinson et al., 1978, p.62).

Post-synthesizer - analog is presented after complete treatment of the target. An example is presented below.

> "Why should I bother to learn to think? Why not let someone else do it, and I can just get the answers?" … As long as you live, you will find yourself in situations that you have never learned about … Throughout this

course in chemistry you will be involved in "thinking training."

An athlete trains to stay in shape for the game. Many of the skills the athlete acquires can be used in daily life. Much the same is true of the study of chemistry. *You will find many uses for your understandings and skills learned in chemistry.* (Bolton et al., 1973, p.11).

5. Level of Enrichment - Is the analogy *simple*, *enriched*, or *extended*?

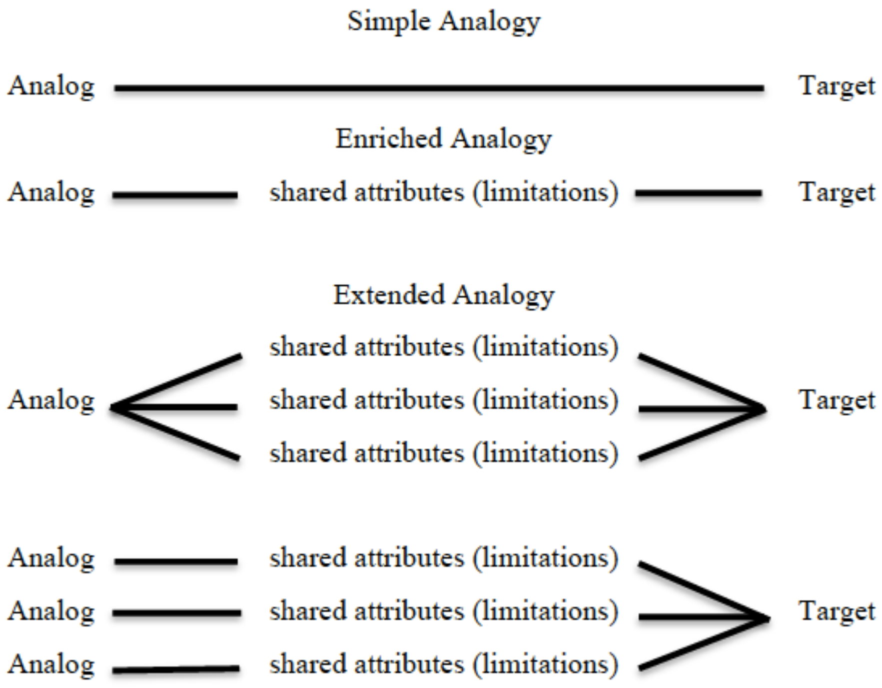
Simple - there is no explanation of the analog-target relationship and the analogy is composed of three main parts, the analog, the target and a connector such as "is like" or "may be compared to."

Enriched - there are varying degrees of shared and unshared attributes of analog and target. An example of enriched analogy with only shared attributes given below.

> Winds carrying sand and soil act somewhat like sandblasting machines. *Sandblasting machines clean the sides of buildings by blowing sand against the stone walls to wear away dirt and stains. Similarly, winds that carry sand and soil wear away the sides of rocks* ... (Jackson, Lauby and Konicek, 1965, p.19).

Extended - various attributes of an analog are used to explain more than one target, or multiple analogs are used to describe a target. An example of extended analogy using two analogs to teach one target is presented below.

> Compare our body and the cells and organs that compose it to a flow tank. *If the amount of water entering the tank is just equal to the amount of water leaving it, the water level will not change*. Such a system is said to be actively balanced, or in *dynamic equilibrium*. *The circus juggler who maintains an active balance as he juggles several objects keeps them in dynamic equilibrium.* Our body processes are regulated so that many such dynamic equilibria are maintained. One of these is the balance of molecules of glucose entering and leaving the bloodstream.



Three levels of enrichment for analogies are presented below.

6. Pre-target orientations - Is the *analog explained*? Is the *strategy identified*?

Analog explanation – analog is explained or reviewed to make it familiar to the learner(s). An example appears below.

> *A president's every action is thought about, talked about, debated, and analyzed… But people also ask what he was like during his childhood and his formative years; do the experiences of those earlier days influence his actions?...* In some respects, <u>Planet Earth is like a President</u>… we know that little evidence of Earth's birth and childhood remains, because… (Flint & Skinner, 1974, p.369).

Strategy identification – the analogy is identified as a cognitive strategy. Here is an example.

> The low-velocity zone, therefore, seems to be a key to the developing picture of Earth's internal processes. The picture has been clarified by measurements of Earth's free oscillations.

> How this is done can be explained by *an analogy* with bells. A bell made of lead or copper- metals that are softer and more plastic – will vibrate only for a few seconds … The same can be done with Earth when an earthquake sets it vibrating (Flint & Skinner, 1974, p.337).

Analog explanation and strategy identification – both are available. An example included below.

> A double helix is like a double coil. *It can be understood by imagining a ladder … this impractical ladder has flexible uprights.* The bottom of the uprights of the ladder are held in place so that they cannot move. Twist the ladder from the top, a double helix is formed.
>
> … The uprights of this special ladder are made of the …. (Biological Sciences Curriculum Study, 1963, pp. 144-5; Welch et al., 1973, p.208).

Absence of both – none is available.

Figure 2.2. A paraphrased detailed summary of the “classification system" developed by Curtis & Reigeluth (1984, pp. 103-114)

First, Curtis (1988) and later Thiele and Treagust (1991, 1992, 1994a) continued the tradition initiated by Curtis and Reigeluth (1984). Although Thiele and Treagust (1991, 1992) have worked on it before, Curtis and Reigeluth’s (1984) classification system was first criticized in detail by Thiele and Treagust (1994a). In their work, they examined the use of analogies in high school chemistry textbooks. Considering that the available classification system might be limited to other analogies inside or outside the science domain, they made the following adaptations and clarifications to Curtis and Reigeluth’s (1984) classification system. They also named this new version classification system as “Analogy Classification Framework.”

First, since they encountered some verbal analogies reinforced by the pictorial target representation, they restricted pictorial-verbal analogy only when the picture(s) represent the analog domain. Second, although they still accepted a stated limitation as a kind of enrichment, they specifically required recording limitations. Thus, they added a new category named “limitations” to the framework to note any general warnings about analogy use or any specific limitations about unshared attributes of analogies. Third, since they worked on textbook analogies, they added a new criterion, “marginalised,” under the category of “position” to indicate analogies positioned in the margin. Fourth, they added another new category called “content” to the framework to get information about the target concept’s content area. Finally, they added another new category called “location” to decide where analogies were most frequently used throughout the curriculum.

After developing these two versions of Curtis and Reigeluth’s classification systems, further studies reported analyses of various science textbook analogies at different grade levels. The majority of these studies (e.g., Akcay, 2016; Dikmenli, 2015; L. D. Newton, 2003; Sangam et al., 2011; Sendur et al., 2011; Thiele et al., 1995, to name but a few) preferred to use original versions of these two directly, with little or no modification. However, some (Orgill, 2003; Orgill & Bodner, 2006) opted for further changes. On the other side, some researchers (e.g., Mastrilli, 1997; Thiele & Treagust, 1994b; Treagust et al., 1992) soon considered Thiele and Treagust’s call for further studies focusing on the use of analogies by teachers and students. They also employed these two frameworks in their studies, as they also have the potential to classify spoken analogies. They did not analyze teacher- or student-generated analogies just like the analysis of textbooks; however, they built their comments on the categories created in these two systems.

Although there are many studies on analogy use in mathematics education, the reflection of this trend, initiated by Curtis and Reigeluth (1984) and maintained by Thiele and Treagust (1991, 1992, 1994a), in mathematics education studies is a little later. To date, only a few studies used either one of their developed frameworks to analyze analogy use in mathematics textbooks. For instance, Unver (2009) and Kepceoglu and Karadeniz (2017) used the Analogy Classification Framework of Thiele and Treagust (1994a) with small adaptations to classify analogies used in textbooks at 9th grade and middle school mathematics levels. On the other side, very few studies used these existing frameworks to analyze analogies used in mathematics classrooms. For example, Unver (2009) used the same framework with little adaptations to classify analogies used by mathematics teachers in her study. Similarly, Ubuz et al., (2013) also used Thiele and Treagust's (1994a) framework with a few modifications to analyze analogies generated by pre-service mathematics teachers during their microteaching events in the university classroom. However, none of these previous studies did worry about evaluating analogies used by both teachers and students in a real mathematics classroom environment. Moreover, they did not go beyond simply reporting the final results of analogy use that they evaluated using frameworks.

2.4 Analogies and Metaphors in Mathematics Education

2.4.1 The Nature and Place of Analogy and Metaphor in Mathematics Education

At one time, analogies, and especially metaphors, were seen as non-mathematical methods (Pimm, 1981). Therefore, the word "analogy" was used very little in mathematics (R. Brown & Porter, 2006), and its use in mathematics teaching has been discussed for a long time (N. Adams & Elliot, 2013). First, Kepler, who called analogies his "most trustworthy masters", later Bernoulli and Leibniz demonstrated the importance of analogy in discovering mathematics very well. More recently, in 1954, George Polya asserted that analogical reasoning is vital to both mathematical insight and pedagogy (Zwicky, 2010, p. 10). Polya (1954) showed to what extent he advocated using analogies in mathematics teaching by dedicating the entire volume to analogy and induction in mathematics. Furthermore, he underlined the place of analogy in mathematics by stressing that the word "analogy" is also borrowed from the language of mathematics (from the name "analogia," meaning proportion to the analogy). In his study, Polya clarified how analogies could provide new suggestions and discoveries and shed light on how analogies could be manipulated in mathematics. Eventually, many mathematicians and mathematics educators agree that analogical reasoning plays a vital role in mathematics learning and teaching.

Analogies used in mathematics (henceforth called mathematical analogies) are not proofs themselves (N. Adams & Elliot, 2013; Zwicky, 2010); however, they are "illustrations" (Pimm, 1981, p. 47), "vehicles of insight" (Zwicky, 2010, p. 11), or "alternative representations of a situation" (Fast, 1997, p.10). "Analogies express the insight; the proof, by contrast, establishes the incontrovertibility of the insight" (Zwicky, 2010, p. 11); hence, discovering proofs requires recognition of analogies between familiar and unfamiliar knowledge (Polya, 1954). Mathematical analogies help simplification of new concepts through the medium of connection to already understood concepts. For instance, when proving inequality $x^x y^y z^z \geq (xyz)^{\frac{x+y+z}{3}}$ for any three positive numbers x, y, z by analogy, it is required to find a simpler one (for example, $(x^x y^y \geq (xy)^{\frac{x+y}{2}}, x, y \in R^+)$ to compare with the original inequality (Fu, 2019).

Since simplification is a type of conversion, which is the basic idea of problem-solving (ibid), analogies are used to find solutions to mathematical problems by comparison (English, 1993; Gick & Holyoak, 1983; Novick & Holyoak, 1991; Polya, 1954). In mathematics education, there are at least three types of comparison that can be utilized with problems: (1) comparison of problems having similar mathematical structures solved with the same strategy (e.g., an equation and a word problem), (2) comparison of problems having dissimilar mathematical structures solved with the same strategy (e.g., work and pipe filling problems) and (3) comparison of the same problem solved with different strategies (e.g., the distance between two points solved with numbers and figure) (Durkin et al., 2017; Richland, 2011; Richland & McDonough, 2010; Rittle-Johnson & Star, 2009)

Analogies draw parallels across different representations (Gentner, 1998, Richland, 2011). In mathematics, these representations may comprise mathematical problems, abstract concepts, graphical or physical manipulatives, or procedures (Richland, 2011; Richland & McDonough, 2010). Analogical reasoning may be based on the comparison of any two representations belonging to the same domain (e.g., propositional logic is like sets), to similar domains (e.g., point symmetry is like a mirror image), or to different domains (e.g., function concept is like a dishwasher) (Vamvakoussi, 2017). These mathematical representations may appear similar or different at a surface or relational level (Richland, 2011; Richland & McDonough, 2010). For instance, equations $x + 1 = 4$ and $x + 2 = 5$ appear quite similar at a surface level since both are first-degree equations with the same unknown (x), similar-sized numbers, and in the same mathematical form. Another instance is that equations $ab = c$ and $n\ (X - y) = G$ appear not similar at the surface level since different symbols form them. However, they have a common relation: variables c and G are the product of the objects or entities a, b and n, $(X - y)$, respectively. Even though both surface features and relations have an essential role for similarity, relational correspondence has a vital role (English & Sharry, 1996).

Mathematical analogies are mainly made between relations among objects, not between objects themselves (R. Brown & Porter, 2006; Polya, 1954). For instance, an analogy is made between the addition of knots and the multiplication of numbers (R. Brown & Porter, 2006). In this example, objects - knots and numbers - are not directly analogous. However, there is an analogy between the laws for adding knots (commutativity, associativity, identity element) and the laws for multiplying numbers. In this sense, for example, $(2 + 3) + 4 = 2 + (3 + 4)$ is analogous to $(2.3).4 = 2.(3.4)$.

In the mathematics education literature, metaphors are defined in terms of analogies or vice versa, and even these two terms are used interchangeably. At this point, it is subtle to distinguish between metaphor and analogy (Aubusson et al., 2006). Pimm (1981) defined metaphors as condensed analogies, and Gentner (1983; 1988) described analogies as relational metaphors. Gentner (1988) emphasized that analogical processes get involved in the utilization of conceptual metaphors such as “love is a journey” (p. 107). Lakoff & Núñez (2000) defined conceptual metaphors as the mechanism by which human beings conceptualize the abstract in concrete terms. They suggested that mathematical thought takes advantage of conceptual metaphors (e.g., conceptualizing numbers as points on a line) and stressed that “mathematics layers metaphors upon metaphor” (p. 7). Further, they differentiated three types of conceptual metaphors concerning mathematics: (1) grounding metaphors, which ground our understanding of mathematical ideas in terms of everyday life (e.g., “classes are containers”), (2) redefinitional metaphors, which allow conceptualizing technical understanding of ordinary concepts (e.g., Cantor’s reconceptualizing terms “more than” and “as many as” for infinite sets) (3) linking metaphors, which allow conceptualizing one mathematical domain in terms of another mathematics domain (e.g., “points in the plane are ordered pairs of numbers or functions are sets of ordered pairs of numbers” (Núñez & Lakoff, 2005, p. 25)).

In general, metaphors are defined as seeing one object or thing as if it were another (Font et al., 2010; Lakoff & Núñez, 2000; Zwicky, 2010). Zwicky (2010) put forth that understanding a metaphor likes understanding certain kinds of mathematical demonstrations. Besides, Zwicky (2010) asserted that both metaphors and certain kinds of mathematical demonstrations are types of analogical reasoning since both want to say that “look at things like this, if you want to understand them”. Recently, many researchers (e.g., Barakaev, 2021; Cangiotti & Nappo, 2022; Font et al., 2010; Lakoff & Núñez, 2000; Núñez, 2000; Núñez & Lakoff, 2005; Zwicky, 2010, to name but a few) have highlighted the nature and place of metaphors in mathematics education.

2.4.2 The Use of Analogies in Mathematics Classrooms

Literature review indicates a growing body of research examining the use of analogies in mathematics classroom contexts (see, for instance, Amir-Modifi et al., 2012; Araya et al., 2010; Bayazit & Aksoy, 2011; Bayazit & Ubuz, 2008; Begolli & Richland, 2013; Cuya et al., 2017; Engle et al., 2002; Espinoza-Vásquez et al., 2017; González, 2015; Hatisaru, 2021, 2022; Richland, 2015; Richland & Hansen, 2013; Richland, Holyoak, & Stigler, 2004; Richland & Richland, 2013; Richland, Zur, & Holyoak, 2007; Sidney, 2020; Sidney & Alibali, 2015; C. A. Thompson & Opfer, 2010; Ubuz, Eryilmaz, Aydin, & Bayazit, 2009; Ubuz, Ozdil, & Cevirgen, 2013; Uyen, 2021; Unver, 2009; Unver-Sezer, 2021; Unver-Sezer & Ubuz, 2018).

Analysis of these studies points out that some focus on analogies developed in mathematics classrooms as a product of analogy-based interventions (e.g., Amir-Modifi et al., 2012; Araya et al., 2010; Begolli & Richland, 2013; Cuya et al., 2017; Richland & Hansen, 2013; Sidney, 2020; Sidney & Alibali, 2015, 2017; C. A. Thompson & Opfer, 2010; Uyen, 2021) or produced in fictitious math classrooms (e.g., Bayazit & Aksoy, 2011; Hatisaru, 2021, 2022; Ubuz, Eryilmaz, Aydin, & Bayazit, 2009; Ubuz, Ozdil, & Cevirgen, 2013). On the other hand, some concentrate on naturally occurring analogies in non-experimental and regular math classroom contexts (e.g., Bayazit & Ubuz, 2008; Engle et al., 2002; Espinoza-Vásquez et al., 2017; González, 2015; Richland, 2015; Richland, Holyoak, & Stigler, 2004; Richland & Richland, 2013; Richland, Zur, & Holyoak, 2007; Unver, 2009; Unver-Sezer, 2021; Unver-Sezer & Ubuz, 2018).

Moreover, it is seen that these existing studies centered around instructional analogies in general and their effects on students' mathematical learning, attitudes towards mathematics, or both in particular (e.g., Amir-Modifi et al., 2012; Araya et al., 2010; Bayazit & Ubuz, 2008; Cuya et al., 2017; Hatisaru, 2021, 2022; Sidney, 2020; Sidney & Alibali, 2015, 2017; C. A. Thompson & Opfer, 2010), their supports (visual supports, for instance; e.g., Begolli & Richland, 2013), teachers' use of linguistics (González, 2015; Richland & Richland, 2013) and linking gestures (Richland, 2015) in the creation of them, or teachers' practices of them in teaching and their structures (Engle et al., 2002; Espinoza-Vásquez et al., 2017; Hatisaru, 2021, 2022; Richland & Hansen, 2013; Richland, Holyoak, & Stigler, 2004, Richland, Zur, & Holyoak, 2007; Ubuz, Eryilmaz, Aydin, & Bayazit, 2009; Ubuz, Ozdil, & Cevirgen, 2013; Unver, 2009; Unver-Sezer & Ubuz, 2018). Among all these studies, as far as is known, only a few studies explicitly investigated students' participatory roles in classroom analogy production (e.g., Engle et al., 2002; Richland et al., 2004) or student-generated analogies (e.g., Sidney & Alibali, 2015, 2017; C. A. Thompson & Opfer, 2010; Uyen, 2021). Similarly, as far as we know, there is just one study (Ubuz et al., 2013) that gives the numerical values of student-generated ones of all classroom analogies without further explanation. Thus far, there is no study that explicitly addresses student-generated classroom analogies in the natural classroom setting and examines their frequencies and features together.

Besides, it is also seen that instructional analogies in mathematics classrooms are used to support understanding of mathematics topics, including specifically the eighth-grade curriculum (primarily algebra and geometry topics; e.g., Engle et al., 2002; Richland, 2015; Richland, Holyoak, & Stigler, 2004; Richland & Richland, 2013; Richland, Zur, & Holyoak, 2007), geometry (e.g., González, 2015), pre-algebra (e.g., Begolli & Richland, 2013; Cuya et al., 2017; Richland & Hansen, 2013; Sidney, 2020; Sidney & Alibali, 2015, 2017; C. A. Thompson & Opfer, 2010), and algebra (e.g., Amir-Modifi et al., 2012; Araya et al., 2010; Bayazit & Ubuz, 2008; Espinoza-Vásquez et al., 2017; Hatisaru, 2021, 2022; Ubuz, Eryilmaz, Aydin, & Bayazit, 2009; Ubuz, Ozdil, & Cevirgen, 2013; Unver, 2009; Unver-Sezer, 2021; Unver-Sezer & Ubuz, 2018; Uyen, 2021) for middle and high school pupils. Finally, among all mentioned studies, merely some algebra studies specifically concentrate on functions (see, for instance, Bayazit & Aksoy, 2011; Bayazit & Ubuz, 2008; Espinoza-Vásquez et al., 2017; Hatisaru, 2021, 2022; Ubuz, Eryilmaz, Aydin, & Bayazit, 2009; Ubuz, Ozdil, & Cevirgen, 2013; Unver, 2009; Unver-Sezer, 2021; Unver-Sezer & Ubuz, 2018).

The current literature explicitly reveals the inadequacy of studies examining naturally occurring instructional analogies on the concept of functions in a regular classroom environment and the deficiency of studies investigating student-generated analogies and their features. Above all, however, it reveals the absence of a study combining both naturally occurring teacher- and student-generated analogies on the concept of functions in everyday classroom contexts.

To date, as far as is known, very few studies have been reported in the literature examining the use of instructional analogies, especially on functions, in mathematics classrooms. However, these studies summarized below provided valuable insight for the present study, as they are the closest studies in the mathematics domain, which combine both analogy and function literature. Alongside these studies, some other studies (e.g., Akkoc, 2006; Ayalon et al., 2017; Ayalon & Wilkie, 2017; Bayazit & Gray, 2006; Elia & Spyrou, 2006; Evangelidou et al., 2004; Hatisaru & Erbas, 2007, 2010, 2020, Vinner, 1983) also provided insight into the use of analogies in function teaching and learning, even though it was not the main focus of their studies.

Ubuz et al. (2009) naturally observed the pre-planned microteaching of five pre-service mathematics teachers (PTs) on ninth-grade functions. They examined PT-generated analogies and assessed the content validity of those analogies. They accepted an analogy as valid when it illustrated the essence and the properties of functions and the structural relations between analog-target pairs.

Analysis of the data corpus revealed that although PTs did not include analogies in their lesson plans, they used analogies during their teaching. Moreover, it disclosed that although they were not restricted to use any specific method in their teaching or received no special formal training on analogies, they preferred to use them. Generally, they used analogies when defining or exemplifying the function concept, composite function, and types of function.

In this study, a total of eight PT-generated analogies were detected. Detailed analysis of these analogies revealed that some were invalid since PTs failed to select epistemologically appropriate analogs illustrating the essence and the properties of the function-related target that they wanted to explain. On the other hand, some were also invalid since they failed to articulate mapping between them, although they chose epistemologically suitable analogs. Besides, they did not discuss the breakdown points of the analogies. Briefly, Ubuz et al. (2009) concluded that PTs have insufficient knowledge about using analogies and were weak in transforming their knowledge to teach functions.

In another study, Ubuz et al. (2013) observed seven PTs' pre-planned microteaching on ninth-grade functions, similar to their previous work, to investigate the nature and extent of analogy use. However, this time, PTs had formal education about both functions and analogies and were restricted to teaching functions by generating analogies. Besides, in this study, Ubuz et al. focused not only on PTs' teaching practices but also their knowledge of functions and analogy (subject matter and pedagogical knowledge) and their beliefs about the use of analogies. Before observations, they applied a knowledge test to all PTs to assess how they define a function and its types and also analogy and its characteristics. After observations, they interviewed all PTs except one to yield data on their beliefs about when to use analogies, who should produce analogies, and the use of analogies in function teaching.

In this study, they evaluated each identified analogy in terms of criteria (analogical relationship, presentational format, level of enrichment, position, and limitations) proposed within Thiele and Treagust's (1994a) framework. In addition to the existing criteria, they explored analogies' epistemological appropriateness, as they did in their previous work. Further, in this study, they evaluated analogies as teacher-generated (generated by the presenter PT) and student-generated (generated by the rest of the PTs).

Observation data indicated that more than half of the analogies were teacher-generated. The overwhelming majority had a functional relationship and were presented in verbal format. Besides, many were presented as an advance organizer at the beginning of the instruction. Furthermore, most were enriched with stated shared and unshared attributes between analog-target pairs. As expected, enriched ones were teacher-generated, and the simple ones were student-generated. Interestingly, although PTs highlighted that an analog and target could not match entirely since they have both shared and unshared attributes, they rarely mentioned the limitations of analogies. Moreover, the vast majority of PT-generated analogies were epistemologically appropriate, and most of the inappropriate ones were student-generated.

Findings of this study revealed that PTs' knowledge and beliefs about function and analogy were strongly associated with their teaching practices. Knowledge test and teaching practice data disclosed that PTs mostly used Bourbaki's modern definition of function in both contexts. Besides, this study found that PTs' knowledge of functions was strongly associated with the epistemological appropriateness of their analogies. Also, this study supported that PTs generated analogies in teaching practices parallel to the value they gave to the use of analogies in teaching functions. Although all PTs greatly valued teaching functions with analogies, they were limited to generating analogies linking functions with other sciences. This study suggested that PTs need knowledge of mathematics and knowledge of interdisciplinary subjects to develop more effective analogies.

In another study, Bayazit and Ubuz (2008) examined the analogy-based teaching of an experienced mathematics teacher on functions and its influence on function concept understanding of his twenty-seven 9th-grade students. They conducted classroom observations to determine the teacher's purpose of analogy use and the content validity of the analogies he generated. Subsequently, they applied questionnaires to provide data on students' conceptual and procedural understanding of function. Then, they interviewed three students they chose to determine whether they could use analogies while solving function problems assigned to them.

Observation data indicated that the teacher used analogies as an advance organizer to prepare his students for the concept function or as an activator to actuate their knowledge while solving function problems. It also revealed that the teacher offered many analogs with no content validity that had no epistemological power to represent the function-related target. When he presented epistemologically appropriate analogs, he did not take any explanatory action or encouraged his students to notice the relations between analog-target pairs. Furthermore, student data analysis showed that these identified limitations in teacher's analogy-based teaching were reflected in students' learning outcomes. Bayazit and Ubuz suggested that only instructional analogies, which only have content validity and show the structural relations between analog-target pairs, support students' understanding of functions. Moreover, they pointed out that an analogy's effectiveness depends on the competence of both teachers and students in carrying out the mapping process.

Espinoza-Vásquez et al. (2017) observed an experienced mathematics teacher's function concept teaching to 9th-grade students. They selected an episode where the teacher used an analogy (washing machine analogy) to support his students' understanding of the function definition. They concentrated teacher's knowledge about function concept and its teaching from the perspective of the Mathematics Teacher's Specialized Knowledge (MTSK) model (see Carrillo et al., 2014, for details about MTSK and its sub-domains). They considered that the knowledge about the definition and properties of the function, and the knowledge about the washing machine analogy and its use in promoting function concept learning, respectively, were parts of the teacher's Knowledge of Topics (KoT) and Knowledge of Mathematics Teaching (KMT), sub-domains of MTSK. They suggested that the teacher's KMT was fed and affected by his KoT, and vice versa.

Observation data supported that the teacher interpreted the function as a process by introducing the washing machine analogy. Besides, it indicated that the teacher successfully described in a verbal and pictorial format both the input/output process and the univalence property of the function by this analogy. However, this analogy lacked highlighting the arbitrariness property of function and was insufficient to show the co-variation of magnitudes. Espinoza-Vasquez et al. asserted that such an analogical representation of the function might cause obstacles to learning ongoing function-related concepts. Besides, they argued that students' individual experiences with the chosen analogy might also lead to misunderstandings about function concepts.

Unver (2009) naturally observed two female mathematics teachers' analogy use in their 9th-grade function unit teaching and examined analogies in the relevant chapter of the textbook used in those classes. The function unit in the study included inverse and compound functions in addition to the current work. Both teachers had no formal training in teaching with analogies, and they were not restricted to teaching functions with analogies. They analyzed the nature of both teacher-generated and textbook analogies according to the mildly adaptive version of the framework developed by Thiele & Treagust (1994a).

Unver's general findings regarding the nature of teacher-generated analogies and textbook analogies used in the classrooms are more applicable for the current study. In her research, she detected a total of 32 (11 from one and 21 from the other) teacher-generated analogies employed for almost all function topics. Analysis of these analogies disclosed that nearly all were enriched or extended, functional, and presented verbally. Besides, the majority had an analog explanation and were employed as embedded activators or post-synthesizers. However, none of them include any statement describing their limitations. Similarly, only one analogy included a statement explicitly stating the applied strategy as an analogy. On the other hand, a comparison of the result of each teacher- and all teacher-generated analogies disclosed that frequencies of all characteristics were reasonably consistent. Similarly, a comparison of all teacher-generated and textbook analogies revealed that only one of the teachers used the classic "function machine" textbook analogy for different target concepts related to function types. Analysis of the analogies generated by each teacher indicated that both had a favorite analogy that they repeated multiple times with different targets.

Unver-Sezer and Ubuz (2018) researched how much teacher-generated analogies in function teaching were remembered by students for a seminar work as an extension of the current study. They interviewed fourteen 9th-graders (sub-sample of the present study) to gain a deeper insight into why they remembered some analogies more than others. During interviews, they asked students to identify an analogy generated by their teacher, and they investigated why they chose it.

Results based on interview data revealed that the most common reason for remembering the selected analogies was that they helped students understand functions. Besides, results also disclosed that students chose these analogies since they were mathematical or contained numerical expressions, based on their favorite topics (for example, business life/commerce, economy, and financial matters), applicable, and attention-grabbing.

Unver-Sezer (2021) interviewed sixteen 9th-grade students from five different classes for a seminar work as an extension of the present study. She investigated students' general views about analogies, what analogies mean to them, and their awareness of the instructional roles of analogies. Students were asked: (1) what an analogy is, (2) what the advantages of analogies are, and (3) what the disadvantages of analogies are. Students' answers to the first question showed the possible benefits of analogy rather than the definition of analogy. Their answers to the second question revealed that they mostly liked their teachers' teaching function-related concepts with analogies. In general, most students found analogies helpful on different grounds. The majority clearly stated that they found analogies valuable because they made it easier to understand and remember the content in general and associated abstract contents with daily life. Their answers to the third question disclosed that most students are also aware of the disadvantages of analogy use. Some students mentioned that they are reluctant to use analogies since they can understand mathematics with numbers, not verbal expressions like analogies. In addition, some stated that analogies confuse their minds, and the use of analogies is not with time and effort.

Bayazit and Aksoy (2011) applied written exams to twenty-two PTs to search for their beliefs concerning students' possible benefits from analogy use and their proficiency in using analogies to teach functions. Based on their written exams, they interviewed three PTs to reveal their actual thinking processes.

Research findings revealed that PTs had strong beliefs regarding the effectiveness of analogies in mathematics teaching and learning. Their beliefs were consistent with the vision of fundamental teaching/learning theories such as constructivism, information-processing theory, and sociocultural theory. Nearly half of the PTs presented analogies as a relation. Many of their analogies were invalid since they were inappropriate to represent function-related target concepts. Moreover, many were used to stress procedures and factual knowledge. Similarly, PTs had difficulty in describing structural relations between analog-target pairs.

Hatisaru (2020) applied a questionnaire to collect data about secondary mathematics teachers' Mathematical Knowledge for Teaching (MKT) functions. Next, Hatisaru (2021, 2022) examined analogies in teachers' responses to define or exemplify functions in this questionnaire. In these two studies, Hatisaru was concerned with understanding the structure of analogy mappings of identified analogies according to Gentner's (1983) structure-mapping theory and the sorts of function conceptions addressed by these analogies. She investigated whether comparisons were an "analogy" (based on only common relational attributes) or an "anomaly" (based on just common object attributes) and whether they addressed functions as "an input-output machine" or "a mapping between two sets." Results of these studies revealed that most of the comparisons were "an analogy," and some were "an anomaly" since there was a surface similarity between analog and target concepts. A detailed examination of the anomalies disclosed that the two teachers used the word "function" in the same everyday sense. Results also demonstrated that the correspondence approach (either the view of a function as input-output machines or mapping between two sets) was dominated, and the covariation approach was absent. Two analogies (child-mother linkage and machine/factory) were popular among all responses to the questionnaire, which were used to stress the relational structures of functions. Hatisaru remarked that while uniqueness was implicit in the input-output machines view, it was relatively explicit in mapping two sets of view. At the end of this study, she deduced that teachers' analogies are possibly a representation of their understanding.

2.5 Factors Limiting the Use of Analogies in a Classroom Context

There is no doubt that analogies are used in classrooms to benefit students; however, there is no guarantee that they will always yield the intended benefit. The research literature (for example, Curtis & Reigeluth, 1983; Donnelly, 1990; Duit, 1991; Harrison & Jong, 2005a, 2005b; Iding, 1997; Newby, Ertmer, & Stepich, 1995; Newby & Stepich, 1991; Nottis & McFarland, 2001; Orgill & Bodner, 2005; Schenke & Richland, 2017; Thiele, 1995; Thiele & Treagust, 1992; Treagust, 1993; Venville & Treagust, 1997) suggests that analogy use in regular classroom contexts is dependent upon various factors. These include student-related factors such as students' cognitive development, familiarity with analog and target domains; teacher-related factors such as teachers' subject matter and instructional knowledge; factors related to the concept included in the curriculum during the study; and intrinsic factors within the analogy itself. Being aware of possible factors can help to understand and interpret better the nature and extent of teacher- and student-generated analogies in the actual classroom context. Although many factors limit the use of analogies, only those most frequently mentioned in the literature and within the scope of this study are discussed below.

2.5.1 Students' Cognitive Development

All students cannot automatically benefit from analogies (even good ones) to the same extent (Guerra-Ramos, 2011; Nottis & McFarland, 2001; Orgill & Bodner, 2005; Richland, 2011). When analogy use is associated with Piaget's cognitive development stages, several studies suggested that low-ability, concrete thinker students benefit more from analogies than high-ability, abstract thinkers (e.g., Bean et al., 1985; Donnelly & McDaniel, 1993; Gabel & Sherwood, 1980; Sarantopoulos & Tsaparlis, 2004). Although accepted by the majority (Thiele & Treagust, 1992), this assertion could be misleading (Orgill & Bodner, 2005). High-ability students may appear to benefit less from analogies since their inclusion may not make sense to these students as they already think formally (Orgill & Bodner, 2005; Thiele & Treagust, 1992; Treagust et al., 1998).

Piaget saw the ability to use higher-order relations (relations between relations) as a hallmark of the formal operational stage. Since reasoning by analogies requires relational similarities rather than object similarities, Piaget argued that it has to be a formal operation skill (Goswami, 1991; Goswami & A. L. Brown, 2006). From this point of view, the claim that concrete thinker students can benefit more from analogies seems to conflict with Piaget's cognitive development since they do not possess the analogical reasoning needed for performing relational mapping (Sarantopoulos & Tsaparlis, 2004).

Although Piaget suggested that analogical skills do not develop until the formal operations stage (at around eleven to twelve years), after Piaget, it turned out that children under three also use analogical reasoning. Goswami and colleagues evinced that as children's knowledge about the world becomes more prosperous and more profound, their performance in analogical reasoning increases with age (see the relational primacy theory of Goswami, 1992; see also Goswami, 2001; Goswami et al., 1998). Alternatively, Gentner and colleagues interpreted cognitive development as a "relational shift" from object similarity to relational similarity (see relational shift theory of Gentner & Rattermann, 1991; see also Gentner, 1988; Gentner & Toupin, 1986; Rattermann & Gentner, 1998). Furthermore, Gentner and Rattermann (1991) proposed that the relational shift would not occur unless children (even adults) have sufficient relational knowledge, and it is not dependent on age, but highly dependent on knowledge. They further argued that the change in knowledge shapes analogical development and explains why analogical ability appears at different times in different domains.

From an instructional standpoint, these partially explain why low-ability students tend to notice correspondences based on object attributes rather than relational similarities (Richland, 2011; Richland & Begolli, 2016; Richland & Simms, 2015; Treagust et al., 1998). However, all of these also explains that in addition to helping low-ability students, analogies can also be used at all developmental levels (Iding, 1997) if a specific instructional guideline is admitted (Zook & Di Vesta, 1991). Therefore, awareness of students' capacities is essential to designing the best learning environments and pedagogical strategies that broaden their interpretation of instructional analogies. Accordingly, in the classroom context, teachers should provide an analogy instruction in which relational commonalities and constraints are apparent (Zook & Di Vesta, 1991) to make analogies beneficial for as many students as possible, whether they are concrete or abstract thinkers.

2.5.2 Students' Familiarity with Analog

There is a common consensus that the analog domain must be within the knowledge of students (Curtis & Reigeluth, 1983, 1984; Gick & Holyoak, 1983; Mastrilli, 1997) or from the "student world" in the words of Thiele and Treagust (1994a) and Treagust et al. (1998), because analog familiarity is an essential prerequisite for students to see the links between analog and target pairs (Duit, 1991; Metsala & Glynn, 1996; Richland et al., 2007). If the analog is unknown to the students, it may cause multiple situations. At best, it may cause the analogy to be meaningless for students (Curtis & Reigeluth, 1984). At worst, it may cause students to be confused and draw spurious conclusions about the target concept (Curtis & Reigeluth, 1983, 1984; Hayes & Tierney, 1980; Iding, 1997; Venville & Treagust, 1997). For instance, Unver (2009) reported a textbook analogy between the social security system (analog) and function concept (target). High school students who have not yet entered business life may not be adequately familiar with the social security system to draw relevant similarities with the target concept, which may pave the way for the stated situations to develop. To avoid these undesirable situations and make the analog domain familiar to the students, a sufficient and relevant explanation of the analog domain can be made at the beginning of the analogy generation process (Curtis & Reigeluth, 1983, 1984; Mastrilli, 1997; Treagust, Stocklmayer, & Harrison, 1994). Whenever possible, the explanation of the analog can be selectively adapted to improve the analogical match with the target (Gray & Holyoak, 2021).

2.5.3 Students' Familiarity with Target Concept

Whether students benefit from analogies may also depend on students' background knowledge about the target concept (Orgill & Bodner, 2005; Richland & Simms, 2015). Since understanding the hitherto unfamiliar (or less familiar) domain with the language of the familiar domain is the hallmark of analogies (Donnelly & McDaniel, 1993), theoretically, analogies are expected to be beneficial for students with little background knowledge about the target concept. This potential benefit of analogies is based not only on theoretical but also on empirical grounds (e.g., Donnelly & McDaniel, 1993; McDaniel & Donnelly, 1996). However, it is theoretically and empirically ambiguous whether the same is valid for students with more extensive background knowledge about the target concept (Donnelly & McDaniel, 2000).

Some researchers argue that analogies may be superfluous (Donnelly & McDaniel, 2000; Venville & Treagust, 1997), or their use may be spurious (Iding, 1997) for students with more extensive background knowledge about the target concept. Although the results of some studies (for example, Donnelly & McDaniel, 1993; Novick, 1988) do not contradict these views, they show that students with more extensive background knowledge may be able to realize and use analogies more quickly than students with little knowledge. However, later Donnelly and McDaniel (2000) investigated this issue in more detail and revealed that although students' being knowledgeable about the target concept is a limiting factor in the effectiveness of analogy instruction, it is not a factor preventing students from making use of analogies. On the other side, it was later proved that the benefit of these students from analogy depends on how they directly relate their prior knowledge to the target concept rather than their level of knowledge about the target concept (for a broader discussion, see Donnelly & McDaniel, 2000). All these findings support the view that analogies will benefit students with both little knowledge and more intensive knowledge about the target concept.

2.5.4 Teachers' Knowledge for Teaching Target Concepts with Analogies

As the people who plan and carry out the teaching process, teachers are primarily responsible for analogies generated in the classroom context (Bayazit & Gray, 2006; Mozzer & Justi, 2013). There is a consensus that the quality of their analogy instruction is related to, among other things, their relevant knowledge bases, particularly their content knowledge on the target concept and pedagogical content knowledge on analogies and their use (Espinoza-Vásquez et al., 2017; Harrison & Jong 2003, 2005b; Hatisaru, 2021, 2022; Hulshof & Verloop, 2002; Mastrilli, 1997; Mozzer & Justi, 2013; Ubuz et al., 2013).

More specifically, building on Shulman's (1986, 1977) content knowledge and pedagogical content knowledge, Ball, Thames, and Phelps (2008) proposed four domains of mathematical knowledge "entailed by teaching" - common content knowledge (CCK), specialized content knowledge (SCK), knowledge of content and students (KCS), and knowledge of content and teaching (KCT). Since current work is in a mathematical context, teachers require these four domains of mathematical knowledge, which are detailed below, for teaching function-related concepts with analogies.

2.5.4.1 Common Content Knowledge and Specialized Content Knowledge

Teachers need to know what they teach, and in fact, nothing could be more fundamental to their competence (Ball et al., 2008; Ball, Lubienski, & Mewborn, 2001; Even, 1989, 1990; Ponte & Chapman, 2008). *Common content knowledge (CCK)* is the most basic level of mathematical knowledge and skill that is not specific to teaching. It includes "declarative knowledge of rules, algorithms, procedures, and concepts related to specific mathematics topic" (Even & Tirosh, 1995, p. 7), such as knowing definitions, properties, and notation of function. Teachers are expected to have CCK at least equal to their colleagues who are experts in that domain (Shulman, 1986). Apart from this, *specialized content knowledge (SCK)*, as the name suggests, is the mathematical knowledge and skill specific to teaching. This knowledge requires going beyond the facts or concepts of a domain (Shulman, 1986) and beyond correct computation (Ponte & Chapman, 2008) or what is taught to students (Ball et al., 1998). It also requires a deep, broad, and thorough understanding (Ponte & Chapman, 2008). For example, Norman (1992) defines teachers' content knowledge of function as involving "a familiarity with a wide spectrum of aspects of the function concept, as well as a depth understanding" with a similar understanding (p. 216).

Simply put, in Shulman's (1986) words, CCK requires "something is so" while SCK requires "why it is so" (p. 9). To give an example from Even's (1990) study, teachers' knowing that functions are univalent and arbitrary is related to CCK. Besides, knowing why functions are defined in this way and being familiar with the historical development of functions that better explains how they came to the present is associated with SCK. As in this example, the demands of teaching the concept of function (like any other mathematical concept) certainly require both bodies of knowledge. Teachers should have mathematical knowledge about functions beyond the level at which they teach (Watson & Harel, 2013).

Perhaps teachers' content knowledge (hereafter both CCK and SCK) is not the only factor influencing their teaching practices, but it is an undoubtedly major component of their instructional decisions (Even, 1989, 1993; Even & Tirosh, 1995; M. Meagher et al., 2019; Sherman et al., 2019). While CCK forms the basis, in part, of teachers' instructional decisions, SCK allows them to make better decisions (Even & Tirosh, 1995). However, in the absence of solid content knowledge may cause teachers to make undesirable or wrong pedagogical decisions. For example, in the study of Even (1993), a teacher (Brian) who used the univalence requirement correctly but also thought that regular graphs such as circles and ellipses are functions (even though they are not, because of not satisfying the univalence requirement), encountered that a circle failed the vertical line test. This situation changed his pedagogical thoughts and decisions, and he decided to use the vertical line test with his students only for linear functions, considering that it is an over-generalized tool.

