A TEACHER’S LEARNING TO IMPLEMENT COGNITIVELY HIGH-LEVEL TASKS TO FACILITATE STUDENT THINKING THROUGH COACHING PROGRAM

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ABSTRACT

A TEACHER’S LEARNING TO IMPLEMENT COGNITIVELY HIGH-LEVEL TASKS TO FACILITATE STUDENT THINKING THROUGH COACHING PROGRAM

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In the present study, one of the purposes was to investigate an in-service teacher's (Aysu) knowledge of the cognitive demand of mathematical tasks in the algebra domain, particularly in the notion of slope, by engaging her in a mathematics coaching program. The second purpose was to examine the changes in the teacher's noticing skills and how her noticing skills progressed through the coaching stages, including planning, enacting, and review. Accordingly, coaching as a professional development model was designed by adopting a teaching experiment methodology. This study is conducted with an 8th-grade mathematics teacher and her students in a classroom environment in a public middle school. Different sources are utilized as data collection tools, such as the classroom sessions, teacher's pre- and post-observation interviews of coaching cycles, design team meetings, students' works, and coach's field notes as audio or video recordings. Data were analyzed by using qualitative methods. The findings revealed development in an in-service teacher's both knowledge of the levels of cognitive demand of tasks and her noticing of students' mathematical thinking throughout the coaching process. It was implied that core features of the coaching program within the teaching experiment which were collaboration, focus on specific content, cyclic process, and research-based materials, had an impact on the teacher's progress in her practices. In that respect, the study has insightful practical and theoretical implications for mathematics teacher educators, policymakers, and scholars in the mathematics education field.

Keywords: Teacher Noticing, Coaching, Slope, Students’ Algebraic Thinking, An In-Service Mathematics Teacher

ÖZ

BİR ÖĞRETMENİN KOÇLUK PROGRAMI VASITASIYLA ÖĞRENCİ DÜŞÜNÜŞLERİNİ İYİLEŞTİRME ADINA YÜKSEK BİLİŞSEL İSTEM DÜZEYİNDEKİ GÖREVLERİ UYGULAMAYA YÖNELİK ÖĞRENİMİ

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Bu çalışmanın amaçlarından biri, bir matematik öğretmeninin (Aysu) matematik koçluğu programına katılımı ile cebir alanında eğim konusundaki matematiksel görevlerin bilişsel istem düzeyleri ile ilgili bilgisini araştırmaktır. İkinci amaç, öğretmenin fark etme becerilerindeki değişiklikleri ve fark etme becerilerinin planlama, canlandırma ve gözden geçirme dahil olmak üzere koçluk aşamalarında nasıl ilerlediğini incelemektir. Bu doğrultuda bir mesleki gelişim modeli olarak koçluk, öğretim deneyi metodolojisi benimsenerek tasarlanmıştır. Bu araştırma bir devlet ortaokulunda 8.sınıf matematik öğretmeni ve öğrencileri ile sınıf ortamında gerçekleştirilmiştir. Veri toplama aracı olarak sınıf oturumları, öğretmenin koçluk döngülerinin ön ve gözlem sonrası görüşmeleri, tasarım ekibi toplantıları, öğrencilerin çalışmaları, koçun ses veya video kaydı olarak aldığı alan notları gibi farklı kaynaklardan yararlanılmıştır. Veriler nitel yöntemler kullanılarak analiz edilmiştir. Bulgular, bir öğretmenin koçluk süreci boyunca hem görevlerin bilişsel istem düzeylerine ilişkin bilgisinde hem de öğrencilerin matematiksel düşünüşlerini fark etmesinde bir gelişme olduğunu ortaya koymuştur. Koçluk programının öğretim deneyi içindeki temel özellikleri olan işbirliği, belirli içeriğe

odaklanma, döngüsel süreç ve araştırmaya dayalı materyallerin öğretmenin uygulamalarında ilerlemesini etkilediği görülmüştür. Bu bağlamda, çalışma matematik öğretmeni eğitimcileri, kural koyucuları ve matematik eğitimi alanındaki araştırmacıları için kapsamlı pratik ve teorik çıkarımlara sahiptir.

Anahtar Kelimeler: Öğretmenin Fark Etmesi, Koçluk, Eğim, Öğrencilerin Cebirsel Düşünüşleri, Matematik Öğretmeni

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LIST OF ABBREVIATIONS

MoNE	Ministry of National Education
NCTM	National Council of Teachers of Mathematics
TAG	Task Analysis Guide
MTF	Mathematical Task Framework

CHAPTER I

INTRODUCTION

A mathematical task is any mathematical activity to enable students to attain a predetermined mathematical idea (Stein, Grover, & Henningsen, 1996). "Mathematical tasks can be examined from a variety of perspectives, including the number and kinds of representations evoked, the variety of ways in which they can be solved, and their requirements for student communication" (Stein, Smith, Henningsen, & Silver, 2000, p. 11). Based on the different nature of mathematical tasks students engage in and the requirements of the tasks, researchers also examine students' mathematical thinking levels while solving tasks involving different cognitive demands. The cognitive demand of tasks is defined as "cognitive processes students are required" to participate in while working on tasks (Doyle, 1988, p.170). The Task Analysis Guide (Stein & Lane, 1996; Stein & Smith, 1998) portrays categorizations of students' mathematical thinking levels and the properties of each level. The guide classifies mathematical tasks into three basic categories: Low-level, High-level, and Unsystematic explorations. Low-level tasks are divided into two layers as memorizations and procedures without connection, and high-level tasks are divided into two layers as procedures with connection and doing mathematics. Compared to low-level tasks on recalling facts and applying a procedure, high-level tasks enable students to create different solutions and hypotheses, test and elaborate on their solutions, and connect prelearned mathematical ideas (e.g., Boaler & Staples, 2008). The third category, unsystematic exploration (Stein & Lane, 1996), refers to a task that might have the potential for higher-level thinking. However, students work with the task by developing an unsystematic approach that leads to the inhibition of the understanding of the concept.

The NCTM's (1991) recommendations for teachers about selecting and implementing high-level tasks named "worthwhile mathematical tasks" (p. 25) showed the importance of applying high-level cognitive tasks. NCTM (2000) kept emphasizing the necessity of worthwhile and high-level mathematical tasks in Principles and Standards for School Mathematics:

> In effective teaching, worthwhile mathematical tasks are used to introduce important mathematical ideas and to engage and challenge students intellectually. Well-chosen tasks can pique students' curiosity and draw them into mathematics... Regardless of the context, worthwhile tasks should be intriguing; with a level of challenge that invites speculation and hard work (p. 16-17).

Based on this assertion, high-level tasks can be considered as gateways for drawing students' attention on tasks and doing mathematics that challenges them to produce mathematical ideas. In line with this, in Turkey, the Ministry of National Education (MoNE, 2018) has recommended that mathematics teachers should use cognitively demanding tasks in order to make students doers of mathematics based on "mediational role of tasks" in teaching and learning mathematics (Johnson, Coles, & Clarke, 2017, p.815). As revealed by previous studies, the use of worthwhile tasks significantly affects students' conceptual understanding and achievement (Tarr, Reys, Reys, Chavez, Shih, & Osterlind, 2008).

Another important feature of tasks is that tasks can be at multiple levels: (a) the task as selected or designed characterizes the potential cognitive level students are intended to engage; (b) the task as set up by the teacher characterizes the boundary of intellectual load for students; c) the task as worked through by students individually and enacted by the teacher and students represents the actual intellectual efforts of students (Smith & Stein, 1998; Tekkumru-Kisa, Schunn, Stein, & Reynolds, 2019; Otten & Soria, 2014); and (d) the task as assessed characterizes the intellectual products students are responsible for knowing and making sense of (Doyle, 1988). The continuum shows that tasks present a window to see what teachers do in actual classrooms, and how this mechanism affects students' understanding. Tekkumru-Kısa and colleagues (2019) point to the significance of the tasks, which “set the parameters for what is possible” in terms of the kinds of thinking students might engage in. Thus, selecting, designing, and modifying tasks for teaching is recognized as an essential aspect of teaching (Tekkumru-Kisa et al., 2019, p.3). Therefore, teachers’ decisions on which tasks will be used in lessons and their knowledge and abilities in setting up and implementing tasks without a decline in intellectual demand are considered to be the crucial issues (Boston, 2013; Chapman, 2013, Smith, Bill & Hughes, 2008; Stein & Kaufman, 2010).

Considering the crucial role of teachers on task nature, some studies aimed to investigate teachers’ capacity to select and enact high-level mathematical tasks (Arbaugh & Brown, 2005; González & Eli, 2015; Graven & Coles, 2017; Jackson, Garrison, Wilson, Gibbons, & Shahan, 2013; Lozano, 2017; Silver, Mesa, Morris, Star, & Benken, 2009; Sullivan, Clarke, Clarke & O’Shea, 2010); Ubuz & Sarpkaya, 2014). The findings of these studies pointed to the difficulties teachers encounter in recognizing task nature and implementing high-level tasks. Specifically, they demonstrated that teachers typically select and categorize tasks depending on the superficial characteristics of tasks such as whether the task includes a real-life context, technology, or diagram and representations; the mathematical content of the task, the length of the task; and task difficulty. The findings showed that teachers do not relate tasks to students’ mathematical thinking (Arbaugh & Brown, 2005; Osana, Lacroix, Tucker, & Desrosiers, 2006; Tekkumru-Kısa, Stein, & Doyle 2020). In other words, previous studies showed that teachers lack essential skills to create, modify, select, and implement high-level mathematical tasks with potentials to promote students’ conceptual learning (Tekkumru-Kısa, Stein, & Doyle 2020).

Given the focus on the enrichment in teachers’ knowledge of mathematical tasks, researchers have started to investigate how this may be possible (Clarke & Roche, 2018; Guberman & Leikin, 2013; Tekkumru-Kısa & Stein, 2015). Providing a guide, such as the Task Analysis Guide (TAG), and requiring teachers to use it when classifying tasks is an effective strategy for enhancing teachers’ capacity and knowledge about tasks (Arbaugh & Brown, 2005; Boston, 2013; Boston & Smith, 2009; Boston & Smith, 2011; Estrella, Zakaryan, Olfos, & Espinoza, 2020). Another strategy is presenting worthwhile tasks to teachers (e.g., Guberman & Leikin, 2013). High-level tasks and how they could be implemented may be analyzed using protocols; the level of tasks may be discussed with other teachers; sample student work may be analyzed, and experience in the field may be scaffolded. The studies further revealed that teachers familiar with worthwhile tasks and participate in related professional development activities have positive perceptions about using high-level tasks. Through professional development activities, they become aware that high-level tasks lead to high-level student understanding, and they can better plan and implement high-level tasks (Boston & Smith, 2009; Parrish, Snider, & Creager, 2022). Although these studies provide a substantial insight into how improvement in mathematics task knowledge improves teachers' planning, initiation, and execution of high-level tasks, they lack a consistent analysis of instructors' actual classroom instruction. Hence, these studies could not provide strong evidence regarding teachers' abilities to maintain academic rigor of tasks during implementation. Therefore, this study aimed to investigate how a practicing teacher's knowledge of the cognitive demand of mathematical tasks changes through planning and teaching in a real classroom environment by an intervention of coaching program. In these coaching program, considering the effectiveness of the TAG in discussing narrative cases for implementing tasks (low or high levels), I used it to evoke teacher's awareness of the cognitive demand of tasks at each level.

In addition, previous studies have primarily focused on mathematical tasks without concentrating on a particular mathematical idea or content (e.g. Chrambalous, 2010; Choppin, 2011; Wilhelm, 2014). Chrambalous (2010) suggested a shift toward teachers' task knowledge on specific tasks within a content strand. Based on this suggestion about the exploration of teachers' capacity to select and implement task design within a specific content, the concept of slope was selected for this study. Apart from necessity to focus on a single idea or topic within the context of mathematical task design, focusing on a single content domain might provide more clear understanding of a teacher's learning of task desing in a more systematic way by eliminating other factors related to task desing such as specialized content knowledge (Wilhelm, 2014). Moreover, a need to focus on tasks regarding the slope concept due to its chracteristics, teachers', and students' conceptualizations on it will be presented in the next section.

1.1. The Concept of Slope

The slope is interconnected to other concepts and disciplines in a complex manner (Peck, 2020), and thus, teachers may struggle to recognize various conceptualizations of slope and select and implement high-level algebra tasks (e.g., Demonty, Vlassis & Fagnant, 2018; Magiera, van den Kieboom & Moyer, 2013; Nagle, Moore-Russo, & Styers, 2017; Rule & Hallagan, 2007; Steele, Hillen, & Smith, 2013; Wilkie, 2016). Slope is considered to be a vital notion to be enlightened through task design.

Slope is composed of sub-constructs as "rate of change, physical property (steepness), geometric ratio (the ratio of rise to run), algebraic ratio (the ratio of the change in y to the change in x), parametric coefficient (the a in the equation, $y = ax + b$)" (Stump, 1999, p. 129), "real-world situations" (Stump, 2001b, p. 81), "determining property (parallel and perpendicular lines), behavior indicator (the line increasing or decreasing or constant), and linear constant (the straightness of a line)" (Nagle & Moore-Russo, 2013, p. 3). On the basis of these sub-constructs and their interconnection with the representations (tabular, algebraic, graphical), some conceptualizations of slope can be expressed by a single representation or several representations (Peck, 2020). As an example of the former, graphs include the meanings of slope as geometric ratio and slope as functional property, in other words, slope as steepness and slope as rate (Tierney & Monk, 2007). In the latter one, slope is demonstrated in tabular and symbolic form as an algebraic ratio. Some studies investigated how middle and high school students learn the subcomponents of slope in multiple contexts (discrete or continuous, static or dynamic) and multiple representations (tabular, graphical, verbal) (Deniz &

Tangül-Kabael, 2017; Nagle, Martínez-Planell & Moore-Russo, 2019; Nathan & Kim, 2007; Peck, 2020).

This complicated nature of slope creates challenges about learning and teaching the notion, which can be classified into two categories: challenges related to the meaning of the slope as a measure, quotient, steepness, and covariational reasoning (e.g., Byerley & Hatfield, 2013; Byerley & Thompson, 2017; Coe, 2007; Lobato & Siebert, 2002; Stump, 2001b; Thompson, 1994; Thompson et al., 2017; Zandieh & Knapp, 2006) and challenges associated with constraints of transition between slope representations as algebraic, tabular, graphical, and verbal (e.g., Ayalon, Watson, & Lerman, 2016; Ellis, 2011; Lobato**,** Ellis, & Muñoz, 2003; Reiken, 2009; Wilkie, 2016; Zazlavsky et al., 2002). Noticing these challenges within a specific content domain (slope) is a key to planning and enacting high-level mathematical tasks (Choppin, 2011). Although teachers' attention and perception on how to support students who had difficulty in understanding mathematical concept behind the task have been less focused, Wilhelm (2014) highlighted that the teachers' conceptions on how to reinforce struggling students was strongly related to maintain the cognitive load of high demanding tasks. Hence noticing of both students' conceptualizations and struggles on slope concept under the algebraic thinking has a main role to select and implement high level slope tasks.

1.2. Teacher Noticing

Teacher noticing has attracted the attention of mathematics researchers and teacher educators following the study of Mason (2002) and van Es and Sherin (2002). Several studies have demonstrated the importance of noticing in enhancing teachers' professional vision of teaching to improve their practices (Goldsmith & Seago, 2011; Jacobs & Spangler, 2017; Star et al., 2011), and in enhancing student thinking (Amador et al., 2021). Many researchers have worked to reveal the elements of the noticing skills and have concentrated on issues about supporting the development of teacher noticing (e.g., Choy, 2016; Güner & Akyüz, 2019;

Prediger, Quasthoff, Vogler, & Heller, 2015; Stahnke, Schueler, & Roesken-Winter, 2016; Star & Strickland, 2008; Tekin-Sitrava, Kaiser, & Işıksal-Bostan, 2021; van Es & Sherin, 2008; Warshauer, Starkey, Herrera, & Smith, 2021). There have been variations in the definition and characterization of noticing among researchers, which led to disparities in its measurement and development. For instance, some researchers developed frameworks for learning to notice within a trajectory (e.g., van Es, 2011), whereas others defined noticing as "how and the extent to which teachers notice children's mathematical thinking" (e.g., Jacobs et al., 2010, p. 171) within differentiated components (attending, interpreting, and responding) (Choy, 2016). Recently, authors have characterized noticing in different contexts by focusing on classroom artifacts, including teacher practice, tools, and students' thinking (Kaiser et al., 2015; Santagata & Yeh, 2016), on different aspects of specific practices during lesson planning, teaching and reflecting (Choy, 2016), or on situation-specific awareness in the moment of teaching (Jazby, 2020; van Es & Sherin, 2021). All these perspectives seemed to agree that noticing is a multidimensional professional vision, skill, or perception. Following the recent contexts for noticing, the present study builds on Doyle's (1988) concept of cognitive demands of mathematical tasks and van Es (2011)'s concept of "learning to notice" which elaborates on how an experienced teacher's noticing changes as they plan, implement, and reflect on mathematics tasks in their own lessons. Van Es (2011) described noticing solely within two dimensions of noticing: What Teachers Notice and How Teachers Notice. These dimensions have four levels: Level 1-Baseline, Level 2- Mixed, Level 3-Focused, and Level 4-Extended. The first dimension is about *what* teachers attend, e.g., whole class learning environment, students as a group, specific students, the teacher, or themselves, and *the topic* of the focus, e.g., pedagogical decision, behavior, or thinking. The second dimension is how teachers interpret what they notice, including both *analytic stance* (evaluating and interpreting) and *depth of the analysis* (providing shreds of evidence or elaborating on their critiques). On the other hand, Jacobs and collegues (2010) developed a framework by paying less attention to the diversity of what teachers see and more attention to how and to what degree teachers notice student's mathematical thinking. In other words, the framework by van Es concentrated on variety of what teachers notice and how they make sense of what they noticed thorugh four levels with a perspective of reflection on action whreas focus of the framework by Jacobs and collegues was on children's mathematical thinking specifically. Within the high-level slope tasks context, one of the purposes of the current study was examine what the teacher attend and how to make sense of the attended issues through not only reflection on action but also reflection in action and planning. In that sense, the efforts to expand the focus of attention beyond a unique emphasis on students' thinking were given to incorporate whole lessons including attending important classroom situations and decision making about lesson continuations of lessons and alternative student-teacher interactions. Hence our focus is both detect variety of what teacher attends during teaching and what the teacher attends to specific students' thinking after the teaching. In that sense, based on Jacobs et al.'s (2010) definition and way of assessing of noticing grounded on specific exclusive emphasis on students' thinking, and two main dimensions (what teacher notice and how teacher notice) through four levels proposed by van Es's was integrated into explorations of noticing of the teacher in the present study. However, some adaptations were employed to understand what elements of teacher noticed while planning, teaching, and reflecting on whole lessons and specific students' thinking and how the teacher made sense of those within cognitively high mathematics tasks context.

Studies on understanding teachers' noticing skills of student thinking of slope indicated that preservice and in-service teachers have struggled to attend to and interpret critical ideas in a sophisticated way (e.g., Callejo & Zapatera, 2017; Lee & Lee, 2021). For instance, Styers, Nagle, and Moore-Russo (2020) found that teachers interpreted ideas related to steepness as linked to real-world situations by isolating its mathematics aspect. Although they were aware of the differences between sample students' thinking, including ratio-nonvisual (geometric ratio) and ratio-visual (algebraic ratio), they could not connect physical situation (static) to the notion of rate of change. Several researchers have focused on enhancing teachers' awareness through video clips, students' artifacts, and high-level activities to broaden students' ideas and enhance the quality of slope instruction (La Rochelle et al., 2019; Walkoe, Sherin & Elby, 2020; Walkoe, 2015). However, despite the fact that these studies yielded promising methods to improve teachers' noticing of students' thinking or mathematics instruction, the methods (use of video-taped lessons or students' written work) might create some concerns about discrepancies between what and how teachers notice in a classroom setting and a setting created by a researcher through videos and written student work. Another concern is researchers' perspective which might affect teachers' attention (Sherin, Russ & Colestock, 2011; Choy, 2016). In other words, researchers mainly focused on how teachers reflect *on action* rather than how they reflect *in action* (Schön, 1991). Teachers need to sustain productive noticing (Spitzer et al., 2011) of task design and students' thinking to perform high-quality instruction. Considering this need, recent studies have begun to expand the boundaries of teacher noticing, including during lesson design (Amador, Males, Earnest, & Dietiker, 2017), lesson implementations, and lesson reflections. If teachers' main aim is to encourage students' mathematical reasoning, they need to attend and elaborate on students' thinking, which emerges from a classroom dialogue or which is evident in written works, using a mathematical and pedagogical perspective before, during, and after each lesson. They also need to pay attention to task design since students' understanding is influenced by how teachers plan, enact and reflect on the designed mathematical tasks (Smith & Stein, 2011).

The skill of deciding is the most challenging skill even for experienced teachers (Lee & Choy, 2017) due to the sophistication of teachers' in-the-moment decision-making process during complicated classroom events (Choy, 2016; Lee & Francis, 2018; Stahnke, Schueler, & Roesken-Winter, 2016). It is important to be aware of students' thinking patterns and to select critical ones from these student thoughts and to give appropriate responses while performing high-level tasks (Van Zoest et al., 2017). Thus, how we can increase teachers' attention, noticing and decision-making skills in ways that will create a more ambitious learning environment as they plan, implement and reflect on high-level algebra tasks in their own lessons are questions waiting to be explored. Coaching is one of the professional development models that can provide an efficient strategy to enrich teachers to implement high-quality instruction (Kraft, Blazar, & Hogan, 2018; Sailors & Price, 2015).

1.3. Coaching as a Professional Development

Previous research has demonstrated that coaching is an effective approach for improving teaching and learning (Ellington et al., 2017; Knapp, Moore & Barrett, 2014; Kraft, Blazar, & Hogan, 2018; Sailors & Price, 2015; Yopp, Burroughs, Sutton, & Greenwood, 2017). Teacher coaching has also been described as a sort of implementation support (Devine, Meyers, & Houssemand, 2013) or an instrument for fostering student learning (Russo, 2004). Coaching styles, on the other hand, are categorized as “responsive” (Dozier, 2006) and “directive” (Deussen, Coskie, Robinson, & Autio, 2007). The former is concerned with reflecting on teacher practice, while the latter focuses on communicating directly with teachers about practice. Three basic coaching approaches, cognitive, content-focused, and instructional, were developed based on diverse views (Barlow, Burroughs, Harmon, Sutton, & Yopp, 2014). These methods share a common premise: coaches are more qualified colleagues who can collaborate with teachers on challenging practices (Cobb & Jackson, 2011). More specifically, mathematics coaching can be defined as a collaborative process that focuses on mathematics content and pedagogy to enhance teachers' practices in the present study.

In general, coaching is site-based, “sustained, individualized, intensive, context-specific, and focused” (Kraft, Blazar, & Hogan, 2018, p. 553) and includes a cyclic process of three main phases: pre-observation, observation, and post-observation (Russell, Correnti, Stein, Thomas, Bill, & Speranzo, 2019, McGatha et al., 2018). In addition, coaching activities vary widely. They may affect the effectiveness of coaching, cause obstacles for researchers, and hinder professional development (Gibbons & Cobb, 2016). In studies, the function of coaches may also be ambiguous. Recent studies have attempted to generate coaching frameworks based on the cyclic process despite the methodological challenges for employing the process as they aimed to provide empirical evidence regarding the affordances of cyclic process in teacher learning. Noticing encompasses not only teachers' attention to classroom interactions, but also their thinking, reflections, and choices in light of what they observed (McDuffie et al., 2014). The nature of coaching is ongoing and intense, and the cyclic process may help a teacher to learn to focus on tasks' nature, anticipating students' responses, making sense of, and reflecting on teaching practice in classrooms with a continuous discussion process. These features can enable coaches to support teachers' noticing. The present study is built on the commonalities between the descriptions and purposes of the three main phases of coaching and boundaries of noticing, including anticipating students' thinking through tasks, implementing (Smith & Stein, 2011), and reflecting on tasks.

As previously stated, the cyclical process consists of pre-observation (planning), observation (teaching), and post-observation (debriefing) phases (Bay-Williams, McGatha, Kobett, & Wray, 2014). During the pre-observation phase, coaches provide suggestions on pedagogy and practice and collaborate with instructors to plan a class. During the observation phase, on the other hand, the coach collects evidence. In the final round of the process, post-observation, teachers are urged to express their thoughts on the implemented lesson (West & Staub, 2003; West, 2009). Based on the cyclical process, numerous coaching models and frameworks have been developed (Campbell & Griffin, 2017; Russell et al., 2019). For instance, Russell and coworkers (2019) developed a mathematics coaching framework that includes the Discussion Process and coaching cycles with a specific emphasis on cognitively challenging mathematical tasks. In particular, the Discussion Process begins with the establishment of a mathematical objective and the selection of an associated task. The phases of pre-observation (planning) conference, lesson observation, and post-observation (feedback) conference comprise the successive parts of the Discussion Process. The pre-observation conference (planning) allows coaches and teachers to explore in greater detail the relationships between tasks, pedagogy, and students' thought processes. During lesson observation (teaching), both instructor and coach collect data on students' thought processes and instructional strategies. The coach's primary responsibility is to identify the strengths and shortcomings of teachers' instruction and students' reasoning. At this point, the differing roles of the coach during instruction can be discussed (West, 2009). In some cases, both the teacher and the coach may teach in the classroom setting. In the post-observation (reviewing) conference, the coach and teacher determine whether the target was met or not while teaching by using evidence-based reasoning, after which they review the aim for the next lesson. During the Discussion Process, predicted student thinking, misconceptions, and associated tools are also taken into account (Smith et al., 2008; Stein et al., 2008). It assists teachers in practicing their responses to students' questions or thinking. This strategy involves in-depth exchanges between the instructor and coach, which may improve the instructor's teaching ability (Russell et al., 2021). According to Walkoe (2015) and Choppin (2011), who suggest a potential reciprocal relationship between the ability to plan and implement high-level tasks and the ability to notice students' thinking, specialized coaching on rigorous tasks has the potential to encourage teachers' expertise in noticing when implementing high-level slope tasks. In this study, this framework was utilized due to its special focus on rigorous mathematics tasks within three cycles and to provide a much-needed theoretical foundation for coaching programs including the nature of teacher-coach interactions. The present study was framed considering the three components of the coaching cycle (Russell et al., 2020) and the triad nature of noticing (Amador et al., 2017; Bakker et al., 2022; Choy et al., 2017), which encompasses the stages of planning, teaching, and reviewing.

Studies have examined various activities or strategies that coaches employ when supporting teacher development (Ellighton et al., 2017; Gibbons & Cobb, 2016; 2017; Gibbons, Kazemi & Lewis, 2017; Hopkins, Ozimek & Sweet, 2017; Mudzimiri et al., 2014; Munson 2017; Neuberger, 2012; Polly, 2012). The studies showed how coaching impacts teachers’ discursive learning and practice from a general perspective on coaching roles. Another line of studies attempted to investigate specific knowledge and skills related to teaching and practice, such as noticing or modeling tasks (Aygün & Işıksal Bostan, 2019; Jakopovic, 2021; Jung

& Brady, 2016; Reinke et al., 2021). Despite this progress, two crucial issues have emerged regarding coaching programs. The first issue is that the mathematics coaching research gives little consideration to systematic and intense data (limited number of cycles of observing teachers' enactment in classrooms). Auletto and Stein (2020) have highlighted the need for in-depth and qualitative research on coaching through substantial observation of teachers' instruction in real classroom environments and its impact on teachers' practices to elicit teachers' noticing. Based on the call for in-depth studies on analyzing the impacts of coaching on teacher's practices in classrooms, the interest in how coaching activities support teacher noticing has increased. This support can be formed via reflecting on high-level tasks, videos, and frameworks that might encourage teacher expertise in noticing students' thinking when implementing high-level slope tasks. In fact, tools such as frameworks can challenge teachers to evalaute the relationship between teaching attempts and student learning and thinking (van Es, Tekkumru-Ksa, & Seago, 2020, p. 37), which helps to maintain an inquiry stance requiring "active teacher participation in meaning making around shifts in practice" (Russell et al., 2019, p.6). The second issue is that studies on coaching programs relied on the effects of these programs on non-specific mathematics topics and notion; however, more information regarding teacher-coach interaction within a specific mathematics domain is also crucial (Russell et al., 2019; Stein et al., 2022) to understand specific coaching activities for specific mathematic domains. Along with postulated needs and suggestions, I intended to portray how coaching activities offer a context to enhance a middle school teacher's noticing skills of algebraic thinking to contribute to the literature on the impacts of one-to-one coaching practices on teachers' noticing skills within a specific content (cognitively high slope tasks) through coaching components.

1.4. Research Questions

The goals of this study are multifaceted: (1) to document the changes in an experienced in-service teacher's knowledge of the cognitive demands of mathematical tasks through her participation in a coaching program, (2) to examine the developments in what an experienced in-service teacher attends to and how she makes sense of her attention through the coaching stages including planning, enacting and review.

The study aimed to address the following research questions:

1. In what ways does the teacher's knowledge of the cognitive demands of the mathematical tasks change following her participation in a coaching program on selecting/adapting mathematical tasks?

2. How does the teacher' noticing of 8^{th} graders' algebraic thinking, specifically slope concept develop through coaching cycles within cognitively high mathematical task context?

1.5. Significance of the Study

Research demonstrates the difficulty mathematics teachers experience in sustaining the rigor of tasks during instruction (Stein, Grover, & Henningsen, 1996; Stigler & Hiebert, 2004). Research also identifies the substantial impact of professional development on practicing teachers' selection and implementation of high-level tasks (Boston & Smith, 2009). These professional development efforts included either student work as classroom artifacts or making teachers to assess the cognitive demands of mathematical tasks and analyze the implementation of mathematical tasks by other teachers (Arbaugh & Brown, 2005; Boston, 2013; Boston & Smith, 2011). Practicing teachers also have a challenge in recognizing Procedure with Connection tasks at the end of the professional development attempts; hence, researchers suggest new designs, including collecting more evidence of teachers' own implementations to enrich teachers' selection and implementation of high-level tasks. Therefore, there is a need to travel into teachers' classrooms persistently to help teachers' learning to select a cognitively demanding task and make sense of interactions between teacher, students, and tasks during the implementation. Such an attempt may contribute to the literature on how activities make teachers recognize high-level tasks and implement those and may give insight into practicing teachers' rationale on algebraic task selections. Hence, teachers' knowledge of the cognitive demand of mathematical tasks is a critical construct in our investigation of teachers' learning through professional activities. In addition, "the focus on tasks help us understand how to support attention to student thinking" (Tekkumru-Kısa, Stein & Doyle, 2020, p.3). In that respect, teacher noticing is one of the conceptual notions that relies on attending and elaborating on essential aspects of instruction, including students' thinking (van Es, 2009) and tasks. While the literature on noticing and professional development has emphasized important aspects of instruction, attention on tasks is missing (Santagata et al., 2021; Tekkumru-Kısa, Stein & Doyle, 2020). Not paying attention to tasks can cause researchers to disregard the context for students' thinking and opportunities to advance student thought may be lost. Given that tasks serve as "a context for students' thinking" (Doyle, 1988, p. 167) and that different tasks elicit different levels of student thinking, teachers' opportunities to attend to student thinking vary according to the type of tasks in which students are engaged (Tekkumru-Kısa, Stein, & Doyle, 2020). Hence, this study aims to contribute to the literature by portraying a practicing teacher's noticing of students thinking in the context of mathematical tasks.

Expertise in noticing is required to discover and interpret instructionally significant aspects in the mathematics classroom because noticing is "the act of focusing attention on and making sense of situation features in a visually complex world" (Jacobs & Spangler, 2017, p. 771). Despite its importance, preservice and practicing teachers have struggled to elaborate on students' thinking or crucial aspects of mathematical instruction (Fernandez & Choy, 2020). Some researchers indicate that novice teachers have less attention to students' understanding as compared to expert teachers selected based on years of experience, students' success, or administrative ideas (Blomberg, Stürmer, & Seidel, 2011; Huang & Li, 2012; Krull, Oras, & Sisask, 2007). For instance, less experienced teachers tend to attend to the superficial characteristic of a classroom environment, such as climate of students' behaviors, rather than specific students' thinking (Star &

Strickland, 2008). Recent studies have shown that even experienced teachers have problems in changing their comments from general aspects of instruction to much more specific aspects of students' thinking and pedagogy (Bonaiuti, Santagata, & Vivanet, 2020) and in responding robustly to students' thinking (Lee & Choy, 2017). Indeed, research reveals that teachers who are seen as experienced have limited awareness of students' algebraic reasoning (Coe, 2007; Styers, Nagle, and Moore-Russo, 2020). Experience alone does not bring about attending to students' algebraic reasoning in depth. Therefore, the present study aims to support an experienced in-service teacher's noticing in the context of mathematics algebraic tasks through a professional development program.

The current study's underlying presumption is that teachers' professional noticing competence is domain-specific (Jacobs & Empson, 2016; Nickerson, Lamb, & LaRochelle, 2017; Walkoe, 2015). Thus, I centered on the slope notion for three main reasons. First, slope has a foundational nature in algebraic and functional thinking (Kieran, 2007) and is interconnected with other concepts such as quotient, measure as relative magnitude, rate of change, and covariation (Byerley & Thompson, 2017), which are also linked to the eleven conceptualizations of slope (Nagle et al., 2012). Most of the textbooks in the USA, Japan and Australia integrate covariational and variational perspectives through examples of tasks and tools (e.g., SimCalc MathWorlds) by emphasizing two varying quantities together. Similarly, the Turkish National Middle School Mathematics Instructional Program (MoNE, 2018) highlights covariational reasoning by emphasizing examples for changing two quantities simultaneously. Thus, it is inferred that teachers who become skilled at noticing students' ideas professionally in slope can aid their students' growth of various slope meanings. The second reason is a call for studying teacher noticing "as it relates to particular mathematical domains" (Dindyal et al., 2021). However, the vast majority of research on teacher noticing focuses on the context of pattern generalizations and functional thinking in the area of algebra rather than slope specifically. Due to the lack of prior research examining in-service teachers' professional noticing in the notion of slope, this study aims to add to the body of knowledge about content-specific noticing. The third reason is another call for studying mathematical task knowledge in a particular content domain because teachers' interactions with task differ depending on their knowledge of particular content domains (Chrambalous, 2010). Studying in a single content domain (slope) might enable to make a more focused analysis of teacher noticing and mathematical task and contribute to the literature by giving an in-service teacher's instances of noticed elements of slope.

Despite the significance of slope in students' thinking and reasoning of mathematics, and teachers' robust noticing skills in practice, teacher preparation has not been a major focus of teacher development programs (Stein et al., 2011). In other words, there is research on teachers' professional noticing of students' thinking on slope tasks with written students' work (e.g., Styers, Nagle, & Moore-Russo, 2020) and attributes of potential instances of student thinking during slope teaching (Van Zoest et al., 2017). However, only a small number of studies has considered supporting teachers to notice students' algebraic thinking (Walkoe, 2014) and slope notion particularly. Previous studies indicated that teachers struggle to notice some of the essential elements of slope teaching and learning slope (Nagle, Martínez-Planell & Moore-Russo, 2019), such as steepness within real-life contexts (e.g., Styers, Nagle, & Moore-Russo, 2020), and many of them have procedural knowledge about "slope" notion (e.g., Byerley & Thompson, 2017). In line with this, teachers have tended to select procedural slope tasks (Zahner, 2015) due to its complex nature, including various conceptualizations and representations. They have incompetency in implementing high-level algebra tasks (Wilkie, 2014). Hence, the field lacks the documentation of the degree to which in-service teachers robustly notice students' mathematical thinking of slope (La Rochelle et al., 2019) within a professional development context. For this reason, this study aims to portray how an in-service teacher's noticing skills are enhanced within the context of highly cognitively demanding slope tasks.

Noticing "is a learnable practice" (Jacobs & Spangler, 2017, p. 772), and studies indicate that teachers can learn to interpret important aspects of teaching (Stockero, 2014) and shift their attention from general pedagogies to the specific aspects of teaching (van Es et al., 2017). To extend students' ideas and improve the quality of instructions on slope, some researchers have attended to develop or elicit teachers' noticing via video clips and students' artifacts, utilizing a framework or high-level tasks (La Rochelle et al., 2019; Walkoe, 2015; Walkoe, Sherin & Elby, 2020). However, although those studies yield promising methods to elicit and improve teachers' noticing on students' thinking or mathematics teaching, they have constraints of sustaining teachers' high quality of instruction and their reflecting on students' thinking. These approaches include only focusing on the way to noticing the specificity of what is noticed (Choy, 2016) by using reflection prompts for classroom videos (Fernandez et al., 2015) on teachers' reflection on action rather than in action (Schön, 1991). Hence, these approaches may not provide evidence of teachers' noticing during actual teaching (Sherin, Russ, & Colestock, 2011) by separating teachers from the observed environment (Scheiner, 2020). In addition, the effects of others' videos on teachers' improvement may be weaker compared to actual implementation (Seidel et al., 2011). Thus, teachers need to sustain productive noticing (Spitzer et al., 2011) of task design and students' thinking to perform high-quality instruction. They should be given opportunities to identify elements of students thinking in the context of highly cognitively demanding tasks within planning, teaching, and reflecting process. Based on these suggestions, the boundaries of noticing were expanded as lesson design (Amador et al., 2017), lesson implementations (Luna & Selmer, 2021; Sherin & Star, 2011; Teuscher, Leatham, & Peterson, 2017), and lesson reflections (Choy, 2015) in the present study. Hence, it is believed that the current study might contribute to the field by examining what an in-service teacher focuses on and how she makes sense of her attention during the three stages of practice as opposed to most studies which focused on the details of what teachers attend to in a general manner (Choy, 2014).

The current study includes a different methodological approach for eliciting and analyzing noticing by expanding the boundaries of noticing and adapting the Learning to Notice Framework (van Es, 2011) within a specific context of coaching embedding mathematical tasks. In most of the noticing studies, researchers asked teachers to reflect on students' work or videos or video clips without concentrating on planning or teaching. Considering the paramount role of productive noticing skills for effective instruction (Spitzer et al., 2011), in the current study, the boundaries of noticing were expanded in lesson planning (Amador et al., 2017) lesson implementations and lesson reflections (Choy, 2015) in order to elicit and enhance teachers' noticing skills. In this respect, two primary dimensions of van Es's framework, what teacher notices and how teacher notices, were adapted. The characteristics of levels seemed to be holistic with these two dimensions and to be associated with reflection on action rather than reflection in action. Furthermore, the study context included mathematical tasks and coaching; hence the teacher's attention to these specific aspects were embedded in the revised framework as an explicit focus. Along with these changes, the framework enables assessing a teacher's noticing from both researcher's and participants' perspectives. Therefore, the newly adapted framework can guide professional developers and researchers to assess teachers' noticing in the context of coaching and highly cognitively tasks.

In order to construct a structure that facilitates collaboration by developing a shared language and set of processes, tools and routines are essential in research-practice partnerships (Tekkumru-Kısa et al., 2020). Tasks and frameworks are practical as they provide analytical lenses to make critiques on students' thinking and pedagogy. In that sense, making teachers analyze instruction by considering relationships among tasks, students' thinking, and pedagogy through a concept or mathematical idea can be effective for a productive conservation between researchers and practitioners and for improving the noticing skills of practicing teachers. In this respect, coaching might be considered a model for research-practice partnerships and a job-embedded mode of professional development (Desimone & Pak, 2016). Specifically, the coaching framework by Russell and colleagues (2019) highlights effective tasks as a tool to increase teacher engagement in discussing teaching practice and student thinking, and it is not specific to any mathematical content domain, topic or idea. Hence another pedagogical tool, Slope Conceptualization Framework (Nagle et al., 2019), is content-specific and is used to increase an in-service teacher's ability in selecting and implementing high level slope tasks with respect to student conceptualizations on slope, and encourage her to elaborate on students' thinking, tasks and her pedagogy. In conclusion, to maximize and portray the potential of coaching in supporting students learning and teachers' improvement (Campbell & Malkus, 2011), the current study has been grounded on the Coaching Framework by Russell and colleagues (2019) framed by rigorous slope tasks within three cycles (pre-observation, observation, and post-observation) with a more specific focus. Hence, the present study might offer fruitful insights into coaching practices within high-level algebraic tasks context through the research-based coaching framework (Kraft et al., 2018). In addition, it might provide much-needed evidence related to one-to-one coaching practices (Auletto & Stein 2020; Cobb & Jackson, 2011; Gibbons & Cobb, 2017) and to what extent and how coaching improves a practicing teacher's noticing skills.

1.6. Definitions of the Important Terms

With respect to the goal and the research questions in the study, there are some important technical terms related to "coaching", "slope", "mathematical tasks", and "noticing". Due to a necessity to identify the meanings of these constructs and terms, all related terms to the current study are constitutively and operationally described in this part of the thesis.

Middle school mathematics teacher: A middle school mathematics teacher is an in-service educator who teaches mathematics to students ages 10 to 14 in public middle schools for five to eight years (elementary or lower secondary schools). A middle school mathematics teacher receives a bachelor's degree from the Elementary Mathematics Teacher Education Program in Faculties of Education. With respect to distinction between novice and experienced teachers, researchers have addressed different criteria and definitions for experienced teacher and to classify types of experience. (Graham et al., 2020). In the current study, similar to Brody and Hadar (2015)'s definitions of experienced and novice, experienced was defined as having more than 10 years of teaching at middle schools. In conclusion, based on these definitions, the participant teacher of the study is an experienced middle school mathematics teacher in a public school.

Mathematics Coaching: Hull, Balka, and Miles (2009) characterize a mathematics coach as "an individual who is well-versed in mathematics content and pedagogy and who works directly with classroom teachers to improve students' learning of mathematics" (p. 8). Mathematics coaching, a kind of professional development, includes the cyclic process of pre-observation, observation, and post-observation. Specifically, Russell and colleagues (2019) identified the Math Coaching Model as "distinctive in its focus on one-on-one coaching that targets planning, enacting, and reflecting on a specific lesson, as well as its focus on core disciplinary teaching practices. In other words, it specifically focuses on building teacher capacity to enact rigorous mathematics tasks that provide opportunities for student reasoning about mathematics concepts" (p. 4). In line with the Model, the current study identifies mathematics coaching as deep and specific conversations about tasks on slope, pedagogy, and eight grade students' thinking with the collaborative teacher during planning, implementing, and reflecting processes. Therefore, mathematic coaching is used to improve an in-service mathematics teacher's knowledge of cognitive demand of mathematical tasks and noticing skills within the context of high-level mathematical tasks on slope.

Teacher noticing: Teacher noticing is seen as an ability to focus on and make sense of key aspects of instructional practices (Goodwin, 1994; Sherin & van Es, 2009) or making sense of students' thinking (Jacobs et al., 2010). Sherin and van Es (2009) indicate teacher noticing as two subskills: " (a) identifying what is important in a teaching situation and (b) drawing on one's knowledge of teaching and learning to reason about the situation" (van Es & Sherin, 2006, p. 215). Based on Jacobs et al.'s (2010) definition and way of assessing of noticing grounded on specific exclusive emphasis on students' thinking, in the present study, these two ideas was adopted to study noticing in the arena of lesson planning, teaching and reflecting. Hence, teacher noticing has two subskills: (a) determining what is key in a situation (specific students' thinking or classrrom events) within the context of mathematical tasks during planning, teaching, and reflecting and (b) making sense of the situation including pedagogical decisions for further teaching or decisions in the moment of instruction.

Mathematical Tasks: A classroom activity (i.e., a problem or a set of problems) draws students' attention on a particular mathematical phenomenon. (Stein, Grover, & Henningsen, 1996, p.460). Within the context of this description, mathematical tasks enable learners to take part in the activity embedded in these tasks and adjust students' conceptions about doing mathematics (Henningsen & Stein, 1997). In the present study, the mathematical tasks are defined as a classroom activity for eliciting students' thinking and promoting students' learning.

Slope: Slope of a linear function is conceptualized as *the geometric ratio, algebraic ratio, physical property, functional property, parametric coefficient, trigonometric conception,* and *calculus conception, real-world situation* (Stump, 1999, 2001b, p.129). Moore-Russo, Conner, and Rugg (2011) have proposed eleven conceptualizations of slope by extending and revisiting eight categorizations by Stump (1999, 2001a, 2001b). Then, Nagle and Moore-Russo (2013) added more conceptualizations, such as *determining property, the Behavior indicator*, and *linear constant*. Nagle, Martinez-Planell, and Moore-Russo (2019) have proposed the idea that slope can be identified by distinguishing between "ways of slope thinking about slope" and "uses of slope" (p. 4). "The ways of slope" was characterized as relations among geometric ratio, algebraic ratio, and functional property conceptualizations. Then, they were combined into linear constant conceptualization, corresponding to the Object stage of slope, while the "uses of slope" include other slope conceptualizations such as parametric coefficient, behavior indicator, physical property, determining property, real-world situation, trigonometric and calculus conception. In the present study, the conceptual framework by Nagle and colleagues (2019) on how students perceive slope was utilized as a tool in the current study to aid a practicing teacher in recognizing a range of task contextual elements, relating it with different conceptualizations of slope, and focusing on student thinking during task design. Hence, the coach and the teacher can discuss students’ slope thinking in relation to tasks by using the tool through the discussion process in coaching.

CHAPTER II

LITERATURE REVIEW

The goals of this study were multifaceted: (1) to document the changes in an in-service teacher's knowledge of the cognitive demands of mathematical tasks through her participation in a coaching program, (2) to examine the changes in the teacher's noticing skills and how the teacher progressed through the coaching stages including planning, enacting and review. This chapter addressed the theoretical and empirical background of the relevant studies and how the current study is situated. Notably, this chapter included four main components: mathematical tasks, slope notion, noticing skills and coaching as a professional development. Studies related to these four components were also mentioned and discussed.

2.1. Mathematical Task and its Importance

A mathematical task is any problem or activity designed to help students engage in a mathematical concept (Stein, Grover, & Henningsen, 1996). Within this definition, mathematical tasks enable students to engage in activities and shape their mathematical perceptions (Henningsen & Stein, 1997). Mathematical tasks can be studied in various ways, including variety of representations elicited, different solving strategies, and the communication need for students (Stein, Smith, Henningsen, & Silver, 2000). In their study, researchers analyzed students' mathematical thinking level while solving tasks involving different levels of cognitive demands. The cognitive demand of tasks is described as intellectual processes required to accomplish the given tasks (Doyle, 1988). To classify the cognitive level of mathematical tasks, make classification of cognitive level of mathematical tasks, Stein and colleagues (Stein & Lane, 1996; Stein & Smith, 1998) proposed the Task Analysis Guide (TAG), which includes three categories:

low level, high level and unsystematic explorations. The first category is divided into two levels: Memorization and Procedure without Connection. Memorization refers to remembering the facts, rules or algorithms; different from Memorization, Procedures without Connection is about practicing procedures and algorithms with or without understanding. Similarly, the second category, high level, is divided into two levels: Procedures with Connection and Doing Mathematics. The procedure with Connection is associated with following a procedure to reason mathematical ideas or connecting ideas. The highest cognitive demand tasks are characterized as Doing Mathematics addressing the connection of different mathematical ideas in a new context and regulation of complex reasoning processes. *Doing mathematics* tasks are basically separated from tasks at *Procedures with Connection* concerning whether a path is implied or not. The last category, unsystematic exploration (Stein & Lane, 1996) refers to tasks that might have the potential for higher level thinking, but students work with the task to develop an unsystematic approach that leads to inhibit understanding of the concept. Detailed criteria for these levels and categories were provided in Appendix A. Compared to low-level tasks, high-level tasks enable students to create different solution strategies, outcomes and hypotheses, test and assert their responses or solutions' ways by connecting with prior learning (e.g. Boaler & Staples, 2008). With respect to the potential of high-level tasks on students' reasoning, there was a call to employ mathematical tasks at a high cognitive demand (MoNE, 2018; NCTM, 2000). Such tasks present multiple entry points of the problem and engage students to use multiple representations and models by building on their existing knowledge (Henningsen & Stein, 1997; Stein & Lane, 1996). In line with this, NCTM (2000) has highlighted the importance of using rich tasks on students' understanding due to their potential to take their curiosity to do mathematics with a challenge in Principles and Standards for School Mathematics. Empirically, Stein and Lane (1996) examined the relation between the cognitive load of tasks in which students engaged and student' mathematical thinking. Based on its findings, the level of cognitive demand of mathematical tasks determines the kind of students' learning. According to Boston and Smith (2016), due to the vast majority of the time, students deal with mathematical tasks,

tasks play a crucial role in students' learning of mathematics. There are two reasons why mathematical tasks can be regarded as the basis for students' learning. First, they can take students' attention, and students can conceptualize the underlying mathematical ideas. Second, given parameters or variables embedded in mathematical tasks, students can operate on mathematical ideas (Doyle, 1983). In addition to the cognitive demand of tasks as a feature another essential feature is that they can situate at multiple levels. These levels were highlighted through Mathematical Task Framework (MTF; Stein, Grover & Henningsen, 1996)

2.1.1. Mathematical Task Framework

As a part of a large-scale project called QUASAR [Quantitative Understanding: Amplifying Student Achievement and Reasoning] (Stein, Grover, & Henningsen, 1996), Mathematics Tasks Framework was developed to underline the relation between the feature of mathematical tasks through different phases (as selected, set up, and implemented), and students' learning. More specifically, the framework emphasizes how students might make sense of mathematics and the level of their mathematics thinking skills (Doyle, 1988; Henningsen & Stein, 1997) through these phases. In addition, factors influencing the way of task preparation, set-up, and implementation related to students' mathematics thinking are mentioned in the framework. Detailed explanations for each phase and factors influencing phases will be explained subsequent paragraph.

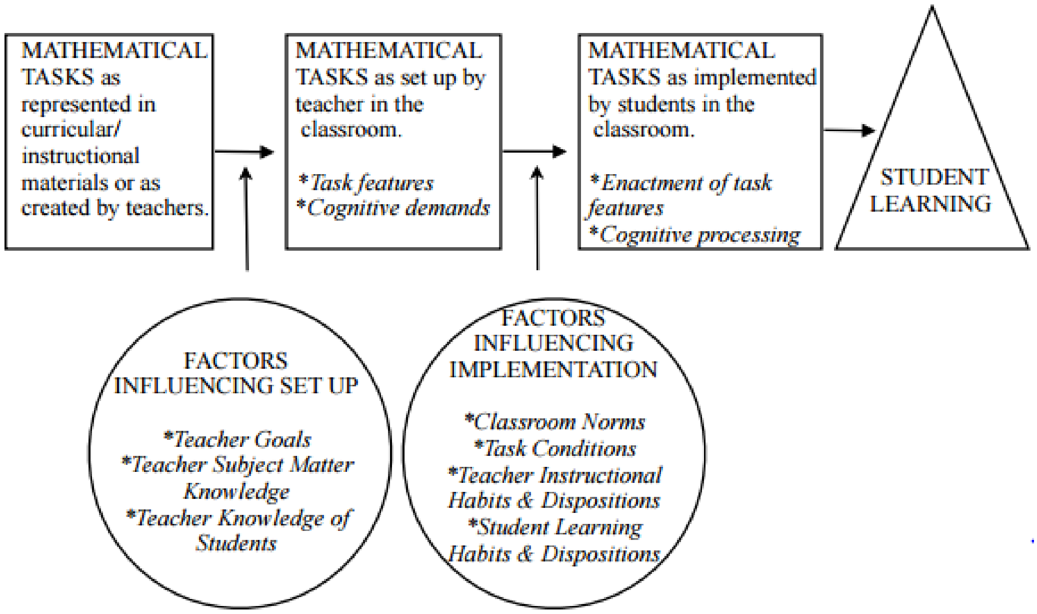


Figure 1. The Mathematical Task Framework (Stein, Grover, & Henningsen, 1996, p.459)

Specifically, in this model (see Figure 1), three phases where a mathematical task goes through are introduced: tasks provided in the curriculum or textbooks, tasks introduced by teachers, and tasks implemented by students. The first phase is related to creating mathematical tasks or selecting or modifying them from curriculum or textbooks (Arbaugh & Brown, 2005). While selecting tasks, teachers' aim, content knowledge, and content and student knowledge are considered agents in this phase. This model informs the second phase by addressing the nature of the task concerning its cognitive demand on students.

Similarly, the set-up phase informs the third phase, including ‘enactment of task features’ and ‘cognitive processing’ related to student-student and student-teacher discussion on the mathematical idea behind tasks in the model. This process is also influenced by factors such as teachers' way of teaching, student affective and cognitive readiness, or classroom norms. That process ends with students' learning. In detail, Smith, Grover, and Henningsen (1996) describe the phase of the task set up and implementation:

> Task set up is defined as the task that is announced by the teacher. It can be quite elaborate, including verbal directions, distribution of various materials and tools, and lengthy discussions of what is expected. Task set up can also be as short as simply telling students to begin work on a set of problems displayed on the blackboard. Task implementation, on the other hand, is defined by the manner in which students actually work on the task. Do they carry out the task as it was set up? Or is the task somehow altered in the process of working through it? (p. 460).

Smith and colleagues (1996) mentioned the characteristics of set-up phase including announcing what is expected, introducing materials and tools, general instructions related to a task or only demonstrating the tasks to students. On the other hand, task implementation is about students' working on tasks. During these phases (set-up and implementation), the question of whether the cognitive demand of tasks is changed or not is aroused (Smith, Grover, & Henningsen, 1996). The question has been enlightened through the findings of some studies revealing that the cognitive demands of tasks differ due to factors related to teachers and students which will be discussed in detail later.

Considering changeability in tasks' demands I argued that students' conceptual development and unexpected situations during enactment or set-up might lead teachers to revise or modify their tasks related to the big ideas after the implementation. Hence, these mechanisms work in harmony until students' conceptual understanding within a cyclic model rather than a linear model that appears in MTF. This is similar to Thanheiser (2017), who highlighted the 'cyclical nature of task design'. Therefore, in the present study, teachers' noticing of task nature as modified or altered based on previous implementation is focused on during the planning and reflecting phases.

Specifically, regarding the arguments about the MTF, Otten and Soria (2014) questioned about implementation stage of the original MTF and divided it into two sub-phases: the working phase and the look back phase based on the theories about the aspects of learning. The former involves students' efforts on tasks individually, whereas the latter involves sharing and discussing ideas in a community. They concluded as follows:

> The arrow directed toward student learning in [new] figure represents the fact that mathematical tasks influence what it is that students learn, with respect to both mathematical content and what it means to do and learn mathematics (Otten & Soria, p. 816).

These two aspects of task implementations guide this study to make students study task on their own then share and discuss their ideas with four-five friends in small groups with strategic help from teachersand then the whole class discussion was handled. However, in the current study, only the teacher's action was taken into consideration while analyzing data. As mentioned before, empirical studies on mathematical tasks have indicated how students' learning varies through these stages of MTF (e.g. Jackson et al., 2013; Silver & Stein, 1996; Tarr, Reys, Reys, Chavez, Shih, & Osterlind, 2008).

Past studies indicate the positive relationships between tasks and students' understanding and achievement (Silver & Stein, 1996; Tarr, Reys, Reys, Chavez, Shih, & Osterlind, 2008). For instance, Silver and Stein (1996) identified the benefits of engaging students with challenging tasks (highly cognitively level tasks) on their conceptual understandings using the MTF. Data were collected from four sites, A, B, C, and D, which differ in whether they select or implement high-level tasks (n=144 tasks). They found that the site, which used high-level tasks and was capable of implementing them, had the highest students' success compared to other sites, which nourished lower levels of cognitive demand. Therefore, with highly cognitively challenging tasks including multiple representations and multiple solution paths beyond remembering and making algorithms or applying rules, students can interpret problems and select and utilize appropriate solutions by organizing their thinking process. The exciting finding of the study is that a few students make progress in conceptual understanding by working with high level-tasks at the set-up phase even if their teachers decreased the cognitive degree of the tasks during the implementation phase. Similarly, Tarr and colleagues (2008) examined relationships between the learning environment and students' high-level abilities such as reasoning and problem solving using a quasi-experimental design at ten middle schools with 2533 students. They found that the environment providing students test their conjectures, explaining their thinking, and using various ways to solve problems contributed to students' success in items requiring reasoning, conceptual thinking, interpreting, and problem-solving. With a five-year longitudinal study, Boaler and Staples (2008) also found that the cognitive demand of tasks at the set-up and implementation phases represent the degree of students doing mathematics. Specifically, Jackson and others (2013) analyzed how the set-up phase relates to students' thinking in the whole class discussion that is a part of the implementation phase. Jackson and colleagues used the MTF (Stein & Lane, 1996) to determine the characteristics of tasks and an expanded version of the Instructional Quality Assessment (IQA) while analyzing video recordings of 165 middle grades teachers' teaching. Congruent with the finding of Silver and Stein (1996)'s work, the authors empirically found that even a small number of students benefited from high-level tasks in the set up phase, regardless of the type of tasks during implementation. As this study indicates the importance of the task nature at the set-up phase, selecting and implementing high-level tasks are other dimensions to increase the number of students who benefit from the mathematical idea through tasks. At that point, teachers' capability to recognize and implement high-level tasks seemed critical, as highlighted in the MTF.

2.1.2. Teachers' ability for recognition and implementation of tasks

Given the paramount effect of tasks on students' conceptualizations of mathematics, teachers' decisions on task selections emerges as an important aspect (Clarke & Roche, 2018). However, studies indicated that teachers had inadequacy of identfying the cognitive demand of mathematical tasks. Studies indicate that teachers select and categorize instructional tasks by attending to the surface-level characteristics of tasks such as having real life context and asking to use diagrams, representations, or technology, or by focusing on the mathematical content, and the length of the task (Arbaugh & Brown, 2005; Osana et al., 2006). Besides they tend to classify them according to the tasks' difficulty and students' level of achievement (Tekkumru-Ksa, Stein, & Doyle, 2020). In addition to these surface-level qualities, teachers think that problems are infused with broad mathematics rather than specific mathematical concepts or solutions (Parrish, 2022). In that sense, the ability to recognize the cognitive demand of the tasks can be seen as the first requirement to increase students’ engagement. Many studies also indicated that teachers either choose a low-level cognitive demand of mathematical tasks before implementation or decrease the cognitive demand while implementing the task (decline in cognitive demand of mathematical tasks) (Chrambalaous, 2010; Henningsen & Stein, 1997; Otten & Soria, 2014). For instance, Silver, Mesa, Morris, Star, and Benken (2009) found that half of the teachers (n=32) in the United States submitted at least one task evaluated as cognitively demanding (out of three tasks) and nearly %30-40 of the tasks submitted were coded as high level. The other finding also revealed that the lessons in teachers' portfolio entries often included activities and tasks situated in broader mathematics content domains, real-life contexts, and needed technology or hands-on materials. However, the authors detected little evidence from their submitted tasks that these innovative tools were being utilized adequately to support students' engagement with cognitively high-demanding tasks in classrooms. Similarly, an analysis of interviews with in-service elementary teachers in Turkey showed that teachers expressed a challenge to prepare tasks at a high level (Bal, 2008). These findings suggest that teachers are not always capable of incorporating worthwhile mathematics activities into the classroom.

Even if preparing/selecting high cognitive demand of mathematical tasks are crucial, it may not result in high-level thinking of mathematical ideas (Boston & Smith, 2016). Thus, both selecting the appropriate tasks yielding high-level thinking and implementing them without decreasing their level are regarded as critical aspects. However, in addition to changes in teachers’ selection of tasks, several authors also showed that teachers’ way of task implementation could vary widely in classrooms (Chrambalous, 2010; Graven & Coles, 2017; Lozano, 2017). For instance, in a study by Sullivan, Clarke, Clarke and O'Shea (2010), it was found that although three teachers were able to design their tasks concerning their goals and students' mathematical thinking, only one teacher could maintain the task at a high level within the implementation process.

To portray factors linked with the level of cognitive demand of tasks during implementation, Henningsen and Stein (1997) examine classroom issues associated with the maintenance or decline of level of the cognitively demanding tasks. For instance, rather than emphasizing concepts, a tendency toward a single correct answer or procedural aspects of the tasks and classroom management concerns results in a decline in high-level cognitive demands of mathematical tasks. On the other hand, scaffolding students' understanding and thinking or creating tasks based on students' prior knowledge are factors that maintain the mathematical tasks at a high level. Several constructs related to teacher and students teaching and learning have also been used to explain the lack or the decline of cognitive load of tasks' implementation. These include teachers' beliefs regarding teaching and learning mathematics (Boaler, 2002; Manouchehri & Goodman, 2000), teachers' orientation toward the curriculum (Remillard & Bryans 2004), as well as their goals and expectations for their students (Sztajn, 2003). Besides teachers' knowledge and conceptions (Garrison-Wilhelm, 2014, Chrambalaous, 2010) are other issues associated with enactment of mathematical tasks. Other plausible causes include students' learning routines when engaged with high-level activities (Doyle, 1983), classroom norms that regulate teacher-student interactions (Herbst, 2006), and various contextual elements (e.g., time constraints, principal expectations). Most of these factors are related to teachers' properties and capabilities that portray a need to improve teachers' knowledge of mathematical tasks. Parallel to the need, development in teachers' knowledge of the cognitive demand of tasks is one of the aims of the current study.

2.1.3. Teacher development in mathematical task knowledge

Given emphasis on enriching of teachers' knowledge of mathematical tasks, researchers have begun to investigate how this might occur (Guberman & Leikin, 2013; Tekkumru-Kısa & Stein, 2015). In the whole process of selecting and enacting tasks, Chapman (2013) states that teachers need knowledge, including understanding the nature of worthwhile tasks and identifying students' needs regarding mathematical tasks to support conceptual understanding. Hence, engaging teachers with highly cognitive mathematical tasks does not guarantee increased knowledge regarding mathematical tasks for teaching. To accomplish this, first, teachers' belief in teaching mathematics through high-level tasks should be taken into account (Chapman, 2013). Second, their capability to make sense of possible students' thinking and their solution strategies while dealing with worthwhile tasks is another issue to be considered (Boston, 2005; Smith, Bill & Hughes, 2008; Stein & Kaufman, 2010). The themes mentioned earlier, including teachers' beliefs or teachers' noticing of students' thinking and task nature, are critical elements for teacher change. In addition to these, teacher practice in their classrooms is core for their sustained change (Clarke & Hollingsworth 2002). Therefore, researchers have decided to nurture teachers' knowledge and beliefs about mathematical tasks' nature. Detailed aspects of those initiatives are explained in the next paragraph.

One productive strategy to enrich teacher capacity on task nature has been professional development attempts to analyze mathematical tasks (Arbaugh & Brown, 2005; Boston, 2013; Watson & Mason, 2007) by utilizing tools such as TAG or MTF. For instance, the QUASAR project indicates that the knowledge regarding properties of the levels of cognitive demand (TAG) supports middle school teachers in recognizing differences among tasks and identifying what students think about that tasks present. Considering the benefits of cognitive demands criteria, Arbaugh and Brown (2005) created a non-threatened learning environment for high school teachers to make them discuss the nature and characteristics of 20 mathematical tasks as initial and final. Collaborating with other teachers and researchers to learn about the criteria of CD led a majority of teachers to think of the nature of mathematical tasks and the relationship between tasks and students' work. The intervention of Arbaugh and Brown in a task sorting activity as initial and final showed an effective way to get insight into how teachers' reasoning on the sorting of change from initial to final displayed variance. In addition, this eight-month study confirmed that learning about the TAG allows teachers to be more conscious of their recognition and selection of high-level tasks. Boston (2013) also confirmed significant relations between teachers' gain in sorting tasks and their experience through a workshop with a mixed methodology approach. Moreover, they found that teachers learned to use descriptors or explanations in the Task Analysis Guide while providing a rationale for sorting high and low-level tasks. In addition, they valued how high-level tasks nurture students' high-level thinking. In contrast, half of the teachers insisted on classifying procedure with connection tasks as low-level since they have an idea that tasks at procedures with connection also present a procedure that corresponds with tasks at procedure without connection. These studies guide me to introduce TAG to the teacher while sorting tasks at different content domains and sample instructional episodes illustrating an implementation of high-level algebra tasks as a workshop before the teaching of linear equations unit as it was hypothesized that teachers should become aware of the variances in tasks and what characteristics they possess to enhance or inhibit students' thinking before successfully selecting and enacting slope tasks. In addition, the teacher was allowed to use the Guide while sorting the algebra tasks during the professional development intervention.