In the same vein, teachers' content knowledge nurtures and influences their choice of analogies (Espinoza-Vásquez et al., 2017). Since an analog should have an intrinsic power to represent the essence and the properties of the target concept (Bayazit & Aksoy, 2011; Bayazit & Ubuz, 2008; Ubuz et al., 2009), a solid understanding of the target concept enables teachers to choose an analog compatible with the target and elaborate on how the target is similar and dissimilar to the selected analog (Nottis, 1999; Treagust et al., 1994). Thereby, solid content knowledge of the target enables teachers to evaluate the strengths and weaknesses of each analogy, select the most appropriate (sound) ones, and use them effectively (Nottis, 1999). However, when teachers use analogies to teach concepts outside of their area of expertise, it is possible to experience a chain of problems, starting with teachers' inability to use analogy effectively (Treagust et al., 1994) and extending to students' lack of understanding of the target concept or creating misconceptions.

2.5.4.2 Knowledge of Content and Students and Knowledge of Content and Teaching

Teachers' content knowledge is immensely crucial to teaching and its development; however, it may not be sufficient alone for teaching (Ball et al., 2008; Even, 1993; Shulman, 1986). Since the aim of teaching is primarily to assist and guide students in their knowledge construction, teachers' knowledge for teaching should also include knowledge about students' common conceptions and preconceptions about particular concepts and what makes those concepts exciting or challenging for students (Ball et al., 2008; Borke, 2021; Even, 1989; Even & Tirosh, 1995; Ponte & Chapman, 2008; Shulman, 1986). At this juncture, *knowledge of content and students (KCS)*, as the name suggests, responds to these demands of teaching and combines knowledge about both students and content. To give an example from Sierpinska (1992), teachers' familiarity with students' difficulties in distinguishing when f(x) represents the name of the function and when it represents the value of the function f and recognizing errors resulting from this without an extensive analysis is an example of their knowledge of content and students (KCS).

Another crucial knowledge required for teaching, as the name implies, is *knowledge of content and teaching (KCT*), which combines knowledge about both content and teaching. In addition to content knowledge, this knowledge is associated with teachers' having to use "the most useful forms of representation (…) the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others" (Shulman, 1986, p. 9). This knowledge is also relevant to teachers' having to array contents for instruction, anticipate the plusses and minuses of using each representation used to teach specific content, and size up mathematical issues in responding to students' varied approaches (Ball et al., 2008; Ponte & Chapman, 2008). For instance, in the example of Sierpinska (1992), while recognizing which symbols related to the function will cause students to have difficulties is about teachers' KCS, deciding what to do about these difficulties is about their KCT.

Taken into account the current work, in addition to content knowledge on functions, these last two domains overlapping with the basic dimensions of Shulman's pedagogical content knowledge are of great importance for teaching function-related concepts (like any other mathematical concepts) with analogies. As stated in Shulman's (1986) notion of pedagogical content knowledge, analogies are essential components of teachers' teaching repertoire. However, teaching by analogy is not straightforward (Richland, 2011) because each analogy necessitates special attention to use so that mathematical concepts (function-related concepts here) are attractive, understandable, and usable by students.

Many demands of teaching target concepts with analogies require KCS. It is apparent from the literature that analogies help students understand complex and abstract target concepts (Gray & Holyoak, 2021; Treagust et al., 1992) only when the analog domain is interesting (Coll, 2015) and familiar to them (Duit, 1991; Metsala & Glynn, 1996; Richland et al., 2007). In this sense, teachers should estimate common student difficulties with a particular math concept (Ball et al., 2008) to decide which target concepts are worth generating analogies. Besides, when selecting an analog, they should anticipate what interests and motivates students (Ball et al., 2008) and choose it from the student world (Thiele & Treagust, 1994a; Treagust et al., 1998). In a similar vein, since analogies allow teachers to correct student misconceptions (Hutchison & Padgett, 2007), teachers should also be aware of common student misconceptions (Ball et al., 2008) arising from selected target concepts or analogy-caused misconceptions (Ubuz et al., 2009) not to reinforce and even eliminate them. Moreover, when discussing analogies, teachers should interpret students' emerging and inchoate thinking (Ball et al., 2008).

On the other hand, the demands of teaching target concepts with analogies also require particularly sound KCT. Since using analogies effectively starts with understanding what analogies are (Glynn, 2015), teachers primarily need to know what analogies are and even "the most powerful" ones, as Shulman (1986) emphasized. Studies investigating teacher-generated analogies on the concept of function (for instance, Ubuz et al., 2013; Unver, 2009) also support this view by underlining that teachers' teaching practices with analogies align with what they know about analogies. Besides, using analogies effectively demands a systematically planned teaching that makes them pertinent to as many students as possible (Treagust, 2015). Therefore, since systematic planning necessitates some guidelines for designing and using analogies in teaching specific concepts (Nashon, 2003), several analogy-teaching models are also proposed. They include, for example, Zeitoun's (1984) *General Model for Analogy Teaching* (GMAT), Glynn et al.'s (1989) *Teaching with Analogy* (TWA) model, and Treagust et al.'s (1998) *Focus, Action, Reflection* (FAR), and Nashon's (2000) *Working with Analogies* (WWA) model. Apart from these models, Gray and Holyoak (2021) recently outlined a general strategy for teaching by analogy, consisting of five principles that can be applied flexibly to topics in a wide variety of domains. While each guideline has some strengths, weaknesses, and differences when considered individually, they all have common recommendations for teaching by analogy. That is, to draw a conclusion and summarize, all suggest that teachers should (1) identify well-understood analogs representing the essence and the properties of hard-to-understand (complex, unfamiliar, or abstract) target concepts (2) map analog-target similarities by explaining mappings fully with the help of visual and verbal supports, discuss relational and object similarities and identify dissimilar features (3) encourage students to discuss analogy and draw conclusions about the target.

Teachers' weighing appropriate time to use analogies and involve students in the process is as important as adhering to specific teaching guidelines. Since "constructing their own analogies helps students take an active role in their learning" (Glynn, 2015, p. 44), it is strongly recommended that teachers encourage students to generate analogies and also train their students in analogy use (Glynn, 2015; Orgill, 2013). Moreover, since generating just one analogy for a specific concept may cause students to think that it is the single explanation for that concept (Orgill, 2013), it is also firmly advised that teachers generate multiple analogies (Harrison & Treagust, 2006; Orgill, 2013). In each of these requirements for effective teaching, there is a blend of knowledge about teaching and content.

2.5.5 Appropriateness of the Target Concept for Analogy Use

Analogies may not be beneficial for all kinds of target concepts (Curtis & Reigeluth, 1983). It is pretty self-evident that the target concepts must be abstract, sufficiently novel, and challenging for analogies to be potentially helpful for students (Duit, 1991; Gick & Holyoak, 1983; Gray & Holyoak, 2021; Harrison & Treagust, 2006; Hayes & Tierney, 1980; Treagust et al., 1992; Venville & Treagust, 1997). Curtis and Reigeluth (1983) indicated that analogies often make abstract and challenging target concepts more exciting and relevant to all students. If the target concept is easy to understand, teachers or students may not bother to generate an analogy (Gick & Holyoak, 1983; Orgill & Bodner, 2005). Even if they do, this time, it may be extra information for students to remember (Gick & Holyoak, 1983). Considering the present study, the concept of function seems suitable for analogy teaching due to its complexity and abstraction. The function is challenging for many students (Cuoco, 1995; Evangelidou et al., 2004; M. Meagher et al., 2019; Sfard, 1992; Sierpinska, 1992), and there are several comments in the literature that it may be beneficial for teachers to use analogies when teaching function concepts, as long as they use valid analogies (e.g., Espinoza-Vásquez et al., 2017; Hatisaru, 2021, 2022; Selden & Selden, 1992).

2.5.6 Intrinsic Factors in Analogy Itself

Since the features of analog and target pairs are never identical (Glynn, 1989, 1994; Glynn et al., 1989; Treagust et al., 1998; Venville & Treagust, 1997), an analog can never fully describe a target concept (Gray & Holyoak, 2021; Orgill & Bodner, 2005; Orgill et al., 2015). This situation may cause students to believe that analog and target share fictitious features that they do not share (Venville & Treagust, 1997). Analog and target domains have shared and unshared features and some shared ones are better than others for the purpose of the analogy (Gentner, 1983; Glynn, 1994). If the number of unshared features is more significant than shared ones, or the shared features are remote, the analogy may be more confusing and harmful than helpful (Curtis and Reigeluth, 1983). Further, if teachers or students intentionally or unintentionally compare the unshared features of analog and target domains, this also leads to misunderstandings (Glynn, 1989; Treagust et al., 1998). For these reasons, it becomes essential for teachers to articulate the shared and unshared features of analogies (Coll, 2015; Venville & Treagust, 1997) and use multiple analogies to explain the target concept (Glynn, 1994; Harrison & Treagust, 2006). Further, it may be advantageous for students to learn how to use analogies (Gentner, 1980; Glynn, 2015; Venville et al., 1994). Students should be trained about what analogies are and how they work. Teachers should support students to generate their own analogies and be aware of the limitations of analogies (Glynn, 2015).

CHAPTER 3

METHODOLOGY

This chapter describes the research design, participants, context, data sources and data collection procedures, and data analysis and interpretation.

3.1 Research Design

In the current study, an interpretive research methodology (Erickson, 1986) was used to explore the nature and frequency of analogies employed by two mathematics teachers and their ninth-grade students while teaching and learning function unit. Interpretive research, with the main idea of understanding and meaning-making, is a framework, widely accepted practice, and set of paradigms within social science research (Bhattacharya, 1998). Several intertwined theories/paradigms of interpretation shape particular research methods. This study is based on the theoretical perspective of constructivism/interpretivism (often called social constructivism combined with interpretivism), into which qualitative research is embedded. Since the identified theoretical framework focuses more on conducting case studies (Bhattacharya, 1998; Bhattacherjee, 2012) and the conditions describing a case study (Merriam, 1998) are fulfilled, the research design for this study is a qualitative case study that used some quantitative data in an interpretive manner.

As Merriam (1998) and Yin (1994) pointed out, the decision to concentrate on this research design stems from a desire for insight, exploration, and interpretation within context. Throughout the study, stress was placed on making constructivist interpretations. As with all interpretive approaches, drawing meaningful inferences from case studies largely depends on observational skills, interpretive abilities, and experiences of the researcher (Bhattacherjee, 2012). In this study, both the researcher and the advisor are highly experienced in functions and analogy. The researcher has been working as a mathematics teacher for many years and teaching the function unit to ninth-grade students for years. Furthermore, in her master thesis, she analyzed textbook and teacher-generated analogies on the ninth-grade function concept under the supervision of the same advisor. The advisor also studied the effectiveness of analogies in teaching and learning function concepts, pre-service teacher-generated analogies on the function concept, pre-service teachers' knowledge and beliefs about function and analogy, and their reflection on teaching practices.

3.2 Participants

Participants of the study consisted of two experienced mathematics teachers (one female and one male) and their 121 ninth-grade students from five different classrooms - two (A and B) of the female teacher and three (C, D, and E) of the male teacher.

The study was conducted in a private high school located in the Anatolian part of Istanbul. There was a total of five ninth-grade classes in the school, and all ninth-grade classes and all ninth graders were included in the study. The school and teachers were purposefully selected for the current study. The school was chosen since it was the researcher's workplace. The teachers were selected because they were teaching math at the ninth-grade level and admitted to using analogies to teach functions. Teachers willingly participated in the research without any coercion.

The female teacher started her career at secondary schools, and from that, she has been teaching in secondary classes for 35 years. She earned a Bachelor's degree in Mathematics and Physics Education from Middle East Technical University, one of Turkey's most competitive universities, in 1982. In the first year of her graduation, she taught physics in a private high school since physics was one of her majors. After that one year, she has only been teaching mathematics in high schools for 34 years. She is highly regarded as an experienced mathematics teacher. She has been teaching in her area of expertise and has been teaching 9^{th}-grade mathematics for 15 years. Besides that, she has considerable expertise in the Advanced Placement (AP) Program and especially AP Calculus AB, roughly equivalent to a first-semester university calculus course, including concepts and skills of limits, derivatives, definite integrals, and the Fundamental Theorem of Calculus. She is the only responsible teacher for giving AP Calculus course for four years in her school.

The male teacher taught in several private schools in Istanbul. His teaching career spanned 18 years (4 years at the middle school and 14 years at the secondary school) in Istanbul's most prestigious private schools. He held a Bachelor of Science degree in 1997 from Bogaziçi University, another most competitive university in Turkey, from the department of Secondary Mathematics Education. He completed a Master of Science with a thesis program at department Secondary School Science and Mathematics Education in 2000. He has been teaching for one year in the school where the study took place.

Table 3.1: Profile of participatory teachers' backgrounds

Teacher	Gender	Major(s)	Number of years teaching	Number of years directly teaching functions concept
Teacher 1	Female	Math. and Phys.	35	34
Teacher 2	Male	Math. Ed. with MS	18	14

Table 3.1 presents a summary of the degrees and teaching experiences of the teachers. A majority of their academic years passed through directly teaching function concepts, and as far as they mentioned, they were using analogies while

teaching function concepts. However, none of them participated in any in-service or particular training setting sight on the function concept and how to use analogies.

There was a total of 121 students (56 female and 65 male students) from five different classes. The number of students in each class ranged from 23 to 25, with an approximately equal number of female and male students (see Table 3.2). The school administration informed that they randomly assigned students per class.

Table 3.2: Number of female and male students in each classroom

Classes	Teacher of the Class	Number of Female Students	Number of Male Students	Total
Class A	Teacher 1	12	11	23
Class B		9	15	24
Class C	Teacher 2	12	13	25
Class D		11	13	24
Class E		12	13	25
Total		56	65	121

3.3 Context

In Turkey, functions are first introduced under “function unit” in ninth grade (ages 15–16). The students had not yet been taught a lesson on functions that school year, but the concept had been covered at a more fundamental level in previous years, and therefore they had unwittingly learned about the function concept.

A centrally developed standard mathematics curriculum was followed in all classrooms in English as the medium of instruction. Although the mother tongue was Turkish among the school population, all students were proficient in spoken and written English, and they could quickly get involved and follow classroom discourses. In all classes, both teachers used the same mathematics textbook, written by T1, followed a departmentally designed syllabus, applied the same quizzes and mid-term examinations, and remained in sequence with their instructional topics (see details in the following section). At the end of each section, they gave the questions under the title of “Exercises” in the book as homework.

In classrooms, students were sitting in individual desks arranged in the traditional row and column style. Seating plans (desk and student arrangements) were class teacher-chosen. Students were in the same arrangement in all lessons for the entire year; however, it was flexible when needed. Everyone was facing the board. The teachers’ desks were in the front of the room, near the board in plain sight of all students, and they were in the same position in each classroom. All classes had a smart board in the middle, one whiteboard on one or both sides, and projectors. Students attended all lessons in their fixed classes.

Teacher 1 (T1) started all lessons by greeting students first and then checking who came to the class and who did not. She used this short period (about five minutes duration) as waiting time to ensure that students were ready to involve and engage in the lesson. When students could not settle down and tune in to the lesson, she reminded classroom norms and her expectations. In such cases, she emphasized her sadness at still repeating which behaviors are allowed in the classroom and which are not. In addition, she articulated the adverse effects of beginning a lesson with negative motivation on learning outcomes and the classroom environment. She also stressed the significant role and positive results of setting and maintaining a positive learning environment. She drew particular attention to her need for motivation for effective teaching.

At the beginning of her teaching, she summarized what concepts students had learned in the previous lesson and made connections between the new and last lesson. She also reported the objectives of the current lesson and what activities to do. She implemented a traditional teacher-centered approach and used direct instruction integrated with other teaching methods. Each lesson started with her lecture, then combined with a PowerPoint presentation, student problem solving, and class discussions. She presented PowerPoint slides only once at the beginning of the unit. Whole class discussions were held when she asked her students to generate analogies.

Apart from the teacher's lecturing, most lessons were devoted to student problem-solving. After demonstrating sample solutions to some textbook problems, she immediately gave students time to work on similar questions from the textbook individually. She checked the correctness of their answers within this period. She asked her students to stop when they found a solution to the given problem and wait for others to finish. When the time was up, she asked volunteers to answer the question on the board. However, sometimes she did not consider the demands of the students, and she decided who should get up to the board. One, two, or sometimes three students solved the questions simultaneously on the board while the rest of the class was watching. It should be pointed out that she sometimes did not give such a waiting time and asked students to get to the board immediately. She enabled all students to participate in this problem-solving phase. She asked the students on the board to read the question first, analyze what was given and missing, and develop a plan to solve it. While implementing the action plan, the whole class and teacher evaluated every solution step. Finally, she asked the student to check the result.

Although she used English as the writing language, she used Turkish as the spoken language. She used the smartboard as a whiteboard or projector screen. She made use of all the learning materials in the prescribed textbook and covered everything in it. She occasionally gave some instances from her own experiences when necessary. She projected the book on the smartboard as a word page and sometimes solved questions using the board marker, sometimes sitting at her desk using the pen of the laptop. In addition to textbook end-of-section exercises, she gave examples similar to those solved in the classroom as homework. However, she did not check whether students did or not.

Her instruction focused on conceptual understanding rather than on teaching procedural knowledge. She warned her students against rote learning and articulated connections between all solution steps and the reasons for applying each step. While introducing any new concept, she connected it to mathematical knowledge learned in past years or during the function unit. Most of the time, she tried to show its connection to daily life by generating analogies. Again, referring to the place of the functions in physics whenever possible, revealing the similarities and differences of significant terms in mathematics and physics, she wanted to show that mathematics does not exist in isolation. She dwelled on particularly the importance of functional terminology and frequently reviewed the definitions of function-related terms. She included articles from mathematics journals in her lectures to show how the function is defined in the literature. Projecting these articles onto the board, she provided brief information without details and references.

Although T1 had a quiet personality, and the teacher-student interaction was upbeat, she could not tolerate any students' fooling and loud or whisper chitchats that distracted her or interrupted her teaching. She often vocalized complaints about certain student behaviors or reprimanded some students in such cases. She was consistently warning the same students in every class. She never put pressure on the students, and students could attend the lesson from their seats without raising a finger and standing up. She praised students with words such as "Well done!" "Quite good!" "Great job!" etc. She motivated students with sentences like "Look! You have started accomplishing!". She emphasized the importance of working to be successful. She also gave verbal feedback to student deficiencies regarding problem-solving, computation, or basic mathematical operations. However, she did so in non-offensive constructive language. She encouraged students to participate in classroom discourses and problem-solving on the board to increase their self-confidence. She used pronouns "we" and "us" to include her in the teaching process. When students indicated a lack of understanding or felt that more explanation was required, she tried to make it easier for them to understand the content or help them to understand some rules, procedures. She sometimes preferred to generate new analogies and adapt or review previously generated ones in such cases. Again sometimes, she explained it by passing through simple examples.

Teacher 2 (T2) started all lessons by greeting students. He gave students time to prepare while getting ready. Meanwhile, he created a warm atmosphere by talking with them about extracurricular issues. Usually, T2 left checking attendance at the end of the lesson. When everybody was ready, he started his teaching. First, he reviewed the previous lesson and previewed the new content briefly before beginning the primary instruction to arouse student curiosity. He implemented a student-centered approach. He used direct instruction integrated with other teaching methods. Each lesson started with his lecture, then continued with group work. He generated analogies to introduce a new concept or make it easier for students to understand, use, or remember functional knowledge. Most of the time, he directed analogies toward the whole class. However, he sometimes used them to benefit individual students or groups who needed more clarification or explanation on function-related concepts by asking questions or requesting repetition of the concept being taught. He covered everything in the prescribed textbook and wrote some exercises from his own experiences when necessary. He explained and wrote everything in English. He never used Turkish, even when chatting with students. He often wrote definitions, explanations, questions, and solutions on the board. However, sometimes he did not want to draw graphs on the board; instead, he projected the textbook on the Smartboard as a word page. Sometimes he used Smartboard software, however, with a board marker and sometimes used it just as a screen or whiteboard. He also integrated a graphing calculator in his teaching and utilized TI-SmartView software. After teaching everything, he gave students time to take notes on the spaces in their textbooks or on their notebooks.

He broke up his lecturing with group work to keep students engaged and encourage their active learning. He reduced direct instruction, allowing students to take ownership of their learning. He underlined the importance of group work in learning and emphasized that no one can learn mathematics and anything by just listening without trying. He demonstrated how to solve basic and challenging exercises and left the others to the students. However, he gave little or no attention to students' defining their thoughts and making conjectures. It is worth mentioning that he never raised any student to the board to solve a question. Students worked in groups of 3 or 4. While students were working collectively, they arranged their desks in squares to see their friends' faces. Students were free after finishing assigned work. T2 left his book with solutions on the teacher's desk as the answer key so that students could check the accuracy of their solutions at any time. During this group work time, he occasionally asked his students to generate analogies collectively. At the end of the group work, the presenter of the groups shared their collectively generated analogies with the class. Only T2 assessed these analogies, and they only received T2's comments and feedback. While students were working, T2 kept his eyes on them without distracting them; however, he made them feel he was ready to assist whenever they needed. He was mobile and did not stay long with a student or group than they needed. When working with students, his interaction was less threatening and facilitator. Students seemed concentrated while working collectively.

The students did not interrupt his teaching in almost all classes and raised their hands when they wanted to ask something. Therefore, he did not repeatedly request students to be quiet or pay attention to what was being taught. When he realized that the students were beginning to wiggle, he waited only a few seconds without doing anything to bring their attention. But he did so calmly and confidently. Immediately after doing this, they stopped wiggling. He also did not engage in unrelated conversations that would disrupt the flow of his teaching. There was a slight buzzing while students were working in groups; however, the classroom environment was generally relaxed and calm. T2 and his students looked happy. There was great humor between him and the students. They were joking, laughing, and smiling together. However, when he started teaching, the interaction between teacher and students became formal. He used pronouns such as "you" and "your" to get their attention and directly involve them in the lesson. When students made constructive comments or solved questions correctly, he often verbally praised them with words such as “Very nice!” “Perfect!” “Nice!” etc.

3.4 Data Sources and Data Collection Procedures

The corpus of the materials collected for the present study included video recordings of lessons (and their transcriptions), field notes, informal conversations (or informal interviews) with teachers, and teachers’ all teaching materials (prescribed textbook and syllabus, and teachers’ notes and PowerPoint presentations).

Teachers were informed that the focus of the study was the use of analogy, and the investigation would continue throughout the function unit. In addition, they were told that the researcher would use the data obtained from the research in her thesis and academic studies and that all participants and school information would be kept confidential. However, with little knowledge about the focus of the study, both teachers were encouraged to teach the concept using their everyday style and were asked not to alter their instruction. Each of the teachers employed their regular mathematics curriculum. No attempt was made to interfere with the order of the unit to be taught, materials to be utilized, or the style of their presentations. Both teachers acknowledged the presence of cameras in their classrooms and willingly shared all their teaching materials.

Classroom observations were made to gather data regarding teachers’ and students’ in-class analogy use during function unit instruction. Data were collected in line with the approval and knowledge of the school administration and the Ministry of National Education for four and a half consecutive weeks (excluding the one-week April Holiday) in the spring term between March and April 2015 (for consent documents, see Appendix A). A total of 91 lessons (45 for T1 and 46 for T2), each lasting 40 minutes, were naturally observed - 22 for Class 9A, 23 for Class 9B, 15 for Classes 9C and 9D, and 16 for Class 9E (see Table 3.3). Variation in the observation frequency indicates each class's length of time to complete the function unit.

Table 3.3: Comparison of the number of observed and unobserved lessons

Teacher	Class	Total Number of Lessons in Weekly-Schedule During Observation Period	Number of Unobserved Lessons	Number of Observed Lessons
T1	9A	30	8	22
	9B	30	7	23
T2	9C	31	16	15
	9D	30	15	15
	9E	31	15	16
Total				**91**

Both teachers allocated about 30 lessons in their annual plans to teach the function unit, of which 28 lessons (13% of the academic year) were foreseen in the national mathematics curriculum (2013) prepared by the Turkish Head Council of Education and Morality. In Turkey, ninth-graders in public schools generally have 6 mathematics lessons in a week. However, schools have an opportunity to make small changes in weekly lesson hours within the rules of the ministry of education. For instance, in the school where the study was carried out, ninth-graders had 7 mathematics lessons per week, and the order of the periods from Monday to Friday was 2-1-1-2-1 for classes A and B; 1-2-1-2-1 for classes C and E, and 2-2-0-2-1 for class D (see Appendix B).

While it was planned to observe all lessons allocated in the annual plan, approximately 75% and 50% of T1 and T2 lessons, respectively, could be observed. Teachers lost lessons for various reasons, such as school events (special day celebrations or ceremonies, seminars, and school trips), common exams, and early departures. In teachers' absences, the researcher did not observe mathematics lessons, although a substitute teacher always was present in the classes – as was the case for T2 classrooms. The researcher acted like this because students were free to study anything and the substitute teacher only sat at the teacher's desk without teaching anything related to functions. Even though T2 lost many more lessons compared with T1, both teachers started and ended the teaching function unit simultaneously. It was probably because; T2 did not spare time to solve every textbook question and did not allow students to solve any on the board.

Data collection was accomplished chiefly through videotaping lessons of T1 and T2. All observations were videotaped during routine classroom periods, typically 40 minutes long, which enabled the researcher to review the lessons in their entirety. During function concept teaching, one camera and one tablet were used in each class to triangulate the speakers and record all utterances with maximum reliability.

Several pilot recordings were made in each classroom before the actual data collection process began to avoid stimulating student and teacher curiosity or potential unexpected situations and ensure that both teachers and students get used to the camera and behave naturally. Pilot recordings were made with two cameras obtained from the school administration and from a math teacher who did not participate in the study. During this preparation phase, to assist in the data collection process, the school administration assigned one of the school staff to record lessons when required. During the pilot, the school staff once had to set up the camera in T2's class, as the researcher had her lecture at that time. Immediately after this lesson, T2 complained to the staff for not leaving the classroom after setting up the camera. T2 clearly expressed his, and his students' discomfort with the school staff's being and observed his lesson from beginning to end. After this process, the researcher realized that the borrowed camera did not record anything in one of the classrooms due to the battery problem and found its sound quality in the existing recordings was low.

After pilot recordings, the researcher decided to set one camera without a battery problem and a tablet together. Each time, she placed them at the back of the classroom to focus on the whole class, and after making sure the recording process had begun, she left without waiting. When this process finished, she took them. Sometimes, the researcher and teachers' lessons overlapped, but teachers and even their students helped set up and remove cameras in these cases. Before the actual study, an observation plan was prepared, showing the teachers' weekly schedules to keep track of the observations and take notes as the work progressed (see Appendix B). Then, it was copied and shared with the head of the mathematics department, the vice-principle responsible for ninth graders, and the school administration to take the necessary precautions against any possible problems. This observation plan was used to note whether the lessons were videotaped or not and, if not, their reasons (see Appendix C).

In addition to video recordings, field notes were taken to produce meaning and understanding of nature and extend analogy use in classrooms. These notes generally consisted of descriptive information (such as any clue for in-class analogy use derived from student conversations with each other or their teachers and students' voice comments before or after lessons) and reflective information (such as thoughts, insights, early interpretations, and assertions arising during this process, questions and concerns regarding fieldwork, and recommendations for future research) with accurate time and data. This information was often obtained before or after lessons when setting up or picking up the camera and watching videotaped recordings after the day's observation was completed. Immediately after getting, raw (or unelaborated) notes were written up in the field diary. Generally, raw notes were fleshed out by handwriting in a complete form with analytic insights on the night of the day of observation.

Informal conversations (or informal interviews) with teachers took place before or after lessons, in everyday situations, whenever and wherever the opportunity arose. They occurred in the staff room, in front of the class gate, corridor, lunchroom, canteen, schoolyard, or school shuttle. The informal conversations were employed to establish an understanding of why teachers generated more analogies in some of their classes, why they generated more analogies at the beginning of the function unit, their beliefs, and experiences regarding function concept teaching, from which sources they drew upon their analogies used in class, what affected their selection of topics in which they created their analogies, why they dwell on some analogies more than others, and why they specifically chose to extend certain analogies rather than others. In general, informal conversations were used to gain a greater in-depth understanding of analogies generated in the classroom context as an additional source to expand and enhance observation data (Swain & Spire, 2020). Dialogs were closer to everyday conversations, but they were purposeful conversations like Burgess (1988) defined. Since the researcher and teachers were colleagues and worked together, conversations flowed naturally to retain trust and rapport. They were not digitally recorded since there was no attempt to capture the whole speech verbatim. Instead, there was an attempt to take factual and brief keynotes on the field diary based on impressions, insights, and interpretations as accurately as possible. Sometimes, teachers initiated conversations, which often included confessions and their self-explanatory details of what happened in the video-recorded lesson. Perhaps this was due to their desire to defend themselves since they knew that the researcher would watch the videos.

All teaching materials used in the classrooms were examined in line with the purpose of the study. The same ninth-grade mathematics textbook, "Mathematics for Highschool 9", written by T1 and published by the school foundation in 2014, was used in all classes. Textbook examination (only for function unit) was employed to understand whom, when, and how the textbook analogies were used in classrooms.

This textbook was scrutinized since it was the primary source of mathematics lessons in all five classes. There are only definitions, explanations, critical notes, comprehension exercises, end-of-chapter, and end-of-unit exercises in it. There are no solutions to any problem or question, and there are spaces below the questions for students to take notes. This textbook closely follows the national curriculum guidelines, and its content table has the same range of contents as the curriculum.

The syllabus guiding the function unit teaching throughout the study was also examined. Teachers followed a departmentally designed syllabus ordered according to the national curriculum and remained in sequence with their instructional topics. Subtopics within function unit involved: (1) function concept and its representations (introduction (definition), functional notation, functions defined by equations, testing for functions, evaluating a function, finding the domain of a function, polynomial function, rational function, irrational function), (2) graphs of functions (vertical line test), and (3) types of functions (constant function, identity function, linear function, piecewise function and its graph, absolute value function, polynomial function, one to one function, horizontal line test, onto function).

Neither T1 nor T2 employed any written lesson plans. However, pictures, figures (or any images), and PowerPoint presentations reflected on the whiteboard were collected from teachers. All these data were assessed in conjunction with the data obtained from classroom observations.

It is worth noting that while collecting data, the researcher worked as a mathematics teacher at the same school - taught prep class and 10th-grade mathematics and 11th-grade geometry, and shared the same teachers' room with the participant teachers. This made it possible to deepen understanding of teachers' everyday teaching practices.

3.5 Data Analysis and Interpretation

Table 3.4: Overview of the process of data analysis

Data Analysis	Steps	Explanation
Phase 1		Organizing and preparing the data corpus for analysis
	Step 1	Transcribing all videotaped data as well as jotting down possible analogies, interpretations, assertions, and any reflections that came to mind, and incorporating transcriptions into a booklet
	Step 2	Compiling, sorting, and arranging the data corpus
Phase 2		Identifying all analogies and assigning codes for each
	Step 1	Reading transcriptions and identifying all possible analogies considering previously noticed analogies during the data collection and transcription processes.
	Step 2	Developing a criterion to define what an "analogy" for this study through discussion with the advisor and re-examining possible ones based on this criterion
	Step 3	Identifying analog-target pairs of identified analogies and assigning codes to each analogy by selecting proper rules and abbreviations
	Step 4	Preparing an analogy index for each class of teachers to check the consistency of the assigned codes

Table 3.4 (continued)

Data Analysis	Steps	Explanation
Phase 3		Coding each analogy according to an elaborated and extended coding scheme (AFF) developed for this study
	Step 1	Coding the features of the analogies employed in the first observed lesson of each teacher considering two existing frameworks by the researcher and her advisor
	Step 2	Developing AFF upon the emergence of new categories, subcategories, criteria, and codes while coding the features of each analogy
	Step 3	Sharing and discussing AFF with the advisor and resolving all disagreements
	Step 4	Re-examining each analogy in order to confirm AFF
	Step 5	Coding each analogy in a series of fourteen passes according to the last version of AFF
	Step 6	Sharing initial coding with the advisor, then recoding some analogies and after checking all coding
	Step 7	Calculation of the frequency and percentages of the analogies that met the outlined features in AFF
Phase 4		Generating assertions from the entire data corpus and testing the validity of the assertions
	Step 1	Examining and categorizing the entire data corpus under assertions
	Step 2	Establishing evidentiary warrant for generated assertions

Although data analysis was carried out throughout the study period, the data corpus was heavily analyzed and interpreted in four phases (see Table 3.4). The first phase, which was carried out after completing classroom observations, included organizing and preparing the data corpus for analysis. At this phase, videotaped lessons were transcribed verbatim by the researcher. These transcriptions incorporated teachers' and students' oral discourses (direct quotations from teachers and students), whiteboard notes (including diagrams, drawings, formulas, projected images, and whatever was written on the whiteboard), teachers' and students' explicit nonverbal behavior cues (tone of voice and gestures) to the extent as they shown on the videotaped recordings, detailed explanations of teachers' teaching methods, and notes describing naturally occurring or unexpected classroom events. While transcribing, possible analogies, early interpretations about the implementation of the analogies, assertions arising during this process, and any reflections that came to mind were also noted.

Since this research analyzed classroom discourse and practices involving analogies used in teachers' function unit teaching, it should be noted that student discourses and practices emerged only when interacting with teachers. In addition, although the primary purpose of this study was not to interpret nonverbal behavior cues, only those associated with analogy use were noted in parentheses in the transcripts. While teachers used various types of gestures during their teaching, only comparative gestures (e.g., teachers' pointing back and forth between a function formula and a function machine) and pointing gestures (e.g., T1's pointing tomatoes and tomato sauce in a visual representing a tomato machine, while describing domain and range in the function) that teachers used to support students' understanding of analogies were focused on and noted on transcriptions. Besides, teachers' and students' different tones of voices (low and high tones, and moments when they were utterly silent) were noted, as they provide information about the classroom context in which analogies were generated. Apart from all these, it should be noted that comments on teacher and student facial expressions were not included in transcripts since cameras were placed at the back of the classroom, directly opposite the whiteboard, in a position that did not allow for monitoring teachers' and students' faces in detail.

For a more effective analysis and meaningful interpretation of the videotaped data, all lessons were transcribed chronologically in line with each teacher's weekly schedules (see Appendix B). After transcriptions were completed, they were incorporated into a booklet and paginated to get ready for the analysis (a two-piece and 389-page booklet for T1 and a one-piece and 195-page booklet for T2). This preparation phase ended with compiling a full set of collected data, and sorting and arranging relevant ones for later use.

It is worth mentioning that the confidentiality of all participants was preserved throughout the study. For this reason, codes were assigned to both teachers and students to keep their identities confidential in transcriptions. Female and male teachers were called Teacher 1 and Teacher 2, respectively, and codes T1 and T2 were assigned to identify them. Next, to satisfy the anonymity of the students when presenting data, codes started with S, followed by the order of the student in the class attendance list like 1, 2, 3, etc., and ended with the name of the student's class (A, B, C, D, or E) were posted to each student. Thus, for instance, code S4B denotes that the student is in Class B and is the fourth student in the class attendance list. In addition, throughout the study, video recordings of classroom observations and their transcriptions were conserved and used for data analysis only by the researcher and her advisor.

Literature concerning the epistemology of the functions (Dubinsky & Harel, 1992), instructional analogies in teaching and learning mathematics (Richland et al., 2004), Gentner's structure mapping theory (1983), and Holyoak and Thagard's multiconstraint theory (1989, 1995) provided a conceptual base for data analysis. The definition of analogy used in the current study was based on Glynn (2015), and an analogy was considered very generally as a comparison of the similarities of a more familiar concept (analog) with another less familiar concept (target). Mapping was defined as a systematic comparison of similar (or shared) attributes (or features) of analog and target concepts. Since each analog and target has dissimilar (or unshared) attributes, a good mapping was expected to indicate both similar and dissimilar attributes of the analog and target. Three components, an analog, a target, and a comparison/mapping between them, were accepted as the basis for generating an analogy, and analogies were determined by marking the units containing these components in the lesson discourses (Richland, et al., 2004).

In the light of the above-mentioned notions, the second phase of analysis began with detecting all possible analogy instances within the 91-videotaped lessons. Transcriptions of each lesson were read line-by-line several times to examine the use of all analogies therein. First reading through transcriptions was primarily to detect: (1) any comparisons between function-related concepts that students were less familiar with and concepts that students were expected to be more familiar with, and (2) any teaching methods that the teachers announced as analogy or implied an analogy by using recurrent keywords such as “same, like, just like, what if, think that and similar” during the class periods. Then, taking into account all these analogy indicators and analogies that were previously noticed in the data collection and transcription processes, all possible analogies generated by teachers and students were underlined with a highlighter, regardless of whether they were actual analogies. Then all transcriptions were read over and over until ensuring that potential analogy units were not overlooked.

Since many comparisons could be made, after lengthy discussions by the researcher and her advisor, it was decided to define “analogy” as an umbrella word containing analogies and metaphors (proverbs and idioms). In other words, any comparison, whether an analogy or a metaphor made between a new function-related concept (target) and an old concept that intrinsically represents the essence of the target concept (analog), was considered an analogy by consensus. Then, each possible analogy was re-examined by the researcher without judging its quality. The decision about the quality of the analogies was left for later evaluation under the feature entitled soundness of mapping.

After analogies were determined, analog-target pairs of each analogy were identified to be used in the following coding steps. Content of function-related title covered in the lesson where an analogy generated was examined in the transcripts and the textbook. Then, the target (or target concept) was decided, considering the selected analog attributes and what was meant to be explained by the analogy. For instance, T1 employed *a machine producing tomato juice from tomatoes* (analog) to explain the *pre-image-image relation*; for this reason, the *pre-image-image relation* was determined as the target concept or target. Once all analog-target pairs were detected, they were written down on the highlighted parts of the transcriptions.

At the same time, codes were assigned to each analogy using proper rules and abbreviations to gather quantitative information about the frequency of all analogies and analogies generated by teachers and students. Therefore, analogy codes were designed to capture by whom, in which class they were constructed, and their occurrence orders in the same class regardless of thinking whether they were a teacher- or student-generated analogies. For instance, the code "T1B2" refers to an analogy generated by T1 in Class B and is the second analogy in the same class. In another example, the code "S4E17" refers to an analogy developed by a student (S4) who is the fourth student in the attendance list of Class E and is the seventeenth analogy in the same class.

When an analog (e.g., *a machine producing tomato juice)* was used to describe different target concepts (e.g., *function concept* and *pre-image/image relation*), each was coded as separate analogies (T1A1 and T1A4, respectively). Similarly, when a little bit modified analog (e.g., *fruit juicer- producing apple juice from apple* or *a machine producing tomato aubergine mush*) was used either with the same target (e.g., *pre-image–image*) or another target (e.g., *function with two variables*), each was coded as two different analogies (S11A16 and T1A19). On the other hand, as expected, if two various analogs (e.g., *a machine producing tomato juice* and *a function machine converting each input to their two times and one more*) were used with the same target (e.g., function concept), each coded with different codes (T1A1 and T1A2, respectively). Moreover, when an analogy appeared more than once with the same analog-target pair (e.g., *a machine producing tomato juice–function concept*) throughout the function unit, the analogy was coded with the same code (T1A1). However, unlike other analogies, hyphen appearance numbers were added to the end of the codes of these repeated analogies from their first appearance. For example, codes T1A1-1 and T1A1-2 indicate the first and second use of analogy T1A1.

After deciding on rules and abbreviations for analogy codes, analogy indexes were prepared separately for the classes of each teacher (see Appendix D). These indexes depicted analog-target pairs of each analogy with their codes and page numbers appearing in the transcription booklets. They served as an organized record of data and a research tool facilitating later viewing or checking coding and analysis of the transcriptions. After constructing initial analogy indexes, analog-target pairs of each analogy and assigned codes were checked several times iteratively.

The third phase of the data analysis included an in-depth examination of identified analogies considering their features. Initially, the first videotaped lesson of each teacher was examined closely by the researcher and her advisor. Later, each of the identified analogies was coded according to a framework based on the original criteria of Curtis and Reigeluth (1984) and its slightly modified version of Thiele and Treagust (1994a). The rest of the analogies were then coded by the researcher through repeated reading of the transcriptions. Since mathematics has some differences from science, and classroom analogies were analyzed instead of textbook analogies in this study, some identified analogies did not fit well into the previous frameworks. During the coding process, iterative coding led to new relevant data, which led to new categories, subcategories, criteria, and codes. Thus, a new, elaborated, and extended version of the two aforementioned frameworks, called here "Analogy Features Framework (AFF)", was developed by the researcher out of necessity and shared with her advisor. After reviewing the decisions of the researcher, her advisor found a few vague points and made suggestions. There were no disagreements to be resolved, so only all ambiguous issues were clarified through discussion. Afterward, the researcher made a few changes in the initial version of AFF and re-examined each analogy to confirm it.

The new framework (as shown in Table 3.5) has fourteen categories. Five (1, 2, 4, 10, and 14) are entirely new, and the other nine (3, 5, 6, 7, 8, 9, 11, 12, and 13) are derived from the aforementioned frameworks by adding, deleting, or adapting their subcategories and related criteria. Data-driven and literature-driven codes with proper abbreviations are assigned to each subcategory, referring here as "features" for systematic evaluation of classroom analogies. Table 3.5, given below, provides an outline of AFF with a complete list of assigned analogy codes and either a teacher- or student-generated analogy identified in the current study for each feature.

Table 3.5: Outline of the Analogy Features Framework (AFF)

Features of analogies with assigned codes	**Description**
<u>1) Participant Structure</u>	who generated analogy
Teacher-generated (T)*	analogy is generated by the teacher
Student-generated (S)*	analogy is generated by the student(s)
*Individually student-generated (Ind-S)**	analogy is generated individually by a student alone (e.g., *function concept is like producing pickle juice from pickle*)
*Collectively student-generated (Coll-S)**	analogy is generated collectively by a group of students (e.g., *function concept is like calculating cab fare*)
<u>2) Classroom Context</u>	contexts in which teachers and students use their analogies
For teacher-generated analogies	
Routine teaching (RouT)*	analogy is generated as a part of teacher's routine teaching (e.g., *function concept is like function machine*)
Lack of understanding (LU)*	analogy is generated following students' demonstration of a lack of understanding (e.g., *function with more than one variable is like a factory*)

Note: *represents completely new subcategories. Underlined categories are entirely new categories consisting of completely new subcategories.

Table 3.5 (continued)

Features of analogies with assigned codes	Description
<u>**2) Classroom Context**</u>	contexts in which teachers and students use their analogies
For student-generated analogies	
At request of teacher (REQ)*	analogy is generated at the teacher's request (e.g., *function concept is like producing paper from wood pulp)*
On a student's own accord (Own)*	analogy is generated by the student's own accord (e.g., *the identity function is like live in a world of one's own*)
3) Location of the Analogy	where analogy is generated in the function unit
<u>**4) Selection of Analog-Target Pairs of Analogies**</u>	sources and topics of the analog domain and the content of the target concept
Sources of analog domain*	from which sources teachers and students naturally select the analog of their analogies
For teacher-generated analogies	
Textbook (TB)*	analog (or analogy) is selected from the textbook (e.g., *univalence requirement of function is like birthday function*)
Teachers' own experiences and/or knowledge bases (Own-Exp)*	analog is selected from teachers' own experiences and/or knowledge bases (e.g., *properties of the domain of a function is like input properties of a machine producing tomato sauce*)
Students' experiences and/or student-relevant conditions (S-Exp)*	analog is selected from students' experiences and/or conditions relevant to students (e.g., *function concept is like water level of a dam by year near to the school campus*)
For student-generated analogies	
Textbook (TB)*	analog (or analogy) is selected from the textbook (e.g., *the piecewise function is like an increase in length of a plant by years*)
Students' own experiences and/or knowledge bases (Own-Exp)*	analog is selected from the students' own experiences and/or knowledge bases (e.g., *pre-image-image in a function is like that in making cappuccino*)
Analogies and examples used by their teachers (T-Exp)*	analog is inspired by the one(s) previously used by the teacher (e.g., *the domain-range relation in a function is like that in a mother's giving birth*)

Note: *represents completely new subcategories. Underlined categories are entirely new categories consisting of completely new subcategories.