Studies also highlight the importance of the way of implementing high-level tasks as well as recognizing them. To accomplish this, for instance, Stein et al. (2000) developed a casebook including various tasks in different content domains of mathematics, criteria for task qualities, illustrative cases of implementation of tasks, and guidelines for discussion on tasks among teachers. Subsequent studies on activities have been based in a similar perspective with slight differences (Boston & Smith 2009; Boston 2013). For instance, Boston and Smith (2009) designed task-centric professional development sessions focusing on selecting and enacting cognitively challenging mathematical tasks with teachers. This initiative has three main aspects: 1) "samples of authentic practice" (Smith, 2001, p. 7), consisting of works of creating solutions for tasks, evaluating specific student work, analyzing instructional events in narrative or video form, 2) samples of practice were linked to ideas about mathematics teaching and learning through

Mathematical Task Framework, 3). Scaffolded field witnesses also gave opportunities for teachers to pertain the principles and ideas discussed in professional development to their classrooms. As a result, the study indicated that 13 teachers out of 18 could maintain the level of challenging tasks during implementation. These teachers also sustained their ability to select and implement cognitively high tasks over time (Boston & Smith 2011). Watson and Mason (2007) reviewed studies addressing teachers' uses of mathematical tasks and concluded that "The fundamental issue in working with teachers is to resonate with their experience so that they can imagine themselves 'doing something in their situation, through having particularized a general strategy for themselves, rather than relying on being given particular things to do" (p.3-4). This perspective sees teachers as an active part of the mechanism for selecting, implementing, and modifying tasks. There is a call to integrate teachers in task design since it has an opportunity to improve teaching and students' understanding (Geiger et al., 2014). Thus, in keeping with similar studies, the present study considers the teacher to be both a partner in task design and an implementer of pre-designed tasks. The research team of this study also hypothesize the importance of the teacher's field experiences as a vital component of this growth process. In addition, as Tekkumru-Kısa, Stein, and Coker (2018) did, I did not explicitly present factors associated with strengthening or limiting students' thinking and asked the teacher to interpret them in classroom videos. Instead, I preferred to see these factors mentioned by the teacher while analyzing video clips of classroom instruction or written classroom cases. This way might demonstrate how the nature of tasks and the reasons behind maintenance and decline in the cognitive demand of high tasks are deeply learned and adopted by the teacher.

Although these studies can give insights into the aspects of fruitful ways to enrich teachers' ability to select and enact tasks, these studies have not investigated teachers' knowledge of tasks' nature within a specific context and content domains. However, studies should focus on a shift toward teachers' task knowledge on specific tasks within a content strand (Chrambalous, 2010). Considering teachers lacking in conceptualizing rate of change, slope, and quotient constructs in the units such as linear equations (Byerley & Thompson, 2017; Coe, 2007) as a part of the algebra strand in middle school grades and necessary pedagogical content knowledge (e.g. Coe, 2007; Nagle & Moore-Russo, 2013) it is inevitable to not expect teachers to be able to select and implement slope tasks. Indeed, studies demonstrated that teachers in Turkey or other countries could not preserve the academic rigor of algebra tasks while implementing them (Otten & Soria, 2014; Ubuz & Sarpkaya, 2014; Wilkie, 2016). Nevertheless, the slope is a foundational notion for other disciplines (Smith et al., 2013) and topics (Teuscher & Reys, 2010; Casey & Nagle, 2016) and the quality of algebra instructions in classrooms is argued as a problem by many researchers. (McCrory et al., 2012). In turn there is a need to develop teachers' task knowledge within "the slope" notion including selecting, presenting and implementing tasks with high cognitive demand to improve students' conceptual understanding of the slope. In the following section, a review of the literature on slope, various conceptualizations of slope and teachers' difficulties with a focus on in-service teachers' development in algebra including slope notion is provided.

2.2. Slope Concept

The slope concept was related to other disciplines, and it is situated in contexts outside of mathematics. This relation could be seen in an application of engineering, such as ramps or ladders and graphics often used in physics and chemistry (Lingefjärd & Farahani, 2018; Planinic, Milin-Sipus, Katic, Susac & Ivanjek, 2012; Smith et al., 2013). Considering the close relationship between slope and other disciplines, the researcher proposed that the concept also has a crucial role in other mathematics concepts and topics. Although the slope notion is typically introduced with linear equations, slope is fundamental for proportionality, rate of change in middle grades (Stump, 1997). While it is base for functions, covariation (Carlson, Oehrtman, & Engelke, 2010; Lobato & Thanheiser, 2002; Teuscher & Reys, 2010), integral, derivative (Bos, Doorman & Piroi, 2020; Dominguez, Barniol, & Zavala, 2017) linear regression, trigonometry

(Bos, Doorman & Piroi, 2020) and lines of best fit (Casey & Nagle, 2016; Nagle et al., 2017a) in secondary schools and undergraduate programs.

Despite its importance and prominence in grounding various mathematical ideas and reasoning, variability in the conceptualization of slopes with multiple representations creates a challenge for learning and teaching (Nagle, Martínez-Planell & Moore-Russo, 2019). The slope of a linear function is conceptualized as *the geometric ratio, algebraic ratio, physical property, functional property, parametric coefficient, trigonometric conception,* and *calculus conception* (Stump, 1999, p.129). In addition to seven categorizations, Stump (2001b) added a new category called a *real-world situation*. Moore-Russo, Conner, and Rugg (2011) have proposed eleven conceptualizations of the slope by extending and revisiting eight categorizations by Stump (1999, 2001a, 2001b). Then Nagle and Moore-Russo (2013) addressed additional conceptualizations, first labelled as *determining property* referring to the role of slope in deciding relationship among lines (as parallel or perpendicular lines). It also includes the idea that a unique line corresponds to a point on a line and the slope of that line given. Second, *the Behavior indicator* addresses whether the line increases, decreases, or is constant. Within this categorization, the absolute value of the slope characterizes the magnitude of the inclination of the line. The last eleventh category *linear constant* property implies the straightness of a line regardless of its region on coordinate axes. Eleven conceptualizations of slope were summarized in Table 1. As seen, the concept of slope has been treated in wide frames and representations.

Table 1. Eleven conceptualizations of slope

Category	Identification of Slope
Physical Property (PP)	An understanding of “Steepness” or “inclinations” of a line
Algebraic Ratio (AR)	Identified as a symbolic ratio between changes in y's and changes in x's: $\frac{y2-y1}{x2-x1}$
Geometric Ratio (GR)	Identified as vertical distance (rise) over horizontal distance (run) of a line
Parametric Coefficient (PC)	Referring to m in the equation of y=mx+b
Functional Property (FP)	Described as rate of change between two variables
Trigonometric Conceptions (TC)	Representing the tangent of a line’s (inclination) angle.
Calculus Conception (CC)	Representing a tangent line (instantaneous rate of change of a function)
Real-World Situation (R)	Including static and dynamic real-life applications
Determining Property (DP)	Representing the characteristics of the lines, such as perpendicular or parallel
Behavior Indicator (B)	Indicating the lines’ direction, increasing or decreasing
Linear Constant (L)	Being constant when the line is straight with an independent of representation

Note: The feature of these conceptualizations was adopted from Moore-Russo et al. (2011, p.9)

Based on these conceptualizations, similar to Carlson and others (2010) examples, Nagle, Martinez-Planell, and Moore-Russo (2019) provided instances of students' understanding of slope in Action, Process, and Object of APOS theory as a theoretical view. They extended and revisited the work of Deniz and Kabael (2017), which addressed eight- grade students' thinking of slope at the Action and Process stage by focusing on only algebraic and geometric ratio conceptualizations. However, Nagle, Martinez-Planell, and Moore-Russo (2019) have proposed the idea that slope can be identified by distinguishing between "ways of slope thinking about slope" and "uses of slope" (p. 4). The ways of slope were characterized as relations among geometric ratio, algebraic ratio, and functional property conceptualizations. Then they became combined into linear constant conceptualization, which corresponds to the Object stage of slope. While "uses of slope" includes other slope conceptualizations, parametric coefficient, behavior indicator, physical property determining property, real-world situation, trigonometric and calculus conception. (see Figure 2).

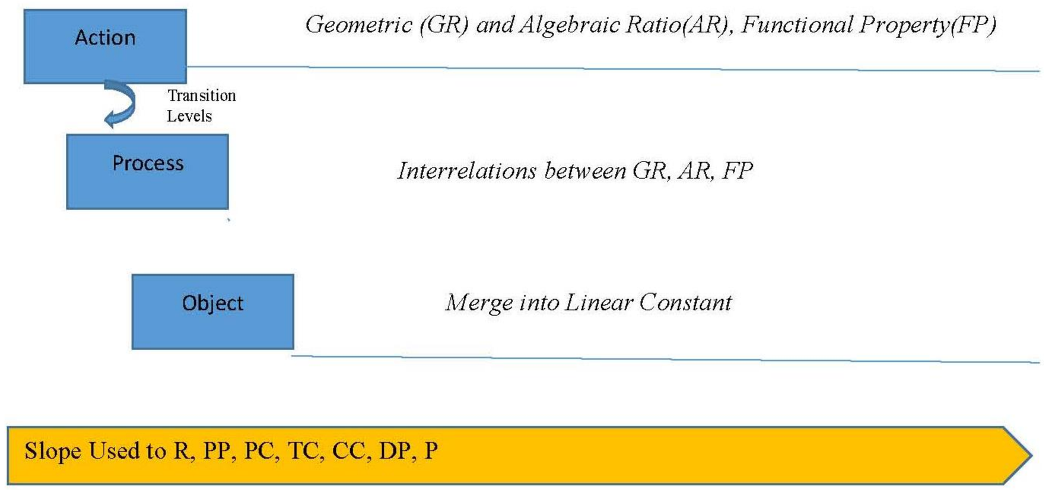


Figure 2. Relations between APOS Theory and Eleven Conceptualization of Slope (Adapted by Nagle et al., 2019, p. 4)

Slope notion is also interconnected with other concepts such as quotient, measure as relative magnitude, rate of change, and covariation (Byerley & Thompson, 2017), which they also connected with the eleven conceptualizations of slope. The quotient is referred to as "A quantitative meaning for quotient entails a multiplicative comparison of two quantities with the intention of determining their relative size." (p. 171). For instance, if the idea that “4/5” being the slope of a line corresponds to the part-whole relationships ("up four over five" in a coordinate plane), the meaning of division is constructed in a non-quantitative way. Measure as relative magnitude involves understanding that the amount of a quantity is changing within the unit (Thompson et al., 2014). This also includes the understanding of the reciprocal relationship of relative size.

Related to the rate of change and covariation, I inferred other meanings for slope from the literature, which are chunky and smooth meanings (Castillo-Garsow, 2012; Thompson & Carlson, 2017) and rate of change as relative sizes (Byerley & Thompson, 2017). Thompson and Carlson (2017) have demonstrated a framework for describing covariation and variation levels. Through upper levels of the framework, chunky continuous reasoning and smooth continuous reasoning are defined (p.441). Chunky continuous reasoning involves thinking that a quantity or variable changes in a determined (completed) interval or chunk, whereas smooth continuous reasoning entails changes in variables occurring within any interval. The idea of smooth continuous variation is defined as,

> The person thinks of variation of a quantity's or variable's value as increasing or decreasing by intervals while anticipating that within each interval the variable's value varies smoothly and continuously (Thompson & Carlson, 2017, p. 440).

For example, a student who emphasizes a specific increment in one variable (1 cm and 2 cm changes on the long side) by considering changes in the other variable ($1cm^2$ and $2cm^2$ changes in rectangle area, respectively) uses chunky continuous reasoning. In contrast, a student who uses smooth continuous reasoning indicates that two quantities vary smoothly within each tiny interval (0.05 cm or 0.3mm). A person engaged in imagining changes as smoothly, simultaneously, and continuously can also reason about chunky continuous reasoning when it is required (Thompson & Carlson, 2017). Hence, smooth continuous reasoning requires more sophisticated thinking than chunky continuous reasoning. The covariational reasoning has emerged while sketching a graph of a situation or equation. Thus, covariational reasoning can be treated as a lens to construct meaning for slope (Smith, 2008) as chunky reasoning and smooth reasoning for slope. For instance, the idea that "4/5" being the slope of a line represent that in every one increment in the x-axis, changes in the y-axis is 4/5 can be an example of chunky reasoning for slope. Whereas, for any sized changes in independent variable yield changes in dependent variable as slope-sized as large represents smooth reasoning for slope. These meanings also are related to sub-components of interiorized ratio (Thompson, 1994) as "ratios as per-one" and "ratio as measure" (Johnson, 2015a).

In addition to the covariational approach, the other related perspective for functional situations is the correspondence approach (Blanton, 2008; Confrey & Smith, 1995; Ellis, Ozgur, Kulow, Dogan, & Amidon, 2016). While covariational perspective involves “looking down,” coordinating changes in one variable with changes in another (Blanton, 2008, p. 32) in tabular representations, correspondence perspective is "looking across" from the independent variable to the dependent variable (Blanton, 2008, p. 32). Covariational perspective can serve as an auxiliary for conceptualizations of Functional property, Geometric ratio, and Algebraic ratio can, whereas correspondence perspective is a base for parametric coefficient (Peck, 2020). Ellis (2011) argued that functional thinking in linear equations should encompass both covariation and correspondence approach and highlighted the transformation of forms and a focus on quantities.

Besides, the rate of change as relative size is associated with the multiplicative comparison of quantities that also includes smooth continuous reasoning, proportionality, the meaning of measure, and quotient. Peck addressed how the rate of change and proportional reasoning is based on the construction of slope understanding empirically. With a lens of covariational perspective on the rate of change, Ellis, Ely, Singleton, and Tasova (2020) implied that 12-year- old students can conceptualize “constant rate as an equivalence class of ratios and viewing instantaneous rate of change as a potential rate” (p. 87) through a teaching experiment on supporting students' algebraic reasoning. Therefore, the meanings for slope are interrelated and differ based on the context that teachers and students face. Furthermore, the concept of slope is associated with multiple perspectives, conceptualizations, representations and constructs. Hence, while selecting and implementing slope tasks, teachers should be equipped with the multifaceted nature of the notion. The tasks in this study were considered through these conceptualizations and perspectives due to purpose of the study which developing both students and the teacher’s learning within context of high cognitive task implementation.

Considering its interrelated meanings, most of the textbooks in the USA, Japanese and Australian textbooks which covariational and variational perspectives are integrated through examples of tasks and tools (e.g., SimCalc MathWorlds) by emphasizing two varying quantities together. Similarly, the Turkish National Middle School Mathematics Instructional Program also highlights covariational reasoning by emphasizing examples for changing two quantities simultaneously. It proposes objectives related to slope: eighth-grade students should be able to explain slope using models and connect the representations of slope with each other (MoNE, 2018). However, "how does this emphasis reflect on students' understanding of slope?" was still a concern for researchers (Nagle & Moore-Russo, 2014). In fact, the actual teaching is far beyond putting the importance of conceptualizing those ideas (Thompson & Carlson, 2017). With a closer look, the national curriculum mentions building covariational reasoning together with proportional reasoning as hidden in grade 6. In grade 7, the covariational approach is not emphasized in pattern generalization tasks. In grade 8, it underlines the various representations while it includes a few conceptualizations of slope (geometric ratio, parametric coefficient, behavior indicator, real-life applications) is mentioned, and the link between these conceptualizations is highlighted with vague statements. Moreover, the curriculum describes no path for learning slope notion within the objectives. Therefore, the question still presents a concern to consider in determining whether the emphasis of covariational reasoning and its relation to the slope is adequately reflected in the classrooms.

2.2.1. Studies on slope conceptualization

Previous research on the concept of slope has mainly revealed how learners and teachers relate different conceptualizations and which ones are preferred or not. A majority of the analysis on slope involved students at the middle and high school level (e.g., Birgin, 2012; Cheng, 2010; Dolores Flores, García-García, & Gálvez-Pacheco, 2017; Hattikudur et al., 2011; Herbert & Pierce, 2005; Lingefjärd & Farahani, 2018; Lobato, Ellis, & Muñoz, 2003; Lobato & Thanheiser,2002; Peck, 2015; Planinic et al., 2012; Stump, 2001b; Tanışlı & Bike-Kalkan, 2018; Teuscher

& Reys, 2010; Zaslavsky, Sela, & Leron, 2002; Walter & Gerson, 2007). On the other hand, participants in severalother studies were university students (e.g., Dolores-Flores, Rivera-López, & García-García, 2019; Hoban, 2020; Lobato & Siebert, 2010; Ivanjek, Planinic, Hopf, & Susac, 2017; McDermott, Rosenquist, & Van Zee, 1987; Nagle, Moore-Russo, Viglietti, & Martin, 2013; Stump, 2001; Teuscher ve Reys, 2010; Wemyss & Kampen, 2013), pre-service teachers (Duncan, 2013; Dündar, 2015; Stump,1999, 2001a) and in-service teachers (Byerley & Thompson, 2017; Coe, 2007; Mudaly & Moore-Russo, 2011; Nagle & Moore-Russo, 2014; Nagle, Moore-Russo, & Styers, 2017; Stump, 1999; Walter & Gerson, 2007). In order to encourage teachers to teach and learn slope effectively, there is a need to synthesise of these existing studies, the nature of the slope, and its relations with other mathematical ideas such as covariational reasoning to identify factors explaining the difficulties with slope.

Studies on slope have also focused on different aspects of their structure due to their multidimensional and complex nature. Researchers have demonstrated that students had challenges in reasoning with representations of slope (Hattikudur et al., 2012; Planinic et al., 2012), connecting among different conceptualizations of slope (Birgin, 2012; Kim, 2007) and relating slope with other notions such as rate of change (Teuscher & Reys, 2010; Nagle & Moore-Russo, 2013) and ratio (Clements, 1985; Stump, 2001).

Regarding reasoning with representations, NCTM pointed out the importance of symbolic, textual, and graphical representations of slope (NCTM, 2009), and Kieran (2007) emphasized a functional-based approach, including relations between variables and multiple representations. However, some studies indicated that students have struggled to reason with different representations of slope. To illustrate, while Birgin (2012) showed that most the eighth graders are unable to switch between graphical to algebraic representations, Zazlavsky et al. (2002) found that even 11^{th}-grade students could not make connections between the algebraic and geometric ratio of slope. Expressly, studies indicated that students are struggling with shifting from graphical representation to the others, such as equations (e.g., Ayalon, Watson, & Lerman, 2016; Reiken, 2009), as well as from tabular (non-uniformly-increasing x values) to the equations and graphics (Ellis, 2011; Lobato et al., 2003). On the other hand, students perform better in identifying slope than y-intercept in graphics within tasks addressing specific values or increments (1 increment on the x-axis) marked for each variable and steepness of the line with no values marked for each variable (Hattikudur et al., 2012; Planinic et al., 2012). For instance, Wilkie (2016a) examined 102, 12-13 years old students' ability to generalize given figures and explain the relationships with multiple representations (textual, algebraic, and graphical). Findings of the study revealed that nearly half of the students could geerate generalizations with mixuse of notations and contextual language of the situation, with one-fifth being able to form algebraic notations. On the other hand, almost half could algebraically depict a real-world situation involving a linear relationship. These findings indicated that students have more difficulty expressing the general formula of the patterns than in real-life scenarios where the rate of change was given. Students had trouble translating given pairs into the graphics by taking discrete points. Students who use the covariation approach while generating rules can create more correct line graphs than students who use the correspondence approach. In contrast to some of Wilkie's findings some studies have also revealed that most students have trouble finding slope in real-life situations (Lobato & Siebert, 2002; Lobato & Thanheiser, 2002) and interpret a slope of the line relating to the constant linear property (Tanışlı & Bike-Kalkan, 2018).

Connecting and reasoning with various conceptualization of slope, studies indicated that middle and high-grade students were challenged to relate geometric relationships with functional properties and linearity (e.g. Aytekin-Kazanç, Acar-Çakırca, Işıksal-Bostan, 2021; Stump, 2001a). For example, Stump (2001a) argued that high school students understand slope better as a measure of rate of change in functional real-world situations than as a measure of slope in physical situations. However, their understanding of slope as steepness and ratio was also limited. Lastly, students think of the slope as a number (fraction) rather than a measure (Lobato & Thanheiser, 2002; Walter & Gerson, 2007). Overall, it was deduced from the findings that most of the students tended to use procedural knowledge of the algorithms of slope, that is, the rule of “rise over run” (Nagle & Moore-Russo, 2013). Hence, students have limited reasoning skills in relating slopes, rates of change, or line positions (Walter & Gerson, 2007). This limited reasoning was related to the fact that students tended to favor of formula-driven conceptions of concepts; even though they were given visual representations of them (Moore- Russo et al., 2012). Apart from students' procedural thinking, rise over run might also lead them to understand that slope is always positive because rise could be perceived as an increase by students.

In conclusion, middle and high graders commonly have difficulty in graphics and tabular representations (x's not increasing by one) as and geometric ratio, functional property (both functional and physical situations), and linear constant conceptualizations. These slightly confronting studies indicate that critical examination of tasks or questions used in the studies on eliciting students' understanding with respect to various representations is necessary due to the complex nature of slope notion representing through various representations and diverse conceptualizations. Thus, it can be inferred that there are discrepancies between students' performance in algebraic thinking within different contexts and across multiple representations due to the complicated nature of slope. Comparatively extensive research on students' misconceptions and their understanding of slope concept in different contexts and across multiple representations, little research (e.g., Peck, 2020) attempted to robustly portray how students connect these subcomponents of slope within multiple contexts.

Besides studies portraying how students understand slope and which representations they use or among which they have difficulty in connecting, studies have also attempted to examine how teachers perceive slope notion and their practices and whether there is a gap between how teachers conceptualize slope and which conceptualizations they used during teaching (e.g., Nagle and Moore-Russo, 2013). There are studies with prospective mathematics teachers (e.g., Duncan, 2013; Dündar, 2015; Stump, 1999) and in-service mathematics teachers (e.g., Byerley & Thompson, 2017; Coe, 2007; Mudaly & Moore-Russo, 2011; Nagle, Moore-Russo, & Styers, 2017; Stump, 1999; Walter & Gerson, 2007). For instance, in Stump's study (1999), most prospective and in-service teachers (n=39) viewed slope as a geometric proportion, and in-service teachers described it as a physical property. A small number of teachers expressed that slope was a rate of change. Some in-service and pre-service teachers had difficulties in tasks involving recognizing variables, interpreting graphs and relating them with slope, and identifying slope as a measure of the rate of change. Their knowledge of slope is dominated by geometric ratio conceptualization; on the other hand, algebraic ratio, trigonometric conceptions, and functional property conceptualizations of slope are less comprehended, and their skill to establish connection and transition with these representations were appeared to be insufficient. In addition to teachers' preference for geometric ratio conceptualization while defining slope, Nagle and Moore-Russo (2013) recently found that incumbent secondary teachers and prospective teachers primarily relate slope notion with behavior indicator and infrequently mention determining property, functional property, and linear constant and trigonometric conception. Stump also found a discrepancy between the conceptualizations that high school teachers preferred to use in their definitions and the conceptualizations they used while instructing. Although their dominant knowledge of slopes was related to "geometric slopes", their teaching was based on the concept of "physical properties" of slopes. In addition, studies by Nagle and Moore-Russo (2012) and Zahner (2015) found a gap between the conceptualizations (such as behavior indicators) usually used by college teachers and those commonly preferred by students. These findings indicated that teachers' intended goal of emphasizing an image of slope to their students is different from their concept image of slope.

In addition to teachers' concept images of slope notions, studies also attempted to portray teachers' understanding of the quotient's meaning (Byerley & Hatfield, 2013), measure (Lobato & Siebert, 2002), steepness (Stump, 2001), and covariational reasoning (Byerley & Thompson, 2017; Coe, 2007; Thompson, 1994; Thompson et al., 2017; Zandieh & Knapp, 2006) that are closely related with slope notion as mentioned before. Regarding quotient, for instance, Byerley and Hatfield (2013) found that only six of 17 prospective teachers can interpret the "20.15 times 0.39 is 7.86" as 7.86 is 20.15 times as large as 0.39. These findings demonstrated that they used multiplicative comparison. However, only one prospective teacher can explain the meaning of division while calculating slope. Hence, they need both quotient and relative size meanings to be able to conceptualize the slope notion. Concerning steepness, Coe (2007) found that teachers struggled to identify the distinction between steepness and slope while interpreting positive and negative slopes.

Besides, many studies stated a lack of covariational reasoning abilities as a major cause of difficulty for students and teachers to grasp the concept of rate of change. (Byerley & Thompson, 2017; Thompson, 1994; Thompson et al., 2017; Zandieh & Knapp, 2006). In another study by Coe (2007), three high school teachers were situated with limited connected ways of thinking about constant, average, and changing rate. In linear tasks, a teacher (Peggy) evidenced ideas related to steepness, using vertical and horizontal change to compare values, while Peggy could not explain the meaning of division for slope. On the other hand, another teacher (Mary) utilized a single way of thinking: graphical interpretations of slope (steepness). Pecky could be able to explain the average rate of change as changeover change, while she could not explain slope as a measure, not a ratio. The author concluded that teachers held distinct approaches while interpreting slope tasks, and they had inadequacy in connecting different mathematical ideas related to slope, such as the multiplicative meaning of slope and chunky meaning of slope. With a large sample, similar to Coe's findings, Byerley and Thompson (2017) stated that most high school teachers (n=251) demonstrated procedural and chunky meanings for slope. Moreover, they had an inadequate understanding of the rate of change as the relative size of changes in any two quantities. Considering covariational reasoning in graphics, Mudaly and Moore-Russo (2011) also concluded that converting a stated situation to an equation was less difficult for teachers than converting to a graph or identifying the gradient.

The more sophisticated thinking related to covariational reasoning within the pattern generalizations context address making generalizations based on figural reasoning rather than arithmetic reasoning (involving both correspondence and covariational approach) (El Mouhayar, 2019) since the figural reasoning involves a complex relationship between the cues (Rivera and Becker, 2008). For instance, El Mouhayar and Jurdak (2013) found that most of the middle school teachers' explanations (n=83) are lacking in terms of generalizing given patterns with different strategies (e.g., constructive, or deconstructive strategy) closely related to advanced covariational reasoning.

Studies also attempted to analyze primary or secondary prospective teachers' professional knowledge for teaching and specialized content knowledge related to the functional relationship, including the idea of slope as rate of change in contiguous and non-contiguous table of values and pattern generalizations (Magiera, van den Kieboom & Moyer, 2013; Nagle, Moore-Russo, & Styers, 2017; Rule & Hallagan 2007). Comparatively, fewer studies were performed with practicing teachers (e.g., Demonty, Vlassis & Fagnant, 2018; Wilkie, 2016). Regarding knowledge of middle graders' teachers, for instance, Wilkie (2016) conducted a study with 105 teachers who taught 8 to 12 years old students. She found that less than half showed adequate pedagogical content knowledge; for the function machine task (two tabular representations of consecutive and non-consecutive pairs of values), only one-quarter of the teacher demonstrated reasonable knowledge of teaching algebra. In line with these findings, less than half were capable of mentioning robust students' thinking as an example, and more than half expressed their hesitation in teaching algebra appropriately. They mostly tend to use correspondence approach rather than covariation. These studies have suggested further studies to design professional learning activities to enrich teachers' knowledge on these construct and conceptualizations and quality of their instruction.

In that respect, some studies initiated to enrich teachers' practice on a slope and their understanding of slope and related notions such as proportionality and covariational and correspondence reasoning by engaging teachers in a collaborative environment with modelling tasks (Gonzalez, 2021; Kertil, Erbaş & Çetinkaya, 2020) using tools such as manipulatives or technology (Walter and Gerson, 2007), providing possible students' misconceptions on this area (Ostermann, Leuders, & Nückles, 2018; Stump 2001b), designing workshops including sample students' work (Derry, 2007) and providing set of algebraic tasks addressing task design principles such as interpreting multiple representations (Swan, 2007). Most of those studies take a situative perspective (Greeno 2003) to enrich teachers' practices. For example, Walter and Gerson (2007) designed a content-focused professional development program to enrich practicing elementary teachers' understanding of slope in a collaborative environment. Considering limited connection on the idea of "rise over run," authors pushed teachers to notice and reason for additive patterns while conceptualizing slope notion. In that sense, two tasks were created to enable teachers to use an alternative approach (additive patterns with rods) to make sense of linear relationships. While three teachers plotted the points, one of the teachers (Lyn) preferred to use rods to create stair-step representations of the given slopes (two-thirds and one-half). In addition, she tunneled other participants to reason for the rule of rise over run and demonstrate on the axes. Cuisenaire rods enable teachers to see recursive relationships between variables by iterating the rods, which are invisible while doing a table with points and different from simply comparing fractions. Then, they can make sense of the meaning of the rate of change as physical property conceptualization of slope in graphics. Different from this study aimed at engaging teachers to use a manipulative (rods), Stump (2001b) also devised a study to improve pre-service teachers' conceptions of slope in real-world situations by introducing them to a framework about various conceptualizations of slope and guiding them to detect students' misconceptions or difficulties. The method course included interviewing high school students and a college student and analyzing the students' responses and difficulties, discussing a framework about the representations of slope: algebraic, geometric, physical, and functional, analyzing a textbook, and creating a series of lesson plans for middle or high scholars. Besides, two tasks focusing on the functional property were introduced. This attempt demonstrated their ability to select and create rich slope tasks in their lesson plans while also their difficulty understanding slope as a measure of steepness and rate of change. All three pre-service teachers became aware of the students' limited understanding of slope as rate of change (functional property) and steepness (physical property and geometric ratio) and students' lack of procedural aspect of slope. Moreover, they develop their capability to select and create slope tasks involving functional, physical, and real-life situations. In contrast, two of them could not emphasize steepness and rate of change meanings of slope during the actual teaching. Hence, the author claimed to use an alternative framework to enrich teachers' pedagogical content knowledge on slope. The study indicates the press for slope as steepness and rate of change rather than focusing on only one conceptualization and also engaging them with a nontraditional and challenging task can increase teachers' understanding of slope and efficient teaching of slope.

Within a more general umbrella for slope notion embedded in functional relationships, few studies investigated how professional initiatives can develop teachers' knowledge of functional thinking by collecting data from their practices (Steele et al., 2013; Wilkie, 2016b). To illustrate, Wilkie (2016b) investigated what extent collaborative learning environment for teachers and interaction between researcher and teacher through five consecutive tasks (five lessons) to support ten middle-school teachers' (5^{th} or 6^{th} grade) professional knowledge for teaching (PCK) and specialized content knowledge (SCK) on pattern generalizations. Specifically, the author revisited the learning progression of pattern generalization, including five tasks with teachers, and then the author, as an expert, co-taught three or four lessons with teachers. After the lesson, the researcher and teachers examined students' work and engagement and made plans for future lessons. The findings of the study indicated that "teachers’ knowledge of the process by which students learn to think functionally (KCS) is challenging to develop even in the context of classroom-based teaching experiments" (p.20) in which teachers were studying with their students within a time. In addition, this study has limitations in measuring teachers' progress in their PCK during the lessons due to collaborative teaching with the expert in actual teaching.

Considering those limitations, the author suggested a longitudinal design addressing “an initial intensive session of collaborative professional development, followed by each teacher's own classroom experimentation over a short time period and later follow-up observations in class"(p.22). In conclusion, these professional attempts suggested utilizing frameworks related to slope and related notions and intense collaboration with teachers as well as in their classroom implementations to enrich their understanding of slope and implement high-level algebra tasks robustly. Consistent with these recommendations, I provide 11 conceptualizations using a conceptualization framework that improves students' understanding of the slopes while observing the teacher's consecutive lessons over a month-long period.

Overall, research indicated both students' difficulties and teachers' lack of professional knowledge for teaching and content knowledge regarding slope and its related constructs such as covariational reasoning. The studies on developing in-service teachers' knowledge of algebraic thinking and slope have a shortage of development programs that do not focus on teachers' practice. In turn, they reported teachers' inadequacy in determining fundamental conceptualizations of slope and how students have better learn of it (Steele et al., 2013; Stump, 1999; Wilkie, 2016) and responding to different students’ thinking and understandings. Moreover, it was stated that teachers also have inadequate knowledge regarding chooosing and implementing the high-level cognitive demand of mathematical tasks (Otten & Soria, 2014; Stylianides & Stylianides, 2008), even for slope tasks (Otten & Soria, 2014; Wilkie, 2016) (Furthermore, teachers were found to be inadequately knowledgeable about selecting and implementing the high cognitive demands of mathematical tasks, even on slope tasks (Otten & Soria, 2014; Stylianides & Stylianides, 2008). Choppin (2011) underlined that teachers who become aware of students thinking are better at implementing tasks with high cognitive demand. To break the chain between the inadequate implementation of high-level slope tasks and nonrelational students' understanding, teachers need to develop robust professional noticing skills (Jacobs & Empson, 2016) that are supposed to be interconnected with knowledge, orientations (Schoenfeld, 2010;

van Es, 2010) and practice (Lee & Francis, 2018). Although teachers' noticing has emerged as a focal point of professional development research (Sherin et al., 2011), there is limited research on supporting teachers to learn how to attend or respond to students' thinking during the enactment of cognitively demanding algebra tasks (Callejo & Zapatera, 2017; Goldsmith & Seago, 2011). Hence the current study aimed to elicit and develop a teacher's noticing of students' algebraic thinking within a highly cognitively demanding task context, which is expected to add literature by developing learning activities for a practicing teacher with a closer look at her instruction. In the following section, a review of the literature on definition and conceptualization of noticing, studies on assessing and developing awareness on understanding researchers' approaches, related studies on teachers' noticing skills on algebra and slope, particularly with a focus on in-service teachers' development of noticing skills of is provided.

2.3. Conceptualizations of Noticing

As the concept of noticing has attracted intense attention from mathematics education scholars over the past two decades, they have tried defining and describing the awareness construct. First, Mason (2002), to separate daily noticing from professional noticing, professional noticing referred to as attention on what someone acts professionally. Compared to Mason, Van Es and Sherin (2002) propose a more detailed explanation for the construct. They defined it as "identifying what is important or noteworthy about a classroom situation; making connections between the specifics of classroom interactions and the broader principles of teaching and learning they represent; and using what one knows about the context to reason about classroom interactions" (p.573). Their focus is on both identifying and interpreting the important events rather than only paying attention to a moment. Later, based on the first aspects of noticing defined by Van Es and Sherin (2002), similar to Mason (2002); Star and Strickland (2008) describe noticing as "what catches their attention, and what they miss-when they view a classroom lesson" (p.111). This approach focuses on examining what a teacher attends to moments, which are regarded as critical, and what a teacher does not attend. Star and Strickland (2008) categorized these moments into five related categories: classroom environment, classroom management, tasks, mathematical content, and communication.

Other researchers have begun to analyze what teachers attend and how they interpret the events attended (e.g., Colestock & Sherin, 2009; Sherin, 2007; Sherin & van Es, 2009). First, Sherin (2007) identified ‘professional vision’ based on the work of Goodwin (1994) concerning two-dimension: ‘selective attention’ and ‘knowledge-based reasoning’. Selective attention is grouped into two main categories: Actor (e.g., teacher, student/s) and Topic (e.g., classroom management, environment, and math topic) that are related to what specific teacher pays attention to. On the other hand, "knowledge-based reasoning" corresponds to the degree of interpreting the moments related to students' thinking. In that sense, this dimension was divided into two main categories: Stance (describe, evaluate and interpret) and Strategy (e.g., questioning students thinking, connecting students thinking with general principles for teaching and learning). Based on the author's explanations of these dimensions as dynamically related to each other, it is argued that the complex relationship between central and sub-components.

In later work, van Es and Sherin (2008) analyze teachers' noticing through four categories: actor, topic, stance, and specificity. They change the category of *strategy* in the work of Sherin (2007) to the category of *specificity* to distinguish what and how the noticing evolved. Based on the literature, van Es (2011) described noticing solely within two dimensions of noticing What Teacher Notice and How Teacher Notice through four levels: Level 1-Baseline, Level 2- Mixed, Level 3-Focused, and Level 4-Extended (see Figure 2). The first dimension attained both *whom* the teachers attend in the video clip that accounted for whether participants focus on the class as a whole, students as a group, particular students, teacher behaviors, or themselves and *the topic* of their focus including issues such as pedagogical strategies or behavior or thinking. The second dimension is how teachers interpret what they notice, including both *analytic stance* (evaluating and interpreting) and *depth of the analysis* (providing shreds of evidence or elaborating on their critiques). Additionally, for each category, the author presents a developmental trajectory of any participants' noticing in order to identify progression in learning to notice over time. At the bottom of the trajectory, the What Teacher Notice dimension is about attention to the whole class environment, general pedagogy, and students' learning how Teacher Notice is concerning forming general conceptions on what happened, providing descriptive and evaluative stance with limited or no evidence. While at the top of it, *What Teacher Notice* presents attention to relationships among particular students' outcomes, teaching strategies, and students' thinking, *How Teacher Notice*, on the other hand, corresponds to the elaborating on important events/ moments, interrelating students' thinking to teaching and learning principles or concepts and providing alternative pedagogical responses. As seen in Table 2, from Level 1 to Level 4, teachers' attention shifts from general classroom issues to the specific student thinking. Teachers' comments vary from descriptive and evaluative to in-depth and interpretive and include alternative solutions for the specific students' thinking.

With a different perspective on the definition of noticing, Jacobs, Lamp, and Philipp (2010) prefer a particular focus on noticing children's mathematical thinking. Then they describe *professional noticing of children's mathematical thinking* (p. 169) as "how, and the extent to which teachers notice children's mathematical thinking" (p. 171). Based on this definition, the authors concentrated on "the extent to which noticing occurs, as opposed to the variety of what is noticed" (Amador, 2020, p. 316). Three interconnected skills as i) attending to the way of learners' solution, ii) interpreting learners' thinking iii) determining how to respond based on learners' thinking form noticing. They also proposed analytic codes for each component: Lacking, Limited, and Robust. Some researchers propose a sequential relationship between these three skills, whilst others argue for a more flexible approach in which teachers' attentiveness and interpretations may occur concurrently. (Superfine et al., 2017). These core frameworks have similarities and differences regarding their definitions and purposes, as explained in the next paragraph.

Table 2. Framework for Learning to Notice Students' Mathematical Thinking (van Es, 2011, p. 139)

	What Teacher Notice	How Teacher Notice
Level 1 (Baseline)	Attend to whole class environment, behavior, and learning and to teacher pedagogy	Form general impressions what occurred. Provide descriptive and evaluative comments. Provide little or no evidence to support analysis.
Level 2 (Mixed)	Primarily attend to teacher pedagogy. Begin to attend to a particular students' mathematical thinking and behaviors.	Form general impressions and highlight noteworthy events. Provide primarily evaluative with some interpretive comments. Begin to refer to specific events and interactions as evidence.
Level 3 (Focused)	Attend to particular students' mathematical thinking	Highlight noteworthy events. Provide interpretive comments. Refer to specific events and interactions as evidence. Elaborate on events and interactions.
Level 4 (Extended)	Attend to the relationship between particular students' outcomes and between teaching strategies and student mathematical thinking	Highlight noteworthy events. Provide interpretive comments. Refer to specific events and interactions as evidence. Elaborate on events and interactions. Make connections between events and principles of teaching and learning. On the basis of interpretations, propose alternative pedagogical solutions.

Both van Es and Jacobs and colleagues have common points in their definitions of noticing in which it is multifaceted and making sense of what is attended is highlighted. However, these researchers "differed on their definitions of noticing and their creation and use of analytic framework" (Amador, 2021). Concerning variants in their design, *Framework for Learning to Notice Students' Mathematical Thinking* is based on utilizing video clubs without editing, whereas the other is grounded on utilizing video cases, which contain specific students' conceptions and misconceptions about a mathematical idea. Regarding variances in the ways of definitions of noticing authors constructed, *Professional noticing of children's mathematical thinking* is grounded on "the extent to which noticing occurs" as compared to *Framework for Learning to Notice Students' Mathematical Thinking*, which depended on "the variety of what is noticed". With a similar aim to identify differences between frameworks, Stockero, Ropnow, and Pascoe (2017) divided the professional noticing studies into two branches, *noticing within an instance* and *noticing among instanc*es with respect to their methodological preferences. In the first group, teachers or PTs were asked to identify and make sense of given specific instances about the students' misconceptions or conceptions and propose pedagogical decisions based on those (e.g., Ulusoy & Çakıoğlu, 2021). In that sense, Jacobs, Lamp, and Philipp (2010)'s work is an example of the first group of studies since specific incidents of critical students' thinking were provided rather than a whole video without selecting any critical moments. In the second group, pre-service/in-service teachers selected an important aspect of the classroom videos, and then they were asked what they had noticed and how they had noticed it. Some studies (e.g., Sherin & van Es, 2009; Star & Strickland, 2008; Walkoe, Sherin & Elby, 2020; van Es & Sherin, 2002) measured what teachers pay attention to while observing the video of instruction. Besides, few studies combined both approaches while designing to elicit and analyze noticing skills of teachers. To illustrate, in Walkoe (2015)'s study, PTs picked three important events from the videos as among instances. Then, teachers were supposed to evaluate specific students' thinking using the "Algebraic Thinking Framework" within instances. Similarly, in the current study, I focused on a teacher's noticing skills in respect of her attention to students' thinking (possible) and her pedagogy, as stated among instances. In addition, I measured a teacher's noticing skills via video cases involving classroom interactions and sample students' work which is critical, as specified within instances. Since I need to assess and develop a teacher's noticing of students' algebraic thinking and detect how teacher attend in lessons without any prompts or incidents grounded in van Es's framework. In addition, the consistency between the purpose of the current study and van Es's framework for portraying teachers' development on noticing and the difficulty in distinguishing these three skills as components of Jacob's framework (Barnhart & van Es, 2015) guide me to use van Es' framework to analyze a teacher's noticing. Although the framework is based on noticing among instances, on developing her attention to specific moments, or ideas, it was also attempted to assess her noticing via specific incidents. Thus, I framed the skills by modifying "A Framework for Learning to Notice Student Mathematical Thinking" (van Es, 2011; Figure 3). And it helped me to examine the trajectory of development of an in-service teacher's what the teacher focused on and to what extent they interpreted the attended issues (van Es, 2011) in the context of high cognitive demand of slope tasks. As decided to be used in other studies, some adaptations have been made in these main frameworks.

Some studies attempted to adapt or modify those frameworks based on their emerging data and research aim (e.g., Amador, Carter, Hudson, 2016; Ding & Dominguez, 2016; Estapa & Amador, 2016; Estapa, Pinnow, & Chval, 2016; Ulusoy & Çakıroğlu, 2021). Considering van Es's framework, these changes were mostly based on adding subcategories to determine a detailed inspection of what and how is noticed. For instance, van Es (2011) focused on who under what dimension, whereas Amador (2016) distinguished "who" from "what" and "who" composed of two dimensions: Teacher and Student. On the other hand, some researchers have also changed the coding schema for three noticing tenets included in Jacobs' framework (e.g., Teuscher, Leatham & Paterson, 2017; Ulusoy & Çakıroğlu, 2021). For instance, while in the original work of Jacobs et al. the extent of evidence for the three-component of noticing (attending, interpreting, and responding) was characterized as *robust evidence, limited or lack of evidence,* Magiera and Zambak (2021) used a different name for pre-service evidence for student justifications and generalizations as *highly focused, partially focused and superficial.*

Apart from the studies in which slight differences were made for only sub-categories, some studies have attempted to revise the main categories or add a new dimension. First, these attempts emerged from the differences in nature of the research questions in the studies, which are concerned with the detailed aspects of participants' noticing (e.g., Van Es et al., 2017). Second, these attempts emerged from the concerns about the boundary of the noticing; in other words, when the noticing would be measured is an issue (e.g., Amador et al., 2017; Scheiner, 2016; Sherin, 2017). Some studies consider noticing during teaching and/or planning (Amador et al., 2017; Bakker, de Glopper, & de Vries, 2022; Kılıç & Doğan, 2021; Luna and Selmer, 2021), and some highlight that teacher's noticing occurs after the lesson while reviewing it (Choy et al., 2017). Considering the moment of noticing while teaching, to illustrate, van Es and Sherin (2021) revisited their prior definition, including "attending" and "interpreting" aspects, and suggested a new aspect, "shaping," based on the existing literature. The last component involves teachers' attempts to make a student's thinking visible rather than advancing one's thinking. Thus, *shaping* is distinguished from the *how to respond* component (Jacobs et al., 2010) concerning boundaries of noticing and teachers' motive to attempt further based on the students' thinking. In other words, *shaping* is measured at the moment of instruction, not after or before any instruction or while observing the video, and *shaping* involves acting to understand one's additional mathematical thinking in a more profound sense. These recent attempts highlight the importance of assessing teachers' practices not only during the reflection on videos but also in planning and implementing the lesson. For that reason, similar to Choy's (2017) attempt to create a framework for an "idealized process of productive noticing" (p.452) through the main stages of practice (planning, teaching, and reviewing), I adapted van Es's framework to three main stages. For instance, in Choy's framework regarding deciding to respond component, in the planning phase, the skills including to "develop and implement a high - level cognitive demand task to target students' confusion about the concept" in the teaching phase, the skills accounting for asking questions aiming to reveal student's thinking about the concept, listening and preparing a reply to student's thinking or reasoning." Last, for reviewing part, *a productive action* is defined as revisiting "the task based on understanding how students may think about the concept" (p. 453). Apart from addressing the framework to analyze data, how to elicit is another concern for noticing studies.

2.3.1. Eliciting noticing

Most studies have elicited prospective or in-service teachers' noticing skills by using whole class videos or video clips of specific students' mathematical thinking or teacher-students discussion (Ding & Dominguez, 2016; Fernandez et al., 2013; Jacobs et al., 2010; Lessig et al., 2016; Llinares & Valls, 2010, Schack et al., 2013; Sherin & van Es, 2009; Warshauer, Starkey, Herrera, & Smith, 2021) and then by asking them about what has been noticed. In those studies, researchers use general or specific questions about elements of videos or students' thinking in video clips. For instance, while Sherin and van Es (2005) generated broad questions such as "what did you notice?" Ulusoy and Çakıroğlu (2020) asked more specific questions "Can you tell me more about why the student defined trapezoids in such a way?" Moreover, in some studies (e.g., Hollingsworth & Clarke, 2017), researchers provided teachers with a set of main elements of teaching such as "Questions and discussions for mathematics learning" or "Tasks for mathematics learning" as a guideline while responding to general questions, "What are your thoughts about what you saw?" (p. 467). In both situations, noticing of the participants has been measured and elicited, aligning with the aim of the study. In addition to these methodological issues, teacher's noticing expertise was also characterized when they reflected on the video or written students' work individually (Doğan-Coşkun, Tekin-Sitrava, & Işıksal-Bostan, 2021; Kılıç, 2016; Star & Strickland, 2008) or in a group (Walkoe, 2015). A few studies combined these two methodological designs (Schack et al., 2013; Ulusoy and Çakıroğlu, 2020). Besides, studies differ in the type of video selected from their classrooms or other's classrooms and unknown students' thinking. Karsenty and Sherin (2017) reviewed five articles conducted in various countries with both prospective and incumbent teachers to identify professional development contexts, including videos. Parallel with the way of using the video mentioned in prior works, the authors aligned those contexts as: “teachers watching their own video and teachers watching a video of unknown colleagues; teachers watching whole lessons and watching selected clips; rubric-based video inspection by teachers leading to a systemized feedback, and teachers' observations that discard evaluations altogether ”(p. 412). To conclude, issues related to methodologies of the studies, including characteristics of the video, including teachers’ own videos, or others, or using frameworks with videos, and interview questions, have varied due to the variations in teaching perspectives (cognitive vs. situated) of researchers and the aim of those.

When studies conducted with incumbent teachers, in particular, are taken into account the common methodology is to make teachers reflect on video clips of their own instruction within a group of other teachers (van Es and Sherin 2008) and written artifacts of students' thinking (Jacobs et al. 2011) or their own and other classrooms videos (Goldsmith & Seago, 2011; Hollingsworth & Clarke, 2017; Santagata & Yeh, 2016). Teachers can attend to specific aspects of the instruction, such as students’ thinking, rather than other issues such as classroom management or students' behaviors after engaging in professional development attempts. Specifically, Hollingsworth and Clarke, (2017) characterize the experiences of teachers with videos as “video as a mirror for teachers providing a visible record of activity in their own classrooms; video as a lens providing an opportunity to re/view video records to consider different levels of detail or different perspectives; and video as a window into other classrooms revealing alternate methods and possibilities” (p. 472).

However, the question of “how they react in the moment of teaching and what aspect of a moment of complex instruction they notice” can be emerged as an issue to enable efficient student learning in classrooms. In that sense there is a need to examine noticing of teachers during the acting of teaching (Nickerson, Lamb, & LaRochelle, 2017; Sherin & Star, 2011; Teuscher, Leatham, & Peterson, 2017). With a different perspective, noticed elements of videos may not be similar with issues noticed during teaching (Sherin, Russ, & Colestock, 2011b). This possible discrepancy present a problem which should be identified or resolved to develop teachers' quality of instruction.

Debate on assessing in- the- moment noticing (see Sherin, Russ, and Colesrock, 2011) inclined researchers adopt new methodologies (e.g., Sherin & Dyer, 2017; Sherin, Russ, & Colesrock, 2011) beyond showing video and asking questions on what is noticed. For instance, Sherin and colleagues (2008) employed an innovative methodology to study the noticing of thirteen teachers in the real l classroom contexts. This methodology relied on what teachers select at crucial moments by pressing the record button of a wearable camera. Teachers can select thirty clips by using the button throughout a lesson. After each lesson, the author asked teachers why they selected the clips and identified the instruction aspects they paid attention to. Other researchers also implemented this perspective in their designs (e.g., Colestock, 2009; Luna et al., 2009; Taylan, 2015). Considering drawbacks of this new methodology, such as possibility of distracting the natural teaching environment, the limited time provided to record the moment, and teachers 'struggle to store issues related to the selected events in their minds (Sherin, Russ, & Colesrock, 2011b), some researchers performed traditional ways to elicit teachers' noticing. Those ways include conducting retrospective interviews immediately after the lesson to ask teachers what they noticed and attracted their attention during their instruction (e.g., Luna & Selmer, 2021; Colestock, 2009) or watching videos of their instruction or video clips (Ainley & Luntley, 2007). Moreover, some researchers have tended to assess teachers' enacted noticing from their recorded instruction by eliminating teachers' perspectives. To sum up, although each methodology has own drawbacks, parallel to my second aim of the current study portraying the teacher's what and how to notice during the planning, teaching and reflecting stages of a coaching program, I used a traditional way to elicit her noticed issues regarding implemented lesson with retrospective interviews since noticing-in-the-moment and noticing-after-the- moment are mutual pairs supporting each other (Bakker, de Glopper, & de Vries, 2022). The studies mentioned above highlight teachers’ lack of noticing important aspects of instructions or students’ thinking yet noticing skills can be improved by appropriate professional attempts and artifacts such as tasks and videos. The next session would explain the characteristics of those initiatives and to what extent teachers’ noticing skills are improved.

2.3.2. Studies on what extent or how teacher noticing is enhanced

Utilizing diverse pedagogical methods to facilitate detecting, trying to make sense of, and making judgments based on specific students' thinking in a variety of classroom artifacts has been the focus of research on increasing prospective and current teachers' ability to observe (Stahnke, Schueler, & Roesken-Winter, 2016). While most of the studies preferred to use video club designs through video clips (Prediger et al., 2015; Star & Strickland, 2008; van es & Sherin, 2008), another line of studies has focused on embedding students' artifacts (Walkoe, 2015), to use frameworks (Warshauer, Starkey, Herrera, Smith, 2021), to scaffold hypothetical learning trajectories, in a variety of context like Lesson Study (Choy, 2016; Güner & Akyüz, 2019), and other professional activities such as using high-level tasks (Hallman-Thrasher, 2017; Kılıç & Doğan, 2021; Luna & Selmer, 2021).

First, the standard artifacts, video clips or instruction videos are used to make teachers attend important events and interpret those events (Sherin & Dyer, 2017) or support them to highlight the events that were found to be crucial. Numerous video-based programs have selected and sequenced others' video clips (e.g., Seago et al., 2004; Walkoe, Sherin & Elby, 2020). Some also provide explicit analytic tools to support teachers in relating issues in cases or whole videos with provided criteria. (Goldsmith & Seago 2012). Other programs have attempted to make teachers discuss their classrooms' video clips in-group discussions. These programs also guide teachers to focus on specific aspects of teaching, such as students' way of thinking or teachers' actions. For example, Borko and colleagues (2015) first asked teachers to interpret students' thinking and then teachers' actions. In another line of studies, authors also make teachers select important cases at first (during teaching), then let them discuss the cases in groups (Sherin & Dyer, 2017). In all of these programs, researchers assumed that teachers learn critical aspects of content, students thinking or teacher actions from video through watching and discussing the video. These attempts also yield substantial learning of students' ideas and develop new ideas to interpret students' ideas (Dyer 2013; Sherin & Han 2004; Sherin 2007) as well as they become attending particular students' ideas and using the strategies discussed in groups to make sense of students' ideas during their own teaching (Borko et al., 2015; van Es & Sherin, 2010).As an example, Sherin and Dyer (2017) make three groups of middle and high school teachers select video clips of three to five lessons before, during, and after the instruction. Making them focus on important students' thinking and prompting them to consider the reason behind these selections enable teachers to create different strategies such as anticipating students' thinking and being ready to capture and respond to these ideas during the teaching. Despite the strength of both types of video experiences, the difference between observing own classroom's video or other teachers' videos reasoned a variation in the extent of the noticing. For instance, Seidel and colleagues (2011) concluded that teachers who observed their own teaching enriched their noticing skills better than those who commented on others' instructions. Another concern for the video-based development program is whether teachers reflect noticed elements through videos into their lessons or not. Sherin and colleagues (2011) revealed a discrepancy between teachers' interpretations of impactful moments in the video clips and the way of acting upon those critical moments at the time of teaching. In that sense, researchers have been inclined to incorporate other pedagogical tools such as frameworks related to instructions or content areas.

Second artifacts including frameworks and noticing frameworks which are used as pedagogical tools to guide teachers in their noticing skills in specific contexts within a video club designs. Findings in those studies indicate that frameworks can help teachers develop their noticing skills of specific aspect of instruction or students' thinking. Those frameworks are used to incline teachers' noticing of students thinking within a particular perspective (Santagata & Guarino, 2011; Stockero et al. 2017; Walkoe 2015; Warshauer, Starkey, Herrera, & Smith, 2021), such as the critical moments as a mathematical opportunity (Specifically, Mathematically Significant Pedagogical Opportunities to Build on Student Thinking (MOST) (e.g., Stockero et al., 2017; Teuscher et al., 2017) or within a particular content domain (Algebraic Thinking Framework, Walkoe, 2015). In addition, in some studies, facilitators used scaffoldings to support noticing. To do so, Hypothetical learning trajectories (HLT) have been used as tools to enrich teachers in making sense of students' thinking and responding to thinking within various content areas such as statistics, fractions, early algebraic thinking, early number sense (e.g., Choy & Dindyal, 2021; Ivars, Fernandez, Llinares & Choy, 2018; Jong, Schack, Fisher, Thomas & Dueber, 2021; Schack et al., 2013).

In addition to frameworks, third artifacts include tasks and letting them discuss expected students' difficulty or understanding related to the tasks within reflection assignments (Kılıç & Doğan, 2021) also indicate that enables pre-service teachers to shift their attention from superficial characteristics to the task nature and students' thinking. Similarly, Hallman-Thrasher (2017) pressed pre-service teachers to write expected students' responses to the tasks, including various content domains. The study's findings indicated that two of three groups of pre-service teachers increase their ability to maintain cognitive demand of the task by responding adequately anticipated, unanticipated correct and incorrect students' answers while studying with two five-grade students. Apart from the positive effect of careful planning, the critical findings of the study are that for the task in the algebra content domain, teachers have difficulty responding to some correct (anticipated and unanticipated) and incorrect students' answers. These findings highlighted both importance of lesson planning to improve noticing skills in the moment action and also equipping teachers with not only incorrect but also correct students' answers.

Although those mediums and different methodological attempts enhance both pre-service and practicing teachers attend to, interpret students' ideas and respond robustly, and give valuable insights on how to enrich teachers’ ability to notice, they struggle with making sense of students' thinking and relation it with teaching and learning principles and making instructional decisions based on those ideas (Fernandez & Choy, 2020). Expressly, studies indicated that teachers find it challenging to interpret and decide pedagogical responses to students' thinking or questions than to attend to students' thinking (e.g., Derry, 2007; Dreher & Kuntze, 2015; Schwarz et al., 2018; Teuscher et al., 2017; Vogler, 2015). In fact, studies indicated that the skill of deciding is the most challenging skill due to complexity of teachers' in the moment decision making (Choy, 2016; Lee & Francis, 2018; Stahnke, Schueler, & Roesken-Winter, 2016) even for experienced teachers (Lee & Choy, 2017). Still experienced teachers should be provided with opportunities for developing ability to notice important events and reasoning about students’ thinking as anticipated or unanticipated during planning, teaching and reflecting process through supportive professional development (Coddington, 2014). At this point, related to the second aim of the present study, it is attempted to examine a practicing teacher’s noticing of students’ algebraic thinking in the context of a high-level mathematical tasks since there is a need for research on practicing teachers’ noticing of students thinking within a particular content domain, algebraic thinking, slope notion, and research on teachers’ development of noticing through efficient professional development. The next session would discuss studies on teachers’ noticing skills on algebraic thinking, slope specifically and unfold which professional attempts been designed to increase teachers’ learning to notice.

2.3.3. Teachers’ noticing of slope and professional attempts

In the last decade, teacher educators have focused on the context-specific characteristics of instruction and students' thinking in a particular content domain. As Dindyal and collegues (2021), “Context is broadly construed as there is noticing research set in mathematical contexts and along content trajectories much in a classroom environment that influences teaching mathematics and teacher noticing" (p.8). Studies are mainly concerned with teachers noticing skills in a broad range of content domains and along content trajectories, including proportional reasoning (Ivars et al., 2020; Son 2013), derivative (Sánchez-Matamoros et al. 2019), pattern generalization (Callejo & Zapatera 2017; Lee & Lee, 2021), algebraic thinking (Walkoe, 2015), early algebra (Fisher et al., 2019), quadrilaterals (Ulusoy & Çakıroğlu, 2021), measurement (Moreno, Sánchez-Matamoros, Callejo, Pérez-Tyteca & Llinares 2021), rational numbers (Pouta, Lehtinen & Palonen, 2020), fractions (Lee, 2021), statistics (Choy & Dindyal, 2021), and early number sense (Schack et al., 2013).

Grounding on the calls for renewals in the algebra teaching and learning (Kaput, 2008) and the need to investigate elements with which teachers should be equipped (Kieran, 2007), in this study, I focused on the slope notion under the algebra branch to support an in-service teacher noticing of student' thinking in the context of high cognitive demand mathematical tasks. Because I centered slope notion on my focus due to paramount role for students to connect other topics (e.g., calculus, functions) and disciplines (e.g., chemisty, physics). Second issue is related its multifaceted nature related to various conceptualizations, and representations. So many challenges encountered by teachers and students regarding slope (Nagle, Martínez-Planell & Moore-Russo, 2019) and there is a difference between their concept image related to slope and the way of teaching (Stump, 2001). Therefore, teachers most likely have incompetency for implementing high level slope tasks. Hence these two issues showed that slope notion is important, yet it is complex. Nevertheless, much of the work on teacher noticing is interested in the context of pattern generalizations and functional thinking in the domain of algebra rather than slope in particular. However, to get insight related to teachers' noticing in the domain of algebra, the next section is devoted to presenting studies related to both elements of what they notice in the pattern generalizations/algebraic reasoning and how activities are developed to reinforce teacher noticing in the domain of algebra and also slope.

Recently, research (e.g., Teuscher et al., 2017; Van Zoest et al., 2017) have closely looked into teachers' professional noticing in the domain of algebra, including the notion of slope, ideas related to functional relationships through pattern generalization contexts. Studies indicated that pre-service and in-service teachers have struggled to attend to and interpret critical ideas in a sophisticated way (Callejo & Zapatera, 2017; Lee & Lee, 2021). For instance, Styers, Nagle and Moore-Russo (2020) explored seven secondary teachers' noticing (ratio, behavior indicator, steepness indicator, determining property, extension to calculus, parametric coefficient, and real life). To gain more insight into the teachers' interpretations of slope, some of the statements in the study were left purposefully unconnected to a particular component. Findings revealed that teachers attend the vocabulary of the given statements to determine which conceptualization is used by students. For instance, they interpreted the word "rate of change" as associated with real-life applications. Besides, they favored the language of "change in y over change in x" over the language of "rise over run". Teachers also attended different subcomponents of slope conceptualizations. Namely, they related the statement that "slope is represented by m in equations and formulas" with value of m in the algebraic form (y=b+mx) and also behavior indicator of a line (i.e., corresponding to whether a line is increasing, decreasing, horizontal, or vertical) without any reasoning provided, interpreted language related to derivatives as only nonvisual. In addition, they interpreted ideas related to Steepness as linked to real-world situations by isolating its mathematics aspect. Although they were aware of the differences between sample students' thinking, including Ratio-Nonvisual (geometric ratio) and Ratio-Visual (algebraic ratio), they lack to connect physical situation (static) to the notion of rate of change. Hence, the author suggests a need to design professional development experiences for teachers involved in observing the teaching of rich slope tasks, including physical real-world situations.

Based on these studies' call on enriching teachers' noticing in the domain of algebra and the crucial role for managing critical moments at the moment of teaching of teachers (Van Zoest et al., 2017), some studies have been interested in designing professional development efforts. Thus, they have focused on enriching teachers' algebraic thinking through presenting video clips, making teachers conduct interviews with students, using frameworks, rich mathematical tasks and sample students' responses (Callejo & Zapatera, 2017; Derry, 2007; Lesseig et al., 2016; Simpson & Haltiwanger, 2016).

To begin with, due to power of videos on teachers' noticing; while in some studies, videos were used as a medium within a collaborative environment (Walkoe, 2015), in some of those, videos presented teachers to identify their noticing skills individually (La Rochelle et al., 2019). Besides, some studies make teachers/pre-service teachers solve first given algebraic tasks and analyze their own solutions and students' solutions (from less sophisticated to the most) in groups (Callejo & Zapatera, 2017). Moreover, for instance Lesseig and colleagues (2016) constructed an interview package to help their prospective teachers' professional-noticing competence of students' mathematical reasoning about linear equations. They discovered that the interview module, which comprised of questions designed to elicit students' mathematical thinking about linear equations, assisted teachers in attending and interpreting the interviews, but not in deciding how to reply.