Table 3.5 (continued)

Features of analogies with assigned codes	**Description**
<u>4) Selection of Analog-Target Pairs of Analogies (continued)</u>	sources and topics of the analog domain and the content of the target concept
Topics of analog domain	
A math-related topic (MRT)*	analog is selected from a math-related topic
Topics involving mathematical computing in daily life (Dly)*	analog is selected from a math-related topic involving computing in daily life (e.g., *function concept is like calculation of water and electricity bills regarding to personnel use with municipality fee*)
Topics involving mathematical calculation with numbers only (Num)*	analog is selected from a math-related topic involving mathematical calculation with numbers only (e.g., *pre-image-image of a function is like input-output of a machine converting each input (x) to their three times and one more (3x+1)*)
An outside-math topic (OMT)*	analog is selected from an outside-related topic
Topics related to the culinary (Cul)*	analog is selected from a topic related to the culinary (e.g., *pre-image-image of a function is like input-output of a machine producing tomato products*)
Topics related to family ties/situations (Fam)*	analog is selected from a topic related to family ties/ family situations (e.g., *univalence requirement of a function is like mother-child relation*)
Topics that are idioms and proverbs (I/P)*	analog is selected from a topic that is an idiom or a proverb (e.g., *the identity function is like an idiom of wearing one's heart on one's sleeve*)
Topic related to industry (just for student-generated analogies) (Indst)*	analog is selected from a topic that is related to industry (e.g., *function concept is like producing paper from a tree*)
Content of target domain	what function-related topic that analogy is related to

Note: *represents completely new subcategories. Underlined categories are entirely new categories consisting of completely new subcategories.

Table 3.5 (continued)

Features of analogies with assigned codes	Description
5) Nature of Shared Attributes	the analogical relationship between analog-target pairs
Functional (F)	analog and target domains shared similar relational structures; in other words, they did the same things (same process), or their elements were related to each other in the same way (same relation) (e.g., *function concept is like a function machine*)
Structural and functional (S&F)	analog and target domains shared both in surface features (just spelling or meaning similarities of the terms) and in relational structures (e.g., *not being a function is like getting a director fired for not specifying the profit of that month*)
6) Presentation Format	format of the analog or analogy presentation
Verbal (Vb)	analogy was explained verbally in words alone, just written on the textbook or the whiteboard (e.g., *the univalence requirement of a function is like the director's obligation to get a profit per month*)
Visual and verbal (Vs &Vb)	analogy was explained verbally and reinforced visually by any visual representing the analog domain or the analogy (e.g., *pre-image-image of a function is like input and output of a function converting each input to its three times and six more*)
Types of visuals	
Static (non-dynamic) visuals*	visuals that had non-movable scenes
Dynamic visuals*	visuals that had movable images
Types of static visuals (T1-T7)*	T1: graph, T2: table, T3: photograph, T4: sketch, T5: diagram, and T6: Arrow diagram, T7: table, Venn diagram, and graph
Instructional features of visuals	
Medium of transmission*	how visuals transmitted (1) drawn on the board, (2) projected on the board, (3) shown from the textbook

Note: *represents completely new subcategories. Underlined categories are entirely new categories consisting of completely new subcategories.

Table 3.5 (continued)

Features of analogies with assigned codes	Description
6) Presentation Format (continued)	format of the analog or analogy presentation
The extent of abstractness of the visuals	the number of details that visuals shared with reality
Realistic (R)*	visuals where objects or phenomena were shown as what they look like in reality with all details.
Partially realistic (PR)*	visuals, where objects or phenomena were shown as what they look like in close to reality and main characters are shown; however, trivial ones were absent
Unrealistic (UnR)*	visuals where objects and phenomena were shown as they look in reality and just depicted them by conventional means of specific graphic characters
7) Level of Abstraction	whether analog and target had an abstract or concrete cognitive level
Abstract/abstract (A/A)	abstract analog and abstract target (e.g., *the identity function is like living in a world of one's own*)
Concrete/abstract (C/A)	concrete analog and abstract target (e.g., *function concept is like acquiring a sweater from the thread*)
8) Position of Analog or Analogy Relative to Target	whether analog or analogy was before, during, or after the presentation of the target
Advance organizer (AO)	analog domain or the analogy was presented before examination of the target concept at the beginning of the instruction
*Advance organizer type 1 (AO^1)**	analog domain or the analogy used to provide background information necessary for the target concept (e.g., *function concept is like function)*
*Advance organizer type 2 (AO^2)**	analog domain or the analogy used to refer back and get the learner to think to previously learned target concept (e.g., *function concept is like calculation of water and electricity bills regarding to personnel use with municipality fee*)

Note: *represents completely new subcategories. Underlined categories are entirely new categories consisting of completely new subcategories.

Table 3.5 (continued)

Features of analogies with assigned codes	Description
8) Position of Analog or Analogy Relative to Target (continued)	whether analog or analogy was before, during, or after the presentation of the target
Embedded activator (EA)	analog or analogy was presented somewhere during instruction, after the introduction of the target concept yet before conclusions were drawn about the target concept (e.g., *pre-image-image of the function concept is like input-output of a machine converting each input to its square*)
Post-synthesizer (PS)	analog or analogy is presented following a complete treatment of the target domain at the end of the instruction (e.g., *pre-image-image of a function is like the input-output of a function machine converting each input to their two times and one more*)
9) Level of Enrichment	the detail of mapping between analog and target domain
Simple (S)	analogies did not specify the shared attributes of analog and target and only stated the target was like analog with no further explanation (e.g., *function concept is like getting pizza from pizza dough*)
Partially enriched (P-En)*	when a simple analogy was accompanied by some statements which partially or explicitly pointing out either of shared and unshared attributes (similarities and dissimilarities) of the analog-target pairs or not explicitly pointing out both at once (e.g., *function concept is like specification profits of a hotel by month*)
Completely enriched (C-En)*	when a simple analogy was reinforced by some statements explicitly pointing out shared and unshared attributes of the analog-target pairs at the same time (e.g., *domain properties of a function is like input properties of a machine producing tomato sauce*)

Note: *represents completely new subcategories. Underlined categories are entirely new categories consisting of completely new subcategories.

Table 3.5 (continued)

Features of analogies with assigned codes	**Description**
<u>10) Extension of Mapping</u>	presence of any extension of mapping between analog and target domains
Extended (Ex)*	the same analogical statements with minor revisions applied to other targets or the same analogical statements with no modifications applied to the same target
*Base (B)**	analogies from which new ones derived throughout the function unit or the analogies of the first appearances of repeated analogies (e.g., *function concept is like input-output of a machine producing tomato juice*)
*Derived analogy (Der)**	an analogy compared the base analog to a different target or compared a modified base analog to a base target or a different target (e.g., *pre-image-image of function concept is like input-output of a machine producing tomato juice*)
Base-repeated analogy (Base-Rep)*	the repetition of the base analogy having the same analog-target pairs with the base (e.g., *function concept is like input-output of a machine producing tomato juice)*
*Derived-repeated analogy * (Der-Rep)*	the repetition of a derived analogy having the same analog-target pairs (e.g., *pre-image-image of function concept is like input-output of a machine producing tomato juice*)
Unextended (UnEx)*	analogical statements appeared once and from which no further ones were derived (e.g., *univalence requirement of a function is like birthday function*)
11) Analog Explanation	presence of any explanation concerning analog
Explained (Exp)	analogies included explicit explanation or with a few words regarding relevant analog attributes (e.g., *pre-image-image of function concept is like input-output of a machine producing tomato juice*)
Unexplained (UnExp)	analogies included no explanation regarding relevant analog attributes (e.g., *function concept is like obtaining carrot juice from carrot*)

Note: *represents completely new subcategories. Underlined categories are entirely new categories consisting of completely new subcategories.

Table 3.5 (continued)

Features of analogies with assigned codes	**Description**
12) Strategy Identification	presence of any strategy identification
Identified (I)	analogies included any statement identifying or referring to strategy as analogy
*Overtly identified (OI)**	analogies included any statement overtly identifying applied strategy as an *analogy* (e.g., *pre-image-image of a function is like buying and selling price of oil products in a factory)*
*Tacitly identified (TI)**	analogies tacitly introduced the applied strategy with phrases other than the word *analogy* (e.g., *univalence requirement of function is like a machine producing tomato juice*)
Unidentified (UnI)	analogies did not include any statement identifying or referring to strategy as *an analogy* (e.g., *function concept is like a function machine*)
13) Presence of Analogical Limitations	presence of any stated warnings or limitations about an analogy
Present (P)	analogies contained some statements regarding analogical limitations
*General (G)**	analogies contained an expression concerning the general limitations of analogy use or analogy generation process (e.g., *function concept is like a function machine converting each input to its two times and one more*)
*Specific (S)**	analogies contained an expression concerning specifically unshared attributes of analog-target pairs (e.g., *domain properties of a function is like input properties of a machine producing tomato sauce*)
Not present (NP)	analogies contained no description regarding analogical limitations (e.g., *the relationship between the dependent-independent variables is like tomatoes-chopped tomatoes relation in a machine producing chopped tomato*)

Note: *represents completely new subcategories. Underlined categories are entirely new categories consisting of completely new subcategories.

Table 3.5 (continued)

Features of analogies with assigned codes	**Description**
<u>14) Soundness of Mapping</u>	epistemological appropriateness
Sound (S)*	analogies included correctly identified analog and target attributes and epistemologically valid mappings (e.g., *not being a function is like getting the director fired for specifying two different profits for a month*)
Unsound (UnS)*	analogies did not include correctly identified analog and target attributes and epistemologically valid mappings (e.g., *domain-range relation of the function is like that in the mother machine)*

Note: *represents completely new subcategories. Underlined categories are entirely new categories consisting of completely new subcategories.

After obtaining its final version, in order to code analogies systematically, tables listing the AFF categories and subcategories in the column and analogies in chronological order in the row were developed (see Appendix F). Two separate tables were prepared for analogies generated in T1 and T2 classrooms. Then each analogy was coded in a series of fourteen passes concerning categories of AFF. In each pass, all analogies were coded considering one category of AFF with its subcategories and criteria under the category. For instance, the category "nature of shared attributes" and the criteria for analogy features "functional" and "structural and functional" under this category were considered, and all analogies were coded as either one of functional or both structural and functional (see Appendix F). All analogies were coded in the same manner, and this initial codification was later shared with the advisor. A random analogy was chosen and evaluated for each pass by the advisor. When discrepancies were identified, they were resolved through discussion, and a consensus was built. Some analogies were recoded in line with the decisions taken here. After repeated passes, the coding phase was completed. With each pass, the reasoning behind each element of the AFF, and what changed, adapted, or remained the same in this new framework compared to previous ones are explained in more detail in what follows.

Pass 1: Coding Participant Structure of Analogies

Previous frameworks were only used to examine textbook analogies employed by authors, so there was no need to consider who else generated the analogies. However, the current study inspected classroom analogies, which could be developed by either or both teacher and student(s). Thereupon, all completed analogy generations (an analog, a target, and a comparison/mapping between them) were scrutinized. Then, it was realized that some were generated individually by the teacher or a student. Some were generated collectively by a group of students; however, no analogies were developed collaboratively by teacher and student(s). Therefore, an entirely new category of "participant structure of analogies" was added to the new framework, and each analogy was categorized as either (1) teacher-generated or (2) student-generated. Then, student-generated ones were classified as (1) individually student-generated or (2) collectively student-generated. This categorization was inspired by that initially presented by Richland et al. (2004).

In order to code the participant structure of each analogy, analogy units were searched for who initiated, completed, and presented analogies. Since teachers individually undertook all these analogy generation processes or just presented completed analogies such as *function like a machine producing tomato products*, all the analogies presented by the teachers were coded as teacher-generated, and they were not categorized further. On the other side, since the analogies presented by students were the product of the individual effort of one student or collective efforts of more than one student, they were coded as either individually or collectively student-generated analogies. To be more precise, student-generated analogies in which several students collectively performed all aforesaid analogy generation processes at the request of teachers during "group work" time were coded as the latter. All the remaining student-generated analogies, in which one student alone performed all analogy generation processes or presented the completed analogy on the student's own accord or at the request of teachers, at a time other than "group work", were coded as the first. Since a student from each group presented all collectively generated analogies, these analogies were coded with the code showing the student presenting the analogy. For instance, the collectively student-generated analogy *the calculation sale price of a cupcake* was coded with S13C18 showing only its presenting student S13.

Pass 2: Coding Classroom Context

Since previous frameworks were only examined written textbook analogies, they did not need to evaluate the circumstances under which analogies were generated. On the contrary, it was vital to gain greater insight into contexts in which teachers and students employed their analogies. Hence, an entirely new category of "classroom context" was put into the new framework, and separate codes were developed for teacher- and student-generated analogies. Although there could be many different situations in which teachers and students can use analogies in a classroom context, general codes were designed in the current study derived from examining discourses in videotaped lessons and considering standard teaching and learning in the mathematics classroom environment and the relevant literature. Teacher-generated analogies were assessed whether they were generated (1) as a part of teachers' routine teaching or (2) following students' demonstration of a lack of understanding. On the other side, student-generated analogies were assessed whether they were generated (1) at the teacher's request or (2) by the student's own accord. Each teacher- and student-generated analogy was coded as either one of two. This classification was inspired by the "student context" offered by Richland et al. (2004).

In the discourses immediately preceding the analogy units, verbal and non-verbal cues were sought to decide whether teachers generated analogies following students' demonstration of a lack of understanding. Verbal cues included situations in which students directly requested or implied audibly further explanation or repetition of the concepts being taught (e.g., "I did not understand! Can you help me?", or "Is this in the book you are talking about?"), where teachers began their descriptions by stating that students gave incorrect or inappropriate answers to their questions (e.g., "No! look...", "No! no! It like this…") or where teachers begin their explanations by pointing out that students did not understand blatantly what was being taught (e.g., "Now look guys! Ok! Let me give you a hint!", "So look! Think about it this way! I am saying this for those who did not understand"). On the other hand, non-verbal cues included situations in which teachers began their explanations after students answered their questions accurately but quietly and in a reserved tone of voice or remained utterly silent without responding. Analogies generated by teachers in contexts including the aforesaid verbal and nonverbal cues were all coded as the latter.

All the remaining teacher-generated analogies, whether preplanned or appeared more spontaneously during instruction, were all coded as the first. Tracing some evidence prepared by teachers, such as PowerPoint presentations, could identify the preplanned analogies. However, it was difficult to clearly distinguish whether teachers were generating analogies when they needed clarification or explanation or were generating analogies when they had difficulty explaining function-related concepts. For these reasons, the rest of the teacher-generated analogies were accepted as part of their routine teaching. For instance, T1 said: "I want to explain it with one more example", and then she generated an analogy of *the univalence requirement of function is like child-mother relation* (T1B25-1). Since T2 developed this analogy to further explain about requirements of being a function, it was coded as the first one.

Analogies generated by students at the request of teachers were coded as the first. For example, S7 developed the analogy of *function concept is like producing pickle* *juice from pickle* (S7B5) at T1's request with the words: "Well, can you think of an analogy example?". Thus, this analogy S7B5 was coded as the preceding one. In addition, when students interrupted the teacher's instruction and generated analogies without any request, all of these analogies were coded as the latter.

Pass 3: Coding Location of the Analogy Through the Function Unit

It was wondered where analogies were most and least frequently employed throughout the function unit. This category was not new and present in one of the existing frameworks; however, it was handled differently in this study. Previous studies divided textbooks into ten deciles by page numbers to decide the location of analogies in the entire book, then assigned each to one of the deciles by page numbers. On the other hand, the location of the analogies in this study was restricted to the chapters of the function unit rather than page ranges. The function unit was covered under 3 chapters (Function concept, Graphs of functions, and Types of functions), consisting of 18 sections in total. Each section was coded with numbers indicating its chapter first, then its order in that chapter (see Appendix E). For instance, code 3.1 referred to the first section in Chapter 3. Later, the discourses just before, during, and just after each analogy unit were examined. The location of each analogy was determined by matching the textbook's content, which is the same as the syllabus. After determining its location, each analogy was coded with identified section codes. For instance, T2 employed an analogy of *not being a function is like a fired man* (T2C24-1) when solving an exercise from the textbook. Later this exercise was searched in the book and found to be within the section of evaluating a function (page 13, question 1a), so the location of the analogy was coded with section code 1.5.

Pass 4: Coding Selection of Analog and Target

Throughout the study, it was known that all analogies were used to explain target concepts that were selected purely from function-related concepts listed under the function unit in the departmentally designed syllabus ordered according to the national mathematics curriculum (see Appendix E). However, there was no clear information about where and which topics teachers and students naturally selected the analogs of their analogies. In the same vein, it is known that previous frameworks (actually only one of them) included a category that classified the content area of the target concept rather than exploring the analog domain. Hence, partially a new category, "the selection of analog and target", was added in the current study to provide information concerning sources and topics of analogs and the content of target concepts. In this study, since it was known that the source of all target concepts is the secondary school mathematics curriculum for 9^{th}-grade, only their contents were examined.

First, to gather general data about where (from which sources) teachers and students invoked their analogies in the ninth-grade mathematics classrooms, two separate codes were developed to examine sources for teacher- and student-generated analogs. Thus, sources of each analog for teacher-generated analogies were coded as one of the three: (1) textbook, (2) teachers' own experiences and/or knowledge bases, (3) students' experiences and/or student-relevant conditions. Besides, sources of each analog for student-generated analogies were coded as one of the three: (1) textbook, (2) students' own experiences and/or knowledge bases, and (3) analogies or examples used by their teachers. These sources were based on preliminary analysis of the entire data corpus (particularly data obtained from observation and informal conversations) and as well as relevant literature, especially findings of the science studies by Mozzer and Justi (2013) and Thiele and Treagust (1994b). The first categorization of teacher-generated analogies paved the way for the formation of student-generated analogies.

In order to code the sources of teacher- and student-generated analogies, discourses just before and during each analogy unit were examined. First, textbook analogies were determined. Later, the ones used in classrooms were identified, and sources of these analogies (both teacher- and student-generated ones) were coded as the textbook (for example, the *birthday function* analogy used in both A and B classes). Next, teacher-generated analogies that employed analogs referencing students' experiences and/or conditions relevant to students were coded as the third one. For example, T2 generated the analogy T2D13, which specifically referred to the water level of a dam close to the school campus, to teach function concept. Thus, this analogy was coded as the third, considering that it was relevant to students. Apart from that, analogies that first presented by the student(s) and then repeated by their teachers by referencing the first user student(s) were also coded as the third. For instance, T2 used the analogy of *function concept is like the calculation annual wage of a footballer* (T2D18), which was initially generated by a group of students in Class E (S17E20). Since T2 used this students-generated analogy and referred to students who used it, this analogy was thought about students' experiences and coded as again the third one. Lastly, all other teacher-generated analogies were considered selected from teachers' own experiences and/or knowledge bases, and they were thus coded as the second one.

In a similar vein, after the textbook-based analogs were determined first, the sources of student-generated analogies whose analogs were inspired by the ones previously generated by the teachers were coded as the third one. For example, the analog of *a mother's childbearing* (S18B24) was inspired by T1's *mother machine* analogy (T1B22). Finally, the sources of the rest of the student-generated analogies were all coded as the second one.

Second, both teacher- and student-generated analogy units were further examined to identify teachers' and students' choice of topics for analogs of their analogies. First, they were determined to exist within (1) a math-related topic (any topic involving direct mathematical computations) or (2) an outside-math topic (any topic including indirect mathematical computations or non-mathematical phenomena). For example, the topics of analogs such as *specification profits of a hotel by month* (T2C3) and *earning money per working hour* (T1B31) were classified as math-related. On the other hand, the topics of analogs such as *producing pickle juice from pickle* (S7B5) and *mother-child relation* (T1B25-1) were coded as outside-math.

Then, math-related and outside-math topics were further categorized according to preliminary analyses of the entire data corpus (again, predominantly videotaped lessons and informal conversations) and research on related literature. Consequently, math-related topics were classified as either (1) those involving mathematical computing in daily life (for example, finance - money matters such as management, creation, and study of money, baking, credit or investment - financial decision making - choosing the best monetary option, making predictions to given situations, and computations for life events or daily routines) or (2) those involving mathematical calculation with numbers only (computation with numbers or application of number properties). For instance, the topic of the analog *calculation of water and electricity bills according to municipality fee plus personal use (*T2E12) was coded as the first. On the other side, the analog *machine converting each input (x) to their three times and one more (3x+1)* (T2C1) was coded as the latter.

On the other hand, outside-math topics were classified as (1) those related to the culinary, (2) those related to family ties/situations, or (3) those that were idioms and proverbs. For example, the topic of analog *a machine producing chopped tomato* (T1B29) was coded as the first one. Besides, the topic of analog *mother-child relation* (T1B25-1) was coded as the second. Apart from these, the topic of analog *wear one's heart on one's sleeve* (T1A24) was coded as the third.

Differently, a fourth criterion, "those related to the industry," was added to the category of outside-math topics in addition to these three criteria for student-generated analogy categorization. For instance, the analog *producing paper from wood* (S13B10) was coded as those related to the industry.

Apart from these, since T1 is female and T2 is male, and both female and male students also generated analogies during function unit teaching and learning, the gender difference was also examined whether it affected the choice of analog topics or not.

Third, to determine the content of the target concepts, all teacher- and student-generated analogies were examined and coded according to 18 sections described in the function unit (see Appendix E). For instance, the content of the target concept, *not being a function,* of the analogy T2C6-1 (*not being a function is like getting fired of the hotel director because of specifying two different profits for a month)* was coded as Section 1.1 as it was related to the univalence requirement and thus the definition of the function. The study, which included the framework mentioned earlier, defined the contents of the target concepts more broadly as it examined the analogies used in all units of the textbooks. However, since only one unit, the function unit, was investigated in this study, the content of the target concepts was defined from narrower sub-headings.

Pass 5: Coding Nature of Shared Analog and Target Attributes

Previous frameworks presented three types of analogical relationships between analog and target: (1) structural, (2) functional, and (3) both structural and functional. However, only two (functional and structural-functional) were included in the AFF since it was thought that analogies in which analog-target pairs only share structural relationships could not potentially incite the essence of functions. Therefore, each identified analog-target pair was examined, and then the analogical relationships between pairs were coded as either functional or structural-functional.

Analogies, in which analog and target domains shared similar relational structures and the analog domain behaved or performed like a function-related target concept, were coded as functional (e.g., classical function machine analogy). On the other hand, functional analogies in which analog and target domains shared surface features (just spelling or semantic similarities of the terms) were coded as structural-functional. For instance, the analogy of *not being a function is like getting a director fired for not specifying the profit of that month* (T2D25) was coded as structural-functional. This analogy was coded as functional since the director was wanted to determine one profit value for each month; likewise, a function was expected to match each input to an output. This functional analogy was also coded as structural since there was a semantic similarity between the two. Namely, in the end, both have an undesirable situation; the director got fired (analog), and the other one was not a function (target). In another instance, the analogy of *identity function is like wearing your heart on your sleeve* was also coded as structural-functional. This analogy was coded as functional since the idiom, *wearing your heart on your sleeve (analog),* means that if you are a transparent person, you turn what you thought or felt into action. Likewise, the identity function (target) returns the same value. This functional analogy was also coded as structural since there was a semantic similarity between the two. Namely, both emphasize the same inside and out.

Further examination of all functional analogies realized that their analog and target domains went through similar stages to attain a particular end (similar process), or their elements were related to each other in a similar way (similar relation). More precisely, the analogs occasionally behaved as a function-related concept (target), which addressed a function as a relation that mapped each input onto only one output or sometimes managed a function as a process that transformed each input into a unique output. Therefore, to determine which attribute of the function they address, all analogies were coded in terms of their analogical relationship as either (1) a process, or (2) a relation. An example of an analogy coded as a process was *a function likes calculating cab fare* (S6E16-1). On the other hand, an example of an analogy coded as a relation was a *function likes comparing different brands of cars according to their gas consumption* (T2C14).

Pass 6: Coding Presentational Format of the Analogy

Previous frameworks presented two formats of analogy presentation under this category. The first one was the verbal (or written) format, where analogy was written and explained in words alone. The second one was the pictorial-verbal format, where the picture(s) (maybe drawings or photographs) of the analog or target domain reinforced the verbal analogy. The present study made several adaptations and clarifications, and several subcategories were added to the existing frameworks under the same category. First, what was meant by the verbal format criterion was clarified. Previous frameworks were used only to examine textbook analogies. Thus, the verbal format was only used to present what was written in words. However, in this study, the AFF was used to examine classroom analogies. Thus, the verbal format was used to present spoken and/or written classroom analogies in words. Second, since visual materials other than the picture can support verbal analogies in today's lessons, the pictorial format criterion for this study was updated. Therefore, the visual format was used instead of the pictorial format in the AFF. Consequently, in the current study, each analogy was categorized as either (1) verbal or (2) visual-verbal. Third, there was no consensus among the existing frameworks on what the pictures would represent in pictorial-verbal analogies. In this study, to avoid confusion, an analogy was considered visual-verbal when expressed verbally and reinforced by a visual representation of the analog domain or the analogy. Lastly, two subcategories of *Types of visuals* and *Instructional features of visuals* were added to examine the visual materials used in more detail.

Next, the discourses during analogy units were examined in order to determine how analogies were presented. Analogies explained orally, written in words in the textbook or on the whiteboard, were coded as verbal. For instance, the analogy of *the univalence requirement of a function is like the director's obligation to get a profit per month* (T2C8-1) was presented orally and written on the board under the title of "Hotel Rule 1", so this analogy was coded as verbal. In addition, verbal analogies reinforced with any visual in the textbook or any visual drawn or projected on the board were coded as visual-verbal. For example, the analogy of *the pre-image-image of a function is like inputs and outputs of a machine converting each input to its three times and six more* (T2C2) was presented orally and reinforced by a diagram showing a machine (see Table 3.8) was coded as visual-verbal.

During coding visual-verbal analogies, it was wondered what types of visual materials (hereafter called visuals) accompanied the explanation of analogies and how they were used in the classrooms. As emphasized by Janko and Knecht (2013) that the types of visuals and their instructional features make significant contributions to the effectiveness of learning, it was thought that these two would contribute to the effectiveness of the generated analogies. From this point of view, the discourses in the analogy units were examined, and all specified visuals during analogy generation were compiled. They were then re-examined and at first categorized in terms of their types.

Table 3.6: Categorization of types of static visuals

Category	Code	Type
Statistical Graphic	T1	Graph
Tabular	T2	Table
Pictorial	T3	Photograph
	T4	Sketch
Diagrammatic	T5	Diagram
	T6	Arrow diagram
Multiple Types	T7	Table, arrow diagram and graph

Static (non-dynamic) visuals having non-movable scenes and dynamic visuals having movable images such as movies, animations, or films were considered as potential visuals for mathematics classrooms. However, since all detected visuals were static, only static visuals were further scrutinized. According to a categorization system modified slightly that of Janko and Knecht (2013), they were classified into their types. Since the original category system consists of items to categorize the types of visuals in geography textbooks and mathematics is the exterior area of geography, some adaptations had to be made considering the visuals obtained from in-class observations. The adapted category system included four main categories and the visual types (T1-T7) under these categories presented in Table 3.6.

There may be many methods for classifying the types of visuals since they can be characterized as a sum of many features such as the medium of transmission, the extent of abstractness, use of color, number of main figures or elements, and the amount of internal details in figures or elements. Considering the related literature, two essential features: (1) medium of transmission, and (2) the extent of abstractness were accepted as instructional features of visuals. These two features were then assessed for each visual together with (1) which analogies (analog-target pairs) were accompanied by the visual, (2) by whom and in which class the visual was applied, and (3) the position of the visual through the function unit. On the other hand, other features: “the use of color, number of the main figures, and the amount of the internal details in figures or elements" were assessed particularly for some visuals, which were examined in more detail.

How the visuals transmitted were determined as the medium of transmission and coded as either (1) drawn on the board, (2) projected on the board, or (3) shown from the textbook. These codes were derived from an examination of the lesson discourses.

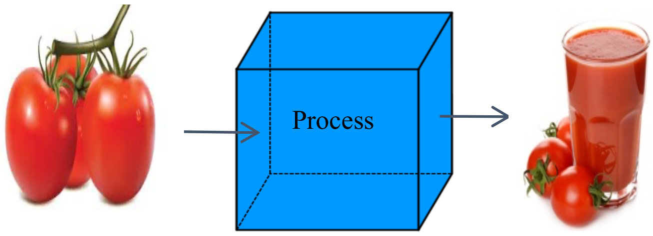
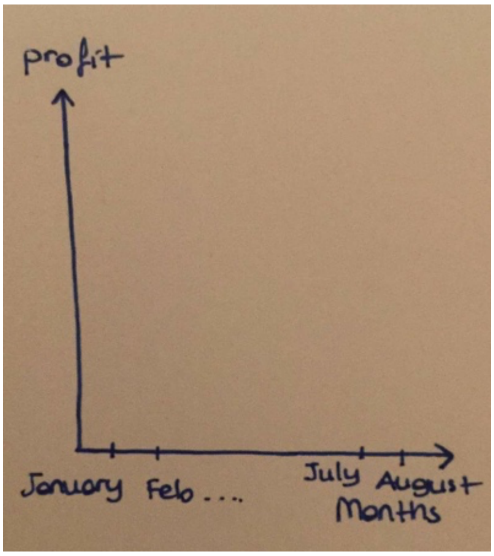
The number of details that visuals shared with reality determined the extent of abstractness of the visuals. Classification of the extent of abstractness was based on the meaning perceived by students. The abstractness of the visuals was coded through the following category developed by Janko and Knecht (2013).

Table 3.7: The extent of abstractness of the visuals

Code	Quality	Description
R	Realistic (iconic or concrete)	Visuals (photographs, drawings, films, and documentaries) showing objects or phenomena as what they look like in reality with all details.
PR	Partially realistic (similar)	Visuals (pictures, drawings, and caricatures) showing objects as what they look like in close to reality and main characters are shown; however, trivial ones are absent.
UR	Unrealistic (logical or abstract)	Visuals (diagrams, graphs, histograms, flow charts, maps, charts, and arrow diagrams) showing objects and phenomena as they look in reality and depict them by conventional means of specific graphic characters.

See Table 3.8 for examples of each the extent of abstractness of the visuals. The first picture was coded as partially realistic because although it included photographs of tomatoes and tomato juice, it was not fully photographed.

Table 3.8: Sample visuals of each coded the extent of abstractness

Abstractness	Visuals
Partially realistic	 Visual 1: "A machine producing tomato juice" analogy
Unrealistic	 Visual 2: "Monthly profit values of the hotel" analogy

Pass 7: Coding the Level of Abstraction of the Analog and Target

Previous frameworks analyzed analogies in terms of the degree of abstraction of analog and target domains. They categorized them as either one of the three possible combinations: (1) concrete/concrete, (2) abstract/abstract, and (3) concrete/abstract.

As anticipated that there was no concrete target domain in this study since the nature of all function-related concepts (targets) was abstract. Therefore, the concrete-concrete condition was not included in the AFF. Consequently, analogies were coded in this study as either (1) abstract/abstract or (2) concrete/abstract under the same category.

The discourses in the analogy units were examined and identified analog, and target domains of each analogy were coded as being either concrete or abstract in nature. As expected, all function-related targets were coded as abstract by definition. An analog was coded as concrete when it was tangible, directly experienced with eyes, ears, and fingers, and was easily associated with everyday life experiences. On the contrary, an analog was coded as abstract when intangible and not directly experienced with eyes, ears, and fingers. In this study, a proverbial and idiomatic analog that was metaphorical and included formulaic language was accepted as abstract. Thus, if the analog domain was concrete, the analog-target relationship was coded as concrete/abstract. For example, the analogy of *a function is like acquiring a sweater from the thread* (S2B13). On the other hand, if the analog domain was abstract, the level of abstraction was coded as abstract/abstract. For example, the analogy of *identity function is like living in a world of one's own* (S18B57).

Pass 8: Coding Position of Analog or Analogy Relative to Target Concept

Previous frameworks examined the position of the analog in relationship to the target and classified analogies in one of the three positions (1) advance organizer, (2) embedded activator, and (3) post-synthesizer. In this study, several points were adapted, clarified, and several new criteria were added in this category of previous frameworks. First, this categorization of previous frameworks required a clear separation of analog and target domains. However, in the present study, it was detected that the whole analogy was used instead of analog somewhere before, during, or after the instruction of the target. Therefore, to avoid confusion, it was decided to examine the position of the analog domain or the analogy relative to the target domain. Second, since the previous frameworks examined textbook analogies, they evaluated the position of the analog relative to the target within the section(s) where the target domain was described. In a similar vein, since the AFF examined classroom analogies in this study, the position of the analog relative to the target was evaluated within the target concept instruction on the same day. Third, since two different types of advance organizers were detected in this study, this position of previous frameworks was divided into two as (1) advance organizer type I (AO^1) and (2) advance organizer type II (AO^2).

Each day's discourse was examined for each class to code the position of the analog domain or analogy in relation to the target domain. An analogy was coded as "advance organizer" if analog or analogy was presented before the main discussion of the target concept. Besides, advance organizers were further investigated and coded either one of two advance organizer types. Ausubel's (1960) advance organizer theory and its two types (expository and comparative advance organizers) provided a conceptual basis for identifying advance organizers. Any new analog or analogy used before the main discussion of a new, unfamiliar target to introduce that target was coded as AO^1. For instance, T1 presented a new analog *(function machine transforming inputs to outputs)* at the beginning of a new target (*function concept)* instruction to introduce that target. This analogy (T1B1-1), which provided the advance organizer criterion, was coded as advance organizer type I since it was entirely new for students.

On the other hand, all other different combinations of old or new analog/analogy and old or new target were coded as AO^2. Namely, any old analog or analogy used before the main discussion of a new target to provide background information for that target was coded as AO^2. Alternatively, any new or old analog or analogy used before the main discussion of a previously learned (old) target to help students remember that target was coded as AO^2. In this position, whether analog/analogy or target was new or previously used, it did not matter. However, the familiarity of one or both analog/analogy and target was key in deciding whether it was AO^2. For instance, T2 presented the analog of *calculating water and electricity bills according to municipality fees plus personal use* before the main discussion of the previously learned function concept (target). Thus, this analogy (T2E12), which provided the advance organizer criterion, was coded as advance organizer type II as students were already familiar with the target concept.

An analogy was coded as an "embedded activator" if the analog domain or the analogy was presented somewhere during the main discussion of the target concept, after the introduction of the target concept, yet before conclusions were drawn about the target. For instance, the analogy (T2C2) was coded as an embedded activator since T2 presented the analog of *input-output of a machine converting each input to its square* immediately after introducing the target concept of *pre-image/ image of the function concept*.

An analogy was coded as "post-synthesizer" if the analog or the analogy was presented following a complete treatment of the target concept at the end of its instruction. In order to code analogies as post-synthesizer, it was examined whether the teachers' discourses in the analogy units contained any evidence that they summarized what they did during target concept instruction. The evidence included some words from teachers such as "I summarize what we did", "Think simple when you get confused", and "Do you understand what we have done so far" followed by a summary of the target concept instruction. For instance, the analogy of *the pre-image/image of a function is like the input-output of a function machine converting each input to their two times, and one more* (T1B21-1) was coded as post-synthesizer. Since T1 used the words "do you understand what we have done so far", which continued with a summary of what they did about the target concept. Moreover, there was other evidence that this analogy was used after T1's completing instruction of the target concept of *pre-image/image* and before starting the new target concept of univalence requirement of the function.

Pass 9: Coding Level of Enrichment

Previous frameworks classified analogies concerning the level of enrichment as (1) simple, (2) enriched, and (3) extended. However, since enrichment and extension were considered different notions, enrichment in the present study was accepted as the elaboration of mapping (explanations about shared and unshared attributes of analog-target pairs, analog domain, and general limitations), not the extent of mapping. Therefore, the category level of enrichment was recategorized regarding the elaboration of mapping as (1) simple (non-elaborated), (2) partially enriched (partially elaborated), and (3) completely enriched (completely elaborated). A detailed description of each of the three new version enrichment levels is shown in Figure 3.1.

In order to code analogies in terms of the level of enrichment, each analogy unit was examined. An analogy was coded as simple (non-elaborated) when the analogy statement contained analog–target pairs and just mentioned a pertinent similarity between pairs such that “target is like analog” or “target is compared to analog” without further explanation. For instance, the analogy of *function concept is like getting pizza from the pizza dough* (S2B20) was coded as simple since only stated target is like analog with no further explanation.

An analogy was coded as partially enriched (partially elaborated) when the analogy statements partially or explicitly pointed out any of the shared and unshared attributes (similarities and dissimilarities) of the analog-target pairs but did not explicitly point out both at the same time (see Figure 3.1).

An analogy was coded as completely enriched (completely elaborated) when the analogy statements explicitly pointed out both shared and unshared attributes of the analog-target pairs simultaneously.

Level of Enrichment	Explanation Details
A simple (non-elaborated) analogy	provides the most basic explanation in a single sentence without further explaining the similarity of analog-target pairs
A partially enriched (partially-elaborated) analogy	provides a partial explanation of shared and/or unshared attributes of analog-target pairs
	provides a partial explanation of the shared attributes of analog-target pairs and an explicit explanation of their unshared ones, or vice versa
	provides explicit explanation of only shared or only unshared attributes of analog-target pairs
A completely enriched (completely-elaborated) analogy	provides an explicit explanation of both shared and unshared attributes of analog-target pairs

Figure 3.1: A detailed description of three levels of enrichment for analogies

Besides, an analogy was coded as enriched when the analogy statement included any stated limitation (general, specific, or both) or an analog explanation or reinforced by a visual representation of the analog domain or the analogy. For instance, all visual-verbal analogies were coded either partially enriched or completely enriched. However, the explicitness of the explanations about shared and non-shared characteristics of analog-target pairs was decisive in determining the level of enriched analogy rather than these details. Similarly, although any stated limitation considered enough to categorize an analogy as enriched, a specific warning explicitly referring to the unshared attribute of analog-target pairs was searched to code an analogy as completely enriched. For instance, the analogy of *domain properties of a function is like input properties of a machine producing tomato sauce* (T1B3-2) was coded as completely enriched since analogy sentences included explicit explanations about both shared and unshared attributes of analog-target pairs. T1 explicitly explained similarities between input-output elements of the machine and domain-range of a function and their processes. Further, T1 pointed out that a function would not use any element not included in the domain set; however, the machine would use anything such as a potato mistakenly mixed in tomatoes.

Pass 10: Coding Extension of Mapping

Previous frameworks discussed extended analogies under the category of the level of enrichment. In these frameworks, (1) if several analogs were used to describe a target, or (2) several attributes of an analog were used to describe a target, the analogy was considered extended. However, in this study, it was first advocated that enrichment and extension were different notions. Second, an extension was interpreted differently in this study. Since the focus of this study was on analogies describing function-related target concepts, there were many analogs used to describe the same target concept as expected. Therefore, this condition was usual for this study and was not considered an extension. Also, using several attributes of analog to describe a target was considered a kind of enrichment, not an extension.

In this study, derivation and repetition of analogies were accepted as an extension. Based on these reasons, a new separate category of extension of mapping was developed in the present study. Then each analogy was classified into one of two descriptors: (1) extended and (2) unextended. Besides, the extended ones were further analyzed in terms of the way of extension and categorized as either one of the four descriptors: (1) base, (2) derived, (3) base-repeated, or (4) derived-repeated analogies.

In order to code analogies extended or unextended, analog-target pairs of each analogy recorded in the analogy indexes were all examined. When it was hard to decide, the discourses in the analogy units were re-examined. Analogies, which appeared once, were not repeated or were not further derived, were all coded as unextended, such as *univalence requirement of a function is like a birthday function*. All the rest were coded as extended. The extended ones were further analyzed, and the originals of the derived or repeated analogies throughout the function unit were coded as the base. The analog and target of the base analogy (hereafter called the base) were referred to as the base analog and base target, respectively. Besides, analogies were coded as derived when one of the following was true: (1) base analog compared to a different target, or (2) a modified base analog compared to a base target or another target. Lastly, analogies, which were the repetition of the base analogy having the same analog-target pairs as the base, were coded as base-repeated. In the same manner, analogies, which were the repetition of a derived one having the same analog-target pairs with the aforementioned derived analogy, were coded as derived-repeated.

As previously mentioned, codes were assigned to each analogy, and while setting these codes, the extension of mapping was also taken into account (see Figure 3.2). For example, an analogy of *the domain properties of a function is like the input properties of a machine producing tomato sauce* (T1B3-1) was coded as extended since it was derived from T1B2 (base) and was used twice with the same analog-target pairs (see 1st condition in Figure 3.2). This extended analogy (also a repeated one) was coded as T1B3-1 and T1B3-2 in its first and second uses, respectively (as mentioned previously). As seen in this analogy, the number at the end was added to show how many times it was used.

1st Condition	Code (Same - number of appearance)	Analog (Same as base analog)	Target (Same as base target)	**REPEATED ANALOGY**
2nd Condition	Code (Different)	Analog (Same as base analog)	Target (Another target)	**DERIVED ANALOGY**
		Analog (Modified from base analog)	Target (Same as base target or another target)	**DERIVED ANALOGY**

Figure 3.2: Coding of extension of mapping under different conditions

Another example, the analogy of *function concept is like specification profits of a hotel by months* (T2C3) was coded as extended and even base, since many analogies were derived from it. For instance, the analogy of *not being a function is like the director's getting fired because of not specifying the profit of that month* (T2C4-1) was derived from this base (T2C3). Since its analog, *the director's getting fired because of not specifying the profit of that month* was modified from the base analog of T2C3, *specification profits of a hotel by months*, and it was used with another target, *not being a function* (see 2nd condition, in Figure 3.2). Moreover, T2C4-2 was coded as derived-repeated since it was a repetition of a derived analogy.

Pass 11: Coding Analog Explanation

Previous frameworks examined both analog explanation and strategy identification under the category of pre-topic (pre-target) orientation and categorized analogies into four descriptors: (1) only the analog was explained, (2) only the strategy was identified, (3) both were done, (3) none were done. However, in this study, the analog explanation was examined under a separate AFF category, and each analogy was categorized as either (1) explained or (2) unexplained.

The discourse on the analog domain was examined to determine if any explanations would reduce analog unfamiliarity. While analog explanations of analogies were coded, this study did not seek a complete definition or description of the analog domain. Instead, a simple phrase containing a few words referring to relevant attributes for analogical transfer, either any visual representation of the analog domain or a simple reminder when the analog was too familiar was sought. For example, the analog domain of the analogy, *the pre-image-image of a function is like a machine producing tomato juice from tomato* (T1B2), was coded as explained. Although the analog itself provided a clue about how this machine works, an extra explanation was sought. Here, this analogy was coded as explained because without going into detail, T1 roughly emphasized that the machine's task was to produce tomato juice from tomatoes and also represented it with a diagram. If an adequate analog explanation was made in the first of two consecutive analogies with the exact analog or analogous derived from each other, analog explanation was not sought in the second analogy and was coded as analog explained. For example, the analog of the analogy, *domain properties of a function is like the input properties of a machine producing tomato sauce* (T1B3-1), was coded as explained since information about its analog was given in the previous analogy (T1B2).

Pass 12: Coding Strategy Identification

Previous frameworks evaluated strategy identification under pre-topic orientation with the analog explanation. However, in this study, analogies were assessed under a new and separate AFF category whether they contained any strategy identification and were categorized as either (1) identified or (2) unidentified. Besides, further analysis of identified ones brought forth two more descriptors: (1) overtly identified and (2) tacitly identified.

The discourse in each analogy unit was searched for any word or phrase indicating analogy generation was about to occur or was occurring. First, analogies that did not identify the cognitive strategy overtly or tacitly were coded as unidentified. Next, analogies containing any statement that overtly identifies the applied strategy as "analogy" were coded as overtly identified. On the other side, analogies that tacitly introduced the used strategy with words or phrases other than the word "analogy" were coded as tacitly identified. The tricky words such as "for example, likes, in daily life, or story" were underlined in all analogy units to create a list of possible analogy indicators. For example, the analogy of *pre-image-image of a function is like buying and selling price of oil products in a factory* (T2D16) was coded as overtly identified in terms of its strategy identification. Since T2 generated this analogy immediately after stating that analogies were daily life examples and helped understand function concepts. Another example was that the analogy of the *univalence requirement of function is like a machine producing tomato juice* (T1B23-1) coded as tacitly identified. T1 generated this analogy, stating that reconsidering the tomato machine example would help the transition to the new concept. Here, the analogy strategy was identified tacitly since the word "example" was used rather than announcing the word "analogy" directly.

Pass 13: Coding Presence of Analogical Limitations

The previous frameworks considered a stated limitation as an example of enrichment. However, one of the frameworks added a separate category of limitations to record general statements on analogy use or specific statements stressing some unshared attributes and categorized it into two descriptors: (1) none and (2) specific. Since general comments of the limitation of analogy use were not detected in the study, the third descriptor of “general” was not included in the category.

On the other hand, in the present study, a general limitation warning of the problems of analogy use was considered the detail of enrichment. In contrast, a specific limitation underlining unshared attributes was sought as a condition for complete enrichment. Furthermore, a stated limitation was analyzed under a separate AFF category of the presence of analogical limitations and categorized within two descriptors: (1) present and (2) not present. Besides, the present limitation was further classified as (1) general or (2) specific.

Discourse in each analogy unit was searched for any statements that included: (1) general limitations of analogy use or analogy generation process and (2) specific limitations related to unshared attributes of analog-target pairs. If the analogies did not include any warnings, they were coded as not present in terms of limitations. Similarly, if they contained any, they were coded as present in terms of limitations. Besides, the stated limitations were further coded as general or specific as specified in points (1) and (2), respectively. For example, in the analogy of *domain properties of a function is like input properties of a machine producing tomato sauce* (T1B3-1), it was emphasized that the machine could only make tomato juice and could not use anything instead of tomatoes. However, the function did not have to use a single object (the arbitrary nature of the function). This analogy was coded specific in terms of its stated limitations, as it emphasized the breakdown point of the tomato machine. In another instance, in the analogy of *function concept is like a function machine* *converting each input to its two times and one more* (T1B21-2), it was emphasized that analogies are delicate to changes. Since a general limitation of analogies was highlighted in this analogy, it was coded as general.

Pass 14: Coding Soundness of Mapping

Unlike previous frameworks, in this study, an entirely new category of the soundness of mapping was developed, and each analogy was classified as either sound or unsound. The category of the soundness of mapping was inspired by the content validity that was initially presented by Bayazit and Ubuz (2008). In this study, the content validity was concerned with whether the analog had epistemological power to represent a function-related target concept and how the knowledge was transferred from analog to target. However, in the present study, it was detected that although the analog domain of some analogies had the intrinsic power to represent the essence of the target concept, the wrong attribute of analog was mapped to the target concept. Consequently, the content validity did not meet the correct identification of analog and target attributes. Since a sound argument is logically valid and based on the true premises (V. A. Thompson, 1996), inspired by the sound argument, it was thought that analogies could also be evaluated in terms of the soundness of mapping in the present study.

The soundness of mapping was evaluated with the presence of two critical aspects. The first one was concerned with correctly identifying relevant analog and target attributes in that correctly identified analog attributes should intrinsically incite the identified function-related target attributes. The second was concerned with the epistemological validity of mappings between previously identified analog and target attributes, which should contain the idea of function (arbitrariness and univalence conditions). Analogies had both mentioned aspects were coded as sound, and the rest were coded as unsound.

Analog-target pairs in each analogy unit were examined to code analogies as sound or unsound. For instance, the analogy of *not being a function is like getting the director fired for specifying two different profits for a month* (T2C6-1) was coded as sound. Since this analogy entailed the idea of *relation* and the analog (getting fired because of specifying two different profits for a month) and the target (not being a function) had addressed this idea. In addition, it was not a function because it wanted to match each domain element to only one range element, but it did not. Similarly, the director was fired because he was asked to specify a profit per month, but he specified two different profits for a month. In another instance, the analogy of *domain-range relation of the function is like that in the mother machine* (T1B22) was coded as unsound. Although this analogy entailed the idea of relation, and the analog domain (mother machine) intrinsically represented this idea, the word "machine" addressed the idea of the *process*. Thus, the analogy was coded as unsound since the analog domain could not be identified correctly.

After each pass, one of the aims of this study was to examine the features of the teacher- and student-generated analogies, frequency counts of these analogies that meet each outlined feature in the AFF were conducted, and their percentages were calculated. In order to interpret how the analogy features differed from teacher to teacher, frequencies, and percentages of T1- and T2-generated analogies were calculated individually. In a similar vein, to interpret how analogies differed from class to class, frequencies, and percentages of both T1- and T2-generated and student-generated analogies in each of the T1 and T2 classes were calculated separately.

The last phase of data analysis and interpretation contained generating empirical assertions separately for teacher- and student-generated analogies. After each pass, the entire data corpus (frequencies and percentages of observed data, transcriptions, videotapes, field diary including field notes and notes regarding informal conversations, and all teaching materials) was reviewed and interpreted according to the focus of that pass. The patterns that emerged within the data set were then organized under an assertion for each pass. Then, the data corpus was reviewed repeatedly to test the validity and reliability of these assertions by triangulation using confirming evidence from different sources of data corpus. For instance, in testing the assertion, the teachers predominantly used analogies at the beginning of the function unit. At first, frequencies and percentages of teacher-generated analogies in all three chapters were examined and compared. Traces of evidence for this assertion were sought in field notes and informal conversations. Therefore, the field diary was reviewed to find any comments made by teachers during informal conversations during the data collection phase. For each assertion confirming evidence and disconfirming evidence, where possible, were sought. Assertions given in the next chapter arrived through repeated cycles of testing and refining findings.

CHAPTER 4

RESULTS

Based on the Analogy Features Framework (AFF) explained in the data analysis and interpretation part, analogy use in mathematics classrooms is reported under two sections of assertions. While assertions in the first section are related to features of analogies used by the two teachers, T1 and T2, those in the second section are about the features of analogies generated by students. Under each assertion, confirming evidence and, whenever possible, disconfirming evidence are provided. A total of 91 (45 for T1 and 46 for T2) lessons were observed to determine the presence of analogies. The frequencies of teacher- and student-generated analogies are shown in Table 4.1 (169 from both T1 and T2 and 46 from the students).

Table 4.1 Numbers of observed lessons and analogy-occurrences in each class

Teacher	Class	Number of observed Lessons	**Total**	Number of Teacher - Generated Analogies	Number of Student - Generated Analogies	**Total Number of Analogies**
Teacher 1	9A	22		27 (82%)	6 (18%)	**33 (100%)**
	9B	23	**45**	58 (71%)	24 (29%)	**82 (100%)**
Teacher 2	9C	15		28 (88%)	4 (12%)	**32 (100%)**
	9D	15	**46**	27 (90%)	3 (10%)	**30 (100%)**
	9E	16		29 (76%)	9 (24%)	**38 (100%)**
Total			**91**	**169 (79%)**	**46 (21%)**	**215 (100%)**

The results of the AFF categorization of teacher- and student-generated analogies for each class are reported in Appendix G.

4.1 Analogies Generated by Teachers

Assertion 1: The Majority of Analogies Constructed during the Function Unit Teaching and Learning were Generated by Teachers

The number of teacher-generated analogies in each lesson ranged from 0 to 15. The total number of teacher-generated analogies in each class is presented in Table 4.1, together with the total number of observed lessons. Of the 215 analogies found in this study, the overwhelming majority (169, 79%) were teacher-generated analogies. While T2 generated almost the same number of analogies between 27-29 in each class, T1 generated a very different number of analogies, 27 and 58.

Data disclosed that T1 employed 58 analogies in Class B; however, the same teacher used only 27 analogies in Class A compared with Class B. In an informal talk, T1 mentioned that she enjoys teaching in Class B more since students in Class B are more interested in math and better at math than students in Class A. Besides, at the beginning of the function unit, T1 had four extra lessons in Class B compared to those in Class A (see Appendix C). When the clues are put together, her employment of much more analogies in Class B seems to be related to her not experiencing stress about catching up with ordinary curricular sequences and enjoying teaching mathematics to students in this class. However, it is tough to comment that T1 generated more analogies because students in Class B are better in mathematics. After all, no difference was observed in this sense. On the other hand, as observed throughout her teaching in Class A, T1 admitted in another informal interview that she had difficulty maintaining discipline in Class A. Therefore, since analogies may require a more interactive classroom environment, her use of fewer analogies in Class A might stem from her having classroom management problems in this class.

There was evidence from the classroom observations that the vast majority of teacher analogies (151 out of 169) were generated as part of teachers' routine teaching (see Table 4.2). It was observed that the teachers used these analogies to introduce new content and to reinforce, review or summarize what was taught. Similarly, they may have used these analogies when their students had difficulty in understanding functions. Because they have years of experience teaching the same unit to 9th graders, they can recognize when extra explanations or different representations are needed. Alternatively, teachers may also have used analogies when they had difficulty explaining some aspects of functions.

Table: 4.2: Occasions that teachers generated their analogies

Teacher	Class	# Teacher-generated analogies employed as part of their routine teaching	# Teacher-generated analogies employed following students' demonstration of a lack of understanding	# Teacher-generated Analogies
T1	A	24 (89%)	3 (11%)	**27 (100%)**
	B	48 (83%)	10 (17%)	**58 (100%)**
T2	C	26 (93%)	2 (7%)	**28 (100%)**
	D	25 (93%)	2 (7%)	**27 (100%)**
	E	28 (97%)	1 (3%)	**29 (100%)**
Total		**151 (89%)**	**18 (11%)**	**169 (100%)**

On the other hand, the remaining analogies (only 18 out of 169) were generated when students audibly requested further explanation or repetition of the concepts being taught, answered teachers' questions with blatantly incorrect or inappropriate answers, answered truthfully but quietly and timidly, or remained utterly silent with no response. On these occasions, teachers most likely used analogies as alternative representations to make it easier for students to understand. For example, after a student (S1E) verbally declared that she did not understand the content, T2 generated the analogy T2E2-2, *specification profits of a hotel by month*, to explain the function concept.

In addition, a close examination of Table 4.2 and transcriptions of classroom observations revealed that analogies were constructed in almost the same situations in each classroom, consistent with the nature of the use of analogies.

Table 4.3: Frequency distribution of teacher-generated analogies in chapters

		Teacher1			Teacher 2				
Chapters	Sections	Class A 27 (100%)	Class B 58 (100%)	**T1 85 (100%)**	Class C 28 (100%)	Class D 27 (100%)	Class E 29 (100%)	**T2 84 (100%)**	**Total 169 (100%)**
Chap.1	1.1 Introduction (definition)	11 (41%)	27 (46%)	**38 (45%)**	18 (64%)	19 (70%)	17 (58%)	**54 (64%)**	**92 (55%)**
	1.2 Functional notation						2 (6%)	**2 (2%)**	**2 (1%)**
	1.3 Functions defined by equations	2 (7%)	1 (2%)	**3 (4%)**			1 (4%)	**1 (1%)**	**4 (2%)**
	1.4 Testing for functions	3 (11%)		**3 (4%)**		2 (8%)	4 (14%)	**6 (7%)**	**9 (6%)**
	1.5 Evaluating a function	2 (7%)	10 (17%)	**12 (14%)**	4 (14%)	3 (11%)		**7 (8%)**	**19 (11%)**

Table 4.3 (continued)

		Teacher1			Teacher 2				
		Class A 27 (100%)	Class B 58 (100%)	**T1 85 (100%)**	Class C 28 (100%)	Class D 27 (100%)	Class E 29 (100%)	**T2 84 (100%)**	**Total 169 (100%)**
Chapters	Sections								
Chap. 1	1.6 Finding domain of a function	1 (4%)		**1 (1%)**		3 (11%)		**3 (4%)**	**4 (2%)**
	1.7 Polynomial function		1 (2%)	**1 (1%)**	1 (4%)			**1 (1%)**	**2 (1%)**
	1.8 Rational function	1 (4%)	4 (7%)	**5 (6%)**	3 (11%)			**3 (4%)**	**8 (5%)**
	1.9 Irrational function		1 (2%)	**1 (1%)**			1 (4%)	**1 (1%)**	**2 (1%)**
	TOTAL	**20 (74%)**	**44 (76%)**	**64 (76%)**	**26 (93%)**	**27 (100%)**	**25 (86%)**	**78 (92%)**	**142 (84%)**
Chap. 2	2.1 Graphs of function	1 (4%)		**1 (1%)**			3 (10%)	**3 (4%)**	**4 (2%)**
	TOTAL	**1 (4%)**		**1 (1%)**			**3 (10%)**	**3 (4%)**	**4 (2%)**
Chap. 3	3.1 Constant function		1 (2%)	**1 (1%)**	2 (7%)			**2 (3%)**	**3 (2%)**
	3.2 Identity function	2 (7%)	4 (7%)	**6 (7%)**					**6 (4%)**
	3.3 Linear function		1 (2%)	**1 (1%)**					**1 (1%)**
	3.4 Piecewise function and its graph	4 (15%)	6 (10%)	**10 (12%)**			1 (4%)	**1 (1%)**	**11 (6%)**

Table 4.3 (continued)

		Teacher1			Teacher 2				
Chapters	Sections	Class A 27 (100%)	Class B 58 (100%)	**T1 85 (100%)**	Class C 28 (100%)	Class D 27 (100%)	Class E 29 (100%)	**T2 84 (100%)**	**Total 169 (100%)**
	3.5 The absolute value function								
	3.6 Polynomial function								
	3.7 One to one function								
	3.8 Onto function		2 (3%)	**2 (2%)**					**2 (1%)**
	TOTAL	**6 (22%)**	**14 (24%)**	**20 (23%)**	**2 (7%)**		**1 (4%)**	**3 (4%)**	**23 (14%)**

Assertion 2: The Teachers Predominantly Used Analogies at the Beginning of the Function Unit

Function Unit was covered under 3 chapters with 18 sections - 9 sections in the first chapter, 1 section in the second chapter, and 8 sections in the third chapter (see Table 4.3). An examination of data given in Table 4.3 revealed that an overwhelming majority of teacher-generated analogies were seen in Chapter 1, far less in Chapter 3, and almost none in Chapter 2.

Further analysis of Table 4.3 revealed at least three out of four teacher-generated analogies in each class, and even all T2-employed analogies in Class D were used in the first chapter. Besides, an investigation of teacher-generated analogy dispersion through Chapter 1 disclosed that a significant amount of the analogies (92, 55%)

belonged to Section 1.1, Introduction (Definition), very few (19, 11%) pertained to Section 1.5, evaluating a function, and the rest belonged to varied sections. In addition, this review revealed that although at least one teacher-generated analogy existed for each section, they were primarily encountered in the early stages of the first chapter and rarely seen towards the end.

These expected results appear to be related to the nature of both function concept and analogy use. As in other mathematical concepts, the definition of function (arbitrariness and univalence requirements) and its essential terms (domain, co-domain, range, dependent and independent variables) are introduced in the early stages of the function unit. Therefore, conceptual understanding, a fundamental principle for learning functions with understanding, can only be established from the beginning of the unit. On the other hand, analogies, which have the power to transfer old knowledge to new knowledge, have also the potential to strengthen students' motivation and attention towards learning functions by helping them think that functions are not isolated concepts. When all these features come together, analogies are expected to aid further conceptual understanding at the very beginning of the function unit. Beyond the nature of function concept and analogy use, the frequent use of analogies at the very beginning of the unit may be a pedagogical tradition that teachers learned from the textbooks, mathematics curriculum, or modelled from their instructors at university or even their high school mathematics teachers.

Moreover, Table 4.3 displayed that both T1 and T2 used little or no analogies when teaching Chapter 2 (Graphs of Functions). T1 and T2 used just 1 and 3 analogies in Classes A and E, respectively, concerning graphs of functions. It may be because teachers do not need to use analogies when they use graphs to represent functions.

Finally, Table 4.3 revealed that teachers used only 23 out of 169 analogies when describing Chapter 3 (Types of Functions). Besides, a detailed examination of the data showed that the teachers did not prefer to use analogies, especially for defining some function types, including absolute value, polynomial, and one-to-one functions. In fact, there is no function type for which both teachers constructed analogies while teaching in all their classrooms. In addition, T1 and T2 exhibited notable differences in their use of analogies in Chapter 3; that is, they preferred to use analogies for different function types. To be more precise, T1 employed analogies to explain function types such as constant, identity, linear, piecewise, and onto functions in all her classes. However, T2 used only 2 analogies to explain constant functions in Class C and only 1 analogy for piecewise functions in Class E.

In brief, based on the data shown in Table 4.3, although T1 used more analogies than T2 did, both teachers generally employed an insufficient number of analogies (only a few or none) when teaching each type of function. It was possibly due to their tendency not to use much analogy when teaching function types in general. Because, in an informal interview, T1 stressed that some function types are just basic mathematical operations, such as piecewise and linear functions, so it is nonsense to allocate time to teach them under a separate chapter. On the other hand, in another informal interview, T2 stated that he spent plenty of time explaining Chapter 1 with analogies. Then, in other interviews, he repeated several times that he could probably never finish the units on time if he taught all the subjects by generating analogies.

Assertion 3: There was a Tendency for Teachers to Extract Analogies that Utilized an Analog from Their Own Experiences and/or Knowledge Bases

Results revealed that analogs (familiar knowledge) of teacher-generated analogies were invoked from three main categories: (1) textbook, (2) teachers' own experiences and/or knowledge bases, and (3) students' experiences and/or student-relevant conditions - Table 4.4 given below shows the frequency distribution of the analog sources of teacher-generated analogies.

Table 4.4: Frequency distribution of the sources of teacher-employed analogs

	Teacher 1		Teacher 2			
Analog Sources	Class A 27 (100%)	Class B 58 (100%)	Class C 28 (100%)	Class D 27 (100%)	Class E 29 (100%)	**Total 169 (100%)**
Textbook	6 (22%)	6 (10%)	0 (0%)	0 (0%)	0 (0%)	**12 (7%)**
Teachers' own experiences and/or knowledge bases	21 (78%)	50 (86%)	25 (89%)	21 (78%)	27 (93%)	**144 (85%)**
Students' experiences and/or student-relevant conditions	0 (0%)	2 (4%)	3 (11%)	6 (22%)	2 (7%)	**13 (8%)**

Analysis of Table 4.4 displayed a general tendency for teachers to extract analogies that employed analogs from their own experiences or knowledge bases much more than the other two sources. It was probably because both teachers had a repertoire of analogies developed by their experiences or reading from function-related materials.

There was evidence that these repertoires of analogies may have been developed during both teachers' long-standing teaching experiences. For example, during informal conversations, T1 stated that the *tomato machine* analogy is an analogy that she always uses while teaching functions. Similarly, T2 was observed to emphasize to his students that the *specification profits of a hotel* analogy always works.

Furthermore, there was evidence that examining additional sources on functions in addition to the 9th-grade textbook could have enhanced their analogy repertoires. For instance, T1 demonstrated that she enhanced her repertoire of analogies by reading from her reference guides, Robert A. Adams' Calculus textbook (1994), and the secondary school mathematics curriculum for grades 9-12 published by the Turkish

Head Council of Education and Morality in 2003. Her analogy repertoire included the *tomato machine* analogy inspired by the "function machine" analogy discussed in these two sources (see Figure 4.1), as she clearly stated in an informal conversation.

Source	Explanation	Related part
Calculus textbook of R.A. Adams (1994)	There was an objective specifically advising ninth-grade teachers to use the "function machine" analogy in a section on the function unit.	Explain function simulating a machine producing output values (f(x)) by a certain rule from some input values (x). Within this framework, make students to find values of f(1), f(2), f(a), f(2x), f(x+1) etc. by using or table of f(x), for a given x value. Make use of examples then clarify identity function, constant function and linear function (p.7)
Secondary school mathematics curriculum (2003)	Function term was defined with the help of the "function machine" analogy.	A function f on a set D into a set S is a rule that assigns a unique element f(x) in S to each element x in D. In this definition D = D(f) (read "D of f") is the domain of the function f. The range R(f) of f is the subset of S consisting of all values f(x) of the function. Think of a function f as a kind of machine that produces an output value f(x) in its range whenever we feed it an input value x from its domain (p.26)

Figure 4.1. Function machine analogy used by two sources

Although analogs were selected chiefly from teachers' own experiences and/or knowledge bases, other sources were also invoked to some extent for analogs. Table 4.5 summarizes textbook analogies, displaying corresponding page numbers in the book in parenthesis, showing by whom (T1, T2, and both or neither), when (in which unit and section), and where (in which classes) they were employed.

Table 4.5: Summary and occurrence of textbook analogies

Chapter	Section	Textbook Analogies	Teacher(s) Used the Analogy (T1/T2/T1+T2/None)
Chapter 1 Function Concept	1.1 Introduction (definition)	Function machine (p.1)	T1 (A+B)
		Earning money per working hour (p.2)	T1 (A+B)
		Birthday function (p.2)	T1 (A+B)
		Calorie burning with a sports activity (p.3)	T1 (A+B)
		Electrical energy use in a house (p.3)	T1 (A+B)
		Price change of a product between the years of 1940 and 1990 decennially (p.3)	None
	1.5 Evaluating a function	Television programs and their audience ratings (p.13)	None
		The length of a plant by time (p.13)	None
		Computation the profit of ticket-sales (p.14)	None
Chapter 3: Types of functions	3.4 Piecewise function and its graph	Fine system in the case of exceeding the speed limit (p.27)	T1 (A+B)

Table 4.5 shows that the textbook contained 10 analogies within the contents covered in the function unit. Analysis of the table revealed that 6 analogies were employed only by T1, the author of the textbook, and any teacher did not use the rest. From

classroom observations, it was detected that the analogies that T1 did not use were in the parts left by the students to work on their own. It was most likely due to T1 thought that the students could understand the remaining parts, including analogies, on their own. It was also possible that she was not even aware that there were analogies in those parts. It should also be no coincidence that she did this in Section 1.5, where she used less analogy (see Assertion 1). On the other side, interestingly, T2 did not use any analogy from the textbook, although he mentioned that the book was well designed and easy-to-use in an informal conversation. It was probably due to either preferring to use his analogies or not paying attention to ones used in the book.

Furthermore, there were examples where both teachers employed analogs that were directly relevant to the students. For instance, T2 referred to obtaining drinking water from a dam near their school and weather conditions in Istanbul to teach functions.

On a few occasions, both teachers appreciated and directly employed some analogies, which their students previously generated. For example, T1 referred to an analog, *every dog barks in his own yard*, used by S11A while teaching piecewise functions in Class B. In another example, in Class D, T2 also referred to an analog, *comparison of two different annual wage formulations of a footballer, employed* by a group of students in Class E. Although in this study, the positive effects of teachers choosing analogs from student-relevant conditions were not explicitly observed, it may be helpful to attract students' attention and involve them in the analogy construction.

Assertion 4: Differences Surfaced in the Topics that Teachers Selected for the Analog Domain of Their Analogies

Teachers' selected topics for the analog domain of their analogies were determined to exist within math-related or outside-math topics. Later, math-related topics were classified as (1) those involving mathematical computing in daily life or (2) those involving mathematical calculation with numbers only. On the other hand, outside-math topics were classified as one of three categories: (1) those related to culinary, (2) those related to family ties/situations, and (3) those were idioms and proverbs. Table 4.6 shows the categories of teacher-selected topics and the frequencies of each category.

Analysis of Table 4.6 revealed that teachers selected most of the analog domain of their analogies (125, 74%) from math-related topics, particularly topics involving mathematical computing in daily life (92, 54%). In addition, findings disclosed that while T1 chose her analogies from all topics, T2 selected only from math-related ones, especially mathematical computing in daily life. Also, interestingly, while T1 chose about half of her analogies from outside-mathematics topics, T2 selected none of his analogies (see Table 4.6).

Table 4.6: Categories and frequency distribution of teacher-selected topics

		Teacher 1		Teacher 2			
		Class A	Class B	Class C	Class D	Class E	**Total**
Topics		27 (100%)	58 (100%)	28 (100%)	27 (100%)	29 (100%)	**169 (100%)**
Math-related topics	Topics involving mathematical computing in daily life	8 (31%)	10 (17%)	26 (93%)	21 (78%)	27 (93%)	**92 (54%)**
	Topics involving mathematical calculation with numbers only	6 (22%)	17 (30%)	2 (7%)	6 (22%)	2 (7%)	**33 (20%)**
Total		**14 (53%)**	**27 (47%)**	**28 (100%)**	**27 (100%)**	**29 (100%)**	**125 (74%)**

Table 4.6 (continued)

Topics		Teacher 1 Class A 27 (100%)	Class B 58 (100%)	Teacher 2 Class C 28 (100%)	Class D 27 (100%)	Class E 29 (100%)	**Total** **169** **(100%)**
Outside-math topics	Topics related to culinary	9 (33%)	18 (31%)				**27** **(16%)**
	Topics related to family ties/family situations	2 (7%)	6 (10%)				**8** **(5%)**
	Topics that were idioms or proverbs	2 (7%)	7 (12%)				**9** **(5%)**
Total		**13** **(47%)**	**31** **(53%)**	**0**	**0**	**0**	**44** **(26%)**

Reviews of selected math-related and outside-math topics disclosed there was little overlap between teachers' topic selections. Close examination on selected math-related topics revealed that T1 selected topics that involve mathematical computing in daily life, such as *earning money, calorie-burning, electrical energy use, traffic infringement fines, and parking and highway payments*, *distance-time relation of a constant-speed car, linear movement*. On the other hand, T2 selected topics that involve mathematical computing in daily life, such as *profit value of a hotel, oil price, water and electricity invoices, cost-profit analysis of goods, interpretation of interest, increase in dollar value, exchange rates forecast*. Furthermore, teachers also employed analogies involving mathematical calculation with numbers only. For instance, both used the analogy *functions like a machine converting each input into an output that is the square of the input* by displaying a machine (picture) and using just numbers and formulas (see Appendix F).

Moreover, close examination on T1-selected outside-math topics revealed that she chose topics related to culinary such as *a machine producing tomato juice, sauce or puree, chopped or mashed tomato, or tomato products from tomato.* Moreover, she selected topics related to family ties/family situations, such as *mother-child relations.* Apart from these, she chose idioms, such as *wearing one's heart on one's sleeve* or a proverb like *every dog barks in his own yard*.

Further examination of the data corpus (classroom observations and informal conversations) revealed similarities between teachers' topic choices and their interests and daily routines. For example, it was known that T1 enjoyed spending time in the kitchen since she brought to school whatever she cooked. It was also known from informal conversations that she paid fines for traffic infringements several times during the study was progressing. Besides, she graduated from the department of Mathematics-Physics Education and was giving IP Calculus lessons, as mentioned before. On the other hand, it was known from informal conversations, T2 took on paying home loans, credit card debt, and invoices. All these suggested that how teachers spend their time outside of school might influence their topic preferences to construct their analogies.

Assertion 5: A Significant Amount of the Teacher-Generated Analogies Were Related to Definition and Properties of Function

Content of the target concepts was classified according to 18 sections listed in the function unit (see Appendix E). Results revealed that most of the teacher-generated analogies (146, 86%) were related to the definition of function and its properties, especially the univalence requirement, pre-image/image, domain-range conceptions, and the properties of the domain (see Table 4.7). Findings also disclosed that T1 generated analogies related to constant, identity, linear, piecewise, and onto functions. On the other side, T2 generated analogies related to constant and piecewise functions. However, as Table 4.7 illustrated, none of them used analogies related to graphs of functions (Chapter 2).

Table 4.7: Frequency distribution of the content of the target domain

		Teacher 1			Teacher 2				
		Class A	Class B	**T1**	Class C	Class D	Class E	**T2**	**Total**
Content	Target	27	58	**85 (100%)**	28	27	29	**84 (100%)**	**169 (100%)**
1.1. Introduction (Definition)	Pre-image /image	1	8	**9 (11%)**	3	8	3	**14 (16%)**	**23 (14%)**
	Properties of domain	2	4	**6 (7%)**	2	2	5	**9 (11%)**	**15 (9%)**
	Domain-range	3	5	**8 (9%)**	1	1	4	**6 (7%)**	**14 (8%)**
	Univalence requirement	5	7	**13 (16%)**	15	10	7	**32 (38%)**	**45 (26%)**
	Dependent-independent variable	2	5	**7 (8%)**			4	**4 (5%)**	**11 (6%)**
	Independent variable					1		**1 (1%)**	**1 (1%)**
	Function concept	7	11	**17 (20%)**	3	2	4	**9 (11%)**	**26 (15%)**
	Input "x"		1	**1 (1%)**					**1 (1%)**
	Domain of the function		1	**1 (1%)**					**1 (1%)**
	Input-function relation		1	**1 (1%)**					**1 (1%)**
	Function with two variables	1	1	**2 (2%)**	2	3	1	**6 (7%)**	**8 (4%)**
Total		**21**	**44**	**65 (76%)**	**26**	**27**	**28**	**81 (96%)**	**146 (86%)**

Table 4.7 (continued)

		Teacher 1			Teacher 2				
		Class A	Class B	**T1**	Class C	Class D	Class E	**T2**	**Total**
		27	58	**85**	28	27	29	**84**	**169**
Content	Target			**(100%)**				**(100%)**	**(100%)**
3.1 Constant Function	Constant function		1	**1 (1%)**	2			**2 (2%)**	**3 (2%)**
3.2 Identity function	Identity function	2	4	**6 (7%)**					**6 (3%)**
3.3 Linear function	Slope of a linear function		1	**1 (1%)**					**1 (1%)**
3.4 Piecewise function	Piecewise function	4	6	**10 (12%)**			1	**1 (1%)**	**11 (7%)**
3.8 Onto Function	Onto function		2	**2 (2%)**					**2 (1%)**
Total		**6**	**14**	**20 (24%)**	**2**	**0**	**1**	**3 (4%)**	**23 (14%)**

Further analyses indicated that some of the teacher-generated analogies regarding the definition and properties of the function were used when describing these contents. However, some were used when describing other contents in subsequent sections (see Appendix F). The use of these analogies at very different times and places throughout the function unit may have resulted from the difficulties in understanding the definition and properties of functions. In addition, the analysis showed that the remaining analogies were used when describing the content to which they related (see Tables 4.3 and 4.7).

Table 4.8: Categorization of teacher-generated analogies for each teacher

Category	Subcategory **Total**	Teacher 1 **85 (100%)**	Teacher 2 **84 (100%)**	**Total 169 (100 %)**
Nature of Shared Attributes	Functional (F)	78 (92%)	57 (68%)	**135 (80%)**
	Structural and Functional (S&F)	7 (8%)	27 (32%)	**34 (20%)**
Presentation Format	Verbal (Vb)	66 (78%)	75 (89%)	**141 (83%)**
	Visual and Verbal (Vs &Vb)	19 (22%)	9 (11%)	**28 (17%)**
Level of Abstraction	Abstract-abstract (A-A)	6 (7%)	0 (0%)	**6 (4%)**
	Concrete– abstract (C-A)	79 (93%)	84 (100%)	**163 (96%)**
Position of Analog Relative to Target	Advance Organizer (AO)	20 (23%)	9 (11%)	**29 (17%)**
	Advance Organizer 1 (AO^1)	7 (8%)	0 (0%)	**7 (4%)**
	Advance Organizer 2 (AO^2)	13 (15%)	9 (11%)	**22 (13%)**
	Embedded Activator (EA)	62 (73%)	75 (89%)	**137 (81%)**
	Post-Synthesizer (PS)	3 (4%)	0 (0%)	**3 (2%)**
Level of Enrichment	Simple (S)	19 (23%)	12 (14%)	**31 (18%)**
	Partially Enriched (P-En)	64 (75%)	72 (86%)	**136 (81%)**
	Completely Enriched (C-En)	2 (2%)	0 (0%)	**2 (1%)**
Extension of Mapping	Extended (Ex)	65 (76%)	72 (86%)	**137 (81%)**
	Unextended (UnEx)	20 (24%)	12 (14%)	**32 (19%)**
Analog Explanation	Explained (Exp)	53 (62%)	66 (79%)	**119 (70%)**
	Unexplained (UnExp)	32 (38%)	18 (21%)	**50 (30%)**
Strategy Identification	Identified (I)	40 (47%)	44 (52%)	**84 (50%)**
	Overtly Identified (OI)	18 (11%)	14(8%)	**32 (19 %)**
	Tacitly Identified (TI)	22(13%)	30(18%)	**52 (31%)**
	Unidentified (UnI)	45 (53%)	40 (48%)	**85 (50%)**
Presence of Analogical Limitations	Present (P)	3 (4%)	6 (7%)	**9 (5%)**
	Not Present (NP)	82 (96%)	78 (93%)	**160 (95%)**
Soundness of Mapping	Sound (S)	83 (98%)	80 (95%)	**163 (96%)**
	Unsound (UnS)	2 (2%)	4 (5%)	**6 (4%)**

Assertion 6: The Vast Majority of Teacher-Generated Analog-Target Pairs Share Similar Function

The analogical relationship between the analog-target pairs of each teacher-generated analogy was coded as either functional or structural-functional. Ultimately, since they were all functional analogies, all analogical relationships were coded as either (1) a process or (2) a relation to determine how they addressed the function.

Analyses of the teacher-generated analogies revealed that the vast majority of analogical relationships were functional (135, 80%) while; far fewer (34, 20%) were both structural and functional (see Table 4.8). Since the nature of the function is compatible with functional analogies, this result is pretty expected. In addition, a similar result was also accurate for both T1- and T2- employed analogies, when they were evaluated independently. Table 4.8 indicated that, except for a few, nearly all T1 analogies and about two-thirds of T2 analogies were only functional due to the behavior or performance of analogs shared by the function-related target concepts. Overall, compared to T1 analogies, there were slightly higher percentages of structural-functional analogies generated by T2 (see Table 4.8).

Further analysis of analogical relations of functions showed that teachers mainly did not explain structural similarities between analog and target concepts (for example, see the analogy of *identity function is like wearing one's heart on one's sleeve*). On the other hand, the analysis indicated that teachers mostly verbally explained functional similarities between analog or target concepts of analogies they used for the first time at the beginning of the unit. Unlike T1, T2 summarized these similarities on the board while verbalizing them for only the hotel analogy (see Figure). However, he only noted the features of the target domain, not the ones belonging to the analog domain, and preferred to express them verbally. Moreover, analysis revealed that both teachers generally explained the similarities between the analog-target pairs in very crowded and scattered sentences instead of stating them directly and clearly.

Table 4.9: Frequency of teacher-generated analogies offered to describe the function as a process or as a relation

	Teacher 1		Teacher 2	
	85		84	
Function	Functional	Structural-Functional	Functional	Structural-Functional
Total	78	7	**57**	27
As a process	**69**	1	**16**	6
As a relation	**9**	6	**41**	21

Although both teachers defined function as a relation, a comparison of T1- and T2-generated functional analogies revealed that most T1- generated ones (69 out of 78) addressed the function as a process (see Table 4.9). In contrast, most T2-generated ones (41 out of 57) addressed the function as a relation. Lastly, a comparison of each structural-functional type of teacher-generated analogies disclosed that nearly all managed the function as a relation (see Table 4.9). This may be because defining the analogical relationship as a process is more concrete or inartificial than defining it as a relation.

Approximately dozens of teacher-generated analogies were detected in the current study. Since all analog-target pairs of teacher-generated analogies shared one of the functional or structural and functional relationships, and they addressed the function as a process or a relation, a reduced sample of eight analogies was presented to illustrate further how teachers might have constructed these relationships. The reduced sample containing each teacher's four most common analogies represented all possible combinations between analogical relationships (functional or structural-functional) and related functional attributes (process or relation). In Table 4.10, all eight teacher-generated analogies were assessed and summarized according to mappable elements of analog and targets, their analogical relationships (functional or structural-functional), and the sort of function attribute (a process or a relation) that they addressed.

Table 4.10: Examples of teacher-generated analogies and analysis of their mappings

Teacher	Analogies	Mappable elements	Analogical relationship	Function attribute
T1	Tomato machine analogy (T1A1-1)	The tomato machine takes tomatoes and produces tomato juice. The function takes input x then transforms to output f(x).	Functional	Process
	Mother- child relation analogy (T1B25-1)	The relation from children to their mothers is similar to the relation from domain to the range of function.	Functional	Relation
	"Wearing one's heart on one's sleeve" analogy (T1B56-1)	Being emotionally transparent is behaving as what thought or feeling. Identity function takes x then converts to f(x)=x.	Structural-Functional	Process
	"Every dog barks in his own yard" analogy (T1B63)	A person is brave when s/he in her/his own protected area. The rule of piecewise function is a selection of appropriate rules on the sub-domains.	Structural-Functional	Relation

Table 4.10 (continued)

Teacher	Analogies	Mappable elements	Analogical relationship	Function attribute
T2	Calculation of invoice amount analogy (T2E12)	Monthly water bills are computed by considering consumers' m^3 of water usage with their fixed fee. Calculation of bills is similar to the process of function.	Functional	Process
	Selecting best car brand analogy (T2C14)	Matching different branded cars with their gas consumptions is similar to matching domain and range elements of a function.	Functional	Relation
	Boss-director relation analogy (getting fired) (T2C4-1)	Boss wants from director to offer exact profits for particular time intervals; otherwise, the director gets fired. To be a function, every input must have an output otherwise will not be a function.	Structural-Functional	Relation
	Constructing a car analogy (T2C22)	While constructing a car, different materials are needed, which is similar to multivariate functions.	Structural-Functional	Process

Observation data verified that T1 generated the *tomato machine* analogy to show that f takes input x then transforms to output f (x), similarly the tomato machine takes tomato and produces tomato juice. The essence of this functional analogy between tomato machine (analog) and function (target) was that both have a similar process that put to use the input to provide output (see Table 4.10).

Moreover, T1 employed another functional analogy, *mother-child relation*, to describe the function as a relation mapping each input onto only one output, which is similar to the connection between mothers and children:

> T1: …"Mother-child relationship" is a crucial example. Think about what we talked about before! Will mothers or children be placed in the domain?
>
> Class: Mothers
>
> T1: Impossible
>
> S14B: Children!
>
> T1: S14B, you are right!
>
> S17B: If not mothers, why children?
>
> T1: No, No! Think again what we said in the tomato example; we cannot get two inputs from one input. If the case, in the domain, will you place mothers or children?
>
> Class: Children
>
> T1: Ok, every child has a mother. But a mother can have more than one child, can't it?

More precisely, if the function was the target (T), and the *mother-child relation* was the analog (A), the relational structure that the teacher wanted to show was a follows:

T1: Functions are relations from domain to range

A1: Mother-child relation is from children to mothers

T2: Every element of the domain is matched with an element of the range

A2: Every child has a mother

T3: Each element of the domain is matched with exactly one element of the range

A3: Every child has only one mother

On inspection of the transcriptions, a few metaphoric expressions (proverb or idioms) were encountered in the analogy use of T1. After further examinations, analogies containing these expressions were categorized as structural-functional. It appeared that T1 employed structural-functional type analogies to map relational structures and to map the spelling and meaning similarities of both analog and target concepts. Here is one of them:

> T1: … It is essential to know symbols. What is our identity function here? $f(x) = x$! It is purely and simply x. It is the same inside and out, isn't it? I can simplify this by saying everything goes to itself. In other words, *wearing one's heart on one's sleeve*. I know that it is not a mathematical sentence; however, it works.

To be precise, before this quote, T1 first defined identity function mathematically as $f: A \rightarrow B, \forall x \epsilon A, f(x) = x \; or \; I(x) = x$, then she rephrased it verbally as "everything goes to itself". Afterward, she extended her explanation with the analogy involving an idiom; *identity function likes wearing one's heart on one's sleeve,* probably to increase the memorability of the process of new knowledge. This analogy was intended to explain that identity function (target) returns the same value; likewise, an emotionally transparent person behaves as what s/he thought or felt, in other words, turn what s/he thought or felt into action (analog). In brief, idiom and identity function emphasized the same inside and out.

In the same manner, T1 offered another structural-functional analogy to define piecewise function:

> T1: … How did we define piecewise function? We defined different functions for different domain intervals. We gave analogy examples for this function type. This morning, one of your friends from Class A (S11A) generated an excellent analogy for piecewise function: "*Every dog barks in his own yard*". I confess it is a vulgar expression. However, it clearly describes the *piecewise function*.

In this analogy, *proverb, every dog barks in his own yard* (analog), and the piecewise function (target) shares a similar relational structure. Namely, the proverb maps every person with their own environment in which they feel valued and have an influence; similarly, the piecewise function maps every element of the domain with the right formula. This analogy was also structural since analog and target have semantic similarities both need to have a proper place to be active. It seems that T1 employed this analogy to make it easier for students to recall piecewise functions.