Another line of few studies employed a variety of tools and ways, such as the Algebraic Thinking Framework within a collaborative environment (Walkoe, Sherin & Elby, 2020; Walkoe, 2015). Rather than pattern generalization context, within the categories of algebraic reasoning: Symbolic Manipulation corresponds to symbolic manipulation and procedures, and Reasoning and Representations concerns reasoning about and with representations of functions, Walkoe (2015) found that in Session 2, the majority of the discussion (63%) of the pre-service teachers have low characteristics with respect to its level of depth whereas in session 6 most of the conversation (89%) at a high level. Walkoe also analyzed shifts in individuals' awareness/thinking within two dimensions of algebraic thinking, Symbol Manipulation and Reasoning about Representations. Pre-service teachers indicate progress in both categories in different ways through weekly assignments. Within the Symbolic Manipulation dimension, the increase in depth from early to the last assignment was slow when looking at the pattern of the Levels 0, 1, and 2. On the other hand, the increase in depth from Level 1 to 2 progressed from early to the last assignment for the dimension of Reasoning and Representations. In other words, while most codes were accounted for Level 1 even in for late assignments in the dimension of Symbolic Manipulation, the majority of depth of thinking reached Level 2 in the dimension of Reasoning and Representations. This finding indicated that teachers can progress in identifying and interpreting students’ thinking within the dimension of Reasoning and Representations. This indicate that a specific framework within video club design related to algebraic thinking was a medium to enrich teachers' attending and interpreting of the sample of students' thinking.

Much of the prior studies’ findings revealed some progress in teachers’ identifying and interpreting students’ thinking on functional relationships. However, some pointed to findings that most of the teachers' weaknesses in interpreting high-level students' thinking related to coordinating the step number and visual patterns, which is accounted for covariational reasoning (Callejo & Zapatera, 2017) and responding to students' thinking (Lesseig et al., 2016; LaRochelle et al., 2019).

To sum up, the extant literature highlights various definitions for noticing, a broad range of methodological issues to elicit noticing skills ranging from standardized tests to observing teachers during instruction, and a variety of tools to enrich teacher noticing such as using frameworks, scaffolding teachers or using video (mainly used). In addition, most of the studies concentrated on pre- or in-service teachers as participants coequally, except for a few studies in which participants consisted of both pre-service and in-service teachers (Lee & Choy, 2017). In particular, studies interested in investigating context-specific noticing ranging from various content domains to different situations, such as Lesson Study. Based on these studies, a relatively few studies attempted to portray teachers' noticing related slope notion (Styers, Nagle, & Moore-Russo 2020), whereas a bit more studies investigated teachers’ noticing regarding algebraic reasoning/functional reasoning and pattern generalization context in particular. Studies indicated that pre-service and in-service teachers have struggled to attend to and interpret the critical ideas in a sophisticated way (Callejo & Zapatera, 2017; Derry, 2007; Lee & Lee, 2021) and revealed both pre- and in-service teachers' inadequacy in responding to students thinking even if various development approach was employed (e.g., LaRochelle et. al., 2019; Luna & Selmer, 2021). Therefore, the field lacks characterizing how to develop in-service teachers' noticing expertise regarding slope notion during instruction.

It is essential to understand students' thinking process, misconceptions, and challenges. (Ball et al., 2008); some core practices are anticipating students' thinking, selecting an appropriate task, and reviewing the lesson (Akyüz, Dixon and Stephan, 2013) to improve quality of teaching. Similar to these practices, as Mason (2002) put it, "noticing is an act of attention, and as such is not something you can decide to do all of a sudden. It has to happen to you, through the exercise of some internal or external impulse or trigger" (p. 61). More specifically Mason (2002) put emphasis on attention on anticipation of students' thinking and envisioned teaching pedagogy is crucial to respond unanticipated or anticipated moments. With regard to at the moment noticing, a number of studies on situating teachers noticing development in the domain of algebra were conducted through video clips or videos of teaching, high-level tasks, and frameworks (LaRochelle et al., 2019; Walkoe, 2015; Walkoe, Sherin & Elby, 2020). Nevertheless, upon closer examination of these researches, they mainly focused on how teachers reflect on action rather than how they reflect in action (Schön, 1991). In other words, what stands out to teachers in the algebra classroom in particular, and how can we help teachers attend to implement high-level algebra tasks and students' thinking in ways that will create a more ambitious learning environment. These questions have remained unanswered entirely yet, and I take a further step toward answering these questions by exploring teacher noticing in the moment of teaching of algebra. The fact that nearly half of the learning opportunities regarding linear equations in classrooms are invisible to students (Van Zoest et al., 2017) indicates the crucial role of teachers in acting. Hence, studies suggested designing further professional activities to provide teachers with expertise to interpret and respond to students' algebraic thinking. In addition to importance of role of noticing during planning and teaching, the results of many studies indicate the power of reflection on lessons implemented considering instruction triangle, tasks, students, and the way of teaching (Fernandez et. al., 2003). Hence it is critical to investigate role of noticing during planning (Choy, 2017), teaching and reflecting (Yang & Ricks, 2012). To leverage the noticing skills in the classrooms, it is also essential to understand how professional development designs take place through these three stages.

Hence one of the crucial attempts to develop teachers' noticing skills is engaging them in planning, teaching and reviewing cycles considering students' conceptions, misconceptions, and task nature (Hallman-Thrasher; 2017; Son & Kim, 2015). Choppin (2011) revealed that teachers' attention on students' thinking influenced teachers' way of implementing highly cognitively demanding tasks. This research, in turn, informs present study on how developing teachers' noticing skills can support teachers' way of implementing highly cognitively tasks and making students engaging high-level tasks. In addition, limited prior research on video clubs provide frameworks related to any content area (Walkoe, 2014) or lesson planning or students' thinking (Santagata, 2011). However, the use of framework as a guide has proven beneficial in teacher reflection on teaching of algebraic thinking specifically (Walkoe, 2014) and in other areas of teacher reflection on practice (Scherrer & Stein 2013). Therefore, as a second issue for developing teachers' noticing, the present study was grounded on a slope conceptualizations framework (Nagle et al., 2019) to make a teacher aware of different conceptualizations, how students understand the conceptualizations along a trajectory and guide her discuss the task design with respect to these conceptualizations. Furthermore, similar to the situated approach to developing teachers' noticing skills (Kaiser, Busse, Hoth, Konig, & Blomeke, 2015), I grounded my study on professional coaching within a highly cognitively tasks context since mathematic coaching possess a cyclic nature of planning, teaching and reviewing that resembles to stages of investigating noticing skills of a teacher. This professional program can sustain ongoing and intense collaboration with teachers on task, teacher pedagogy and students' thinking that likely enabled teachers learning from enacting high level tasks and provided development to teachers within their classrooms (Fennell, 2017). These features of coaching most likely address a gap of traditional teacher development programs regarding aspects of reform-based approaches (Desimone & Pak, 2016). The next section would characterize definition of coaching, different models and frameworks related to coaching program and coaching activities.

2.4. Coaching

In some definitions regarding teacher coaching, it is characterized as a form of implementation support (Devine et al., 2013) or a tool for developing student learning (Russo, 2004). The ways of how coaches situate themselves also has been identified as "responsive" (Dozier, 2006) and "directive" (Deussen et al., 2007). The former stance focuses on teacher self-reflection on instruction (Ippolito, 2010), whereas the latter stance is concerned with a direct message about practice for teachers. Based on Kraft, Blazar, and Hogan's (2018) meta-analysis of the definition of coaching, they concluded that in some studies, coaching was defined as collaborating with peers, whereas more often, coaching was grounded on enhancing teachers' learning with an expert. In accordance with these many viewpoints, Gallucci, Van Lare, Yoon, and Boatright (2010) characterize coaching as "inherently multifaceted and ambiguous" (p. 922).

Three main coaching models—cognitive, content-focused, and instructional—were created based on various definitions. (Barlow, Burroughs, Harmon, Sutton, & Yopp, 2014). These designs possess a common basis, "namely that coaches are more accomplished colleagues who can work with teachers on problems that are close to practice". (Cobb and Jackson, 2011, p.19). Cognitive coaching (Costa & Garmston, 2002) is related to assessing individual behaviors or changes by focusing on what a teacher implied or said. This attempt is concerned with reflexive thought of teachers and coaches, enhancing them to set up goals in the process of self-assessment. Paraphrasing and asking reflective questions to elicit teachers' understandings or beliefs were used to both make their thinking more visible to both coaches and teachers themselves. Instructional coaching (Knight, 2007) relies on a partnership among teachers and coaches while planning the lessons. Like cognitive coaching, instructional coaching focuses on understanding teachers' points of view or beliefs through conversations among teachers and coaches. It is founded on seven principles, including “equality, choice, voice, dialogue, reflection, praxis, and reciprocity, and four elements in the classroom environments: behavior, content, instruction, and formative assessment” (Knight, 2011, p. 18). Desimone and Pak (2017) argue consistencies and inconsistencies between key features of effective professional development (*content focus, active learning, duration, collective participation and coherence (p.4-5)*) and instructional coaching. Content-focused coaching (West & Staub, 2003) depends on a particular content domain approach. Coaches scaffold teachers’ development in ambitious instructional practices with a focus on the way of students’ learning and pedagogical principles about the content (Coburn & Russell, 2008). Teachers are engaged in activities that include shreds of evidence of students' work or understanding to make teachers notify students’ mathematical understanding or misconceptions. Coach also needs to assess teachers’ practices, knowledge, belief about learning and teaching mathematics and disposition toward mathematics to detect teachers’ needs. In addition, coaches take equal responsibility for effective student learning with teachers (West & Staub, 2003).

At some point, there are also overlapping ideas among the three models (Mudzimiri, Burroughs, Luebeck, Sutton & Youp, 2014). The first idea suggests a cyclic process of pre-observation, observation and post observation (Carr, Herman, & Harris, 2005); the second is related to the similar "underlying assumptions about the knowledge base or skill set for asking questions" (Yopp et al. 2017, p.2). These knowledge and skills are issues to trigger teachers to change. On the other hand, the differences between those models appear due to their philosophical approaches (Mudzimiri et al., 2014). To be specific, in the cognitive coach model, coaches enhance teachers to be aware of and elicit their views. In a content-focused coach approach, coaches highlight teacher knowledge and students' learning in specific mathematical content, while in instructional models, coaches build partnerships and close relationships with teachers. On the other hand, although it was argued that coaches are more accomplished in some points than the teachers (e.g., Campbell & Malkus, 2014; Jackson et al., 2011) the knowledge base of coaches (Yopp et al., 2017) might create a difference among coaching models. Based on this, it is not assumed that coaches have a higher level of content knowledge than the teacher in instructional coaches and cognitive coaching models when compared with content model.

Grounding on these multiple definitions of coaching and models, mathematics coaching studies have newly begun (e.g., Obara, 2010), Hull, Balka, and Miles (2009) characterize the mathematics coach as “an individual who is well-versed in mathematics content and pedagogy and who works directly with classroom teachers to improve students’ learning of mathematics” (p. 8). Mathematics coaching relies on changes in teachers' mathematical practices. We used a hybrid of these two approaches, content focused and instructional coaching, to structure the coach's work with teachers and understand the changes and gains in teachers’ instructional practices within a particular content. As these models give general properties related to coaching, scholars have begun to conceptualize how teachers and coaches learn through frameworks and practices in recent works.

2.4.1. Recent coaching models and frameworks

Although there are several coaching models in education, the main features of those are vague or not well defined (Gallucci et al., 2010). Some researchers have begun to articulate coaching practices to understand better how coaching supports teacher learning (Gibbons & Cobb, 2016; Killion, 2008). For instance, Gibbons and Cobb (2016) hypothesized five key coaching practices the coach engaged in during content-focused coaching planning. Those coaching practices were: “(a) identifying long-term goals for teachers' development, (b) assessing teachers' current instructional practices, (c) locating teachers' current instructional practices on general trajectories of teachers' development, (d) identifying next steps for teachers' development, and (e) designing activities to support teachers' learning" (p. 246). For the last practice, the authors listed designing activities as co-teaching, modeling, observing, and debriefing after the lesson. They also identified two forms of knowledge needed to enact these practices: (a) knowledge of effective teaching mathematics and (b) knowledge of general teachers' learning progressions of ambitious instructional practices. These knowledge and practices clarify what coaches might need to have to support teachers' and students' learning. Differently, some conceptual frameworks attempted to figure out how the coaching cycle enables teachers' and coaches' learning in detail (Campbell & Griffin, 2017; Russell et al., 2019).

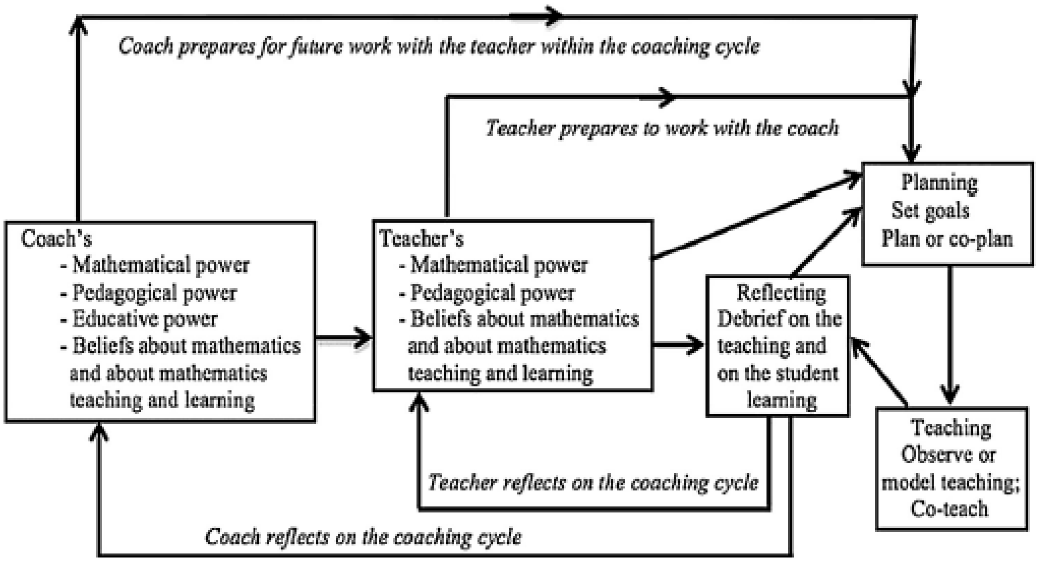


Figure 3. Conceptual framework of coach and teacher co-learning through the coaching cycle (Campbell & Griffin, 2017, p.3)

Figure 3 presents the model highlighting the bidirectional relations between the three powers including mathematical, pedagogical, and educative power (Jawaorski, 2001) and beliefs of teachers and coaches and debriefs on the lesson (Zaslavsky & Leikin, 2004). This dual cultivation makes coaches set new goals about teachers' needs and makes teachers design a new instructional plan for effective students' thinking. Although their debriefing about students' thinking is emphasized in the model, it is not clear whether the coach considers students' needs as well as teachers' needs. Moreover, these models have not traditionally linked observations about coaching practices with analysis of teaching.

The recent TN+IFL Math Coaching Model (Russell et al., 2019) portrays the main elements of the coaching framework, including roles of coaches. These three elements are a set of 3 Key Coaching Practices; the coach-teacher discussion process; and an inquiry stance. The three key coaching practices are: “(1) deep and specific discussions of the instructional triangle, (2) establishing mathematics and pedagogical goals, and (3) evidence-based feedback” (Russell et al., 2019, p. 5). Each of the three key coaching practices takes a role at a particular point during the coach-teacher discussion process (Figure 4), which is an an upgraded and more refined version of the plan, execute, and reflect coaching cycle. The model indicated that coaching as professional development improved mathematics teaching and led to conceptual understanding of students. The Coach- Teacher discussion process (Figure 5) in the model consisted of pre-observation, observation, and post conference phases, which were similar to the cyclic process of West and Staub (2003). Two of the key coaching practices (1 and 2) were utilized during the pre-observation, observation and post observation phases, while evidence-based feedback (3) became paramount during the lesson analysis within the post-conference phase.

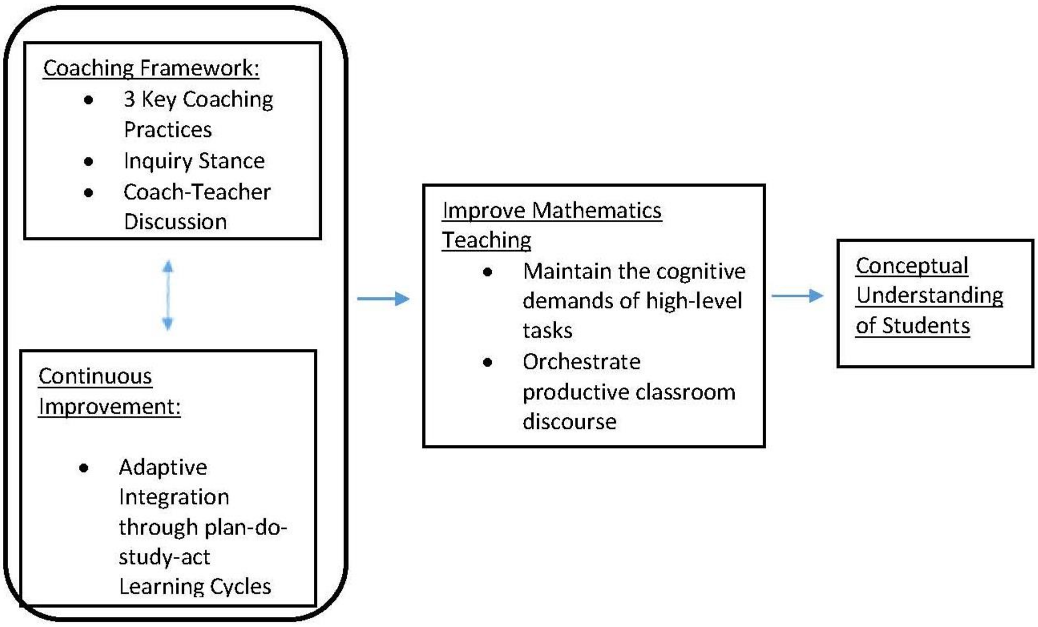


Figure 4. The Coaching Model taken from a part of the model proposed by Russell and colleagues (2019, p.6).

The highlighted text in Figure 4 represents three critical practices in the model. Throughout the coach-teacher discussion process, the coach maintains an inquiry stance. According to Russell and colleagues (2019), taking an inquiry approach entail employing observations and queries in lieu of direct instruction. "The inquiry stance stems from…the need for active teacher participation in meaning making around shifts in practice" (p. 6). The model differed in focusing on one-on-one coaching on specific lessons and particular teaching practices: enacting high-level mathematical tasks. In addition, rather than utilizing co-teaching and modeling in lessons as coaching activities, strategic and limited coach help was emphasized (see Figure 5). However, it has common elements with other models, such as including the cycle of planning, enacting and reviewing, and addressing coach and teacher learning through the cycle. The current study was built upon this model since it highlights particular teaching practice: enacting-high level mathematics tasks aligned with the focus of the current study enriching a teacher's implementing high level tasks.

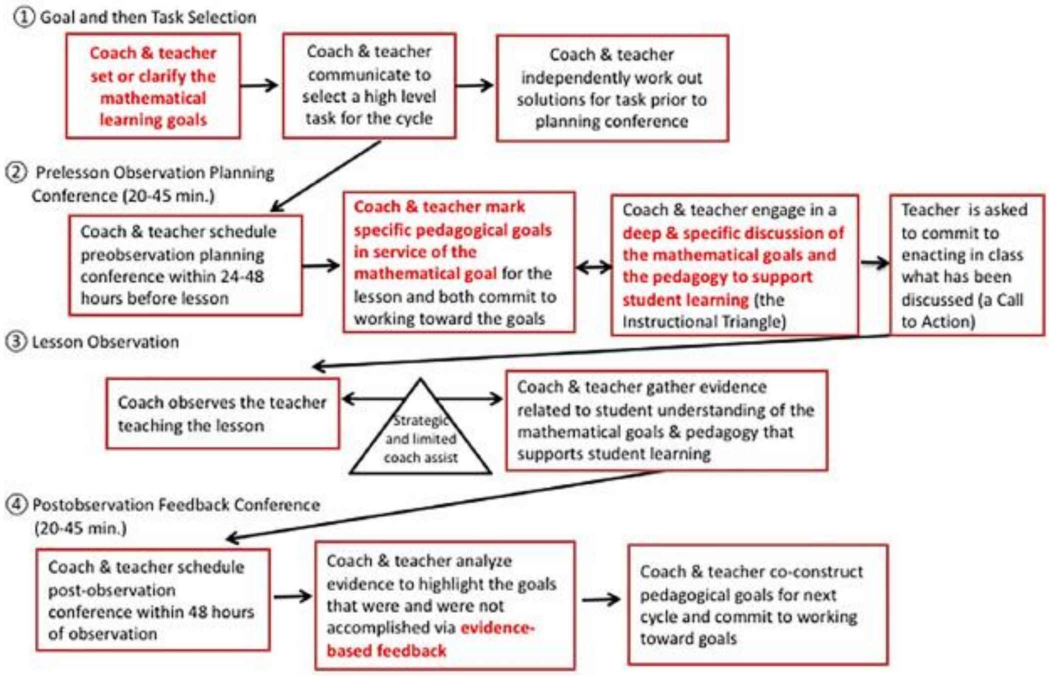


Figure 5. Coach-Teacher Discussion procedure through coaching cycles (Russell et al., 2019, p.8)

In particular the Discussion process begins with setting mathematical goal and select a task associated with the goal. It is hypothesized that utilizing a framework on slope conceptualizations gives a rationale for both teacher and coach of tasks designs of slope tasks. The subsequent steps of the process consist of a pre-observation (planning conference), a lesson observation, and a post-observation (feedback conference). Pre-lesson conference (planning) enables coach and teachers to discuss the relations among tasks, pedagogy and students' thinking in a more detailed way. Throughout the discussion process, anticipated students' thinking, misconceptions and related tools are also considered (Smith et al., 2008; Stein et al., 2008). It helps teachers to rehearse for responding students' ideas. During lesson observation (teaching) both teacher and coach collects data of students' thinking and pedagogy and the main of role of the coach is noting strength and weaknesses of teachers' instruction as well as students' thinking. While West (2009) argued the variances of coach's role during the teaching. These included that the coach may teach, or teacher may teach or they teach together. At last, during the post observation (reviewing) discuss whether the goal is attained or not with reasons based on the evidenced then they look over the goal of the next lesson. This process involves high depth conversations between teacher and coach that could add to teacher's capacity to teach (Russell, et al., 2020). In addition to deep in substance, Russell and colleagues have conjectured other principles with regards to discussion process composing of specific in content and the context of the instruction triangle (pedagogy, mathematics and students' learning).

Although the model and principles of coaching programs are varied through studies, this program has high effectiveness on students and teacher learning. Due to challenges in learning new instructional strategies (Obara, 2010) or methods and negative beliefs about the effectiveness about the new strategy, teachers might resist applying different strategies or making changes in their practices (e.g. Bengo, 2013). However, some argue that mathematic coaching has potential to address this point. Since mathematics coaching empowers teachers learning, teaching and beliefs with ongoing and intense support in the moment of instruction as well as out of instruction time.

Over the last decade, mathematics coaching has gained impetus to outline the coaching activities and roles and their impact on teacher and student learning with a practice and research lens (Ellinghton et al., 2017). Regarding students gain, research on mathematics coaching reports a positive influence on student achievement in many the countries, for example, Australia, Canada, the Netherlands, the United Kingdom, and the U.S. (e.g., Blank, 2013; Campbell & Malkus, 2011; Ellington, Whitenack, & Edwards, 2017; Harbour, Adelson, & Karp, 2016; Harbour et al., 2018; Teemant, 2014). To illustrate, Campbell and Malkus (2011) with a 3-year randomized control study, elementary graders in schools with coaches had significantly higher scores on their states' high stakes standardized mathematics achievement than did elementary students in schools without coaches.

Concerning teachers' gain, Kraft and colleagues (2018) indicated a "large positive effect of coaching on teachers' practice [0.49 SD]" despite considerable variations in programs [0.33 SD] in their meta-analysis focusing on a limited number of mathematics coaching programs. However, the authors struggled with the maintenance of the effect while scaling up. Despite the mixed finding, a growing body of mathematics education research has shown the positive impact of coaching on teacher practices (e.g., Auletto & Stein, 2020; Ellington et al., 2017), teachers' mathematical knowledge for teaching (e.g., Knapp et al., 2016; Yopp et al., 2014). Some studies also indicated that coaching increased teachers' self-efficacy (e.g., Bruce & Ross, 2008; Taylor, 2017) and beliefs about mathematics teaching and learning (e.g., Bengo, 2016; Hopkins et al., 2017; Yopp et al, 2014). Whereas some studies showed no effect on instructional improvement or teacher change (Olson & Barrett, 2004; Saclarides & Lubienski, 2021). This contradiction between studies might be related to differences in characteristics of the coach and teachers' orientations in those studies. Specifically, concerning coaching impacts on teachers' knowledge on teaching and beliefs, Knapp and colleagues (2016) investigated teachers' development of mathematical knowledge for teaching (MKT) through coaching and they concluded that teachers get benefit from coaching prompting about their use of technology and teaching of geometry. In regard to teacher beliefs, Bengo (2016) reported changes in teachers' beliefs and practices with the help of coach. The author concluded that the coach needed to understand teachers' beliefs in order to select appropriate activities and rapport.

Regarding coaching impacts on teachers' practices, for instance, Ellington and colleagues (2017) portrayed findings of cases to illustrate how coaches supported teachers' practices or assessments. Surveys and observations are used to collect quantitative and qualitative data. To collect data on cases, Coach B was visited 6-8 times on-site each year, whereas Coach B was visited three times in the second year of the study. In terms of students' achievement, the results indicated that sixth and seventh graders whose teacher engaged in coaching activities had higher scores than those whose teacher did not. The engaged variable was not statistically significant for the Grade 8 ($p = 0.248$) or Algebra I ($p = 0.903$) SAA test levels. In terms of teachers' beliefs, teachers who are highly engaged in activities believed that students need to make sense of mathematics. However, data also point out that simply providing coaches could not change teachers' views on teaching and learning mathematics. Hence, the authors argued that the positive relationships between high-quality discussions with coach and teacher change. The cases also highlighted that Coach B and Teacher P maintained the whole class discussion and adapted tasks while planning the lessons together, whereas Coach A and Teacher K focused on assessment practices in preparation for the national achievement exam. Therefore, differences in teachers' demands, resources, and administrative support caused a difference in how they support teachers' practices with their students. To analyze coaching impacts on observable mathematical teaching expertise of teachers who engaged in activities with coaches, Auletto and Stein (2020) employed regression analysis to understand relationships between observed mathematical teaching expertise, mathematics learning, and students' self-efficacy and 298 upper elementary teachers' learning through collaborations with coaches. Hundred and eighty-seven teachers' lesson implementations lasting 1-hour were observed three times throughout the year to measure teachers' change in instructional practice over time. Findings identified that teachers who interact more regularly with a mathematics coach, either by observing the coach or having the coach observe them, exhibit greater advances in their observable mathematical teaching ability from one year to the next, compared to teachers who engage less routinely with a coach. The striking finding related to the effectiveness of teacher development models. The authors concluded that only coaching mapped onto observed changes in teachers' practice compared to the other forms of professional development, including inquiry-based professional learning in which teachers worked with their peers. Based on these findings, many have claimed that assisting teachers to reflect on their practice and providing activities could enhance their instructional practice and make students flourish. However, helping and selecting appropriate activities to improve instruction is difficult. Therefore, a growing body of research on mathematics coaching figured out the components of effective coaching. It classifies them as relating to the coach's abilities, roles, characteristics, specific activities and practices.

The research regarding the way of being an effective coach suggested acquiring common skills (Hull, Balka, & Miles, 2009; Knight et al., 2015; Kraft et al., 2018; Obara, 2010). Those skills or knowledge of being an effective coach have been listed as content knowledge (Bengo, 2016; Chval et al., 2010), pedagogical content knowledge (Obara, 2010), and communication and leadership skills (Knight, 2007). Related to the knowledge and skills, Knight and collegues (2015) also argue that “it may be most important that coaches understand how to move through the components of an effective coaching cycle that leads to improvements in student learning” (p. 18). According to Knight and colleagues (2015) as well as common skills, teacher should understand when and how to use these skills to improve teachers’ learning.

Specifically, Mudzimiri and colleagues (2014) claimed that coaches need to make the latest research findings visible for both teacher learning and effective instruction. Bengo (2016) also indicate that the coach needs to convince some teacher about the effectiveness of the new strategy, discern the way of coaching models and strategies concerning the needs, and collocate enough time to administer the cyclic process through prolonged succession. In addition, while Mudzimiri and colleagues (2014) stress the importance of knowing how to effectively communicate with teachers, while other scholars argue that coaches must also be aware of the politics of coaching during times of policy and curricular change (Coburn & Woulfin, 2012), such as by attempting to comprehend the needs and wants of school principals (Campbell & Griffin, 2017; Huguet et al., 2014). The issues as mentioned earlier also consisted of the eight aspects of coaching knowledge of Sutton and colleagues (2011). These eight issues are teacher learning, development, practice, student learning, assessment, communication, relationships, and leadership. Based on these issues, coaches must be equipped with knowledge related to pedagogy and content and how to develop teaching practices, knowledge, or beliefs and learn how to negotiate with teachers and principals in sustained, context specific and focused ways. Although many have debated the role and characteristics of coaches for effective coaching programs as “individualized, intensive, sustained, context specific, and focused” (Kraft, Blazar,

& Hogan, 2018, p. 553), the field knows little about what coaching practices contribute to teaching improvement (Russell et al., 2020) and activities due to vague explanations regarding coaching practices, insufficient evidences concerning coaching effects, or inconsistency among coaches.Thus, the present study designed particular coaching activities based on the Coaching Framework parallel with the second aim of the study. Following this part will be a presentation of specific coaching program activities conducted in previous research.

2.4.2. Studies concerning coaching activities and effects on teachers' learning

Contemporary research has concentrated on determining “productive coaching strategies” (Gibbons & Cobb, 2017, p.1) and the role of coaches during the interaction with groups of or individual teachers. Gibbons and Cobb (2017) reviewed previous literacy and mathematics coaching studies to identify potentially productive activities utilized while working with groups of teachers: *engaging in the discipline, examining student work, analyzing the classroom video,* and *engaging in lesson study* (p.5). On the other hand, *these studies regarded co-teaching* and *modeling the instruction* as productive activities while working with individual teachers. Although *debriefing challenges of implementation* met all five characteristics of compelling professional developments, this activity had not been involved due to inadequate evidence of effectiveness of the activity in teacher learning and teaching. With this activity, the reflecting phase may provide fewer learning opportunities because of insufficient attention (Campbell & Griffin, 2017; Saclarides & Lubienski, 2021). Lastly, whereas intense coaching studies have frequently mentioned the *cyclic process as* a strategy (McGatha et al., 2018), authors argued that it was not regarded as an activity since it did not meet the criteria of intensive and ongoing. In fact, Campbell and Griffin (2017) studied 21 coaches in eleven school districts; coaches stated that they rarely use the cyclic process due to lack of time.

Recently, studies have attempted to document how different coaching activities arouse effective teacher changes (Campbell & Griffin, 2017; Fennell et al., 2013; Giamellaro & Siegel, 2018; Gibbons & Cobb, 2017; McGatha, 2008; Mudzimiri et al., 2014; Polly, 2012; Yopp et al., 2019; Wilder, 2014) and their effects on teachers' practices. For instance, Polly (2012) documented that coaching through various activities, including planning, task selection, and co-teaching, supported four in-service teachers to ask more challenging and probing questions, whereas post-lesson feedback was not regarded as direct support. On the other hand, post lesson feedback on the types of questions teachers asked enhanced their use of different questions. Olson and Barrett (2004) analyzed three first-grade teachers' practice and reflection during five lessons over three weeks within a teacher development experiment (Simon, 2000). The authors adjusted different coaching approaches to support teachers' professional growth. Those approaches include cognitive coaching, reflection on the lessons, co-teaching, modeling, and authentic tasks. Those activities, however, did not enable teachers with traditional beliefs about mathematics teaching and learning to apply the intended teaching practices. Then the authors proposed a new approach," evoking teachers' pedagogical curiosity to make students implement mathematics reform suggestions. They revealed that when they notice that students are able to create mathematical ideas and build relationships with the ideas, they might start to reason students' answers.

Concerning modeling and co-teaching, to illustrate Ellighton and colleagues (2017), the coach modeled the lesson initially, and then the teacher gradually took more responsibility for teaching throughout the study. During the co-teaching episodes, the coach introduced the topic to the class; the teachers assisted with whole-class discussion and maintained classroom norms. Some studies examined the effect of coaching by utilizing co-teaching within coaching cycles (Jackson, 2011; Jung & Brady, 2016; Saclarides & Harbour, 2020). To exemplify, Saclarides and Harbour (2020) investigated the structure of one-on-one coaching to support one first-grade teacher with one school-based instructional coach for differentiating the instruction. The study took two and a half weeks. It consisted of two planning meetings (19-34 minutes), four observed co-taught lessons (21-

27 minutes), and two reflection meetings. The findings of the study revealed that much of the coach-teacher talk has a medium depth about differentiation. To increase the depth of the talk, the authors suggested using protocols to make dyads guide and focus on their conversations. In addition to using protocols, Cobb and Jackson (2011) suggested specific instructional practices such as pressing teachers to expect possible students' thinking to support them to orchestrate productive whole-class discussion.

As an alternative to face-to-face interaction with teachers, Güler and Çelik (2022) utilized e-mentoring as a professional development which is a particular case of mentoring/coaching teachers via digital tools to meet with them at different locations at the same time. They reported the effect of e-mentoring was enhanced through video-recorded lessons and video clips of a group of novice middle school teachers' (n=6) lesson analysis skills via a pre-post-test design. Four video clips from a video-recorded lesson concerning height to any side of a parallelogram were used as data collection tools for pre and post-tests. Besides a video-based collaborative environment, they presented teachers with the Lesson Analysis Framework consisted of four components: “Identify the lesson goal, analyze students thinking and learning, construct hypotheses about the effects of teaching on students' learning, use analysis to propose improvements in teaching” (Santagata & Guarino, 2011, p.134). The coaching activities are selecting three lesson plans, orchestrating discussions related to videos of others, and taking teachers as partners in the study. Teachers are asked to create two lesson plans as groups and two as individuals. The finding showed that the intervention significantly enhanced the lesson analysis skills of novice mathematics teachers. This suggests that perspectives such as e-mentoring can effectively improve teachers' lesson analysis skills.

In overall, studies differ in their selections of coaching activities including modelling, co-teaching, coaching cycles and using models. However, there is also a need to more clear understanding of coaching practices (Gibbons & Cobb, 2016) through sharper vision for coaching (Russell et al., 2020). In addition, although most of these small-scale qualitative studies provide empirical evidence related to the effect of the variety of coaching activities on teachers' practice, teachers' development on specific aspects of teaching or any particular content domain was overlooked. Thus, two of most critical characteristics of the coaching process is context specific and intense (Kraft, Blazar, & Hogan, 2018) are missing. In order to give insight about specific context of teaching practices via coaching, the current study is grounded the on coaching framework (Russell et al., 2020a) focusing on enacting high level mathematical tasks through three stages, planning, teaching and reviewing as coaching cycles. It is believed that coaching program within a specific context might enrich teachers' knowledge of cognitive demand of tasks. In that sense, the first aim of the current study is to examine impact of coaching program on an in-service teacher's knowledge of cognitive demand of mathematics tasks. In fact, coaching cycles are beneficial however more empirical evidences is needed to portray how and what extent teacher learning is enhanced to discuss its effects on teachers' learning. Considering this gap in one to one coaching practices, Russell and colleagues (2020) devised a research based framework and coaching is framed by a view of crucial role of on teachers' capacity planning, teaching and reflecting on high level tasks (Russell et al., 2020; Stein et al., 2008) to maintain students' learning in it. Grounding on this gap and the framework, the second purpose of the study is to examine the changes in the teacher's noticing skills and how the teacher progressed through the coaching stages including planning, enacting and review. Hence it is believed that this study likely to contribute the coaching and noticing literature by investigating how an in-service teachers' noticing skills on algebraic thinking within high level mathematical tasks context through coaching cycles.

Specifically, few studies examine coaching's effect on teachers' noticing skills (e.g., Jakopovic, 2021; Munson, 2020; Reinke, Schmidt, Myers, & Polly, 2021). Investigating more closely how coaching can support teachers' development in noticing, Reinke, Schmidt, Myers and Polly (2021) illuminated how coaching moves within three coaching cycles could reinforce two elementary prospective teachers' responding skills. At the final observation, prospective teachers could ask probing questions to the students who gave correct answers compared to the students who did not give correct answers. Ten coaching moves emerged, such as Naming/highlighting teaching and Prompting for interpreting student thinking. During meetings, the most frequent coaching move was directing suggestions for the following lessons. The least frequent one was pointing out the missed opportunity. The author concluded that these different coaching moves enable prospective teachers to elicit and interpret students' answers. However, the study also pointed out that pre-service teachers were not competent in responding to wrong and correct answers, which might be due to limited coaching moves about critical moments they missed. Jakopovic (2021) investigated an elementary novice teacher's noticing after observing two coaching cycles through semi-structured interviews with the teacher, teacher coach bilateral talks, and classroom observations. Findings indicated that the teacher's focus shifted from the organization and logistical issues to students' mathematical ideas due to the dyad's targeted engaging students with productive solutions. In a more profound sense, the study documented the three practices of planning and reflecting phases: developing mathematical goals, planning and adapting mathematical tasks/lessons (planning), and examining student thinking (reflecting). These studies indicated that teachers' noticing skills, from general to specific, were enriched with coaching cycles, yet the number of coaching cycles is limited, and how to respond to students' thinking is still challenging for teachers. Therefore, to detect progress in teachers' noticing skills, there should be more cycles than two to determine at which aspects of teachers are struggling and observe their progress through a process of teaching a mathematical idea, topic or unit. In that sense, this study will how a teacher's noticing is changed through cycles having ongoing, intense and focused features.

2.5. Summary of the Literature

Scholars indicated that both selecting and enacting tasks at high cognitive demand is critical for conceptual understanding (Tarr et al., 2008). However, more findings have revealed that teachers typically select and categorize tasks with respect to their superficial characteristics of tasks such as either including real life context, technology, diagram or representations, mathematical content, length of the text, task difficulty without providing rationale for students (Arbaugh & Brown, 2005; Osana et al., 2006; Tekkumru-Kısa, Stein & Doyle 2020). One productive strategy to enrich teacher capacity on task is to provide a guide namely Task Analysis Guide (TAG) to make teachers to use it while classifying task (Arbaugh & Brown, 2005; Boston, 2013; Boston & Smith, 2009; Boston & Smith, 2011; Estrella, Zakaryan, Olfos & Espinoza, 2020) and using worthwhile tasks (Guberman & Leikin, 2013). With regard to launching and implementing tasks, activities have include analyzing illustrative episodes of implementation of tasks at high cognitive demand, using protocols to discuss the level of tasks with other teachers and analyzing sample students' work, using MTF and experiencing the practice scaffolded. (Boston & Smith, 2009; Parrish, Snider, & Creager, 2022). Considering effectiveness of TAG on mathematical task knowledge and activities related to narrative cases for implementing high level tasks (Tekkumru-Kısa et al., 2020) I used them to evoke an in-service teachers' awareness of cognitive demand of tasks at each level in MTF.

Slope is considered as important notion to be enlighten through teacher task design since it has a complex nature of interconnectedness of other concepts and disciplines (Peck, 2020) and students' and teachers' struggle to recognize various conceptualizations of slope and teachers' difffciulty in selecting and implementing high level algebra tasks (e.g. Demonty, Vlassis, & Fagnant, 2018; Magiera, van den Kieboom & Moyer, 2013; Nagle, Moore-Russo, & Styers, 2017; Rule & Hallagan 2007; Steele et al., 2013; Wilkie, 2016). In that sense mumerous calls for algebra classroom change have been made (Kaput, 2008). Among these calls attention on student thinking is pressed. Noticing is a core aspect of teaching expertise (Goldsmith & Seago 2011; Jacobs and Spangler, 2017; Star et al., 2011) for exploring how teachers attend and interpret student thinking (Sherin & van Es, 2009). Specifically, learning to notice of the crucial elements of slope including its conceptualizations, representations, and relations between those is a key to plan and enact high cognitively mathematical tasks (Choppin, 2011). Frameworks related to noticing differ in terms of the way of defining it and purpose of the studies. In addition, a broad range of methodological issues to elicit noticing skills ranging from standardized tests to observing teachers during instruction, and a variety of tools to enrich teacher noticing such as using frameworks, scaffolding teachers or using video (mainly used). In particular, studies interested in investigating context-specific noticing ranging from various content domains to different situations. Based on these studies, a relatively few studies attempted to portray teachers' noticing related slope notion (Styers, Nagle & Moore-Russo 2020), whereas a bit more studies investigated teachers' noticing regarding algebraic reasoning/functional reasoning and pattern generalization context in particular. It is important to learn what and how a practicing teacher notice in slope due its fundamental concept for other mathematical notion and disciplines (Nagle, 2019).

In recent years, coaching has emerged as a promising area for professional growth of teachers. Hence studies have been interested in providing much-needed evidence for the effects of one-on-one coaching and/or groups of teachers. They importantly examine the various activities/ or strategies coaches enact when supporting teacher development (Aygün, 2016; Mudzimiri et al., 2014, Gibbons & Cobb, 2016, 2017; Neuberger, 2012, Polly 2012, Munson 2017; Hopkins, Ozimek & Sweet, 2017; Ellighton et al., 2017; Gibbons, Kazemi & Lewis, 2017). However, the research on "how coaches might work with individual teachers in their classrooms and what constitutes high-quality coach professional development is limited" (Cobb & Jackson, 2011; p. 19). Similarly, Gibbons and Cobb (2017) argued a paucity of research on activities carried out one-on-one in classrooms with teachers. Therefore, it is suggested to investigate how successful coaching strategies and activities are carried to enrich a teacher's learning. based on three components of the coaching cycle (Russell et. al., 2020) within context of high mathematical tasks and the triad nature of noticing (Choy et al., 2017; Amador et al., 2017; Baker et. al., 2022); I explored a teacher's noticing skills about students' algebraic thinking during planning, enacting and reviewing with an eye toward teachers' opportunities to select, enact and modify highly cognitively slope tasks. The framework of Russell was selected since coaching is framed by rigorous mathematics tasks within three cycles with a more specific focus. In that respect, first aim of the current study is to document the changes in an experienced in-service teacher's knowledge of the cognitive demands of mathematical tasks through her participation in a coaching program. Second one is to examine the changes in what an experienced in-service teacher attends to and how she makes sense of her attention through the coaching stages including planning, enacting and review.

CHAPTER III

METHODOLOGY

This chapter includes the methodology of the study consisting of five main sections, which are design of the study, sampling and selection of the participants, implementation, data collection, data analysis, and trustworthiness. The goals of this study were multifaceted: (1) to document the changes in an in-service teacher's knowledge of the cognitive demands of mathematical tasks, (2) to examine the changes in the teacher's noticing skills and how the teacher progressed through the coaching stages including planning, enacting and review. To these ends, we sought to answer the following research questions:

1. In what ways does the teacher's knowledge of the cognitive demands of the mathematical tasks change following her participation in a coaching program on selecting/adapting mathematical tasks?

2. How does the teacher's noticing of 8^{th} graders' algebraic thinking, specifically in slope concept develop through coaching cycles within cognitively high mathematics task context?

3.1. The Design of the Study

The primary purpose of the present study, in a broad sense, was to understand the nature and development of a middle school mathematics teacher's knowledge of cognitive demand of tasks and noticing skills in students' algebraic thinking within a coaching program. The aim of the study aligned with the principles of the coaching program which requires creating a collaborative environment in which researchers and a teacher are working on selecting and implementing cognitively demanding slope tasks by combining theory and practice. Hence to investigate the nature of development of the teacher's knowledge of cognitive demand of tasks and noticing of students' algebraic thinking including slope notion, the teaching experiment method was employed. In that respect, in the current study, it is attempted to report learning process of an in-service teacher. The brief information regarding teaching experiment methodology, relations with frame of the current study and principles of this methodology and how to adapt coaching activities through the design are presented in the next paragraph.

Experimentation in design research is investigating, adapting, and enhancing the local teaching theory. Together with retrospective analysis, researchers employ these activities to learn about the new practice. The outcome of the design study is essentially what the researchers have learned in terms of empirically informed hypotheses about how the intervention functions. Each design experiment consists of three distinct phases: (1) experiment preparation, (2) teaching experiments, and (3) retrospective analysis (Gravemeijer & van Eerde, 2009). Throughout the initial phase, some preliminary design will be conducted, but the actual instructional desicions will be formulated, amended, or altered during the classroom teaching experiment. Regarding the function of the preliminary design, it resembles Simon's (1995) analogy of a "travel plan". Preliminary design is a starting point for the studies, and it includes hypothetical plan for teaching experiment. In order to create a travel plan for the present study, it is critical to specify the potential instructional goal of the current study and which activities are going to be selected (Note: These goal, activities and reason for activities will be explained in the preparation procedure of the first teaching experiment, and revisions and adaptations were made on it as a preparation for second teaching experiment). The second part, the teaching experiment, aims to test the research hypotheses and generate hypotheses (Steffe & Thompson, 2000). While testing hypotheses, in contrast to clinical interviews, the nature of the teaching experiment is a flexible design that consists of a series of teaching episodes (in a cyclic nature) designed to help researchers understand the long-term growth of students in various professions (Cobb & Steffe, 1983; Steffe & Thompson, 2000; Cobb, 2007). Similar to Yackel, Gravemeijer, and Sfard (2011), the main objective of a teaching experiment is to gain insight into the mathematical reasoning growth of learners. In other words, focus of the teaching experiment is solely not on differences between beginning and end point of the conceptions or knowledge of participants but also it deals with how a student/s learning progresses throughout the experiment (Steffe & Thompson, 2000). Thus, researchers can observe and detect progress in learners' learning and make sense of their conceptualizations of the object with ongoing analysis between sessions. That shows that the ability to observe and analyze successful learning is independent of the researchers' original understanding of learning and how to encourage it. In other words, each intervention is updated based on the researchers' more recent understanding of learner's learning and the way of supporting it (Simon, 2018). That is, there is gradual growth in the researchers' understanding of both instructional design and learning. The third one is retrospective analysis refers to analysis for comparing hypothesized learning or principle and actual learning or outcome. These main processes will be explained by relating goal of the current study in the next session.

The coaching program having a cyclic nature of planning, teaching and reflecting as an iterative nature of teacher learning models (Carr, Herman, & Harris, 2005) possesses a noticeable resemblance to the iterative character of teaching experiment. The core idea behind the coaching program is that the teacher and the coach and other researchers jointly design, test, monitor, and improve inventive mathematics instruction and learning thorugh a series of teaching sessions (Steffe, 1983). The primary goal of teaching episodes in teaching experiment is to test, generate and revise hypotheses with respect to learners' thinking or understanding. In this sense, the researcher/coach assumes two vital roles: posing crucially important questions and designing situations in which learners can actively participate and assess how learning happens in teaching episodes (Steffe, 1991) to revisit the previous hypotheses based on this analysis. By adopting the above-mentioned crucial roles, I was both the coach and researcher in this study. The role is required to "continually postulate possible meanings that lie behind students' [teacher in this case] language and actions'' (Steffe & Thompson, 2000, p. 277). In that sense, continually, the researchers are constrained by the gap between expected (hypothesized) and observed learning of the teacher in selecting and implementing high level algebra (specifically slope) tasks and reflecting on implementation and students' thinking. Therefore, the formative analysis of each (micro) coaching cycles (planning, teaching and reflecting) of two-hour lessons were utilized.

Another characteristic of desing research is that it possesses prospective and reflective component in each teaching experiments (Steffe & Thompson, 2000). While implementing envisoned learning (prospective component) the researchers test their conjectures with actual learning (reflective part). This reflective analysis guide researchers to create new hypotheses, refute or modify them (Bakker, 2019). Even if the teaching experiment involves more than one lesson, reflection can be performed after each lesson. This type of analysis may result in modifications to the original lesson plan for the following class. Coaching has a cylic nature of planning, implementing and reflecting of each lesson or lessons in a week. In that sense, analysis of these micro cycles of the current study including both the teacher's instruction and her comments about planning and reflecting of two-hour lessons at three times in a week can guide to refute initial conjecture and generate new ones. The findings of such an analysis mostly informed a new cycle (Bakker, 2018) and collective analysis of multiple micro-cycles and the macro cyle (Study 1) also informed the next micro cycles within teaching experiment and the next macro cycle (Study 2) between teaching experiments. As a result, a coaching model consisting of the phases planning, teaching, and reflecting was devised and used as a teaching experiment in order to give professional expertise in noticing within the context of high-level mathematics tasks. Moreover, the current study involved two macro cycles involving three phases: preliminary phase, experiment and micro cycle analysis and retrospective analysis over the couse of two years. Figure 1 locates design of the study within the three-phase teaching approach: preliminary, implementation (experiments) and assessment phases that is also aligned with five coaching practices (Gibbons & Cobb, 2016). The first iteration (Study 1) was planned as an evaluation of the effectiveness and practicality of practices in a coaching program on cognitively demanding mathematics tasks on a teacher learning by utilizing formative assessment. In this regard, an eight-grade mathematics teacher and students in this teacher's classroom participated in the study. As a second iteration (Study II) the main was to investigate contribution of the coaching program to the teacher. For this aim, another eight-grade mathematics teacher and students in this teacher's classroom was selected. Finally, in the summative evaluation phase (Dixson & Worrell, 2016), the focus was to evaluate of the effectiveness of coaching on teachers' learning.

In this manner, coaching studies might provide an alternative to the problematic approaches in teacher development and to innovation in education. In addition to the fact that teachers are the primary agents and gain a strong feeling of ownership at the end, the iterative nature of the coaching makes them potentially effective. Specifically, series of teaching experiment also enable to test the coaching model (Russell et al., 2020) with respect to the learning outcomes of the teacher. To do so, critical elements of development of noticing within the cognitively demanding algebraic (slope) tasks context such as utilizing videos, tasks, students thinking, and practices of coaching program were hypothesized before the study began and revised after the first year teaching experiment and implemented in the second year teaching experiment.

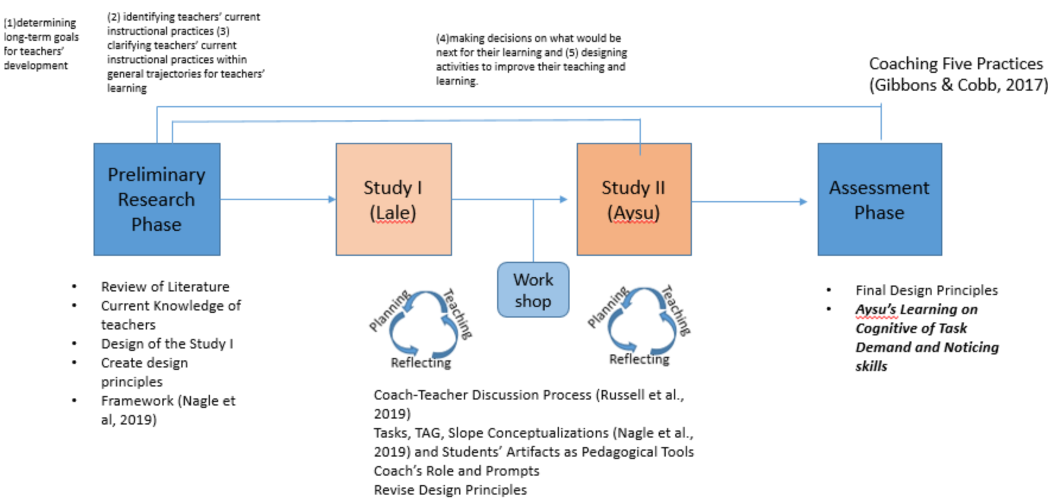


Figure 6. Overview of the design of the study

3.2. Context and Characteristics of the Participants

The purposeful sampling method was utilized in the current study to provide in-depth information about the core issues which are teacher's noticing skills and knowledge of cognitive demand of tasksThe purposeful sampling method is based on the assumption that "the investigator wants to discover, understand, and gain insight and therefore must select a sample from which the most can be learned" (Merriam, 1998, p. 61). The purposeful sampling method requires "the researcher [to] establish in advance a set of criteria or a list of attributes that the units for study must possess" (Patton, 1990, p. 69). In this manner, some criteria for selecting participants to find better responses to the research questions in the present study were identified. These criteria are as follows: willingness to collaborate with the coach for an at least two-month period, current way of teaching linear equations and slopes in traditional and algorithmic methods, their enthusiasm to learn and teach new instructional methodologies, being an experienced teacher (as having more than 10 years of teaching at middle schools) and lastly, having a similar school context where teachers are professional members. The first criterion linked to long-term participation in the study so that the researcher could obtain more comprehensive data about the teacher's development in noticing skills and knowledge of mathematical tasks and efficiency of the coaching activities. In that sense, the teachers who were available to participate as colloborater in this study for at least two months were chosen. The duration in the criterion was determined with respect to the duration of the workshop before coaching program began and approximate time period for implementing hypothesized tasks. The second criterion was related to limited knowledge about cognitive of mathematical tasks and unproductive concept images and meanings for slope that the teacher possessed, and the way of instruction was based on algorithmic and traditional approach. This criterion would guide the researchers in a way of developing the teacher's learning of selecting and enacting mathematical tasks and noticing skills through the cocaching cycles. The third criterion was about the teachers' enthusiasm to learn about new ideas related to algebraic thinking, slope specifically and understand detailed aspects of instructional artifacts tasks, tools (e.g., softwares, manipulatives). Because this study was grounded on the teacher's development in noticing skills of important events and students' thinking and knowledge of cognitive demand of tasks. Therefore, the teachers who were eager to learn about task design in algebra, specifically in slope notion, and test the new-learned materials in an eight-grade classroom. The fourth criterion was related to being an experienced teacher since the skill of noticing of important events is the most challenging skill due to complexity of teachers in the moment decision making even for experienced teachers (Lee & Choy, 2017) and their struggles to identify and implement high level tasks (Boston & Smith, 2009). At that point, although novice and experienced teacher are not distinguished from each other with respect to their skills on implementing high level tasks without decreasing its cognitive demand, the reason for the focus on experienced teachers in the present study was related to intend on eliminating novice teachers' possible weaknesses on classroom management (Wolff et al., 2017) that might create a challenge for researchers to analyze efficiency of coaching program on implementing cognitively highly mathematical tasks. Therefore, studying with experienced teachers might present rich data. The fifth criterion was connected with having a similar school context where teachers are professional members due to comparing efficiency of coaching activities between first and second teaching experiment. As mentioned earlier, in order to be able to test and revise conjectures about students' thinking on slope and the teacher's learning of noticing critical issues related to task and student (possible) thinking as a researcher, in addition to similar chracteristics of participating teachers, similar school contexts where they taught becomes an crucial point to be considered. At first, four teachers in different midle schools were identified with respect to their general inclinations to learn and try out reforms and trends in mathematics educations and be teaching 8th graders. Based on these main criteria of the study, four teachers were interviewed about their orientations to learn and test new ideas, current ways of teaching slope and their meanings for slope conceptualizations, and their approach to the nature of coaching in depth. Based on the interviews' analysis, two out of four teachers who possessed all attirubutes above selected to participate in this study whose demographics and brief information about their use of technology are given in Table 3.

Table 3. Teacher participants' overall characteristics

Characteristics	Aysu (Study II-Main Study)	Lale (Study I)
Gender	Female	Female
Education	PhD in MED (the topic of the thesis: surface area and volume of cylinders) Bachelor Degree in MED from a Public University	MS in MED (the topic of the thesis: translations of vectorsBachelor Degree in MED from a Public University
Teaching Experience	14 years	12 years
Number of Students	34	31
Technology usage	None (in the domain of algebra)	Medium (in the domain of algebra)
Seminars taken regarding the criteria of mathematical tasks	No	Yes

*MED: Mathematics Education Department

The pseudonyms, "Aysu" and "Lale" were used instead of their real names. Both of them were female, and they mentioned that they follow the sequence in the national textbook, yet they highlighted that small changes were made in their plan with respect to students' questions from other textbooks. Despite their tendency to apply the sequence in the textbook in general, I realized that they were willing to change their views when exposed to convincing arguments in a collaborative environment. Both, teachers had graduated from the same public university, and they had MS degrees in mathematics education program. In that sense, it can be stated that they had similar educational backgrounds. Aysu had fourteen-year teaching experience in middle schools. Thirty-four students were enrolled in her classroom. Aysu got a PhD degree in 2018 with a study on investigating classroom practices in the domain of geometry with seventh graders. Although she stated that she knew the basic features of the computer software such as Geogebra, she highlighted that no computer software had been used for teaching algebra in her classrooms. On the other hand, Lale had twelve-year teaching expertise as a middle school mathematics teacher. There were thirty-three students in her classroom. As part of her master's degree requirements, Lale conducted a classroom teaching experiment in her own class in 2008. In addition, she participated in the redesign of the Turkish Middle School Mathematics Curriculum (MoNE, 2013) and the development of middle school mathematics textbooks in accordance with the Turkish mathematics education reform movement. Although Lale was already familiar with the implementation of a learning progression of the concept of slope to a certain extent, Aysu had no experience with such an intervention about the concept of slope, and she lacked knowledge of the criteria of the demands of mathematical tasks. As mentioned earlier the current study situated as a part of a larger project and the main study's finding including Aysu' learning was reported in this study. The information about the teacher Lale who was participant of the year I study were provided since the activities of coaching and procedure of the second-year study was informed by the analysis of first year of teaching experiment that was aligned with the iterative nature of teaching experiments.

The information regarding students who enrolled in classrooms of these two teachers were also identified. Students from two eight-grade classrooms (14-15 ages) at two different public middle schools (grades 5–8) located in central Turkey participated in the current study. There were 31 (60% female, 40% male) students in the first study (4 of those were inclusive students) and 34 in the second study-main study (56% female, 44% male; 2 of those were inclusive students). The students who enrolled in these schools ranked at various levels of achievement from low to high. Almost 1000 students with a low socioeconomic background were studying in these schools.

3.2.1. Planning procedures of teaching experiments in the first year

Parallel with the main aim of the current study to create a learning trajectory for an in-service teacher's in selecting and implementing high level slope tasks literature-derived hypotheses about teacher learning informed the design and implementation of this professional development. and the nature of algebraic thinking through the lens of covariational reasoning perspective served as foundation for the design of the preliminary plan. The conjectures provided in this teacher professional development was mostly related to the focus of this study which was knowledge of selecting and implementing high level tasks and that would assist practicing teacher in analyzing students' thinking by highlighting particular algebraic thinking and slope conceptualizations. In that respect, a task repository related to algebra domain, specifically slope notion was created to increace the teacher's knowledge of cognitive demand of slope tasks and noticing skills of students' algebraic thinking in the context of high-level tasks.

In order to create a task repository, as suggested by Gravemeijer and Cobb (2006), the first attempt was related to determine and clarify learning goals for slope in eighth grade. To begin with, the researchers started to work on sequencing of various slope conceptualizations and related notions or reasoning with slope and adaptations of prior work on instructional sequence (Deniz &Tangül Kabael, 2017). Lale also got engaged in some of these meetings to satisfy the state of mutual work of researcher and teacher that is a common feature of design research (Anderson & Shattuck, 2012). Furthermore, the related objectives in The Turkish Middle School Mathematic Curriculum (2018) about slope notion under linear equations unit and textbooks and objectives in other countries were examined. The Turkish Middle School Mathematic Curriculum (2018) allocated 30 hours for the topic of linear equations including the concept of slope, 10 hours for inequalities in the eighth grade and 40 hours for proportional reasoning and equations in the seventh grade. It was planned to implement in February and March 2019 in eight weeks. Therefore, six to eight weeks is a reasonable and realistic period of time to conduct learning activities designed to improve students' learning of slope conceptualizations and the teacher's attention to this concept. The objectives related to the concept of slope and linear equations in eight-grade according to the National Mathematics Program (MoNE, 2018) were as follows:

Students should be able to:

- solve equations in the form of y=ax and y=ax+b
- define coordinate systems and show pairs of points
- identify how two quantities having a linear relationship with each other vary simultaneously by using tables and equations
- sketch linear equation graphs
- create and interpret equations, tables and graphs of real-life contexts including linear relationships
- explain the slope of lines by using models, build links between slopes and linear equations and their graphs

Based on these six objectives given above, The Turkish National Middle School Mathematics Instructional Program emphasizes covariational reasoning by stressing varying of two quantities as it is highlighted in curriculum materials in the other countries such as USA, Japan and Australia. Researchers continued to inquire how this concentration affects pupils' knowledge of slope (Nagle & Moore-Russo, 2014). In reality, the training goes well beyond emphasizing the significance of comprehending those notions (Thompson & Carlson, 2017). A closer examination of the national curriculum reveals that the development of covariational and proportional reasoning is concealed in the sixth-grade curriculum. The covariational method is not stressed in the seventh-grade pattern generalization problems. In eighth grade, the curriculum emphasizes the various representations of slope while mentioning a few conceptualizations of slope (geometric ratio, parametric coefficient, behavior indicator, and real-world applications) and highlighting the connection between these conceptualizations with ambiguous statements. In addition, there is no path for learning the slope concept outlined in the objectives. Determining whether the emphasis on covariational reasoning and its relationship with slope is appropriately reflected in classrooms is, therefore, a matter of concern.

A cross-analysis of curriculum objectives and related literature on slope and related notions was carried out in order to determine the big ideas in slope notion. This detailed analysis indicated several big ideas:

(1) rate of change, (2) physical property (steepness), (3) geometric ratio (rise over run), (4) algebraic ratio, (5) parametric coefficient (the a in the equation, $y = ax + b$), (6) trigonometric ratio (the tangent of the angle that a graphed line makes with the x-axis), and (7) derivative of a function (Stump, 1999), (8) real life application (Stump, 2001), (9) determining property, (10) behavior indicator, (11) linear constant property (Moore-Russo, Conner, and Rugg (2011); covariation and variation (Confrey & Smith, 1994; Lobato, Ellis, & Muñoz, 2003; Thompson, 1994b; Thompson and Carlson, 2017); rates as measures of intensive quantities (Stephan et al., 2015; Lobato et al. (2003); slope-as-steepness; slope-as-rate (Tierney & Monk, 2007); correspondence perspectives (Blanton, 2008).

Concerning these notion and relationship between slope conceptualizations (Nagle et al., 2019), domain specific theory of Realistic Mathematics Education and Emergent Perspective was used as theoretical underpinnings to interpret tasks and students' thinking (Gravemeijer & Cobb, 2006). The retrospective analysis and initial version of the tasks were not demonstrated since these aspects are beyond the current study. However, the revised learning sequence of these tasks and additional tasks guided the implementation in the teaching experiments in the second year as a serving of a task repository for the teacher. In this task repository, there were also low-level algebra tasks gathered from national textbooks or literature. The teacher's learning process were divided into four main cycles (phases) based on dimensions of Algebriac Thinking Framework (Walkoe, 2015) in order to specify the shifts in teacher's learning. In order to not distinguish these four phases (cycles) with micro-cycles of coaching and macro cycles, these cycles could be seen as meso-cycles. McKenney and Reeves (2012) asserted that "several micro-cycles of activity are combined, e.g., in reporting, or before major decisions are made, thus creating meso-cycles" (p.78). These four aspects of Algebraic Thinking are interralated, in turn reporting the teacher's learning in these phases have potential to enable readers to follow the teacher's learning in a sequence of these four branches of algebraic thinking, especially for slope notion. Furthermore, these meso cycles might provide the researchers to think major decisions on teacher learning through these four-learning focus. The main goals and sources of the tasks of the cycles are shown in Table 4. The final version of tasks is given Appendix 1. Detailed explanation of tasks in each cycle regarding learning focus would be explained in the section 3.5.

Table 4. The learning focus of tasks in four cycles

Cycles	Learning Focus	Sources of Tasks
1 (Symbolic Manipulation)	Eliciting ideas on proportionality and unit rate and connecting them to students' prior knowledge about variables in equations and unknowns	(Blanton & Kaput, 2005; Asquith et al, 2007)
2 (Exploring Ralationships)	Generalizing the patterns in geometric figures, where independent variables increase by one and relate symbols with words. In addition, the tasks provided opportunities to build on covariations including the coordination of directions and discuss the steepness of lines. Analyzing the given linear and nonlinear situations with their graphs and sketching the graphs of dynamic situations. Eliciting ideas on coordinating directions and the amount of changes in one variable with the changes in other variables. (Covariational Reasoning, Parametric Coefficient, Functional property)	(van de Walle, 2013; Carlson et al., 2002; Radford, 2008)
3 (Connecting Repr)	Converting between multiple representations of functions if rate of change is clarified in the real life scenario (relating algebraic form with rate of change emphasized in the scenario), and converting multiple data points (table) where the relationships between variables were proportional and non-proportional) to algebraic and graphical forms.	(Wilkie, 2016)

Table 4. (continued)		
4 (Algebra as Tool)	Measuring the steepness of lines (dynamic triangle model), discussing the rate of change in coordinate planes (geometric rate of change) and elaborating on negative and positive slopes. Posing problems in which the graph was provided to elaborate on coordinating the amount of rate of change. Creating graphs of lines Discussing non-linear multiple data points to generalize them (coordinating the direction and the amount of rate of change)	(Byerley & Thomspon, 2017; Erbaş et al., 2016)

In addition to hypothesized tasks related to algebra domain, feature of professional development on recognition and implementations cognitively demanding mathematics tasks, related professional development activities were analyzed. Based on the analysis, the work of Boston and Smith (2009) which focused on creating a learning environment for teachers to select and implement highly cognitive tasks was considered as a main guide. The environment consists of samples of authentic practice, cognitive conflicts (Swan, 2007) between teachers' previous belief and knowledge and new conceptions about teaching and learning mathematics including ideas related Mathematical Task Framework, field experiences supported through critical questions and feedbacks. In that respect, practice-based materials (Silver, Clark, Ghousseini, Charalambous, & Sealy, 2007) including teachers' own classroom and other teachers' classroom artifacts were also employed. The final feature of the coaching program was related to the practices of the coaching program and role of the coach. In that sense the iterative nature of coaching (pre-conference, conference, post-conference) (West & Staub 2003), and partnership approach to coaching (Knight, 2007) were considered as another characteristics for initial principles before the first iteration of a coaching program. Although the concept of principle was conceptualized as various meanings such as value, prediction, criteria and heuristics (Bakker, 2018), in the current study principle had a meaning of prediction (Greeno, 2016) and guideline or heuristic (Van den Akker, 2013). Then, a teaching experiment was conducted with an experienced in-service mathematics teacher (Lale) in the first year of the larger study as a first iteration of a coaching program. The detailed aspects of those activities will be discussed in the teaching experiment 2 by associating to the coaching cyclic process. Based on ongoing and retrospective analysis, conjectured plan of the research team was changed. In the following section revisions to the conjectures about teacher learning of implementing high level tasks were discussed.

3.2.2. Teaching experiment for teacher learning and revisions to the conjectures during and after the study-1

The teacher and the researchers conducted debriefing sessions in order to assess teacher's learning with respect to the researcher and the teacher's own perspective. During a coacing cycle and once it was concluded, and the classroom videos, audio recordings of teacher and student interactions, video recording of teacher-coach discussions and meetings of the design research team were analyzed. The design features formed the foundation for our professional learning materials, and role of the coach. In order to achieve our vision for a change in the teacher practice in the classroom and conceptions about mathematics tasks, we needed to refine each feature for optimum efficacy of coaching program. This helped the researchers to revisit the principles and related activitites to have a better case for the subsequent experiment. Besides Van den Akker (2013) stated that a design principle is more than just a directive (either "do this" or "don't do that"); it comes with theoretical justifications or empirical underpinning and aims to accomplish. In conclusion, the gap between design elements and learning outcomes, and the rationale and the goal for the principles were presented to portray initial conjectures, argument for them and observed outcome (Table 5).

In particular, in between coaching cycles, the design team made conjectures/principles about the type of activities, the teacher's perception of coaching, the time of the activities, how the teacher moved from decreasing the level of slope tasks with limited understanding of its various conceptualizations to selecting and maintaining cognitive demand of high-level slope tasks by relating them to various conceptualizations of slope notion. For instance, after the completion of the first cycle and watching a high-level geometry task implementation through a TIMMS video, the following dialogue took place between the teacher (T), the coach(C) and the coach and the advisor (A) in a debriefing session:

> *C: What is your opinion related to the first cycle of the study and the video you observed?*
>
> *T: I think they are all very efficient. In the video, I realized that the teacher gave hints to those who have difficulty continuing in front of the board. I can do it in my lessons, too. The task we selected is very efficient I think, in previous years, I started with a balanced scale to find the unknowns. The criteria in TAG was useful to select tasks; for instance, asking posing problem was high level. We can go like this, by considering slope and its conceptualizations. We discussed manipulating symbols. Now, we can prepare tasks for the understanding of unit rate in a given statements. These statements could include negative and positive slope and their differences in graphics.*

In this debriefing session, the teacher assessed the tasks which she and the coach suggested by relating criteria in TAG and commented on a sample implementation of high-level geometry tasks. The teacher and the coach engaged in a great number of similar debriefing sessions before and after the implementation of each lesson. One of the revisions to the instructional sequence by her was the inclusion of more problems including real life situations. Her suggestion to any inclusion of tasks is dominated by her previous experience in slope teaching, which were about connecting functional property and parametric coefficient. The envisioned learning trajectory by researchers was also built on students' making sense of changing and constant variables in given real-life situations beginning with unit rate. Whereas plenty of tasks similar in nature that the teacher suggested to include also might lead students practice fluency in parametric coefficient and understand the rule of rise over run procedurally. This indicated that the coach's envisioned trajectory and the teacher's thought/conceptions about the task sequence contradicted many times during the experiment. In addition, it was deduced that she analyzed the video by focusing on the teacher behavior rather than the type of questions the teacher asked and the sequence of students' responses.

Based on her comments about the tasks and activities of the coaching program, the coach and the advisor negotiated some changes. The following dialogue between the coach and advisor was carried out:

> *A: Pressing on some tasks and slope ideas are important in that case. In some situations, the coach should insist on implementing or removing some of them.*
>
> *C: She also did not seem to use different conceptualizations of slope while assessing the task potential. Therefore, consistently asking her rationale while selecting tasks and slope conceptualizations is needed.*
>
> *A: Yes, TAG and slope conceptualizations should be highlighted during task selection or reflection on task implementations. Our topic is slope in the algebra domain, so while selecting the videos, we can select them from algebra domain.*
>
> *C: Yes, the study also increased her knowledge of teaching of algebra, particularly, slope, so noticing the critical moments in the video or evaluating the issue in narrative cases will both contribute to her noticing on issues related to implementing high level tasks and specific students' understanding of slope. In that sense, it is better to select cases from algebra domains, and maybe, beginning with a workshop on criteria of TAG and narrative cases of implementing high- and low-level tasks within various content domains is a critical decision for the next teaching experiment. In that respect, activities should focus on teaching and learning slope during the coaching period.*

In the dialogue above, the research team decided that focusing on certain tasks and ideas is needed to improve students' and the teacher's understanding of slope since the teacher might select tasks without any critical analysis of learning progression of students and without negotiating with the coach. In that sense, the role of the coach during the discussion process should be reflexive based on the teacher's perspective of the coaching program and her previous knowledge of teaching slope and knowledge of the content (slope). At that point, a decision of the research team was changing the role of the coach from receptive to responsive to press teacher to interpret students' idea and task nature by using TAG and the frameworks related to slope and covariational reasoning (see Table 5, second principle). Another revision for the next teaching experiment was related to sequence of the coaching activities. The advisor and the coach had anticipated that a workshop on cognitive demand of mathematical tasks within a longer duration before the coaching on slope instruction began would support the in-service teacher's understanding of the cognitive demand of tasks and factors influencing the demand of the tasks during implementation.

In these debriefing sessions, the researchers discussed the teacher's learning about the task selection and implementations, and researchers revised first conjectures about teacher learning. The excerpts provided above can be viewed as evidence of this, even though numerous other modifications were made during many debriefing sessions. The other expected condition for teacher changes and the actual situations related to those expected conjectures was given in Table 5. As aforementioned, the first inconsistency between the hypothesized conjecture and actual one was tried to be handled by insisting on using the selected particular tasks (the ways of selecting tasks were explained in the previous heading) which design team decided on. This was related to the coach's role of putting pressure on the teacher to give rationale for the task selection by using TAG and to relate task context with slope conceptualizations. Another issue is the mismatch between the hypothesized efficiency of sample videos and narrative cases selected from TIMMS and Stein and collogues' work (2009). Then, it was decided to use particular examples of implementations of high and low levels on the linear equations or algebraic thinking during the coaching process. The workload which included discussing the factors that have impact on the decrease or increase in the level of the cognitive demands of mathematical tasks, examining each classroom teaching and planning for the next lesson needed a remarkable amount of time and effort. Hence, we shifted some of the professional activities to be used in the "before coaching" sessions rather than "during coaching" sessions. Hence the workshop was designed before second study began. It had activities on sorting mathematical tasks by using Smith and Stein categories, discussing about the aspects of Mathematical Task Framework and discussing narrative and video cases of implementation of tasks. Those tasks were at different content domains. Another issue was about the gap between the teacher's decision on sequence of the task and research team decisions regarding task selections (third principle) although it was hypothesized that collaborative environment of coaching program would eliminate disaggrement on issues related to task selections. This situation demonstrated that teacher change was more complicated than simply applying a new artifact such as tasks, perspective provided through the professional development that appreciated by the professional growth model of Clarke and Hollingsworth (2002). Regarding fourth principle, the research team and the teacher agreed on the necessity of the coach's prompt during the actual instruction due to observing of shortcomings of the hypothesized principle regarding the role of the coach as a non-participant observer. Therefore, it was decided that the role of the coach during teaching was providing strategic and technical help at a moment to create critical learning opportunities for students.

Table 5. Expected and actual issues for the teacher's learning in the study I

Justifications and Aims for Principles	Principles related to the Teacher Learning	Actual Observations on the Teacher Learning
• Because students thinking and frameworks can allows teachers to select and implement high level tasks (Walkoe, 2014; Choppin, 2011)	• Teacher can select high level tasks with respect to students' previous understanding and notice different slope conceptualizations of students and tasks by using TAG	• The teacher struggled to select appropriate slope task to advance students' thinking and faced difficulty in implementing high level slope tasks.
• Because "samples of authentic practice" (Smith, 2001, p. 7) is beneficial in teachers' learning of cognitive demand of tasks (Boston & Smith, 2009)	• Using videos that include instances of applying high and low level could be beneficial in enriching teachers' mathematical task knowledge	• The teacher focused on teacher's behavior in those videos with a general claim such as "calling out some students who have difficulty and giving hint is good"

Table 5. (continued)

• Because the coach and the teacher have common goal for enriching students' thinking	• A consensus on which task will be selected or sequenced could be reached between the design team and the teacher to prepare or select cognitively demanding tasks	• In some points, the consensus could not be reached.
• Because the main role of the coach during teaching phase as data gathering and observing the teacher (Bay-Williams, McGatha, Kobett, & Wray, 2014)	• Teacher can enact cognitively demanding slope tasks even if the role of coach is a non-participant observer during teaching	• Being a non-participant observer during instruction limited the coach to make visible critical instances to the teacher and the teacher sometimes deviated from what was planned before the lesson

Based on this discrepancy between what is expected and what is observed (Table 5), research team began to revise prior conjectures or set new conjectures about teacher learning. Research-based knowledge from the previous studies on teacher learning in professional development and the nature of algebraic thinking through the lens of covariational reasoning perspective served as foundation for the design of our intervention. The core activities and principles on enriching teachers to select and implement high level tasks mentioned before were included. One of those was the work of Boston and Smith (2009) focusing on creating a learning environment for teachers to select and implement highly cognitive tasks based on the Mathematical Task Framework. Second issue was utilizing field experiences supported with critical questions and feedbacks. In that respect, practice-based materials (Silver, Clark, Ghousseini, Charalambous, & Sealy, 2007) including teachers' own classroom and other teachers' classroom artifacts were employed.

Specifically, after the first teaching experiment, to gather teacher's awareness solely on algebraic thinking, it was decided to select some instances of implementing high level algebra tasks and students' thinking in the first teaching experiments. In addition, both low- and high-level tasks were included in early stages of the study intentionally to create a cognitive conflict about cognitive demand of tasks through planning, enacting and reflecting stages. Besides coaching practices (Gibbons &Cobb, 2017) and elements of coaching framework on mathematical tasks (Russell et al., 2020) were used to shape principles for coaching program in the current study. Both highlight that professional learning happens in situations that are intensive, ongoing and reflective, connected to practice and student learning whereas the latter focuses specifically on cognitively demanding mathematics tasks. Guided by these recent studies and revised principles, our professional learning activities were developed around a set of design principles.
These conjectures were as follows:

a. focus primarily on how teachers' knowledge of mathematical task demand and noticing skills change in the context of high-level mathematical tasks

b. embed potential opportunities for engagement in inquiry with coach about students' thinking, pedagogy and mathematical task (e.g. pressure for explanations and interpretations)

c. use practice-based activities on slope that challenge teacher's key mathematical idea behind task and possible student-teacher discourse on the task,

d. collect artifacts (sample students' thinking, work, classroom episodes) to challenge the teacher's previous conjecture about the task and students' thinking and to support her in discussing the related pedagogy.

e. provide the frameworks related to slope conceptualizations used in developing envisioned learning sequence to highlight mathematical task characteristics associated with students' possible learning progression.

f. provide strategic and limited assistance during teaching

g. utilize a reflexive role of being directive or responsive with respect to the teacher decision on task selection

These principles given above were related to design elements for teacher growth including specific pedagogical tools and feature of coaching program within the context of cognitively demanding mathematics tasks. Teaching experiments might provide "powerful tools they can use in their classrooms, especially for designing tasks and modeling students' mathematics" (Norton, 2008, p. 286). In the current study, tools such as conceptual frameworks on slope and TAG, research-based learning outcomes and sample tasks were used in our discussions with a participating teacher to assist her in modeling students' mathematics on the selected tasks and implementing tasks without decreasing its demand. Hence, the principles of coaching in the current study combined with teaching experiment methodology (Lamb &Geiger, 2012) are organized within three phases: pre-observation (planning), observation (teaching) and post-observation (reviewing) since the essential nature of coaching program (Russell et al., 2020) includes cyclic model of three mentioned phases, and crucial expertise development includes reflection and enaction (Bakkenes, Vermunt & Wubbels, 2010). In the remainder of this chapter, how the researcher stimulated learning activities as conjectured within these three phases by involving a teacher is explained.

3.2.3. Teaching experiment in the second year (Study-2)

The second teaching experiment was carried out with another teacher with tasks studied and sequenced with a class of eighth graders. For each teaching experiment, the researchers, in collaboration with the teacher, designed roughly

30 hours of lesson in the domain of slopes and linear equations according to the teaching and learning process of the concept of slope (Nagle, Martinez-Planell, and Moore, 2019). Lessons were taught three times a week, and each lesson lasted around 80 minutes. The first researcher took on the role of a coach as clinical supervisor to foster and study the changes in the teacher's noticing of the students' thinking during the preparation, action and reflection phases and her practice in the context of high cognitive algebra tasks. This methodology resonated with content-focused coaching, which is one of the methods of professional development, (Darling-Hammond et al., 2009) since it involved an ongoing and in-depth collaboration between teachers and researchers for the purpose of allowing the teachers to view their current practices in a particular topic (linear equations and slope) and improving practices. This professional development model incorporated planning, enaction and reflection. Our conceptual lenses in our study were based on the developmental trajectory of the Noticing Framework (van Es & Sherin, 2011). This provides the developmental process of noticing in two dimensions - what she notices and how she notices it -and on four levels. We used these noticing levels as a lens to study and improve the teacher's practice, noticing skills and beliefs. Implementation procedures of the teaching experiment will be discussed under the Mathematics Coaching session since the experiment framed the phases and principles of the mathematics coaching and framework of mathematics coaching. The Mathematics Coaching as Professional Development section including core elements of the coaching and the procedures of implementing the experiment following principles of effective coaching was provided in the subsequent section.

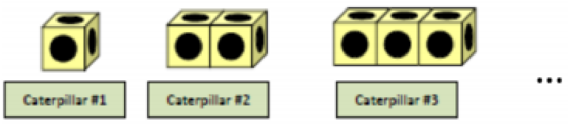
3.3. Coaching as a Professional Development Program-Preliminary Phase

As mentioned before, before a teaching experiment, assessing the learner's readiness and current knowledge is vital to locate the teacher's current knowledge and practice of envisioned learning trajectories of teachers and to revisit the hypothesized activities. In a similar perspective, Gibbons and Cobb (2016) proposed five practices of coaching: (a) determining long-term goals for teachers' development, (b) identifying teachers' current instructional practices, (c) clarifying teachers' current instructional practices within general trajectories for teachers' learning, (d) making decisions on what would be next for their learning and (e) designing activities to improve their teaching and learning. Based on Gibbons and Cobb's (2016) five key practices for conducting content-focused coaching, the goal was defined as to improve the teachers' selections and enact high-level tasks without decreasing their complexity and to improve the teacher's noticing skills throughout the coaching process.

After the identification of the goal, parallel with Gibbons and Cobb's second practice, open-ended questions and tasks were employedto determine the teacher'scurrent concept images and concept definitions, her thoughts about teaching the concept of slope and meaning for slope (Thompson, et al., 2014). I preferred to use concept image and meaning for slope rather than knowledge since meaning is associated to a person's current understanding whereas knowledge is related to a collection of "declarative facts" (Byerley & Thompson, 2017, p. 170). It was believed that this brief information about underlying aim of the interview questions and tasks could make reader to understand that the intention of the researcher was to document the teacher's current meaning for slope and concept image for slope conceptualizations rather than the declarative facts which the teacher's mastered related to slope teaching. (A sample task was provided in Figure 6). Moreover, 14 tasks, which were used to assess the teacher's rationale on task sorting, four were at the level of doing mathematics, five were at the level of procedures with connection, three were at the level of procedures without connection and one was at the level of memorization (Smith and Stein, 1996, p.346; Arbaugh and Brown, p. 87-88). Samples of those tasks are given in the Appendix B. Therefore, the clinical interviews on mathematical task knowledge and the teacher's concept images and meanings for slope teaching and the description of slope teaching were conducted. The interview questionsbased on these three themes in were presented in Appendix C. The next section was devoted for findings related tothe teacher's knowledge of cognitive demand of tasks and her current meanings and images for slope and slope teaching.

3.3.1. The way of the teacher's instructions of slope before coaching and meanings for slope notion

One day my little niece saw a clump of wriggling spotted caterpillars on the branch of a tree. Later she made her own collection of caterpillars with linking blocks and stickers. The first caterpillar was made with 1 block and 6 stickers. The second caterpillar was made with 2 blocks and 10 stickers. She continued to add to her collection:



i) How many stickers are needed for the next caterpillar's spots (Caterpillar #4)?

ii) How many stickers are needed for Caterpillar #7?

iii) How many stickers are needed for Caterpillar #17?

iv) For any caterpillar number you are given, how do you find the total number of stickers needed for its spots?

(The task from Wilkie, 2019, p. 24)

First, solve this task, and then respond to the sub-questions above.

1. What are the common responses to this question by the students?
2. Indicate other possible responses from students.
3. If some of students responded like: ''*Number of caterpillars multiplied by 4, and then add 2 for each*'' What would you ask next?
4. In the equation form of this question, what corresponds to 4 in the equation?

Figure 7. A sample task to assess Aysu's meaning for slope

In order to detect Aysu's current way of teaching of slope she asked to explain preferred task selections for slope concepts and the sequence of the tasks. Findings revealed that she preferred representing linear functional relationship in algebraic symbols starting with a real-life context. In addition, she mentioned about her way of instruction which based on physical property conceptualization and the formula (rise over run). This indicated that her concept image on slope and instructional decisions were parallel in terms of limited use of various slope conceptualizations.

In addition, I proposed a question about the types of slope representations, which students rely on most. She stated that:

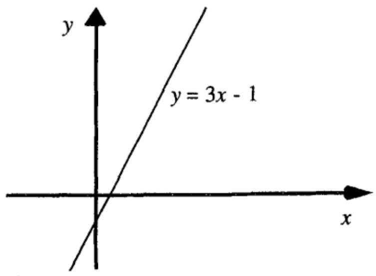
> Students often use the symbols or the rule of rise over run. I support this, too, It is very important to express the situation algebraically. I always want them to draw a graph because I think the graph is important since it is one of the multiple representations, but students use the graph less. But when I ask them to plot the graph of the given equation or situation, they draw the graph fast. Given the situation, I want them to express it algebraically first, and then graph it.

As can be understood from the above verbatim, she was aware of the fact that common use of representations by students included algebraic ratio and rise over run rule. Although she acknowledges the importance of the graphic display as I understood from her way of using graphics and algebraic representations, she could not combine graphics with functional property and geometric rate of change meanings of the slope. Hence, she used graphics figuratively rather than operatively. The next section would illustrate how she conceptualize slope and relate to other meanings of slope.

3.3.1.1. Meanings for slope

Teacher responses about the meaning of slope were categorized into eleven conceptualizations of slope (Stump, 1996; Nagle; Moore-Russo, 2011). She mainly defined slope as 1) ratio between the difference in the y-coordinates divided by the difference (rise) in the x-coordinates for two points (run), 2) tangents of line’s angle of inclination 3) slant of the objects such as ramps or stairs. The teacher mentioned algebraic ratio, physical property, trigonometric conception, geometric ratio and real-world static situations as meanings of slope concept. As seen in these definitions, she did not mention other conceptualizations of slope functional property, determining property, behavior indicator, linear constant and transitions between those conceptualizations.

Besides, the slope is linked with other constructs and notions. In fact, sub-components of interiorized ratio (Thompson, 1994) as "ratios as per-one" and "ratio as measure" (Johnson, 2015a) are related to the idea of chunky and smoot meanings of slope notion. In order to specify how Aysu defines slope by associating it with measure, the meaning of 4/5 and meaning of division in slope formula were questioned. She replied that “Because we need to calculate the ratio between rise over run, we use division operation”. Then, she continued that “4/5 represent slope and the ratio between vertical and horizontal change”. She also commented “it is a degree for steepness of the line; for instance, 3/5 is less steep than 4/5”. The responses demonstrated that Aysu conveyed a chunky, non-multiplicative meaning for slope, and she did not conceptualize slope as measure rather than a separate number indicating horizontal and vertical displacement. She also insisted on meaning of slope as an indicator of steepness. In order to characterize her understanding of smooth and chunky meanings of slope in depth, I used the task provided in Figure 7. She substituted x with a and a+1 to find out the increase in y (given below). Then, she got 3. She uses the same approach to find the increase in y as x increases a a+h. However, she could not link between results of a and b, and in turn, she could not generate a notion of unit rate of change including vertical change as a function of horizontal change. Hence, her conceptualization of slope was dominated by chunky rather than smooth meaning. In addition, she had difficulty in conceptualizing slope as functional property in which a unit of change in input yield corresponds to a fixed change in output (m).



a. What is the increase in y as x increase 1?

b. What is the increase in y as x increases a to a+h?

Figure 8. The Task adapted from Stump (1999, p.136)

Y=3a-1
Y'=3(a+1)-1
Y'=3a+2

Figure 9. Aysu's solution in the task (see Figure 7) for part a.

Based on her responses in the task (Figure 7), it was observed that although Aysu could express linear relationships by referring to varying of two variables with a constant rate, she seemed to consider the distinction between linear equations and pattern generalizations, whereas two topics have the same mathematical idea of rate of change, rate and slope. Her dominated representation in the given situation in which linear relations were implied or given was algebraic ratio. To illustrate, the dialogue between the coach and her was:

C: How did you get 4n+2?

A: I counted the total number of stickers every time a cube is added. Then, I looked at the difference between stickers such as 6 stickers for one cube and 10 for two cubes. When I subtract 6 from 10, I got 4. Hence, I got 4n. For the first cube, I put 1 instead of n, , so I added 2 to get 6., so the expression is 4n+2.

C: If you think of the given situation, what is the meaning of 4 and 2 in this expression?

A: 4 is the amount of increase, and 2 is a number that I got after calculation.

As seen in the dialogue below, she did not connect slope and constant (or y-intercept) with the corresponding situations such as 4 is the number of stickers on four faces, and these four faces increase by 4 when each one of the cubes is added. She focused only on numbers to establish the equation and conceptualized x as an unknown rather than a variable. Thus, she struggled to identify what variables are represented in the given context. The coach pushed her to relate the equation to the given situation where number of the cubes and number of the stickers varied at the same time to notice the indeterminacy of the functional relationship using physical structure of the pattern. Then the dialogue emerged as:

> *C: What if the cubes were separate? How can you generalize the number of stickers with respect to number of cubes?*
>
> *Aysu: If there are 3 caterpillars, then 4 stickers will disappear…. If 4,then 6 stickers will disappear. Ummm…. if I want to create an equation, it should be 2n-2. 2n-2 stickers should be removed since these edges are joint. Then, the formula would be the same since I subtract 2n-2 from 6n.*

Although she used correspondence approach emphasizing the relation between corresponding pairs of variable values described as input and output idea, she could not relate this relation as covariational approach which points to changes in one variable with changes in another variable between and within the variables. Hence, her understanding seemed recursive not relational. Her interpretations also confirmed that she had difficulty in understanding smooth meaning of slope within covariational reasoning perspective, and she struggled to elaborate on the notion of the unit rate. That also confirmed why she could not propose the functional property conceptualization of slope while defining slope. In addition, she typically selected and categorized tasks by attending to their surface level characteristics such as length of the tasks, utilizing diagram or representations, real-life context and general mathematical idea behind the task. Furthermore, she tended to classify tasks with respect to task difficulty based on her previous experiences. These were an indicators of Aysu's lack of knowledge of slope and inability to sort task affordances with students' thinking.

Aysu's inability to recognize mathematics tasks with respect to their cognitive demand and her limited understanding of slope notion and instructions were taken into considerations to enrich selecting and implementing high level slope tasks. To support Aysu, based on Russell and colleagues' framework, a set of conjectures about teacher learning derived from the literature guided the coaching and the activities. These ideas and works led the process of how knowledge of mathematical tasks and the concept of slope support the teacher in selecting, enacting and modifying the tasks without losing the complexity of their demands. In this respect, before the coaching cycles, a workshop was utilized to improve the teachers' ability to classify the given tasks based on their cognitive demands and identify the elements for maintaining or declining the level of high-level tasks since teachers' ability to identify the cognitive demands of the tasks was assumed as the first step for an effective implementation of high-level tasks (Boston and Smith, 2011; Arbaugh and Brown, 2005). The Mathematical Task Framework (MTF), developed by QUASAR researchers (Stein et al, 1996), provided as the theoretical basis for the planning and selection of professional development experiences. Other written documents that were central to the professional learning experience were the Task Analysis Guide (Appendix A), and the book titled Implementing Standards- Based Instruction: A Casebook for Professional Development (Stein et al., 2000). The guide included the criteria of the tasks at each of the four levels of demand - memorization, procedures without connections, procedures with connections and doing mathematics (Stein et al. 2000, p.343) and the attributes that maintained or decreased the level of the cognitive demands of the high-level tasks during enacting (Stein et al. 2000, p. 16). The book included a set of narrative cases that present empirical patterns of task set-ups and their implementations by teachers in the classroom settings. The study conducted by the QUASAR project team showed that the criteria for the cognitive demands of the tasks and narrative cases help teachers distinguish the types of students' thinking that the tasks provide and identify how teachers' pedagogical decisions during the implementation of mathematical tasks influence the demands of the task.

In this vein, at first, the participating teacher was asked to determine the level of the same 14 mathematical tasks by using the Task Analysis Guide (TAG) criteria and to compare her decisions about the categorization of these tasks before and after the Task Analysis Guide was introduced. Secondly, two empirical cases regarding the implementation of the tasks from the book, one of which is about algebraic symbolization and the other one about data analysis, were used to discuss the elements of implementation of high-level tasks based on TAG. The former is an example of the implementation of high-level tasks at low level, whereas the latter is an example of the implementation of high-level tasks at high level. These professional development sessions lasted 15 hours for 10 days. In fact, studies indicated that teachers succeeding in selecting high-level tasks might not be able to maintain the level of complexity during instruction (Arbaugh & Brown, 2005; Tekkumru-Kısa &Stein, 2015; Boston and Smith, 2011). Therefore, in each session during the coaching, the teacher was continuously guided to evaluate the cognitive demands of the tasks as selected, enacted and modified related to the linear equations and the concept of slope in particular. After the workshop coaching sessions began. The next section would portray underlying principles of the coaching program and how they were utilized within the context of the teaching experiment.

3.4. Mathematics Coaching as a Professional Development (Teaching Experiment-Study-2)

The process of the study was designed based on a coaching model (see Russell et al., 2020, p.152). The Math Coaching Model included three components: "(1) a coach development framework that specifies our method to train coaches, (2) a coaching framework that specifies key coaching practices and routines, and (3) an ethos of continuous improvement that informs how coaches are trained to use disciplined inquiry cycles to adaptively integrate the coaching model into their diverse local contexts." (Russell et al., 2020, p.152). Since the aim of the study was not to examine how coaching improves a teacher's practice and not to train coaches, only coaching framework was focused. As seen in Figure 2, the Coaching

Framework included three key coaching practices: (1) deep and specific discussions of the instructional triangle (student thinking, mathematics and pedagogy), (2) establishing mathematical and pedagogical goals, and (3) evidence-based feedback, inquiry stance and discussion process. Within a broad perspective, the model indicated that coaching improved mathematics teaching with respect to cognitive demand of tasks as enacted and students' thinking and needs. Then, this will lead to conceptual understanding of students.

According to Russell et al. (2020), taking an inquiry stance involved using noticing, wonderings or suggestions that were open to discussion rather than giving direct instructions. The inquiry stance was triggered by "the need for active teacher participation in meaning making around shifts in practice" (p. 6). In that respect, the coach had an analytic and interactional style to document what Aysu noticed about classroom practice and mathematical tasks addressing specific students' thinking and raise questions or wonderings about the elements of instructions and planning.

Specifically, Russell and colleagues (2020) claimed that three key coaching practices were subsumed through The Coach-Teacher discussion process. The process begins with the coach and teacher selecting a high-level task and determine students' possible thinking while solving it. The subsequent steps of the process consist of pre-observation, a lesson observation and post conference phases, which were similar to the cyclic process of West and Staub (2003). In the current study, the process began with the subsequent steps highlighted in the model, therefore the phases of maintaining the goal and selecting/adapting the task and identifying students' possible thinking was embedded during the pre-conference stage. Two of the key coaching practices (1 and 2) were utilized during the pre-observation, observation and post observation phases while evidence-based feedback became paramount during the lesson analysis within the post-conference phase.

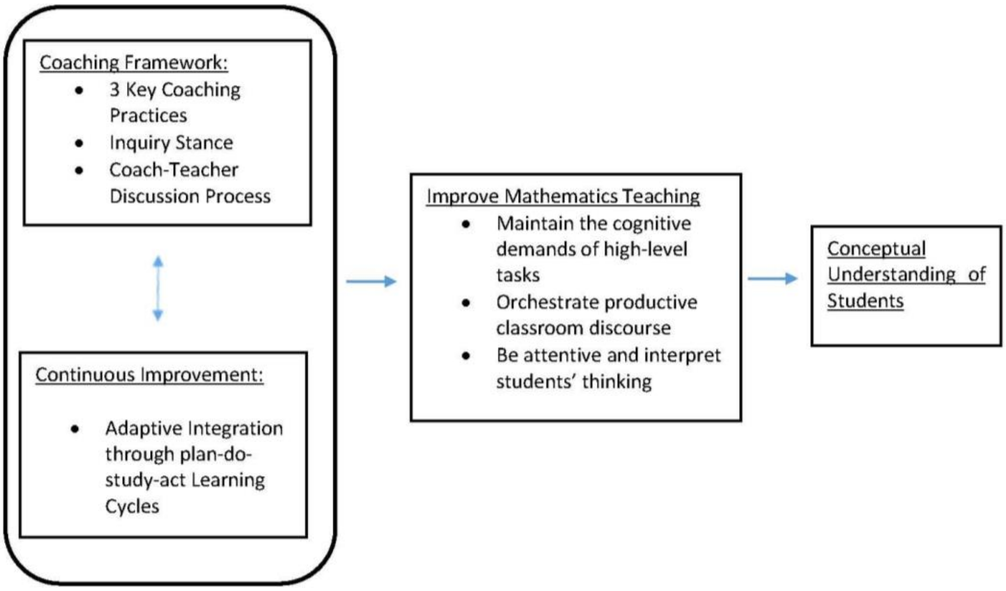


Figure 10. The Coach Development Framework taken from a part of the TN+IFL Math Coaching Model proposed by Russell et al. (2020, p.6).

The work of Coaching as Professional Development (Figure 10) was intended to improve teachers' quality of instruction by focusing on mathematical task knowledge The current study was built upon this model since it highlights particular teaching practice: enacting-high level mathematics tasks aligned with the focus of the current study enriching a teacher's implementing high level tasks. More specifically, on slope concept in particular, the teacher's noticing skills related to students' thinking, task and pedagogy and increasing academic rigor of the whole class discussion was a concern in the current study. In that respect, the model and goal of current study were matching with each other. The sequential progress between the coaching framework and elements for mathematic teaching shown in the model including maintaining the level of high demand tasks, employing productive classroom discussion and attending and responding to students' ideas and students' conceptual understanding were considered as a central target for this study. Based on the principles and conjecture of the study explained above, three main features of the coaching practices were applied. The first feature was the activities used in coaching sessions selected to represent "sample authentic tasks and their applications" (Smith, 2001, p.43). It was assumed that it might give the teacher the opportunity to learn about practice by examining the elements of the practice. These activities included solving slope tasks, analyzing sample works of students, examining instructional cases in written or video forms and interpreting artifacts from the teacher's own classroom. Second feature was that episodes of practice were linked to broader ideas about mathematics teaching and learning through Task Analysis Guide and the Mathematical Task Framework, and also instances of students' thinking were linked to more specific ideas on slope conceptualization and to its relationship with other interconnected mathematical constructs. Third, coached field experiences helped the teacher apply ideas and principles that emerge from coaching sessions to her own classroom. The last two features of this professional attempt had a great deal of ongoing work when compared to the nature of the development initiative of Boston and Smith (2011) where they observed teachers once and collected only three of the instructional tasks used by the teachers.

Since the tasks alone did not provide adequate support for the teacher, research-based frameworks on slope were utilized in the context of coaching. The frameworks by Thompson and Byerley (2017) related to covariational reasoning within various levels and by Nagle, Martínez-Planell and Moore-Russo (2019) concerning students' cognitive development from the action stage to the object stage in various slope conceptualizations were used as pedagogical tools for the research team and the teacher. Studies indicated how avoiding meaningful linking among different slope conceptualizations and between those conceptualizations and covariational reasoning produced barriers for learning the "rate of change". (Thompson & Byerley, 2017; Nagle et al., 2019). The aim of the design was to handle this issue. The process of the coach-the teacher discussion was explained in the following section.

3.4.1. The Coach-Teacher Discussion process

The process begins with selecting a high-cognitive demand tasks and then consisting of students' possible thinking and conceptualizations. In the domain of teacher change in teaching algebra, experiencing algebra tasks based on a functional approach rather than traditional equations-based approach before teaching them encouraged teachers to shift the teaching algebra from traditional to a reform-based (Steele et al. 2013). Therefore, the tasks that were selected or developed for this study were based on this functional approach and were shared with the teacher before her own classroom experimentation. However, the teacher was encouraged to select, modify, or create tasks according to the main goal of each lesson. In order to support the teacher to identify the main goal of the lessons and facilitate the teacher's understanding of the functional approach in the domain of the concept of slope, professional readings about various slope conceptualizations and covariational reasoning were provided (Nagle, 2019; Thompson & Carlson, 2017). While analyzing the mathematical ideas of the task, the coach also guided Aysu to explain relationships between cognitive demand of the task, context of the task and anticipated mathematical idea of students (Stein, Engle, Smith, & Hughes, 2008; Stein et al., 2009).

In the observation phase, the teacher was also guided to pay attention to the students' thinking in order to increase the level of the cognitive demands of the tasks and improve the quality of instruction. The coach also provided some technical and strategic cognitive help. Technical help included copying activity sheets, teaching how to use Geogebra for algebra, whereas cognitive help included asking key questions to students or giving students time to express and elaborate on their ideas in order to show teacher how and when to press on students' thinking. Reflecting on their own teaching and the students' thinking were regarded as other important ways to increase teachers' noticing skills and develop new knowledge. In the light of this suggestion, reflecting on the classroom practice was the main source for the post-observation phase. Thoroughout three phases, the coach-teacher discussion process in the Framework was expanded by highlighting the specific role of teachers and coaches more about teaching and learning of slope concept. Overall Table 6 summarized the roles of the teacher and the coach during the discussion process. Furthermore, detailed information is provided below regarding the pre-observation, observation and post-observation phases of the discussion process. Although the roles of the coach and the teacher mentioned in the Table 6 could seem to be identified before the study, based on continuous improvement and cyclic nature of coaching program and variations in teachers' needs through the mini-cycles, variations in role and activities also appeared during the process of the study.

Table 6. Roles of the teacher and the coach during the discussion process

Pre-Observation	**Description of the Work**
Teacher	Clarifying the Learning Goal Selecting a High Level Mathematical Task Working out possible responses from students
Coach	Working out possible responses from students Providing conceptual or technical support (the use of Geogebra for algebra, Covariational Thinking Framework and slope conceptualization, video or written cases of other classrooms)
Teacher& Coach	Engaging in a deep discussion with the coach regarding mathematical goals and pedagogy
Observation	
Teacher	Teaching the Lesson Gathering evidence related to student understanding
Coach	Observing the teacher while teaching the lesson Limited and strategic assistance (asking a question to the students) Gathering evidence related to student understanding
Post-Observation	
Teacher	Analyzing the cognitive demands of the task and possible reasons for the decrease in the level of the tasks and maintenance of their complexity. Reflecting on what is seen in the video clips.
Coach	Creating video clips from the lessons that highlight critical incidents about students' thinking and teacher reactions. Developing conjectures about noticing skills and practices of the teacher and students' learning.
Teacher& Coach	Analyzing the evidence to highlight the goals that were accomplished or not.

The core activities regarding the current study was explained by using The Coaching Framework in teacher-coach discussion process below. Specifically, selection and organization procedures of activities for pre- and post-observation of the sessions were executed as in Figure 14. As seen in Figure 14, before starting the teaching experiment (year 2), tasks were tentatively prepared based on the revised learning progression after the first teaching experiment and previous literature on symbolic manipulation, quantitative reasoning and linear equations. Several tasks enable researchers to assess how teacher select tasks and rationale behind her selections and improve previous learning trajectory on slope. In addition to task selection, the specific activities during planning and reflecting were established. Based on the analysis of the teacher's progress in planning, teaching and reflecting, new activities were decided and implemented in the planning phase; however, the core coaching principles were unchanged. These specific activities would be provided in the section of planning, teaching and reviewing stages.

3.4.1.1. Pre-Observation (Planning)

Before the pre-observation phase, teachers were asked to select a high-level task/s for the next lesson and then solve the task and specify the students' possible thinking or misunderstandings related to the task and the algebraic thinking that the task provided. These attempts were related to practice of *anticipation* for Aysu to "make an effort to actively envision how students might mathematically approach the instructional tasks(s) that they will be asked to work" (Stein et al., 2008, p.322). However, practices during the previous implementation and her comments about the way of implementing the task for further instruction indicated that she struggled sequencing and connecting various students' thinking. In that sense, the five practices for employing productive discussion including *anticipating, monitoring, selecting, sequencing and connecting* were provided for her with an illustrative case in the article of Stein and colleagues (2008). This illustrative case on proportional reasoning helped her move from unsystematic show-and-tell strategy to effective strategies to challenge students. Combining ideas in the illustrative cases with frameworks on covariational reasoning and slope conceptualizations also promoted Aysu in discussing the variants between students' algebraic thinking. However, she still had inability to address to mathematical ideas embedded in the tasks and how to extend the students' ideas. To handle this, researchers decided to demonstrate a sample of various students' thinking embedded in the task (mostly high level) and ask her to reanalyze the task context and mathematical idea behind it rather than directly explain the main idea of the task. To illustrate, in Cycle 4, she had difficulty in extending students' thinking from chunky slope meaning to the smooth slope meaning. To eliminate this, samples on teachers' various types of reasoning about slope, quotient, and rate of change from the work of Byerley and Thompson (2017) were demonstrated to her to help her understand that different reasoning requires different understanding in slope.

To conclude, in the pre-conference phase, the researchers concentrated on teacher's knowledge of the cognitive demand of the selected tasks and on how she connected the new task to the previous one. The coach also clarified how she planned to implement the task in the classroom. This pre-conference phase helped shed some light on what the teacher noticed and how she noticed them and the teacher's technical, pedagogical and tasks-related needs. Based on the teacher's decisions or deficiency in task modification, use of technology or instructional issues, the coach was able to offer the tasks to the teachers, and then they made changes in the tasks together or gave suggestions regarding how to implement the task and how to integrate technological tools such as virtual manipulative and Geogebra software into teaching.

Specifically, the first purpose was to increase teacher attention on nature of task itself with contradicting examples (high- and low-level tasks in Cycle 1), whereas the second purpose was to shift a teacher's focus from the whole class understanding to students' algebraic thinking by relating it to contextual of tasks through four cycles. By conducting a teaching experiment, the teacher was engaging with tasks directly through a coaching professional development. I offer the following vignette as an illustration of a typical planning episode to provide the reader with some indication of the coach's activities while planning the episodes, including the kinds of tasks and questions she asked. For instance, planning an episode for the task (Figure 10) in cycle 2 were illustrated to show how the teacher and the coach discuss it. The teacher implied that there were indecisions about the sequence of the tasks and how to make the tasks suitable for eighth graders. At that point, the coach suggested omitting some items, which could be too difficult for the students' level. Then, the coach asked which item/s or bottle/s could be removed in the activity sheet. At last, the teacher had decided to omit some of the cases of bottles which include three stages to sketch their graphs such as item 3 in the original task (see Figure 11).

Figure 11. Original task (Adapted from Carlson et al, 2002)

3. Aşağıda verilen variller eş musluklardan akan su ile doldurulmaktadır. Zamana bağlı olarak 1-2-3-4-5 numaralı varillerdeki su yüksekliğini veren grafiklerini çiziniz. Aşağıdaki soruları cevaplayınız.

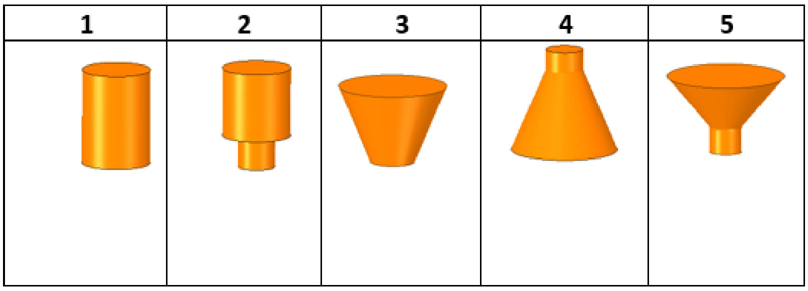


Figure 12. The task modified by the teacher

3.4.1.2. Observation (Teaching)

In each mathematics coaching sessions in each cycle, the coach observed the teacher's lessons for two hours. The teacher implemented a high-level task in the eighth-grade class, which was discussed and modified in the pre-observation phase. The coach and the teacher took notes about the students' misunderstandings, unexpected correct responses and students' thinking separately, and a mini-conversation was held with the teacher on what she noticed without interrupting the flow of the instruction when possible. Furthermore, the coach collected the works of the students and took notes about factors influencing the maintain or decreasing the cognitive demands of the tasks, and the quality of the teacher in linking the ideas of the students and quality of her questions in order to determine the needs of the teachers and students. For instance, the coach took notes about an unexpected students' thinking (illustrated Figure 12) about graphs for three different situations.

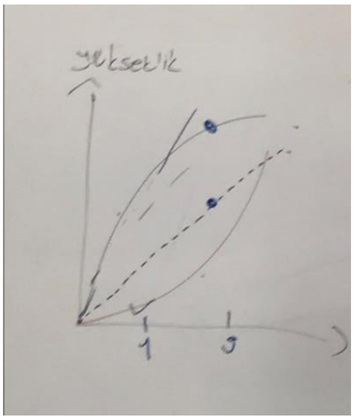


Figure 13. An unexpected student's answer for Task B (2nd Cycle)

In addition, the coach provided limited and strategic assistance for the teacher during the instruction. For instance, one of students was confused about the relationship between the sign of the slope and its graphical representation. The teacher wanted to clarify her ideas by using technology; however, she could not provide technical help for the student to use Geogegbra since class discussion was going on at the same time. Therefore, the coach helped the student sketch the graph with Geogebra after the student stated the algebraic form of the equations, which she wanted to view their graphs.

3.4.1.3. Post Observation (Reflecting)

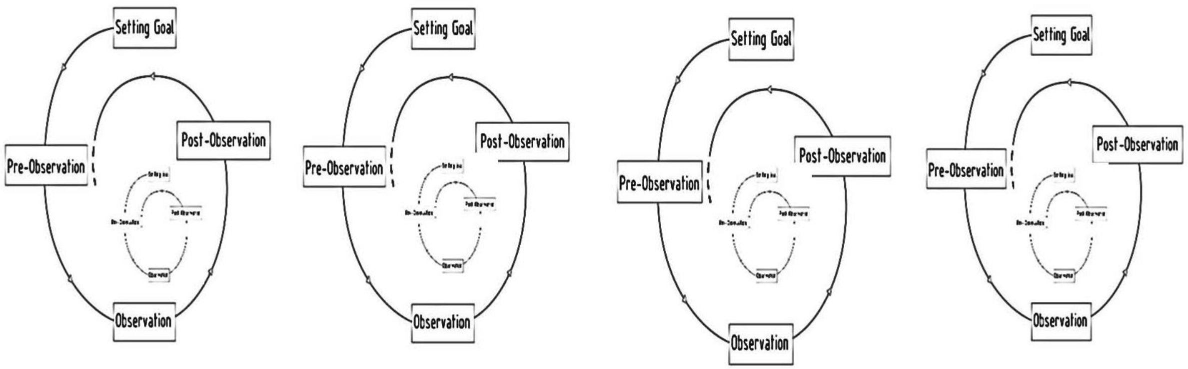
Before the post-conference session with the teacher, the teacher was encouraged to take notes on what she noticed about her teaching and the students' algebraic thinking. During the post-conference session, video clips were shown including the events that the teacher mentioned before, and the teacher was asked to evaluate her actions and the students' thinking in order to assess how the teacher responded to and interpreted students' thinking in detail. To illustrate, video clip of the unexpected students' idea given above (Figure) and her responses to the idea was shown to Aysu. She reflected on the student's focus on a single quantity without thinking two quantities together. Moreover, she was asked about her performance in terms of the quality of instruction including cognitive demands of the tasks, students' discussions, the teacher's questions, linking of ideas and pressing them for discussion in order to gain insight into her judgments on these issues. For instance, in cycle 3, she stated that the students could not produce productive discussions on geometric representations of the slope. Then, she stated that it was necessary to improve students’ learning by adding a similar task with extra sub-questions such as “why do we divide rise over run in geometric representations?”, “what is the meaning of the ratio?” Therefore, giving the teacher the opportunity to raise her concerns and self-criticism related to her previous lesson provided insight into what and how she noticed the students’ algebraic thinking. In addition, video clips were prepared including the critical points of the lessons such as students’ correct answers, misunderstandings or unexpected conversations among students and teacher’s unproductive orchestration of the discussion in order to discuss the points that the teacher did not mention. This would encourage teachers to reconsider the critical events and her actions that improve her noticing skills on implementation of the mathematical tasks and students’ algebraic thinking.

3.5. Data Collection Procedure

To analyze the development of the noticing skills of an elementary mathematics teacher, the researcher designed the implementation in three sections: before, during and after the content-focused coaching (CFC). Overall, the process of the study consisted of (1) professional development activities and pre-interviews before the CFC, (2) four phases consisting of pre-observation, observation, and post-observation in a cyclic manner during the CFC and (3) post interviews after the CFC. The process of the study is depicted in Figure 15. In the process, four cycles were highlighted and the description of each cycle was given in Table 4 including the number of tasks and categories of algebraic thinking in each cycle (Kieran, 2007). Through the cycles, at least twelve coach discussion process was employed.

Cycle 1 **Cycle 2** **Cycle 3** **Cycle 4**

- Understanding of the Teacher's slope conceptualizations
- Task Sorting Activity
- Analysis of written cases that highlight high level task implementations
- Analysis of a TIMMS video illustrating the enactment of a high level geometry task
- Analysis of Eleven conceptualizations of the slope (Moore-Russo et al., 2011) and Task Analysis Guide
- Presenting Linear Equation Tasks from the first phase of the study



- Interviews on coaching experience
- Analysis of a case related to high-level algebra task implementations

PRE PCD (15 hours) **DURING PCD (7 weeks)** **POST PCD (2,5 hours)**

Figure 14. The process of the design of the study

Table 7. Number of tasks and duration in each of four cycles

Mathematics Cycles	General Descriptions related to Algebraic Thinking	Duration	# of Tasks
Cycle 1: Manipulations of symbols and procedures	Symbolic manipulation, identifying the elements of algebraic expressions	10 hours	6
Cycle 2: Exploring Relationships	Looking for patterns (use of covariation or correspondence of variables expected)	8 hours	3
Cycle 3: Connecting Representations and Reasoning About Representations	Connecting reps (word and graphs; table, equations, word and graph), reasoning about representations (equations and graph), reasoning about slope of linear function (algebraic, geometric, functional, physical conceptualization of slope)	8 hours	4
Cycle 4: Algebra as Tool	Reasoning about reps (geometric rate of change and linearity); modeling with graphs and non-linear relationships	8 hours	4

The model has four macro stages/cycles which include several micro cycles of planning, teaching and reflecting. These micro cycles are launched in every two-hour implementation of the tasks. Before the coaching sessions began, Aysu's current knowledge regarding slope and nature of mathematical tasks were assessed. In line with the teacher's knowledge before any intervention began, the workshop on Mathematical Task Framework and criteria in TAG was utilized. The researcher identified Aysu's mathematical task knowledge during the coaching cycles and pre-coaching period. Through the four cycles, her noticing skills were also assessed in each mini-cycles including pre-conference, observation and post-conference. Underlying algebraic idea of four cycles and duration of those was provided in Table xxx. Each macro cycle consisted of at least three tasks with respect to main mathematical goal of the instructional sequence. In order to detect the teacher's changes in noticing four-time points throughout the coaching/teaching experiments are identified. As explained before, these time points were determined according to Algebraic Thinking Framework (Walkoe,

2015) including "manipulation of symbols and procedures, exploring relationships, generalizing and formalizing, using algebra as tool, reasoning about and with representations, and connecting representations" (p. 528). Detailed explanations concerning mathematical ideas of the tasks and reasoning behind selecting those in each cycle was given in the following section.

As seen in Table 7, the first cycle included six tasks designed for two main purposes. At first, considering the component of manipulations of symbols and procedures in the Algebraic Thinking Framework, tasks for manipulations of symbols is essential to encourage the teacher and students to develop ideas related to equal signs, unknown, variables and multiplicative thinking. In that respect, cycle 1 could be seen as precursor for developing slope notion. Second, it was aimed to enable the teacher to experience variances between students' thinking in low- and high-level tasks and address how her knowledge of task levels in each cycle of tasks as planned. Hence, the level of tasks in this cycle included both high- and low-level tasks.

Tasks (n=3) in the second cycle were designed and conjectured to make students explore covarying relationships in the context of pattern generalizations and graphics of real-life situations. The cycle was designed according to the idea of exploring relationships, a dimension in Algebraic Thinking Framework. Third cycle includes tasks (n=4) which are about connecting representations and conceptualization of slope, whereas tasks in the fourth cycle (n=4) were designed to help students generalize, making conjectures and using algebra as tools.

3.5.1. Data collection instruments and sources

The collaboration between the teacher and the researcher during the study included: a questionnaire on previous experiences and beliefs about teaching and learning algebra, an initial professional learning session led by the researcher about various slope conceptualizations and mathematical tasks, a joint planning meeting (pre-observation) to incorporate the proposed tasks or develop new tasks in the teaching sequence, classroom experimentation with tasks, co-teaching with the teacher, or researcher observation during the four cycles (observation) and post-lesson debriefings after each experimentation (pre-observation) and the final interview. The data were obtained from the observation notes taken by the in-service teacher and the coach. The field notes of the coach were used to describe the instructions and evidences of classroom episodes and the experience the teacher had. The observation notes and the reflections on the lesson episodes of the teacher were investigated to identify what she noticed about the students' thinking and the nature of the enacted task and how she interpreted her own teaching. Table 8 presents an overview of the types of data collected in each stage of the study.

Table 8. Overview of data collection process and sources of data

Data Collection	Methods	Sources of Data	Where Data are Used
Initial	Teacher's Background	Interview on previous experience of teaching algebra, knowledge of teaching algebra, beliefs regarding teaching and learning mathematics, knowledge of mathematical tasks.	A part of Coaching Practices
	Student Survey	Questionnaire (n=30) on a variety of tasks including pattern generalizations and graphical and symbolic presentations of slope.	

Table 8. (continued)

Main	Teaching experiment (eighth grade-linear equations, slope) ↓ Professional development and Planning ↓ Classroom experimentations including planning, ↓ teaching and ↓ reflecting ↓	Samples of students' works, video clips of the lessons, mathematical tasks that are enacted or selected, narrative cases audio or video recording of the interview researcher's journal, video recordings of the teaching audio or video recordings of the interview	First Teaching experiment Research Question 1 Research Question 1 Research Question 2
Final	Teacher Interview Student survey	Audio and video recorded Questionnaire (n=30) on a variety of tasks including pattern generalizations and graphical and symbolic representations of slope	

3.5.1.1. Data collection tools before the coaching program

Gibbons and Cobb (2016) proposed five practices of coaching: (a) determining long-term goals for the development of teachers, (b) identifying teachers' current instructional practices, (c) focusing on teachers' current instructional practices within general trajectories for the learning of the teachers, (d) making decisions on what would be next for their learning and (e) designing activities to improve their teaching and learning. Based on Gibbons and Cobb's (2016) five key practices for conducting content-focused coaching, the goal was identified as to improve the teachers' selections and enact high-level tasks without decreasing their complexity and noticing skills of the teachers throughout the coaching process. After the identification of the goal, six tasks were provided to determine the teachers' current level of mathematics knowledge of the topic domain of linear equations, inequalities and slope conceptualizations. These tasks were adapted from other studies about teachers' knowledge of teaching algebra (Vaiyavutjamai & Clements (2006); Stump, 1999; Wilkie 2019). A sample task was provided in Figure 6.

Table 9. Sample interview questions to elicit the teacher' current knowledge of task and slope

Current Knowledge of Aysu related to the	Sample Interview Questions
Nature of Mathematical Task	• How do you classify these 14 mathematical tasks? Give your rationale. • What do you think about the impact of mathematical tasks on students' learning? Do you think that you are critical while selecting mathematical tasks? Can you give an example?
The way of implementation "linear equations unit"	• To what extent did you follow the textbook? • Explain your teaching in previous years. Which tasks did you select? • How would you integrate technology into the "linear equations" unit in the 7th grade mathematics curriculum?
Teacher Knowledge regarding Conceptualizations of Slope	• What is the meaning of a slope? How can you represent it? • The slope formula is rise over run. What is the meaning of the division?

3.5.1.2. Data collection tools during the coaching program

The main data sources in the current study were individual clinical pre- and post-interviews for each micro cycle, the coach the teacher discussions, and field notes of the coach and the teacher. In the following subsection, the functions and design of each data source were explained. I mentioned all the details about how I used each data collection tool in the section of procedures of teaching experiment (see section 3.2)

3.5.1.3. Pre-Observation interview protocols

In the pre-observation phase, the teacher and the coach designed the lesson plans. This phase focused on both understanding the teachers' noticing skills, mathematical task knowledge, and enriching the teachers' knowledge of the misconceptions of the students and on how to enact the tasks without losing their complexity by proposing detailed questions and responses. In line with this focus, considering the first research question on the teachers' mathematical task knowledge through the cycles, I created interview questions for each pre-observation conference based on the thinking through a Lesson Protocol (Smith, Hill and Hughes, 2008) suggesting ways of lesson planning in a collaboration to prompt the teacher to focus on how to design and implement high cognitive slope tasks. Thus, the teacher's mathematical task knowledge, and her rationale behind the cognitive demand of the tasks as selected/adapted (R.Q. 1) would be assessed. In addition, in this phase, it was focused on the teacher's noticing skills on algebraic thinking while planning the tasks (as a part of the research question 2). The second research question also focused on examining how Aysu attended key contextual features, key mathematical relationships of the task and issues related to maintain academic rigor of the task during task as launch and task as enacted which are the main components of rubrics of Jackson and colleagues (2013). To some degree the question related to these three main themes are about to understand Aysu's mathematical task knowledge. However, in order to delve into her rationale behind adapting tasks and sorting those with certain levels of cognitive demands, questions related to those issues were added under the mathematical task dimension. In addition to questions related to the nature of the mathematical task, students thinking on slope notion and mathematic pedagogy are considered as the other critical dimension to understand what she notices and how she notices students' thinking within high level mathematics task context. The themes of interview questions were constructed by means of the idea of the instructional triangle including pedagogy, students' thinking and math content (Russell et al., 2020). Eliciting and extending probing questions were also added to understand to what extent and how she connected mathematics task nature and key mathematical ideas regarding slope and to identify how the teacher interpreted various slope conceptualizations by linking with students' thinking and feature of the task. Thus, the questions asked associated with her rationale behind task selection or adaptation and classifications of lessons according to the criteria in TAG, changes in the features of the tasks and noticed elements regarding the task, mathematical idea, specific students' thinking and pedagogy. Some of the questions for each dimension mentioned were provided in Table 10.

Table 10. Sample interview questions in pre-observation

Mathematical Task Knowledge Dimension
• What is the mathematical goal of the task? • What is the level of the task? Why? • What are the key contextual features of the task? • What is the key mathematical idea behind the task? • How do you maintain the rigor of the task during the set-up and enactment process? • While selecting/modifying the tasks, what did you attend? Explain specifically. • What will you do if a group of students could not proceed? • What will you do if a group of students finish the task early? (Adapted from • Smith, Bill & Hughes, 2008) • How do you plan to deal with multiple students' thinking for the task?
Students' Thinking Dimension
• Which of these methods, strategies or thinking do you think your students will use? • How can you detect students' possible misconceptions regarding the task? Will this task or extra questions solve this confusion or misconception? How? • How does the contextual feature of the task relate to slope conceptualizations framework? Probing- How do you elicit or improve students' understanding of these mathematical ideas?

Table 10. (continued)

Pedagogy
• What resources or tools will the students have to use in their work and help them reason through the task? • What will you do if one group finishes the task almost immediately? How will you extend it? • While planning your strategies or tools, what did you focus on most? Why? What elements of your planned lesson did you attend? Give examples.
Eliciting and Extending Probing Questions
• You constructed the sub-questions, which asked to make tables, graphs and equations. Can explicitly stating representations in this order hinder students' high-level reasoning? Why? How can you eliminate this? • What if our main goal is to make students use both …. and …. conceptualizations? What characteristics should the task possess? Why? What do you think about this task and the students' thinking of "geometric rate of change" while conceptualizing the concept of slope?

3.5.1.4. Observation protocol

The researcher observed the mathematics teacher as part of mathematics coaching during four main cycles including 18 lessons each of which lasted 80 minutes. Throughout all these lessons, the researcher monitored the teacher to observe how she conducted the co-designed lessons, collected evidences of the expected or unexpected students' thinking and took notes of the teacher's questions and of how and when she pressed students to justify or elicit their answers. In that respect, the researcher applied an observation protocol and took field notes to determine either students' progress or struggling and what Aysu attended to and how she responded to those moments. In addition to observed elements of classroom episodes with respect to researcher perspective, in order to understand what the teacher noticed during teaching, she was also encouraged to share the events that she noticed with the coach in a short period. In the end, the field notes included the aspects of the teacher's noticing as perceived by the researcher and the issues the teacher noticed during the instruction as she stated. Furthermore, the coach took notes on how and when the coach provided technical or cognitive support during the teaching experiments. Overall, observation of lessons is the central of the study to elaborate on both students' and teacher's progress and struggling and generate conjectures about students' learning and teaching practices that point to effective and ineffective mechanism in coaching model provided in these mini cycles.

3.5.1.5. Post-Observation interview protocol

The third instrument used to facilitate Aysu's learning process consisted of systematic, reflective interviews with the teacher conducted between lessons. Following a predetermined sequence of question types, we conducted stimulated recall interviews with the instructor after each lesson. Similar with pre-observation interview questions' themes, the themes of these interview questions (see Table 11) were constructed by means of the idea of the instructional triangle including pedagogy, students' thinking and math content (Russell et al., 2020). The interviews began by asking the teacher questions such as "What did you notice during the enactment?, Was there any point that you did not expect? What could be the reason why students had an understanding or an idea like that?, How did you take action to respond to the students or the whole class?"What she was seeing and what was attracting her attention while teaching was also intended to understand with these retrospective interview questions immediately after the teaching. (e.g., Luna & Selmer, 2021; Colestock, 2009). In addition to the teachers' own memories of the critical events, we prepared video clips (Ainley & Luntley, 2007) or written works of students including different correct answers or misconceptions to discuss the issues in the relevant segment of teaching or written works of students. The coach asked eliciting and extending questions such as "What happens here? What do you observe while watching this segment? What do you think about your action? What reasons can explain this understanding or misconception of the student?". These questions would stimulate the teacher to think aloud, and thus make her consider the lesson in depth (Meijer et al., 2002). In addition, we asked the teacher to identify and evaluate the level of the students' thinking by using the framework of Nagle and colleagues on students' cognitive development of slope conceptualization. Besides, in the post observation phase, I held discussions with her about how this lesson guided the next lesson such as selecting or designing the tasks. These questions enabled to understand what Aysu attended to while teaching and how she evaluate difference or coherence between planning and teaching and how they interpreted the moments she noted or captured and the moment the coach presented. A sample of the questions in the post-observation phase was presented below (Table 11).

Table 11. Sample interview questions in post-observations

Retrospective Questions on what and how she noticed during teaching

- What events attracted your attention while teaching?
- Based on your noticed elements during the instruction, what action did you take? Please give examples.