On examination of the records regarding T2 analogies, it appeared that some functional analogies were used to emphasize how to calculate the images in case pre-images were given, some of which are as follows:

> T2: … *"calculation of invoice amount"* will be a better analogy… Now, think about your water use in your house and the bills you have to pay for this service. So, you know that the more water consumption means, the more money. You also understand that you have to pay the government or private companies for electricity, Internet, and telephone services. It is an actual life condition, isn't it? Think that in summertime although you are not at home, maybe you are in Bodrum, the invoice keeps on coming because the government says that you should pay some amount for each month whether you use it or not, ok! How much is the fixed price here? …

Calculation of invoice amount analogy was categorized as functional analogy since both analog (calculation of invoice amount) and target (function) do the same things. When considering a function; f (x) (output) dependent on the value of x (input); likewise, considering calculation of invoice amount; the invoice amount depends on the amount of use. Briefly, this functional analogy conveyed information about process occurring in function. In the same vein, T2 generated the next functional analogy:

> T2: … In Class E, I wanted students to construct their own analogies. While they were working, I realized that they were choosing unreal values. For instance, think that "I have a car and it uses 1 liter for 1 km". Is it logical? Is

it Ferrari? No, No, it is too much, 1-liter for 1 km. Your selected numbers must be rational. When constructing an analogy, I think that if I use $1m^3$, how much will I pay at the end of the month? When deciding the best car brand, I should think about its gas consumption for 1 km

This quotation reveals *that selecting the best car brand* analogy is functional. It points out that matching different branded cars with their gas consumption is similar to matching domain and range elements of a function.

The Boss-director relation (getting fired) analogy was an extended, the most famous, and the most repeated analogy of T2. It was also a structural-functional analogy since there was an attempt to map relational structures to map meaning similarities. To be more precise, in this analogy, T2 mapped *not being a function* (target) to *getting fired from the job* (analog). As it is seen obviously, two unfortunate cases were linked as given below:

T2: … for example, assume that you are the boss and I am the director. You want me to guess monthly or daily profit, ok! Hımmm. Maybe you want yearly, Ok! For example, you requested me to think about the profit for June. Then, I couldn't guess, and I said, "I don't know". In the face of such a situation, what are you going to do? Are you going to fire me?

Class: (all students) Yes! Yes!

T2: You all are going to fire me... Imagine that you are asking me again about the profit for June. To answer your question as "it may be ten thousand or two hundred dollars", what are you going to do? Are you going to fire me?

Class: Yes!

T2 employed another structural-functional analogy to introduce multivariable functions:

If $f(x, y) = x^2 + 5y$ then, find the value of f(3,4)? (T2 wrote on the board).

T2: How many variables do I have in this set? In other words, how many unknowns are there in this function?

Class: 2

T2: 2, yes! Thank you. There are 2 unknowns. This is here, and that's here. We talked before there may be 2 unknowns in a function or a factory. Now, imagine that you are planning to construct a car. You require too many variables to build it. Think that what are auto parts? These parts may be a car steering wheel, a motor, four-car tires, etc. Do you understand?

As you can see, you may have too many variables, which is also the same for any function. For instance, (x, y), you need to construct just two materials, and x is 3; y is 4 enough for this function. Can you have 3 inputs?

Class: Yes!

T2: Of course, it may be three. For example: $\mathrm{x}, \mathrm{y}, \mathrm{z}\ldots$ you can also have too many variables.

Constructing a car analogy was generated to stress taking out raw materials, car steering wheel, motor, car tires, etc. what needed to build a car, and processing them to give out product, car. This analogy was functional since there was an attempt to map functional attributes. Raw car materials were mapped to function inputs, x and y, and the product, car, was mapped to function output, $f\ (x, y)$. However, this analogy was not merely functional; it was also structural since analog-target pairs have spelling similarities and share the same numbered materials/variables. Thus, for these reasons, this analogy was classified as a structural-functional analogy.

Assertion 7: The Teachers Tended to Use Verbal Analogies in the Lessons

Analysis of the presentation format of teacher-generated analogies is denoted in Table 4.8. Although three formats (visual, verbal, and visual-verbal) were foreseen in which analogs or analogies could be presented in mathematics classrooms, the current study detected only two formats (verbal and visual-verbal). Of the 169 analogies pointed out in Table 4.8, the overwhelming majority (141, 83%) was in a verbal format, and the remainders (28,17%) were in a combination of visual and verbal formats. This may be because, as Curtis and Reigeluth (1984) emphasized, the teachers did not need an extra visual as analogies already had visualization powers on their own.

Moreover, data suggested that neither analogs nor analogies were represented in the visual format without any verbal explanation. In fact, verbally presented analogies were reinforced by visual components used to help illustrate analogs or the relationships between analogs and targets. Although static and dynamic visuals were foreseen as potential visual components in mathematics classrooms, in the current study, it was observed that both of the teachers employed merely static visuals. To be more precise, static visuals with non-movable scenes were used; nonetheless, dynamic visuals with movable images such as movies, animations, or films were not employed. Teachers may not have used dynamic visuals, most probably because they wanted to visualize something that the students can easily imagine in their minds.

Employed static visuals were categorized in terms of their types, as mentioned in the methodology chapter. Table 4.11 provided the frequency of T1- and T2-generated visual-verbal analogies concerning types of accompanying visuals (graph, table, sketch, diagram, and arrow diagram) under specified categories (graphic, tabular, pictorial, and diagrammatic). An examination of the types of visuals revealed that diagrammatic visuals (mostly diagrams and very few arrow diagrams) were the most frequently employed ones by both teachers since they achieved 50% of the total number of visuals assessed in the current study. Diagrams are the only common visuals used by both of the teachers. The most frequently used type of visual was different for each teacher. For T1 and T2, these were diagrams and graphs, respectively. Similarly, the second most frequently utilized type of visual was table for T1, diagram for T2. Other visual types other than the ones mentioned were used very little or not at all (see Table 4.11).

Table 4.11: Frequency distribution of analogies according to employed visual types

	Categorization						
	Graphic	Tabular	Pictorial	Diagrammatic		Multiple Types	
	Graph	Table	Sketch	Diagram	Arrow diagram	Table, arrow diagram and graph	**Total**
T1	0 (0%)	4 (21%)	2 (11%)	9 (46%)	2 (11%)	2 (11%)	**19 (100%)**
T2	6 (67%)	0 (0%)	0 (0%)	3 (33%)	0 (0%)	0 (0%)	**9 (100%)**
Total	**6 (22%)**	**4 (14%)**	**2 (7%)**	**12 (43%)**	**2 (7%)**	**2 (7%)**	**28 (100%)**

Further analysis of Table 4.11 depicted that T1 used by far the most significant number of visuals (more than two times what T2 used). The gap between the number of visuals utilized by T1 and T2 could probably be because T1 used too many analogies in Class B and chose visuals from the textbook (see Table 4.12). Table 4.12 displayed more details regarding all selected visuals: (1) for which visual-verbal analogies (analog-target pairs) they were accompanied with, (2) by whom and in which class they were applied (3) when they were utilized through the teaching function unit, (4) their medium of transmission, and (5) their types and extent of abstractness.

As seen from Table 4.12 given on the next pages, T1 utilized half of the visuals by selecting from the textbook. In addition, T1 preset and projected the other half of the visuals and sketched just a few visuals on the whiteboard during her lessons. On the other part, T2 did not select any analogies from the textbook; thus, he did not employ any visuals from the book. T2 mostly preferred to sketch visuals (numerically two-thirds). Besides, he chose to project readily prepared visuals (numerically one-third) on the whiteboard.

Table 4.12: Details about teacher employed visuals

Used by	Used in	Analog	Target (Unit)	Visual type	The extent of visual Abstractness	Medium of transmission
T1	9A (2 times)	Function machine (Converting each input to their two times and one more)	Function concept (1.1)	Diagram	Unrealistic	Projected on the board
T1	9A	Function machine (Converting each input to their three more)	Function concept (1.1)	Diagram	Unrealistic	Projected on the board
T1	9A (2 times)	A machine producing tomato juice	Function concept (1.1)	Diagram	Partially Realistic	Projected on the board
T1	9B	A machine producing tomato juice	Pre-image-Image (1.1)	Diagram	Partially Realistic	Projected on the board
T1	9B	Input properties of a machine producing tomato sauce	Properties of the domain (1.1)	Diagram	Partially Realistic	Projected on the board
T1	9A -B	Earning money per working hour	Function concept (1.1)	Table, Venn diagram and graph	Unrealistic	Selected from Textbook (p.2)
T1	9A-B	Birthday function	Univalence requirement (1.1)	Venn Diagram	Unrealistic	Selected from Textbook (p.2)

Table 4.12 (continued)

Used by	Used in	Analog	Target (Unit)	Visual type	The extent of visual Abstractness	Medium of transmission
T1	9A -B	Calorie burning with a sports activity	Domain-range (1.1)	Table	Unrealistic	Selected from Textbook (p.3)
T1	9A-B	Electrical energy use in a house	Domain-range (1.1)	Table	Unrealistic	Selected from Textbook (p.3)
T1	9B	Function machine	Function concept (1.1)	Diagram	Partially Realistic	Projected on the board
T1	9B	Function machine (Converting each input to their square)	Function concept (1.1)	Diagram	Unrealistic	Projected on the board
T1	9B	Function machine (Applies the given rule)	Function concept (1.5)	Sketch	Partially Realistic	Drawn by the teacher on the board
T1	9B	Dichotomizing apple	Onto function (3.8)	Sketch	Partially Realistic	Drawn by the teacher on the board
T2	9C-D-E	Machine	Pre-image-image (1.1)	Diagram	Partially Realistic	Projected on the board
T2	9C-D-E (In 9E 2 times)	Specification profits of a hotel by month	Pre-image-image (1.1)	Graph	Unrealistic	Drawn on the board
T2	9D	Increase in dollar value	Pre-image-image (1.1)	Graph	Unrealistic	Drawn on the board
T2	9D	Water level of Istanbul's dams by year	Function concept (1.1)	Graph	Unrealistic	Drawn on the board

Moreover, analyses revealed that almost all visual-verbal analogies were offered when the function concept was introduced in the early stages of the chapter (except for T1's one use in Chapter 3). This was most probably related to teachers' efforts to supply a strong background for functions or take students' attention by showing them that the concept is not as isolated as they thought.

Evaluation of visuals concerning their extent of abstractness disclosed that the majority (18, 64%) was unrealistic visuals (graphs, tables, diagrams, and Venn diagrams). Only 10 (36%) were partially realistic visuals (diagrams, sketches). However, no realistic visuals were used (see Table 4.13).

Table 4.13: Frequency of visuals according to their abstractness

	Abstractness of Visuals			
Teacher	Realistic	Partially Realistic	Unrealistic	**Total**
T1		7 (37%)	12 (63%)	**19 (100%)**
T2		3 (33%)	6 (67%)	**9(100%)**
Total	**(0%)**	**10 (36%)**	**18 (64%)**	**28 (100%)**

Table 4.14: Samples of T1 visuals according to their extent of abstractness

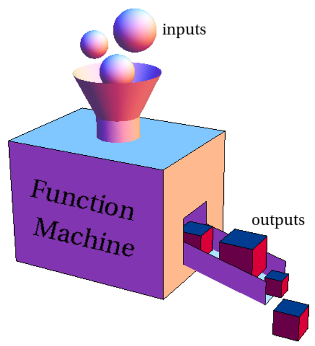
Quality	Visuals
Partially realistic	 Visual 1: "Function machine" analogy

Table 4.14 (continued)

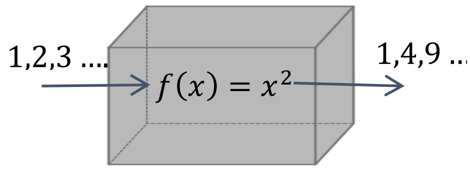
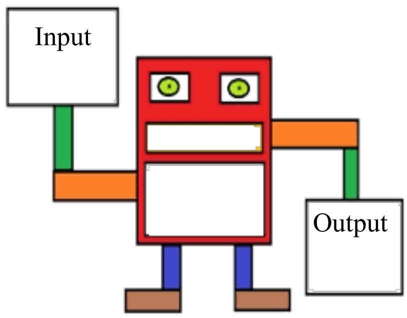
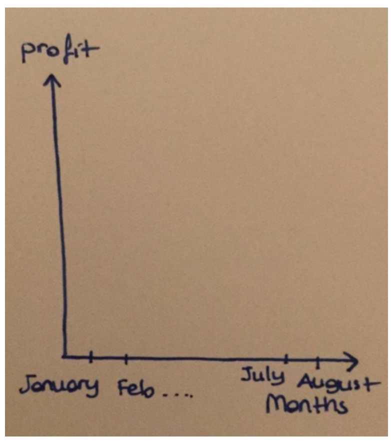
Quality	Visuals
Unrealistic	 Visual 2: “A function machine converting each input to its square” analogy

Table 4.15: Samples of T2 visuals according to their extent of abstractness

Quality	Visuals
Partially Realistic	 Visual 1: “Function machine” analogy
Unrealistic	 Visual 2: “Monthly profit values of the hotel” analogy

In addition, a further concentration on the abstractness of T1 and T2 employed visuals revealed some similarities and differences. Both teachers mostly used unrealistic visuals, such as the second visuals given in Tables 4.14 and 4.15.

Suppose visuals are evaluated in a spectrum from unrealistic to realistic; T1 used visuals closer to reality, while T2 used visuals closer to unrealistic. T1 used partially realistic visuals, which were more lifelike drawings, or including a few photographs such as tomato and tomato juice (see Table 3.8). However, T2 used partially realistic visuals lacking details and more caricatured, such as see visual 1 in Table 4.15.

Although the number of visuals varied from class to class, a detailed examination of transcriptions indicated that the number and teachers' use of those visuals in their own classes were fairly consistent regardless of the classroom. Nevertheless, there were some differences in how analogies and visuals were presented in different classes of the same teachers. For instance, although T2 projected the same diagrams in his classrooms, some differences were observed during his presentation of those diagrams (see Table 4.16).

Table 4.16: Analogy drawn between a machine and a function in classes of T2

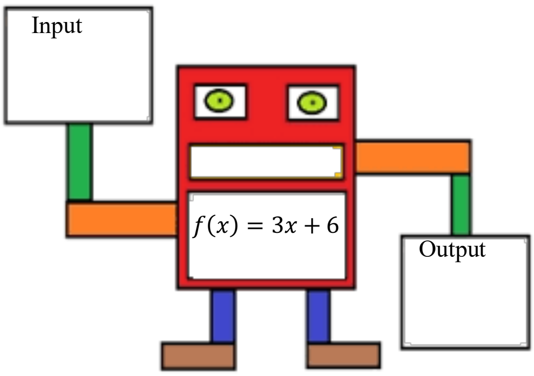
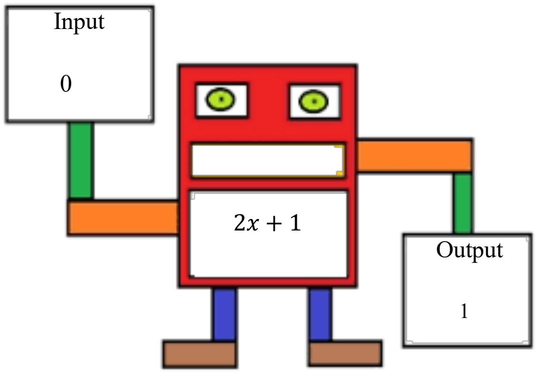
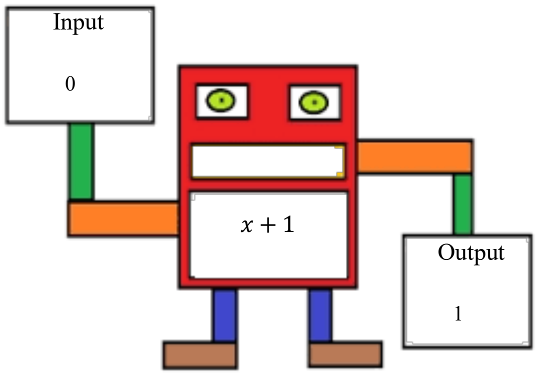
Analogy explained to students:	Function Machine
In Class C Just, we want to tell f (x) is equal to the function of x, 3x + 6. You know, we have an input 0, it is going to 6 and we can show 0 → 6. You know 0 is going to 6. What about 1? 1 is going then? 9, ok! Now, as you can see here: we have an input, an output, and a function. You can generate any function. What is your function here? f(x) = 3x + 6? Then, how many inputs can you have? You will have infinitely many.	

Table 4.16 (continued)

Analogy explained to students:	Function Machine
In Class D You have an input and an output. As you can see, it is a function machine, and it is here. What are the inputs? 0,3, aren't they? What else? -5 You may continue, of course. You may have many inputs. What is your function here? $2x + 1$! What are your outputs? 1,7 What else? -9 and you may continue, or you may have many more like these.	
In Class E You have a machine, as you can see. You have an input, and you have an output in the machine, don't you? Now, you have a function, $x + 1$. We have a sample here like $0{,}5, -4$ and many more. You have outputs here $1{,}6, -3$ and have, of course, many. As you can realize, x is input and y is output.	

Before a meticulous inspection, at first glance, they looked almost the same according to such features: use of colour, number of main figures or elements, and the amount of internal details in figures or elements in that (1) all three represented an anthropomorphized two-handed machine, which had two boxes instead of left and right hands of a human, (2) just the same colours were used for all three diagrams, (3) there were scripts inside the boxes; as "input" in the right box and "output" in the left box, and (4) there were algebraic expressions written inside the body of the machines. However, after a detailed examination, a few differences between these diagrams were detected, such as (1) numbers were written additionally below to scripts "input and output" in diagrams used in Classes D and E, and (2) different algebraic expressions were written inside the machine bodies; like $f(x) = 3x + 6$ in Class C, $2x + 1$ in Class D and $x + 1$ in Class E. In Class C, T2 used numbers 0,1 as inputs and 6, 9 as outputs. He did not write inside the diagram. As mentioned here, the teacher used different algebraic expressions to define functions. However, to decide whichever was the correct symbolization of the function, Robert A. Adams's calculus book, both teachers' bedside books (as it was observed), was scrutinized:

There are several ways to represent a function symbolically. The squaring function, which concerts any, input real number x into its square x^2 could be denoted:

> (a) By a formula such as $\mathrm{y} = \mathrm{x}^2$ which uses a dependent variable y to denote value of the function
> (b) By a formula such as $\mathrm{f(x)} = \mathrm{x}^2$which defines a function symbol f to name the function; or
> (c) By a mapping rule such as $\mathrm{x} \rightarrow \mathrm{x}^2$. (Read this as "x goes to x^2")
>
> Strictly speaking, we should call a function f and not f (x), since the latter denotes the value of the function at the point x. However, as is common usage, we will often refer to the function as f (x) in order to name variable on which f depends.

As disclosed, T2 represented a function machine by a diagram to map a function and a machine saliently. However, examining the explanation and Figure 4.2, it could be seen that T2 symbolized function and told about function improperly. In fact, he could correct symbolization in an orderly: (1) the output of the machine by $\mathrm{f}(x) = 3x + 6$, (2) rules of the functions by $2x + 1$ and $x + 1$. Maybe, he did not misapply the expressions $2x + 1$ and $x + 1$ by writing down the body of the machine; however, they should be corrected by: (1) adding the word "rule" like: RULE: $2x + 1$ and RULE: $x + 1$, or (2) rewriting like: f: $3x + 6$, f: $2x + 1$, f: $x + 1$.

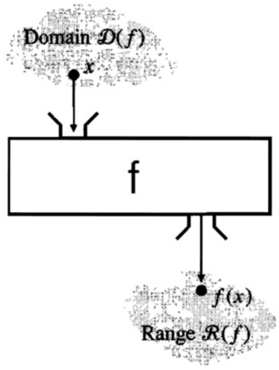


Figure 4.2: A function machine taken from "A Complete Course Calculus" of R.A. Adams (1999)

Moreover, keeping on examination of the teachers' presentation of visuals revealed one more exciting difference in that some of the diagrams were presented with verbal cover stories. T1 described diagrams in an everyday context, such as a machine producing tomato juice from tomatoes. However, T2 described analogies as more artificial and more story-like, probably to attract students' attention. For instance, an analogy illustrated the interactions of the cast of the characters (boss- director):

> T2: For example, you are a boss, and you have a director, ok! You have inputs. What are your inputs? Months, for example, June, July, and August. Think that you are the director of a hotel, which is in Bodrum. As you know, in the wintertime, you don't have many consumers. In the summertime, you have many, of course. At this point, you wonder something. That is your graph (he drew the graph on the board). These are months. Months are inputs here such as January, February and it is going like that March, April, and July, June here. August here. September likes that… These are your profits.
>
> Now, you are asking the director, "What will be our profit?" Think again, you are the boss, and I am the director. You are asking me about the profit in July, ok? Then, I am answering, as "I don't know". What are you reflecting on in my answer? Are you going to fire me?

Here, you are asking my forecast regarding profit value in July (excerpt from observation of Class C).

Assertion 8: Predominantly Concrete-Abstract Nature of Teacher-Generated Analogies Used During Teaching of Function Unit

The level of abstraction was coded into two possible combinations: (1) concrete/abstract (where analog was concrete and the target was abstract) and (2) abstract/abstract (where both analog and target were abstract). As anticipated, out of 169 analogies, nearly all teacher-generated analogies (163, 96%) demonstrated a concrete analog domain for an abstract target domain. The remainder, scarcely any teacher-generated analogies (6, 4%), comprised an abstract analog for an abstract target.

Based on the data presented in Appendix F, results indicated that T1 offered very few analogies (1 in Class A and 5 in Class B) comparing an abstract analog to an abstract target. T1 employed 6 abstract-abstract analogies in her classes; however, four out of six were repeated ones. Briefly, she employed just two different abstract/abstract analogies: (1) *Wear one's own heart on one's sleeve* analogy (used once in Class A, four times in Class B) and (2) *Every dog barks in his own yard* analogy (used once in Class B). In each of these cases, the analog concept was idiomatic or proverbial that considered being abstract. Although they were accepted as abstract, the fact that T1 did not explain what these proverbs and idioms mean might have been due to her assumption that they were concrete. On the other hand, T2 preferred to employ solely concrete/abstract analogies.

Both teachers tended to ascribe human characteristics to functions or to choose analogs in the human context. For instance, T1 presented identity functions as *wearing one's heart on one's sleeve*, and similarly, T2 offered *not being a function* as *being fired the job* that is belonged the human features. Teachers may have employed these analogies to reduce the degree of abstraction of the target concept.

Assertion 9: The Teachers Mostly Presented Analogies as Embedded Activator

Position of the analog or analogies in relationship to the target was classified as one of the three positions (1) advance organizer, (2) embedded activator, and (3) post-synthesizer. Advance organizer ones were further coded as (1) advance organizer type I (AO^1) and (2) advance organizer type II (AO^2).

An examination of the 169 teacher-generated analogies revealed that by far the most significant number of analogies (137, 81%) was presented with the main discussion of the target concept as an embedded activator (see Table 4.8). Most of the other analogies (29, 17%) were presented as an advance organizer (7, 4% AO^1 and 22, 13% AO^2), and scarcely any analogies (3, 2%) were presented as a post-synthesizer. The frequent use of analogies as embedded activators may be relevant to teachers' attempts to strengthen students' understanding of the target concepts during their main discussions. In addition, the infrequent use of post-synthesizer was perhaps due to the teachers' use of analogies as embedded activators so often that they did not need re-summation.

These positions were also examined to decide whether they varied from teacher to teacher or from class to class. Results revealed that the percentages of T1's and T2's positioning analogies as advance organizers, embedded activators, and post-synthesizers were almost identical. To be more precise, when comparing, T1 employed 20 (74%) analogies out of 27 and 42 (73%) analogies out of 58 as an embedded activator in classes A and B, respectively. Moreover, T1 presented only 7 (26%) analogies in Class A and 13 (22%) analogies as an advance organizer in Classes B. Lastly, she offered none in Class A and merely 3 (5%) analogies in Class B as post-synthesizer (see App. F). In the same vein, T2 employed 27 (96%)

analogies out of 28, 24 (89%) analogies out of 27, and 24 (83%) analogies out of 29 as an embedded activator in Classes C, D, and E, respectively. The remainder exclusively was advance organizers (1 in Class C, 3 in Class D, and 5 in Class E). None of the analogies were presented as post synthesizers in any of the classes.

Further examinations disclosed that only T1 employed unique (entirely new) analogs or analogies before the main discussion of unfamiliar (new) targets as advance organizer type I (AO^1), most probably to introduce and guide students to think about function-related target concepts. An example of one of those analogies is given below excerpt.

> T1: The function concept is related to relation and the cross product. You know the relation is the subset of the cross product. If we have two sets, if we define AXB, relation must be a subset of this cross product. The direction is essential. Do you remember the direction of this part? The name of the first set is "domain"; the name of the second set is "range". Now we will talk about the *function machine*. If we put something inside this one, we get something (T1 projected function machine on the board).
>
> T1: We call this one function machine. For example, think that we have a machine and we have some tomato, we want to produce tomato juice. The rule for this machine is it will produce tomato juice, or we can define for other things (from observation of Class B).

On the other side, both teachers employed old analogs or analogies before the main discussion of new targets as an advance organizer type II to provide background for those targets. An example of these is the following:

> T1: Yes, guys, in the last lesson, we gave a task to the tomato machine and wanted it to produce tomato sauce. We learned the terms "input and output". Do you remember? Today, we will give another example and learn about new terms such as domain, codomain, and range (from observations of Class B).

Alternatively, both teachers employed new or old analogs before the main discussion of previously learned (old) targets as advance organizer type II to help students remember those targets. An example of one of those analogies is presented below.

> T2: We started functions with examples from daily life. Today, we will continue to generate similar ones. Consider that you pay a fee to the municipality when you use water and electricity. However, the fee is calculated based on the amount of water you use at home. If you do not use anything, you must also pay a flat fee (from observations of Class E).

Occasionally, advance organizers can be presented at the end of lessons and just before the main discussion of the new target as a teaser like short announcements to report to students what they will learn in the next lesson. However, this technique was never employed by T1 and T2. Instead, both teachers used advance organizers at the beginning of the lessons before the main discussion of the targets.

T1 and T2 employed analog or analogies with the main discussion of the target concept as an embedded activator, most probably when they had difficulty clarifying some aspects of the target. An example is included below.

> S8E: Teacher! What is $5x + 7$? (While T2 was making drills regarding functions)
>
> T2: x is what here?
>
> …
>
> T2: Why do you multiply the fixed number by a number? Think that “7” is fixed and plus 5 times x. Ok, how can I explain it! Ok! Think that you use 1m^3 of water in each month and pay 7 TL fee for every month. What will be the charge for that month? I should pay if I am using 10 m^3 of water which is like $h(x) = 7 + 5.10 = 57$ TL. I should pay a fee every month. Did you understand? Now, I used an analogy (from observations of Class E).

Alternatively, teachers presented analogies as embedded activators, most probably when they realized students required an alternative representation or extra clarification or when they responded to teachers' questions with inaccurate or improper answers. An example of one of those analogies is presented below.

> T1: It is another excellent example for function, but here is mother my new input? Or children?
>
> Students: Mother! Child! No mother! No child! (They are thinking aloud altogether)
>
> S17B: Are they twins?
>
> T1: Ahh, no!
>
> S2B: Are they quin?
>
> T1: Hahhh… not so of course!
>
> S22B: Teacher! What do you mean by "mother machine"?
>
> T1: "Mother machine", actually now I want a function between child and mother.
>
> Students: Is it father?
>
> T1: Look at me! I want to give you a hint.
>
> S13B: Is mother input? Is father output?
>
> T1: Mother is input, No! There is one crucial point related to a function. I will turn back to that tomato example, and then we open up the subject slowly.
>
> S2B: Teacher, we passed the child from tomato, and now we are passing tomato from the child.
>
> T1: We will see together. I am assigning a duty to the machine, and now the machine's duty is to produce tomato juice! But, its task is not making tomato

sauce simultaneously, now what did it begin now? Just be careful!

S2B: We assigned a duty

T1: I assigned a duty to the machine to produce not more than one product using the input (from observations of Class B).

Moreover, both of the teachers rarely or did not use an analog or analogy after complete treatment of the target concept as post synthesizer. Excerpts of those analogies were presented below:

T1: In the book, there is a summary of definitions of what we have defined basically. We said, "A function is like a machine, it has an input, and an output, a function relates inputs to outputs. There will always be three main parts: input, relationship, and output. Its example, we have some elements for inputs 0,1,7, 10, and then we should multiply these domain elements by 2, and then we get the elements of output (from observation of Class A).

Assertion 10: Except a Few Analogies, None of the Teacher-Generated Analogies Were Completely Enriched

Based on their level of enrichment, three types of teacher-generated analogies were identified: (1) simple (completely non-elaborated), (2) partially enriched (partially elaborated), and (3) completely enriched (completely elaborated). Analysis of these analogies revealed that 31 (18%) were simple, 136 (81%) were partially enriched, and 2 (1%) were completely enriched, with a clear preference for partially enriched analogies (see Table 4.8). Although similar percentages of partially enriched teacher-generated analogies appeared in T1 and T2 classes, slightly more simple ones emerged in T1 classrooms (see Appendix G).

A detailed examination of these three types of analogies suggested that teachers most likely preferred simple analogies when the relationship between analog and target was distinctly visible and further mapping or explanation was unnecessary. Teachers used simple analogies while reminding an elaborated analogy used in the previous lesson or the same day. In most such cases, teachers left analogical transfer to students by referring to previously elaborated analogies. For example, simple analogy, *payment according to the parking meter* (T1A26 -2), explaining identity function was reused since the same analogy was partially elaborated in the previous day (T1A26-1). In the same vein, teachers also used simple analogies when the analog domain of the simple analogy was adjusted from an analogy used immediately before the aforementioned analogy. For instance, T2 employed a simple analogy of *specification profits of a hotel monthly, annually, by winter/summertime or whatever time* (T2E3) for clarifying properties of the domain set, just right after elaborating another analogy of *specification of a hotel by June* (T2E2-1) explaining pre-image and image relation. Obviously, it is seen that T2E3 was derived from another elaborated analogy, T2E2-1; hence no additional explanation was required.

In addition, there were some simple analogies; for example, *a function is like a function machine* (T1B1-1, T1A2) or *a factory* (T2E21), both teachers did not mention similarities and dissimilarities between a machine/factory and a function-related target. This strengthens the possibility that they become mechanical clichés that teachers utilize without explaining the idea beyond them. T1 did not elaborate on these analogies in the best-case scenario because she employed another elaborated analogy, *a machine producing tomato juice*, just after T1B1-1 and T1A2. On the other side, there is no consistency in T2's use of this analogy; sometimes, he explained the mapping between analog and target, sometimes just showed the picture of a machine without any explanation, and other times used its most basic way as a simple analogy. But from informal interviews with him, it is known that he thinks the function machine analogy is stereotyped.

Further, simple analogies also involved metaphoric expressions from time to time, such as T1's description of identity function with the idiom, *wear one's heart on one's sleeve*, or again T1's description of piecewise function with the proverb, *every dog barks in his own yard*. Teachers may not have provided an extra explanation for these analogies, including idioms and proverbs, because they think these idioms and proverbs are already known or defined themselves.

Lastly, unlike those with little instructional value, simple analogies were employed to draw students' attention to functions such as T1's description of *students like a noisemaker machine* (T1B41). It displayed that these analogies were used with motivational purposes as understood from students laughing loudly and their wordings: "it is nice and directly relevant to function concept".

Table 4.8 shows that over three-quarters of total teacher-generated analogies were explained to some extent (partially enriched), and just a few of them could be described in great detail (completely enriched). An examination of partially enriched analogies disclosed that they enriched with varying degrees of similarities and dissimilarities (shared and unshared attributes) of the analog-target pairs, general limitations of the analogy method, and analog explanation. As mentioned in the methodology chapter, all listed details were adequate to categorize an analogy as partially enriched. However, especially descriptions pointing out similarities and dissimilarities of mappings were searched to classify an analogy either partially or completely enriched. The vast majority of the teacher-generated analogies except nine of them were enriched without stating any limitations. Merely two out of nine analogies (T1B3-1 and T1B3-2) touched on misleading aspects of mapping besides an explicit explanation of shared attributes (completely enriched analogies). Actually, this two times-recurring analogy, *properties of the domain are like input properties of a machine producing tomato sauce* (T1B3), was explained with different limitations in each time. In this analogy, T1 explained that every function could only use a defined input to get an output; in just the same way, a machine can use only tomato to get tomato juice. T1 discussed a limitation of the analogy (T1B3-1); namely, a function can define more than one procedure to get different outputs. However, the aforementioned tomato machine is designated only for pressing tomato juice, so it cannot carry out the task by using potato or aubergine as input. Later, T1 briefly discussed another limitation of the same analogy (T1B3-2). Namely, a function has to use all defined inputs and cannot use anything else; however, it is possible that someone can mix a few potatoes or aubergine into tomatoes, and the same machine can press them and produce impure tomato juice.

Ideally, analogies should have been explained completely when they were used; however, the results of the study showed that this did not come true. This situation may be because teachers do not want to spend time explaining, as they sometimes emphasize that analogy causes a waste of time. The teachers most likely explained more some analogies, which they had previously experienced with other students and felt more competent.

Assertion 11: The Teachers Mostly Used Extended Analogies

Table 4.17: Details of teacher-generated analogies' extension of mapping

Teacher	Class	**NEx**	**Ex**	Base	Der	Base-Rep	Der-Rep	**Total**
T1	9A	**9**	**18**	3	12	3	0	**27**
	9B	**11**	**47**	5	26	8	8	**58**
T2	9C	**4**	**24**	3	17	0	4	**28**
	9D	**4**	**23**	3	18	1	1	**27**
	9E	**4**	**25**	3	19	2	1	**29**
Total		**32**	**137 (100%)**	**17 (13%)**	**92 (67%)**	**14 (10%)**	**14 (10%)**	**169**

Depending on the "extension of mapping," teacher-generated analogies were categorized as either extended or not extended. Then, extended analogies were further classified as a base, derived, base-repeated, and derived-repeated to understand how analogy extension occurred. Analyses revealed that throughout 169 teacher-generated analogies, the overwhelming majority (137, 81%) was extended while the remainder (32, 19%) was not extended (see Table 4.8). Further analysis revealed that approximately 13% of the extended analogies were base analogies, 67% were the derivatives of the bases, and the remaining 20% were a repetition of the base or the derived ones (see Table 4.17).

Table 4.17 showed that T1 and T2 generated an average of 14 and 7 different (unique) analogies respectively in their classes, and they tended to use almost similar analogies in each of their classes. Detailed analysis of all teacher-generated analogies discovered that T1 and T2 referred remarkably to a few base analogies, which they had previously introduced at the beginning of the function unit. Thus, actually three base analogies (*tomato machine*, *function machine*, and *mother-child relation*) of T1 and two base analogies (*function machine* and *specification profits of a hotel*) of T2 were observed in the majority of the lessons examined in the present study. These base analogies were sometimes employed in consecutive lessons and sometimes used multiple times in the same lesson periods. The *tomato machine* analogy, which compared the attributes of *a machine producing tomato juice* (analog) to *function concept* (target), was the most frequently used one in all T1 classes. However, this tomato machine did not merely produce tomato juice all the time; instead created interchanging tomato products such as tomato sauce, tomato puree, mashed tomato, or sometimes a much more general one: tomato products. In addition, sometimes, rather than tomato machines, tomato fabrics or just fabrics whose operating processes were utterly the same were used.

On the other hand, an analogy comparing *specification profits of a hotel by month* (analog) to *function concept* (target) was the most used base analogy in T2 classes throughout the function unit. This analogy was repeated two times only in Class E; other times, its slightly changed version analog was compared to a new target concept. The modifications on its analog domain varied from analogy to analogy. Sometimes, the time interval: "one month" mentioned in its analog was replaced with other periods, "daily, weekly, yearly, summertime or wintertime". Sometimes, more adaptations were made on the base analog. For instance, the analog of T2C4 was derived from the common base analogy T2C3 (*function concept is like* *specification profits of a hotel by month*). It compared *getting fired of the hotel director because of not specifying the profit value of that month* (analog) to *not being a function* (target).

As results revealed, T1 and T2 specifically preferred to extend certain analogies rather than others. This may be because both teachers only extended analogies that they thought were powerful and unique only to them. There was evidence from classroom observations and informal interviews that T2 mainly used some unique ones since he constructed them for the first time and never encountered them in any textbook and never heard before. Moreover, the aforementioned analogies became T1's classic for the function concept, and she feels to master them every year with 9th grades.

Assertion 12: A Considerable Number of Analogies Included Analog Explanation

Analysis of all teacher-generated analogies revealed that a considerable number of them (119, 70%) contained some analog explanation while the rest (50, 30%) contained none (see Table 4.8). Most included at least a verbal analog description and sometimes had verbal explanations accompanied by visual representations of the analogs. For instance, both teachers presented function machine analogs with figures. While all analogs require some explanation, some were found to need further explanation in particular. For instance, an analogy (T1B2) compared *pre-image and image of a function* to *input and output of a machine producing tomato juice from tomatoes*. As described before, this analogy was supported by a visual representation of a tomato machine, probably to ensure analog familiarity (see Visual 1 in Table 3.8). However, three tomatoes entered the machine in the visual and produced one glass of tomato juice with three more petite tomatoes. At first glance, the words "a machine, tomatoes, tomato juice" and the visual representation of the tomato machine were offered all information required to understand the analogy. However, it should be clarified what kinds of the machine was used and how many kilograms of tomatoes were needed to produce how many liters of tomato juice. Maybe T1 just wanted to give the students a rough idea of the pre-image/image in this analogy without getting stuck with the shape. However, it was possible that students could not picture the situation in their minds similar to the teacher.

Moreover, a few analog explanations were detected that caused some problems. For example, when T2 attempted to relate the univalence property with the director's responses to the boss question about the profit value of July (T2E2-2), T2 was exposed to student questions about the reality of the profit values. In another instance, when T1 started to use *mother machine* analog for univalence property of a function (T1B23), one student asked what T1 referred to with that word. After students were confused about the analog, T1 probably hesitated to go on with it then brought it back to the previously used analog *tomato machine* for the same target. Moreover, at the end of the lesson, she repeated the same analogy by altering the analog *mother machine* as *mother-child relation* (T1B25-1).

On the other hand, it was disclosed that some analogies had no further analog explanation, most probably because teachers assumed they were familiar to students. When T1 introduced the analogies, such as an *identity function is like wearing one's heart on one's sleeve* or *a piecewise function is like every dog bark in his own yard*, did not attempt to explain these analogs. Most probably, this may have been a result of students' not asking anything about the meaning of the analogs, which were idiom and proverb.

Additionally, it was also found that when using some repeated or derived analogies, teachers did not offer any explanation other than just reminding the name of the base analogy. This most probably happened because they thought that the students remembered the old analogy. For instance, T1 explained tomato machine analog many times for different function-related targets. When she introduced an analogy of mashed tomato for function input x, she wanted students to remember the tomato machine analogy. Then, she directly wanted them to think of mashed tomato instead of tomato as an input.

Assertion 13: The Teachers Varied to Identify Their Applied Strategy

Of 169 teacher-generated analogies, roughly 50% included several statements identifying the cognitive strategy. Among them, merely 19% overtly identified the applied strategy as "analogy," while the rest 31% tacitly introduced it with phrases other than the word "analogy" (see Table 4.8). Phrases that signal that the analogy generation was about to occur or just occurred and extracts of their use are listed in Table 4.18 below.

Table: 4.18: Some phrases used by teachers to refer the analogy presence

Phrases	Extracts
"example", "for example", "analogy example"	"x cannot be input every time! It is just a symbol, and it can be altered. For example, this time, I don't want to put tomato inside the machine, and I want to use mashed tomato instead of tomato as an input. Can you see that the logic is the same?" (T1B38)
"suppose that" "assume that"	"Suppose that you are boss and I am the director of a hotel in Bodrum. You know, we have to get a profit value for every month of a year. Now, inputs are months, and outputs are profits. Right now, we are comparing profit values of months." (T2D3)
"… is like …", "it likes…" "… likes …"	"We have an output and a function here. It is like a machine. We can generate any function. What is your function here?" (T2C1)

Table: 4.18 (continued)

Phrases	Extracts
“in daily life”, “daily life example/case/analogy”, “in daily life for example” “examples of functions in daily life”, “analogies in daily life”	"We have talked about piecewise functions. Let's think of an example for piecewise functions... There is a daily life case, which we meet almost every day. For instance, you left your car in a parking lot for two hours. How much would you pay for parking? 10 TL? Can you think how much you would pay for an extra one more hour? …." (T1B60-1)
“story”	“Remember that I generated daily life examples… Now, think that there is a bakery and you are selling something. Cost-profit is your input-output pairs. Ok! Now, it’s your turn! Maybe you can generate better examples than mine. Start generating your own story with your friends” (T2E14).
“comparison”	“I am still generating analogies. Whenever talking about new content, I make this kind of comparison. Let’s continue with a new one. There is a birthday function matching each person with his/her birthday…” (T1A5)
“ real life example/condition/situation”	“I have a good analogy for you… It is a real-life condition. What is a municipality? You pay the invoice for your water or electricity usage at your homes, don’t you? Assume that, in the summertime, you are not at home, but you have to pay the invoice without consuming anything…. (T2C13).

Analysis disclosed that both teachers introduced their analogies in much the same way, and they frequently prefaced their explanations with phrases such as “for example, assume that, suppose that,” which were all other versions of saying: “Let’s clarify the content with an analogy” (see Table 4.18). Teachers also used connectors such as “likes, is like” to suggest that the analogs and targets are similar.

Additionally, they used phrases such as “example, daily life case/situation, story, comparison, and real-life example/condition/situation” as synonyms of analogies or to stress features of analogy strategy. Here is one of them:

> T2: You know we started the function concept. If you remember the last lesson, I gave daily life examples, but today you will generate similar examples cooperatively with your friends. Please remember my daily life examples, analogies, which are “benzetme” in Turkish. They are all precisely daily life examples (from observation of Class E).

Another example of those is presented below.

> T1: We took some tomatoes and put them inside the machine. Then that machine converted tomatoes to tomato juice, didn’t it! Ok, all right, can you think of a new *example, analogy*? (from observation of Class B).

It seems that teachers did not differentiate the term “analogy” from “example,” and they used these two terms interchangeably or used both at the same time (analogy example). Further, most probably, they were viewing the words “example, daily life or real-life example, and even story” have the same meaning as the word “analogy”. However, this was quite normal since they stated they had not received any specific training in analogy use. Likewise, none of them clarified what they wanted to convey with the word analogy, probably because they also did not know how to construct or use an analogy as a learning tool.

Assertion 14: Teachers Failed to Mention Where the Analogies Break Down

Examination of all teacher-generated analogies disclosed that limitations of only nine analogies were mentioned. In addition, it was found out that two of these analogies (T1B3-1 and T1B3-2) referred to a single analogy and repeated twice in the same class, and another two (T2C12 and T2D16) referred to a single analogy, but this time this analogy was used in different classes of the same teacher (see Table 4.19).