Mathematical Task Knowledge Dimension

- What did you think about the enactment of the task at a high level? Why did you think the high-level task maintained its level during the lesson? What were your evidences here?
- Did you change your task nature? If yes, in what ways? If not, please explain why?

Students' Thinking Dimension

- What strategies did the student use to solve the problem? Did the students try different approaches?
- Did you need to elaborate on some mathematical ideas in which students had difficulty? If yes, explain. If no, explain your further action?

Pedagogy Dimension

- In what ways did the questions you asked extend the students' thinking?

Eliciting and Extending Questions

- Based on this specific episode of the lesson [characteristics of the moment was explained above] what did you notice? How do you interpret this condition?

3.6. Pilot Study

The data collection instruments and procedures were evaluated by means of a pilot study. Pilot study could give a chance to reconsider the data collection procedure and to give insight into and clues for the final version of data collection instruments. In this respect, a pilot study was conducted with an in-service teacher (Ayşe) at a public school. She had eight years of experience in mathematics teaching, and her classroom consisted of 12 students. She had a master's degree in mathematics education program, and she was willing to enact new approaches in teaching algebra. During the initial professional learning session with the tasks at various levels of cognitive demand, although she could explain her reasoning behind clarifying the task with the corresponded level in a way to align with the Smith and Stein criteria, she could not change her focus solely on the correct answers provided by the students so she could not maintain the cognitive demand of high-level tasks during her instruction. This might indicated the influence of the teacher's personal domain including knowledge, beliefs and attitudes (Clarke and Hollingsworth 2002) towards the changes in teaching. Because she had a perception that students should give correct answer to all questions immediately. Then she could not maintain the cognitive demand of the high level tasks. Furthermore, she claimed that her weaknesses in concentrating on an issue within a long time period. These two elements gave researchers an insight that some belief scheme of teachers or other factors such as lack of concentration could be a strong barrier to learn a new instructional method. In that respect, the research team decided to create or select situations including contradicting cases which might make the teacher change her beliefs, perceptions or knowledge by constructing cognitive conflict. Furthermore, the research team negotiated about these unpredictable factors to come a decision about it. Another issue for the teacher struggle might be related to effectiveness of the professional activities of the pilot study. In other words, the close relation with teachers' knowledge and practices also indicated that failure of professional activities in changing in teacher practice. Then these initiatives could not help teachers build new knowledge, beliefs, or attitudes and in turn improve students thinking. In that respect, we in the first and second stage of the study, in addition to activities related to mathematical tasks, we designed more activities aimed to evoke the teacher's attention in various aspects of slope conceptualization. It was hypothesized that viewing various slope conceptualizations and findings of other studies related to the functional approach in algebra as external experiences of participating in professional activities would attract the teacher's attention and change the praxeologies on the students' learning in algebraic thinking.

Data collection tools and processes were evaluated based on the quality of the responses of the teacher based on the knowledge of slope tasks and issues on students' learning that were noticed. The research team realized that asking what was noticed in the previous lesson could not lead the teacher to give details about her interpretations of the important events. Hence, it was decided to add specific questions in the pre- and post-observation interview protocols. To sum up, the pilot study highlighted the importance of alternative ways in handling unpredictable outcomes or studying with human. It also allowed the coach reinforce her time management skills and collaborative skills.

3.7. Data Analysis

Changes in the teachers' knowledge (research question 1) were assessed using a task-sort instrument. Task-sort responses were coded as correct or incorrect based on four levels of cognitive demand. Besides, her responses were analyzed qualitatively to identify the teacher's learning of the cognitive demand of tasks and providing rationale for the level of tasks. The rationale provided by the teacher was used to assess whether task classifications were consistent with the descriptors in the Task Analysis Guide [TAG]) or not. In that respect, codes of Boston (2013) were adapted with respect to data collection tools and procedure of the current study. Boston identifies teachers' changes in knowledge of cognitive demand by examining whether they use labels and criteria in the TAG by comparing their answers on pre and post-tasks instrument, in the current study, in order to understand changes in Aysu's responses, I looked at her responses before and during the coaching. The reason not to identify Aysu’s responses after the coaching is to understand how ongoing coaching activities affect language while giving rationale for selection of tasks and their cognitive demand. Thus, in order to respond to the first research question about the nature of variations in Aysu’s knowledge of cognitive demand of tasks through coaching cycles, I determined three main components that refer to Aysu’s knowledge of the level of cognitive demand of tasks. Codes included: (1) specific use of categories from the TAG (e.g., memorization, procedures without connections, procedures with connections, and doing mathematics); (2) statements providing a rationale for task sorting by comparing characteristics of low and high level (e.g., "low-level tasks contain diagrams"); and (3) prominent language used by the teacher through four cycles. The first and second codes corresponded to her responses to nature of tasks by associating levels and criteria of each level in TAG with her generalizations or assumptions regarding task level. Third one is about statements reflecting the emergent language used during the conversations of planning sessions. Findings related to the first and second code were presented as a whole, whereas the third one was presented separately. In summary, changes in the teacher’s knowledge were assessed by using a task-instrument, and these changes were connected to teachers’ experiences in the coaching sessions through video analysis of planning sessions.

The second goal of this study was to analyze Aysu's noticing skills regarding students' algebraic thinking within cognitively high mathematical task context during the planning, teaching, and reviewing phases. To do so, teaching video and interview data were examined for this study to develop a preliminary understanding of noticing levels and elements regarding what and how she noticed in the phases of planning, teaching and reviewing. This analysis involved two phases: (a) identifying noticing levels b) characterizing those instances within two broad categories of noticing (what and how to notice) by inductive and deductive analysis process. To ensure the reliability of the coding procedure, the first researcher of the study and another researcher in the mathematics education program coded 25% of the selected transcripts of the planning, teaching and reflecting phases from the coaching. The data were coded with the help of *the framework for learning to notice student mathematical thinking* (van Es, 2011). This framework (see Table 3.3.) presents an understanding of a trajectory of development of noticing by two-dimension, what and how teachers notice through four levels. Those levels are Level 1 (Baseline), Level 2 (Mixed), Level 3 (Focused), and Level 4 (Extended). Within the classification of Level 1 noticing, for ‘what is noticed", features were related to attending to the whole class environment, behavior, and learning and teacher pedagogy. Along the continuum, at the most sophisticated level, Level 4, features were related to attending to relationships between specific students' thinking and ways of teaching. Similarly, within the classification of Level 1 noticing, 'how is noticed' consists of general comments on elements of instruction with little or no evidence.

On the other hand, at the end of the trajectory, the feature for Level 4 noticing includes interpretive comments about what is noticed with evidence and making connections between events and principles of mathematics teaching and learning. Specifically, 'how is noticed' characteristics include "teacher stance" corresponding to comments as descriptive, evaluative, and interpretative. Despite *descriptive statements* referring to comments about what was observed without any judgmental position, *evaluative ideas* establish for judgmental comments with limited or no making sense of the events. Lastly, *interpretive* comments include ideas related to the reasoning of the events and their relationships with the principle of learning and teaching.

Table 12. Framework for Learning to Notice Students' Mathematical Thinking (van Es, 2011, p. 139)

	What Teacher Notices	How Teacher Notices
Level 1 (Baseline)	Attend to whole class environment, behavior, and learning and teacher pedagogy	Form general impressions about what occurred. Provide descriptive and evaluative comments. Provide little or no evidence to support analysis.
Level 2 (Mixed)	Primarily attend to teacher pedagogy. Begin to attend to a particular students' mathematical thinking and behaviors.	Form general impressions and highlight noteworthy events. Provide primarily evaluative with some interpretive comments. Begin to refer to specific events and interactions as evidence.
Level 3 (Focused)	Attend to particular students' mathematical thinking	Highlight noteworthy events. Provide interpretive comments. Refer to specific events and interactions as evidence. Elaborate on events and interactions.
Level 4 (Extended)	Attend to the relationship between particular students' outcomes and between teaching strategies and student mathematical thinking	Highlight noteworthy events. Provide interpretive comments. Refer to specific events and interactions as evidence. Elaborate on events and interactions. Make connections between events and principles of teaching and learning. On the basis of interpretations, propose alternative pedagogical solutions.

The main focus of the present study was not only on the teacher's noticing skills about the students' thinking as in the Van Es (2011)'s noticing framework, but also on how noticing skills evolved through these levels during three stages of learning from practice (planning, teaching and reflecting) in the context of high cognitive algebra tasks. Therefore, the dimension of "what they notice" and "how they notice" was revised by combining mathematical tasks and students' mathematical thinking to respond to the second research question. In addition, what and how teachers notice during planning, acting and reflecting phases were identified in each phase. The revised noticing framework used to assess noticing skills of the teacher and planning high-level algebra tasks with sample quotations from the study is given in Table 13.

Table 13. Revised noticing framework during the pre-observation phase (planning of the high-level algebra tasks)

	What Teacher Notices	How Teacher Notices	Representative Responses
L1	Attend to students' thinking (possible) and teachers' pedagogy related to tasks in general	Form general impressions of what is planned	*Aysu:"The task incudes real life example,s so I am going to implement this"(Cycle 1 interview)*
L2	Primarily attend to teacher pedagogy. Begin to attend to particular students' mathematical thinking related to the tasks.	Form general impressions and highlight important events.	*Aysu:" Students could easily form algebraic form of the given problem, so its cognitive demand decreased. There is a need to prepare more challenging problems to make them consider relation of the context with the algebraic form" (Cycle 1 interview)*

Table 13. (continued)

L3	Attend to particular students' mathematical thinking utilized with the task.	Provide interpretive comments Refer to specific events and interactions as evidence	*Aysu: Most of the students begin to solve problem by making a table. Then, they use a graph. For the high level tasks, we should give them chance to select their own representations.* *Coach: Why is this important?* *Aysu: We can observe which representations were used by each student and their struggles in representations, so the telephone task is suitable for this, they need to decide their own representations, and then, I will force them to discuss each representation which they begin with to solve the question, for instance, how they reach the solution by using table, graph or equation, ….. I should ask: if we want to begin with a graph, how can we represent the situation?" (Cycle 4, interview)*
L4	Attend to the relationship between students' mathematical thinking and the task.	Highlight noteworthy events related to the task and students' learning Refer to specific events and interactions as evidence. On the basis of interpretations, propose alternative pedagogical decisions.	*"Students get to use the language of rate of change. This interpretation is a memorized explanation for middle and low level students, when I ask what the meaning of slope in the context and graph is; they could not add conceptual meanings such as for unit changing in x will yield changes in y as a slope or rate of change. However, in the empty coordinate system, we will provide a line passing through the first region and only x-axes. Then, I will ask them to pose a problem, and then, I will ask if x changes by 3, y changes by 2 and what the meaning of that in their written context is. I deliberately give rate of change as rational rather than integer. (Cycle 4 interview)*

Furthermore, I began with an initial set of codes from Van Es (2011), but updated them as needed to account for this particular data corpus. In addition to applying this set of convergent codes to the data, I viewed the data with an eye for additional noteworthy patterns that arose in the types of moments collected by Aysu or in the manner she described them.

Although there is no common consensus about the distinction between types of units and definition of the unit of analysis (Strijbos, Martens, Prins, and Jochems, 2005), based on the current research purpose, I selected sentences (in a paragraph) and discourse in the instruction attributed to the meaningful interval as the unit of analysis and coded them. Van Es authored the Framework of Learning to Notice Student Mathematical Thinking, which stems from three main components: the focus of what is noticed, strategies to analyze what is noticed, and the detail in which teachers can describe what was noticed. Although the relation was admitted, I agree with Stockero and Van Zoest's (2013) view that the noticing that teachers engage after the instruction is different from the one they engage during the instruction. As Stockero and Van Zoest claimed, during teaching, teachers tend to recognize moments whether they are mathematically or pedagogically significant or not. In tandem, teachers decide how these moments could be handled due to limited time for analyzing and interpreting students' thinking. In this respect, 'what is noticed' was characterized as 'what is noticed that triggered the pedagogical response' similar to the idea of Luna and Selmer (2021). In that sense, for teaching, the coding scheme by Stockero and Van Zoest (2013) involving five types of critical teaching moments which are extending, contradiction, confusion, sense-making, and incorrect mathematics were modified based on the data. Moreover, it was noticed that some of the instances were distinct from these five sub-codes during the coding process due to the coach's presence during instruction and slight differences in defining the noticing construct. I was trying to detect whether an event provided an opportunity for students and the teacher and what the teacher attended to. This led me to add additional subcodes to what triggered the teacher to attend/act while teaching: a) eliciting b) coach's action, and c) conceptual understanding.

To conclude, for the planning part within the dimension of “what she notices”, "Students' Possible Understanding and Task", “Specific Moment of the Instruction (Confusion)” and “Students’ Algebraic Thinking & Task Nature” codes emerged. First one was related to the elements of the mathematical idea behind the task, contextual feature of the task and students' possible understanding for further instruction, and the "Specific Moment of the Instruction or Student Thinking” code corresponded to noticed issues related to a specific moment of the previous instruction, another teacher’s implementation or a student thinking which could be high or low correct thinking or incorrect. These noticed issues could be generated by the coach or the teacher; thus, the abbreviation of [CI] referred to an event or idea which is initiated by the coach, whereas [TI] corresponded to an event or idea which is initiated by the teacher.

For the teaching part, because of the aforementioned reasons, codes were formed as “correct answer, extending, sense-making, mathematical contradiction/confusion, conceptual understanding, eliciting and coach action”. The correct answer code is related to the tendency of the teacher to get a correct answer without any elaboration. The descriptors of subsequent theme in the teaching sessions were described in details below:

> *Extending*: It occurs when students make a comment or ask a question, but goes beyond the mathematics that the teacher had planned to discuss.
>
> *Sense-making*: It occurs when students are trying to make sense of the mathematics, and the teacher ask "how" questions to enable students to make sense of an idea.
>
> *Mathematical contradiction/confusion*: It occurs when students do not reason their response or face a challenge to make sense of an idea, or the teacher creates a situation contradicting with each other to make students discuss it.
>
> *Conceptual understanding*: It occurs when the teacher helps the whole class develop more profound understanding of the slope concept. Based on the algebraic thinking framework, it is divided into two: reasoning with representations and connecting representations.
>
> *Eliciting:* Making students ' existing mathematical ideas visible to others without connecting it or showing/telling the mathematical idea.

> *Coach action:* Teachers take action, including revoicing or extending the coach' questions to make students discuss or justify any mathematical ideas (p.136-139).

As it is seen, there were explanations for each dimension of "what triggered her to act" in the moment of teaching for "what she notices" in the teaching dimension. For instance, *eliciting* is similar to the Van Es's (2021) notion, shaping referring to teachers' attempts to make a student's thinking visible to whole class rather than advancing one's thinking. On the other hand, confusion, extending, sense-making and conceptual understanding are different from eliciting in that they require a bigger effort.

For the reviewing part, the "Specific Moment of Instructions" code was regarded as a comment about instances of teacher pedagogy, communication between students-teacher or coach interventions, whereas the "Whole Class Understanding" code was related to the general comments about the mathematical idea behind the enacted tasks and students' responses to the task. "Specific Students' Thinking" code emerged from her comments about the specific students' thinking or specific students' thinking in their written works that were dismissed due to limited time or ignored by the teacher.

Table 14. The list of coding categories used in data analysis

	What the Teacher Notices	How the Teacher Notices
Planning	Students' Possible Understandings & Task [TI&CI] Specific Moment of the Instruction or Student Thinking [CI &TI] Students' Algebraic Thinking & Task Nature [CI]	• Analytic stance (Descriptive, Evaluative, Interpretive) • Course of action (creating or modifying the task)
Teaching	Short correct answers Eliciting Sense-making Extending Particular students' Confusion/contradiction Coach's Prompt Conceptual Understanding (	• Course of Action (Talk moves Chapin, O'Conner and Anderson, 2009) modifying/adding task)
Reviewing	Whole-Class Understanding [TI] Specific Moment of Instruction [TI&CI] Specific Students' Understanding [TI & CI] Students' Algebraic Thinking & Task Nature [CI]	• Analytic stance (Descriptive, Evaluative, Interpretive) • Course of Action (refining or adding a task)

*TI: Teacher Initiated, CI: Coach Initiated

3.8. The Researcher's Role

This study involved the participation of a researcher. I (first researcher) was the coach who worked with the teacher to select the tasks, explore her utterances as she interacted with the students over time and debrief her after each task implementation. In my professional role, although I had little experience working with teachers in a professional development setting. I had worked with elementary school teachers in the previous two years in a project that was aimed to design a learning sequence for number sense. Besides, I am a research assistant at a public university, and I have observed a number of pre-service mathematics teachers while teaching and have given feedback based on the criteria including their pedagogical content knowledge, specialized content knowledge, discourses and materials. Working with a teacher, Lale, in the first cohort of the study, also developed my knowledge of how to design activities for teacher learning related to mathematical tasks and slope conceptualizations. In addition, I had the chance to improve my collaboration and time management skills.

In addition to my professional adequateness for the role of a coach, my other roles were active listener, encourager and collaborator during the planning, enacting and reflecting phases of coaching as professional development. Moreover, the role of the researcher was to monitor and assess the learning of teacher and students regarding slope conceptualizations within the unit of linear equations during the coaching cycles and phases. My role also included improving their learning with technical and cognitive support in a classroom environment and carrying out interview sessions with the teacher as explained in Section 3.2.1.

3.9. Trustworthiness

In order to evaluate quality of qualitative findings, trustworthiness has become a concern (Lincoln & Guba, 1985). It incorporates four criteria: credibility, transferability, dependability and confirmability. These four dimensions were interconnected with each other. First, credibility refers to what extent to interpret participants' original data plausibly (Graneheim & Lundman, 2004; Lincoln & Guba, 1985). It deals with the "quality of data and the soundness of the reasoning that has led to the conclusions" (Bakker & van Eerde, 2015, p. 444). A qualitative study is considered credible if issues related to prolonged engagement in participants, peer debriefing, triangulation and member checking are handled. In this study, the researcher took a role for designing tasks with teachers, classroom experimentations and revisiting the conjectures and principles regarding teachers' learning on the use of tasks and students' understanding of slope for two years. In this process, the researcher made plenty of observations and interviews with teachers and students through two main cycles in two consecutive years, which made it possible to engage in real classroom environment and construct trust relationship with teachers congruent with inquiry nature of coaching. In addition, a variety of tools (e.g., student and teacher interviews, field notes, classroom observations, debriefing sessions of the design research team) were used to collect data, all of which contributed to the development of the study's principles. In this process, the research design team, consisting of the researcher, the advisor, and the collaborating teacher, developed and tested hypotheses concerning student learning, teacher practices, and teacher development. These tools enabled the researcher to discuss data with the teacher and make comparisons of data across multiple sources (Patton, 2002). In particular, in the process of data analysis, both researcher's and participants' perspectives (Tzur et al., 2001) were considered while determining the noticing level of the teachers and teachers' interpretations of nature of tasks.

Second, transferability deals with issues related to how findings inform further studies' context (Bakker & van Eerde, 2015). To be able to infer the nature of the experiment or context of the study for subsequent study's context, thick description (Lincoln & Guba, 1985; McKenney & Reeves, 2012) should be established. These descriptions were concerning data collection process, data collection instruments, data analysis procedure, the role of coach and teacher, context, characteristics of the participants (Creswell, 2012) in detail. In this respect, I aimed to give detailed explanations about each procedure and description of participants and context. To illustrate, the frameworks related to slope conceptualizations, covariational reasoning, coaching, which criteria is taken into consideration while selecting the participants, how the current knowledge or perceptions of teachers satisfy these pre-determined criteria, data analysis procedure including how the codes and subcodes were generated with respect to Noticing Framework and other studies' analysis frameworks were presented. In addition, sample of coded segments were provided to illustrate the consistency between quotations and given codes.

Third dependability is concerned with consistency between the data and interpretations of data (Merriam 1998) and replicability of the findings (Lincoln & Guba, 1985). Exhaustive reporting of the methodology and methods permits others to determine which method was used (Shenton, 2004) and coherence with the method and findings. To ensure dependability, the researcher documented design of the study, the methodology and methods, the details of data collection (e.g., field notes, memos), and reflective analysis of the work (Shenton, 2004) which is the appraisal of the coaching practices in the current study. In addition, dependability can be demonstrated through peer examination to gather additional perspective concerning the analysis process and research questions (Creswell & Miller 2000). In the study, the researcher and a peer have been working on teachers' noticing on mathematical teaching and learning and have been examining data analysis tools and conflicts on coding scheme, data and the frameworks (Noticing and Mathematical Tasks Frameworks) were discussed and continued until reaching a high percentage of agreement (inter-rater reliability). Moreover, the advisor and the coach (the research team) negotiated about level of cognitive demand of tasks on basis of Smith and Stein (1998) criteria and the coach also reached training materials of Quality of Instruction Assessment (IQA) (Boston, 2012) Rubrics, Checklists and Tasks. By the help of the materials, the coach could check and reinforce her knowledge related to cognitive demand of tasks.

Last, confirmability is related to stating perspective of the researcher clearly. It deals with "report on the steps taken both to manage and reflect on the effects of their philosophical or experiential preferences and where necessary" (Moon et al., 2016, p.3) and demonstrating to what extent characteristics of the data are shaped through the lens of the researcher. First, researcher's beliefs, dispositions, and conjectures are needed to be mentioned (Miles & Huberman, 1994, Patton, 2002). Second, Shenton (2004) put importance on explaining how the codes, categories and theories generated from the data in the methodology section. This process typically involves comparing evidence from many sources to clarify a code or viewpoint (Creswell 2012). Thus, to minimize the researcher's bias on interpretations, the role of the coach and teacher were explained in detail throughout all stages of the study. Besides, different sources are utilized as data collection tools such as the classroom sessions, teacher's pre- and post-observation interviews, design team meetings, students' works, and coach's field notes as audio- or video recording. The data gathered from each source was used where necessary while reporting the result.

CHAPTER IV

FINDINGS

This chapter will present the findings related to an in-service teacher's knowledge about the cognitive demand of mathematical tasks and how they improve her noticing skills in the context of the coaching program process. The chapter's sections are organized in the order of the research questions of the present study. In order to understand an in-service teacher's knowledge about the cognitive demand of tasks, findings are presented with respect to the level of the cognitive demand of mathematical tasks (Smith & Stein, 1996) by comparing them before and during coaching activities. Then the levels of the teacher's noticing through four primary coaching cycles are provided. Each coaching cycle has three components: planning, teaching, and reflecting. Specifically, dimensions of "what she notices" and "how she notices" in the Learning to Notice Framework (van Es, 2011) are provided with the evidence of the teacher's statements and teaching transcripts. In the abbreviations used in the quotations below, T and A refer to the teacher, Aysu. C refers to the coach, while S corresponds to her students. For instance, S3 refers to a student labeled with the number 3.

4.1. Aysu's Specific Learning about Cognitive Demands

Aysu's pre-task-sort responses and task sort process during the coaching were analyzed to determine the nature of Aysu's learning in identifying the level of tasks and her ability to use criteria given through TAG to describe a feature of the low and high-level tasks before and during four coaching cycles. In that respect, her knowledge regarding mathematical task's demand was presented through three aspects: i. identifying the level of cognitive demand, ii. Establishing rationale for task levels and iii. Using of new terminology.

4.1.1. Identifying the level of cognitive demand

Data in Table 15 illustrate the changes in the teacher's task classification responses over time by a number of correct sorting. Before the TAG, she classified 14 mathematical tasks (Appendix B) with respect to their content, difficulty level for students, length of the context of the tasks, and congruence with the curriculum. TAG was introduced in order to make her elaborate on four levels and notice which certain characteristics of the tasks possess different mathematical thinking. The coach also guided her to analyze the affordances or constraints offered by the tasks by scaffolding questions such as "what are the differences between the tasks? (Task E and F involve operation with fractions, or Task G and H involve converting between fractions, decimals, and percentages) or “consider the tasks concerning students' thinking? or embedded mathematical idea?"

Table 15. Analysis of the number of tasks correctly chosen by the teacher

Level of Cognitive Demand	Task Analysis Guide Provided (Before CDP)		During CDP	
	#of Tasks	# of Tasks correctly identified	# of tasks	# of Linear Equation Tasks correctly selected
DM	5	3 (%60)	9	5 (%56)
PWC	5	3 (%60)	7	5 (%71)
Total	10 (%100)	6 (%60)	16 (%100)	10 (%63)
PC	3	2 (%67)	2	1 (%50)
M	1	1 (%100)	1	1 (%100)
Total	4 (%100)	3 (%75)	3 (%100)	2 (%67)

*PWC: procedures with connection, DM: Doing Mathematics PC: procedures without connection M: Memorization

Concerning the level of "memorization", it was seen that Aysu was proficient in identifying low-level tasks. In particular, with regard to "procedures without connection", Aysu categorized fewer "procedures without connections" tasks incorrectly at pre-coaching than she did during the coaching (%67 and %50 respectively). When the distribution of these tasks (at the level of procedures without connection) was considered through four cycles, most of those tasks were sorted as being different from the level of procedures without connection in Cycle 1. The teacher's decision referred to her overgeneralizations of low-level algebra tasks, including using virtual manipulatives and asking students to create equations based on real-life situations. Although Aysu’s detailed explanations indicated that she was capable of using the criteria of TAG (Task Analysis Guide, 1998) while sorting tasks in various content domains of mathematics, she slightly struggled to determine the level of low-level tasks in the domain of algebra tasks in specific. On the other hand, 63% of high-level tasks were sorted appropriately during the coaching sessions while the ratio was also similar to the pre-coaching task sorting session. However, during the coaching cycles, she was able to sort high-level tasks as high despite some confusion about determining whether the task at level of doing mathematics or procedures with connection. However, during the pre-coaching cycles, she tended to ignore the critical mathematical idea behind the task and focus on the procedure. In turn, she typically tended to recognize the high-level tasks as low. Therefore, it was concluded that coaching activities help her select and adapt to high-level tasks during the subsequent cycles in the coaching process. The above data suggested Aysu’s improvement in the teachers' task-sort responses over time. However, I was also interested in how Aysu’s reasoning changes regarding task level to understand characteristics of her knowledge of cognitive demand of mathematical tasks. For this reason, I now shift to look at her rationale while sorting tasks at pre-coaching and during four cycles of coaching.

4.1.2. Establishing the rationales for task levels

Her rationale while classifying tasks was provided through four levels of mathematical tasks, namely, *doing mathematics*, *procedures with connection*, *procedure without connection*, and *memorization*. In particular, the teacher successfully classified some of the *Doing Mathematics* tasks as high during pre-coaching and coaching. To illustrate, she was able to correctly sort Task P as the level of doing mathematics. Her rationale for this decision depends on the ideas that unambiguous mathematics actions (calculating percentages or finding arithmetic mean) were not provided, and the task enabled the students to understand a given complex situation and devise multiple solutions. Moreover, at one of the coaching sessions, she was able to evaluate the task (given in Figure 15) as doing mathematics since the task offered multiple representations of tables, equations, and graphs to make generalizations, and the students needed to discuss the differences between those representations. Moreover, she implied that if the task was not well-structured, students would need to decide on a solution strategy using appropriate experience and knowledge (such as slope conceptualizations of linear constant).

Calculate the height of the person whose humerus is 47.5 cm. You can create mathematical models using the statistical relationships of different bones with the human height. For this, you can use the information given above in the TATVEM database.

A	B	C	D	E	F
1	150	30,5	39,6	39,6	20,5
1	154	30,2	38,4	38,4	21,7
1	150	31,5	38,6	38,6	20,4
1	155	31,9	38,9	38,9	19,6
1	150	32,1	38,6	38,6	20,3
1	154	30,6	39,8	39,8	20,2
1	151	33,9	39,4	39,4	22,1
1	152	32,4	38,4	38,4	21,8
1	147	33,1	39,8	39,8	21,8
1	159	34,7	41,0	41,0	22,8
1	153	35,3	42,1	42,1	22,0
1	172	37,2	44,8	44,8	24,1
1	153	32,5	41,2	41,2	21,5
1	165	34,4	42,4	42,4	22,9
1	154	33,5	41,2	41,2	22,4
1	157	32,8	39,8	39,8	21,5
1	144	31,4	38,1	38,1	20,5
1	164	34,3	42,6	42,6	22,8
1	143	31,3	36,2	36,2	21,6
1	160	36,6	41,9	41,9	24,0
1	152	33,2	40,7	40,7	22,4
1	150	31,7	38,5	38,5	21,6
1	154	33,7	41,6	41,6	22,6
1	162	34,0	41,8	41,8	25,4

Şekil 1. Türkiye Ali Tıp Veri Merkezi (TATVEM) Veri Tabanı [Anahtar, A sütunu 1: erkek, 2: kadın, B sütunu boy (cm), C sütunu kaval kemiği (cm), D sütunu uyluk kemiği (cm), E sütunu pazı kemiği (cm), F sütunu ön kol kemiği (cm)]

Figure 15. A Doing Mathematics task (Erbaş et al., 2016, p. 98) discussed in a coaching session of the 4th Cycle

Although her approach and reasoning in classifying mathematic tasks are operative in analyzing the characteristics of doing mathematic tasks, this overgeneralization might lead her to decrease the cognitive demand of the tasks at the level of *doing mathematics*. Therefore, it is said that this difficulty continues while classifying the *Doing Mathematics* task in Figure 15 as procedures with connection during one of the coaching sessions. The rationales the teacher provided were the presence of the pathway using algebraic or graphical representations of the given situation and offering little ambiguity to accomplish the task.

Mr. Ali saw the advertisements of two telephone companies for their monthly fee. Company A offers telephone service for a fixed fee of 20 TL per month and 0.10 TL for each minute of talk. Company B has no fixed monthly fees, but each minute of talk costs 0.35 TL. Which company do you think Mr. Ali should choose? Explain your solution mathematically.

a. Express in your own sentences what you understand from the question.

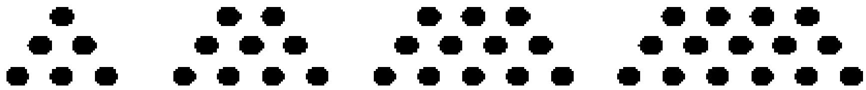
b. Which company do you think Mr. Ali should choose? Why?

Figure 16. A Doing Mathematics task discussed in a coaching session of the 4th Cycle

When the teacher incorrectly classified a doing mathematics task given in Figure 16 as having procedures with connection at the pre-coaching task sort, her rationales indicated that: (1) the task does not require much cognitive effort; (2) the task includes making calculations, overlooking the opportunities for developing mathematical connections (i.e., functional thinking) and the understanding embedded in the task. In addition, she classified the Lemonade task (Task H) as an example of doing mathematics at the level of procedures without connection. Her rationale was that the task requires students to compare two ratios similar to Task C (provided in Appendix B), and the procedure is explicit and robust. It might indicate that she was better at recognizing high-level tasks on the slope during the coaching. However, her struggle in distinguishing procedures with connection and doing mathematics still continues. Her reasoning on the distinction between the level of procedure with connection and doing mathematics is that a non-explicit path and ambiguity in the context always lead to doing mathematics tasks. However, she merely seemed to evaluate the task associated with the context of the task rather than relating the context to the mathematical idea.

Manipulatives/Tools: Counters

For homework Mark's teacher asked him to look at the pattern below and draw the figure that should come next.



Mark does not know how to find the next figure.

A. Draw the next figure for Mark.

B. Write a description for Mark telling him how you knew which figure comes next.

(QUASAR Project—QUASAR Cognitive Assessment Instrument—Release Task)

Figure 17. A Doing Mathematics task in pre-coaching

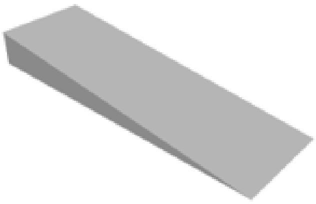
She correctly classified 3 out of 5 *procedures with connection* tasks before coaching sessions. For instance, she considered Task G a "procedure with connection" task (Figure 17). She assessed that using the blocks allows students to produce their invented strategy rather than the general formula of finding averages. However, while sorting and interpreting Task B, she was confused about whether the level of its cognitive demand was procedures with connection or procedures without connection. She stated that this confusion emerged due to being unable to make sure whether the task included more than one mathematical step, such as subtracting a given number and finding the ratio to answer the problem, should increase the level or not. Then she continued:

> When I considered my solution strategy, it consisted of only arithmetic; however, when I considered the students' thinking, I realized that they may have had trouble with the meaning of better player was injured and decide on finding ratios of what. At that point, understanding the context is a challenging issue for them.

The argument above demonstrated that the teacher might avoid making overgeneralizations such as that complex contextual situations increase the students' mathematical thinking or they only lead students to consider the implied path with extra mathematical steps such as multiplying or subtracting two quantities before comparing the ratios. At that point, it was inferred that she could not distinguish the tasks possessing meaningful contextual situations from those with typical contextual situations concerning their cognitive level.

2. Engelli Rampası Eğimi belirlenirken, tekerlekli sandalye kullanıcıları, yürüme zorluğu yaşayan yaşlılar, bebek arabası kullanan yayalar ve görme engellilerin de kullanacağı düşünülerek mümkün olan en az eğim dikkate alınmalıdır.

Yükseklik 51 cm – 100 cm arasında ise rampanın eğimin en fazla (%8) olması beklenmektedir.



a. Siz mühendis olsanız ve rampa yapmak isteseniz bu rampanın özellikleri ne olabilir?

b. Koordinat düzleminde gösterilmek istense bu durumu nasıl gösterirdiniz? %8 yolun eğimi dışında ne anlama gelmektedir?

When determining the slope of the disabled ramp, the least possible slope should be taken into account by considering that it will also be used by wheelchair users, the elderly with walking difficulties, pedestrians using baby carriages and visually impaired people.

If the height is between 51 cm and 100 cm, the slope of the ramp should be at most 8%.

a. If you were an engineer and wanted to build a ramp, what would be the features of this ramp?

b. How would you show this in the coordinate plane? What does 8% mean other than the slope of the ramp?

Figure 18. A Procedures with Connection task discussed in a coaching session of the 4th Cycle

During the coaching session, she indicated that the task (Figure 18) was parallel with the level of doing mathematics. She stated that the students were supposed to indicate the ramp in the coordinate plane and construct the geometric ratio and functional property conceptualizations. Furthermore, she considered that the task required students to generate various responses regarding the location of the lines on coordinate axes, enabling them to explore and conceptualize mathematical relationships (geometric ratio, determining property, physical situations). Her explanations were in line with the criteria in TAG and the important mathematical idea of the task; however, she did not realize that the task offered which representations they used, and the concept "slope" was given. Hence the task corresponds to procedure with connection not doing mathematics as she stated.

On the contrary, she classified one of the procedures without connection tasks as procedures with connection both before and during coaching. Her reason was that the tasks included "explanation" and "make use of diagram". During coaching sessions, she missed two tasks while classifying those as a procedure with connection. Her reason for this decision was that the task allowed students to understand the algorithm's logic while finding unknowns in equations via virtual manipulatives and creating equations of a given context that requested understanding of the situation. Based on these explanations, she overgeneralized the mathematical idea embedded in the tasks without considering the students' grade level and prerequisite knowledge. She also tended to interpret creating equations as a more challenging issue if given through real life situation than if provided through a pattern of numbers. This perspective is very interesting because she seemed to be disoriented due to her inability to distinguish between the way students think and her way of solving problems. Moreover, these wrong classifications are assumed to claim her deficiency in manipulating x as a variable or parameter. No "Memorization" tasks were misclassified during pre-coaching and coaching. Parallel with the first aim of this study, the prominent language of Aysu used while classifying the tasks also was analyzed. Related findings were given in the next section.

4.1.3. Use of new terminology

It was also identified where the specific criteria for high-and low-level tasks prevalent in the teacher's interview data arose during discussions throughout the coaching to identify Aysu's knowledge of cognitive demand of tasks. Aysu consistently identified the presence of a procedure as a feature of low-level tasks and insisted on providing justifications and posing problems as an essential requirement for high-level tasks. Evolving criteria were frequently expressly stated by the coach during the discussion of tasks that were selected and enacted (i.e., "What do you think about these different strategies? In what order do you discuss these strategies during the whole class discussion process?" [video transcript, cycle 2]; "What is different about S4 and S7's strategy? [audio transcript, cycle 1]; "How is the strategy connected to slope conceptualizations? [video transcript, cycle 4]". During Cycle 1, Aysu discussed multiple strategies and the meaning of algebraic thinking for solving the six tasks (see Appendix B). The coach made explicit moves to enable connections between strategies and the contextual feature of the tasks. (i.e., what do you think about the task context and algebraic idea (structural relationships between variables) of Task A?). Aysu's comment during the comparison of the tasks illustrated that Aysu concentrated on the critical mathematical idea of tasks. For instance, she commented on how students' thinking differs while working on Task D as follows:

> Students might find the relationship between two variables and maybe create a table for the transition to ordered pairs. They will be confused about the meaning of the result. Some of those might say that—2x+9 is correspondence to two times of a number plus 9. I do not know if they can say that x is a variable. They also might state that if we know the value of x, we can find the value of 2x+9

This excerpt indicated that she mentioned multiple solutions and strategies by the students. Moreover, Aysu stated that technology is a criterion for procedure with connection tasks before implementing the corresponding task. Nevertheless, after the implementation, she changed her idea and stated that the use of technology could be a tool for only practicing their prior knowledge. Moreover, she tended to classify the tasks that required problem posing at a high level.

In cycles 2 and 3, the teacher was provided with resources to enable her to interpret various conceptualizations of slope and its relations with covariational reasoning. Aysu was prompted to evaluate the context of the task and related mathematical ideas. Including multiple representations, making generalizations, building connections of multiple strategies, and slope conceptualizations were identified as a feature that made the task high-level. Cycle 3 differed from cycle 2 as rationalized by her idea that utilizing technology was one of the crucial characteristics that increased the task demand. To illustrate in cycle 3, Aysu commented:

> When we used technology, we can demonstrate differences between ax+b a/x+b for the student who said that they are similar in identifying linear equations. In addition, students can relate to and discuss the positive and negative slopes and the changes in the slopes on the graph with technology.

Based on the excerpt above, she connected determining property conceptualization to the multiple representations via technology. She discussed the effect of technology on making students discuss the characteristics of linear and nonlinear graphs and their equations.

Similarly, in Cycle 4, she continued to elaborate on the high cognitive task nature by associating it with the characteristics of the use of technology, connecting multiple representations, and a higher level of students' conceptualizations of slope (linear constant as free of representations). While she focused on technology as a tool for transition level of geometric ratio conceptualizations (by means of determining property, connecting algebraic and geometric ratio) in cycle 3, in cycle 4, she emphasized technology as characteristic of high cognitive demand tasks as an enabler for understanding slope as "the change in outputs is the rate of change (m) times the change in inputs" and a tool for solving a situation. In other words, her purpose of inclusion of technology changed with respect to the trajectory of slope understanding. In conclusion, using TAG descriptors through all cycles made the teacher develop ideas related to the tasks' nature. However, the prominent language used in classifying tasks before and after implementation might give detailed evidence of her understanding of relations between the mathematical idea and the context of the tasks. To summary, the data revealed that Aysu improved her knowledge of cognitive demand of tasks with respect to correct classifications of tasks, appropriate and in-depth rationale for task sorting and prominent language consistent with TAG and particular slope conceptualizations. The next section would present the findings related to second aim of this study portraying Aysu's noticing skills through four coaching cycles.

4.2. Noticing Skills in Coaching Macro-Cycle 1

Aligned with the second research question, the distribution and proportion of Aysu's noticing levels in three phases during Cycle 1 were presented in Table 16. To do so, teaching video and interview data were examined to develop a preliminary understanding of noticing levels and elements regarding what and how she noticed in the phases of planning, teaching and reviewing. Each unit of analysis was coded with four levels of noticing within three aspects of coaching cycle (planning, teaching and reviewing). Based on the table, her attempts were mainly seen grounding on Level 1 and Level 2. She could not demonstrate the characteristics of Level 4 noticing during Cycle 1.

Table 16. Distributions of noticing levels through planning, teaching, and reviewing of the tasks in Cycle 1

Level of Noticing	Planning/%	Teaching/%	Reviewing/%
Level 1	12/29%	10/27%	8/31%
Level 2	13/54%	22/59%	13/50%
Level 3	4/17%	5/19%	5/19%
Level 4	0	0	0
Total	100%	100%	100%

Specifically, the frequency of what she notices and how she notices is given in Table 17 through three components, which are planning, teaching, reviewing.

Table 17. Frequency of what and how she notices in Cycle 1 through planning, teaching, and reviewing components

Planning	What she Notice	f	How she Notice
Level 1	Possible Students' Mathematical Ideas[TI]*	4	General and descriptive assertions for the task affordances and constraints
	Student Algebraic Thinking & Task Nature [TI]	8	
	-General Feature of the Task (CD of task)	2	
	-General Mathematics of the Task	2	
	-General Pedagogy	1	
	-Relate Task and Student' Thinking	3	
Level 2	Students Algebraic Thinking &Task Nature[TI]	3	Evaluative stance and Descriptive Stance
	-Related to Context and Students Idea	2	
	-Teacher Pedagogy	1	
	Possible students' Mathematical Ideas [TI]	3	
	Specific Students' understanding [CI]*	4	
	Students' Algebraic Thinking & Task Nature [CI]	2	
	Specific Episode of the Instruction [CI]	1	
Level 3	Student Algebraic Thinking & Task Nature [TI]	2	Probing questions Sequencing the Ideas Modifying the Task Utilizing Technology Adding the Task
	Specific Student Thinking [CI]	2	
Teaching	**What she Notice**	**f**	**How she Notice**
Level 1	Students' Confusion	6	Asking yes/no or short answer questions Restating the phrase in the tasks without opening it for discussion
	Correct Answer	4	

Table 17. (continued)

Level 2	Eliciting Students' Ideas	5	Revoicing the idea without elaborating it.
	Confusion/Questions/Vague Statement	4	Making explanations
	Building Conceptual Understanding	3	Asking high level questions without connecting students' ideas
	Sense-making	2	
	Coach's Prompt/Action	3	Yes no questions
	Extending	1	Making explanations
Level 3	Students' Confusion/Questions/Vague Statement	1	Probing questions Pressing students' to justify or falsify thinking
	Building Conceptual Understanding	1	Using additional representations
	Sense Making	1	Connecting previous
	Coach's Prompt/Action	1	students' work
	Extending	1	Modifying the task Sequencing and linking among different ideas Using technology
Reviewing	**What she Notice**	**f**	**How she Notice**
Level 1	General Aspects of the Instruction	3	Describing with general comments
	Specific Moment of Instruction	2	
	Teacher Pedagogy	3	
Level 2	Whole Class Understanding	4	Adding a task
	Specific Students' Thinking	6	Descriptive and
	Specific Moment of Instruction [TI]	3	Evaluative Stance
Level 3	Specific Students' Thinking	2	Sequencing students' thinking or strategies
	Specific Moment of Instruction	3	Interpretive Stance Adding/Modifying the task

CI: Coach Initiated, TI: Teacher Initiated

Based on Table 17, she attended to aspects of the practice such as students' algebraic thinking and task nature, the contextual feature of tasks, mathematical idea of the tasks, students' expected ideas, extending, connecting students' ideas, confusion, and specific moments of instruction. These aspects highlighted by the teacher and the coach appeared to be essential aspects of the practice. The teacher's comment was described as descriptive and evaluative in terms of how to respond. It showed that the teacher met a challenge in making sense of these critical situations through three components. The next section would provide instances of what and how Aysu notices in planning phase in Cycle 1.

4.2.1. What and how Aysu notices in the planning phase (pre-observation) in Cycle 1

The second aim of the study was to understand Aysu's noticing skills through coaching cycles within planning, teaching and reflecting phases. Consistent with this aim in this section findings related to what Aysu notice and how Aysu notice in the planning phases of Cycle 1 were provided. Aysu's noticing varied mostly between levels 1 and 2 in the planning phase. 12 out of 29 instances were at level 1, and 13 instances related to her noticing were related to Level 2. Four instances were coded as Level 3. Throughout the planning sessions of Cycle 1, it was decided to give her freedom to select tasks from among the suggested tasks, use her selected tasks, and sequence them. This decision was given to establish intense collaboration with her, reinforce her to feel like being a part of the study and feel responsible for the student's learning and enable her to notice whether the selection and decisions of the tasks reached the intended goal or not. The following section showed what and how she notices across four levels, namely from Level 1 to Level 4

4.2.1.1. Level 1-Baseline Noticing

How she demonstrated her noticing at Level 1 while commenting on the tasks' nature and how to structure the teaching plans were introduced in the following section. The presentations of the examples based on Aysu's expressions from the planning meetings of cycle 1 were given in the next paragraph.

For the first two-hour teaching, she agreed to begin with the essential elements of algebraic thinking since she believed that these tasks were beneficial for the students to remember the way of finding unknowns and identities and help students while dealing with unknowns and the function of parentheses as procedures. Then, Task A (a&b) was provided to her to analyze its level with respect to its cognitive demand, consider how students react to tasks, and the students' possible thinking. She attended to the students' general mathematics thinking and general pedagogy. Her ideas related to the nature of the task and the students' possible mathematical ideas were as follows:

> *C: What are the students' possible answers for Task A, part a?*
>
> *A: They use only calculations. They subtract two numbers; then, they find the unknowns.*
>
> *C: What did you think about part b regarding its cognitive demand and students' thinking? (Task A, part b)*
>
> *A: It is good to see the algebraic expressions as an example to remember the pre-learned facts such as x2-81=(x-9)(x+9). With this example, asking "what is equation and expression" is a good starting point to teach how these two things are different. Students will see multiple examples.*
>
> *C: How could you add probing questions for these two tasks? What is the idea regarding algebraic thinking behind these tasks?*
>
> *A: I think it is enough. It is only about memorizing the rules.*

As seen in the dialogue, she could not propose any suggestions for making changes to the task nature and determine the key mathematical idea embedded in the subsection of the task. Hence, she could not state the possible relational reasoning for the quantities in the equations due to her tendency to look at general features of the task, and she mentioned asking a probing question (e.g., *what is equation and expression).* Besides, she mentioned the importance of daily language (as context) in introducing the equations. However, she did not offer details of this need in learning algebraic thinking in the domain of “manipulating of the variables” and how daily life examples are embedded through these tasks. In that sense, her noticing level is at the lowest level due to her broad and limited approach in terms of connecting students’ learning and the nature of the task. Specifically, her attention was solely on the students’ struggle with procedures and general characteristics of the given tasks rather than the task affordances or constraints with respect to the students’ higher cognitive thinking. In other words, she did not elaborate on how the task were related to the intended mathematical concept, such as the notion of slope and relational understanding of quantities. Concerning how she noticed, her comments had little relation to the students’ learning and were descriptive and evaluative. She did not elaborate on any opportunities offered to increase the students’ algebraic thinking and the branch of algebraic thinking utilized in the task. Hence, her comments were solely derived from her noticing of Level 1.

She also noticed the general features of the tasks and the mathematical idea behind the tasks from a general perspective. Specifically, she attended to the students’ easiness with the procedures and struggled with the relational understanding and cognitive demand of the task. Before noticing the elements regarding the students’ thinking on the notion of unknowns, parameters, and variables in the eighth grade in the further task enactments, she recalled Task E as low level, and she was tempted to conclude that the students would recall the structure of the equations, expression, and identity through the task. Like Task D, she evaluated Task E with regard to her general claim about the students’ algebraic thinking. In conclusion, noticing the issues regarding the struggles or easiness that the students encountered shaped her rationale on whether the task was included or why the task should be situated through the learning progress. However, the vague and general claims about the students’ thinking led her to make vague and general comments about the nature and sequence of the tasks. In that respect, her explanations possessed the characteristics of Level 1 noticing.

She also attended to connecting the general features of the tasks and the students' possible thinking or struggle. For instance, it was decided that a one-hour lesson devoted to Task D creating an algebraic expression of the given situation is implemented in the classroom since similar tasks commonly took place in textbooks, and Aysu confirmed that teachers are used to utilizing this kind of task in the classroom. To investigate the students' thinking in this kind of task, encourage Aysu to notice what the students think and how their thinking is shaped by the selected task and measure what and how she noticed, the coach directed her to implement the task. In addition to the coach's guidance, Aysu's reason for including the task in the instructional sequence was that it was needed to add such tasks to encourage the students to write correct equations of the given situation. However, she could not interpret how this needs to be associated with the students' understanding of the variables and relationships between quantities in detail. It also indicated that she noticed the superficial characteristics of the task that consisted of real-life context. She had general comments about the students' challenges in structuring the equations of the given situations. Therefore, her general and descriptive comments justified that her noticing was at Level 1 (e.g., *Students cannot write the equation because they do not understand the situation given in the problem. They are constantly faced with such questions* [tasks involving real-life situations], *so it would be good to have an example*)

Similar to her idea on that typical real-life problems were a challenge for students, she also attended to the nature of the task (using virtual manipulatives) and cognitive demand of the task (Task B) by relating it with the students' conceptual understanding in general. For instance, she argued that Task B (about finding values of the unknowns by using virtual manipulatives) would be used to portray the underlying relations in the algorithmic approach via technology, and she continued as follows:

> *C: What did you think about Task B and its sequence in the learning progression?*
>
> *A: The first task (Task B) is needed to get higher thinking; -procedures with connection since students will learn why they made the application, for example, - 8 becomes as (right side of the equation) as +8 if we changes its sides in a equation.*

As seen in the excerpt above, she mentioned that students could portray the underlying idea of the recalled rule (operation/s will be reversed for the other side of the equality) by using relational reasoning with Task B. She considered using virtual manipulatives and making students reason their recalled algorithmic way, referring to changing the operations in reverse to find the unknown. Although she mentioned critical mathematical thinking, including equivalence, variable and solving equations, she did not evaluate how those ideas evolve through the learning progress and relate to the mathematical ideas embedded in the previous task (Task A) and this one (Task B). In line with this, she assessed the task level as "Procedures with connection" without any supportive arguments for this claim. Although her comments had an evaluative stance in nature, she did not mention how technology enables students to generalize the way of finding the unknown. Similarly, she attended to the descriptive characteristics of the tasks (posing problems) by associating them with their potential to facilitate high-level students' thinking. To illustrate, she evaluated that task C aimed to make students pose a problem based on given algebraic equality. The way she interpreted the task is given as follows:

> The other task (Task C) is good; after they learn to find the unknown, they will consider what is the meaning of this equality in the form of ax+b=cx+d. They will create the context; I think this is important. Posing problems requires high-level thinking.

She made general and descriptive comments about the task with respect to the embedded ideas or practices that the coach mentioned in earlier meetings, such as "the meaning of equality". She also evaluated the potential of the task by overlooking the task affordances. Hence, her comments were regarded as bearing the characteristics of Level 1 noticing.

4.2.1.2. Level 2 Mixed Noticing

Findings related to how she demonstrated her noticing at Level 2 while commenting on the tasks' nature and how to structure the teaching plans were introduced in the following section. She attended to contextual of the task and students' thinking. Specifically, after the coach's prompt for Task B and the level of students' thinking, she proposed that she had had some dilemmas with respect to the level of eighth graders on algebraic thinking and task affordances. Therefore, she added a question, "what about your generalizations in division and multiplications?". At that point, the coach presented the question, "what about taking the square or square root of both sides." Regarding how she notices, she accepted this suggestion to apply in the classroom without mentioning the order of the task and any justification for adding these questions to increase the cognitive load of the task. It revealed that these additions and relations contained elements of a Level 2 noticing.

In addition to the cognitive demand of the task, she also attended to specific students' relational thinking of algebra. To illustrate, the coach demonstrated a student's thinking process regarding Task A, which included the structural relationships between quantities in the addition operation. Then she had an "Aha moment" and evaluated this piece of the students' thinking as building up relations with quantities in the operation of addition. Hence, she was able to attend to high-level student's algebraic thinking in general comments (e.g., it is good; this thinking is a base for algebra) and mentioned that the task does not only require calculations and she stated: "in the lesson, students should be funneled into such thinking". That was also an indicator of her assertions on pedagogical and teaching decisions since she did not highlight how this piece of reasoning is generalized congruent with the algebraic thinking category: Manipulation of symbols and procedures. Hence her noticing was considered as Level 2. Regarding how to notice, she described the students' thinking and made generic comments about it. In that sense, the coach asked, "based on these examples, which generalizations could be made if you think about multiplication and division?" Then she replied as: "*...when the operation changes such as an addition to multiplication;, students should be able to realize that it is needed to add and subtract with the same number in both sides whereas in multiplication/division operations, students will reach to the generalization that quantities will be divided or multiplied by the same number while structuring the relations between quantities".* This question led her to make attend to and evaluate the affordances of the sub-question related to the task despite her confusion about unit changes between quantities based on the operations on both sides. In order to eliminate this confusion, the coach suggested that she reconsider the difference between the student's thinking in the case of subtraction rather than addition and the principle of "doing the same operation to both sides". Then, she stated that, "*I actually did it for generalization; why do a decrease and an increase depend on this given operation?*" Therefore, it can be said that she had little elaboration about further task/question suggestions and how the role of structural relationships extends students' understanding of algebraic thinking.

Regardless of the coach's prompts or suggestions, Aysu attended to specific students' confusion and added to the task. She revealed the idea that students should discuss multiplicative thinking in a given equilibrium when a unit rate is a rational number rather than an integer. The time of the proposed idea is the break time of two consecutive lessons; thus, the coach could not make an additional comment and elaborate on this idea with the teacher. However, this idea could be related to Aysu's noticing of the students' difficulties in dealing with rational numbers procedurally or her noticing of the content corresponding to multiplicative thinking and algebraic reasoning. Even if the second case was proper, she could not seemed to relate the students' challenges with the task. Overall, her comments were limited in addressing specific aspects of the student's thinking and the nature of the task. Thus, her remarks were evaluative; in turn, her noticing level was regarded to be Level 2.

Moreover, she began to pay close attention to the task affordances related to the students' learning, thinking, and pedagogical decisions. For instance, she was able to make evaluative assertions about Task C in terms of implementing Task C and the students' possible solutions. The conversation concerning the nature of Task C and planning of ideas was given below:

> *C: However, if we analyzed the task, the form is as follows: ax+b/c=dx-e/f. What did you think about this?*
>
> *A: Yes, they will need to consider the meaning of division,*
>
> *C: What did you mean by the meaning of division?*
>
> *A: I do not know, equal sharing, for instance*
>
> *C: Ok. What misconceptions might students have about these two tasks?*
>
> *A: For the first task, students generally made correct calculations. If the equation is complex, such as in the second task, they had a problem with rational numbers and made mistakes in "cross multiplication."*
>
> *C: How can we present a remedy for this challenge?*
>
> *A: I will make them recall the rule. In the second task, students generally build on the unknowns, such as I had a number, I got two times this number, then I divided by 2.*
>
> *C: What else?*
>
> *A: Maybe they create a context; I do not know the amount of money in my pocket. I put two more Turkish liras then I shared with my two sisters. Maybe like this*
>
> *C: I will show you two samples from the students' answers. Let's investigate.*
>
> *A: I realized that some create a problem based on equal sharing and the number of groups.*
>
> *C: What else?*
>
> *A: I would let them think in a context, so I will not accept the answer as "twice a number plus three divided by 5"; I will say, "Think in another way."*

For task C, she realized from sample student responses that they could create different contexts using partition and measurement division. However, she could not state the students' possible answers concerning unit values or affordances of the task considering the idea of rate of change. Overall, Aysu attended to the students' ideas and task features related to the concepts in specific points; however, she could not attach students' algebraic thinking to the nature of the task in a straightforward way. Hence, her comments showed the elements of Level 2 noticing.

She also attended to specific students' responses and discussed the instructional decisions, including probing questions or adding conditions for the tasks to enable students to justify their responses. How she planned to implement Task E is given in the following dialogue:

> *A: Before I watched the episode, I did not think to ask additional questions for ax+b=cy, such as the possible values of x and y and the link between x and y. I realized that I began linear equations without establishing a connection with the previous topic. I manipulated x as only unknowns.*
>
> *C: In addition to additional questions for the item of ax+b=cy, what about other items? What would you expect from your students?*
>
> *A: I can ask students to give real-life examples for each item and make them discuss possible values for each of the given items.*
>
> *C: Every item has a solution or not?*
>
> *A: Aaa yes, for instance, for b, there is no solution. I could make them discuss the possible values for x's or y's.*
>
> *C: Ok, what about adding an item such as 3(x-4); if you want students to write real-life stories, what did you expect?*
>
> *A: Similar to the item 2x+9, two bags of apple plus 9 kilos. I do not need to add this.*

Although her suggestion to use real-life stories could enrich students' algebraic reasoning, she created stories with quantities that suggested fixed rather than varying amounts. She could not see the advantages of the item of 3(x-4) to expand the idea of changing quantities as fixed unknowns. On the other hand, her vision was that it was beneficial to discuss possible values for x and y to make students consider the changes in x concerning the changes in y. In this respect, her comments mainly included a description of the activity and her evaluation of the items in the task and the students' understanding of the changing values in the context of ax+b=y provided some evidence. However, those pieces of evidence were evaluative and initiated by the coach. On the other hand, she was able to propose alternative instructional decisions that contained real-life scenarios. However, her evidence on her decision seemed general and lacked elaborated ideas. Thus, this mixed approach demonstrated that her noticing exemplified Level 2 characteristics.

In addition to the attention to sample student's solutions, she also focused on the students' possible solution strategies. For instance, she attended to the students' thinking by presenting possible correct student solutions for Task C. Her comments on potential students' answers are as follows:

> *C: What do you think about potential student solutions for Task D?*
>
> *A: Students tend to use integers rather than rational numbers, so students begin by representing an unknown quantity as x and its multiplies.*
>
> *C: What else?*
>
> *A: Some of the students will begin by giving the first unknown given in the situation (in this case, the amount of box of A). Moreover, some establish an equation for the amounts of quantities in boxes, and some students will establish an equation for money paid for the amounts of quantities by multiplying it with its price.*

Based on the dialogue between the coach and teacher, she tended to relate the anticipation of students' algebraic reasoning with symbolic manipulations rather than students' informal approaches and relational understanding of the given (AX+BX+CX=D) context (Johanning, 2004). This indicate that Aysu attended to the students' general ideas and task features related to the concepts. She attempted to explain her decisions and thoughts broadly and disconnectedly. The attribution of Level 2 was supreme due to her general and evaluative claims on task features and students' understandings. Hence, she mostly tended to routinely link the mathematical ideas embedded in tasks and the students' thinking of algebra. At that point, she could not identify an inconsistency between her rationale while sorting the task at a high level and expected students' answers.

4.2.1.3. Level 3-Focused Noticing

Findings related to how she demonstrated her noticing at Level 3 in Cycle 1 while commenting on the tasks' nature and how to structure the teaching plans were introduced in the following section. Regarding Level 3, she proposed a change for sequences of the tasks D and E based on her experiences concerning the students' struggles with a relational understanding of quantities and their limited idea of the meanings of algebraic notations and systems (Task A) that are related to the students' algebraic thinking and the nature of the task. Then, she evaluated both tasks as high level. The former is "procedures with connection," and the latter is "doing math". With respect to what she noticed, she attended to the order of the tasks with regard to their cognitive demand. The conversation about this issue is as follows:

> *A: Asking students to make inferences between numbers in this way pushed them to think at a higher level.*
>
> *C: How is Task E related to what you said?*
>
> *A: What I mean is that the student needs to compare the meaning of equation and identity and x as a single entity or x has any numbers. In fact, there is a continuous relationship like Task A between the given variables. But when I solved Task D, I realized that it was easier for me to think of x as unknown.*

She gave little detail about the students' possible thinking on both tasks although she attended to the nature of the tasks with respect to the students' thinking. The coach provided the implementation of Task E by a teacher. Them, her interpretations changed based on the elements she noticed elements in the given segments of the teacher's actions in which the teacher added questions to make students associate possible values of x and y with the idea of linear relationship. Moreover, she used previous student thinking in Task A as evidence while interpreting both tasks. Hence, it was inferred that the coach's move and her interpretations of the students' algebraic thinking in specific were the impetus for analyzing the order of the tasks. Then, she was able to reason the sorting of Task D and E as high levels, and she compared the two tasks with respect to their cognitive load. Her justification was that creating equations based on unknowns required less cognitive thinking than building a relationship between variables, parameters, and unknowns and making a criticism about possible or impossible values for the given equations and expressions. At that point, her noticing had elements of both Level 2 and 3.

Apart from attending to the order of the task, she also attended to adding a task to enable students to connect mathematical ideas. In that respect, Task F (Figure 19) was selected by the teacher. She stated:

> I think it is important for students to think multiplicatively while determining the relations between variables to start this topic. I actually got this idea from our previous discussion with you. When we look at the unit, rate is important; it is necessary to combine the previous learning with the next learning.

Her suggestion on the task (see Figure 19) was about making a connection between quantitative reasoning and linear relationships that could be considered as a bridge for conceptualizing quantities varying linearly although she could not make detailed elaboration on the students' possible answers and representations. Hence, her comments about the suggested task could be regarded as a sign of Level 3.

If it takes 3 glasses of water and 2 glasses of rice to make rice pilaf, how would you describe the relationship between water and rice?

Figure 19. Task F, which selected by Aysu

In addition, she attended to a specific pedagogy for Task E. After watching the video clip of the implementation of the format of ax+b=y, she became aware of other meanings of the letters other than unknown. Then she discussed the item d in Task E as:

> If they think that p and s are different, then we'd better add an explanation. Or are we waiting for their interpretation? Do they firstly say that we cannot say p is equal to s? In that case, I will ask in what condition these two become equal for every case? Or in which case it could be asked.

Based on this comment, it can be inferred that she began to discuss specific pedagogy related to the meaning of equivalence and symbols rather than the general features of the task or general mathematical idea or related topic. She highlighted that they might understand a symbol as a fixed value that can be chosen arbitrarily, so they could not understand how these two quantities are equal. Then she proposed further pedagogical decisions to make students consider the relationship between two quantities. Although her comment could be regarded as Level 4, however, her pedagogy related to the task (asking questions of *in what condition these two become equal for every case? Or in which case it could be?*) seemed to be general in some respects because these questions could not guide students to think symbols as variables and identify that variables are changing together.

4.2.2. What and how Aysu noticed in the teaching phase (observation) in Cycle 1

The second aim of the study was to understand Aysu's noticing skills through coaching cycles within planning, teaching and reflecting phases. Consistent with this aim in this section the findings related to what Aysu notice and how Aysu notice in the teaching phases of Cycle 1 were provided. Aysu's noticing varied between levels 1 and 2 through the teaching phase in Cycle 1. Ten out of 37 instances were at level 1, and 22 instances related to her noticing were related to Level 2. Five instances were coded as Level 3. For the manipulating of the symbols and procedures as an algebraic thinking branch, the items related to the meaning of the equality and variable were mainly selected for Cycle 1. The findings of this episode were obtained from the implementation of the tasks. The findings from Aysu's lesson were presented, and her noticing during the teaching phase was revealed.

4.2.2.1. Level 1-Baseline Noticing

How she demonstrated her noticing at Level 1 with respect to what she attended a need for students during teaching and expository comments related to her noticed issues were introduced in the following section. The presentations of the examples based on Aysu's practices in lessons from the teaching meetings of cycle 1 were given in the next paragraph. Regarding Level 1, the most coded segment was related to her attention to correct answers from some of the students. For instance, the one-hour lesson was devoted to Task B, which was about finding the values of the unknowns by using virtual manipulatives. She began the lesson by introducing the virtual manipulative feature and asked students: how do they find the unknown? Some students began to maintain balance by operating with the constants one by one (e.g., S4's attempt in Figure 20) whereas some students dealt with the unknowns by thinking of x as an object or plenty of x as an object. Moreover, some students utilized the rule "changing operation and clustering xs in one side versus constants on the other side" (e.g., Figure 21), which could be related to the strategy called "unwinding" and they performed arithmetic computation rather than algebraic manipulations (Nathan & Kim, 2007). In addition, S23 considered 2x+4 as an object rather than unknowns and constants as separate. She acknowledged the two groups' thinking by letting them demonstrate on the board whereas she missed the S23's thinking. Then she asked: "why was the operation reversed when the xs or constants was carried out the other side of the equality? One of the students stated that, "Actually, I'm adding and subtracting from one side because it stays in balance," and the teacher concluded the lesson by explaining what the student stated. According to the students' correct answers about the value of x, she seemed to rely on the fact that they were able to reason the rule or the unwind strategy with algebraic thinking. However, she did not press students who used the unwind strategy to justify their thinking, and her evaluation depended only on the students' correct answers. Thus, her attempt to take a few students' responses might not reflect what other students think. Hence, her performance and issues not being attended could be a sign of noticing of Level 1.

Figure 20. S4’s strategy for solving the equations by using virtual manipulatives

Figure 21. S7’s solution for Task B with the unwind strategy

Similarly, she continued with the b part of Task A, which was about whether a given situation has equivalent relationships and reasoning behind this decision. She mainly attended to correct answers of students without pressing them to justify and review alternative explanations. She directed students to consider the possible values of x in the given equations where x could be any real number. The whole class discussion is provided below.

S5: Is it not in equilibrium because it is an identity?

T: Why would it be in balance?
Three students: because they are identical.

S7: Even if we write an infinite number instead of x, won’t the same equation give the same result?

T: So even if I give root three instead of x, will it provide equivalence?

S1: If we consider it an expansion rather than a number, won’t the bracketed product give us another general result?

T: If we express it better, we will use it instead of any number.
S4 and S5: we can say a set of real numbers (Task B, item b)

The dialogue between her and students revealed that she mainly pressed students to prove their claims by substituting any numbers for given balanced or unbalanced situations. It appeared that she noted specific students' thinking that was related to the multiple value interpretation of literal symbols (literal symbols as generalized numbers), but her action was limited due to limited space given students to discuss the meaning of literal symbols as a specific number, generalized number or variable. It can be considered that Aysu's noticing in this dialog was Level 1.

Concerning the reliance on the correct answers, she also did not attend to any other students' correct ideas, which she did not expect. For example, she set up the boxes task (Task D) for the students; she listened to all student answers and made them show the answers on the board that was assumed to be taken as an action based on the coach's recommendations. She did not mention the students' possible answers relied on "guess and check" in the planning session, and she did not acknowledge this informal approach of some of the students in the classroom. Therefore, she did not lead students to make a link between formal and informal approaches. She listened to all correct answers, and most of the students reached a solution by giving x as the boxes' amount.

In addition to her focus on correct answers, her attention on dealing with multiple students' limited/incorrect was also observed. She corrected the mistake by directing students with leading questions like "… isn't it, right?" Besides, she did not manage to orchestrate the students' incorrect and limited thinking or questions. For instance, the first two hour-lesson was devoted to the meaning of equality in algebraic forms, algebraic expressions, and finding the unknown in a given algebraic form. Based on the coach's emphasis on the meaning of equality, she claimed that students should be able to manipulate the symbols and numbers in a given equality rather than perceiving that the result comes out after the equal sign. In that respect, she began the task by asking about the meaning of equality. She attended to a student's confusion about the meaning of the equality of the variables; however, she did not take any action during instruction, and she claimed that “*I thought we’d come back to this topic, the timing was early*.” Although she admitted that this confusion should be considered in time, she could not elaborate on the exact time to open this issue to the whole class and provide any detailed way to overcome this misconception. The following dialogue describes how Aysu led the activity and what she did to help the students recognize the meaning of the equal sign.

A: What is the meaning of the equal sign?

S1: It includes unknowns

S4: Both sides of the equation are equal

A: It didn’t mean much to me

S2: It can be a computation on both sides

A: Does it have to be a computation on both sides?
Some of the students: No

S6: The result will be, the answer will come

S7: The number may be unknown…

S3: I think it can’t be unknown

S1: Like 2x=5x can’t be?

S3: Yes, for example, having x and y

S4: How can it not be x=y? Why not?

A: Did you say that we could not write this as 2x+6=2y

S3: I could write this; since I have a number in your example. But we could not say, x=y

A: We will talk later on this.

In the episode below, it seemed that the students’ ideas of equivalence relation were grounded on both an operational and a relational approach. She asked students to make them discuss the meaning of the equal sign and what the left and right equivalence relation includes. However, she pointed out a student’s operational meaning of equality as “being equal of both sides” by implying its incorrectness. Specifically, she did not acknowledge a student's idea of the fact that 2x=5x could not reach any solution. In contrast, she noticed the students' idea that x=y could not be written as equality. Although she noticed the students' limited relational understanding of the function of equality, she could not continue to arrange the students' ideas to delve into different aspects of them and ask extra questions. She did not also attend to a student's relational understanding of equal sign; she accepted the students' ideas as vague and continued the lesson with the question of "what is the equation". Based on illustrative incidents, it is possible to argue that Aysu's level of noticing in this conversation was Level 1 because she was unable to attend to the students' mathematical thinking and encourage them to clarify their various mathematical ideas regarding the meaning of the equal sign and literal symbols.

4.2.2.2. Level 2-Mixed Noticing

How she demonstrated her noticing at Level 2 with respect to what she attended a need for students during teaching and expository comments related to her noticed issues were introduced in the following section. The presentations of the examples based on Aysu's practices in lessons from the teaching meetings of Cycle 1 were given in the next paragraph.

Regarding Level 2, she attended to a student's thinking by explaining the idea to the whole class. The conversation between the students and her is given below.

> *A: Let's move on to the next question.*
>
> *S1: There is no balance. They are not equal.*
>
> *A: Why not?*
>
> *S1: On one side, all increased by 3 times, but on the other side, only 3 times of x was taken.*
>
> *A: Your friend thought holistically and thus says 3 times (x+5)*
>
> *S4: If we give a number, for example, if we give 2?*

S7: 21=11

A: Can I say these are equal?

S6: If I put it in common parentheses and if there was 15 on the other side, that would be equal.

A: Your friend thought the other way around here. Ok, let's examine the other one.

As seen in the dialogue, she was able to attend to some of the student's mathematical thinking and gave little chance to the students to explain their mathematical approaches. Specifically, she acknowledged a student's idea of x+5 as an object (structural operation of algebra); on the other hand, she did not give time to other students to elaborate on the reasoning. She seemed to recognize that a student had different thinking and wanted to explain her thinking to the whole class without using any probing or guiding questions. Therefore, it can be considered that Aysu's noticing in this dialog was Level 2.

She also attended to a need for eliciting students' thinking by prompting them to explain with no follow-up questions. This attempt could be related to the coach's emphasis on a need to prompt students to demonstrate their mathematical thinking explicitly. In addition to the students' attempts to present their thinking, the coach also advised that she might pay attention to distinct students' answers or strategies while monitoring small group discussions to demonstrate those to the whole class. After the group's discussion on Task F, she allowed students to write mathematical relations, which they built on the board (see Figure 22).

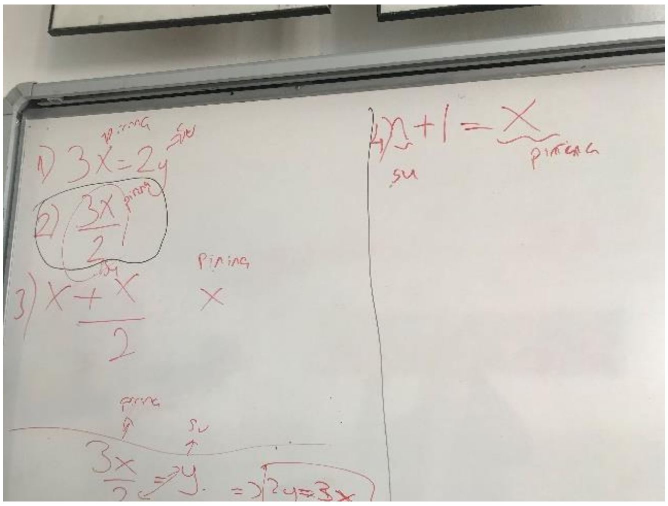


Figure 22. Students' responses on Task F

The dialogue presented below indicates how she directs the students' attention towards item 3 on the board.

> *T: Well, think about it this way, I will use 3 glasses of water for 2 glasses of rice, or you always increase it, I want to reduce it, I will make a glass less, maybe I will use a glass of rice or I will use a glass of water, how will it be?*
>
> *S12: we said, now, rice is 2 cups of rice, if there are 2 cups of rice, I take half of it, then I collect it, then I subtract 3 glasses of water. Let's divide a glass of rice into one, let's divide a half or even 0.5 with one, I find 1.5 water. I always found that*
>
> *S13: Depending on the number of cups, you found the amount of rice*
>
> *T: You said you will add half as much as the rice itself each time, and there is a total of a group, yes, you thought so, you made the switch, you said that if you add half as much water as the rice itself, this ratio will not deteriorate, neither you nor you. You said the same way, right?*
>
> *S6: 2 times the rice and take 1*

As shown in Figure 22 she wanted the students to write their ideas on the board and called a student in a group to demonstrate their equations. On the other hand, the excerpt illustrates what and how she attended to a group of students' ideas (item 3, see Figure 22). She listened to elicit the students' thinking by letting them explain with no follow-up questions. Moreover, she did not issue a challenge for other students to participate in the reasoning of S12's argument and provided space to discuss the differences between item 3 and the others. Hence, her attempt was coded as Level 2.

She also attended to a need for the student to build conceptual understanding by asking probing questions without making students elaborate on those ideas. After discussing the meaning of the equal sign, by considering the discussion with the coach, she tunneled the students' to find the unknowns without making any calculations for the sub-questions of Task A (Figure 23). The dialogue between students and her is given below:

i. 674-389=664-380.. (D/Y)
ii. 5.84=10.168...(D/Y)

Figure 23. Sub items in the Task A

S2: Based on S2's idea on the 7-□ = 6-4, 6 is 1 less from 7 then space should be more than 1 from 4. If I take 674, then compare with 664, there are ten more; hence 380 is wrong. It should be 379.

A: Could you change 389?
S2 and some students: Yes

A: How?

S3: It must be 390

A: What changes could you make to satisfy the balance?

S4: There are 9, so 664 must be 665.

A: If you consider the operation of addition instead of subtraction. What would you think?
S3 and S2: Decrease

A: What about multiplication? Think as 674x389=664x380. What would you consider if you think that the numbers 674 and 664?

S2: Not for multiplication and divison

A: Ok, if you consider the operation of multiplication. In question ii, what would you say? Did you think the difference as S2 said?

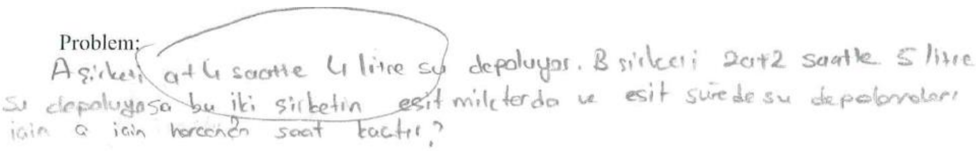
S3: Yes, when I doubled, for instance, 168 doubled by 84, then I doubled 5 with respect to the sequence of what S2 said; however, it is not valid for this situation

S2: Your claim and mine are different

S4: Something like that, if we add something, we need to subtract; in the same way, if we multiply by two, we need to divide the other side.

In the excerpt, although Aysu asked high-level questions, she did not orchestrate the discussion of the students, or she did not let the students justify their claims. For instance, although she wanted students to consider how to deal with the difference between quantities when operation changes, the students tended to explain the difference or multiples by beginning with only the right side. In addition, they focused on "decreasing by one and increasing by one" rather than building relationships considering the quantities on both sides. Although she attempted for students to consider other quantities in the given equivalence relation and the difference between the changes in addition and subtraction or division and multiplication, she could not clarify and categorize the students' responses. Some of the students could not follow the discussion. Overall, her instances of noticing were mostly regarded as Level 2.

She also attended to some students' thinking by probing and guiding them to create a problem of the given equivalent relations. She missed some of the students' interesting responses. Aysu started the lesson by warning students not to use such a structure, "*2 times five more of a number divided by seven is equal to 3 times five minus of the same number divided by 9,"* while creating the problem in line with the coach's suggestion. In a small group discussion, she made students think about the meaning of "a" in the context and the meaning of equivalence and division. Some students attempted to create context by defining a+4 and 2a+2 and dividing with 4 and 5 by focusing on two different meanings of division: the partitive and measurement. The teacher made students explain the difference between the two problems provided in Figures 26 and 27. However, she could not make students explain these two situations by using different tools such as drawing of the given situation. She expected them to realize the differences among the cases by letting students read their problems constructed by themselves. At that moment, the coach suggested teacher propose a simpler problem by thinking of the division meaning. In that sense, she guided students to reconsider whether the number of groups or the number of objects in each group was asked. However, she could not manage the whole class discussion, and she assumed that all students understood. On the other hand, she helped students revisit their problems by asking such questions: "what is the meaning of the a? and what is the problem sentence?" in small group discussion. As emphasized before, she could not acknowledge limited and high-level students' responses for the whole classroom discussion. For instance, the problem structure created by one of the students (given in Figure 24) lacked appropriate units although the answer could be seen as an opportunity to discuss the relation between time and the amount of water (the notion of rate of change)


Problem;
A şirketi a+4 saatte 4 litre su depoluyor. B şirketi 2a+2 saatte 5 litre su depoluyorsa bu iki şirketin eşit miktarda ve eşit sürede su depolamaları için a için hereninin saat kaçtır?

Company A stores 4 liters of water in a+4 hours. If company B stores 5 liters of water in 2a+2 hours, what is the time (hours) spent for these two companies to store water in equal amounts and in equal time?

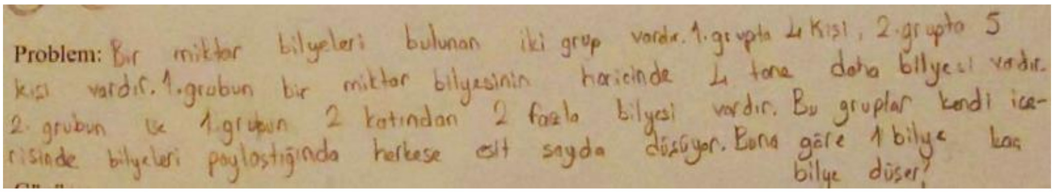
Figure 24. S5's problem context

The x cargo vehicle of a company engaged in intercity cargo transportation traveled a+4 kilometers in 4 hours. The cargo vehicle of the same company traveled 2a+2 km in 5 hours. The speed of these two cargo vehicles are equal. After the two cars moved, they drove up to a and stopped at a gas station. How many kilometers is it from the starting point to the gas station?

Figure 25. S21's problem context

Two groups of schools will go to the cinema. From the first school, *a* number of students and 4 guests are going to the cinema. The cinema attendant places the students in each hall in groups of four. The second group includes twice as many people as the first group and 2 guests. The attendant places the second group in the halls in groups of five. Since the two groups reserved an equal number of rooms, how many people are there in the first group, excluding the guests?

Figure 26. S20’s problem context



Problem: Bir miktar bilyeleri bulunan iki grup vardır. 1. grupta 4 Kişi, 2. grupta 5 kişi vardır. 1. grubun bir miktar bilyesinin haricinde 4 tane daha bilyesi vardır. 2. grubun ise 1. grubun 2 katından 2 fazla bilyesi vardır. Bu gruplar kendi içerisinde bilyeleri paylaştığında herkese eşit sayda düşüyor. Buna göre 1 bilye kaç bilye düşer?

Figure 27. S10’s problem context

There are two groups with some marbles. There are 4 people in the first group and 5 people in the second group. The first group has 4 more marbles in addition to some marbles. The second group has x2+2 marbles more than the first group. When these groups share marbles among themselves, everyone gets an equal number of marbles. How many marbles does each person get?

Therefore, it could be said that she had an inability to notice a difference in students’ thinking; in turn, she could not sequence the students’ thinking and make them discuss. She only noticed the students’ problems which were constructed on two distinct meanings of division that were discussed with the coach; however, she could not create a discussion on why these two problems differ from each other. This indicates that she noticed what the coach stated in the planning and reviewing stages regarding the students’ possible understandings. However, she could not notice different students’ high levels and limited answers related to the relational meaning of the equations and invariant relationship between quantities.

At that point, her actions in the classrooms seemed to possess the features of Level 2.

In addition, she also elicited the students' confusion by revoicing the idea and making students find the correct answer by questions required short responses. For instance, she let students discuss the differences between the given expressions or equivalent relations (Figure 28). The following dialogue describes the interaction between Aysu and the students:

p+12=s+12
2x+9=0
2x+9=y
2x+9

Figure 28. Some items in Task E in the teaching phase from the 1st Cycle

A: What do you think about these?
S23 and S24: I think they are not equal as p and s are different unknowns. Because p and s are different, we'll give different numbers, so it won't be the same.

A: What else?

S6: We thought p and s could be the same numbers but different representations.

S7: So nine is like three squared or like all real numbers or not at all. There are three possibilities.

A: Why can't I give 1 and 1?

S9: No, why did he say p and s and call them both p or s?

A: Is it stated that p and s are different from each other in the question?

S8: But we give values, we go through values

A: Can't I find y as one while x as 1 in an equation with two unknowns?

S23: Maybe, it will be as S7 said then.

A: Did you say p and s are two different variables in the question?

S10: Why not, for example, we could say x=1 in balance scales?

A: Let's go from the balance model. I have 12 kilos each; I put such things that I wrote p and s into the equation
S9: Then what if we say that there is 1 kg of apples in one pan (for p) and there is 1 kg of other types of apples (s)

A: Let's say orange, not another kind of apple

S4: So, is an orange equal to an apple?

S6: No, if you look at it in terms of grain, we looked at it in kilograms and said it's equal.

A: I provide what I write here? Without thinking of different representation
Some of the students: All real numbers can be

As seen in the discussion above, the students are confused about the invariant relationships between quantities due to assigning fixed values for the variables and viewing literal symbols as an object. She acknowledged and attended to the students' ideas and arguments. One of the students gave a clue by saying the balance scale, and then she tunneled students to think about the operational relationship between p and s, which does not stand for any objects. On the other hand, a student argued that orange and apple could not be the same due to her attempt to emphasize apple as orange as the number of objects instead of a focus on the weight of the things. She attended to this kind of idea by prompting them to change it, and she did not ask any follow-up or specific questions. Thus, she could not interpret the students' reasoning, and she seemed to hold a challenge about how to remedy students' the misunderstanding of literal symbols. In spite of explaining the correct answer to fix the confusion, she was able to attend to and interpret the students' misunderstandings in terms of how they might think what p and s stand for (single, multiple or an object). Therefore, her level of noticing could be Level 2.

Similarly, she also elicited the students' confusion/misconception by revoicing the idea without probing questions. For instance, she began the lesson by discussing a table (see Figure 29) created on the notion of additive thinking rather than multiplicative thinking by some of the students. Students claimed that cups should be increased by one; if we increase rice from 2 to 3, water must increase by 3 to 4.

Some of the students said, "No;" one of the students said that "think if I doubled 2 then I must double 3, so when I look at your table, four corresponds to 5, not 6". Then she concluded this discussion by confirming the explanation. Although Aysu elicited the students' misconceptions and demonstrated them to the whole class, she did not query why additive thinking did not work for this situation and did not press groups to justify their reasoning on different ideas.

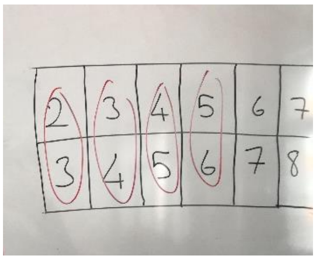


Figure 29. Demonstration of the students’ misconception on multiplicative thinking through table

Another issue that she attended to a need for students to ponder the conceptual ideas of linear relationships with additional representation (table) through questions with short-answer. She did not rely on no follow-up or specific questions.

> *S1: I can say that 2x+9=0 is an equation, and x is 4.5 when I think 2x+9=y if y is 0; x should be 4.5.*
>
> *A: What else?*
>
> *S2: We could say that x is equal to y-9/2.*
>
> *S3: It means that x takes values according to y*
>
> *A: Only with respect to y?*
>
> *S4: No, y changes according to x*
>
> *A: Okay then, give the values (she created a table on the board)*
> *Some of the Students: if we give x as one, then y is 11, giving two is 13*
>
> *QS6: Hocam, let’s give value for y at first.*
>
> *A: Okay, give. When we look at this table, we can say that x and y can take any values, real numbers.*

She tunneled students to consider x and y as varying quantities. She determined whether the students' responses and explanations were sufficient and accurate. She did not issue a challenge to students to participate in the reasoning of their arguments. Hence, her comments were coded as Level 2.

4.2.2.3. Level 3-Focused Noticing

How she demonstrated her noticing at Level 3 with respect to what she attended a need for students during teaching and expository comments related to her noticed issues were introduced in the following section. The presentations of the examples based on Aysu's practices in lessons from the teaching meetings of cycle 1 were given in the next paragraph

Regarding Level 3, she elicited students' thinking by adding a sub-item to the tasks. Before the enactment, she made an emphasis on the students' struggle to interpret the algebraic situations (that could be a proportional situation) including rational numbers and make calculations with rational numbers rather than integers. Then, she declared that she might use an additional equivalent situation in which the ratio is not an integer. However, efforts to negotiate these aspects of instruction were never explicitly planned by the coach and due to limited time for discussion, the coach did not provide protocols to be utilized to detect the teacher's idea but the coach encouraged her to ask additional questions and feel free to deviate from the planned route. She added sub-questions provided in Figure 30 during implementation. The discussion around these questions are given below.

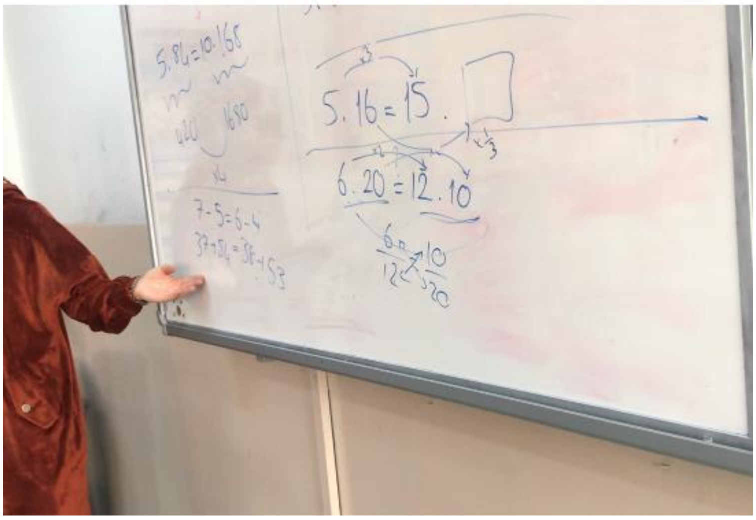


Figure 30. Creating a new task to discuss multiplicative thinking and the unit rate as rational numbers

A: 5x16=15x?

S3:16/3

A: Did you do an operation?

S3: No, 15 is 3 times 3 then the question mark is 16/3.
S2 and S6: We just thought of it as times.

A: 6.20=12.100 Well, what can you say about this equivalence?

S2: 5 times, not the other

S4: 10 times, the other is bigger.

S5: It would be okay to have 10 instead of 100.

S8: Actually, we can make it 200.

A: How do the changes in the factors affect the equality? I mean, it affects them in terms of times, right? Why?

S7: Doesn't it mean that the ratio will not change, no matter how much I expand and simplify one side so that the ratio does not change?

A: How about we convert it into proportion as you said?

S5: Wouldn't it be like inverse proportion?

A: How so?

S5: As one increases, the other must decrease.

S2: If it is 12 out of 6?

A: How are ratio and proportion shown?

S2: In multiplication form

A: Just in the form of multiplication?

S6: Wouldn't it be in the form of a fraction?

A: Can you convert this multiplication expression into a fraction?
Three students: If it is 6 in 12, what is it in 20?

A: Look (she wrote the equivalence). Later, you developed cross-multiplication. What does this actually mean? There is a times relationship between them.

The dialogue and the teacher’s questions shows that she attempted to connect the idea of algebraic reasoning and notions of unit rate and rate of change with multiplicative thinking by using cross multiplication. At that point, her emphasis in the lesson could be regarded as Level 3. However, the dialogue between the students and her indicates that she demonstrated the relations between invariant functional relationship rather than scale factor.

At last, she attended to a need for students to ponder big algebraic thinking by asking high-level specific questions. This attempt could be related to the coach’s suggestion on using probing questions to challenge the students regarding specific algebraic thinking. For instance, the coach suggested that creating a discussion environment in regards to the meaning of unit rate and ratio for the given cup of the rice and water (Task F).

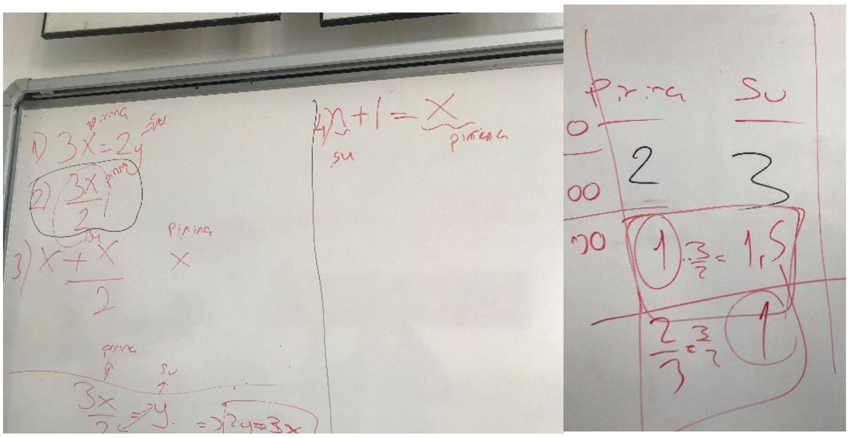


Figure 31. The students’ responses on Task F

During instructional time, she asked students to conceptualize unit rate with respect to one cup of rice and water, and this question challenged students and some of them realized the relation between the equation (3x=2y; substituting x for 2/3 or y for 3/2) and unit rate (Figure 31). The attempt could be seen as a sign of noticing of Level 3.

4.2.3. What and how Aysu noticed in the reflecting phase (post-observation) in Cycle 1

The second aim of the study was to understand Aysu’s noticing skills through coaching cycles within planning, teaching and reflecting phases. Consistent with this aim in this section findings related to what Aysu notice and how Aysu notice in the reflecting phases of Cycle 1 were provided.

While Aysu was implementing the lesson plan, the coach observed this lesson and took notes about the students’ work and the teacher action, and her instant comments are a sign for her noticing.

After teaching was completed, the mini-reflecting meeting about what attracted her attention was scheduled immediately at the school. In the reflecting phase on the day later of teaching, Aysu was asked to reflect about the incidents in the lesson. The interview protocol included what worked well, what did not function, what difficulties were encountered, what was observed about the students' work, discussions, and understanding, the cognitive demand of the enacted task, as well as what should be changed to improve the lesson and the rationale for the changes?

4.2.3.1. Level 1-Baseline Noticing

How she demonstrated her noticing at Level 1 while commenting on the tasks' nature enacted and elements of prior instruction were introduced in the following section. The presentations of the examples based on Aysu's expressions from the reflecting meetings of cycle 1 were given in the next paragraph.

Regarding Level 1, Aysu attended to diverse issues including comparison with previous practice, time, interest of the students towards the tasks, their orientation and the students' mathematical learning in general. For instance,

> I thought they would be interested in technology, I was wrong. Some students do not want to participate in the lesson" (Task B-interest of students)
>
> They realized it themselves, I really liked it (multiplication division), and they also came up with good ideas. It was beautiful this way. I remind you of the concept of equation, I was starting with scales, I was asking what it means to be in balance, but we did not make such a generalization." (Task A-practice, students' action)
>
> The students are generally good; They answer the questions immediately, at least think about the questions, and listen carefully to each other's answers (Task D, students' orientation)

In those quotations, Aysu focused on the students' interest and their success instead of the issues such as the students' mathematical thinking and task affordances. Moreover, she evaluated the quality of the lesson through general and evaluative statements, (e.g. "The lesson was good" and "Students are successful to reach the result"). However, her justifications were far from specific students' understanding related to the algebraic thinking, which the task promoted. Due to her general reflection, her noticing her was considered as evidence of Level 1.

4.2.3.2. Level 2-Mixed Noticing

How she demonstrated her noticing at Level 2 while commenting on the tasks' nature enacted and elements of prior instruction were introduced in the following section. The presentations of the examples based on Aysu's expressions from the reflecting meetings of cycle 1 were given in the next paragraph. Regarding Level 2, she shared her opinions about effectiveness of the teaching and the task and whole class thinking and made some suggestions to improve it. The following section shows what and how Aysu noticed during the reviewing phase.

First, she reflected on the previous implementation of similar tasks and the students' mathematical thinking. She addressed how the students could link their thinking with further mathematical ideas. The following excerpt shows what and how Aysu noticed during the reviewing phase of Task A.

> In previous years, I didn't teach that way. They were always making long computations, actually, I thought they would have a hard time finding this relation, but they discussed productively. I did not highlight the relation between variables in either sides. However, I appreciate the multiplicative thinking and it would be better if we connect to the children later, if a student sees that it is doubled, she can make better solve equations It will make our job easier". (Task A)

Regarding how to notice, the data revealed that she tried to interpret the students' mathematical thinking without providing specific ideas related to the students' works. For example, she reasoned on how students create relational understanding of quantities but she did not provide interpretation about the importance of this kind of thinking in terms of algebraic thinking. She mostly made evaluative and general comments (e.g., the students were able to generalize this relation of the quantities with other context, this is surprising, they discussed effectively). Thus, her noticing was regarded as characteristics of Level 2.

Secondly, she attended to specific students' easiness and confusion about the task and her decision to remedy this. For instance, she argued:

> The students said that p=s could not exist, I tried to ask questions by emphasizing two different variables, but I did not understand why they still had difficulties. I thought for a while, then how can I solve this. It occurred to me that a student said that there may be different types of apples in the same weight or unit prices of apples. (Task F)

She reflected on the students' confusion about equivalence of two objects and literal symbols and on how to remedy this. She could not verify a mathematical difference in whether students were seeing variables as an object substituted on a range of values versus an object that maintain for a specific fixed quantity. Therefore, she could not argue the reason of the students' challenges in regards to their limited understanding for the variables and her way of instruction. However, she pointed out to a crucial mathematical issue, and the students' thinking required her to reconsider to find a remedy for this confusion. Thus, this attention represented the elements of Level 2 noticing.

Thirdly, she attended to whole class thinking relating the cognitive demand of task as enacted and to offer a new task to attain a higher mathematical idea.

> Task D was supposed to be more complex. I said that its level is procedures with connection, but I think the level was low. The students got the answer right away. We told them to use the information they did not use to make it difficult and write a question, but they wrote and solved it quickly, starting from the unknown. I can answer the question of how I could raise the level as follows. I could not do anything about this question. Instead, a different question can be asked. For example, there are crossing the street or river questions that they can answer more relationally or there may be more complex questions.

She attended to the link between the students' work and the level of the task based on the students' efforts. She referred to a decline in the task during implementation by focusing on the students' quick answers. She identified an inconsistency between the students' cognitive level and task affordances, and it can be verified that she was able to detect the possible factor in decline in the task during enaction with respect to the task itself and the students' cognitive efforts to accomplish a solution. However, while she was describing the teaching episode, she did not elaborate on the students' understanding related to literal symbols as unknowns or variable and what she meant by stating “complex real life algebra problems”. Hence, her noticing was at Level 2 and adopted an evaluative approach.

4.2.3.3. Level 3-Focused Noticing

How she demonstrated her noticing at Level 3 while commenting on the tasks’ nature enacted and elements of prior instruction were introduced in the following section. The presentations of the examples based on Aysu’s expressions from the reflecting meetings of Cycle 1 were given in the next paragraph. Regarding Level 3, she attended to specific students’ easiness by interpreting underlying its reasons based on the cognitive demand of the task and the students’ algebraic reasoning.

> I thought that with the use of technology, the students would be able to make sense of the rule of numbers on one side and unknowns on the other side, or why -6 passes to the other side as +6 by solving this equation; but most students understood this state of equilibrium or the meaning of equality. Presumably, the first tasks we gave included the meaning of relationality and equality, so they had no difficulty in this task. The children already had conceptual learning there and they already knew by heart how they could find the unknown. Therefore, they did not have any difficulties and we can take this task out.

Although her interpretation was based on whole class discussion and multiple students’ thinking, she was able to attend to how the students’ understanding of the relational meaning of equivalence relate with solving equations. She gave detailed comments about this relation by interpreting both the students’ cognitive process and the nature of the task. Hence, her noticing possessed the characteristics of Level 3.

In addition, she interpreted her way of instruction, pedagogical choices and their relation with the students’ learning. To illustrate:

> It was important for students to work among themselves first and then show each example on the board. Doing this instead of just the correct answer allowed students to see and comment on different answers. For example, the answers of x+x/2 or x+1 were different answers. As planned, while the students generally focused on how much rice there was in a glass of water, I also asked them about the amount of water that corresponded to a glass of rice, which led them to the

> concept of unit ratio. It is also important to ask them to explain by using a table. I think some students understand the correlation more easily from the table. I'm not so sure, but did they understand why the multiplicative relation is necessary by using this table?... They also filled in the table by establishing a relation in the form of one more. Perhaps I should have asked additional questions or given more time; I could have elaborated more on my question. For example, I could have asked them to add the details of porridge or raw; or I could have directed them to the unit ratio.

As the excerpt below was completely considered, Aysu's observation was deemed Level 3 because she focused on the students' responses and attempted to recognize how they conceptualize mathematical ideas. She referred to particular instances of the students' thinking and identified the significant occurrences. (e.g., "students' answers such as x+x/2 and x+1" and "unit rate"). She noticed different students' thinking and confusion and elaborated on what should be done to make students who use additive thinking realize the inappropriateness of the thinking for multiplicative situations. She tried to provide reasoning about noteworthy events and focused on her limited attempts for this kind of thinking during teaching. Yet she could not propose efficient actions to handle the misconception since her questions (2 cup of rice for one cup of water is flake or not when compared to other) were still vague and less demanding.

Overall, data revealed that in the first cycle, the teacher generally did not visibly attend closely to the potential of the task and take opportunity of different students' thinking. The coach also did not provide a list of all correct student answers, the characteristics of the task and what factors have effect on decrease and increase on task implementations. Analysis of the first cycle shows to what extent the teacher get benefit of the activities grounding on the teacher's noticed elements in teaching and the evidences on the students' learning of symbolic manipulation. In some points, although she reflected on the students' high-level thinking or confusion, she failed to take this opportunity while planning the next lessons and action. At that point, the coach insisted that teacher consider how she can respond to the students' correct answers or confusion.

4.3. Noticing Skills in Coaching Macro-Cycle 2

The distribution and proportion of Aysu's noticing levels in three phases during Cycle 2 are presented in Table 18. Based on the table, her attempts are mostly seen to ground on Level 1 and Level 2. She demonstrated the characteristics of Level 4 noticing during the Cycle 2 in some point. Aysu's comments on the nature of the task, the students' possible answers and alternative pedagogical decisions provided evidence of a mixed level noticing. 12% of the comments were at Level 1, 48 % of the comments were at Level 2, 34 % of the comments were at Level 3, and 6 % of the comments were at level 4.

Table 18. Distributions of noticing levels through planning, teaching and reviewing of the tasks in Cycle 2

Level of Noticing	Planning/%	Teaching/%	Reviewing/%
Level 1	3/15%	5/16%	1/5%
Level 2	12/60%	12/39%	11/50%
Level 3	5/25%	12/39%	8/36%
Level 4	0/	2/6%	2/9%
Total	100%	100%	100%

Specifically, the frequency of what she notices and how she notices is given in Table 19 through three components, which are planning, teaching, reviewing across four levels.

Table 19. Frequency of what and how she notices in Cycle 2 through planning, teaching, and reviewing components

Planning	What she Notice	f	How she Notice
Level 1	Possible Students' Mathematical Ideas[TI]*	1	General and descriptive assertions for the task affordances and constraints
	Student Algebraic Thinking & Task Nature [TI]	2	
	-General Feature of the Task (CD of task)		
	-General Mathematics of the Task		
	-General Pedagogy		
	-Relate Task and Student' Thinking		
Level 2	Students Algebraic Thinking &Task Nature[TI]	6	
	-Related to Context and Students Idea	3	Evaluative stance and Descriptive Stance
	-Teacher Pedagogy	3	
	Possible students' Mathematical Ideas [TI]	2	
	Specific Students' understanding [CI]*	2	
	Students' Algebraic Thinking & Task Nature [CI]	1	
	Specific Episode of the Instruction [CI]	1	
Level 3	Students' Possible Understandings [TI]	2	Probing questions Sequencing the Ideas
	Specific Students' Understanding [CI]	1	Modifying the Task Utilizing Technology
	Student Algebraic Thinking and Task Nature [TI]	1	Adding the Task
	Student Algebraic Thinking and Task Nature [CI]	1	

Teaching	What she Notice	f	How she Notice
Level 1	Short correct answers	4	Asking yes/no or short answer questions Restating the phrase in the tasks without opening it for discussion
	Particular Students' confusion	1	

Table 19. (continued)

Level 2	Eliciting Students' Ideas Confusion/Questions/Vague Statement Building Conceptual Understanding Sense-making Coach's Prompt/Action Extending	2 1 1 3 1 4	Revoicing the idea without elaborating it. Making explanations Asking high level questions without connecting students' ideas Yes no questions Making explanations
Level 3	Confusion/Questions/Vague Statement Building Conceptual Understanding Sense Making Coach's Prompt/Action	2 6 3 1	Probing questions Pressing students' to justify or falsify thinking Using additional representations Connecting previous students' work Modifying the task Sequencing and linking among different ideas Using technology
Level 4	Students' Confusion Extending	1 1	Probing Questions Pressing for Justification
Reviewing	What She Notice	f	How She Notice
Level 1	General Aspects of the Instruction Specific Moment of Instruction [CI] Teacher Pedagogy	1	Describing with general comments
Level 2	Whole class understanding[TI] Particular students' thinking [TI] Specific moments of instruction [CI]	5 1 5	Adding a task Descriptive and Evaluative Stance
Level 3	Specific students' thinking [TI] Specific Moment of instruction[CI &TI] Particular students' thinking [CI]	2 2 4	Sequencing students' thinking or strategies Interpretive Stance Adding/Modifying the task
Level 4	Specific students' thinking [CI]	2	Elaborating on students' thinking, task and slope framework Utilizing technology

Based on Table 17, she attended some aspects of the practice including the students’ algebraic thinking and the nature of the task, the contextual feature of the task, the mathematical idea of the task, the students’ expected ideas, extending, connecting the students’ ideas, confusion, and the specific moment of instruction. These aspects highlighted by the teacher and initiated by the coach appeared to be related to the important aspects of practice. With regard to how to respond, the teacher’s comment was regarded to be evaluative in common and in some extent as interpretive. This shows that the teacher began to realize the important aspects of instruction and task design.

4.3.1. What and how Aysu noticed in the planning phase (pre-observation)

The second aim of the study was to understand Aysu’s noticing skills through coaching cycles within planning, teaching and reflecting phases. Consistent with this aim in this section findings related to what Aysu notice and how Aysu notice in the planning phases of Cycle 2 were provided.

The coach and the teacher discussed the nature of the given tasks and adapted these with respect to the students’ cognitive development in slope conceptualizations (Nagle et al, 2017) and Covariational Framework (Thompson & Carlson, 2017). The first task related to pattern generalizations offered to support students to operate x as a variable, discuss the meaning of the rate of change in the patterns and distinguish between the approaches of correspondence and covariational. The last two tasks were about covariation and variation in the given situations. The teacher’s reasoning on the nature of the tasks with respect to cognitive demand and the students’ algebraic thinking process through the tasks was discussed with noticing levels below.

4.3.1.1. Level 1-Baseline Noticing

How she demonstrated her noticing at Level 1 while commenting on the tasks’ nature and how to structure the teaching plans were introduced in the following section. The presentations of the examples based on Aysu's expressions from the planning meetings of Cycle 2 were given in the next paragraph.

At the beginning of the planning process, the coach suggested the first task (Rivera and Becker, 2008) to support the students in terms of the experience of relating both variables from a geometric sequential growing pattern with a covariation and correspondence approach (Smith, 2008). In addition, to make students create and interpret the scatterplot of the linear relationship, the students supposed to sketch the graph of the situations. A conversation between the coach and Aysu is as follows:

> *C: What do you think of this task (Task A; cycle 2 see appendix)?*
>
> *A: It's like a pattern finding question. Geometrically, children can also tell the increase by looking at the figure.*
>
> *C: What do you think about adding this question to our app and why?*
>
> *A: I think we should add. It would be nice for them to know that finding patterns is actually looking at the relationship between variables. In terms of relating to the slope. We also want them to draw graphs at the end.*
>
> *Coach: Why did you specify the chart? Can you explain?*
>
> *Aysu: Algebraic, it is important in terms of graphic transitions.*

In terms of what to notice, Aysu attended to the general mathematical aspects of the task and context of the task. Based on her evaluation concerning the sub-question of the task, she gave general comments (e.g., *It is important to use graphics*). Her comments were mostly descriptive and evaluative so the features of Level 1 emerged. Moreover, she attended to the potential of the task with a loose and general description of the high algebraic thinking (e.g., *Geometrik olarak çocuklar figure bakarak da artışı söyleyebilirler)*. Specifically, she was able to attempt to elaborate on the mathematical idea of the given task that is *contextual generalization* referred to as "the relationship between a quantifiable aspect of an item and its position in the sequence" (Wilkie, 2020, p.321). However, it lacked details on this type of generalization and other generalizations such as constructive or deconstructive generalizations and the nature of the task related with those types of student understanding. Therefore, she possessed Level 1 noticing.

4.3.1.2 Level 2-Mixed Noticing

How she demonstrated her noticing at Level 2 while commenting on the tasks' nature and how to structure the teaching plans were introduced in the following section. The presentations of the examples based on Aysu's expressions from the planning meetings of cycle 2 were given in the next paragraph.

Aysu anticipated the students' possible responses to the tasks in a limited way and she evaluated the tasks context by considering the coach's prompting. As illustrated, the dialogue below is provided.

> *C: Considering the place of this task and linear equations and slope, what can you say about the necessity of this task?*
>
> *Aysu: Students see this task in 6th and 7th grades. Actually, I had not established a relationship between this question of generalizing the number of steps and linear equations, so I never thought of discussing such a question in the 8th grade. But looking at your problem in detail, I can say that this task is important for the transition. It would be nice for them to know that finding a pattern is actually looking at the relationship between variables. That's why, I can tell this is the third level. Students will be asked to think in more detail, and they will focus on the relation of the symbolic meaning of what they write with the number on the sides.*
>
> *C: How did you teach in these classes?*
>
> *A: I was always teaching based on numerical values.*
>
> *Coach: What could be the answers from the students?*
>
> *Aysu: Their previous knowledge is always in the following direction: For example, they look how much the number of sides has always increased, and then y= x… they give a value to x; they decide which number they will write plus or minus. For example, if the number of steps is one, one is written. Depending on how many edges they counted, for example 3, then they say +2, and the equation becomes x+2.*
>
> *Coach: What else?*

> *Aysu: Some focus on the difference, always say 2 more, but cannot write the equation.*
>
> *Coach: How can they see the relationship between the number of steps in the pattern and n+2?*
>
> *Aysu: Two sides are added or they can always show it. As the number of steps in n increases, other sides are added.*
>
> *Coach: What else?*
>
> *Aysu: I cannot think of any more.*

As seen in the conversation, she could not propose strong evidence of the reasoning for selection of the task. She noticed the point that the task was not acknowledged by her and other mathematics teachers previously and implied to use the task to bridge a gap between the notion of slope, linear relationships and pattern generalizations. She was able to associate mathematical idea of the task with the broad mathematical notion of slope and rate of change and referred to noteworthy points. However, she missed some of important mathematical thinking of the students in the task, and her elaboration on the way of extending the students' thinking was limited. To illustrate, her statement in the dialogue above shows that she gained the sense of the fact that delving into the meaning of the rate of change in the pattern generalizations requires high mathematical thinking. However, she provided undetailed justifications for her reasoning related with the demand of the task and types of algebraic thinking. Her comments possessed the elements of Level 2 noticing.

She also highlighted the targeted concept behind the sub-component of the task and the students' possible conceptions. She purported that:

> With the graphic representation, they discover how the number of sides that change or remain constant between steps is reflected in the graph. For example, it gets steeper in a hexagon. This may facilitate their association with the rate of change. It may also be important to start off by asking how they should display the rate of change. They can evaluate the points relative to each other. In one, the distance between the points is too much and in the other, it is less. Students can be asked to interpret here the number of sides that change in unit step. I think the use of Geogebra is visually important. It allows them to see the position of the

> points more clearly. But of course each should be asked to locate the dots individually.

In this episode, Aysu described specific details about the notion of rate of change as a concept (…to interpret rate of change and steepness in the graph displays) and students' general confusion on graphs (reasoning of the rate of change with graphs) by relating these aspects with her previous experience and current knowledge on students built in collobaration with the coach. However, she did not mention specific students' misconceptions regarding the meaning of rate of change in the graphs. Since her comments were solely descriptive and evaluative, her noticing was at Level 2.

She attended to specific students' thinking which the coach had presented to her.

> C: The other thing Sena said is 2n-(n-2); couldn't 2n and n-2 be found here from the image? In the class, Sena and other students could not explain. What do you think?
>
> T: 3+n-1. Actually, she explained, but I didn't know how to direct them. Other students did not understand. For 2n-(n-2), there is no explanation other than numeric. Let's not dwell too much on it.

Based on the excerpt above, it was revealed that she acknowledge that S7's explanation might be different from other students however she might also find this generalizations by trial and change. Based on her evaluative comments on a specific student's thinking her noticing was coded as Level 2.

4.3.1.3. Level 3-Focused Noticing

How she demonstrated her noticing at Level 3 while commenting on the tasks' nature and how to structure the teaching plans were introduced in the following section. The presentations of the examples based on Aysu's expressions from the planning meetings of cycle 2 were given in the next paragraph.

After the implementation of part a of Task A, she attends to the students' difficulty in explaining how they construct their generalizations and her decisions for the next teaching. She mainly increased her noticing with the help of the students' limited understanding and students' high-level thinking. Moreover, she indicated that the coach's suggestion on emphasizing the conceptualization of the patterns without use of arithmetic and questions on the sequence of different students' thinking and strategies were an auxiliary force to interpret students' thinking and how to take action in teaching. The dialogue between the coach and the teacher illustrates how she planned the next lesson related to Task A.

> *T: It can be better if we constantly emphasize where n+2 comes from. Let's give the order from the beginning like this, because Sena's is top level.*
>
> *C: What will they say?*
>
> *T: They will say 2 increments. We will say where the increment comes from on the figure. Let's see what they will say. We can ask what has changed and what remained constant. For example, they can say N is the base number. One base and two bases. If they see this, they will see it more easily in the hexagon.*
>
> *C: Actually, I saw it differently. Let's go systemically. How do we provide this? So, do you think there is a need to rank these answers?*
>
> *T: Let's go over it again. They can see it as you say. Let's make a table and consider it separately.*

The number of triangle (n)	Number of bottom and top edges	Constant Edges	Circumference of the Pattern
1	1	2	1+2 (n+2)
2	2	2	2+2
3	3	2	3+2

Figure 32. The table constructed by Aysu to enable students' understanding of meaning of "n+2"

> *They will see where n+2 comes from there. How else do you see this n+2? We can say "Where are the changing places in the shapes? Think about it this way". How much does n increase and what does 2 mean in n+2? I think something different will come out on the square. It will be enough if we say that Sena's is numerical. 3 is constant, she adds one, and then obtains 3+n-1.*

C: What do you think? Sena also made an explanation about where 3 comes from and where n-1 comes from.

T: This is not what we want; we want what they say as n+2 in the pattern.

C: If we consider 3+n-1, isn't there an explanation like n+2? (in terms of relating the figure to the equation)

T: Yes, there is. Sena explained. The point is that this is higher level, so we can ask it after n+2.

C: I think we'll decide based on their reactions. Maybe if Sena can make a comment, we can discuss it.

T: It would be better that way.

As illustrated above; regarding what she noticed, she focused on the learners' mathematical thinking and further pedagogical decisions about the flow of the lesson. She referenced notable particular events and attempted to make sense of the students' mathematical comprehension. Regarding how to notice, she showed how students would think and make sense of the terms of the algebraic expression. She mentioned the questions, which she asked in an order and what actions such as constructing table to be applied (Figure 32). Her remarks were predominantly interpretive and elaborative. Therefore, her noticing was accepted as evidence of Level 3.

It included comments regarding connection between big conceptual ideas of unit by adding/modifying an instructional task. To illustrate, she claimed that:

…. It is possible to give the verbal explanation and graph first, and then proceed with the table and equation. How does the graph emerge? If they learn how to draw a graph and in which relationships, they will understand why and what they draw in the future. They know the meaning of what they are doing while drawing the correct equation. ..They can see it like free fall in physics. I drew a state change graph, for example. They could say that this is a state change graph. These tasks can be used for example. Mapping the graph to the given state first. They could then plot the graph of the given situation. The relationship between the two variables we just talked about (covariational reasoning levels) can be difficult to draw considering the instantaneous rate of change for more than one compartment in some bins. It may be sufficient to include only linear relationship and one nonlinear relationship.

In that sense, she was able to offer tasks to make students discuss the meaning of the graph of the situations including linear and non-linear relationships. The coach prompted her to comment on her decision with respect to the framework related to objectification of slope representations and Covariational reasoning. She declared that: “In slope, the graph is always used last; in order to show kids how the two variables change, students think of these levels [levels of covariational thinking]. It is easier to combine them later based on the geometric and algebraic ratio representation of slope.” Although her comment was limited in terms of detailed aspects of the conceptual frameworks with the elements of the task, it demonstrated that she gained insight into how to combine the task choice and conceptual frameworks. Hence, her idea of adding the task without the coach’s help and her rationale with the coach’s prompt could be a sign for Level 3. The next quotation also presented how she highlighted the modification of the task presented by the coach. Similarly, she was able to attend to specific students’ answers to enhance them to reason why they demonstrate two quantities covarying in the graphs in that way.

> Yes, drawing is important, but it is also important to ask students “how did you do it? How did you draw?” They probably give a value to make an explanation, but this is also important because they can also draw from memory. Let's add one more question: Explain how you drew on the drawing.” Let's leave room for them to explain verbally in two sentences.

In addition to sample student works including high and limited thinking, she gets the benefit of the frameworks related to slope conceptualization and levels of covariational reasoning, sample student works and the coach’s probing questions.

4.3.2. What and how Aysu noticed in the teaching phase (observation) in Cycle 2

The findings of this episode obtained from the implementation of the second cycle included the main mathematical practices of “generalizing the patterns in geometric figures, where independent variables increase by one, relating symbols with the situations (correspondence and covariation approach) and demonstrating the coordination of directions. The findings from Aysu's four lessons were presented, and her noticing levels during the teaching phase were revealed. Overall, 16% and 39% of her decisions and the discourse in the classroom were characterized as level 1 and 2, respectively. On the other hand, 39% and 6% of her attempts were related to Level 3 and 4, respectively.

4.3.2.1 Level 1-Baseline Noticing

How she demonstrated her noticing at Level 1 with respect to what she attended a need for students during teaching and expository comments related to her noticed issues were introduced in the following section. The presentations of the examples based on Aysu's practices in lessons from the teaching meetings of cycle 2 were given in the next paragraph

Regarding Level 1, she solely relied on the students' correct answers, and she did not extend the students' thinking, create discussions on what they realize on different students' answers and ask leading questions. For instance, in the following portion of the lesson, she missed the opportunity of the first student's answer to take the students' attention for the meaning of n and +2 associated with its growing edges and number of triangles or steps.

> *T: You established the relationship between the number of triangles and the perimeter. You said that it always increases by 2. Let's show this on the board.*
>
> *S1: I actually found a formula when I first counted the number of steps and the circumference. Then, when I look at it, in the first step, 1 plus 2 is three, and in two triangles, it should be 3 times 2, but when 6 two sides are common, it becomes 4.*
>
> *T: So, did you think that way at first and then say n+2?*
>
> *S1: No, actually I saw n+2. When I thought how I can explain it, my explanation was the other one.*
>
> *T: Well, what is the relationship between the number of steps and the number of triangles?*
> *Two or three students said they are equal.*

T: Then, instead of n, I can say the number of steps or the number of triangles. Another student stood by the blackboard

Ebru: If we look at the number of common sides, there is a common number of sides in the form of zero in the first, one in the two, and two in the third (n-1).

S4: Since there are two each, the common side must be 2n-1

T: In 2n-1, what should you do with that 2? You should put it in parentheses.

In the next part of the conversation provided above, apart from missing of delving into the students' correct answers, she attended to the students' confusion by stating the correct idea, restating the questions, restating the relevant rule and giving time to reconsider the question. In addition, she tended to correct the misunderstanding procedurally. To illustrate, S4 wrote 2n+4 for the perimeter of any number of hexagons instead of 4n+2 (2n+2n+2). Understanding of S4 seemed to have confusion about determining amount of change in perimeter with changes in the number of polygons, and she probably checked whether the equation was correct for only two stages. But, she seemed to respond to the student's incorrect equation by providing unspecific guidance such as "*count the edges again, you missed some of them or you mixed the place of the numbers*". Hence, her performance was coded as Level 1 noticing.

4.3.2.2. Level 2-Mixed Noticing

How she demonstrated her noticing at Level 2 with respect to what she attended a need for students during teaching and expository comments related to her noticed issues were introduced in the following section. The presentations of the examples based on Aysu's practices in lessons from the teaching meetings of cycle 2 were given in the next paragraph

Regarding noticing at Level 2, although she acknowledged different students' ideas by eliciting their thinking, she could not extend the students' thinking by demonstrating additional activities or prompt students to generate alternative solutions. For instance:

S4: Actually, when we subtract the common sides, we find the perimeter.

T: What are you actually doing here?

T: You are explaining the reason why you did it in the first place.

T: Anyone else doing anything?

T: We started with numbers.

T: Come and explain on the board.
(S7 wrote 2n-(n-2) on the board)

T: What is 2n and what is n-2?

Sena: For example, we reach the correct result when we replace them with 2 and 3.

T: Well, why did you write 2n and n-2? What do they mean?

Sena: As my step count increases, the number I subtract increases as well, which is normal.

T: Okay, but your friends are still wondering where that 2 came from. Secondly, where did n-2 come from?

T: If it provides this, can we say that there is no (n+2) relationship or there is a relationship that you do not see?

Aysu could not orchestrate the discussion provided above, in other words students could not create ideas through teacher's questions. Although she focused on different students' answers and she queried the students with questions such as: "What is the meaning of 2n and n-2; how could we connect this formula and n+2?, she might not know the underlying meaning of this deconstructive generalization (2n-(n-2)). Therefore, limited and vague attempts to lead students to interpret the underlying meaning of the generalization indicated her limited noticing of particular students' understanding. Hence, her attention level was considered to be Level 2 due to her complicated decisions on the sequence of instructional actions.

In addition, although she challenged students by asking probing questions (why and how?), she seemed to not create efficient follow-up questions to highlight the notion of geometric ratio and functional property conceptualization of the slope. Then, she gave responses to her own questions.

Figure 33. S13's demonstration for the rate of changes for the three situations (number of polygons versus perimeter)

T: Did it change as double? What is the relationship between the numbers of steps? In the form of 2 times. Here while 2 is constant, it has changed as 4 times with the number of steps. If I don't see this, it's already fixed, can I see the rate of change here? Here is the number of steps and here is the circle. You said 1 changes. Where is that change?

S2: Distance between two points (Joined two points to form a diagonal to each other)

T: But didn't we talk about the fact that the diagonal of a square whose side is 1 unit is equal to root 2 when processing square roots?
S3: The perimeter of the next step is one more than the perimeter of the next step.

T: I don't understand. How can we get the changes you mentioned from here?

S22: That of the triangle increases one by one and that of the square increases two by two. That of hexagon increases four by four.

T: How would you show this on the graph?

S12: I don't know if it has anything to do with the distance between two points, but the hexagon increases four by four, but when we remove the two sides, what remains is 4. That is the case in the triangle. We said two sides are constant. It has three sides. When we remove the two sides, what remains is 2.

T: Okay, we talked about it. If we do not see these, what kind of a difference is there between them? Here we said 1 to 1, here we said 1 to 2, and here we said 1 to 4.

S13: When we subtract 2 from the number of sides, we find the coefficient that comes before.

T: What is its function for me?

S13: This helps us find the distance between.

T: What distance? Are you talking about the distance between points?

S13: When I say distance, I mean the difference between the point coming from one point and the second step.

T: Come and show me.

S13: Since it is from here to here in the square as well, it will be from here to here in the hexagon. It goes like 4 2 1 (Figure 32)

T: Now this is nice! What does it tell me again?

S13: Between the point emerging in 1 step and the point emerging in 2 steps…I cannot explain.

T: It shows the difference between the variables that are formed according to the number of steps. What has a constant rate of change every time within a triangle, within a square, and within a hexagon? What is constantly changing? The number of steps. Be careful. While the number of steps in all of them changes by 1 unit, how much has the perimeter of the triangle changed in this one? It has also changed one unit, but when we look at the square, how much has perimeter changed while the number of steps has changed by 1? 2 units. How much has the perimeter changed as the number of steps in the hexagon changes by 11? 4 units.

This conversation reveals that she added probing question to elicit the students' thinking. Although she attend to guide the students to relate the rate of change with the parametric coefficient of the equation within a correspondence approach, she seemed to give the correct answer while directing the students' attention to the covarying quantities through a covariational approach. For instance, in S13's explanation of the meaning of the rate of change covariationally (Figure 33), she tended to concentrate only on the difference in y-axis although S13 identified that "each step is associated with four edges" verbally. She most probably demonstrated her understanding of functional property conceptualization of slope without dynamic geometric, algebraic imagery of what it means. At that point, she did not attempt to guide the students to try to connect the notion of rate of change with the context and perceive what it means geometrically as an initial step. Moreover, she attended a students' confusion (line 4) about the meaning of rate of change by correcting it. Therefore, she missed the opportunity to increase the students' understanding on how two quantities vary. This limited focus on students' thinking was considered as evidence of Level 2 attention.

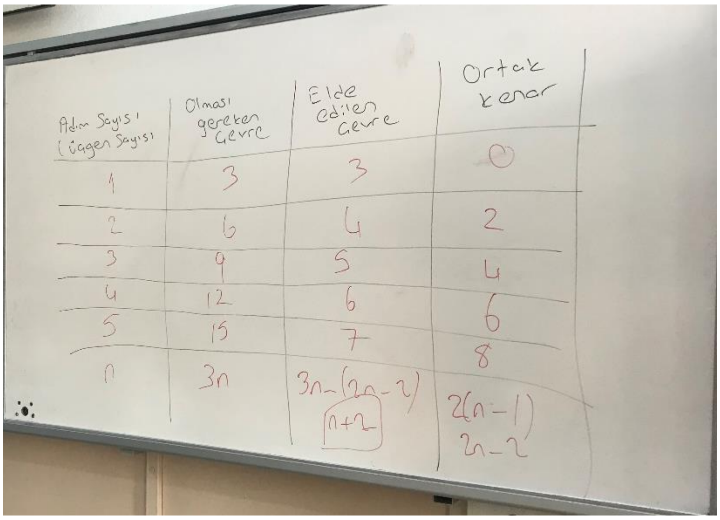


Figure 34. Table created by Aysu to demonstrate the deconstructive generalization

She also focused on the students' confusion by using additional representation. She directed students to calculate the perimeters of triangles as if they were separate and the perimeter of the triangles jointed and wrote the number of edges jointed to fill the table. Then the dialogue continued as follows:

T: Now let's look at the table. What should be the perimeter? n is the number of steps.

S5: Always 3 times, so 3n.

T: Does anyone disagree?

T: How about the common side? 0 out of 1, 4 out of 2. That is, 2(n-1). What to do next?

S9: We subtract from 3n.

T: Exactly. So, now it is clear why we did 3n-2(n-1). How do we reach the perimeter we want when we look at the numbers?
Five students: We get it by subtracting the perimeter obtained from the required perimeter.

As the excerpt implies, although she the queried students to realize that while subtracting common edges from the perimeter of separate triangles they reached the perimeter of the jointed triangles, she directed students to create symbolic notation by manipulating the numbers, which demonstrates her emphasis on arithmetic generalizations rather than a covariational approach (Figure 34). Therefore, it shows that she used the table as the justification of why 2(n-1) is subtracted from 3n by attributing numbers for some steps. Whereas the table could be a step to demonstrate the relationship between the number of steps (its position) and common edges of the triangles jointed (any value for the corresponding position), it enabled the students to construct the explicit rule. Therefore, her noticing characteristic for this episode was Level 2.

4.3.2.3 Level 3-Focused Noticing

How she demonstrated her noticing at Level 3 with respect to what she attended a need for students during teaching and expository comments related to her noticed issues were introduced in the following section. The presentations of the examples based on Aysu’s practices in lessons from the teaching meetings of cycle 2 were given in the next paragraph

Regarding Level 3, she mainly prompt the students to extend their understanding to big conceptual ideas such as deconstructive and constructive generalizations and covariational reasoning. Following is an example of Aysu’s action to ponder the idea of constructive generalization and the notion rate of change.

T: Let's talk about n+2 again. Where do I see +2?

S4: 3, 5, 7, 9. Always two more.

T: Okay, we talked about that. You looked at the relationship between the numbers. So, how do you show it on the figure? For example, where is n or +2 on the figure?

S5, 7, 8: We did not understand. Shall we call the number of common sides as n?

T: I need to count the common side in this case. We say n+2. Is there a subtraction?

S8: No, no

T: If we proceed from the sides, is there a place (side) associated with the number of steps or a place that changes or increases, for example?
Two students: Yes

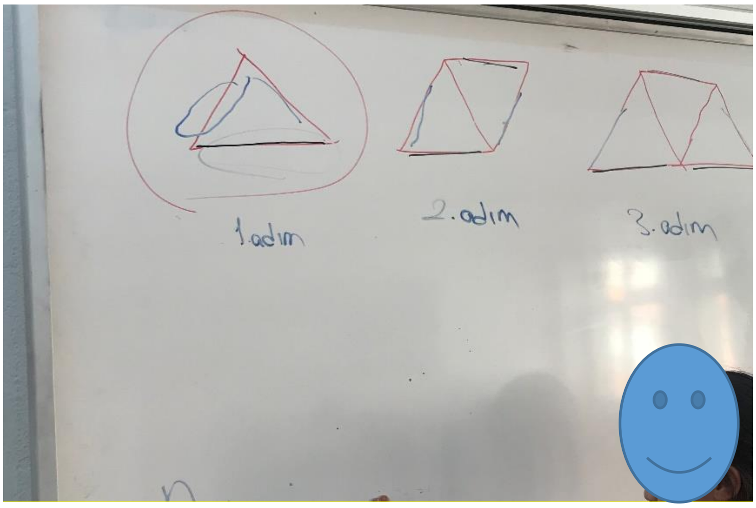


Figure 35. Demonstration of the constructive generalizations on the board

T: Come on, someone show me.