Table 4.19: Some general and specific limitations of analogies stated by teachers

Analogy Code	General	Specific	Limitation
T1B3 -1		✓	T1: "If we put the tomato into this machine, we will get only tomato juice. This is very important; if we put the potato into this machine, we cannot get tomato juice anymore. However, sometimes a function can produce different kinds of outputs. In order to fulfill this, the machine needs two processes; however, this machine is set up to get only single output".
T1B3 -2		✓	T1: "We assigned a task for this machine as transforming tomato into tomato juice. We cannot fulfill the task by putting some potato or aubergine into the machine, can we? … First, for a function, we have to define its inputs well and use all of them. We have to be careful when selecting tomatoes from the field for this machine. We can't mix any potato or aubergine into tomatoes since; we have to use all input, don't we? So as we can produce what we want".
T1B21-2	✓		T1: "Our function machine transforms into … Always, you have to define your analogies perfectly. When thinking of this machine analogy, if you say that it would be like this, all rules that you determined would change. Is it clear?"
T2C12	✓		T2: "For here, I have an analogy. Think! There is a factory, which has an input and an output. For example, oil is your input, and you can produce something from oil. I don't know what it will be… I am aware that my analogy isn't creative enough… But I know that for each input, we might have an output… the factory is my function here".

Table 4.19 (continued)

Analogy Code	General	Specific	Limitation
T2D16	✓		T2: “Look at my analogies. They’re daily life examples… I have a factory, and I have to explain everything. I am using oil, and I get some products from oil… Maybe this analogy isn’t a good one since I didn’t mention the inputs and outputs in detail… it is my fault. Sorry for that”.
T2D19	✓		T2: “Here, it is again “factory analogy.” However, I used numbers instead of oil and the product differently here. I have values, but I don’t have a story for this analogy, do you understand?”
T2D22	✓		T2: “For instance, f(x-3) means what? It means that your old input (x) is now three less (x−3). Did you understand? Now, you still have a clear function with three less, so you can easily guess your output. On the other side, assume that you have a factory. As you know, conditions affect everything in daily life, which means you cannot find a fixed function. If you have a factory, you have so many variables. Do you understand? In mathematics, it is easy, as you see. But in daily life, it is not very easy”.
T2E2-2	✓		T2: “Let’s imagine that you are the boss and I am the director of a hotel in Bodrum. You want me to guess profit for June”. (T2 drew a month-profit graph on the board). T2: “That’s your profit (T2 signed profit value on the graph). In June, you have much more profit in dollars…” SE1: “This hotel cannot get such a high profit, am I right? T2: It is an analogy! We just imagined, and I know it is not actual data!”
T2E14	✓		T2: “Think that you have a factory. In this factory, you have an input and an output. As you guess, production of something in this factory will cost you to get profit. This was my analogy; you can also generate your own analogies. I had no story for my analogy, but you have to state a story for your analogy. You will do better than mine!”

Further examination of the details of stated limitations revealed that nearly all were presented as a general statement. Students were warned about possible shortcomings of analogy construction or lacking analogy features. For example, T2 stated the limitations of the analogies (T2C12, T2D16, T2D19, and T2E14) before requesting students to construct their own analogies. In highlighting the limitations of the aforementioned analogies, T2 suggested that the wrong thing with the analogies was that they failed at giving details concerning analog domains. Afterward, he confessed that the stated limitation was his fault and called his imperfect analogies “noncreative”. In another case, T1 referred to a general limitation, which declared that analogies are delicate to changes when explaining the analogy T1B21-2. In remained cases, stated general limitations were referred to lacking features of analogies as mentioned. For instance, T2 stated a limitation about a lacking feature of an analogy when one of his students (SE1) questioned the reality of numbers used in the analogy T2E2-2. In another instance, T2 stated another limitation, implying that an analog and target cannot overlap perfectly. In this analogy (T2D22), T2 mentioned that any change on function concept (target) could be easily made; however, the same change couldn’t be easy for the analog domain.

It is also possible that a limitation can be used to warn students concerning misleading aspects of a particular analogy. However, out of all nine identified limitations, only two specific statements pointed out where a particular analogy (T1B3) broke down (see Table 4.19). In highlighting the limitations of the analogy at different times, T1 suggested that breakdown points with the analogy were that the function machine could not discriminate what you put inside it and could not produce anything other than tomato juice (discussed earlier in this chapter).

It is seen that teachers implicitly talked about the general and special limitations of analogies. They did not explicitly mention that analogies may have some limitations as well as strengths. This may have been because, at best, teachers wanted their students to benefit only from the strengths of analogies or thought their students could see the limitations themselves, which seems unlikely. At worst, and most likely, teachers themselves were unaware of the limitations of the analogies they established. They did not even know that any analogy would have special and general limitations.

Assertion 15: Mostly, Teachers Successfully Employed Analogies to Illustrate Function-Related Concepts

Depending on the “soundness of mapping,” teacher-generated analogies were categorized as either one of sound or unsound. Results documented that amongst 169, the vast majority (163, 96%) had sound mappings while the remainder (6, %4) had unsound mappings, which confirms that mainly teachers successfully employed analogies to illustrate function-related concepts. For this reason, to understand how some analogies were not mapped correctly, a further examination of unsound mappings was done. It was found that 3 of 6 unsound mappings referred to a single analogy, which was employed once in Class C and repeated two times in Class D (T2C23, T2D23-1, and T2D23-2). In other words, a total of 4 (2 different from each teacher) unsound mappings were detected.

As mentioned in Pass 14 in Chapter 3, the soundness of mapping is concerned with the correct identification of analog and target attributes and the epistemological validity of mappings between the identified analog and target attributes. Nevertheless, these two critical aspects for soundness of mapping were not evident in the four analogies that T1 and T2 used. An example of one of those analogies is presented below.

> T1: Now, I want to show you another example, “Mother machine”. It is also an excellent example of functions. But first, you have to decide what input is. Which one of them, mother or children?
>
> S22B: Teacher! I wonder that what you meant by the word “mother machine”.

> T1: I want a function between mother and children…
>
> S13B: In this function, “mother” is input, and “father” is output
>
> T1: “Mother is input!” No! Not correct! Now, think about the rule of functions!
>
> …
>
> T1: Ok, children! Every child has a mother, but a mother can have more than one child, can’t it?

Mother machine analogy *(T1B22)*, as T1 called it, was offered to explain the domain-range relation of functions. As seen in the dialog, T1 most probably intended to emphasize the idea of the function concept as a relation doing matching between elements of two sets. However, contrary to what was intended, identified analog (*mother machine)* stressed the idea of the function as a process converting inputs to outputs. T1’s expressing the analog with the word “machine”, with or without thinking, may have caused a mental image of “input-output” in the students' brains. Perhaps this is why students thought that the mother was input and the children were outputs since the mother is the one who gave birth to the children, and the children were born. Thus, this analogy was accepted as unsound since the analog domain could not be identified correctly and violated the soundness of mapping aspects. Another example of violating soundness of mapping is included below.

> T1: (to S17B) Son, don’t make noise! Listen carefully! Well, you become a noise machine. Noise is your function.

Here, T1 sarcastically used this analogy at the beginning of the lesson. However, since this *noise-machine* (T1B41) analogy did not state the idea of input, output, and transformation correctly, it was considered an unsound analogy.

The remaining two analogies (factory and specification profits of a hotel for two months) were accepted as unsound since their analogs were not identified correctly.

Incidentally, these two unsound analogies were employed to illustrate the multivariate functions, and here is one of them.

> T2: Now, look at the board! $f(x,y) = 2x - 3y + 5, f(2,1) = ?$ (He wrote on the board)
>
> T2: … How many unknowns here?
>
> S19C: 2
>
> T2: 2! How many variables? 2! Can you have 3 variables? Four? or five? Class: (shouting) Yes, of course.
>
> T2: You also have two variables here. Remember factory analogy! Now, think that you are dealing with a factory. Do you have only one variable? Can you have many variables for the factory?
>
> Class: (shouting) Many
>
> T2: Yes! Many, it likes factory having many conditions such as economic conditions, dollar, weather conditions, etc.

T2 used *factory* analogy to remind function concept and introduce multivariate functions. T2 wanted students to remember *the factory* analogy, which was previously described as a process transforming inputs to outputs. Then, he extended *factory* analogy with the intent to describe multivariable functions. However, T2 did not clarify the relation between specified conditions (economic conditions, dollar, weather conditions, etc.) and the factory; in other words, he could not define the analog domain properly. Moreover, T2 could not explain adequately the relations between factory and the idea multivariable functions. Due to these reasons, *the factory* analogy was accepted as an unsound analogy. The other unsound example appears below.

> S6D: Teacher! Can you help me?
>
> T2: Sure! You have two variables there, ok! Now, think that x is 0 and y is

2, which means you will put 0 and 2 instead of x and y.

(He wrote: $f(x, y) = 2x + 4y, f(1,2) = 2.0 + 4.2 = 0 + 8 = 8$)

T2: Remember my hotel analogy, but it is a bit different now. This time, it likes that you want the director to specify profit value for not one month for two months! Because you have two variables, ok!

T2 extended a previously used *hotel analogy* (as he called it) to help his student understand multivariable functions at the student's request. This analogy, however, was not an appropriate analogy for the functions of multiple variables because s*pecification profits of a hotel for two months* analogy (T2D26) could be used only for defining a function with one variable. Profit values, for instance, months June, July, or August, represent entities, not different variables. His described analog did not incite the idea of multivariable functions so that this analogy could be a sound.

4.2 Analogies Generated by Students

Assertion 1: Except for in Class B, Students Generated Only a Few Analogies in the Observed Lessons

The total number of analogies observed in 91 lessons was 215; nearly four-fifths (169, 79%) were teacher-generated, and only one-fifth (46, 21%) were student-generated (see Table 4.1). Results indicated that 30 (65%) of these 46 identified student-generated analogies were from T1 students, while the remaining 16 (35%) were from T2 students. As it is seen obviously, T1 students generated nearly twice as many analogies as T2 students did.

The uppermost reason for this considerable difference seems to be related to the participant structure of student-generated analogies. As Table 4.20 illustrates, in all

T2 classrooms majority of the student-generated analogies (100% in Class C and 67% in Classes D and E) were collectively generated between multiple students. In contrast, in T1 classes, entire student-generated analogies were individually generated. Furthermore, this difference might stem from T1's often encouraging her students to create analogies right after her; conversely, T2's encouraging his students during what he called "group work" time. In addition, this difference may also be because only one analogy was requested from each collective work, while there was no limit to the number of analogies generated individually.

Table: 4.20: Participant structure of student-generated analogies

Teacher	Class	# Individually student-generated analogies	# Collectively student-generated analogies	# Student-generated Analogies
T1	A	6 (100%)	0 (0%)	**6 (100%)**
	B	24 (100%)	0 (0%)	**24 (100%)**
T2	C	0 (0%)	4 (100%)	**4 (100%)**
	D	1 (33%)	2 (67%)	**3 (100%)**
	E	3 (33%)	6 (67%)	**9 (100%)**
Total		**34 (74%)**	**12 (26%)**	**46 (100%)**

There was evidence from the classroom observations that most of the student-generated analogies were produced at the request of teachers. In contrast, the rest were generated by students' own accord (see Table 4.21).

Table: 4.21: Occasions that students generated their analogies

Teacher	Class	# Student-generated analogies employed at the teacher's request	# Student-generated analogies employed by students' own accord	# Student-generated Analogies
T1	A	1 (17%)	5 (83%)	6 (100%)
	B	21 (88%)	3 (12%)	24 (100%)
T2	C	4 (100%)	0 (0%)	4 (100%)
	D	3 (100%)	0 (0%)	3 (100%)
	E	6 (67%)	3 (33%)	9 (100%)
Total		**35 (76%)**	**11 (24%)**	**46 (100%)**

Further analysis of the results shown in Tables 4.20 and 4.21 revealed that students produced only a few analogies individually by their own accord. As expected, they only generated analogies collectively at the request of teachers. In addition, it was observed that students produced analogies by their own accord when they wanted to support teachers' teaching and exemplify teachers' explanations. Here is one of those analogies:

> T2: This graph does not belong to a function. Can you tell me why it is not a function?
>
> S4E: It is not a function since one input has more than one output.
>
> T2: Ok! Thank you. Let's look at another graph. This one also does not display a function for the same reason, does it?
>
> S4E: It likes that a child cannot have two mothers.
>
> T2: Yes! You are perfect! Nice!

Table 4.22: Frequency distrewzibution of student-generated analogies in chapters

	Classes of T1			Classes of T2				
	Class A	Class B	**Total**	Class C	Class D	Class E	**Total**	**TOTAL**
Section	6 (100%)	24 (100%)	**30 (100%)**	4 (100%)	3 (100%)	9 (100%)	**16 (100%)**	**46 (100%)**
1.1 Introduction (definition)	4 (67%)	18 (75%)	**22 (73%)**	4 (100%)	3 (100%)	6 (67%)	**13 (82%)**	**35 (76%)**
1.2 Functional notation			**0 (0%)**				**0 (0%)**	**0 (0%)**
1.3 Functions defined by equations			**0 (0%)**				**0 (0%)**	**0 (0%)**
1.4 Testing for functions			**0 (0%)**				**0 (0%)**	**0 (0%)**
1.5 Evaluating a function			**0 (0%)**				**0 (0%)**	**0 (0%)**
1.6 Finding domain of a function			**0 (0%)**				**0 (0%)**	**0 (0%)**
1.7 Polynomial function			**0 (0%)**				**0 (0%)**	**0 (0%)**
1.8 Rational function			**0 (0%)**				**0 (0%)**	**0 (0%)**
1.9 Irrational function			**0 (0%)**				**0 (0%)**	**0 (0%)**
TOTAL	**4 (67%)**	**18 (75%)**	**22 (73%)**	**4 (100%)**	**3 (100%)**	**6 (67%)**	**13 (82%)**	**35 (76%)**
2.1 Graphs of function			**0 (0%)**			1 (11%)	**1 (6%)**	**1 (2%)**
TOTAL	**0 (0%)**	**0 (0%)**	**0 (0%)**	**0 (0%)**	**0 (0%)**	**0 (11%)**	**1 (6%)**	**1 (2%)**

Table 4.22 (continued)

	Classes of T1			Classes of T2				
Section	Class A 6 (100%)	Class B 24 (100%)	**Total 30 (100%)**	Class C 4 (100%)	Class D 3 (100%)	Class E 9 (100%)	**Total 16 (100%)**	**TOTAL 46 (100%)**
3.1 Constant function		4 (17%)	**4 (14%)**				**0 (0%)**	**4 (9%)**
3.2 Identity function		1 (4%)	**1 (3%)**				**0 (0%)**	**1 (2%)**
3.3 Linear function			**0 (0%)**				**0 (0%)**	**0 (0%)**
3.4 Piecewise function and its graph	2 (33%)	1 (4%)	**3 (10%)**				**0 (0%)**	**3 (7%)**
3.5 The absolute value function			**0 (0%)**				**0 (0%)**	**0 (0%)**
3.6 Polynomial function			**0 (0%)**				**0 (0%)**	**0 (0%)**
3.7 One-to-one function			**0 (0%)**			1 (11%)	**1 (6%)**	**1 (2%)**
3.8 Onto function			**0 (0%)**			1 (11%)	**1 (6%)**	**1 (2%)**
TOTAL	**2 (33%)**	**6 (25%)**	**8 (27%)**	**0 (%0)**	**0 (0%)**	**2 (22%)**	**2 (12%)**	**9 (22%)**

Assertion 2: Vast Majority of Student-generated Analogies were Observed in Chapter 1

As mentioned before, the function unit was covered under 3 chapters (function concept, graphs of functions, and types of functions) consisting of 18 sections (see Appendix E). Data shown in Table 4.22 indicated that considerable proportions (35, 76%) of the student-generated analogies were encountered in Chapter 1, much less in Chapter 3 (9, 22%), and only one (S4E28) in Chapter 2. In addition, Table 4.22 suggested that at least three out of four student-generated analogies in each class and even all in Classes C and D were seen in Section 1.1, Introduction (Definition), in the first chapter. Moreover, when compared all student-generated analogies belonging to Chapter 3, a relatively large number (8, 27%) were presented in T1 classes. Students in T1 classes preferred to use analogies while learning about constant, identity, and piecewise functions. On the other hand, just one T2-student (S4E) used an analogy while learning about functions by his own accord.

As expected, data revealed that where teachers generated more analogies, their students generated more (see Tables 4.3 and 4.22). This was related to the fact that teachers encouraged their students to produce analogies individually or collectively where they themselves produced more analogies (Section 1.1 for both teachers). In addition, it was observed that students generated analogies, albeit a little, by their own accord (especially in Chapter 3) without any request from the teachers. This may have been due to teachers highlighting the general positive aspects of the analogies and praising their students who contributed an analogy many times.

Assertion 3: There was a Tendency for Students to Extract Analogies that Utilized an Analog from Their Daily Life Experiences and/or Knowledge Bases

Analogs of student-generated analogies were invoked from three main categories: (1) textbook, (2) students' own experiences and/or knowledge bases, and (3) analogies or examples used by their teachers - Table 4.23 given below displays the frequency distribution of the analog sources of student-generated analogies.

Table 4.23: Frequency distribution of the sources of students employed analog

	Teacher 1		Teacher 2			
Analog Sources	Class A	Class B	Class C	Class D	Class E	**Total**
Textbook	0 (0%)	1 (4%)	0 (0%)	0 (0%)	0 (0%)	**1** **(2%)**
Students' own experiences and/or knowledge bases	5 (83%)	18 (75%)	4 (100%)	3 (100%)	8 (89%)	**38** **(83%)**
Analogies or examples used by their teachers	1 (17%)	5 (21%)	0 (0%)	0 (0%)	1 (11%)	**7** **(15%)**
Total	**6** **(100%)**	**24** **(100%)**	**4** **(100%)**	**3** **(100%)**	**9** **(100%)**	**46** **(100%)**

Analysis of Table 4.23 revealed a general tendency for students to draw analogies used an analog from their own daily life experiences more than examples or analogies previously used by their teachers or the textbook. Since analogy is a teaching tool and even a learning tool, it was expected and entirely natural that students selected their analogs more from their own experiences and/or knowledge bases. An example of one of those analogies is included below.

> T1: … k is not a function.
>
> S20A: Since an element in a domain cannot go to two aspects of a range simultaneously.
>
> …
>
> T1: I am explaining it another way for those who did not understand what we talked about before. Let's think that we put some tomato inside a machine and we want this machine to produce tomato and carrot juice simultaneously.

S11A: As for me, a citrus juicer is the best example of being a function. Again, I would put an apple inside the machine to get apple juice since I always do so at home.

S22A: My mother puts everything inside the citrus juicer.

T1: There is no such thing.

S22A: Why? My mother does! She gets mixed fruit juice.

…

Although analogs of student-generated analogies were primarily selected from students' own experiences and/or knowledge bases, other sources were also invoked to some extent for their analogs. The textbook contained 10 analogies within the contents covered in the function unit presented in the previous section (see Table 4.5). Analysis of all student-generated analogies disclosed that solely one analog, *increase in the length of a plant by years*, was selected from a textbook analogy by S18B (see Table 4.23). An examination of related observation data disclosed that T1 had left her students some parts of the textbook for their self-study, including the aforesaid analogy. It once again proved the importance of each analogy in the book.

Also, in a few cases, students used analogs adapted from analogies used by their teachers (see Table 4.23). For instance, S18B selected *a mother's childbearing* analog from T1's *mother-child relation* analogy. Even another student in the same class, S17B, repeated T1's this analogy with almost the same analogy components. Other instances, S22A, S17B, and S23B, have selected analogs of their own devised analogies from T1's *tomato machine* analogy. In the same vein, S3B developed his analogy with the analog, *a machine converting whatever input into the same output*, from T1's *function machine* analogy. Lastly, only one student (S4E) from T2 classes employed an analog; a *hotel makes the same profit for two different months*, from T2's *hotel* analogy.

In general, results disclosed that the same analogies of the teachers always inspired the students. One reason may be that the teachers repeated the same analogies or used their derivatives many times.

Assertion 4: Differences Surfaced in the Topics that Students Selected for the Analog Domain of Their Analogies

Students' selected topics for the analog domain of their analogies were determined to exist within math-related or outside-math topics. Later, math-related ones were categorized either as (1) those involving mathematical computing in daily life or (2) those involving mathematical calculation with numbers only. On the other hand, outside-math topics were categorized as one of four categories: (1) those related to culinary, (2) those related to family ties/family situations, (3) those were idioms or proverbs, or (4) those related to the industry. Table 4.24 shows the categories of student-selected topics and the frequencies of each category.

Table 4.24: Frequency distribution of student-selected analog topics

		T1 Classes		T2 Classes			
		Class A	Class B	Class C	Class D	Class E	**Total**
Topics		6 (100%)	24 (100%)	4 (100%)	3 (100%)	9 (100%)	**46 (100%)**
Math-related topics	Topics involving mathematical computing in daily life		1 (4%)	4 (100%)	3 (100%)	7 (78%)	**15 (33%)**
	Topics involving mathematical calculation with numbers only		1 (4%)				**1 (2%)**
Total			**2 (8%)**	**4 (100%)**	**3 (100%)**	**7 (78%)**	**16 (35%)**

Table 4.24 (continued)

Topics		Class A (T1 Classes)	Class B (T1 Classes)	Class C (T2 Classes)	Class D (T2 Classes)	Class E (T2 Classes)	**Total**
		6 (100%)	24 (100%)	4 (100%)	3 (100%)	9 (100%)	**46 (100%)**
Outside-math topics	Topics related to culinary	4 (67%)	8 (33%)				**12 (26%)**
	Topics related to family ties/family situations		2 (9%)			2 (22%)	**4 (9%)**
	Topics that were idioms or proverbs	2 (33%)	1 (4%)				**3 (6%)**
	Topic related to the industry		11 (46%)				**11 (24%)**
Total		**6 (100%)**	**22 (92%)**	**0**	**0**	**2 (22%)**	**30 (65%)**

Results disclosed that there were some discrepancies in the frequency of these topics. When comparing the frequency of two main categories (math-related or outside-math topics), most analogs (30, 65%) were selected from outside-math topics. However, the most substantial distinction occurred between the topic choices of T1 and T2 students. Namely, T1 students selected almost all analogs from outside-math topics, particularly those related to culinary and industry, whereas T2 students chose math-related topics, especially those involving mathematical computing in daily life (see Table 4.24).

As mentioned in the previous section, T1's and T2's commonly used analogies - *tomato machine* and *specification profits of a hotel by month* - were also related to culinary and mathematical computing in daily life. Reviews of topics that teachers and students selected for the analog domain of their analogies revealed the similarities between their choices. There was even evidence from classroom observations that teachers also noticed these similarities. An example of one of those is presented below.

> T1: Function likes a machine producing tomato juice from tomato. So can you think of an analogy?
>
> S7B: Producing pickle juice from the pickle
>
> S10B: Producing carrot juice
>
> S9B: Obtaining French-fried potatoes
>
> …
>
> S9B: Producing paper from tree
>
> T1: You selected all from vegetables. You don't have to do it. Let's choose them from another subject.

This example also disclosed that T1 stimulated her students to generate analogies different from hers. Upon this request, students selected topics related to the industry for their analogies, as given below.

> S13B: Producing paper from wood.
>
> T1: Producing paper from wood! Yes, it is an excellent analogy; anything else? That's to say that the machine has a duty to convert an input to output, doesn't it?
>
> S2B: Producing paper from wood pulp.
>
> S17B: Obtaining thread from cotton
>
> …

These excerpts displayed how effective their teachers' and even friends' analogies were in choosing topics in which students developed their analogies. Students were so influenced by the topic choices of others, perhaps because they did not fully understand how to construct analogies.

Furthermore, disparities also surfaced in female and male students' selection of topics. Female students selected topics involving mathematical computations in daily life such as *the length of a plant, the price of a cupcake in a bakery, income, and expense of designing a hairdresser salon, and airplane fuel consumption*. On the other hand, male students selected topics involving mathematical computations within everyday life or their special interests such as *deciding the cheapest car park, budgets of famous car brands, and selection of the best contract for a football player, cab fare, and time-gas consumption relation of a car*.

In addition, a detailed examination of students' selection of outside-math topics revealed that male students constructed almost all these analogies. They employed analogies related to culinary such as *producing fruit, carrot, apple, tomato, and pickle juice, getting pizza from pizza dough*. In addition, they employed analogies related to the industry, for instance, *producing paper from tree/wood/wood pulp, getting shoes from leather, obtaining thread from cotton, acquiring sweaters from the thread, getting knitting from wool, and attaining glass from sand*. Both female and male students selected the analog *mother-child relation* related to family ties/ family situations. Lastly, one female student (S18B) selected the analog *living in a world of one's own* that was an idiom, and one male student (S11A) selected the analog *every dog barks in his own yard* that was a proverb. These results suggested that there was a slight overlap between selected topics of female and male students. Based on these results, although it is not possible to comment on the exact reasons for the differences between the topics in which male and female students choose their analogies, it can be said that students preferred topic familiarity when they constructed their analogies.

Assertion 5: A Significant Amount of the Student-Generated Analogies Were Related to Definition and Properties of Function

Content of the target concepts was classified according to 18 sections listed in the function unit (see Appendix E). Results disclosed that a significant amount of the student-generated analogies (36, 78%) were related to the definition and properties of the function, including univalence requirement, and pre-image/image, domain-range conceptions (see Table 4.25). Findings also revealed that T1 students generated analogies related to constant, identity, and piecewise functions. On the other side, T2 students developed analogies regarding one-to-one and onto functions. However, as Table 4.25 demonstrated, none of the T1 and T2 students used analogies related to graphs of functions (Chapter 2).

Table 4.25: Frequency distribution of the content of the target domain

		Classes of T1			Classes of T2				
		Class A	Class B	**T1**	Class C	Class D	Class E	**T2**	**Total**
Content	Target	6	24	**30 (100%)**	4	3	9	**16 (100%)**	**46 (100%)**
1.1. Introduction (Definition)	Pre-image /image	3 (50%)		**3 (10%)**					**3 (7%)**
	Domain-range		1 (4%)	**1 (3%)**		1 (33%)		**1 (6%)**	**2 (4%)**
	Univalence requirement	1 (17%)	1 (4%)	**2 (7%)**			1 (11%)	**1 (6%)**	**3 (7%)**
	Function concept		16 (67%)	**16 (53%)**	4 (100%)	2 (67%)	6 (67%)	**12 (75%)**	**28 (60%)**
Total		**4 (67%)**	**18 (75%)**	**22 (73%)**	**4 (100%)**	**3 (100%)**	**7 (78%)**	**14 (87%)**	**36 (78%)**

Table 4.25 (continued)

		Classes of T1			Classes of T2				
		Class A	Class B	**T1**	Class C	Class D	Class E	**T2**	**Total**
		6	24	**30**	4	3	9	**16**	**46**
Content	Target			**(100%)**				**(100%)**	**(100%)**
3.1 Constant Function	Constant function		4 (17%)	**4 (14%)**					**4 (9%)**
3.2 Identity function	Identity function		1 (4%)	**1 (3%)**					**1 (2%)**
3.4 Piecewise function	Piecewise function	2 (33%)	1 (4%)	**3 (10%)**					**3 (7%)**
3.7 One-to-one function	One to one function						1 (11%)	**1 (6%)**	**1 (2%)**
3.8 Onto Function	Onto function						1 (11%)	**1 (6%)**	**1 (2%)**
Total		**2 (33%)**	**6 (25%)**	**8 (27%)**	**0**	**0**	**2 (22%)**	**2 (13%)**	**10 (22%)**

Further analyses indicated that almost all student-generated analogies, except for one analogy, were used when describing the content to which they related (see Appendix F). Only the analogy, *univalence requirement of functions likes a child cannot have two mothers* (S4E28) used when learning about graphs of functions (see Assertion 2). This was likely because many of the student-generated analogies were produced at the request of teachers, and teachers requested students to generate analogies about what they were currently teaching (see Assertion 1).

Table 4.26: Categorization of student-generated analogies employed in T1-2 Classes

Category	Subcategory **Total**	In T1 Classes 30 (100%)	In T2 Classes 16 (100%)	**In T1-2 Classes 46 (100 %)**
Nature of Shared Attributes	Functional (F)	26 (87%)	14 (88%)	**40 (87%)**
	Structural and Functional (S&F)	4 (13%)	2 (12%)	**6 (13%)**
Presentation Format	Verbal (Vb)	30 (100%)	16 (100%)	**46 (100%)**
	Visual and Verbal (Vs &Vb)	0 (0%)	0 (0%)	**0 (0%)**
Level of Abstraction	Abstract-abstract (A-A)	3 (10%)	0 (0%)	**3 (7%)**
	Concrete– abstract (C-A)	27 (90%)	16 (100%)	**43 (93%)**
Position of Analog Relative to Target	Advance Organizer (AO)	1 (3%)	0 (0%)	**1 (2%)**
	Advance Organizer 1 (AO^1)	0 (0%)	0 (0%)	**0 (0%)**
	Advance Organizer 2 (AO^2)	1 (3%)	0 (0%)	**1(%)**
	Embedded Activator (EA)	29 (97%)	16 (100%)	**45(98%)**
	Post-Synthesizer (PS)	0 (0%)	0 (0%)	**0 (0%)**
Level of Enrichment	Simple (S)	24 (80%)	7 (44%)	**31 (67%)**
	Partially Enriched (P-En)	6 (20%)	9 (56%)	**15 (33%)**
	Completely Enriched (C-En)	0 (0%)	0 (0%)	**0 (0%)**
Extension of Mapping	Extended (Ex)	15 (50%)	6 (38%)	**21 (46%)**
	Unextended (UnEx)	15 (50%)	10 (62%)	**25 (54%)**
Analog Explanation	Explained (Exp)	5 (17%)	8 (50%)	**13 (28%)**
	Unexplained (UnExp)	25 (83%)	8 (50%)	**33 (72%)**
Strategy Identification	Identified (I)	26 (87%)	14 (88%)	**40 (87%)**
	Overtly Identified (OI)	22 (74%)	8 (50%)	**30 (65%)**
	Tacitly Identified (TI)	4 (13%)	6 (38%)	**10 (22%)**
	Unidentified (UnI)	4 (13%)	2 (12%)	**6 (13%)**
Presence of Analogical Limitations	Present (P)	3 (10%)	1 (6%)	**4 (9%)**
	Not Present (NP)	27 (90%)	15 (94%)	**42 (91%)**
Soundness of Mapping	Sound (S)	14 (47%)	15 (94%)	**29 (63%)**
	Unsound (UnS)	16 (53%)	1 (6%)	**17 (37%)**

Assertion 6: The Overwhelming Majority of the Student-Generated Analog/Target Pairs Share Similar Function.

The analogical relationship between the analog-target pairs of each student-generated analogy was coded as either functional or structural-functional. Eventually, since they were all functional analogies, all analogical relationships were coded as either (1) a process or (2) a relation to determine how they addressed the function.

Table 4.27: Frequency of student-generated analogies offered to describe the function as a process or as a relation

	T1-Students		T2 -Students	
		30		16
	Functional	Structural-Functional	Functional	Structural-Functional
Function as Total	**26**	4	**14**	2
A Process	**25**	1	**5**	0
A Relation	**1**	3	**9**	2

Of the 46 student-generated analogies surveyed in the current study (see Table 4.26), the vast majority of analogical relationships were functional (40, 87%), while far fewer were structural-functional (6, 13%). When comparing student-generated analogies in T1 and T2 classes separately, it would seem that there was no difference between the proportions of functional and structural-functional analogies (see Table 4.26). However, there were some differences in whether they addressed the function as a process or a relation. As displayed in Table 4.27, almost all functional analogies generated by T1-students addressed the function as a *process*; however, most of the functional analogies generated by T2-students offered the function as a *relation*. Further, almost all structural-functional analogies generated by both T1 and T2 students presented the function-related concepts as a *relation*.

Further examination about the analogical relationship of teacher- and student-generated analogies and how they addressed the function conception confirmed that students used analogies with the same properties as their teachers'. For instance, after T1 used the *tomato machine analogy, which was functional and addressed the* function as a process, her students mainly employed analogies with the same properties as this analogy. Her students used analogies, for instance, *making a cappuccino* (S21A), *producing pickle juice* (S7B), and *getting a steel cooker from iron* (S17B). As it is seen obviously, these analogies were functional and addressed the function as a process. That is, they transformed inputs (espresso roast coffee, steamed milk, and foam for cappuccino, pickle including such as tomato, vinegar, and salt for pickle juice, and iron, nickel, chrome and other materials for steel cooker) into outputs (cappuccino, pickle juice, and steel cooker) as a function does.

Another instance, after T2 used the *specification profits of a hotel by month* analogy, which was functional and addressed the function as a relation, his students predominantly used analogies with the same properties as this analogy. His students employed analogies, for example, *deciding the cheapest car park* (S11C), *selection of the best contract for a football player* (S5D), and *time-gas consumption relation of a car* (S6E). These stated analogies were functional, and the elements of analog and function-related targets related to each other in the same way. Namely, *deciding the cheapest car park* analogy pointed out matching cars with tariffs of car parks is similar to matching the elements of domain and range of a function. Similarly, selecting *the best contract for a football player* analogy implied matching footballers with different contracts. The *time-gas consumption relation of the car* analogy suggested matching elapsed time and gas consumption during the elapsed time interval. There may be many reasons for this but it may also be due to the fact that students do not fully understand the subject or they do not know how to construct an analogy.

In addition, a few metaphoric expressions were detected in the analogies generated by T1 students, as in T1's analogies. These analogies were classified as structural-functional since their analog meaning and behavior were typically attributed to a function-related concept. As an example of these kinds of analogies, *a piecewise function is like every dog barks in his own yard* can be given. A student of T1 (S11A) initially generated this analogy, which T1 also used. The last section explained in detail why this analogy was structural-functional (see Assertion 6).

Another example was identity *function (target) is like living in a world of one's own (analog)* generated by S18B. In this analogy, the idiom (*living in a world of one's own), meaning that* someone's concerned only with own thoughts and being unaware of what is happening around, was used as an analog. This analogy was functional since it indicated an analog that behaved like the target. In other words, the identity function did matching between x and f (x) in a similar way in the idiom did matching from someone to his/her world. This analogy was also structural because it was planned to map analog and target spelling and meaning similarities. That's to say, an identity function is f (x)=x, and similarly, someone was matched with a world of someone's own. In this analogy, as it is seen obviously, the function was addressed as a relation.

On examination of the structural-functional analogies, two further ones generated by a T2 student (S4E) were detected. A further investigation on these two analogies revealed that the T2-drawn graphs inspired S4E while T2 was teaching the vertical line test. For example, in the first instance, T2 explained why the graph did not represent a function and stated that it is not a function since one input goes to two outputs. Thereupon, S4E exemplified, in this case, *a graph not defining a function* (target) *is like a child cannot have two mothers* (analog). This analogy was classified as functional since both analog and target behaved similarly. Besides, it was classified as structural since there were some common words in both analog-target pairs. The graph matched *one element of the domain* with *two elements of the range*, in the same way, in the analog same numbered elements, namely, *one child* and *two mothers* matched.

Assertion 7: All Student-Generated Analogies Were Presented in Verbal Form

As shown in Table 4.26, all 46 student-generated analogies were presented verbally, and none had a visual component. This may be likely related to the fact that the students spontaneously established all their analogies, either voluntarily or at the teachers' requests. Another possible reason might be that the students did not think or feel the need to represent their analogies with extra visuals since their teachers already supported their teaching with visuals.

Assertion 8: Students Predominantly Used Concrete-Abstract Nature Analogies

Since analogies provide links from familiar to unfamiliar, it was anticipated that most of the analog-target pairs would be concrete/abstract. If any, there would be very few abstract-abstract analogies. As expected, results confirmed that most student-generated analogies (43, 93%) demonstrated concrete analogs for abstract targets. The remainder (3, 7%) compared abstract analogs to abstract targets (see Table 4.26). As also expected, all these results are pretty similar to those of teachers.

Based on data offered in Appendix G, results denoted that only T1 students from Classes A and B generated analogies comparing an abstract analog to an abstract target; in each of these cases, the abstract analog was idiom or proverb. T1 students offered 3 analogies; however, one of three was a repeated analogy. Actually, they employed just two different analogies: *every dog barks in his own yard* (used twice by S11A) and *live in a world of one's own* (employed once by S18B). On the other side, T2 students preferred to use merely concrete/abstract nature analogies. All these results are quite similar to those of teachers.

Assertion 9: Students Tended to Present All of Their Analogies as Embedded Activator

An examination of the 46 student-generated analogies disclosed that nearly all (45, 98%) were presented as embedded activators (see Table 4.26). In such a position, the analogs or analogies were presented with the main discussion of the target concept.

The frequent use of student-generated analogies in this position could be due to the attempts of teachers to enrich the analog-target relationship when analogies were presented as embedded activators. To be more precise, as is known, students primarily generated their analogies at the request of teachers. Classroom observations also evidenced that teachers tended to encourage students' analogy generation to reinforce student understanding after introducing the target concept during the main discussion of the target.

While almost all student-generated analogies were presented as embedded activators, only 1 out of 46 was presented as an advance organizer type II (AO^2). As seen in the excerpt below, the aforementioned analogy was not a new one (repeated analogy). It was employed at the request of T1, referring back to the last lesson and providing background for the new target concept.

> T1: In the last lesson, we learned about piecewise functions. Your friend (S11A) used a very precious analogy to explain the piecewise function. I really liked it. Please, can you repeat that analogy?
>
> S11A: Every dog barks in his own yard!

Assertion 10: Students Mostly Used Simple Type of Analogies

Out of 46 student-generated analogies, 31 (67%) were simple, 15 (33%) were partially enriched, and none were completely enriched (see Table 4.26). A detailed examination of simple and partially enriched ones revealed that simple ones were used extensively by T1-students; partially enriched ones were used extensively by T2-students. The frequent use of partially enriched analogies by T2 students was most probably due to T2's encouragement to explain their own generated analogies.

A further examination of these two types of student-generated analogies disclosed that simple ones were probably used when the relationship between analog-target pairs was apparent and additional mapping was considered unnecessary. Because students often generated analogies similar to those created by their teachers or friends, they may have produced simple ones without further explanation. For instance, a student of T1 (S7B) developed a simple analogy *producing pickle from pickle* to explain function concept immediately after T1 generated a partially enriched analogy, *tomato machine* explaining function concept. In fact, the analog-target relationship of these two stated analogies was similar since both of them pointed out obtaining products (outputs) from raw materials (inputs). However, since there was no further explanation regarding this relationship, the understanding was left to others. Alternatively, they were far more likely not even aware that they needed additional descriptions about analogies.

Additionally, focusing on simple analogies revealed that some of them comprised metaphoric statements like teachers', such that one student (S11A) described piecewise function with the proverb *every dog barks in his own yard*, and another student (S18B) explained identity function with the idiom *live in a world of one's own*. Perhaps, students did not need to explain further because they thought everyone understood these proverbs and idioms clearly.

Results also revealed that nearly one-third of student-generated analogies were explained to some degree. An examination of these analogies disclosed that they were enriched with varying details, including shared and unshared attributes of analog target pairs or any misleading aspects of the analogy. Although these analogies were not completely explained, they were partially explained and contained at least an explanation referring to a relationship from the analog domain to the target domain.

Further examination upon partially enriched analogies suggested that students might have chosen to develop simple analogies when teachers did not validate their analogies or when they felt that simple ones were not apparent enough. An example of one of those analogies is offered below.

> S11A: A function likes a fruit juicer.
>
> S22A: My mother puts everything into the fruit juicer.
>
> T1: You cannot do that.
>
> S11A: Why we don't? I meant that a function likes a fruit juicer and inputs can be defined as mixed fruits so that outputs will be mixed fruit juice.
>
> …

Results also revealed that almost all student-generated analogies were enriched without stating any limitations or misleading aspects. Merely four of them (S21B14, S8B17, S17B25-4, and S5D20) clearly touched on misleading aspects of analogy mappings besides explicit explanations of shared attributes.

Assertion 11: The Students Equally Used Extended and Not Extended Analogies

Analyses revealed that nearly half were extended analogies of 46 student-generated analogies surveyed in the current study, while the other half were not extended ones (see Table 4.26). Further analysis on extended analogies revealed that 24% were base analogies, 62% were their derivatives, while the other 14% were their repetitions (see Table 4.28).

Table 4. 28: Details of student-generated analogies' extension of mapping

Teacher	Class	NEx	Ex	Base	Der	Base-Rep	Der-Rep	**Total**
T1	9A	**0**	**6**	1	4	1	0	**6**
	9B	**15**	**9**	2	6	1	0	**24**
T2	9C	**4**	**0**	0	0	0	0	**4**
	9D	**2**	**1**	0	1	0	0	**3**
	9E	**4**	**5**	2	2	1	0	**9**
Total		**25**	**21 (100%)**	**5 (24%)**	**13 (62%)**	**3 (14%)**	**0 (0%)**	**46 (100%)**

Detailed analysis of all student-generated analogies revealed that students referred mainly to a few base analogies. Mostly they derived other analogies from these bases and sometimes just repeated the bases. A further examination of base analogies and the transition to their derivatives and repetitions disclosed that sometimes they were derived by the same student and sometimes by other students; however, they were repeated continuously by the same students. For instance, two base analogies *producing pickle juice from pickle* (S7B5) *and producing paper from wood* (S13B10) were derived by students who did not first employ them. On another occasion, a base analogy of *a child cannot have two mothers* (S4E28) was used by S4E to explain the univalence requirement. Next, the aforementioned analogy was derived and a new one *a case that a child has two mothers* (S4E34) was used by the same student to explain onto functions. Lastly, in the base analogy *every dog barks in his own yard* (S11A28) describing piecewise functions, and *another base analogy cab fare* (S6E16) explaining the function concept repeated by the same students who first employed them. This may be an indication that students own these analogies.

Moreover, a further examination on derived analogies discovered that students did not always derive their analogies from student-employed bases but also base analogies previously introduced by teachers. For instance, analogies (S17B52, S3B55, and S18B24) derived from T1's *function machine analogy* (T1B1) and other analogies (S11A16, S22A17, and S21A18) derived from again T1's *tomato machine analogy* (T1A1). In another instance, the analogy of *two months the hotel makes the same profit* (T2E2) used to explain *not being a one-to-one function* was derived from T2's *hotel analogy.* This situation reveals the effect of all classroom analogies on student-generated analogies.

Assertion 12: The Students Infrequently Engaged in Analog Explanation

Analysis of all student-generated analogies revealed that only 13 (28%) contained some analog explanation while the remainder (33, 72%) had none. When compared analogies generated by T1 and T2 students, T2 students more frequently explained the analog domains (see Table 4.26). The most likely reason for this is that T2 asked, each time insistently, his students for more explanations about the analogies they generated analogies.

Observation data suggested that students mostly drew analogies from their daily life experiences, which most probably resulted in rare analog explanations. In addition, data revealed that student-generated analogies derived from teachers' previously used ones did not contain any analog descriptions. For example, one student of T2 (S4E) generated an analogy; *a hotel making the same profit for two months is like not being a one-to-one function,* whose analog domain derived from T2's previously used analogy (*specification profits of a hotel by month*). Students probably didn't need to explain more, as their teachers had already talked enough about the base analogy. Results also disclosed that analogies comprised some metaphoric expressions, such as every *dog barks in his own yard* (S11A28-1 and S11A28-2) and *lives in a world of one's own* (S18B57)*,* did not contain any analog explanations.