S9: Let's count the bases (indicated with a black pen; Figure 23) one in the first step; two in the second step; then this n is added from two sides each time, and plus 2 comes from here.
(Three or four groups said they got it)

T: What did you understand?

S12: The common sides will not be counted anyway. If we look at the outer sides, the lower bases and the upper bases progress with the same number of steps. Two sides are always added from the sides.

T: Anyone else with a different idea?

It could be argued that Aysu's level of noticing in this conversation was Level 3, as she was able to attend to the students' reasoning and encourage them to explain the rationale behind their answers. Since she focused on the mathematical reasoning of specific students and probed their thinking, she responded to them in a way that revealed their reasoning, prompted them to reconsider with a different mathematical approach, and expressed Level 3 noticing. She also seemed to sequence the students' thinking based on previous students' confusion on reasoning for the constructive generalization (Figure 35) and the coach's prompting for the need to order the mathematical approaches from simple to complex. Because n+2 could be seen as a precursor of conceptualizing the rate of change and constants of the growing pattern, she attended to elicit the corresponding students' thinking (S9) by asking probing questions (why, how) and guiding students through connection with explanations. Then the conversation continued as follows:

> *S6: (3+ n-1) 3 is the constant number. I directly start with 3. We can keep 3 constants here. When I subtract one from my step count, I always find the remaining number*
>
> *T: What did your friend think and formulate this equation? Explain.*
> *Three students: She always keeps the first triangle constant and then subtract one from each step.*
>
> *T: Why did she take out one?*
>
> *Two students: On the common side*
> *S7: Do they each have one common side?*
>
> *S12: No, n-1*
>
> *S6: I did not understand how.*
>
> *Coach: It is like that when you write it. Does that mean the same thing when your friend wrote? (The teacher indicated it as n number of steps on the board)*
> *Students: It means that the number of common sides is not included in the perimeter.*
>
> *S7: If I am going to find the perimeter, why would I add the number of the sides that are not included in the perimeter?*
>
> *T: Listen to what your friend is saying ….*
>
> *S7: Wouldn't 3+n come, teacher?*
>
> *S8: We remove the common ones. They are removed from the other sides.*

S6: Oh, I found it, let me explain. I actually matched the common side with the other side.

S9: So, isn't n-1 the number of overlapping sides?
Sean: No, it's not.

T: What do you mean then Sena?

S6: The correlation between the side remaining after matching and the number of steps.

T: For the quadrilateral, I want you to establish a similar relationship between the shape and the equation.

Apart from the previous student's example of deconstructive generalization (2n-(n-1)), in this excerpt, it was clarified that she was engaged in and acknowledged the coach's prompt regarding the student's idea of constructive generalization (3+(n-1)). After she queried the students with regard to the meaning of n+2 with the number of edges of triangles in the pattern, she then continued to encourage the students to interpret the meaning of 3+ (n-1) associated with any edges of triangles in the pattern. Students were able to demonstrate +2 by associating the constant edges of each triangle, and they relate changes in the number of bases of the triangles with the number of triangles. In that sense, she tried to promote the students to reach the meaning of the rate of change in the given context. Then, some of the students were able to elaborate on how 3+ (n-1) relate with edges of the growing pattern. In that respect, it could be said that Aysu seemed to identify that this response is interesting and wanted to shape the interaction (Van Es & Sherin, 2021) and the coach's suggestions on how to the sequence students' answers in order to enable and extend their thinking on both deconstructive and constructive generalizations. Therefore, it was argued that Aysu's noticing was level 3 since she was able to attend to the students' thinking and extend their understanding. She did not automatically provide correct answers to the students but she rather gave them reasonable time to justify their answers through an interpretive stance. She noticed that the students tended to use the arithmetical routine way (partial automatized by covariational approach and correspondence

approach) when constructing the equation; and she tried to support the students who had different answers to explain their thinking in detail.
She also attended to a student's immature understanding of the slope as "a measure of steepness "and delved into the students' understanding of geometric ratio and functional property conceptualization of the slope.

C: I am curious about one thing. You said it gets closer to the y-axis. Why does it get closer to the y-axis?

S: The number of steps is constant, and the difference gets bigger.

C: The difference between what?

S: Between points.

C: What does it have to do with the y-axis then?

S: I need to go higher as I go one step further.

S: Like a ladder, the more the number of steps, the more the number.

C: As the number of steps increases, does the steepness increase? What do you mean by step?

S: It has the steepness of a ladder with 2 steps. It has the steepness of a ladder with a long step.
T: So what do you mean?

S3: I think our teacher probably means the height of the steps.

S2: Yes, I couldn't explain clearly.

T: Is it just the step height? Could it be another factor?

S3: No, it can't.

S12: What about the depth or width?

T: What is the effect of depth?

S12: Doesn't it increase the steepness?

S5: But how? Isn't a 10-step staircase steeper than a 3-step staircase?

T: A good question! Think about it and let's discuss it in the next lesson. We talked about the height of the steps. Can you think of something else?

It is possible to argue that Aysu's noticing in this communication was Level 3 because she was able to attend to the students' reasoning and encourage them to extend their thinking on the idea related to slope as a measure of the steepness. In spite of some of students' appeal to height, a single quantity, regardless of considering varying quantities together, they were able to relate steepness with a physical property that seemed to require geometric understanding to conceive the corresponding physical entity (e.g., road, ladder, and mountain) without relating the rate of change as a number with the entity. Based on this student's idea, she told whole class to create pro and co- arguments based on this relation. However, she clearly seemed to decide not to give time students to justify their reasoning or take into consideration an alternative pedagogy. It might show that she thought that the understanding of "slope as a measure of steepness" requires high cognitive thinking which should be brought up later. Her statement after the lesson (Since we will discuss this in our upcoming lessons, I remembered that the geometric meaning comes later, so I left it here.) confirmed this claim. Hence, her interpretive approach indicates that her noticing level was Level 3.

She also attended to the coach's suggestion on eliciting the students' thinking for Task C given in the Figure 36.

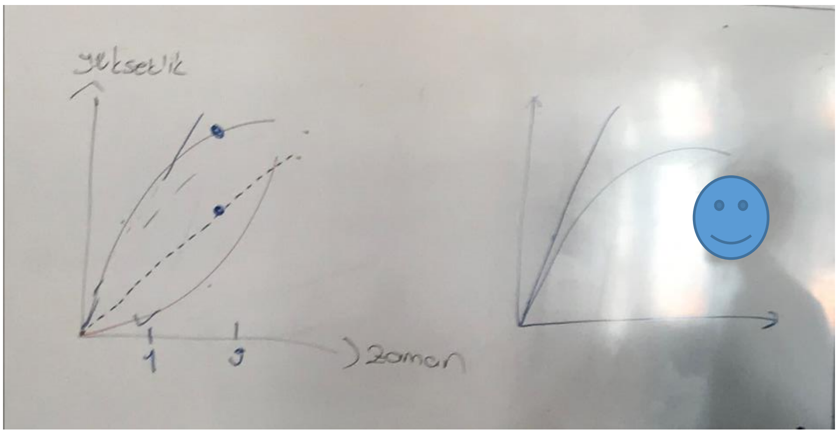


Figure 36. A student's sketch for Task B

S10: Let's say this is the time. In the time interval of 1 minute, if our graph is as below, if it were linear, it would progress in the same way. If it's like red, the line will fill up less in the unit time frame. I thought this was more appropriate. If we chose this, the container would have to narrow since the line would move further towards in a time interval.

S3: There is no one who does not know that the small area will fill fast. I guess we will get more height because the small area will fill faster.

S10: We start to get less. I mean the linear one and the line are bigger because normal base is bigger.

S12: OK, but then if we do it as you have drawn, as it grows again, there is a height greater than linear. At the same time, when we say this is 5 minutes, it should be here in the linear, but it is here in your drawing.

S5: I think you drew the linear inaccurately.

S10: Why wrong?

S5: Because if it were linear here, it would be better.

R: Let's listen.

S5: The volume of the figure here is expanding larger than normal, so the linear and xx.this is not there, but here. It's like this is rising like this.

T: Draw next to it.

S5: It's like it's not there but here. It's rising like this.
S10: No, I thought this way, I thought if the container was flat. I did not think according to this container. If our container was completely flat, I would act accordingly as in Figure 1.

T: You thought of a flat container, but according to which base did you think it is flat? If it was as wide as the top, then how would it be according to what you said?

S10: If we draw according to a container, if we accept it as a container, wouldn't it be like I drew? If we act according to it? Not the lower base or the upper base.

S12: But we have to think about it; maybe it will be less in the linear one. We focus on bilateral relationship

T: But I can't establish any linearity relationship here, right? All I can say here is, how does it fill up when it's down? What happens when you go up?
….

S7: The base drawn in black is a little narrower, and it will fill faster. It gets wider and fills up more slowly

R: How do you see it slowing down on the graph?

S7: If we put a time interval, while it is here at the beginning, it will normally come here, and it will be less here.
T: Draw the other graph. Why isn't it the case in red?

S6: Here it accelerated more, but in the end, it should be the slowest. The width at the end is larger, so it needs more water.

S17: Teacher, I thought like this at first. I tried with numbers. I thought like this: in a second, for example, let's say you extend it 3 centimeters and let the water rise by 3 centimeters. Then, at 2, if it were linear, it should have been 6, but when we think like this, because the volume of the container is constantly growing, I thought it should be at a lower height than 6. While it should normally be 9 in 3d, it should be at a lower height than 9, but there should also be a less increase than the increase here, because it is constantly growing. For example, because it I thought that it would increase less here than it did there.

As seen in the dialogue, S10 explained her way of sketching of the graph of the third image by comparing the constant rate of change of the first image. However, she appealed to the single quantity to justify her reasoning and she had a memorized way to sketch linear and nonlinear graphs without thinking of changing the variables together. Then, after a student came to the board and reasoned on S10's thinking, he revealed an argument, which contradicted with the sketching (Figure 36). Then, the teacher pushed other students to think which sketch was reasonable if the radius of the top was considered as the first image's radius of bottom. Some of the students were able to understand that linear graphs could not indicate the location or characteristics of nonlinear graphs. However, some of them had difficulty in reasoning why S10 constructed the graph by taking a linear graph as a reference. Hence, her attempt could be a sign of her noticing at Level 3 rather than Level 4 due to lacking in proposing alternative decisions. This attempt might include activities concerning comparison of average rate of change of concave down graphs and rate in linear equations. After she received various ideas from students for their sketches, she sequenced the students' idea from general to more abstract. Based on her selection of the students' ideas in the class, she seemed to understand that most of the students were at the level of *gross coordination of the values* (height is increasing when the volume or time is increasing without identifying the amount of change) and they used *gross* *quantification* (perceptual or memorized idea) while describing their ideas. She discussed those ideas at different layers in class in an order. Meanwhile, a few of them used the idea related to *extensive quantification* (changes in the dependent variable with successive unit increments in independent variable). Hence, the coach also insisted on building up a discussion about the way of students' sketching of non-linear graphs, which utilized a thinking of extensive quantification and their justification. Hence, Aysu pressed the students to explain their reasoning to the whole class and asked probing questions to elicit their thinking to be visible for others. Then, the teacher directed the students' attention to what happens within intervals and identifying the rates with respect to volume (time) and height (chunky-continuous covaration).

It can be argued that in terms of practicing the tasks in the classroom environment, coach's prompts and an illustrative case of the teacher's questions and the written response of students related with the tasks are key factors that impact on the teacher's noticing. She achieved a gradual non-linear increase in attending and evaluating the cognitive demand of tasks, the sequence of the tasks and students' specific thinking at the end of the Cycle 2. Moreover, she became familiar with which mathematical idea requires higher thinking and was aware of the strong relations among the pedagogical decision, students' thinking and tasks.

4.4. Noticing Skills in Coaching Macro-Cycle 3

To investigate what and how Aysu noticed algebraic thinking during these coaching cycles and to develop her noticing skills, the tasks were prepared by the coach and the teacher. The tasks in the cycle were related to connecting and reasoning with representations. In contrast to Cycle 1 and 2, in Cycle 3, Aysu's noticing skills were improved at three phases. The table demonstrating the frequency of the instances with the given attribute indicates the presence of high level noticing (Level 3 and 4). We also assessed what and how she noticed through planning-teaching-reviewing cycles and the corresponding noticing levels as shown in the Table 20.

Table 20. Distributions of noticing levels through planning, teaching and reviewing of the tasks in Cycle 3

Level of Noticing	Planning/%	Teaching/%	Reviewing/%
Level 1	0/	2/6%	1/5%
Level 2	4/24%	13/41%	6/30%
Level 3	13/76%	12/38%	9/45%
Level 4	0	5/15%	4/20%
Total	100/%	100/%	100/%

Specifically, the frequency of what she notices and how she notices is given in Table 21 through three components, which are planning, teaching, reviewing across four levels.

Table 21. Frequency of what and how she notices in Cycle 3 through planning, teaching, and reviewing components

Planning	What She Attend	f	How she Attend
Level 2	Students Algebraic Thinking &Task Nature[TI] -Related to Context and Students Idea -Teacher Pedagogy		Evaluative stance and Descriptive Stance
	Possible students' Mathematical Ideas [TI]	2	
	Specific Students' understanding [CI]*	2	
	Students' Algebraic Thinking & Task Nature [CI]		
	Specific Episode of the Instruction [CI]		
Level 3	Students' Possible Understandings [TI]	4	Interpreting Probing questions
	Specific Students' Understanding [CI]	3	Modifying the Task Adding to the Task
	Student Algebraic Thinking and Task Nature [TI]	3	Probing the questions Utilizing Technology
	Student Algebraic Thinking and Task Nature [CI]	3	

Table 21. (continued)

Teaching	What She Attend	f	How She Attend
Level 1	Short correct answers	1	Asking yes/no or short
	Particular Students' confusion	1	answer questions
			Restating the phrase in
			the tasks without
			opening it for
			discussion
Level 2	Eliciting Students' Ideas	4	Revoicing the idea
	Confusion/Questions/Vague	2	without elaborating it.
	Statement		Making explanations
	Building Conceptual	2	Asking high level
	Understanding		questions without
	Sense-making	2	connecting students'
	Coach's Prompt/Action	2	ideas
	Extending	1	Yes no questions
Level 3	Confusion/Questions/Vague	3	Probing questions
	Statement		Pressing students' to
	Building Conceptual	4	justify or falsify
	Understanding		thinking
	Sense Making	3	Using additional
	Coach's Prompt/Action	2	representations
			Connecting previous
			students' work
			Modifying the task
			Sequencing and
			linking among
			different ideas
			Using technology
Level 4	Students' Confusion	1	Probing Questions
	Coach's Prompt	1	Pressing for
	Conceptual Understanding	2	Justification
	Extending	1	
Reviewing	**What she Notice**	**f**	**How she Notice**
Level 1	Specific moment of instruction	1	Describing with
	[CI]		general comments
Level 2	General Aspect of Instruction [TI]	5	Adding a task
	Students' Thinking [TI]	1	Descriptive and
			Evaluative Stance

Table 21. (continued)

Level 3	Specific Students’ thinking [TI] Specific Moment of instruction [CI] Whole Class Confusion [TI] Specific Moment of Instruction [TI]	2 3 1 3	Sequencing students’ thinking or strategies Interpretive Stance Adding/Modifying the task
Level 4	Specific Students’ Thinking [CI] Students’ Struggle [TI]	2 2	Elaborating on students’ thinking, task and slope framework Utilizing technology

One of the aims of the present study is to explore what and how Aysu noticed during coaching development cycles. Therefore, the next section is devoted to presenting what and how she noticed in Cycle 3 through planning, teaching and reflecting phases.

4.4.1. What and how Aysu noticed in planning phase (pre-observation)

The teacher’s reasoning on the nature of the tasks with respect to cognitive demand and the students’ algebraic thinking process through the tasks was discussed with noticing levels below.

4.4.1.1 Level 2-Mixed Noticing

Aysu attended to describe the students’ possible answers with a limited focus. For instance, she explained different students' answers for Task A, including the 'correspondence' approach, and she indicated that the students would justify the constant unit rate by looking at the rate between the quantities. However, she could not provide a detailed explanation of how students got the idea and the possible misconceptions. In that sense, her approach was evaluative for how to notice, and her comments lacked interpretations. Those explanations might indicate that her attention was on getting correct answers from the students rather than specific students' thinking, such as additive and multiplicative relationships. Although the coach emphasized the need to make students discuss the graphs generated by themselves, she mentioned a student's tendency of creating create the line passing through plotted points without any interpretations. In that respect, contrasting to level 3 noticing, she highlighted the students' possible responses, and she could not analyze and interpret these students' thinking and propose an alternative decision for them. Hence, her noticing was regarded to be Level 2.

Aysu also attended to further teacher pedagogy while task set-up and task implementations. For instance, for task B, she highlighted a need to make students read the situation, force them to create diagrams or explanations for each variable in the equations. Moreover, she asserted that "I will ask extra questions to solve the problem with other ways". Although these assertions are descriptive in nature, she gave different students' possible solutions regarding the task so her highlighted issues related to pedagogy are crucial. Therefore, it is coded as Level 2 due to her evaluative comments related to pedagogy.

4.4.1.2. Level 3-Focused Noticing

Aysu attended to the students' confusion by taking alternative pedagogical decisions such as probing questions, ordering students' thinking, and using additional representations. How she proposes alternative ways to handle the students' confusion on proportional and non-proportional relationships and their understanding of dynamic triangles on the graphs is provided below.

> *A: Although the students can find the rate of change from the table and write the equation, they have difficulties in displaying it in the graph. We noticed this in Task A. For this reason, it is very important to ask students to produce a solution by using the notation they want, without stating the table, equation and graphic form. For Task C, it may also be important to ask some students to draw the graph using Geogebra. Students are comfortable doing algebraic and unit rate [functional property] representations. It is important to provide linear constants as geometric ratio and the transition between algebraic ratio and geometric ratio.*
>
> *C: What do you think might be asked?*
>
> *A: First, those who created a table may be asked how they did it. Then, the meaning of the unit ratio can be asked, and then it could be asked what will*

happen when different points are selected on the graph. Students will probably still get a single point proportionally.

She was able to provide evidence on the students' limited understanding of slope conceptualizations, and she included a description of how she will support the student's progress on geometric ratio. To be specific, she mentioned that she included probing questions (Grafik üzerinde değişim oranı nasıl gösterebilirsiniz? Farklı noktalar alınsa nasıl olur? How can you show the rate of change on the chart? What if different points are taken?) And how to use of Geogebra to enhance the students' understanding. I characterize Aysu's comments as Level 3 noticing due to her orientation to evaluate the students' possible answers while at Level 4, she needs to give detailed explanations on making associations among conceptualizations of linear constant, geometric ratio and functional property.

She also attended to the coach's suggestion on emphasizing the geometric rate of change by adding/modifying the tasks and sub-questions.

C: What do you think of Task D? A static stationary state. I gave it on purpose. It is important that they focus on the variables that make up the geometric ratio.

A: Yes, the geometric ratio is given later than the others. It is higher level. In this question, it is important that they think about what the slope depends on, without putting a grid behind it. Then let's not tell them to pass through the origin and place it on the coordinate plane. Let them do as they wish, and this way they see that the slope will not change. Let them enter equations both during the lesson and on Geogebra. They can see whether the equation they say and the line they think are the same.

C: A student was able to say "the rate of change in the unit".

T: He said "change in unit" but he didn't say in which unit. We don't know which unit he means. Therefore, additional questions are needed, such as "Is it the y change in x or x change in y?" For example, what does 7/4 mean? We must ask this.

In the excerpt above, she was able to attend to big ideas behind the task, including behavior indicator and geometric rate of change, and she proposed some probing questions to guide students to make sense of the geometric rate of change. She proposed the details of the students' thinking and provided evidence while evaluating the task. Due to her detailed analysis of the nature of the task concerning the students' thinking on slope, she possessed the characteristics of Level 3 noticing. Since she did not mention the slope conceptualization framework related to the idea behind the task and how her question would lead students to conceptualize the geometric rate of change, her noticing level could not be regarded as the most sophisticated level of noticing, Level 4.

She also highlighted the need for probing questions to make students build connections between slope conceptualizations. She tended to make students elicit their understanding of transition between geometric ratio, algebraic ratio, and parametric coefficient conceptualizations.

> *What does y=ax+b change rate mean in the equation? For example, what does the rate of change mean at 3y=ax+b? How is it reflected on the chart? What does it mean when it is shown on the graph as a triangle? These should always be given to students. Students should be able to explain what they relate to what. They should also be able to see the differences in graphs and equations. Where do we use graphs? Where do we use equations?*

Although she mentioned the need for transition between representations, she seemed to focus on connecting representations rather than reasoning with representations, especially with graphics. Therefore, her attempt was considered as Level 3 due to the limited comments on a high slope understanding, such as understanding algebraic notations on graphs by using imaginary triangles. The following section presents Aysu's noticing in the teaching phase for Cycle 3.

4.4.2. What and how Aysu notice in the teaching phase (observation) in Cycle 3

The following section heading presents Aysu's noticing in the teaching phase for Cycle 3 through four levels: Level 1, Level 2, Level 3, and Level 4.

4.4.2.1. Level 1-Baseline Noticing

She solely attended to the students' short correct answers without extending or elaborating on them since she had the goal of summarizing the lesson. Hence, she spent the last five minutes summarizing the lesson by asking critical questions with short answers.

In addition, she seemed to attend to the students' misconceptions by restating the phrase in the questions without opening it for discussion. The dialogue between her and a group of students who had misconceptions on the rate of change as constant (Task A) is as follows:

> *Aysu: [Approaching the group] asks you about the rate of change between the 25th and 50th seconds, provided that the fuel consumption rate is constant; you said 50 is twice as much as 25, so it doubles the speed, but think about it again. Do you think it's true?*

She also attended to guide the whole class or particular students to a specific idea or answer with leading questions. The following excerpt is an example of the situation:

> *S13: Y/23 already gives us the remaining fuel, so we don't need to make 20 minus; let's say x is the remaining fuel directly.*
>
> *S21: How?*
>
> *S21: Oh, wait a minute.*
>
> *T: What does this give you (y/23)? The consumed fuel, so then shouldn't we subtract it from 20?*
>
> *S21: Yes, I understand.*

In contrast to Level 2 noticing, a Level 1 noticing takes actions to guide students to specific responses with leading questions or to orient students who have confusion to listen to correct solutions without elaboration on these issues.

4.4.2.2. Level 2-Mixed Noticing

She attended to elicit particular students' strategies to ponder big conceptual ideas (reasoning with representations of geometric ratio) without elaborating on it.

T: How many units is that?

Students: 1 unit, this is 6000.
This is 2 units, here?

Students: 12 units.
This is 3 units.

Students: This is 18 units

T: How many units are there in between each time?
Students: 6000

T: You saw the progress of 6000 units at a time. You have seen that Y=6000x.

T: A friend of yours thought of 24,000 and the point 4. How is 6000 found?
S5: By dividing 24,000 by 4.

T: Yes.

She attended to revoice a particular student's thinking without creating a discussion environment in order to demonstrate an invariant relationship between quantities. Since she attended primarily the students' mathematical thinking and the teaching slope, and began to move from the class as a whole to specific students' ideas her noticing is characterized as Level 2. Instead of providing probing questions, visual support or extra time to support their interpretations, the teacher solely tended to get correct answer from a student and then continued with the lesson. She adopted an interpretive stance as well.

She attended to contradiction regarding slope conceptualizations but did not attend to particular students' answers and asked probing questions to make students justify their thinking. The excerpt below illustrates how she managed the contradicting issue related to the "rise over run" rule.

T: How do we compare? y/x or x/y?
A few students: dividing x by y.
A few students: dividing y by x.

T: Why?

S13: We are looking at the change depending on x.

T: Yes, exactly. When we look at it, we see that we constantly compared the changes. Therefore, we need to call the ratio of the change in y to the change in x. As you will see in the examples, we have always focused on the differences.

At Level 2, the teacher continued to prompt the students to give short and quick answers, but she also began to highlight noteworthy events, in this case the type of rate of change. In contrast to Level 3, she oriented towards the correct answer rather than the co-arguments given by the students (x in y ye bölümü); therefore, she could not lead the discussion to make students analyze their thinking.

Moreover, the teacher attended to build a conceptual understanding regarding reasoning with representations of slope conceptualizations by guiding them to specific answers.

T: Now let's come to the coefficients in the equations in the question -What does -23 mean? There is a sign in front of it.

S13: It gives the remainder.

S12: Actually, it decreases backwards.

T: Do you remember that in the previous questions we discussed the graphs of descending from the sea level. While one of your friends took it as distance, your friends in the other group took it as remoteness. When the graphs were drawn, we observed that the directions changed. What does the minus mean?
A few students: direction

T: Yes, if not, the rate of change is still the same, that is, 23.

S23: What does -23 mean?

S12: Going backwards starting from minus

T: What will the direction of the chart be? Why?
Two or three students pointed downwards and said it would decrease.

T: Yes, as one increases, the other decreases. If you have noticed here, as the road has increased, the fuel has decreased.

In this case she attended to make students use slope to describe the behavior of the line (Behavior indicator conceptualization). However, the students could not justify negative and positive slopes using the dynamic triangles since they might have been at an early level of conceptualizing the algebraic ratio and functional property. Moreover, the students could not justify why those relationships were held by grounding on geometric ratio. In that sense, action on not giving time to students to respond to question, such as, “What is the direction of the line? Why?,” might be a sign of that the teacher could not be aware of the process of the students' learning of slope. In fact, in cycle 4, she admitted that “In fact, this example allowed the students to learn without thinking about the direction of the graph in a decreasing state. I guess I couldn't let them have the necessary discussions there. They conceptualized the graph of the decline in that way because they actually thought of the situation as one-sided as if there was time.” These comments also confirms that she could not respond to the students' limited thinking. Due to her failure to primarily examine particular students’ mathematical thinking on slope conceptualizations, her noticing level was Level 2.

4.*4.2.3. Level 3-Focused Noticing*

She also attended to select appropriate follow-up problems to make students building connections among multiple conceptualizations of slope. Following is an example of connecting the functional property with geometric ratio understanding.

T: So, you said that while x is changing one by one, y is changing by 6000. How do you show this increase/change by shifting between any two points?

S5: We find the intersection points. If we continue intermittently, we find the difference between two points, for example, on the y-axis, and then we find the ratio and proportion.

T: Come and show me.

T: When your friend looked at the graph, he said that as one unit in x changes, the change in y is 6000. [Specifying the points on the graph] Let's determine the points in the 2nd and 6th seconds. How can we prove this geometrically?

S5: If we bring them in the same direction, we will not see this axis (x-axis). We will see the y-axis. Then, we will do the same for the other axis.

C: If we shift the first line you drew, will the distance change?
S5 and S7: No, it won't.

T: Now can you think of the same for the x-axis?

S5: Yes.

T: If the unit points were not given, how would the distance be calculated?
S5 and S3: We know the values numerically, and so, we can subtract them from each other.

T: So what did you prove this way?

S5: We have shown what we previously found with numbers between two points, the distances on the horizontal axis and the vertical axis.

T: So a change of 4 units corresponds to a change of 24 units. Do I see 6 times again?
A few students: Yes

T: How do I find 6 starting from 24 and 4?

S13: If one has increased by 4 units and the other by 24 units, there is 6000 times between them.

T: What is the rate of change in a unit then?
Five or six students: 6000

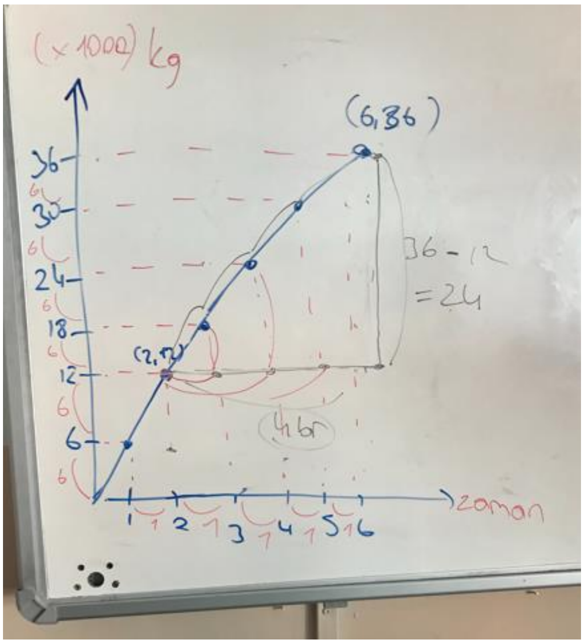


Figure 37. S5's demonstration of connecting functional property and geometric ratio in the graphic.

Aysu's noticing in this activity was rated as Level 3 because she not only listened to the students' responses regarding a measure of steepness, but also prompted them to reveal their mathematical reasoning through questions. Rather than focusing on the class as a whole, she focused on particular students and their mathematical reasoning (Figure 37). She encouraged the students to provide more justifications for their arguments by posing questions such as the following: "How could you confirm your argument?", "Why did you think of steepness remaining same, decreasing or increasing?". Because she created an environment to promote the students to share multiple solutions and justify their arguments, her noticing was considered to be Level 3.

She attended to make sense of conceptual ideas related to different conceptualizations of the slope by sequencing the students' strategies and ideas. To illustrate, Aysu attended to various the students' ideas and strategies (Figure 38) regarding covariational, correspondence reasoning and algebraic ratio, parametric coefficient and functional property conceptualizations of slope.

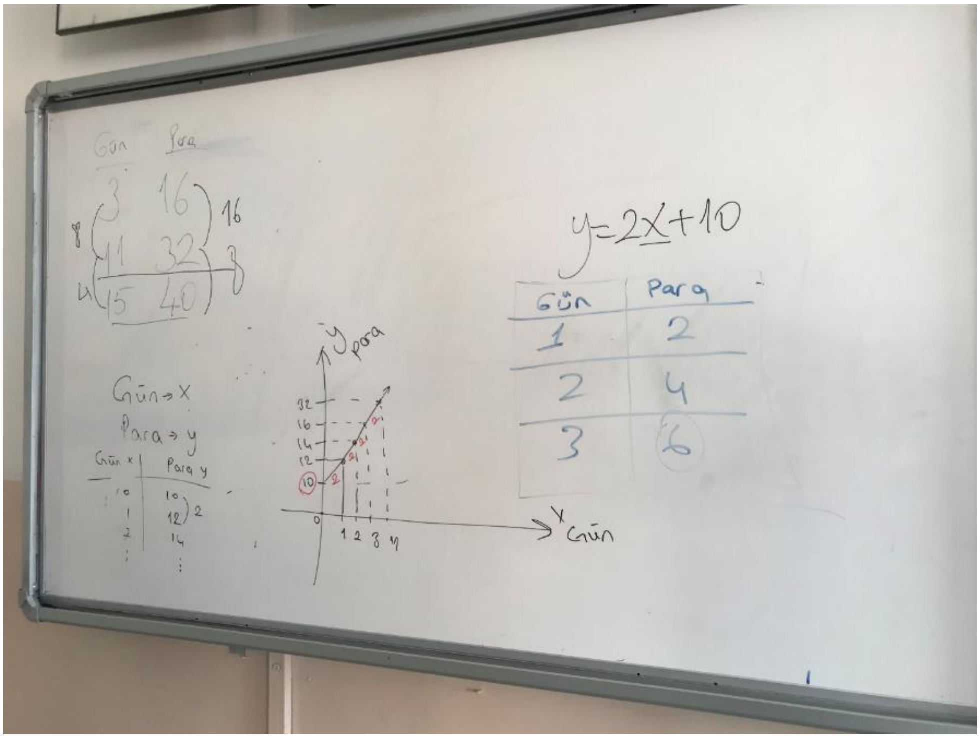


Figure 38. Students' thinking strategies for Task C

Aysu queried the students to explain how they created mathematical solutions regarding the amount of money for changing in each day. Most of the students used the covariational approach while working on collection of pairs of quantities to find the rate of change although they found y-intercept with the idea of both covariational and correspondence approaches. She discussed how students make connections between algebraic notion and the unit rate related to parametric coefficient, functional property, and algebraic ratio. Then she shifted the discussion on the transition of the algebraic and geometric ratio by focusing on the students' strategies in which they used a collection of pairs and graphics. The dialogue between the students and Aysu below portrays how she continued the lesson.

> *S19: In the chart, we first started at 10 at the rate of change and advanced one by one.*

> *S21: I thought like this. From day zero, I progressed 2 by 2. On day 3, it was supposed to be 6 liras, but according to the table, it was 16, so we thought we had 10 liras from the start.*

T: But we just want you to consider these given points on the graph and find the rate of change or the point where the y-axis intersects. How would you do?

S21: If we take this point first and compare the difference between them, we get 2.

T: Can you explain?

S21: First of all, if we place the points 3,16 and 11, 32, we find the distances between these two points. We can find 2 from here.

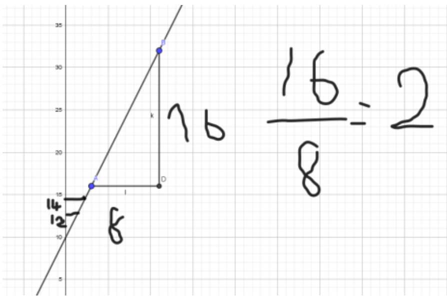


Figure 39. S22's solution for (using functional property conceptualization) y-intercept

A: How about we use the other point instead of these points?
S13, S7, S4: It would be the same

A: Why the same?

S14: Because triangles are similar to each other, at the same rate.

A: At the same rate?

S3: Differences between
S7: Distance between points

A: Are you saying the ratio does not change?
S7, S3, S15: Yes.

A: How do you find 10 on the chart this way?

S22: We thought like this. If we consider the point 3 to 16, I proceed by one unit and reach 14, I proceed one more unit and reach 12, and if I proceed one more unit, I can find that it cuts x at 0 and y at 10.

T: If I didn't think about this one-by-one reduction, how else would you find 10?

T: Now I know the ratio of 16/8. If this ratio has to be the same everywhere and if I create a new triangle, what must be the change in 3 units of y between 0 and 3?

Two students: 6

T: In this case, if I subtract 6 from 16, I get 10.

This excerpt indicates that her attempt seemed to shift the students' understanding on algebraic and functional property to transition on algebraic and geometric ratio by focusing the students on the dynamic image of slope. Although it demonstrated that she was able to sequence particular students' responses beginning from algebraic ratio to geometric ratio, she could not realize that some of the students tended to find y-intercept grounding on the functional property conceptualization in graphics rather than geometric ratio understanding (see Figure 39). In that sense, due to her missed action to particular students' ideas (S22) and failure to create a discussion for conceptualizing the geometric ratio, her noticing level is at Level 3 instead of Level 4.

She also attended to discuss different students' responses, then she created an environment to encourage the students to reason on different solutions with the help of technology (Figure 40). In this case, she connects her observations with central teaching features such as classroom discourse to enable students to reason with geometric ratio and geometric rate of change conceptualizations.

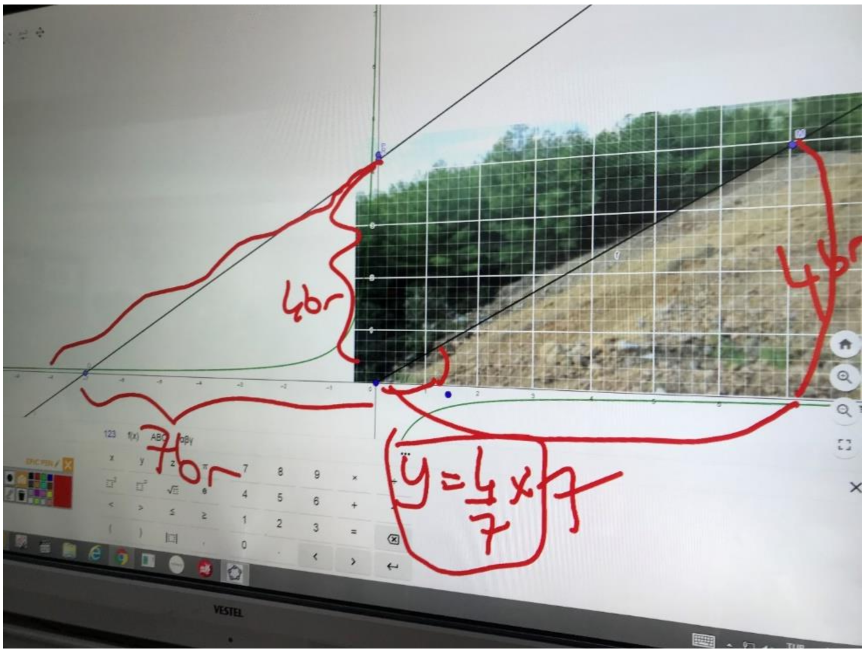


Figure 40. The students’ demonstrations of the road (given as picture) on coordinate axes (Task D)

In Figure 38, it could be seen that the students created different lines with the same steepness for the static situation (in this case, it is a road, then the teacher made them realize that geometric ratio is four over 7 with dynamic triangles. Her attempt to take into account the students' thinking (different lines) and make students conceptualize the geometric ratio is what distinguishes noticing at Level 2 from Level 3. Therefore, her attempt was coded as Level 3 noticing.

In addition, she attended to. contradicting the issues related to slope conceptualizations. For instance, she queried the students about in what conditions slope of the road was increased, decreased, and maintained. In addition, she asked the students to confirm their claims by comparing the slopes. To measure such "steepness" might only require the use of a memorized formula (rise over run), but to relate it with a physical property, understanding necessitates a geometric understanding to relate the number with the entity and justify it.

T: In what ways you can increase the physical entity steepness

S5: We can increase the vertical length

S18: We can decrease the horizontal length

T: How did you decide these answers are correct?

S18: We can increase numbers then we can compare the ratio between the previous and the last.

T: Come and show on the board. What else?

S23: We can create triangles, which represent the road.

T: Justify by sketching on the board

After that, the students explained their claims by showing visuals and giving numbers for the physical entities. She continued the lesson:

T: If I want to conserve the slope? What can I do?

S10 and s13: We can increase both or decrease

T: In what amount? Please give concrete examples

S12: We can add the same numbers to both lengths

T: How?

S12: For instance, vertical length is 8; horizontal length is 4. If we add 2 to both sides; new lengths are 10 to 6. So, they are the same… oh, not

T: Yes; S12 said if we add 2s to both sides we could not get similar ratio. Did you agree?

S13, s17, s19, s21: Yes, we could not

T: Why?

S13: Because we need to change the lengths with respect to its ratio.

T: What did you mean by saying with respect to its ratio?

S14: If we want to change the lengths, for instance 4:2; 6:3; 8:4

T: What did s14 do while giving these ratios?

S12: We get the same slope or ratio which is 2.
T: If we want to add a number to both sides as S12 tried, how can we do this rather than finding the equivalent ratios

S5: Add to both sides?

T: Only adding? What about subtracting?

S3, S6: adding and subtracting are possible

T: Yes, let's examine these on this example.

S12: If we increase 4 to 6; we need to increase 2 to 3.

T: Yes; how much did you add to these two lengths?

S5: 2 for the vertical length, 1 for the horizontal length

T: What did you think about S5's argument?

S13 and S14: The ratio between of quantity of change have also same ratio, which is 2.

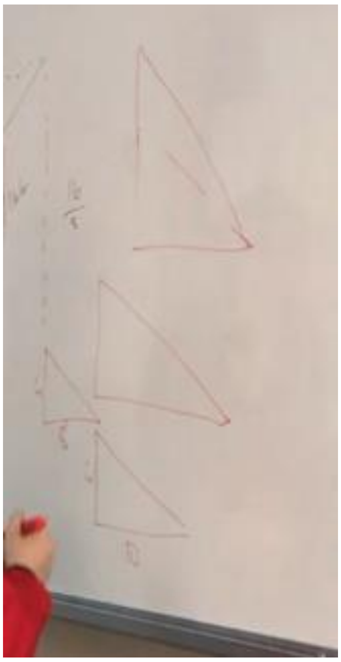


Figure 41. S12's confusion about the multiplicative relationship between the measure of length and height

She made students construct the meaning for the slope as multiplicative comparison of height to length to enrich their understanding of geometric ratio. In that respect, Aysu acknowledged S12's confusion about changing variables in a static situation and created a learning environment (Figure 41) to discuss whether the argument was valid or not (adding the same units to both sides). However, she funneled the students to build awareness of the invariance of slope to any particular pair of points and covariational approach with a consideration of changes in the amounts of pairs. Although her questions were beneficial for funneling and extending students' thinking to the geometric ratio, teacher pedagogy is limited compared to the characteristics of Level 4 noticing. Hence, instead of level 4 noticing, her noticing level was regarded to be Level 3.

4.4.2.4. Level 4-Extended Noticing

She attended to handle particular students' ongoing confusion. The episode below is an illustrative example of how she attends a particular student's confusion regarding determining the rate of change in non-proportional contexts with one point using Geogebra.

> *T: Those who found 16/3, can you explain how you found it?*
>
> *S12: We found the first point and divided the values by each other.*
>
> *T: Do you think this is true?*
>
> *S12 and other students: No, this is wrong.*
>
> *T: Yes, we talked about why it is wrong in the previous lesson. Now what does it mean to say that the relationship between two variables is linear?*
>
> *S15: It means the rate of change is constant.*
>
> *S16: In the graph that will be correct.*
>
> *T: Let's draw it correctly on Geogebra. One of you, please. Others, please watch. Tell us how you drew it.*
>
> *S8: I clicked on the line. I drew a point and then another point on that line.*
>
> *T: Why do you think the program asked for the second point?*
>
> *S10: To determine the direction.*
>
> *T: If not, could I not draw the second point?*
>
> *S28, S13 and S14: It would have been possible, but there would have been a lot of lines. Which one do we want?*

T: What does that mean? Come and show how many straight lines emerge from one point.

S2: Infinite

S3: Many

T: Okay, we find the second point for direction. We all understand that. Well, one point was enough in the other lesson; wasn't there the second point do you think? How did it happen?

S3: It was the same whether we looked at one point or two.

T: Think while you are drawing; couldn't we draw another line by keeping the ratio the same?

S5: Zero goes through the origin.

T: What does that mean?

S5: There is nothing at zero.

T: Let's show it on the graph. Tell me about a point. Will it be enough? How would you determine the second point? Or is it necessary to determine it?

S16: Yes, it is necessary. It is 0.0 point.

S13: There may be other points 2-12,000 and 1-6000 points.

The episode above indicates that her noticing has the characteristics of Level 3 and also included the teacher's action to support the students' progress to determine the invariant relationship between quantities by using any two pairs of points rather than one point. In the reflection, the teacher discussed how she helped the students understand proportionality and linearity and made interpretations about the students' way of thinking. She claimed that the students tended to look at the changes in the y-axis rather than those in the x-axis, due to one unit increment for the independent variable.

In addition, she attended to particular students' conjectures and the coach's prompt or told the students to use Geogebra. To illustrate, the teacher attended to particular students' contradiction about the positive and negative slope and extended their understanding of the dynamic image of slope in the graphs using technology (Figure 42). The following is an example of her initial attempt to press the students to justify negative and positive slopes beyond the visual reference of the line "going down or up".

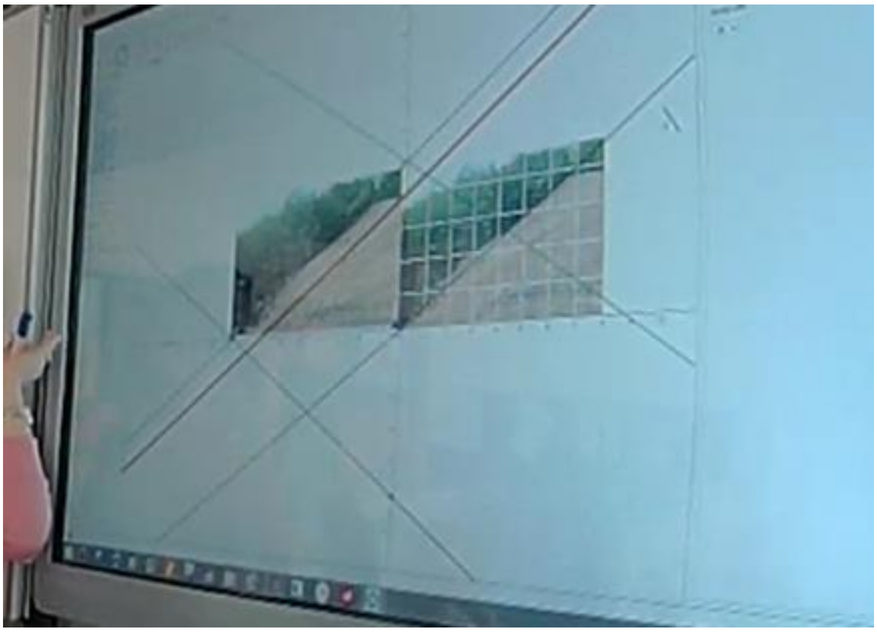


Figure 42. An illustrataion of how Aysu use geogebra

C: Well, if we look at it as positive and negative instead of increasing and decreasing, which ones are positive and which ones are negative? Why?

S13: According to the arms, red and black are negative, others are positive.

C: How would you determine if the decision was not made according to the arms?

S15: For example, let's consider green. As x increases, y decreases, so there is an inversely proportional relationship; so, it's negative

T: What can you say about other lines?

S14: For example, while x increased in red, y also increased, so it is directly proportional; therefore, it is positive.

T: I think this discourse about increase and decrease is related to the content. For example, we say that the gasoline is decreasing, and you draw the graph by decreasing.

S15: But this is related to our perspective. If we read the graph reversely, there will be more fuel in the tank as the car does not move. We do not decide on increase and decrease according to a single point of view.

Based on the discourse created by the teacher, she became aware of particular students' limited understanding of negative and positive slopes. Hence, her characteristic of noticing could be Level 3. Two additional characteristics of her attempt emerged at Level 4. First, she used probing questions and made students demonstrate and justify negative and positive slope by using GeoGebra. Second, her solution to students' struggle could be the first appropriate attempt to make the students' discuss the relationships between two quantities rather than considering only the dependent variable as increasing or decreasing.

She attended to connect two or more modes of slope conceptualizations.

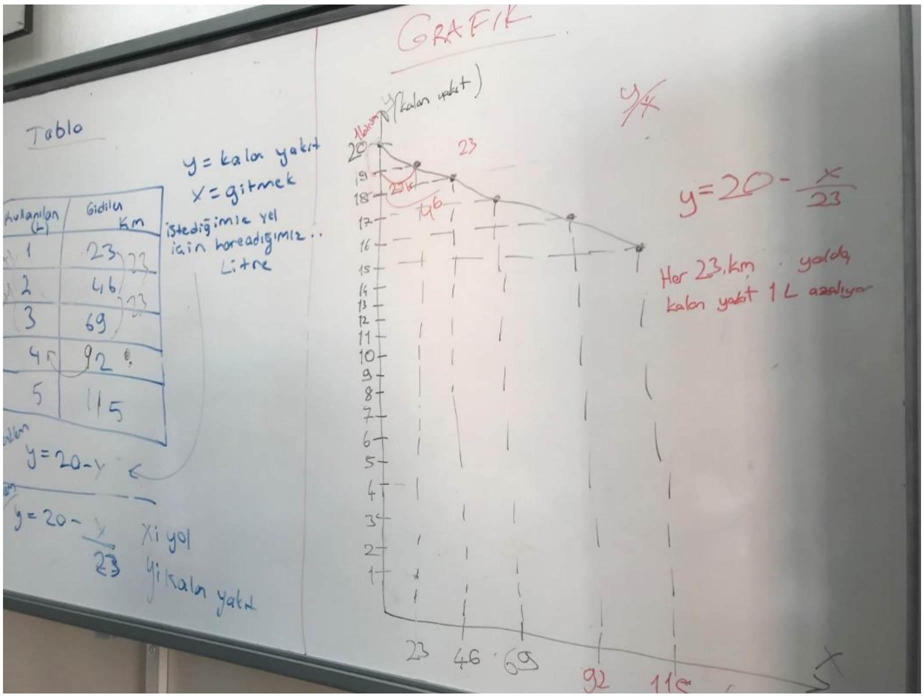


Figure 43. Students' conceptualization of slope, namely functional property, algebraic ratio and covariational reasoning for Task A.

She began to elicit an understanding of students who used tabular representations of the situation. After that, she led the students who used algebraic notation to connect the functional property with algebraic ratio. In addition, she made the students discuss the varying quantities in a graphic display. Regarding Level 1 and 2, the teacher acted based on the students' thinking beyond eliciting those ideas such as elaborating on them. Although her path that depends on the students' various solutions seemed to direct the students to memorize the fact related to slope conceptualization such as parametric coefficient and algebraic ratio, the path might be considered as a precursor for connecting among algebraic ratio, parametric coefficient and functional property. Therefore, her attempt was characterized as Level 4.

4.4.3. What and how Aysu notice in reflecting phase (observation) in Cycle 3

The next heading presents Aysu’s noticing in the reflecting phase for Cycle 3 through four levels: Level 1, Level 2, Level 3, and Level 4.

4.4.3.1. Level 1-Baseline Noticing

She attended to her action on a specific moment of instruction, which has potential for a missed mathematical opportunity. To illustrate, a student who uses a geometric understanding of rate of change to find the y-intercept asked the whole class to find the y-intercept with dynamic triangles rather than focusing on the students' idea. However, she explained how the y-intercept would be found by dividing rise with run in the triangle sketched by the teacher. The related segment of the instruction was provided by Aysu to reflect on it. She responded as follows:

> I asked every question I had to ask. I think there was no problem; I think the students understood how to create a dynamic triangle and find the y-intercept.

Based on this descriptive and evaluative comment, she was aware of the need to extend a student's geometric understanding of rate of change of slope by asking the question, "If we know that the rate of change, if we create a triangle here, how do we find the y-intercept?" The coach asked her to view the segment of this moment of instruction to enable her to analyze her action resulting in a decline in the cognitive rigor of the question. However, she emphasized general and descriptive aspects of the students' learning. Therefore, the general and descriptive analysis of the moment of instruction indicated her low noticing skills (Level 1).

4.4.3.2. Level 2-Mixed Noticing

She attended to point out her practice in a general and disconnected way by using the students' various answers as a piece of evidence.

> *A: I think we asked everything we should in the lesson. For example, the students were able to explain the proof for why we say there is linearity or why the ratio found never changes in different ways.*
>
> *C: What were they? Can you give an example of the students' answers?*
>
> *A: One group said that the rates won't change because the rate of change is fixed; the other group proved that the unit changes in the graph are the same or the unit changes are the same when they take different points. The lesson went really well.*

Although she attended some aspects of the moment of instruction and the students' learning on linearity and rate of change concept in proportional situations, her emphasis was grounded on evaluative and general approach in some aspects of the moment of instruction (e.g., Students were able to justify their reasoning on linearity, and the lesson went well). She could not distinguish among various students' answers, and she also missed pointing out some of the students' incomplete sketches of the graphs of proportionality and possible reasoning behind the students' mathematics. Therefore, her noticing has evaluative in nature, thus coded as Level 2.

She attended to add a task to make students think mathematically highly. Although the proposed task has multiple entry points and extends and/or enriches the students' thinking on slope conceptualizations, she was able to give little emphasis concerning the students' possible answers for the task and vague connection between the framework on students' understanding of slope and task affordances.

> What is the equation of the line that passes through the point (-2,4) and has a slope of -5? Our previous questions included situations with more than one point or dynamic states whose y-intercept and slope were known. This question would be a little different and nice in that sense. Students will try to solve it algebraically

As seen in her comments above, she seemed to recognize the difference between this task and previous ones concerning its nature. However, her reasoning on this discrimination between this task and the others was solely based on the limited nature of the characteristics of the tasks in terms of details concerning the notion of slope and linear relationships. Moreover, she evaluated the task based on her knowledge on the previous subject matter and experience on students' struggle to generalize the equation y-y1=m (x-x1) from the formula y2-y1=m (x2-x1) explicitly. However, she did not give evidence on the affordances of the task concerning algebraic ratio and its relation with the functional property. It seemed that she favored algebraic ratio regardless of a ways slope conceptualization. Due to her being primarily evaluative with some interpretive comments (e.g., Students tend to solve algebraically, this is a good question), we coded her attempt as a Level 2 noticing.

She attended to describe particular the students' thinking with respect to their common use of representations.

> *A: S13 continuously solves using a table; most students explain their solutions using algebra. I didn't think that they would adopt such different approaches.*
>
> *C: Can we say that it is enough to use one of them? So, should we say that students should always solve using a table?*
>
> *A: I think it's good that they're free; sometimes they will need to use tables and sometimes graphics; it is enough to be aware of them.*
>
> *C: How will they know? Can you give an example?*
> *A: When there is too much data, they should use a graph. When a value is given, they have to express it algebraically. In order to generalize, for example. They need to think about this.*

Based on her comments, although she attended to a way of students' solutions, she could not propose evidence on why those students rely on these specific representations and how to shift the students' use from table or algebraic notation to graphics and/or no representations. She seemed to consider that students automatically begin to use representations interchangeably without referring to different stages of slope conceptualizations and necessary teacher actions. She began to analyze particular students' thinking; however, she failed to provide reasons for those thinking congruently with the slope conceptualization framework. Due to her descriptive comments about the students' preference in representations, her noticing exhibited the characteristics of Level 2.

4.4.3.3. Level 3-Focused Noticing

She attended to elaborate on students' particular thinking with its sequence for further discussion in the class.

> The students took the first point and thought that there was a proportional relationship; we never really thought about it; we looked at 16 and 32 and expected them to say twice as much. Then some students found their difference from the table and found the ratio, and the result was 2; then they went back 2 by 2 for the y-axis; some of them thought 2x and added 3 instead. They saw from the table that they had to add 10 to reach that value [correspondence approach]. Some specified graphic usage points. They took two points and then reduced y by 2 in one advance on the graph. We can first start with the misconceptions and then discuss those who used tables. There's also the replacement. It would be appropriate in this order for algebraic and geometric ratio combinations. It would be good for them to also see the rate of change on the graph.

She sequenced the students' thinking from misconceptions to a more complex understanding of slope conceptualization. She stressed the geometric rate of change and transition between algebraic ratio and geometric ratio as high compared to the understanding of algebraic ratio itself. Due to her elaborations on various students' thinking with slope conceptualizations and not giving details about mathematical thinking and teacher pedagogy, her noticing skills were considered to be Level 3.

Similarly, she attended to interpret a particular students' struggle on slope conceptualizations. To illustrate, S10 struggled to make sense of the line of two varying quantities however he can create algebraic notation and explain the rate of change as words. She stated that:

> S10 is one of our most successful students. But when we told the student to deliberately start from the graph and do the equation he had set up, I saw that he could not do it. In other words, while the student was creating his graph, he knew that the total gasoline would decrease, and he did not know how much change there would be in each unit. It means that students are seriously struggling with graphics. It is not good to give in order.

Based on her comment about specific students' understanding, we claim that she could connect the framework on slope understanding and the specific students' difficulty. Although she commented on the students' struggle and a link between the framework and specific students' limited understanding, she could not tell the solution for this challenge. Hence, this noticing possessed many characteristics related to the evidence of students' understanding (Level 2) and fewer characteristics related to solutions for the problem (Level 4). Therefore, we coded this reflection as Level 3.

She attended to make connections in the moment of instruction where the coach interfered with teaching pedagogy and the moment of whole class students' struggle. This moment of instruction was related to the whole-class struggle on explaining geometric rate of change while using graphics. She agreed with the coach that the task should include graphic representations rather than pairs of values in tabular to enable students' understanding of transition among algebraic ratio, geometric ratio, and functional property.

> *Aysu: When we started with this type of graphs, we allowed students to create different triangles.*
>
> *C: Did you mean similar triangles?*
>
> *Aysu: Yes. Now they will have to use the graph; I previously thought it was not necessary, but now I see that it is necessary.*
>
> *C: Is there any situation that you think was important during the lesson?*
>
> *Aysu: When I said, "Come prove it; show on the chart; what is the meaning of 4/7?"I understood that we show the students that 4/7 is preserved with dynamic triangles (parallel lines or lines with negative and positive slopes), and we are trying to make them say what the geometric meaning of slope means. I realized*

that I should focus on unit ratio. What does it mean to be 7/4? This is also important. Some students are confused.

C: Yes, unit change is important; considering the levels of covariational thinking, can we give another interval instead of an interval in the form of a unit ratio?

Aysu: Sure; there are infinite numbers.

C: We can discuss this in class, but is it high level?

Aysu: They know that.

Although her interpretations were based on the difficulty that the whole class had in understanding of geometric ratio and geometric rate of change instead of specific students' understanding, it is essential to interpret the future instruction based on whether the idea was taken as shared to proceed with a more sophisticated understanding of slope notion. She also admitted the students' confusion on the meaning of steepness "birim x teki y deki değişim or y deki x teki değişim". She highlighted the need to discuss the meanings of the things to identify the meaning for the road's inclination. Hence, she needs to eliminate such confusion and press students to interpret these two different unit rates. In addition, she claimed that she understood the reason for the coach's action during instruction, and she claimed that based on the coach's question, she noticed the need to ask probing questions, which is important for the slope conceptualizations. Whereas she made only interpretive comments about the students' struggle and the coach's attempt in Level 4, she could not propose alternative pedagogical decisions grounding on the frameworks. Hence, those statements indicated that she was at Level 3.

4.4.3.4. Level 4-Extended Noticing

She attended to reflect on her practice related to particular students' understandings and the context of the task.

> The fact that the students expressed it as increasing or decreasing is actually due to our emphasis. Our use of the word “descending” in the previous question may have led to this situation. In addition, students can express the situation as

> decreasing or increasing as it will be easier for them to interpret a variable. In any case, besides discovering the algebraic reflections of the lines, it can be questioned what they see as increasing and decreasing in positive and negative relationships. Those who cannot make an explanation about two variables may be informed that they cannot see the decreasing graph as increasing, and we can give the dynamic situation that we give proportionally by using a different context. Thus, they learn that they need to look at both variables, not just one variable. When variables such as time and duration are given, students may be focusing on a single variable. Therefore, they may be making interpretations by looking at the decrease, increase or the regions according to a single variable.

As asserted above, she realized that using time or days as an independent variable made students focus only on the dependent variable. Although "time" suggests a motion as an independent variable going from one point to the other might support conceptions of continuous variation of a quantity's value, it might not be sufficient for quantifying variation. In that sense, Aysu could not make this reasoning behind the students' inability to conceive the changing in gas within the magnitude of time and their focus on the dependent variable as decreasing. However, she was able to attend to change types of variables embedded in the task. At that point, she realized that variables such as gear rotation and the number of feet (variables in a task provided by the coach) differed from the variables such as time. These noticing elements have both evaluative natures informed by analysis and substantive interpretation. Hence, this noticing is coded as Level 4 instead of Level 3.

She attended to the details of the students' thinking and/or struggle and future support for those students. To illustrate, she took the opportunity of the students' hypothesized incorrect answer, which focuses on increasing or decreasing the length and height with the same quantity resulting in the same slope. She utilized Geogebra in order to handle the misconceptions and make the students understand the unit ratio and relation between the changes in the x and y-axis. In that sense, adding 2's on both sides could not give the same slope; Aysu highlighted that students need to see that the changes in lengths must be utilized with the same ratio, not with the same changes if the ratio is different from 1. She offered to use Geogebra to see a fixed value for slope to make students see that length changes should also be in the same ratio. At that point, she highlighted that students would need to observe what would happen to the changes in vertical length and what would happen if the length of horizontal length is changed from 1 to 2. The coach prepared a file including the dynamic triangles to demonstrate how shifts in x and y-axis affect the slope (see Figure 44.). The dialogue between Aysu and the coach is given below.

C: Students will find that a change in x will be proportional to a change in y. What does this ratio mean? Equal to what?

T: It will be 2.5. That is, the slope.

C: Can we also ask them to observe what happens with a change in Y? Will it be a rate again? What happens to the rate?

T: I don't get it, let me think about it, I can do it algebraically. Let's say 2y=5x. It will be more difficult than the previous example (y=2x); when y is 1, x is 2/5; when y is 2, x is 4/5. When they are subtracted from each other, it becomes 2/5. In fact, the ratio is maintained between the changes. But these numbers are very difficult for students. When we bring y from one to six, we can say, "Guess where x comes from". Let them deal with integers; the numbers are intimidating when they are complex. They focus on the difference better.

C: Then, one of the sub-questions is to ask how much y changes when you bring a from 1 to 2; the other would be to ask how much x changes when you change y from one to six.

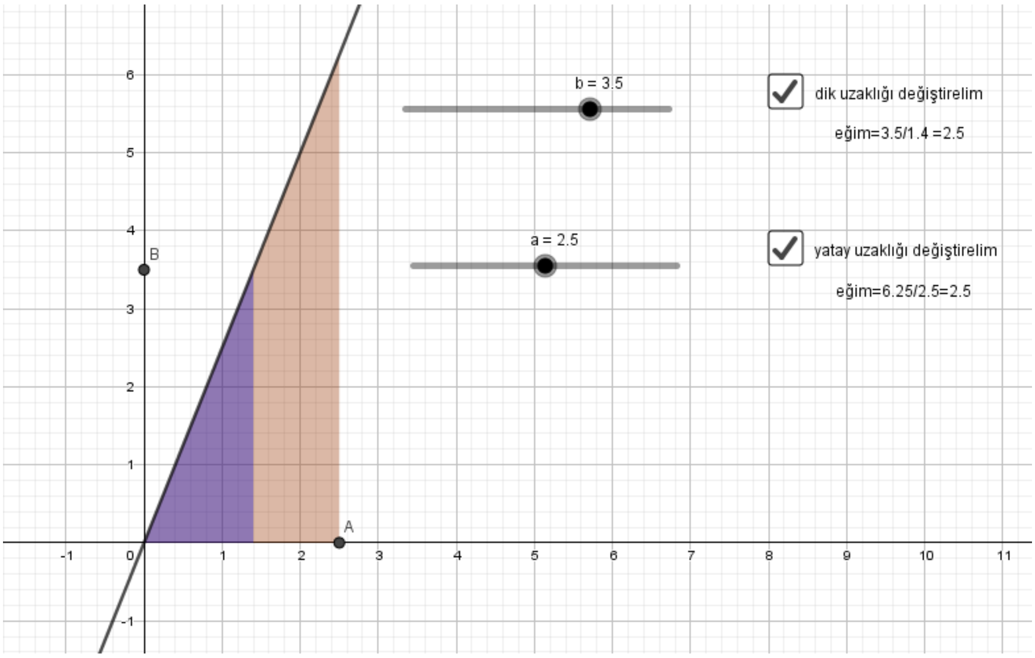
Aysu: Exactly, yes. They can also focus better on how they are changing together. It is also important to emphasize that the slope must remain constant. An activity should be prepared to help them see how the slope changes when the horizontal height remains constant and the vertical height changes, without actually drawing triangles."

C: We can actually ask the question of how much change does any change in x causes in y. As smooth covariational reasoning.

Aysu: Yes, but isn't this question high-level? We can do this by doing a few more tasks.

C: I see, you are saying that the students have not yet fully established the connection in between.

Aysu: It is enough if they give numbers and discover the ratio between them.



1. If we change A's position from 1 to 2 (with the slider called a changed from 1 to 2). What did you observe in the changes in vertical length (you can observe the B point while changing the slider from 1 to 2)?

If you did not know the B point, would you be able to you find that the changes in horizontal length in the case of that change in vertical length is 2.