On the other hand, observation data confirmed that students had to explain some analogs to eliminate analog unfamiliarity when describing analog attributes. For instance, S3B first used an analog; the *identity function is like a machine whose all inputs and outputs are the same again.* A few minutes later, the same student mentioned briefly that input and output would be the same. Then, S3B required to explain more about the machine (analog) and said that, for example, the number 1 goes inside the machine, and this machine doesn't do anything on the number, so 1 goes outside as unchanging.

Assertion 13: Strategy of Student-Generated Analogies Identified by the Teachers

Out of 46 student-generated analogies, the vast majority (40, 87%) included statements identifying the cognitive strategy. Among them, 30 (65%) overtly identified the strategy as "analogy," while the remaining (10, 22%) tacitly introduced it with phrases other than direct the word analogy.

It is worth saying that teachers made strategy identification of almost all these analogies. Teachers identified the strategy of these analogies in the same way as they did when generating their analogies (see Table 4.18). Analysis of classroom discourses disclosed that teachers mostly referred to analogy strategy when requesting students to develop their analogies and giving feedback to students about their employed analogies. T1 used the phrase "example"; on the other side, T2 used the phrases “story, the story of functions, analogy story, and example” when they did not directly use the word “analogy”.

Only two times, students identified the strategy of the analogies. In one instance, a student of T1 (S11A) stated, “the best example is fruit juicer for functions” to stress the similar traits of fruit juicer and functions. Examination of the spotted case disclosed that the student used “example” instead of “analogy”. In another instance, a student of T1 (S11B) prefaced his analogy of *increase in the length of a plant by years* (S11B61) with the phrase “for instance” after T1 requested her students to exemplify real-life situations for piecewise functions. Classroom discourse analysis revealed that the student used the phrase “for instance” to mean “I am illustrating it with a real-life situation”. As can be seen, there was no clear indication that students knew what an analogy is or how to construct an analogy. However, this was quite normal, considering their teachers were unaware of what analogies really were.

Assertion 14: Students Failed to Mention Where the Analogies Break Down

Students failed to state the limitations of the analogies that they employed. Only 4 (7%) limitations were remarked for the 46 student-generated analogies examined in the current study. While it is possible to point out general limitations on the harms of analogy use, a detailed analysis of all identified limitations revealed that they were all for particular analogies. For instance, one of the students of T1 (S17B) constructed an analogy (S17B25-4), *a child cannot have two mothers,* during T1 was explaining the univalence requirement of functions. After giving details about shared attributes of analogy, another student (S10B) realized a breakdown point of the same analogy and mentioned that a child could also have a wet nurse or a stepmother. In another example, a T2 student (S5D) generated an analogy (S5D20), *selecting the best contract for each football player,* to explain the function concept. In addition, the same student suddenly realized a breaking point for this analogy and said that every contract was made for an energetic football player. However, he added that every football player has the possibility of injury. Here, students most likely questioned the compatibility of the chosen analogs with real life. Again, perhaps they were not aware that they cited limitations of analogies.

Examination of these analogies revealed that two of the student-generated analogies (S21B14 and S8B17) contained teacher-stated limitations. For instance, one of the students of T1 (S21B) stated the analogy *function likes acquiring a book from the pape*r when T1 requested her students to exemplify function concepts with their analogies. It is worth noting that although a student generated this analogy, the teacher directly mentioned the misleading aspects without letting the student find them himself. In highlighting the limitation of the analogy, T1 stressed that although defined function had only one stage to transform raw materials into products, in that analogy, more than one stage could be needed to convert paper into a book. Another example occurred when another student of T1 (S8B) used an analogy *attaining glass from sand* to clarify the function concept. Similarly, T1 warned the student to think about all phases of this operation. Then she underlined that at first other materials must be accompanied to sand, thus all would be the primary raw materials of the process, and next, they could turn into other compounds up to becoming glass.

Assertion 15: In All Classes, Except in Class B, Students Mostly Generated Sound Analogies

An examination of the 46 student-generated analogies revealed that 29 (63%) were sound while 17 (37%) were unsound (see Table 4.26). Further analysis disclosed that almost all were sound in all classes, except Class B (see Appendix G).

As mentioned before, the soundness of mapping is concerned with the correct identification of analog and target attributes and the epistemological validity of mappings between these identified analog and target attributes. Nonetheless, these properties were not evident in many of the student-generated analogies. For instance, preparing her students for the idea of function concept, T1 wanted her students to generate their own analogies for functions of one variable. Thereupon students generated analogies such as *obtaining French-fried potatoes* (S9B7), *attaining trousers from fabric* (S2B18), *producing shoes from leathe*r (S8B19), and *getting pizza from the pizza dough* (S2B20). As it is seen, all these analogies entailed the idea of a function as a process transforming inputs to outputs. However, they were all considered unsound since their identified analogs containing more than one variable did not present the attributes of the target concept (function of one variable). When considering *French-fried potatoes,* someone needs more than a potato and needs at least oil and salt or maybe other spices. Again, in addition to the fabric, some yarn, a zipper, and one or more buttons are needed to obtain a trouser. Once again, more than leather is required to produce a pair of shoes, such as a couple of shoestring, tread, and insole. Lastly, more than pizza dough is needed to make a pizza, such as olive, sauce, corn, sausage, or more. Amazingly, T1 did not warn students about their analogies were not appropriate for functions of one variable and she even confirmed these analogies. This may be because either T1 did not examine the student-generated analogies too much or was not competent in evaluating the appropriateness of these analogies. T1 even mentioned that she was planning to use the S2B20 analogy, which was an analogy that was accepted as unsound. This strengthens the second possibility even more.

In that vein, students continued generating further analogies to exemplify the function concept. However, similarly, some of the identified analog attributes did not correspond to the specified target attribute (being a function of one process). Each of the analogies, *producing paper from the tree* (S9B9), *producing paper from wood* (S13B10), *acquiring book from paper* (S21B14), *getting steel cooker from iron* (S17B16), and *attaining glass from sand* (S8B17), were epistemologically appropriate to illustrate functions as a process transforming inputs to outputs. However, they were all examples of composite functions containing more than one process. Considering student-generated analogies, for instance, analogy S9B9, the tree must be chopped down to get row wood to produce paper from the tree. At this stage, wood is not enough to make paper (S13B10); raw wood must first be turned into wood pulp containing cellulose fibers, lignin (natural glue), and water. Next, chemicals separate lignin from the cellulose fibers and water (pulp mixture). Afterward, cellulose fibers must be separated from water, and a mat (cellulose fibers without water) is obtained. Terminally, the mat is compressed into a roll of paper. When considering remaining analogies (S17B16 and S8B17), iron is not enough alone to produce a steel cooker; thus, nickel and chrome must accompany iron. Similarly, sand is not sufficient to get a glass; thus, soda and iron must accompany sand. In the same vein, there are even many procedures to get a steel cooker and attain a glass from these mixtures. While the students were producing all these analogies, T1 warned the students that producing a book from paper required more than one process. However, she just endorsed others without any warning. This situation seems to show that T1 did not examine student-generated analogies very much or did not know about other procedures or processes.

Nevertheless, there were similar situations in the student-generated analogies for function types. Four student-generated analogies were offered to clarify the idea of a constant function. Yet, it was sound that merely one of them, the *out-of-order number machine for every entered number, gives the resulting number one* (S3B55). The remaining three analogies, *producing tomato juice from tomato* (S17B52), *obtaining tomato sauce from tomato* (S23B53), and producing *pickle juice from pickle* (S9B54), were unsound since each of them could not be a constant function. Moreover, one, three, and one student-generated analogy were offered to clarify the ideas of the identity, piecewise and onto functions, respectively. Only two repeated analogies referring to one analogy, *every dog barks in his own yard* (S11A28-1/2), were sound. The remaining analogies, *living in a world of one's own* (S18B57), *increase in length of a plant by years* (S18B61), and *a case that a child has two mothers* (S4E34) were unsound. Since they violate the correct identification of analog and target attributes, they did not represent the properties of the target concepts. The analogy S18B57 was inappropriate to illustrate identity function (defined as f(x)=x) since it has a meaning of being insensitive to the outside world not representing the identity function definition. Another analogy, S18B61, was inappropriate to exemplify a piecewise function (a function defined on a sequence of intervals) since, in this analogy; only one function for an interval was defined. However, this analogy can be an example of linear functions. Lastly, analogy S4E34 was inappropriate to explain onto functions (defined as its image equal to its range) since, as mentioned previously in analogy, mother-child relationship; a child cannot have two mothers; thus, this analogy other than onto function it is not a function. All these may indicate that the students could not choose the appropriate analogs, probably because they did not understand the target concept.

CHAPTER 5

DISCUSSION, CONCLUSIONS, AND IMPLICATIONS

Based on the results described in the previous chapter, the main findings related to the research aims and questions are discussed and concluded in this chapter. Afterward, the limitations of the study are reviewed, and implications and recommendations for future research are made.

5.1. Discussion and Conclusions

The primary purpose of this study was to portray how two ninth-grade mathematics teachers and their 121 ninth-grade students from five different classrooms employed analogies while teaching and learning function unit in the natural classroom settings and the features of teacher- and student-generated analogies. More specifically, two research questions shaped the current study: (1) how are the features of teacher- and student-generated analogies in the ninth-grade function unit? (2) how do the features of analogies employed in the ninth-grade functions unit differ from teacher to teacher and from class to class?

In this section, in the following two parts, the results described in the previous chapter are discussed to explore features of teacher- and student-generated analogies and how they change from teacher to teacher and from class to class.

5.1.1 Features of Teacher-generated Analogies and Their Comparisons from Teacher to Teacher and from Class to Class

As stated in previous studies (e.g., Richland et al., 2004; Ubuz et al., 2013), it is evident from the data that the vast majority of all identified analogies in mathematics classes in this study were teacher-generated. This may be likely because teachers are people who plan and carry out the teaching process, and the unequal teacher/expert–student/novice relationship, as noted by Richland et al. (2004). In addition, data revealed that each teacher employed a nearly equal number of analogies in each of their classes except in Class B. T1 may have used more analogies in Class B since she spent more time teaching the function unit with this class than with Class A, she did not have any classroom management problems in this class as far as it was observed, and as it was known, she generally enjoyed teaching mathematics to the students in this class. These reasons suggest that the time required to use analogies, as Newby et al. (1995) noted, and teachers' concerns about losing control of classroom management might affect their pedagogical decisions, from whether to use analogies to how much to use. Moreover, findings disclosed that the teachers mainly employed analogies as a part of their routine teaching, which supports the idea that the teachers observed in this study believe in the potential and power of analogies to teach function-related concepts. In addition, when all the data are analyzed, unlike in some studies (e.g., Orgill & Bodner, 2005; Sarantopoulos & Tsaparlis, 2004), it cannot be concluded that analogical explanations should be offered to students with low or high academic success.

Results revealed that the overwhelming majority of teacher-generated analogies in all classes were related to the definition and properties of function and were used at the beginning of the function unit, especially in Section 1.1. Many studies (e.g., Akkoc & Tall, 2005; L. L. Clement, 2001; Bardini et al., 2014; Elia, Panaoura, Eracleous, & Gagatsis, 2007; Elia, Panaoura, Gagatsis, Gravvani, & Spyrou, 2008; Gagatsis et al., 2006; Hatisaru & Erbas, 2017; Panaoura, Michael-Chrysanthou, Gagatsis, Elia, & Philippou, 2017; Panaoura, Michael-Chrysanthou, & Philippou, 2015), pointed out that students generally have problems presenting an accurate contemporary definition of the function. Analogies are powerful tools to improve conceptual understanding and provide the ability to draw inferences from well-understood concepts to novel ones (Gray & Holyoak, 2021). Since both participating teachers have been teaching functions for years and employed analogies as a routine part of their teaching, they are likely aware of the difficulties of functions and the potential of analogies. Therefore, there is nothing more natural than using analogies when describing the concept of functions and introducing its essential components (arbitrariness and univalence requirements) and fundamental terms (domain, co-domain, range, dependent and independent variables) more intensively. Of course, their use of analogies at the very beginning of the unit may also stem from a pedagogical tradition learned from textbooks, mathematics curricula, or from their university professors, or even high school mathematics teachers.

Similar to Thiele and Treagust's (1994b) and Mozzer and Justi's (2013) findings, this study indicated that teachers predominantly drew upon analogies that employed analogs from their repertoires formed mainly by their previously experienced analogs in the past years or by their reading from function-related materials more than the textbook or student-relevant conditions. In fact, teachers seem to instinctively choose analogs that can easily connect with the target they want to teach without being unaware of the possible side effects of choosing analogs from their own experiences and/or knowledge bases. Although in this study, the positive impacts of teachers choosing analogs from student-relevant conditions were not explicitly observed, it may be helpful to attract students' attention and involve them in the analogy construction. Since analog familiarity is an essential prerequisite for students to see the links between analog and target pairs (Richland et al., 2007), if teachers develop analogies employing analogs from their own experiences and/or knowledge bases, they should somehow ensure that students become familiar with them.

An inspection of the analog domain of teacher-generated analogies revealed some variances in the choice of topics in which teachers produced analogs of analogies. While in both classes, T1 selected half of all her analogs from math-related and the other half from outside-math topics, interestingly, T2 chose analogs only from math-related topics in all his classes. However, both chose the more significant part of their analogies from the subcategory of mathematical computing in daily life. Another interesting finding of this study was that teachers' interests, daily routines, and the problems they seek solutions to in everyday life directly affected their choice of topics. These findings naturally suggest how teachers experience the world influences their topic preferences in constructing their analogies. Moreover, results depicted disparities in the topic selection of female and male teachers. However, since many factors could lead to this, findings are not adequate to associate this disparity only with the gender difference.

Gentner's (1983) structure mapping framework advocated that relational commonality is essential to analogy but not object commonalities. Consistent with Gentner's (1983) recommendation and parallel to the findings of Hatisaru (2021, 2022), Ubuz et al. (2013), and Unver (2009), this study revealed that all teacher-generated analogies were functional. Very few were structural simultaneously, which shared similar semantic similarities. These results are pretty expected since the nature of the function is compatible with functional analogies.

Moreover, results regarding functional analogies disclosed that although both teachers defined function as a relation, most T1 analogies addressed the function as a process, while most T2 analogies addressed it as a relation. As Hatisaru (2022) found, this shows the consistency between the idea of function given in teacher-generated analogies and the most recent 2017 mathematics curriculum (see Section 2.1.1) and the degree to which the curriculum influenced teachers in their analogy generation. In the same vein, this also illustrates the contradiction between the notion of function presented in T1-generated analogies and the curriculum. Besides, analogical relations of functions showed that teachers mainly did not explain structural similarities between analog and target concepts. This suggests that it is not a coincidence. Most likely, teachers thought that students were familiar with the analog domain and could easily see the relationship between analog and target.

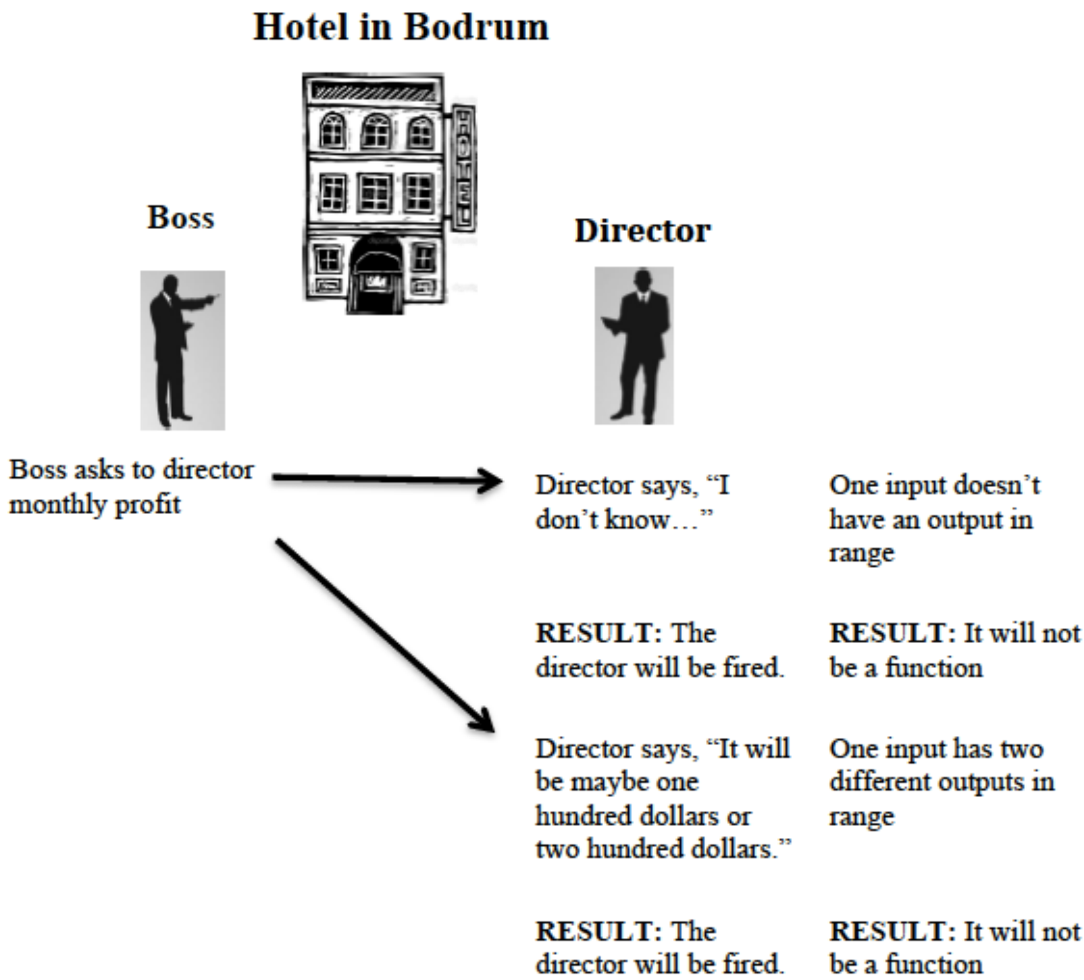


Figure 5.1: Suggested analogy mapping for T2's hotel analogy

On the other hand, results revealed that both teachers generally explained the similarities between the analog-target pairs in very crowded and scattered sentences instead of stating them directly and clearly. This is probably due to teachers' unplanned use of analogies, not developing analogies in their repertoires, and maybe not knowing exactly how to construct them. This suggests that teachers need to precisely and clearly articulate the similarities between analog-target pairs, so they do not negatively impact students' learning, as Bayazit and Ubuz (2008) reported. It may be even more effective if teachers explain the analogical relationship between analog and target by illustrating the sharing features as in Glynn's (1995) diagram representation (see Figure 2.1). It may be challenging to apply precisely the same in practice. However, teachers should somehow draw attention to corresponding attributes of analog and target and ease comparisons (Richland et al., 2007; Richland & McDonough, 2010). Related attributes of analog and target may be aligned spatially, written in corresponding colors, and combined with visual representations (Gray & Holyoak, 2021) and gestures (Richland et al., 2007; Richland & McDonough, 2010). Considering these recommendations, for example, the analogical relationship in the hotel analogy of T2 can be explained as given in Figure 5.1.

The findings of the study denoted that all analogies were in a verbal format. However, very few (primarily those employed at the very beginning of the unit) were in both visual and verbal form. This may be because both teachers tried to make the concept of function more understandable at the beginning of the unit. As Curtis and Reigeluth (1984) suggested, nothing else may be needed when sufficient explanation is given about the analogical relationship. However, as they added, visual components can strengthen student understanding. In fact, there are some studies reporting feedback in this direction. For instance, Bean et al. (1990) emphasized that students' visualizations may be improved by combining pictorial and verbal analogies. Similarly, Begolli and Richland (2013), who manipulated videotaped mathematics instruction to test the role of visual representations in instructional analogies, noted that making analog and target visual (versus verbal) improved participants' awareness of similarity and benefit from structure mapping opportunities. Beyond this, Kubricht et al. (2015) found that animated analogs provided more automatic transfer than diagrammatic or merely verbal analogs. These suggest that supplying visual representation (not just pictorial representation) may help students grasp the function-related concepts and maintain an analog-target relationship. On the other hand, findings indicated that both teachers utilized mostly unrealistic static visuals. Although how this impact students' learning is unknown, perhaps not using realistic static visuals such as photographs can prevent students from focusing on unnecessary details in the photographs. However, neither teacher seem to have consciously chosen the images they used for the visual representation. Moreover, the most exciting difference the results showed was that T1 described visuals in everyday context; on the contrary, T2 explained more artificially and story-like. Although implications are unknown, a story-like narration of analogies may lead students to perceive analogies as purely fictional and think that the target does not have the stated features in reality. In addition, the chronological arrangement of the same analogies revealed that the verbal presentations of the teachers improved each time. In other words, while making more crowded and complex sentences at the beginning, there was a slight improvement in each subsequent use compared to the previous one. This suggests that the analogies in teachers' repertoire can improve somewhat with experience. However, teachers should not leave them to chance and develop them urgently, considering that each analogy has an unobservable effect on student learning.

As expected, results showed that almost all teacher-generated analogies employed concrete analogs to explain abstract targets. Similar to Mozzer and Justi's (2013) findings, in this study, both teachers very likely selected concrete entities such as machines, fabric, or people, which are more familiar and meaningful to students, to diminish the degree of abstraction. However, although T1's use of idiomatic or proverbial analogs, which are considered abstract, to describe abstract target concepts, appears to cause any students misunderstandings, the consequences are still not fully known. Therefore, it may be better for teachers to briefly explain these metaphorical expressions to avoid possible negative consequences.

Findings disclosed that the great majority of all teacher-generated analogies in all classes were presented with the main discussion of the target as an embedded activator, most probably whenever teachers felt that students required further explanations. Results of other studies examining textbook analogies (e.g., Akcay, 2016; Curtis, 1988; Curtis & Reigeluth, 1984; Dikmenli, 2015; L. D. Newton, 2003; Orgill and Bodner, 2006; Thiele & Treagust, 1994a) and Unver (2009) examining teacher-generated analogies in 9th-grade mathematics classes found that most were presented as embedded activators. On the other hand, Ubuz et al. (2013) found that analogies were often presented before discussing the target as an advance organizer and were rarely presented as embedded activators. Results of all previous studies and the current study confirm Curtis and Reigeluth's (1984) deduction that "the most effective placement of analogies appears to be as either advance organizers or embedded activators" (p. 115). The frequent use of analogies as advance organizers or embedded activators may be due to attempts of teachers or authors to make the target concept more student-friendly and facilitate the analog-target relationship. However, even though it has been a long time since the call of Orgill and Bodner (2006), there has not been a study investigating the effects of these different positions on analogy use and target concept learning.

The study results revealed that almost all teacher-generated analogies were either simple or partially enriched. Teachers' use of simple analogies may be because they think a further explanation is unnecessary when the analog-target relationship is apparent or previously explained or when the analogies are stereotyped (like the function machine analogy). In practice, it will not be reasonable to expect teachers to thoroughly explain, or at least explain, as if it is used for the first time when they repeat or use the derivatives of a previously described analogy. In that sense, it is pretty reasonable to use the analogy in these cases as simple. However, teachers should explain more about analogies that they think are stereotyped or the relationship between analog-target pairs is apparent because their understanding and students' understanding may not be the same. On the other hand, both teachers' use of partially enriched analogies might be because they do not know precisely what analogies are and how to implement them as teaching tools. This suggests that teachers should first know the features of completely enriched analogies and how to construct them during teaching functions.

Furthermore, the findings of the study demonstrated that the vast majority of the teacher-generated analogies were extended from base analogies (*tomato machine*, *function machine*, and *mother-child relation* from T1 and function machine and specification profits of a hotel from T2) that teachers experienced in previous years, as far as they mentioned. In addition, results revealed that *tomato machine* and *specification profits of hotel* analogies were the most extended analogies which were unique to teachers, again as far as they mentioned. Teachers' extension of specifically these analogies may be related to their experience that they helped students understand functions and made it easier for teachers to explain functions. It may be linked that they were unique to teachers. In addition, it may be related to their feeling more comfortable and competent when using these analogies as they are in their repertoire. Again, these possibilities underline the need to develop and qualify the analogies in teachers' repertoire (Treagust et al., 1992) to describe functions.

Curtis and Reigeluth (1984) and Glynn (2015) recommended clarifying the analog domain before presenting the target concept, which will ease students' understanding of the analogy. Furthermore, Orgill and Bodner (2006) and Thiele and Treagust (1994a) suggested an analog explanation to prevent problems arising from analog unfamiliarity and incorrect attribute transfer. Likewise, Mastrilli (1997) emphasized that if students have little or no knowledge of the analog domain, it means nothing how effective an analogy is. Similarly, Gray and Holyoak (2021) underlined the need first to understand why the analog behaves as it does since learning from an instructional analogy occurs by transferring information from analog to target. Consistent with these strong recommendations for analog explanations, the study results showed that many teacher-generated analogies included some analog explanations; however, they also pointed out some difficulties encountered due to insufficient analog explanations. In similar cases, students were observed to question the authenticity of the analogs or what exactly teachers meant by these analogs. This suggests that teachers should not leave the analog domain to students' own interpretations, as students cannot see through their eyes. Teachers should choose non-fiction analogs that students can easily understand, and they should evaluate the analog knowledge of students to deactivate potential sources of misrepresentations (Zook & Di Vesta, 1991). In practice, the analog explanation may not necessarily be a full-fledged description; however, teachers should somehow ensure that students become familiar with as much as necessary. As Curtis and Reigeluth (1984) mentioned, when teachers feel the analog domain is unfamiliar or complex to students, they should describe, review, and at least remind the relevant components of analog explicitly (Gray & Holyoak, 2021). The extent to which teachers can achieve this is related to both their knowledge of content and teaching (KCT) and knowledge of content and students (KCS), as defined by Ball et al. (2008).

The findings of the study disclosed that about half of the teacher-generated analogies identified the implemented strategy either overtly with the word "analogy" or tacitly. Curtis and Reigeluth (1984) pointed out the connection between analog explanation and strategy identification. They asserted that identification or explanation of the cognitive strategy might be unnecessary if the analog domain is explained before presenting the target concept. However, it cannot be assumed that students are aware that an analogy is generated or understand what analogy actually is even if their teachers have clearly explained the analog in full with its relevant attributes. Because students may not be still familiar with the analog domain despite their teachers' explanations, may not have enough knowledge of the target concept, or may not be able to understand why teachers map the attributes even if they know the relevant attributes of both. These possibilities suggest that it may be best for teachers to identify that they have generated an analogy, explain what the analogy method is (Glynn, 2015; Harrison & Treagust, 1993; Orgill & Bodner, 2006), and train students in analogical reasoning (S. Brown & Salter, 2010; Treagust et al., 1996). But what teachers know about analogy and how well they dominate analogical reasoning is debatable, as Unver (2009) detected before. In fact, it is enough for teachers to make these explanations once, and no one expects them to repeat them repeatedly in every analogy. Likewise, they do not have to give a direct dictionary definition of analogy; they can roughly define analogy in a way that students can understand. But in general, teachers should be aware of the cognitive load imposed by their explanations (Gray and Harrison, 2021) and make explanations accordingly. How teachers will achieve this is up to their foresight and knowledge.

Gray and Holyoak (2021) mentioned, “in general, it is challenging (and often impossible) to find an analog that perfectly instantiates every relation in the target concept, and that introduces no extraneous information.” In other words, “every analogy breaks down at some point” (Glynn, 1994, p.7). Therefore, good mapping should indicate breakdown points where attributes are not shared (Thiele & Treagust, 1991). In the present study, since the nature of analogy use has some limitations (general limitations) (Gray & Holyoak, 2021; Thiele & Treagust, 1994a), and every analogy has unshared attributes (specific limitations), these two kinds of limitations were examined. Consistent with Ubuz et al. (2013) and Unver (2009), the present study showed that teachers implicitly mentioned a few limitations. This is most likely because teachers may not be aware of the limitations of analogies and, of course, the distinction between the specific and general limitations. Even if they did, they might not be conscious of the need to address the limitations of analogies since they often focused on similar attributes. In this sense, if teachers systematically preplan their analogies, the possibility of pointing out the limitations of analogies will increase (Nashon, 2003).

This study revealed that the vast majority of the teacher-generated analogies correctly identified analog-target attributes, and there was epistemological validity of mappings between analog and target attributes. This suggests that both teachers were relatively successful in using analogies on the concept of function compared with pre-service teachers who participated in the studies of Bayazit and Aksoy (2011), Bayazit and Ubuz (2008), and Ubuz et al. (2013). Detailed examination of analogies that did not meet the aforementioned requirements showed a conflict between entailed and intended function ideas of T1 analogies. On the other hand, it indicated a problem in identifying the correct analog and target attributes of T2 analogies. However, Orgill and Bodner (2004) and Unver-Sezer and Ubuz (2018) mentioned that students are interested in and recall what their teachers do in the classroom; hence, teachers must be conscientious when using analogies. Bayazit and Ubuz (2008) evidenced that analogies without content validity could not establish students' understanding of the function concept. To avoid possible unfavorable side effects of analogy, teachers should correctly select analogs whose nature overlaps with the target concept and take care to construct epistemologically valid mappings. Most importantly, this also suggests that teachers require knowledge about functional-related contents and interdisciplinary subjects to effectively use analogies, as indicated by Ubuz et al. (2013).

In short, the study results showed that although the features and frequency of analogies differ from teacher to teacher, both teachers generally used the same analogies in their classrooms. This may be because teachers automatically used an analogy in other classes after using it for the first time in a class, as well as maybe because, in general, they encounter similar situations due to following the same order and solving the same examples in each class.

5.1.2 Features of Student-generated Analogies and Their Comparisons from Teacher to Teacher and from Class to Class

Only one-fifth of all identified analogies were student-generated, of which two-thirds were generated by T1 students and another one-third by T2 students. These results may be related to T1's encouraging her students to create analogies individually without a limit on the number, while T2's asking his students to develop one analogy collectively only during the periods he called "group work." In addition, results indicated that most student-generated analogies were constructed at the request of teachers. This shows how effective teachers are in students' analogy generation as those who guide classroom teaching. In fact, as found in Richland et al. (2004), teachers in this study also seemed to act from a constructivist point of view, desiring students to be actively involved in the analogies. However, they could not do it thoroughly, perhaps because they do not have adequate information about the essential components of an analogy, how to apply it, and what special attention should be paid to it. This situation raises the issue that teachers should know how to guide students while constructing their analogies.

Results indicated that almost all student-generated analogies in each class were seen in Chapter 1 (Section 1.1) related to the definition and properties of the function in the same way as teacher-generated ones. This showed that where teachers generated more analogies, their students generated more. This is, of course, related to students' construction analogies at the request of teachers. However, teachers especially asked students to create analogies at the very beginning of the unit that may be related to their thinking that the definition of function and its properties are vital for learning the function unit and that students' active participation will strengthen this.

The study revealed a general tendency of students to draw analogies using an analog from their own experiences and/or knowledge bases more than analogies or examples previously used by their teachers or their textbook. This may be due to the fact that teachers' requesting students to generate their own analogies naturally pushes students to choose analogs from their own experiences and/or pre-existing knowledge. On the other hand, results showed that students selected some analogs from teachers' classic -most repeated and most derived - analogies that they employed during function unit teaching. Students sticking to teacher-generated analogies may be a sign of their inability to create a deep understanding of the target concept.

An examination of the analog domain of T1 and T2 student-generated analogies revealed that T1 students decided on almost all analogs from outside-math topics, particularly those related to culinary and industry, whereas T2 students chose math-related topics, especially those involving mathematical computing in daily life. Since commonly used analogies of T1 and T2 (*tomato machine* and *specification profits of a hotel by month* analogies) were also related to culinary and mathematical computing in daily life, it is possible to say that students may have been inspired by the content of the analog domain of teachers' analogies. This may indicate that the students did not fully understand the target concept, or they did not understand how an analogy is constructed. On the other hand, consistent with the results of Pittman (1999), the study results disclosed a slight overlap between selected topics of female and male students. However, since many factors could lead to this, findings are not adequate to associate this disparity only with the gender difference.

Examination of functional and structural-functional student-generated analogies revealed that while almost all functional analogies generated by T1 students addressed the function as a process like T1 analogies, T2 students addressed it as a relation like T2 analogies. In addition, a few metaphoric expressions were detected in structural-functional analogies generated by T1 students like T1 analogies. These results may indicate that students modeled the analog-target relationship of their teachers' analogies, which is less likely. They might also suggest that they just copied without realizing the analog-target relationship.

As expected, results disclosed that all student-generated analogies were presented verbally, and none of them was reinforced with visuals. This may be most likely because students often created unplanned, spontaneous analogies at the request of their teachers. It seems like they probably did not need to show analogies with visuals anyway. But maybe if the teachers asked the students to explain the analogy they created with visuals, students could better understand the analog-target relationship.

Moreover, the current study results showed that almost all student-generated analogies employed a concrete analog to explain an abstract target. All abstract/ abstract analogies that used an abstract analog (idioms or proverbs) and an abstract target were generated by T1 students as T1 did. All again supported the idea that students may have developed or just copied analogies that had the features of their teacher's analogies.

Besides, results disclosed that nearly all student-generated analogies were presented as embedded activators, as did T1 and T2. This is highly anticipated because teachers employed analogies during the main discussion of the target concept and simultaneously encouraged students to generate analogies.

Further examination of student-generated analogies revealed that about two-thirds were simple and generated by T1 students, while one-third were partially enriched and constructed by T2 students. However, unlike T1 students, T2 students' collectively developing analogies may be naturally have forced them to explain their analogies.

Differently, the findings also revealed that students enriched their analogies with varying details without any teacher triggering. Students might have chosen to improve simple analogies when teachers did not validate their analogies or when they felt that simple ones were not apparent enough.

Besides, findings showed that nearly half of the student-generated analogies were extended. Interestingly, an examination of extended analogies revealed that the same students repeated base analogies continuously, which may indicate that these students have embraced these analogies. This examination also disclosed that students derived their analogies from student-generated analogies and teacher-employed analogies, which shows how the effects of all classroom analogies on student-generated analogies.

Results of the study showed that student-generated analogies infrequently included some analog explanations. This may be related to the fact that students did not need an extra analog explanation because they had already chosen analogs from their own experiences or from their teachers' previously used analogies, with which they were already familiar. In addition, unlike analogies generated by T1 students, analogies generated by T2 students had the analog explanation more frequently. This may be because T2 requested, each time insistently, his students for more explanations about their own generated analogies.

In addition, results showed that teachers identified the strategy of almost all student-generated analogies. However, interestingly, although there was no clear indication that students knew what an analogy is or how to construct an analogy, they were always expected to understand "analogy" when teachers used words such as examples or real-life situations instead of the word analogy. However, it is pretty natural since teachers seem not to know the difference between analogy and these words and use them unconsciously.

Results of the study revealed that students stated only two breakdown points. However, it seems that they only questioned the real-life compatibility of the chosen analogs, most probably without being aware of the limitations of analogies. This was reasonably expected since teachers did not stress that every analogy has limitations. This data suggests that teachers should first know the limitations of analogies, discuss the limitations of analogies with their students (Ubuz et al., 2009), and even encourage them to find the limitations of their analogies.

The current study revealed that, except for some analogies in Class B, all student-generated analogies in T1 and T2 classes were sound and included correctly identified analog and target attributes and epistemologically valid mappings between identified analog and target attributes. Most of the analogies in Class B were unsound and violated the correct identification of analog and target concepts. Interestingly, however, T1 did not warn her students, except a few, whether their analogies appropriately represent function-related concepts. This may be because T1 was unaware of whether these analogies were sound or unsound or completely overlooked them at the time. In addition, students could not select appropriate analogs may be because they could not understand analog or target concepts well. Alternatively, since the effectiveness of analogies in learning functions depends on students' ability to carry out the mapping process between analog and function-related concepts (Bayazit & Ubuz, 2008), it may be because students could not carry out this process. The worst of all, it may have resulted from students randomly constructing their analogies, just modeling the structure (for instance, input-output relation) of analogies generated by their teachers.

All of these suggest that requesting students to generate their own analogies without teaching them analogical reasoning can be risky for them not to understand the analog-target relationship and even make different judgments about the target concept.

Overall, differences were observed between the features and frequency of analogies generated by T1 and T2 students, most likely because of disparities in teachers' pedagogical practices and individual differences in students.

5.2 Limitations of Study

Despite every effort to be as complete as possible throughout the present study, several limitations need to be kept in mind in interpreting the results of this study. First of all, teachers participating in this research graduated from two distinguished universities in Turkey, taught in one of Turkey's leading educational institutions, had many years of experience in teaching function concepts, were interested in professional development, and were full of self-confidence. Similarly, students participating in this study attended one of Turkey's leading private schools and received training under the guidance of selected teachers. Although teachers and students in this study have not generated effective analogies in line with their features as mentioned above, features and effectiveness of analogies generated by teachers from different schools in Turkey or other countries with varying levels of content and pedagogical knowledge, values, and skills, and their students from diverse backgrounds and raised in dissimilar environments may vary. Analogies should be viewed as a natural product of diversity among teachers and students (Mastrilli, 1997). While this study offers valuable data on the analogy use of teachers and students and the features of both teacher- and student-generated analogies, it should be noted that they are only suggestive of a much broader range of possibilities that may only be encountered in many other classroom settings. For this reason, attention should be paid to its generalizability to teachers and students in other schools in Turkey or other countries.

Second, in this study, teachers’ use of analogy was limited to the function unit only. Perhaps they could have used more effective analogies in other mathematics content if they had been given a choice.

Third, the lesson hours in which the teachers taught the function unit differed. Although it is commonly challenging to create ideal conditions, different results could be obtained if T1 could start teaching the function unit simultaneously in both her classes and if T1 and T2 could teach the unit in the same class hour.

Fourth, it should be clarified that both teacher- and student-generated analogies emerged just when teachers and students shared aloud them with the class during teachers’ function unit teaching. While focusing only on the analogies created by the talkative students with this application, who knows, perhaps the analogies developed by the students and teachers together or by students while working collectively in a low voice were overlooked.

Fifth, the fact that the researcher was a colleague of the participating teachers and the teachers knew the main focus of the study might have caused them to pay more attention to the analogy use and explain the function-related concepts more carefully. Likewise, it may also have caused students to think the video recordings would somehow affect their mathematics performance grades.

Sixth and finally, although pilot recordings were made before the actual data collection process not to affect the data collection process adversely, the presence of cameras may have affected teachers’ teaching of the function unit, students’ reactions to these teachings, and their classroom behaviors. However, as the study progressed, the effects of the camera on teachers and students were most likely to have diminished.

5.3 Implications and Recommendations for Future Research

The findings of this study with two experienced mathematics teachers and their 121 ninth-grade students can be used to inform teacher education (stakeholders such as pre-service and in-service teachers, mathematics teacher educators, and policymakers) on the use of analogy in the natural classroom setting and the features of used analogies. Both teacher- and student-generated analogies discussed in this study will provide a source for pre-service and in-service teachers to structure and present their analogies. Namely, assertions and interpretations regarding the features of analogies reported in this study will assist pre-service and in-service teachers in improving or refining their analogy use, deepen their understanding of function concepts and help them foresee possible analogy-caused student misconceptions. Besides, proposing the following recommendations will shed light on pre-service and in-service teachers' implementation of analogy strategy in mathematics classrooms.

In light of the findings of this study and other studies, it is first recommended that teachers should have a repertoire of sound analogies that will help them increase their ability to create effective analogies during classroom practices. It is then suggested that they should use these analogies effectively in their everyday teaching to maximize their potential benefits and minimize their potential limitations.

To put the first suggestion into practice, teachers should know what analogies and sound analogies are. The analogies discussed in the current study will help teachers understand what analogies and their key components are and will guide them to develop sound analogies, including correctly identified analog and target attributes and epistemologically valid mappings. Teachers' analogical repertoire can be supported by examining manuscripts that document sound analogies for various mathematics subjects.

To implement the second recommendation, teachers should know how to use these analogies effectively in their everyday teaching. Teachers can achieve this by being aware of theoretical teaching models or frameworks and preplanning systematic analogy presentations based on them. These models include, for example, Zeitoun's (1984) *General Model for Analogy Teaching* (GMAT), Glynn, et al.'s (1989) *Teaching with Analogy* (TWA) model, and Treagust et al.'s (1998) Focus, Action, Reflection (FAR), Nashon's (2000) *Working with Analogies* (WWA) or Gray and Holyoak's (2021) more recently outlined five-principles analogical approach to teaching.

At this juncture, if teachers are open to professional developments, teacher preparation courses and in-service teacher training come into play for both suggestions. In such courses and training, attention should be drawn to analogies, and brief information should be given about what analogies and sound analogies are, their importance, advantages and disadvantages, and systematic teaching models or frameworks. Teachers should then be supported to develop a repertoire of sound analogies and practice these analogies so they can use them effectively.

For example, both groups of teachers may be asked to create their own analogies in a written or verbal form, then share them with other teachers and discuss the soundness of these analogies together. In addition, although it is not possible with in-service teachers, pre-service teachers may be asked to plan sample lessons in which they will use analogies on different mathematics topics, present these lessons to other teachers and the instructor, and evaluate them together. On the other hand, in-service teachers may similarly plan sample lessons and exchange these plans with their peers to assess each with written feedback. Alternatively, in-service teachers may only share their analogy repertoire of analogies they experienced or plan to experience in identified mathematics topics and allow them to comment on these analogies. Apart from these, case study videos may be watched by both groups of teachers, and face-to-face discussions may be made about features of analogies. In this sense, the framework developed in the present study can guide in determining the flow of these discussions, and interpretations regarding this study's findings can determine the discussion's course. Instead, teachers may watch case study videos online, and by answering the online questionnaire at the end of the video, they may be made to realize what sound analogies are and how they can be used effectively.

Furthermore, the findings of this study reveal a need for systematic use of analogies for analogies to be effective during classroom practices. Although it is not a prescription, based on the aforementioned systematic frameworks and the results of the current study, it can be suggested that teachers should (1) decide where and when it is better to use analogies, (2) select analogs considering students' everyday experiences and/or knowledge bases and related attributes of target concepts, (3) ensure that students are familiar with both analog and target concepts, (4) describe entirely the relevant shared and unshared attributes of analog-target pairs and mappings shared attributes between them verbally or both verbally and visually (decide when either is necessary, taking into account students' needs), (5) help students understand what an analogy strategy is with its advantages and possible disadvantages and assist them in recognizing analogies, and (6) engage students in analogy generation processes (invite students to find the shared and unshared attributes of analog-target pairs), and encourage them to generate their own sound analogies.

The extent to which teachers will carry out these suggested steps is a sample of their mathematical knowledge entailed by teaching (CCK, SCK, KCS, and KCT) as proposed by Ball et al. (2008) and a reflection of their attitudes and beliefs on analogy use. That is, teachers' having well-founded content knowledge (CCK and SCK) about the target concept, assuming knowing the analog domain well, may increase the possibility of choosing analogs compatible with the target and easily expressing correct shared and unshared attributes of the analog-target pair. At this point, mathematics courses and mathematical education courses at universities are responsible for developing content knowledge about target concepts as well as the teachers themselves. Instead of rote learning, teachers should be supported with lectures, materials, and questions to encourage learning by understanding. On the other hand, teachers' well-founded knowledge of student learning and the analogy method, in addition to content knowledge (KCS and KCT), along with their positive attitudes towards analogy strategy and beliefs that they are effective, may increase the likelihood of establishing more effective analogies. However, it may not be possible to do much with teachers' knowledge about the analog domain. By emphasizing the importance of the analog domain only in the aforementioned courses or training, teachers should be encouraged to choose analogs that students and they know well.

On the other side, interpretations and assertions related to student- and teacher-generated analogies reported in this study and the suggestions mentioned above for analogy implementation may provide a source for textbook authors about how to integrate analogies into their textbooks while guiding on how to present analogies in textbooks. Textbook authors should overtly map and articulate the limitations of analogies, considering that textbook analogies may inspire teachers in their teaching and may be a primary source for students when they cannot reach their teacher.

Similarly, interpretations and suggestions about analogy use may provide a source for curriculum designers. The time required for analogy use, which emerged as one of the limiting factors to the effective use of analogies, should be rearranged by curriculum designers to allow teachers to use analogies properly.

Furthermore, the new framework developed in this study for the classification of both teacher-and student-generated analogies during function unit teaching will assist future researchers in continuing to build and test the analogy features proposed here or examine the use of analogies in other mathematics contents other than functions or non-math contents.

To date, this is the only study to examine how both ninth-grade mathematics teachers and their students use analogies while teaching and learning the function unit in natural classroom settings and the features of these analogies. Therefore, it is unclear how both teacher- and student-generated analogies compare to analogies generated by teachers with varying degrees of experience and their students in other schools in Turkey or other countries. This will be an exciting topic for future research.

Further research may focus on the effects of students generating analogies individually and collectively. There is also a need to explore the impact of teachers' explaining the function unit by repeating or modifying the same analogy or generating various analogies. It is also necessary to investigate analogies that teachers develop by quoting from real-life or analogies that are artificially generated and told like a story on students' future recall. Similarly, the effects of teachers' clear explanation of the analogy strategy to their students on the later analogy use of students need to be explored. In addition, the impact of different types of visual analogies on students' learning can be investigated. Besides, when analogies are systematically preplanned and unplanned, the features of the resulting analogies can be explored and compared. Finally, developing manuscripts documenting sound analogies for function-related concepts based on assertions and interpretations drawn from the data in the current study will be among plans immediately after this study.

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APPENDICES

A. Consent Documents

T.C.
BEYKOZ KAYMAKAMLIĞI
İlçe Milli Eğitim Müdürlüğü

Sayı : 51550526/405.01/4769045 23/10/2014
Konu: Ted İstanbul Koleji Vakfı Özel Ortaokulu(Analiz)

KAYMAKAMLIK MAKAMINA

İlçemizde faaliyetde bulunan Ted İstanbul Koleji Vakfı Özel Ortaokulu Matematik Öğretmeni Emel Ünver SEZER'in, aynı zamanda ODTÜ Fen Bilimleri Enstitüsü Orta Öğretim Matematik Eğitim alanında doktora eğitimine devam ettiği ilgi yazı ile belirtilmektedir.

Adı geçen öğrencinin tez çalışmasında kullanmak üzere, Matematik alanında fonksiyon konusunda benzetim yönetimi kullanımı ve öğrencilerin fonksiyon konusunu öğrenmeleri ile bu alandaki öğrenme yaklaşımları arasındaki ilişkiyi incelemek üzere ve detaylı analiz etmek üzere okulun Hazırlık, 9. ve 10.sınıfların matematik dersinde ders esnasında izleyip, gözlemlerini görsel olarak (kamera yardımıyla) kayıt altına almak isteği Müdürlüğümüzce uygun görülmüştür.

Makamlarınızca da uygun görülmesi halinde olurlarınıza arz ederim.

Kazım BOZBAY
Müdür

OLUR
23/10/2014

Süleyman ERDOĞAN
Kaymakam

Beykoz ilçe Milli Eğitim Müdürlüğü
Gümüşsuyu Mah.Hükümet Konağı 34820 Beykoz/İstanbul
Santral :0216 331 1620---331 2741 dahili :(137-138-140-148)
Faks:0216 331 27 01 E.posta: beykoz34@meb.gov.tr

Bu evrak güvenli elektronik imza ile imzalanmıştır. http://evraksorgu.meb.gov.tr adresinden 6fee-16dc-3aeb-9702-3652 kodu ile teyit edilebilir.

Gir.Tar:23/10/2014
Sayi:

T.C.
BEYKOZ KAYMAKAMLIĞI
İlçe Milli Eğitim Müdürlüğü

Sayı : 51550526/405.01/4786685 23/10/2014
Konu: Analiz

TED İSTANBUL KOLEJİ VAKFI ÖZEL LİSESİ MÜDÜRLÜĞÜ'NE

İlgi:a) 14/10/2014 tarih ve 158 sayılı yazınız
b) 17/10/2014 tarih ve 405/4635727 sayılı yazımız
c) İl M.E.M.'nün 21/10/2014 tarih ve 405/4716470 sayılı yazısı
d) 23/10/2014 tarih ve 405.01/4769045 sayılı kaymakamlık onayı

Okulunuz Matematik Öğretmeni Emel Ünver SEZER'in Matematik alanında fonksiyon konusunda benzetim yönetimi kullanımı ve öğrencilerin fonksiyon konusunu öğrenmeleri ile bu alandaki öğrenme yaklaşımları arasındaki ilişkiyi incelemek üzere ve detaylı analiz etmek üzere okulunuzun Hazırlık,9.ve 10.sınıfların matematik dersinde ders esnasında izleyip,gözlemlerini görsel olarak (kamera yardımıyla) kayıt altına alma isteğine ait ilgi (d) Kaymakamlık Oluru ilişikte gönderilmiştir.

Bilgilerinizi ve gereğini rica ederim.

Necla ÇAĞLAYAN
Müdür a.
Şube Müdürü

Eki:1 adet

Beykoz ilçe Milli Eğitim Müdürlüğü
Gümüşsuyu Mah.Hükümet Konağı 34820 Beykoz/İstanbul
Santral :0216 331 16 20---331 27 41 dahili :(137-138-140-148)
Faks:0216 331 27 01 E.posta: beykoz34@meb.gov.tr

B. Observation Plan

1st Week- 10-13 March 2015

	MONDAY	**TUESDAY (10 March, 2015)**	**WEDNESDAY (11 March, 2015)**	**THURSDAY (12 March, 2015)**	**FRIDAY (13 March, 2015)**
1 (08:25-09:05)	**T1-9A** T2- 9C (08:50-09:28)	T2- 9C		T2- 9E	**T1-9B** T2- 9E
2 (09:12-09:52)	**T1-9A** T2- 9D (09:33-10:11)	T2- 9C		**T1-9A** T2- 9E	
3 (09:59-10:39)	**T1-9B** T2- 9D (10:16-10:54)	T2- 9E		T1-9A	
4 (10:46-11:26)	**T1-9B** T2- 9E (10:59-11:37)	T2- 9E	T2- 9E		T2- 9D
5 (11:33-12:13)		T1-9B	T1-9B	**T1-9B** T2- 9D	T2- 9C
6 (12:53-13:33)		T2- 9D	T1-9A	**T1-9B** T2- 9D	T1-9A
7 (13:40-14:20)		**T1-9A** T2- 9D	T2- 9C	T2- 9C	
8 (14:27-15:07)				T2- 9C	

NOTE: Differences in lesson periods for Mondays are shown on the timetable.
Time intervals are shown in grey for T1, green for T2, and yellow for both (T1 and T2).

2nd Week- 16-120 March 2015

	MONDAY (16 March, 2015)	TUESDAY (17 March, 2015)	WEDNESDAY (18 March, 2015)	THURSDAY (19 March, 2015)	FRIDAY (20 March, 2015)
1 **(08:25-09:05)**	**T1-9A** T2- 9C (08:50-09:28)	T2- 9C		T2- 9E	**T1-9B** T2- 9E
2 **(09:12-09:52)**	**T1-9A** T2- 9D (09:33-10:11)	T2- 9C		**T1-9A** T2- 9E	
3 **(09:59-10:39)**	**T1-9B** T2- 9D (10:16-10:54)	T2- 9E		T1-9A	
4 **(10:46-11:26)**	**T1-9B** T2- 9E (10:59-11:37)	T2- 9E	T2- 9E		T2- 9D
5 **(11:33-12:13)**		T1-9B	T1-9B	**T1-9B** T2- 9D	T2- 9C
6 **(12:53-13:33)**		T2- 9D	T1-9A	**T1-9B** T2- 9D	T1-9A
7 **(13:40-14:20)**		**T1-9A** T2- 9D	T2- 9C	T2- 9C	
8 **(14:27-15:07)**				T2- 9C	

NOTE: Differences in lesson periods for Mondays are shown on the timetable.
Time intervals are shown in grey for T1, green for T2, and yellow for both (T1 and T2).

	MONDAY (23 March, 2015)	**TUESDAY (24 March, 2015)**	**WEDNESDAY (25 March, 2015)**	**THURSDAY (26 March, 2015)**	**FRIDAY (27 March, 2015)**
1 (08:25-09:05)	**T1-9A** T2- 9C (08:50-09:28)	T2- 9C		T2- 9E	**T1-9B** T2- 9E
2 (09:12-09:52)	**T1-9A** T2- 9D (09:33-10:11)	T2- 9C		**T1-9A** T2- 9E	
3 (09:59-10:39)	**T1-9B** T2- 9D (10:16-10:54)	T2- 9E		T1-9A	
4 (10:46-11:26)	**T1-9B** T2- 9E (10:59-11:37)	T2- 9E	T2- 9E		T2- 9D
5 (11:33-12:13)		T1-9B	T1-9B	**T1-9B** T2- 9D	T2- 9C
6 (12:53-13:33)		T2- 9D	T1-9A	**T1-9B** T2- 9D	T1-9A
7 (13:40-14:20)		**T1-9A** T2- 9D	T2- 9C	T2- 9C	
8 (14:27-15:07)				T2- 9C	

NOTE: Differences in lesson periods for Mondays are shown on the timetable.
Time intervals are shown in grey for T1, green for T2, and yellow for both (T1 and T2).

	MONDAY (30 March, 2015)	**TUESDAY (31 March, 2015)**	**WEDNESDAY (1 April, 2015)**	**THURSDAY (2 April, 2015)**	**FRIDAY (3 April, 2015)**
1 (08:25-09:05)	**T1-9A** T2- 9C (08:50-09:28)	T2- 9C		T2- 9E	**T1-9B** T2- 9E
2 (09:12-09:52)	**T1-9A** T2- 9D (09:33-10:11)	T2- 9C		**T1-9A** T2- 9E	
3 (09:59-10:39)	**T1-9B** T2- 9D (10:16-10:54)	T2- 9E		T1-9A	
4 (10:46-11:26)	**T1-9B** T2- 9E (10:59-11:37)	T2- 9E	T2- 9E		T2- 9D
5 (11:33-12:13)		T1-9B	T1-9B	**T1-9B** T2- 9D	T2- 9C
6 (12:53-13:33)		T2- 9D	T1-9A	**T1-9B** T2- 9D	T1-9A
7 (13:40-14:20)		**T1-9A** T2- 9D	T2- 9C	T2- 9C	
8 (14:27-15:07)				T2- 9C	

NOTE: Differences in lesson periods for Mondays are shown on the timetable.
Time intervals are shown in grey for T1, green for T2, and yellow for both (T1 and T2).

6-10 APRIL MID-BREAK

	MONDAY (13 April, 2015)	**TUESDAY (14 April, 2015)**	**WEDNESDAY (15 April, 2015)**	**THURSDAY (16 April, 2015)**	**FRIDAY (17 April, 2015)**
1 (08:25-09:05)	**T1-9A** T2- 9C (08:50-09:28)	T2- 9C		T2- 9E	**T1-9B** T2- 9E
2 (09:12-09:52)	**T1-9A** T2- 9D (09:33-10:11)	T2- 9C		**T1-9A** T2- 9E	
3 (09:59-10:39)	**T1-9B** T2- 9D (10:16-10:54)	T2- 9E		T1-9A	
4 (10:46-11:26)	**T1-9B** T2- 9E (10:59-11:37)	T2- 9E	T2- 9E		T2- 9D
5 (11:33-12:13)		T1-9B	T1-9B	**T1-9B** T2- 9D	T2- 9C
6 (12:53-13:33)		T2- 9D	T1-9A	**T1-9B** T2- 9D	T1-9A
7 (13:40-14:20)		**T1-9A** T2- 9D	T2- 9C	T2- 9C	
8 (14:27-15:07)				T2- 9C	

NOTE: Differences in lesson periods for Mondays are shown on the timetable.
Time intervals are shown in grey for T1, green for T2, and yellow for both (T1 and T2).

	MONDAY	TUESDAY (10 March, 2015)	WEDNESDAY (11 March, 2015)	THURSDAY (12 March, 2015)	FRIDAY (13 March, 2015)
1st Lesson (08:25-09:05)		Selçuk İNCE- 9C Video (+)		Selçuk İNCE- 9E Video (-)	Selçuk İNCE- 9E Video (-) Preparation for the exam Canan GÜLER-9B Video (+)
2nd Lesson (09:12-09:52)		Selçuk İNCE- 9C Video (-) Class Work		Selçuk İNCE- 9E Video (-) Canan GÜLER-9A Video (+)	
3rd Lesson (09:59-10:39)		Selçuk İNCE- 9E Video (+)		Canan GÜLER-9A Video (-) **Chemistry Exam**	
4th Lesson (10:46-11:26)		Selçuk İNCE- 9E Video (-) **Literature Exam**	Selçuk İNCE- 9E Video (+)		Selçuk İNCE- 9D Video (-) Preparation for the exam
5th Lesson (11:33-12:13)		Canan GÜLER-9B Video (+)	Canan GÜLER-9B Video (+)	Selçuk İNCE- 9D Video (-) Canan GÜLER-9B Video (+)	Selçuk İNCE- 9C Video (-) Preparation for the exam
6th Lesson (12:53-13:33)		Selçuk İNCE- 9D Video (+)	Canan GÜLER-9A Video (-) **English Exam**	Selçuk İNCE- 9D Video (+) Canan GÜLER-9B Video (-)	Canan GÜLER-9A Video (+)
7th Lesson (13:40-14:20)		Selçuk İNCE- 9DVideo (+) (1.11 Dk. Class Work) Canan GÜLER-9A Video (+)	Selçuk İNCE- 9C Video (-) **English Exam**	Selçuk İNCE- 9C Video (+)	
8th Lesson (14:27-15:07)				Selçuk İNCE- 9C Video (+) Class Work	

NOTE: Differences in lesson periods for Mondays are shown on the timetable.
Time intervals are shown in grey for T1, green for T2, and yellow for both (T1 and T2).

	MONDAY (16 March, 2015)	TUESDAY (17 March, 2015)	WEDNESDAY (18 March, 2015)	THURSDAY (19 March, 2015)	FRIDAY (20 March, 2015)
1st Lesson (08:25-09:05)	Selçuk İNCE- 9C Video (-) Canan GÜLER-9A Video (-) (08:50-09:28)	Selçuk İNCE- 9C Video (-)		Selçuk İNCE- 9E Video (-)	Selçuk İNCE- 9E Video (+) **Canan GÜLER-9B Video (+)**
2nd Lesson (09:12-09:52)	Selçuk İNCE- 9D Video (-) Canan GÜLER-9A (09:33-10:11) Video (-) **Maths Exam** **(2nd Term -1st Mid-term Examination)**	Selçuk İNCE- 9C Video (-) **Biology Exam** **(2nd Term -1st Mid-term Examination)**		Selçuk İNCE- 9E Video (-) Canan GÜLER-9A Video (+)	
3rd Lesson (09:59-10:39)	Selçuk İNCE- 9D Video (-) Canan GÜLER-9B Video (+) (10:16-10:54)	Selçuk İNCE- 9E Video (+)		Canan GÜLER-9A Video (+)	
4th Lesson (10:46-11:26)	Selçuk İNCE- 9E Video (-) Canan GÜLER-9B Video (+) (10:59-11:37)	Selçuk İNCE- 9E Video (+)	Selçuk İNCE- 9E Video (-)		Selçuk İNCE- 9D Video (-)
5th Lesson (11:33-12:13)		Canan GÜLER-9B Video (+)	Canan GÜLER-9B Video (-)	Selçuk İNCE- 9D Video (+) Canan GÜLER-9B Video (-)	Selçuk İNCE- 9C Video (-)
6th Lesson (12:53-13:33)		Selçuk İNCE- 9D Video (+)	Canan GÜLER-9A Video (-)	Selçuk İNCE- 9D Video (+) Class Work Canan GÜLER-9B Video (-)	Canan GÜLER-9A Video (+)
7th Lesson (13:40-14:20)		Selçuk İNCE- 9D Video (+) Canan GÜLER-9A Video (-)	Selçuk İNCE- 9C Video (-)	Selçuk İNCE- 9C Video (+)	
8th Lesson (14:27-15:07)				**Selçuk İNCE- 9C Video (+)** **Class Work**	

NOTE: Differences in lesson periods for Mondays are shown on the timetable.
Time intervals are shown in grey for T1, green for T2, and yellow for both (T1 and T2).

	MONDAY (23 March, 2015)	**TUESDAY (24 March, 2015)**	**WEDNESDAY (25 March, 2015)**	**THURSDAY (26 March, 2015)**	**FRIDAY (27 March, 2015)**
1st Lesson (08:25-09:05)	Selçuk İNCE- 9C Video (-) Canan GÜLER-9A Video (+) (08:50-09:28)	Selçuk İNCE- 9C Video (+)		Selçuk İNCE- 9E Video (-)	Selçuk İNCE- 9E Video (-) (He is absent) Canan GÜLER-9B Video (+)
2nd Lesson (09:12-09:52)	Selçuk İNCE- 9D Video (-) Canan GÜLER-9A Video (+) (09:33-10:11)	Selçuk İNCE- 9C Video (-)		Selçuk İNCE- 9E Video (-) Canan GÜLER-9A Video (-)	
3rd Lesson (09:59-10:39)	Selçuk İNCE- 9D Video (-) Canan GÜLER-9B Video (+) (10:16-10:54)	Selçuk İNCE- 9E Video (+)		Canan GÜLER-9A Video (-)	
4th Lesson (10:46-11:26)	Selçuk İNCE- 9E Video (-) Canan GÜLER-9B Video (-) (10:59-11:37)	Selçuk İNCE- 9EVideo (+)	Selçuk İNCE- 9E Video (+)		Selçuk İNCE- 9D Video (-) (He is absent)
5th Lesson (11:33-12:13)		Canan GÜLER-9BVideo (+)	Canan GÜLER-9B Video (+)	Selçuk İNCE- 9D Video (+) Canan GÜLER-9B Video (+)	Selçuk İNCE- 9C Video (-) (He is absent)
6th Lesson (12:53-13:33)		Selçuk İNCE- 9D Video (+)	Canan GÜLER-9A Video (+)	Selçuk İNCE- 9D Video (-) Canan GÜLER-9B Video (-)	Canan GÜLER-9A Video (+)
7th Lesson (13:40-14:20)		Selçuk İNCE- 9D Video (-) Canan GÜLER-9AVideo (+)	Selçuk İNCE- 9C Video (+)	Selçuk İNCE- 9C Video (+)	
8th Lesson (14:27-15:07)				Selçuk İNCE- 9C Video (-)	

NOTE: Differences in lesson periods for Mondays are shown on the timetable.
Time intervals are shown in grey for T1, green for T2, and yellow for both (T1 and T2).

4th Week- 3

	MONDAY (30 March, 2015)	TUESDAY (31 March, 2015)	WEDNESDAY (1 April, 2015)	THURSDAY (2 April, 2015)	FRIDAY (3 April, 2015)
1st Lesson (08:25-09:05)	Selçuk İNCE- 9C Video (+) Canan GÜLER-9A Video (+) (08:50-09:28)	Selçuk İNCE- 9C Video (+)		Selçuk İNCE- 9E Video (+)	Selçuk İNCE- 9E Video (-) (Exercises) Canan GÜLER-9B Video (+)
2nd Lesson (09:12-09:52)	Selçuk İNCE- 9D Video (-) Canan GÜLER-9A Video (+) (09:33-10:11)	Selçuk İNCE- 9C Video (-)		Selçuk İNCE- 9E Video (-) Canan GÜLER-9A Video (+)	
3rd Lesson (09:59-10:39)	Selçuk İNCE- 9D Video (+) Canan GÜLER-9B Video (+) (10:16-10:54)	Selçuk İNCE- 9E Video (-)		Canan GÜLER-9A Video (-)	
4th Lesson (10:46-11:26)	Selçuk İNCE- 9E Video (+) Canan GÜLER-9B Video (+) (10:59-11:37)	Selçuk İNCE- 9E Video (+)	Selçuk İNCE- 9E Video (+)		Selçuk İNCE- 9D Video (-) (Exercises)
5th Lesson (11:33-12:13)		Canan GÜLER-9B Video (+)	Canan GÜLER-9B Video (+)	Selçuk İNCE- 9D Video (-) Canan GÜLER-9B Video (+)	Selçuk İNCE- 9C Video (-) (Exercises)
6th Lesson (12:53-13:33)		Selçuk İNCE- 9D Video (+)	Canan GÜLER-9A Video (+)	Selçuk İNCE- 9D Video (+) Canan GÜLER-9B Video (+)	Canan GÜLER-9A Video (+)
7th Lesson (13:40-14:20)		Selçuk İNCE- 9D Video (-) Canan GÜLER-9A Video (+)	Selçuk İNCE- 9C Video (-)	Selçuk İNCE- 9C Video (+)	
8th Lesson (14:27-15:07)				Selçuk İNCE- 9C Video (-)	

NOTE: Differences in lesson periods for Mondays are shown on the timetable.
Time intervals are shown in grey for T1, green for T2, and yellow for both (T1 and T2).

6-10 APRIL MID-BREAK

	MONDAY (13 April, 2015)	TUESDAY (14 April, 2015)	WEDNESDAY (15 April, 2015)	THURSDAY (16 April, 2015)	FRIDAY (17 April, 2015)
1st Lesson (08:25-09:05)	Selçuk İNCE- 9C Video (+) Canan GÜLER-9A Video (+) (08:50-09:28)	Selçuk İNCE- 9C Video (+)			
2nd Lesson (09:12-09:52)	Selçuk İNCE- 9D Video (+) Canan GÜLER-9A Video (+) (09:33-10:11)	Selçuk İNCE- 9C Video (+)			
3rd Lesson (09:59-10:39)	Selçuk İNCE- 9D Video (+) Canan GÜLER-9B Video (+) (10:16-10:54)	Selçuk İNCE- 9E Video (+)			
4th Lesson (10:46-11:26)	Selçuk İNCE- 9E Video (+) Canan GÜLER-9B Video (+) (10:59-11:37)	Selçuk İNCE- 9E Video (-) Class Work	Selçuk İNCE- 9E Video (+)		
5th Lesson (11:33-12:13)		Canan GÜLER-9BVideo (+)	Canan GÜLER-9B Video (-)		
6th Lesson (12:53-13:33)		Selçuk İNCE- 9DVideo (+)	Canan GÜLER-9A Video (+)		
7th Lesson (13:40-14:20)		Selçuk İNCE- 9D Video (-) Class Work Canan GÜLER-9A Video (+)	Selçuk İNCE- 9C Video (+)		
8th Lesson (14:27-15:07)					

NOTE: Differences in lesson periods for Mondays are shown on the timetable.
Time intervals are shown in grey for T1, green for T2, and yellow for both (T1 and T2).

D. Sample of Indexes Recording Analogies Generated by Teachers and Their Students

CLASS A

Date: 10.03.2015-7th Lesson

Code	Analog	Target (Content area of the target domain)	Page
***T1A1-1**	A machine producing tomato products	Function concept (1.1)	-
***T1A2-1**	Function machine	Function concept (1.1)	-

Date: 12.03.2015-2nd Lesson

Code	Analog	Target (Content area of the target domain)	Page
*T1A1-2 **(T1: "Last day we started with tomato fabric analogy")**	A machine producing tomato products	Function concept (1.1)	22
*T1A2-2	Function machine	Function concept (1.1)	23
T1A3	Earning money per working hour	Function concept (1.1)	23
T1A4	A machine producing tomato juice	Pre-image- Image (1.1)	26
T1A5	Birthday function	Univalence requirement (1.4)	27
T1A6	Tomato fabric	Dependent-independent variable relation (1.1)	29
T1A7	Calorie burning with a sports activity	Domain-range relation (1.1)	30
T1A8	Electrical energy use in a house	Domain-range relation (1.1)	30
T1A9	Function machine	Function concept (1.1)	34

Date: 13.03.2015-6th Lesson

Code	Analog	Target (Content area of the target domain)	Page
T1A10	A machine producing tomato juice	Domain-Range relation (1.3)	63
T1A11	Tomato fabric	Dependent-independent variable domain-range relation (1.3)	65

T1A12	Mother- child relation	Univalence requirement (1.4)	66
T1A13	A machine producing only tomato juice from tomato	Univalence requirement (1.4)	67
S22A14	Tomato fabric	Univalence requirement (1.4)	68
T1A15	Waiting to receive tomato and carrot juice from a machine producing only tomato juice	Univalence requirement (Not being a function) (1.4)	69
S11A16	Fruit juicer (Apple-apple juice)	Pre-image- Image (1.4)	69
S22A17	Fruit juicer (Mixed fruits- mixed fruit juice)	Pre-image- Image (1.4)	69
S21A18	Making cappuccino	Pre-image- Image (1.4)	69

Date: 19.03.2015-2nd Lesson

Code	**Analog**	**Target (Content area of the target domain)**	**Page**
There is NO ANALOGY			

Date: 19.03.2015-3rd Lesson

Code	**Analog**	**Target (Content area of the target domain)**	**Page**
T1A19	A machine producing tomato-aubergine mush	Function with two variables (1.5)	125

Date: 20.03.2015-6th Lesson

Code	**Analog**	**Target (Content area of the target domain)**	**Page**
There is NO ANALOGY			

Date: 23.03.2015- 1st Lesson

Code	**Analog**	**Target (Content area of the target domain)**	**Page**
There is NO ANALOGY			

E. Order of the Contents of the Function Unit

Chapters	Sections
Chapter 1: Function Concept	1.1 Introduction (definition)
	1.2 Functional notation
	1.3 Functions defined by equations
	1.4 Testing for functions
	1.5 Evaluating a function
	1.6 Finding domain of a function
	1.7 Polynomial function
	1.8 Rational function
	1.9 Irrational function
Chapter 2: Graphs of Functions	2.1 Graphs of functions
Chapter 3: Types of Functions	3.1 Constant function
	3.2 Identity function
	3.3 Linear function
	3.4 Piecewise function and its graph
	3.5 The absolute value function
	3.6 Polynomial function
	3.7 One to one function (1-1)
	3.8 Onto function
	3.1 Constant function

Code	Analog	Target (Location of the Analogy)	Nature of Shared Attributes	Presentation Format	Level of Abstraction	Position of Analog Relative to Target	Level of Enrichment	Extension of Mapping	Analog Explanation	Strategy Identification (Clue for Analogy)	Presence of Analogical Limitations	Soundness of Mapping
	Teacher – based (T) Student-based (S)		Functional (F) Structural and Functional (S&F)	Visual (Vs) Type (T1-T7) Abstractness (R/PR/UR) Verbal (Vb) Visual and Verbal (Vs &Vb)	Abstract-Abstract (A-A) Concrete–Abstract (C-A)	Advance Organizer (AO^1) / (AO^2) Embedded Activator (EA) Post-Synthesizer (PS)	Simple (S) Partially Enriched (P-En) Completely Enriched (C-En)	Extended (Ex) B/Der/Rep/Der-Rep Not Extended (NEx)	Evinced (E) Not Evinced (NE)	Evinced (E) Not Evinced (NE)	Evinced (E) General (G) Specific (S) Not Evinced (NE)	Sound (S) Unsound (UnS)
T1B1-1	Function machine	Pre-image-Image (1.1)	F	Vs &Vb (PR) (T6 was projected on the board)	C-A	AO^1	S	Ex (Base)	E	NE	NE	S
T1B2	A machine producing tomato juice	Pre-image-Image (1.1)	F	Vs &Vb (R) (T6 was projected on the board)	C-A	AO^1	P-En	Ex (Base)	E	NE	NE	S
T1B3-1	Input-output properties of a machine producing tomato sauce	Properties of the domain (1.1)	F	Vs &Vb (R) (T6 was projected on the board)	C-A	AO^1	C-En	Ex (Der) (Derived from T1B2) (T1: Look at the second example)	E (Explained while generating the former one T1B2)	NE	E	S

Code	Analog	Target (Location of the Analogy)	Nature of Shared Attributes	Presentation Format	Level of Abstraction	Position of Analog Relative to Target	Level of Enrichment	Extension of Mapping	Analog Explanation	Strategy Identification (Clue for Analogy)	Presence of Analogical Limitations	Soundness of Mapping
	Teacher – based (T) Student-based (S)		Functional (F) Structural and Functional (S&F)	Visual (Vs) Type (T1-T7) Abstractness (R/PR/UR) Verbal (Vb) Visual and Verbal (Vs &Vb)	Abstract-Abstract (A-A) Concrete– Abstract (C-A)	Advance Organizer (AO^1) / (AO^2) Embedded Activator (EA) Post-Synthesizer (PS)	Simple (S) Partially Enriched (P-En) Completely Enriched (C-En)	Extended (Ex) B/Der/Rep/Der-Rep Not Extended (NEx)	Evinced (E) Not Evinced (NE)	Evinced (E) Not Evinced (NE)	Evinced (E) General (G) Specific (S) Not Evinced (NE)	Sound (S) Unsound (UnS)
T1B4-1	Function machine converting each input to their square (T1: "This is also again the function machine")	Pre-image-Image (1.1)	F	Vs &Vb (UR) (T6 was projected on the board)	C-A	EA	P-En	Ex (Der) (Derived from T2B1) (T1: "We'll look at the another example. "	E	NE	NE	S
T1B3-2	Input-output properties of a machine producing tomato sauce	Properties of the domain (1.1)	F	Vb	C-A	EA	C-En	Ex (Der-Rep)	E	E- Example, Analogy	E	S
S7B5	Producing pickle juice from pickle	Function concept (1.1)	F	Vb	C-A	EA	S	Ex (Base)	NE	E-Teacher mentioned "Example, analogy"	NE	S

Table 2: List of a

Code	Analog	Target (Location of the Analogy)	Nature of Shared Attributes	Presentation Format	Level of Abstraction	Position of Analog Relative to Target	Level of Enrichment	Extension of Mapping	Analog Explanation	Strategy Identification (Clue for Analogy)	Presence of Analogical Limitations	Soundness of Mapping
	Teacher – based (T) Student-based (S)		Functional (F) Structural and Functional (S&F)	Visual (Vs) Type (T1-T7) Abstractness (R/PR/UR) Verbal (Vb) Visual and Verbal (Vs &Vb)	Abstract-Abstract (A-A) Concrete–Abstract (C-A)	Advance Organizer (AO^1) / (AO^2) Embedded Activator (EA) Post-Synthesizer (PS)	Simple (S) Partially Enriched (P-En) Completely Enriched (C-En)	Extended (Ex) B/Der/Rep/Der-Rep Not Extended (NEx)	Evinced (E) Not Evinced (NE)	Evinced (E) Not Evinced (NE)	Evinced (E) General (G) Specific (S) Not Evinced (NE)	Sound (S) Unsound (UnS)
T2C1	Machine converting each input to their three times and six more	Pre-image-image (1.1)	F	Vs &Vb (PR) (T6 was projected on the board)	C-A	EA	P-En	Ex (Base)	E (Diagram)	E- "… is like"	NE	S
T2C2	Machine converting each input to their square	Pre-image-image (1.1)	F	Vb	C-A	EA	P-En	Ex (Der) (Derived from T2C1)	E	NE	NE	S
T2C3	Specification profits of a hotel by month	Pre-image-image (1.1)	F	Vs &Vb (UR) (T1was drawn on the board)	C-A	EA	P-En	Ex (Base)	E (Graph)	E-"For example"	NE	S
T2C4-1	To get fired because of not specifying the profit of that month	Not to be a function (1.1)	S&F	Vb	C-A	EA	P-En	Ex (Der) (Derived from T2C3)	E	NE	NE	S

Code	Analog	Target (Location of the Analogy)	Nature of Shared Attributes	Presentation Format	Level of Abstraction	Position of Analog Relative to Target	Level of Enrichment	Extension of Mapping	Analog Explanation	Strategy Identification (Clue for Analogy)	Presence of Analogical Limitations	Soundness of Mapping
	Teacher – based (T) Student-based (S)		Functional (F) Structural and Functional (S&F)	Visual (Vs) Type (T1-T7) Abstractness (R/PR/UR) Verbal (Vb) Visual and Verbal (Vs &Vb)	Abstract-Abstract (A-A) Concrete–Abstract (C-A)	Advance Organizer (AO^1) / (AO^2) Embedded Activator (EA) Post-Synthesizer (PS)	Simple (S) Partially Enriched (P-En) Completely Enriched (C-En)	Extended (Ex) B/Der/Rep/Der-Rep Not Extended (NEx)	Evinced (E) Not Evinced (NE)	Evinced (E) Not Evinced (NE)	Evinced (E) General (G) Specific (S) Not Evinced (NE)	Sound (S) Unsound (UnS)
T2C5	Not to get fired, director should specify a profit value for each month	Univalence requirement (Rule 1) (1.1)	S&F	Vb	C-A	EA	P-En	Ex (Der) (Derived from T2C3)	E	NE	NE	S
T2C6-1	To get fired because of specifying two different profits for a month	Not to be a function (1.1)	S&F	Vb	C-A	EA	P-En	Ex (Der) (Derived from T2C3)	E	NE	NE	S
T2C7	The boss wants a clear data for each month	Univalence requirement (Rule 2) (1.1)	S&F	Vb	C-A	EA	P-En	Ex (Der) (Derived from T2C3)	E	E- "… in daily life"	NE	S
T2C4-2	To get fired for not specifying the profit	Not to be a function (1.1)	S&F	Vb	C-A	EA	P-En	Ex (Der-Rep) (Derived from T2C3)	E	NE	NE	S

		Teacher 1 (85)		Teacher 2 (84)		
		Class A	Class B	Class C	Class D	Class E
Category	Subcategories	27(100%)	58 (100%)	28 (100%)	27 (100%)	29 (100%)
Nature of Shared Attributes	Functional (F)	26 (96%)	52 (90%)	16 (57%)	19 (70%)	22 (76%)
	Functional and Structural (F&S)	1 (4%)	6 (10%)	12 (43%)	8(30%)	7 (24%)
Presentation Format	Visual (Vs)	0 (0 %)	0 (0 %)	0 (0 %)	0 (0 %)	0 (0 %)
	Verbal (Vb)	18 (67 %)	48 (83%)	26 (93%)	23 (85%)	26 (90%)
	Visual and Verbal (Vs &Vb)	9 (33%)	10 (17%)	2 (7%)	4 (15%)	3 (10%)
Level of Abstraction	Abstract-abstract (A-A)	1 (4 %)	5 (9%)	0 (0 %)	0 (0 %)	0 (0 %)
	Concrete– abstract (C-A)	26 (96 %)	53 (91%)	28 (100%)	27 (100%)	29 (100%)
Position of Analog Relative to Target	Advance Organizer (AO)	7 (26%)	13 (22%)	1 (4%)	3 (11%)	5 (17%)
	Advance Organizer 1 (AO^1)	2 (7%)	5 (9%)	0 (0%)	0 (0%)	0 (0%)
	Advance Organizer 2 (AO^2)	5 (19%)	8 (13%)	1 (4%)	3 (11%)	5 (17%)
	Embedded Activator (EA)	20 (74%)	42 (73%)	27 (96%)	24 (89%)	24 (83%)
	Post-Synthesizer (PS)	0 (0 %)	3 (5%)	0 (0%)	0 (0 %)	0 (0%)

		Teacher 1 (85)		Teacher 2 (84)		
		Class A	Class B	Class C	Class D	Class E
Category	Subcategories	27(32%)	58 (68%)	28 (33%)	27 (32%)	29 (%35)
Level of Enrichment	Simple (S)	9 (33%)	12 (21%)	3 (11%)	5 (19%)	4 (14%)
	Partly Enriched (P-En)	18 (67%)	44 (76%)	25 (89%)	22 (81%)	25 (86%)
	Completely Enriched (C-En)	0 (0%)	2 (3%)	0 (0%)	0 (0%)	0 (0%)
Extension of Mapping	Extended (Ex)	18 (67%)	47 (81%)	24 (86%)	23 (85%)	25 (86%)
	Not Extended (NEx)	9 (33%)	11 (19%)	4 (14%)	4 (15%)	4 (14%)
Analog Explanation	Evinced (E)	14 (52%)	39 (67%)	21 (75%)	21 (78%)	24 (83%)
	Not Evinced (NE)	13 (48%)	19 (33%)	7 (25%)	6 (22%)	5 (17%)
Strategy Identification (Clue for Analogy)	Evinced (E)	14 (52%)	26 (45%)	11 (39%)	21 (78%)	12 (41%)
	Not Evinced (NE)	13 (48%)	32 (55%)	17 (61%)	6 (22%)	17 (59%)
Presence of Analogical Limitations	Evinced (E)	0 (0%)	3 (5%)	1 (4%)	3 (11%)	2 (7%)
	Not Evinced (NE)	27 (100%)	55 (95%)	27 (96%)	24 (89%)	27 (93%)
Soundness of Mapping	Correct Transfer of Attributes (CTA)	27 (100%)	56 (97%)	27 (96%)	24 (89%)	29 (100%)
	Incorrect Transfer of Attributes (InCTA)	0 (0%)	2 (3%)	1 (4%)	3 (11%)	0 (0%)

Table 2: Results of AFF categorization of student-generated analogies in each classroom

		Classes of Teacher 1 (30)		Classes of Teacher 2 (16)		
		Class A	Class B	Class A	Class B	Class A
Category	Subcategories	6 (100%)	24 (100%)	4 (100%)	3 (100%)	9 (100%)
Nature of Shared Attributes	Functional (F)	4 (67%)	22 (92%)	4 (100%)	3 (100%)	7 (78%)
	Functional and Structural (F&S)	2 (33%)	2 (8%)	0 (0 %)	0 (0 %)	2 (22%)
Presentation Format	Visual (Vs)	0 (0 %)	0 (0 %)	0 (0 %)	0 (0 %)	0 (0 %)
	Verbal (Vb)	6 (100%)	24 (100%)	4 (100%)	3 (100%)	9 (100%)
	Visual and Verbal (Vs &Vb)	0 (0 %)	0 (0 %)	0 (0 %)	0 (0 %)	0 (0 %)
Level of Abstraction	Abstract-abstract (A-A)	2 (33%)	1 (4 %)	0 (0 %)	0 (0 %)	0 (0 %)
	Concrete– abstract (C-A)	4 (67%)	23 (96%)	4 (00%)	3 (100%)	9 (100%)
Position of Analog Relative to Target	Advance Organizer (AO)	1 (17%)	0 (0 %)	0 (0 %)	0 (0 %)	0 (0 %)
	Advance Organizer 1 (AO^1)	0 (0 %)	0 (0 %)	0 (0 %)	0 (0 %)	0 (0 %)
	Advance Organizer 2 (AO^2)	1 (17%)	0 (0 %)	0 (0 %)	0 (0 %)	0 (0 %)
	Embedded Activator (EA)	5 (83%)	24 (100%)	4 (100%)	3 (100%)	9 (100%)
	Post-Synthesizer (PS)	0 (0 %)	0 (0 %)	0 (0 %)	0 (0 %)	0 (0%)

		Classes of Teacher 1 (30)		Classes of Teacher 2 (16)		
		Class A	Class B	Class C	Class D	Class E
Category	Subcategories	6 (100%)	24 (100%)	4 (100%)	3 (100%)	9 (100%)
Level of Enrichment	Simple (S)	5 (83%)	19 (79%)	1 (25%)	2 (67%)	4 (44%)
	Partly Enriched (P-En)	1 (17%)	5 (21%)	3 (75%)	1 (33%)	5 (56%)
	Completely Enriched (C-En)	0 (0 %)	0 (0%)	0 (0%)	0 (0%)	0 (0 %)
Extension of Mapping	Extended (Ex)	6 (100%)	9 (38%)	0 (0%)	1 (33%)	5 (56%)
	Not Extended (NEx)	0 (0%)	15 (62%)	4 (100%)	2 (67%)	4 (44%)
Analog Explanation	Evinced (E)	1 (17%)	4 (17%)	3 (75%)	1 (33%)	4 (44%)
	Not Evinced (NE)	5 (83%)	20 (83%)	1 (25%)	2 (67%)	5 (56%)
Strategy Identification (Clue for Analogy)	Evinced (E)	3 (50%)	23 (96%)	4 (100%)	3 (100%)	7 (78%)
	Not Evinced (NE)	3 (50%)	1 (4%)	0 (0%)	0 (0%)	2 (22%)
Presence of Analogical Limitations	Evinced (E)	0 (0 %)	3 (12%)	0 (0%)	1 (33%)	0 (0 %)
	Not Evinced (NE)	6 (100%)	21 (88%)	4 (100%)	2 (67%)	9 (100%)
Soundness of Mapping	Correct Transfer of Attributes (CTA)	5 (83%)	9 (37%)	4 (100%)	3 (100%)	8 (89%)
	Incorrect Transfer of Attributes (InCTA)	1 (17%)	15 (63%)	0 (0 %)	0 (0 %)	1 (11%)

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EDUCATION

Degree	Institution	Year of Graduation
MS	METU Secondary Science and Mathematics Education	2009
BS	METU Elementary Mathematics Education	2006
High School	Anatolian Teacher Training High School, Amasya	2001

FOREIGN LANGUAGES

Advanced English

PUBLICATIONS

1. Unver-Sezer, E. (2021) What does “analogy” mean for students? https://eera-ecer.de/ecer-programmes/conference/26/contribution/51700/

2. Unver-Sezer, E., & Ubuz, B. (2018). Why do students remember some teacher-generated analogies more than others? https://eera-ecer.de/ecer programmes/conference/23/contribution/44439/

3. Unver, E. (2009). Analysis of analogy use on function concept in the ninth-grade mathematics textbook and classrooms Unpublished Master’s thesis. Middle East Technical University.

Wood painting, walking, dealing with plants, watching documentaries, and exploring new places.