2. Observe changes in horizontal length if you move b slider from 0 to 5. What is the change in vertical length?

Figure 44. A ggb file created by the coach

This attempt to prepare a task with the coach indicates that she reflected on her practice related to the constant rate of change. The teacher discussed how she helped the students who did not understand the unit ratio between the variables rather than constructing equivalent ratios and reflected on her practice with the help of the coach. Therefore, this reflection contains many of the features of Level 4 noticing.

One of the aims of the present study is to explore what and how Aysu noticed during coaching development cycles. Therefore, the next section is devoted to presenting what and how she noticed in Cycle 4 through planning, teaching and reflecting phases.

4.5. Noticing Skills in Coaching Macro-Cycle 4

In order to make students make connections and reason with representations, the tasks were prepared by the teacher and the coach in Cycle 4. Table 22 demonstrating the frequency of the instances with the given attribute indicates the presence of high levels of noticing (Level 3 and 4). Specifically, it was also assessed what and how she noticed through planning-teaching-reviewing cycles and corresponding noticing levels shown in the Table 23.

Table 22. Distributions of noticing levels through planning, teaching and reviewing of the tasks in Cycle 4

Level of Noticing	Planning/%	Teaching*/%	Reviewing/%
Level 1	0	2/6%	1/3%
Level 2	3/20%	9/27%	9/29%
Level 3	12/80%	16/48%	16/52%
Level 4	0	6/18%	5/16%
Total	100%	100%	100%

Specifically, the frequency of what she notices and how she notices is given in Table 23 through three components, which are planning, teaching, reviewing across four levels.

Table 23. Frequency of what and how she notices in Cycle 4 through planning, teaching, and reviewing components

Planning	What She Notice	f	How She Notice
Level 2	Possible students' answers Students' Possible Understandings & Task [CI]	2 1	Evaluative stance and Descriptive Stance
Level 3	Students' Possible Understandings & Task [TI]	4	Interpreting Probing questions
	Students' Possible Understandings & Task Nature [CI]	5	Modifying the Task Adding to the Task Probing the questions
	Specific Moment of the Instruction (Confusion) [CI]	3	Utilizing Technology

Teaching	What She Notice	f	How She Notice
Level 1	Eliciting Confusion/Contradiction	1 1	Asking yes/no or short answer questions Restating the phrase in the tasks without opening it for discussion
Level 2	Eliciting Students' Ideas Sense-making Extending Confusion/contradiction Coach's Prompt Conceptual Understanding	2 2 2 1 1 1	Revoicing the idea without elaborating it. Making explanations Asking high level questions without connecting students' ideas Yes no questions
Level 3	Confusion/Questions/Vague Statement Conceptual Understanding Coach's Prompt/Action Extending Sense making	3 4 4 3 2	Probing questions Pressing students' to justify or falsify thinking Using additional representations Connecting previous students' work Modifying the task Sequencing and linking among different ideas Using technology
Level 4	Confusion/Contradiction Extending Coach's Prompt	2 0 4	Probing Questions Pressing for Justification

Table 23. (continued)

Reviewing	What She Notice	f	How She Notice
Level 1	Specific moment of instruction[CI]	1	Describing with general comments
Level 2	Whole class understanding[TI]	5	Describing and evaluating teaching pedagogy
	Specific Moment of Instruction [CI]	4	
Level 3	Specific Moment of Instruction [TI]	3	Sequencing students' thinking or strategies
	Specific Student Understanding [CI]	3	Interpretive Stance Adding/Modifying the task
	Specific Moment of instruction [CI]	4	
	Students' Algebraic Thinking &Task Nature [CI]	3	
	Students' Algebraic Thinking &Task Nature [TI]	3	
Level 4	Specific Students' Understanding	2	Elaborating on students' thinking, task and slope framework
	Specific Moment of Instruction [CI]	3	Utilizing technology

One of the aims of the present study is to explore what and how Aysu noticed during coaching development cycles. Therefore, the next section is devoted to presenting what and how she noticed in Cycle 4 through planning, teaching and reflecting phases.

4.5.1. What and how Aysu notices in the planning phase (pre-observation) in Cycle 4

The next heading presents Aysu's noticing in the reflecting phase for Cycle 3 through four levels: Level 2, Level 3, and Level 4

4.5.1.1. Level 2-Mixed Noticing

She attended to a specific mathematical idea that would be given through Task C that was characterized by the coach. Related dialogue is given below:

> *C: For example, I have a straight line, it cuts at x and y axis in a place I pinned. I've shifted it one unit to the right. What's the point it cuts at y? The slope will be the same.*
>
> *T: If it shifted one unit to the right, it goes down.*
>
> *C: Are you sure? If I come to the left at the X-angle, how much change will there be?*
>
> *T: If I shift one unit, it changes as much as the slope.*
>
> *C: What happens if I shift as much as N?*
>
> *T: n*3.04. I thought what would happen if one unit changed, 3.04 changes and if two units change, 3.04 changes.*
>
> *C: But what I mean, what will happen if the rates of change are thought of as x and y axes. I mean the graphical meaning is also important. They should be generalizing this too; In fact, they should not think of it as a ratio-proportion.*
>
> *T: Nice, we can talk about it. We need to locate it. We say change in one unit, or can we not look at the change in two units? Can't we reach a conclusion? There may be questions like this.*

This segment of the dialogue indicates that Aysu made an evaluative comment regarding the idea (for any changes in x-axis would yield changes in m*n in y-axis) provided by the coach (e.g., Yes, this is good, it can be considered in the lesson). It seemed that she provided related questions (what about one and two units' changes?) corresponding to the idea of algebraic ratio and correspondence approach rather than the geometric ratio and smooth continuous covariational reasoning. In contrast to Level 1 noticing, she interpreted the mathematical idea with judgmental comments, and we coded her comments as Level 2.

4.5.1.2. Level 3-Focused Noticing

She attended to the students' possible mathematical understanding. For instance,

Mr. Ali saw the monthly fee advertisements of two telephone companies. Company A offers telephone service for a fixed fee of 20 TL per month and 0.10 TL for each minute of talk and 8GB Internet. Company B has no fixed monthly fees, but each minute of talk costs 0.35 TL with 8G Internet. Which company do you think Mr. Ali should choose? Explain your solution mathematically.

Figure 45. A Task in the planning phase from the 4th Cycle

> *Aysu: Most of the students begin to solve the problem by making tables, then they use graphs. We should give them a chance to select their own representations for high-level tasks.*
>
> *Coach: Why is this important?*
>
> *Aysu: We can observe which representations each student used and their struggles in the use of other types of representations. So the telephone task is suitable for this, they need to decide their own representations; then I will force them to discuss each representation which they begin with to solve the question; for instance, how they reach to a solution by using a table, graph or equation, ... I should ask: if we want to begin with a graph, how can we represent the situation?"*
>
> *Coach: Why?*
>
> *Aysu: We saw in the previous lesson that students tend to use tables rather than graphs; they might have problems with graphs if we want students to give the meaning of slope as rate of change and build covariational reasoning. Graph is the main tool to develop it.*

This dialogue between the coach and Aysu shows her interest in the students' understanding of representations and task affordances. She interpreted the students' challenges in elaborating on two varying quantities smoothly and continuously in a graph. Afterwards, she commented on the changes in the task context that referred to asking the students to create their [students] own mathematical models rather than creating tables, equations, and graphs, respectively, as an explicit path. Therefore, her interpretive and detailed comments were considered as evidence of noticing at Level 3. Although she mentioned the noteworthy aspect of the task and the students' thinking, she made insufficient elaborations on pedagogical decisions related to task implementation or probing questions (e.g., How can we represent the situation with graphs?). Therefore, instead of Level 4, her noticed issues had the characteristics of Level 3 noticing.

4.5.2. What and how Aysu notices in the teaching phase (observation) in Cycle 4

The next section is devoted to presenting what and how she noticed in Cycle 4 through planning, teaching and reflecting phases.

4.5.2.1-Level 1-Baseline Noticing

Aysu attended to getting a single correct answer by asking yes/no or short answer questions. Moreover, she attended to particular students' confusion by explaining the mathematical idea behind the confusion. For instance,

> *T: What kind of lines are these?*
>
> *S15 and S10: Parallel lines*
>
> *T: Why parallel?*
>
> *S15: Because the rates of change are the same.*
>
> *S19 and S19 and S5: Yes.*
>
> *S18: How?*
>
> *T: When parallel, the rates of change are the same because the coefficient of x in the equations is the same.*

Based on the dialogue above, she seemed to guide the whole class or particular students to a specific idea or short answers rather than promote them to elaborate on their thinking or a particular student’s question (S18's question). She did not lead the students to think about the reason for the lines being parallel in a way that is different from stating short oral answers. Moreover, she gave a accurate general explanation in reply to the student's unexpected question. In that respect, this move was coded as Level 1 due to the superficial stance on student thinking.

4.5.2.2. Level 2-Mixed Noticing

She attended to elicit the students' thinking without elaborating or extending it. The first episode is related to her action associated with no discussion on the power of representations. However, she did not support the students to use representations interchangeably, yet she emphasized that using one of these representations to solve a task is sufficient.

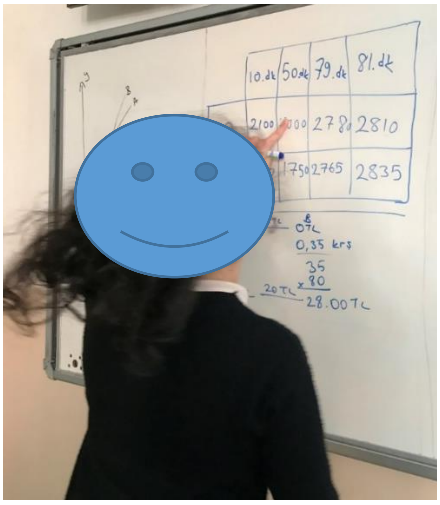


Figure 46. S6's use of table to solve Task B

S6: I found it by trying. I first tried 50 one by one; B is less than A, but after a while, Month B exceeds Month A and I also found it to show it [pointing to the table]

T: So why did you look at 79 and 81?

S6: I actually did it in my table; I didn't give them here.

S17: How did you find that they intersect at 80?

T: She didn't know that anyway; her table was more crowded, she tried them all, and got it. She shortened it here; she tried 100 and 200, for example, when she saw that it passed, she decreased the numbers and tried 75, 76, and 77.

T: While S17 is creating the table on the board, could one of you enter the equations on the Geogebra? What will happen? Will it be as S6 says?

S6: But it's easier than the equation.

S19: Is it easier?
T: Everyone has a different way of solving the question; she finds it easier that way.

S17: This is a waste of time.

Although she elicited a student's (S6) use of the context of tabular representations of the relations by asking probing questions, she did not extend S6's thinking with probing questions to guide S6 to reason what would happen if the other representations such as graphs or equations were considered by utilizing the technology. Moreover, some of the students (e.g., S6, S19) argued about the solution's applicability (provided in Figure 46), then she acknowledged that S6's solution was different from the rest of the class. Nevertheless, she did not extend S6's thinking which used the correspondence approach without linking any slope conceptualizations, and she did not bring up the idea of differences among representations for discussion to support uniting geometric, algebraic, and numeric understandings by using different representations.

She attended to connect the modes of slope to ensure the conceptual understanding of slope.

S5: The day we removed it from below the ground floor.

T: What does "in ax+b format" mean?

S7: It shows that it did not pass through the origin.

T: What do you mean it doesn't pass through the origin? Where is -4 on the graph?

S6: In the sense that it started from b.

S4: Constant term; the point where it intersects the axes.

T: Yes, right.

The dialogue above indicates that she wanted the students to connect geometric ratio, algebraic ratio and real-life situations by asking questions (e.g., what is the meaning for real-life situations in the context of y=ax+b). After getting the students' answers, she evaluated whether it was correct. She tried to attend to the students' answers and respond to them; she was limited in revealing their mathematical thinking. In addition, she did not seem to consider the main mathematical idea behind the task related to the meaning of unit ratio and geometric ratio notion in a non-homogeneous graph. Hence, Aysu's noticing, according to van Es's adapted framework, is at Level 2 (van Es, 2011).

In the excerpt below, she attended to build the idea of "ratio as a measure" (Simon and Blume, 1994) and the geometric ratio by modifying the task (Task C). In addition she attended to particular student's (S5) confusion by explaining the meaning of the ratio as a measure of given attributes.

> *T: No points were given in the question. Your examples are all integers. What would you think if it was said that "As x increases by 4, y increases by 5?"*
>
> *S5: We were already doing it by proportion. When it is 4/5, there is a change; x goes 5 by 5, and y goes 4 by 4.*
>
> *C: Do you agree?*
>
> *S9: I don't know.*
>
> *T: We said that while x or day changes by one unit, accordingly the floor increases by one, right? You said that you take the rate of change as one [shows on the graph]. It means that we're going up four floors in five days. Remember where the rate of change comes from? We were writing it as the coefficient of x. We reach it by looking at the unit change. If four floors are to be climbed up in five days and if I ask how many floors are climbed up in a day, wouldn't it be 4/5?*

Based on S5's explanations, S5 simply wrote a number at the top and another number at the bottom without considering the unit rate of change and the ratio mentally. However, she did not seem to remedy the students' confusion due to her tendency to respond to her question. In reflection, she explained that she attempted to make the students discuss the geometric rate of change whereas she had not attained her goal considering pedagogical decisions during enactment. Hence, her act was classified as Level 2. In contrast to Level 1 noticing behavior, exhibiting level 2 noticing behavior implies taking actions to make sense of mathematical ideas.

In addition, she guided to build conceptual understanding with the help of the coach's prompt in the instruction; the episode given below illustrates how she attended to the coach's question.

> *A: What are the similar and different parts of the graphs you see on four screens? What must be the same and what must be different?*
>
> *S5: Doesn't the slope have to be the same?*
>
> *C: Do you agree?*
>
> *S3: The vertical or horizontal ratio must not be distorted.*
>
> *T: Are the slopes the same in all four graphs? How?*
>
> *T: We had the road question. We talked about how you would transfer it to the coordinate system in different ways. We showed the same slope in different coordinate systems. What do you remember?*
>
> *S4: I do not know (Task A)*

She attended to the coach's prompt by adding probing questions (e.g., what is the new ratio? and why did you think like that?) she did not take any action such as using Geogebra to illustrate how slope is used to describe behavior (behavior indicator) and determine relationships (determining property conceptualization). In that sense, using probing questions did not make students justify their understanding or enable productive discussion. In contrast to Level 1 noticing, Level 2 takes action to make sense of or extend the students' thinking. To do so, I coded this segment of the noticing as Level 2.

4.5.2.3. Level 3-Focused Noticing

Below is illustrated how she attended to the students' confusion, questions and vague statements related to the intersecting points of the lines.

S10: If it was 0.35, both would not meet [lines would not intersect]

T: Yes, your friend said that too. Let's imagine that they are both 0.35 TL.

S15: Let's consider 100

T: Let's consider the graphs of the lines.

S10: No, they would have continued equally.

S7 and S8: I don't understand.

S15: It would be like this (He drew two parallel lines on the board without showing them on the coordinate system)

T: What kind of lines are these?

S15 and S10: Like parallel lines

T: Why parallel?

S15: Because the rates of change are the same.

S19, S19, S5: Yes.

T: How can we prove your idea for those who do not understand? Let's use Geogebra

S21: Shall we write the equation?

C: Tell the equation to your friend.

S15: 0.35x+20 and the other 0.35x

T: What happened now?
Most of the class said they did not intersect.

In this interaction between the teacher and the students, it seemed that the teacher became aware of the students' confusion at the transition level of geometric ratio conceptualization of slope. Aysu both attended to elicit S10's idea stating that two lines have a similar slope and do not cross if they are parallel to each other and help other students who struggled to identify why they had a similar slope when they were parallel. In contrast to Level 2 noticing, she used the students' ideas to make them ponder big mathematical ideas in that she seemed to connect the conceptualization of the geometric and algebraic ratio. Hence, her attribute was coded as Level 3.

She also attended to build a conceptual understanding of the geometric ratio with different dynamic triangles. To do so, she made the students create different triangles (see Figure 47) and discuss whether they had different slopes and assign variables to explain the relationships.

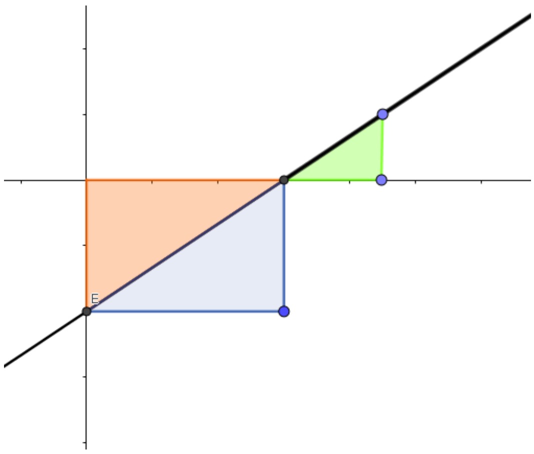


Figure 47. Dyamic imagery of triangles

T: If we didn't move one step from there and if we looked at the -4 4 triangle, couldn't I see the rate of change? [Looked at the big triangle]. We extended it like this. -4 here, and we said 4 here.

S9 and S10: We can see.

C: Draw

S9: I couldn't do it.

T: What are you trying to say?

T: How do you name the axes?

S3: I named x-axis as time, and y-axis as speed.

T: Then you have completed the part called coffee. Can you draw that too? If you stretch it down, where it cuts the y-axis is negative, right? Then how are you going to explain speed here?

S3: I spelled the names of the graph incorrectly. I guess I should have said speed.

T: Which part will be the road, and which part will be the speed?

S3: [correcting the drawing] I think it might be like this.

T: You think that it's not the road it took here; you think of zero as starting behind the starting point. What do you think about this idea? How do you feel about the part of the graph where coffee was spilled? Does anyone think of a different content?

S4: Would there be a plant and time graph?

T: You mean the height of the plant. Then, how would you explain that negative part?

She pressed the students to create different triangles as the mental imagery below and above the line. This Level 3 behavior is distinct from a noticing at Level 2 in that she focused on the students' mathematical thinking and guided them to think differently while they mostly tended to locate the triangles in a familiar way that moved up and down the line with unchanged direction such as blue and green triangles. On the other hand, Aysu missed one of the mathematical opportunities that support student's attempt to sketch another line intersecting with the L1 with the angle of 90 that includes the idea of the slope of the perpendicular lines. That is what distinguishes her noticing at Level 4 from noticing at Level 3.

She also attended to elicit the students' strategies and thinking by taking action, such as elaborating on these students' strategies and making other students elicit and connect these strategies. To illustrate, in the excerpt below, she attended to S8's idea, building upon the geometric ratio rather than algebraic ratio.

T: I've seen several solutions; there are students who said that they saw the relationship using a graph. Who was the one who started with the graph? I think it was your group, Beyzanur.

S8: But teacher, I started with the equation.

T: Okay, but you had an idea about how to draw. Come to the board and tell us about it first.

S8: [She drew graphs with intersecting y axes] The reason we made A like this is because it has a fixed fee, so we started from y and drew it linearly because it progressed by 0.10 TL. We started B from the origin because there was no fixed fee and the fee for each minute was 0.35 TL; but after a while, we thought that if we draw a linear line, it will exceed A because it increases by 0.35 TL..

T: Why did you think that it would be more than A?

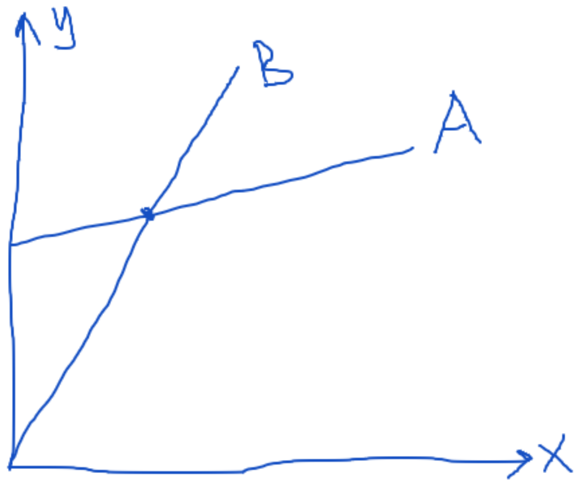


Figure 48. S8’s thinking on for Task B

S8: How can I say? There is an increase of 0.10 TL each time and an increase of 0.35 TL.

S4: There is a difference of 0.25 TL.

T: How did you reflect those increases on your graph?

S8: I thought that A is closer to x while B is closer to y.

T: Did anyone else think so?

S3: We thought like that but could not move forward.

She acknowledged the idea of beginning with graphs rather than tabular or algebraic expressions. Although some students were aware of the fact that at some point there is a point in which the money to be paid, which is offered by companies, is equal, she made other students think about determining property and behavior indicator conceptualization in order to construct the mental scheme (graphs) for the situation rather than algebraic notation. However, Aysu continued the lesson by asking questions such as "what is the meaning of the intersecting point? How could we find this point? Why do you equalize these two equations?" to make students combine algebraic ratio and geometric ratio conceptualizations. All these attempts indicates that she attended to specific students' answers and made the students connect these ideas. However, she missed elaborating on these strategies and making students extend their ideas. For instance, she missed the opportunity to ask S5 or the whole class about the way of finding the intersecting point excluding the use of algebraic notations and Geogebra to enrich their understanding of geometric ratio conceptualization and funneled the students to connect vertical change as the slope times horizontal change (20=m*x, as saying that the change in outputs (20) is the rate of change times the change in inputs (mx)). Hence, her noticing level was regarded to be Level 3, not Level 4.

From a researcher's perspective, she also attended to extend the students' conjectures based on the coach's prompt, questions, statements, or suggestions during the instruction. On the other side, with the perspective of Aysu, she also declared that "I always forget to tell them to try to support what they said in Geogebra. During the lesson, we can use Geogebra in this part of the video, your suggestion allowed me to ask additional questions to test what students think." During post-interview for the lesson. In addition, the dialogue below demonstrates how she changed the flow of the instruction based on the coach's suggestion about the use of Geogebra and the probing question of the coach (What about if the equations have constant?).

S6: But the idea that it must be smaller than that is wrong. They intersect at some point.

T: Oh yes, come and explain this on the board.

C: We can use Geogebra.

T: Your friend added a small number to the section where the constant term is. What did you observe?

Most of the class: Yes.

S6: They intersected on the negative side, that is, in the 3rd Region.

C: Can only constant term and non-constant be equated? What would happen if both of them had constant terms or not?
S5: It doesn't matter if it's a constant term.

S7: Yes.

S6: But if there is no constant term, they do not intersect.

T: Are you sure? Let's look at the Geogrebra.

S: Oh yes, they intersect. They both pass through the origin.

Aysu attended to the coach's suggestion, supported the students to verify their conjectures using technology, and modified the task by adding additional questions. In that sense, in contrast to Level 2, she attended to make sense of the students' thinking via probing questions. She asserted that " A student's question of "Why not?" and your saying "let's use technology" made me turn to this at that moment, but I couldn't think of many additional questions from here." As she stated, she missed some mathematical ideas related to functional property conceptualization where *b* represents the initial height (the *y* intercept) and *mx* represents rise (vertical change) given as the rate of change *m* times the change in inputs *x* (or horizontal change) to generalize this situation (the relation between differences of y's and mx's). This action was coded as Level 3 rather than 4.

She also attended to make sense of the mathematical ideas behind the task. The dialogue is given below as an example.

T: First, you show it. Then, let's move on to Geogebra.

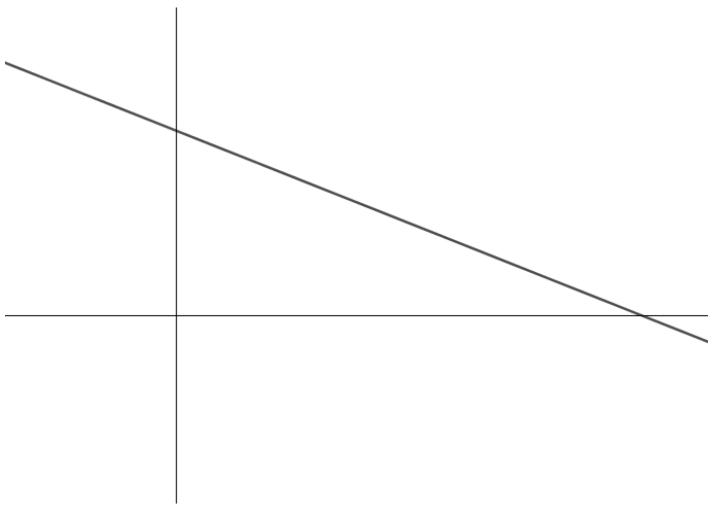


Figure 49. S5's answer for Task B

C: Will it be 50 to 4? Let's give names to the axes.

S5: Height of ramp; the horizontal length of the ramp (gave names in the coordinate planes)

T: Does anyone have a different graph? For example, I thought of something different.

S12: That's what I thought too.

S6: In ours, y is the vertical axis and x is the horizontal axis. We need to write 56/700 on the axes.

C: Your teacher asked a question. Can't a ramp be drawn differently?

S8: It can be drawn.

In this example, she was aware that most of the students tended to sketch the graph in Figure 49 and other possible sketches were not taken into consideration. She also mentioned the students' difficulty in manipulating different lines while conserving their slope. Both of these perspectives indicates that she attended to build the students' understanding of the difference between steepness and slope of the line.

4.5.2.4. Level 4-Extended Noticing

She attended to extend the students' thinking on linear constant conceptualization by asking why they could not reach a fixed slope of any two pairs. In addition, with the independence of representations, she helped the students to consider slope as a constant property unique to straight figures. The dialogue between the students and her indicates how she managed the episode:

S1: We could take a piece of data from the table provided.

S5: But I look at the data and I cannot find any constant.

Aysu: How could you get this? What is the meaning of no-constant?

S5: I subtract two y's and x's then I find ratio, but this is not same for other two set of data. For example, in this thing 150 154 the difference was four 1.6, right, this is the distance between 41.2 and 42.5. The ratio is not the same when you take the other points

T: What does that mean?

S5, and S15: It means non-linear, no specific ratio

S5: The distance ratio between points is always different from each other, so they are different.

T: What are we going to do then?

S6: The height is four times as length of length of humerus. If I multiply the given length with 4, I can find result. I took a point and compared it; I would say approximately.

Aysu: What about decimals? Is this solution ok for everyone?

S7: No, since we could give a rough estimation.

S8: No

T: Can you see what the next point might be? Can you shorten it?

Two students: No

S7: Maybe we can plot some piece of data in the graph to see the trend of the data.

Aysu: That's interesting. Yes, some of you can use the Geogebra. Now we have identified the points. How will we think from now on?

S7, S8, and S15: We can pass it right.

Aysu: Why?

Regarding how she noticed, she attempted to comprehend the rationale behind the students' reasoning, and she made the students explain their statements and connect different students' ideas by pressing students to consider alternatives to their solutions. This action consisted of multiple moments in which students proposed ideas that might have been regarded as objects of inquiry regarding the linear constant conceptualization and smooth and continuous covariation. Her actions could be regarded as mainly interpretive and specific, referring to the students' ideas since she invited them to consider the task context and mathematical decisions and explain their reasoning. Hence, her actions indicated a shift in Aysu’s noticing from Level 3 to 4.

In addition, she also attended to the coach action during the instruction. The dialogue among students, the teacher, and the coach is provided below.

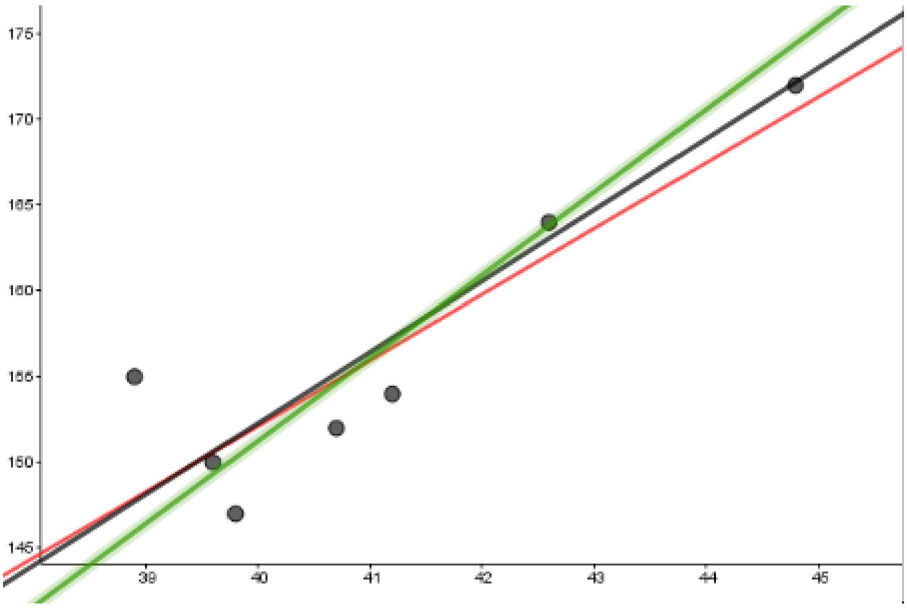


Figure 50. The Coach and students' sketches of the informal "best fit of line."

C: Where does it pass through? (Chooses any point) Will it pass like this or that? (Passed straight lines through a specified point or any two points)

S9: Doesn't it pass through 150, from that point? (Drew the line shown in black)

S12: Let it pass through the origin.

T: What else?

S11: Doesn't it matter if it covers the points and passes through the origin? Wouldn't it be too little to pick a spot?

T: Why?

S14: We need to consider other points so that we can find the location of the line.

C: Does it need to go through the origin?

T: Can't I draw correctly even though it doesn't go through the origin?

S11: When we looked at it, I thought that the range was too low. It could be between zero and 145.

T: By looking at the intervals, we see that 145 is actually far from zero, so our goal is not to pass through the origin. So what is it?

S15: Draw a line

S11: I wanted to come to the closest point and pass it through. I drew the closest one (drew the green line)

T: Why the closest? What do you mean?

S13: That is, the line is almost the same distance from those points.

The coach initiation to sketch line by considering a pair of point and support students to estimate the place of the line activated teacher attention on eliciting students' linear constant conceptualization. The students began to sketch the line by considering one or two pairs of points, and then their responses shifted from it to considering all the points. The tudents' responses indicates that they mostly used linear constant conceptualization. It seemed that she supported the students in transferring the knowledge on slope conceptualization while locating the line. The coach's question that adjusted her action to consider the students' thinking and included support for how to place a line is what differentiated noticing at Level 4 from Level 3.

4.5.3. What and how Aysu noticed in the reflecting phase (post-observation) in Cycle 1

The next heading presents Aysu’s noticing in the reflecting phase for Cycle 4 through four levels: Level 2, Level 3, and Level 4.

4.5.3.1. Level 2-Mixed Focusing

She attended to the whole class of students' confusion by mentioning the adding Task A for instruction. She made this claim by considering the students' limited answers for the homework, including the geometric rate of change and the rate of change in static situations of slope conceptualization. Although she claimed that the students would understand the rate of change in static situations, there is no need to add the second one, which mathematical idea was similar to Task D in the third cycle. She solely emphasized the students' blank and limited answers for the items. She evaluated her instruction in a general way, for instance; “It means we passed this place quickly, more questions are needed, and students need to solve more questions.” Although she became aware of the students' difficulty in the geometric rate of change, she did not mention the instances of possible future instruction to enable conceptual understanding.

In that sense, while conceptualizing the geometric rate of change in graphs, the coach mentioned the model of the geometric rate of change in graphs. The dialogue between her and the coach is provided below.

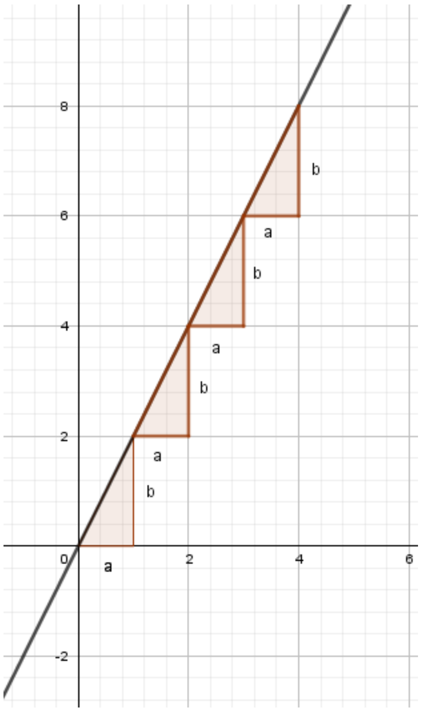


Figure 51. The model was created based on the work of (Nagle et al., 2019)

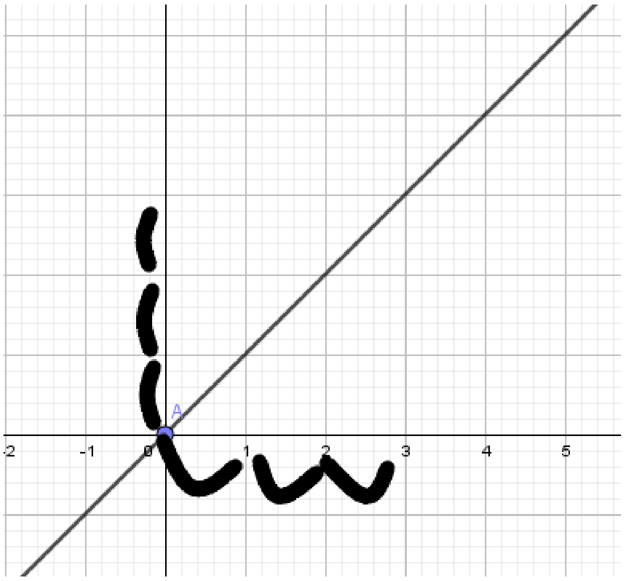


Figure 52. Aysu’s demonstration of the unit rate of change on graphics

C: What did you think about this demonstration? You said that students are confused while demonstrating the rate of change in graphs and interpret it as words.

A: That is good, I can use it.
C: What about the a and b?

A: In the previous task, it was 4/7

C: What about the unit rate of change? What is a and b?

A: a is 1, and b is the slope.

C: Yes, what do you think about this demonstration and your previous task? How could we make the students to create this model?

A: In fact, I said: yes x’s increasing by one and y’s increasing by the ratio. I used the axes to demonstrate it. In a similar, I can show this.

As exemplified above, the coach prompted the teacher to use the model inclass time and compare this moel and her model used in previous lessons (Figure 52). She could not distinguish between the idea of the "unit" rate of change emphasized in the model used in the lecture and the idea behind this model. She demonstrated her willingness to use this model with general statements such as "I can show this [the model] in the class". As general sentences indicates, she did not explain the model's underlying meaning, including the dynamic imagery for the transition level of functional property (Nagle et al., 2019). Since she partially explained the mathematical idea behind the model without linking this idea with pedagogical decisions for further instructions, her noticing was characterized as Leve l2.

4.5.3.2. Level 3-Focused Noticing

She reflected on the coach's questions or prompts during instruction. The related excerpt is as follows:

> When you keep saying come, show, let's use Geogebra, it is actually important to elicit how they think. For example, the student can use different meanings of "increasing and decreasing" while explaining negative and positive slopes.

As seen in the excerpt, she mentioned the critical role of eliciting the students' understanding with how and why questions. At some point, she mentioned that she missed delving into the students' responses with probing questions; rather, she concentrated on accepting the students' answers as completed. In that sense, she declared how the coach's intervention helped her ask additional questions. She was able to provide evidence related to what she noticed (the coach's intervention) and how the coach's action shaped her noticing the students' mathematical thinking on positive and negative slope in this case.

In some points, she was able to attend to important aspects of the instruction and the students' strategies or make inferences about it, which the coach mentioned. The following dialogue is an example of how she attended to a specific moment of instruction given by the coach.

> *A: Actually, I found it sufficient to say that they intersect somewhere. S5's solution was nice to think about*
>
> *C: Let's watch the current interaction in the classroom where S5 explains her answer on the video.*
>
> *A: So, as I said, it was nice that S5 used graphics.*

> *C: Did she reach a conclusion on graphics about the intersecting point? At that time, most students said that they equalized the equations.*
>
> *A: She did not reach a solution, we explained it by using equation, then we saw that they intersect at one point in Geogebra.*
>
> *C: Can't they be solved if dynamic triangle is created on the graph for them to solve rather than using equations?*
>
> *A: Oh yes, it can be solved, why I did not ask… I do not know, it should have been asked. So it didn't fit me or the students, it seems like the geometric ratio part should be focused on more.*

As seen in the dialogue above, she was able to attend to a particular students' thinking and with the help of the coach's prompting, she became aware of the need for effort to make students build reasoning with representations, especially for the geometric ratio. The characteristics of Level 4 include how the teacher will support the students' progress in this area whereas she only made interpretive comments about the students' struggles in this area. In that respect, I characterize Aysu's action as Level 3 noticing.

4.5.3.3. Level 4-Focused Noticing

She attended to a specific moment of instruction. To illustrate, she interpreted a specific segment of the last instruction as given below.

> In the last question, the students asked questions like "Will it pass through one point? What if it passes through two points? Where should the line pass through?" Their questions guided my questions. I thought that they would draw a line from the closest point, but the fact that they discussed this among themselves gave direction to the questions I would ask. They produced solutions with technology without using slope and they saw the points and wondered what kind of lines there could be. I did not expect these. (Task D)

She attended to a piece of instruction relating to the fact that the students took reference of multiple conceptualizations of slope while placing the informal line of the best fit, precisely that of linear constant and behavior indicator. In addition, she stated that technology makes them compare the informal line of the best fit that they hypothesized and the line of the best fit provided through Geogebra. She did not only interpret the students thinking, but she also started to connect her observations to central teaching features, such as classroom discourse. In that sense, her attempt was coded as Level 4.

She also attended to specific students' struggles such as transferring the rate of change into graphs on homogeneous axes.

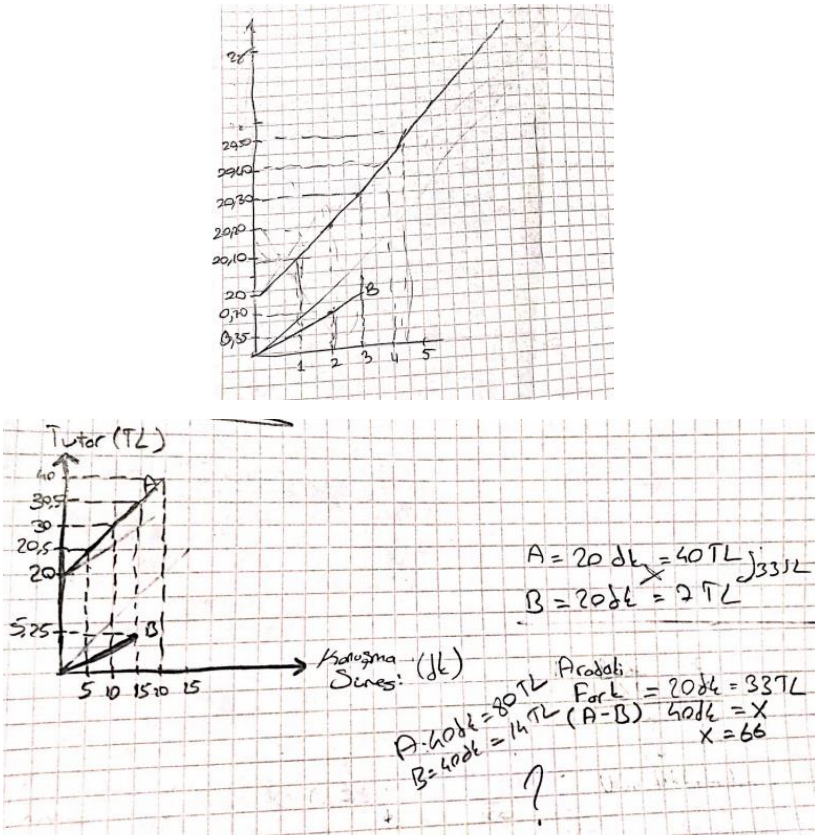


Figure 53. Specific students’ limited solutions for the Task B

C: What do you think of these student answers?

A: I never saw these answers during the lesson, it escaped my attention. They got the units wrongly. When the difficulty continues, I will tell the child that I cannot see the same rate. What happened to the change? Maybe then he'll realize that he doesn't take numbers [on x and y axes] with equal parts

C: How then is the straightness of the line? Do you think they are drawn correctly?

A: No, it's not drawn anyway, they can't draw it correctly because they change the distance between points on axes.

C: How do you think we can make them realize what they did wrong?

A: We can provide millimeter paper. Geogebra can be thought of as a way of demonstrating an intersection point. It can actually be led to S8's thinking by looking at the rates of change [linear constant] when considered on a blank sheet of paper.

C: Does it matter if they make sense of these lines that will intersect at a point before using Geogebra
A: Maybe, but its okay to see through Geogebra. We expect them to reach a generalization that the rate of change should be different from each other to intersect.

Based on the dialogue above, she connected the students' ideas on the unit rate of change and one of the students' ideas on linear constant property. The teacher analyzed two students' lack of sense of "linearity" connecting their tendency to create graphs with unit increments. The teacher also discussed how she would help the students, such as asking probing questions (e.g., you demonstrated 0.1 and 0.35 increment with same units, are they the same? What about the distance between 20 and the origin, is it reasonable to take this interval like that?). Therefore, it is coded as Level 4.

4.6. The Shift in Noticing in Cycles

The graph below was created with respect to the percentage of each level for each cycle to portray how Aysu' noticing levels changed through coaching cycles

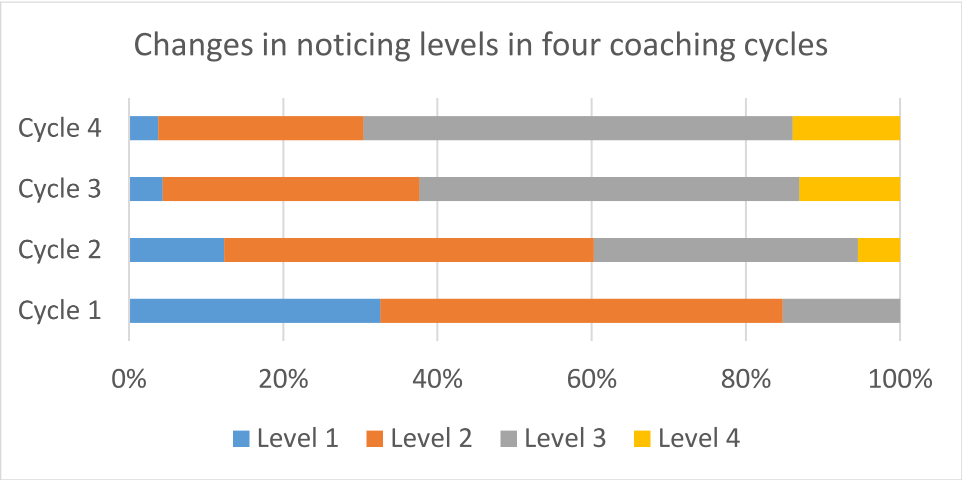


Figure 54. Distribution of Aysu's noticing thorugh cycles with respect to four levels

As seen in Figure 54, Aysu's Level 3 and Level 4 noticing increased in contrast to Level 1 and 2 when the cycles shifted from 1 to 4. This might be a significant sign of the positive effect of the coaching program on the teacher's noticing skills.

CHAPTER V

CONCLUSIONS AND DISCUSSION

In the present study, one of the purposes was to investigate an in-service teacher's (Aysu) knowledge of the cognitive demand of mathematical tasks in the algebra domain and the notion of slope, particularly by engaging in a mathematics coaching program. The second purpose was to examine the changes in the teacher's noticing skills and how the teacher progressed through the coaching stages, including planning, enacting, and review. Based on the purposes of the study, this chapter consists of five parts: a discussion on the findings related to the teacher's progress in knowledge of cognitive demand of mathematical tasks through the coaching program when compared to her responses before the coaching cycles began, and a discussion on the findings related to her development in noticing of students' thinking within the context of rich mathematical tasks through planning, teaching and reviewing. The third part is related to a discussion of the essential features and activities of coaching that influence knowledge of cognitive demand and noticing of Aysu. The subsequent parts include implications for the mathematics coaching model for slope task design, noticing framework adapted and educational practices related to mathematical tasks in textbooks, limitations of the present study, and suggestions for further studies.

5.1. Aysu's Development in Knowledge of the Cognitive Demand of Mathematical Tasks

One of the critical goals of the current study was to examine the mathematical task knowledge of a practicing teacher, Aysu, in terms of the nature of mathematical tasks as sorted and selected before and during the coaching planning sessions. Aysu's rationale for the tasks classifications was discussed based on the task levels in TAG, and her rationale before and during coaching was compared.

The data revealed that Aysu was proficient in identifying low-level tasks in both stages of the study, before and during the coaching. In particular, all *memorization* tasks were recognized correctly by her. With respect to *procedures without connections* tasks, she could classify them mostly correctly before the coaching program compared to her performances during the coaching program. When the distribution of these tasks (at the level of *procedures without connection*) was evaluated throughout four cycles, it was seen that most of these tasks were used in the current study through Cycle 1(Cycle 1 consisted of both low and high-level algebra tasks in the dimension Symbolic Manipulation and Procedures). Although Aysu's extensive explanations revealed that she could employ the TAG (Task Analysis Guide, 1998) criteria for classifying tasks in various topic domains of mathematics, she had some difficulty determining the level of low-level algebra tasks. The reason why she classified low-level tasks in Cycle 1 incorrectly might be related to the decisions of the research team about the nature of tasks in Cycle 1 and the subsequent three cycles. In other words, in Cycle 1, the research team decided to include a wide variety of cognitive demand tasks (low and high tasks) in order to create a cognitive conflict for teachers regarding the task level, whereas only high-level tasks were utilized in the subsequent cycles. Although this pre-set decision on task levels was not highlighted by the coach, the negotiation between the coach and Aysu on selecting and implementing high-level tasks indicated that she believed all of the tasks during coaching are high level. Thus, even when classifying low-level tasks, she was inclined to evaluate them as high level tasks. Although her tendency seemed to be a barrier to assess her knowledge of cognitive demand of tasks, this overgeneralization could lead her to revisit her responses on tasks' cognitive demand. She was "surprised" by the level of student participation and work that was not expected by considering the potential of the task before the lessons. Because of this, she paused to reconsider and adjust her beliefs about the level of tasks. In this respect, providing high and low level tasks to be implemented for the teacher use had a potential for triggering their cruosity or doubt (Swan, 2007; Olson & Barrett, 2004; Watson & Mason, 2007). In that sense, keeping in mind that even the major aim of the coaching program was to select and enact high-level tasks, it was believed that designing an earlier stages of coaching (Cycle 1 for the current study) including both low and high level tasks before moving to the later stages having multiple coaching micro-cycles that aimed to use only high-level tasks was beneficial. To be specific, it could promote the teacher to reevaluate the tasks' level by assocating her initial classification (intention) and actual implementation (student activity) throughout the coaching stages. The other feature of tasks in Cycle 1 was related to the type of algebraic thinking of manipulation of symbols and procedures (Walkoe, 2014), including conceptualizing unknown, variable, and meaning of equality (Kieran, 2007). Although the main focus of the present study is the slope notion, as one of the categories of algebraic thinking, Symbol Manipulation was also used. Since it is the basis of functional thinking, it was decided to begin with tasks related to this category of algebraic thinking. In this respect, the reason for the inability of the teacher to recognize low-level tasks in Cycle 1 compared to other cycles might be her limited knowledge of a concept (Chrambalous, 2010), specifically algebraic thinking in the category of Symbol Manipulation. For example, she admitted that she never had an idea of manipulating symbols by considering two quantities in an equation. This comment might also be related to her tendency to see symbol manipulation as primarily procedural and avoid teaching it conceptually, as Walkoe (2014) discussed.

With respect to high-level tasks, she could sort more than half of the tasks correctly before and during the coaching. Although there are no vast differences between the teacher's performances before and during the coaching program concerning the ratio of the level of tasks classified correctly, her knowledge of the discrepancy between high and low-level tasks differed during the coaching program. In other words, unlike before the coaching period, she experienced a challenge in labeling high-level tasks as a procedure with connection and doing mathematics throughout the coaching program. However, before the coaching program, she often tended to disregard the essential mathematical ideas underlying the high-level tasks. She sorted them as low-level tasks. Thus, it can be said that she was more proficient in recognizing high-level tasks during the coaching period compared to before coaching. This finding is consistent with previous research indicating that there were significant differences between teachers' pre- and post-workshop rationale on task classification and sorting performances (Arbaugh & Brown, 2005; Boston, 2013; Watson & Mason, 2007). Consequently, it was determined that coaching activities assisted Aysu select and adapt to high-level slope tasks despite the fact that she struggled to identify the potential difference between procedure with connection and doing mathematics tasks. To sum up, for low and high levels, she was able to use the criteria in the TAG, and she was more able to identify the level of tasks.

Boston (2013) found that practicing teachers still struggled with the idea of *procedures with connections* at the end of the intervention since they recognized them as low level without identifying the critical mathematical idea of the tasks. It could be concluded that Aysu was competent in viewing the potential of *procedures with connections* tasks to support meaningful mathematics understanding. The reason might be the nature of the notion of slope taken as a focus of the current study that includes various conceptualizations, representations, and mathematical ideas that encouraged Aysu to classify slope tasks as high level (at least at the level of procedure with connection). For this reason, when discussing the level of tasks, Aysu might be more likely to work harder to relate the context of tasks with students' thinking. Thus, she could recognize high-level tasks and provide appropriate interpretations regarding task nature. Nevertheless, she struggles to recognize the differences between the two levels of high tasks mentioned before. This is similar to Pettersen and Nortvedt's (2017) findings that teachers have difficulties differentiating between high-level tasks. Aysu’s confusion about distinguishing between these two levels might be related to her continual mismatching between students' prior knowledge and the context of the task. This relationship is crucial because teachers' conceptions of students' prior knowledge influence the concentration of their instruction (Schwartz et al., 2007) and their rationale on task affordances before the instruction. While she admitted the importance of students' prior knowledge and stated that lessons should be developed based on prior knowledge, she may not differentiate between task affordances and what and how mathematical ideas will be developed through the task. For instance, relating tabular representations with graphical representations requires high-level thinking of students to conceptualize algebraic and geometric ratios. However, the mathematical idea for relating algebraic ratio, geometric ratio, and functional property is beyond the idea of relating geometric and algebraic ratios (Nagle et al., 2019). Hence while she evaluated task potential, she sometimes disregarded these advanced ideas embedded in the tasks and focused on task contexts relating to students' ideas with a limited understanding of advanced slope conceptualizations. In that respect, she could not decide on the task potential as doing mathematics or procedure with connection.

The second reason for the difficulty might be level descriptions in the TAG, which are inadequate operationalizations of the cognitive demands of tasks. For instance, in high level tasks, relative terms, such as "some degree of cognitive effort" and "considerable cognitive effort," are used in the level descriptions to define the difference between procedures with connection and doing mathematics. The meaning of the quantity of effort could be different for different people. Thus, Aysu might have experienced a challenge associating slope conceptualizations with vague criteria in the TAG for doing mathematics. Aysu mentioned the necessity of more specific criteria for the "doing mathematics" task to distinguish between procedures and connection. This issue was also raised by Osana and colleagues (2006), whose work indicated that the TAG might not be a robust instrument as they hypothesized. In addition to vagueness in TAG criteria, this challenge also might be related to structure dilemma (conflict) (Barbosa & de Oliveira, 2013) which refers to the degree of openness in tasks. This can be associated with “as much a function of the task outcome as it is the structure” (Sullivan et al, 2018, p.93). In fact Klein and Leikin (2020) found that teachers declared that they have most familiar with tasks having multiple strategies, rather than tasks having multiple outcome and investigation tasks in which students can approach in different ways as initial. Klein and Leikin argued this result by associating teachers' tendency of use this type of tasks that can be regarded as closed tasks when compared to others with their familiarity of these kind of tasks. In the current study, in a similar way Aysu tended to sort tasks including multiple strategies as procedure with connection (e.g. Task B in Cycle 4) whereas she classified tasks with open-start and multiple outcomes as doing mathematics (e.g. Task E in Cycle 4). Therefore, in addition to contrasting cases for high and low level, contrasting cases for doing mathematics and procedures with connection thorugh different types of open tasks (multiple strategies, multiple outcome or investigation tasks) might be embedded through coaching activities.

An in-depth analysis of her knowledge on classifying tasks showed that before coaching, Aysu had overgeneralizations, namely "posing problems is high level" and "rules or making generalizations are low level". Specific overgeneralizations might be due to overlooking the underlying mathematical ideas or connections embedded in the tasks and noticing the superficial characteristics of the algebraic tasks before coaching sessions. This reasoning is similar to the reasoning of practicing teachers while misclassifying high-level tasks in the study of Boston (2013). In addition, Aysu tended to sort tasks with respect to the perceived difficulty of the mathematical content by students. She maintained the former rationale that posing problems is a high level activity while classifying tasks through cycles. This idea might have stemmed from the given sample task characterized as high level (posing a problem for a given situation in the workshop). However, in Cycle 2, the latter rationale that a task including a pattern generalizations context, which leads students to apply the pre-learned rules on manipulating the numbers to find a general formula appears to have been eliminated as well. The reason for that might be the demonstration of different students' high-level thinking on pattern generalizations. Hence it can be said that rather than articulating the level of the task, demonstrating possible sophisticated high level students' thinking through coaching made the teacher to reevaluate the task potential in a more critical perspective.

Interestingly, before coaching, she sorted a task on finding a percentage of a number in a real-life context as a low-level task. In contrast, she classified a task on creating equations of the given situations (unknowns) that includes real-life context as high level. Her different responses for tasks with similar characteristics (real-life context) are related to differences in the content domain of the tasks. The former is about number sense, while the other relates to algebra. She might have thought that students are good at real-life contexts of operations from elementary to middle-grade schools. However, putting x for unknowns begins much more lately when compared to finding percentages. Moreover, in parallel with other studies' findings, she stated that understanding the context of the text of the tasks in algebra is hard for students since they do not know how they can begin to give unknowns. This belief might have caused her to think that real-life problems require high-level thinking. This perspective is fascinating because she seemed disoriented due to her inability to distinguish between the way students think and her way of solving problems. Therefore, if she could solve the task quickly, she was more eager to classify it as a low-level task. Moreover, other wrong classifications were due to her deficiency in manipulating x as a variable or parameter and advanced ideas in slope conceptualizations.

In conclusion, the data revealed that pre-service teachers' ideas of the cognitive demand of tasks evolved from emphasizing the superficial qualities or procedural components of tasks to linking students' learning to the cognitive demand of tasks. The emergent and prominent language used in teachers' task-sort responses during coaching indicates that teachers have gained a greater understanding of how high-level activities assist student learning. Thus, it could be hypothesized that three factors account for Aysu's enhanced conceptual models for the nature of tasks: the TAG used in the study, her perspective on minimizing the gap between intended design and enacted design (Johnson et al., 2017) by relating issues to pedagogy, and the coach's insistence on reflection on high-level task implementation and its relations with students' slope thinking by using the framework (Nagle et al., 2019). In this way, Aysu might have shown the strong relationship between the mathematical concept underpinning the task and the teacher's role by giving reference to the TAG. Her perceptions about the nature of mathematical tasks transitioned from "designed" to "enacted", and there were shifts from general comments about the contextual aspects of tasks to the detailed descriptions of collaborative student thinking, task and pedagogy in later cycles. Lastly, students' thinking is a critical indicator for efficient task implementation during teaching (Stein et al., 2009; Tarr et al., 2008). Aysu's justifications on task levels might also have stemmed from the actual implementations of tasks in classroom. Because the result of the current study suggested that the evidences which was collected by teacher Aysu related to students' thinking and specific episodes of instruction can lead the teacher to compare the cognitive demand of a task as selected and the task (same as before instruction or modified during instruction) as enacted. In addition students' anticipated and unanticipated thinking could guide her to consider later tasks based on prior experience iteratively. In that sense, classroom-based experiences might be beneficial for in-service development in conceptualizing relationships in the context of tasks, mathematics ideas, students' prior knowledge, and responsive skills to students' thinking to some extent. Finally, engaging in high-level activities as learners and coach's emphasis appear to have also enabled Aysu to consider the features and characteristics of tasks that afford the potential for high-level slope thinking and reasoning.

5.2. The Progress in Teacher Noticing with respect to Three Phases

Another aim of the current study was to examine the changes in the teacher's (Aysu) noticing skills and how the teacher progressed through the coaching stages, including planning, enacting, and review. The next section was devoted to the discussion of findings related to *what Aysu noticed and how Aysu noticed* in each three coaching phases, i.e., planning, teaching and reflecting.

5.2.1. Theteacher's noticing in planning phases

In the planning phase, Aysu and the coach discussed expected student responses, cognitive demand of tasks, sequence of tasks, expected student thinking, and task affordances. In terms of what she noticed, the findings revealed that she focused on a variety of issues, including equipment and facilities such as geogebra or virtual manipulatives, time, pedagogy, tasks, and students' mathematical reasoning and comprehension. Her noticing varied among Level 1, Level 2, Level 3, and Level 4. Diversity in issues she focused on is seen through each cycle; however, in early cycles, regarding how she noticed, her comments were general and evaluative. She did not interpret the sequencing of students' thinking, possible probing, and prompting questions to elicit and extend students' thinking in spite of her main focus on the cognitive demand of the task and contextual features of the task. This indicates that she could not robustly relate the task's contextual features and students' thinking. As also stated in previous studies (Star, Lynch & Perova, 2011; Vondrova & Zalska, 2013), it can be difficult for the teacher to detect the mathematical aspects of the tasks or the teacher can attend to possible students' thinking and strategies; however, they experience a challenge in relating these strategies with essential characteristics of the problem (Fernandez, Llinares & Valls, 2012). However, as the coaching cycles continued, the teacher began to relate the contextual aspects of the task to students' thinking. Most of the comments of Aysu progressed from Level 1 and 2 to Level 3 to Level 4 in later cycles. She interpreted students' needs and difficulties by relating them to task sequences and task affordances. She also tried to propose alternative pedagogical decisions related to task implementations. Similarly, Choy (2017) found that pre-service teachers could consider changing tasks depending on students' difficulties in fractions.

Consistent with what Sullivan, Clarke, and Clarke (2013) emphasized, Aysu stated that recognizing the task level is not enough; modifying and changing the sequence of the tasks with respect to students' needs or difficulties and the lesson's goal is also required for effective teaching. This belief might have been evoked by the coach who asked Aysu to reflect on task context and possible student thinking by encouraging her to consider task design. This might have helped to direct the teacher's attention to task design and student thinking.

5.2.2. A middle school mathematics teacher’s noticing in teaching phases

The data of the current study indicated that Aysu listened to the thoughts of different students, did not take action to manage the discussion, and relied on correct answers without clustering its demand. Hence, the level of her noticing skills was classified as Level 1 and Level 2 mostly in the early cycles of the coaching program. The finding parallels with other studies' findings that revealed both pre and in-service teachers' inadequacy in responding to students thinking (e.g., La Rochelle et al., 2019; Luna & Selmer, 2021) since their ability to notice during mathematics teaching was low (Jacobs, Philipp, & Sherin, 2011, p. xxvii). This is not a surprising finding since even if teachers are good at classifying tasks, they have trouble maintaining the academic rigor of the task during implementation. Similarly, although Aysu explained why tasks were high-level tasks, she could not attend and respond to students' thinking related to tasks. Hence, the study indicated that noticing students' thinking and maintaining high-level thinking without decreasing the level of tasks are interrelated. In fact, Choppin (2011) found a unidirectional relationship between noticing and mathematical task implementation. Many classroom attempts are considered as lower-level noticing because of attending to more general aspects of the lesson (Erickson, 2011). The second reason could be her belief on her responsibility for raising students to become successful in national assessments. Brown and colleagues (2011) also argue that teachers mostly relate their responsibility with school or national decision on the assessment. In this respect, she paid attention to showing the correct way of solving tasks and correcting wrong solutions of students immediately to help them be successful in the exams. Therefore, she faced challenges in noticing the essential aspects of instruction.

Aysu had a more than ten years of professional teaching experience, and although she did not exert effort to managing the classroom or students' behavior like novice teachers or pre-service teachers (Güner & Akyüz, 2020), she had difficulty shifting her noticing from listening to students' thinking and making general conclusions to delving into students' thinking or advancing them. While previous studies indicated that experienced teachers are better at noticing mathematics learning, congruent with this study's findings, some studies showed that experienced teachers are also inadequate in noticing (Goldsmith & Seago, 2011; Kazemi & Franke, 2004; van Es, 2011). This finding indicates that teaching is complex, and even experienced teachers might have difficulty enriching students' thinking. Thus, rather than being experienced, other characteristics of teachers are likely to interfere with effective decision-making during teaching. These characteristics may include teachers' knowledge, assets, and perspectives that influence their teaching-related decisions and behaviors (Dreher & Kuntze, 2015; Schoenfeld, 2010). In addition to these characteristics mentioned by the scholars, Lee and Francis (2017) claimed that specialized content knowledge and responsive skills such as eliciting students' thinking and engaging students in exploring alternative strategies also have a relation with noticing and effective teaching. In conclusion, Aysu's limited responsive skills and specialized content knowledge and content knowledge might be the most plausible reasons for her limited pedagogical responses to students' thinking. One of the crucial indicators for this claim is that at the beginning of the coaching program, she perceived algebraic thinking as isolated from the notion of variables and the meaning of changing variables and she had limited meanings for varios slope conceptualizations. In another perspective, based on the close relationships between maintanence of cognitive demand of high level tasks and the ability of noticing student' thinking during instruction (Choppin, 2010) and the current study's frame including teacher noticing in the context of mathematical tasks, it might be inferred that there is a close relationships between teachers' mathematical knowledge for teaching and sustaining the cognitive load of complex tasks (see Wilhelm, 2014).

Specifically, at Level 1, she tended to correct the wrong answers given by the students, and she did not take any action on different students' thinking or understanding. Her attempt can be regarded as descriptive and evaluative due to the lack of evidence for her effective classroom decisions. At Level 2, Aysu focused on both the students' thinking and the main goal of the task and forced the students to explain their reasoning without elaborating on it. Although she attended to some of the students' confusion and difficulty, and extended and elicited students' ideas, her action in these situations were mostly not robust. For instance, although she was aware of a misconception, she did not act to correct it immediately and could not take any role in managing productive discussion among students. In Level 3, although her attention was on extending and making sense of students' thinking and incorrect student' answers, she was also inclined to build conceptual understanding of connecting slope conceptualizations and reasoning of those. Different from Level 2, she took actions such as the use of technology and asking some probing questions. Moreover, she was able to interpret different students' ideas and open these ideas to the whole class. As for Level 4 noticing, she attended to the coach's prompt for making sense of students' thinking. The coach's actions seemed efficient in Aysu's pedagogical decisions for students' misconceptions and extending students' ideas.

To sum up, with respect to the "what teacher notices" dimension, she seemed to attend to more situations in which students gave correct and incorrect answers, extending and eliciting students' ideas and building conceptual understanding in subsequent cycles. Regarding "how teacher notices" compared to early cycles, her action shifted from listening to different students' answers, making explanations for correct and incorrect answers, and making evaluative comments about students' thinking to modifying tasks to increase cognitive demand, asking more probing questions to orchestrate discussion or asking advancing questions, and eliciting and sequencing students' various works. These shifts are likely to be related to her interpretative stance on students' thinking and her pedagogy. It can be stated that although she implemented many tasks, the improvement of her noticing skills in teaching experiences might take time. Ongoing and intense intervention for teachers' development within a considerable period, like coaching in this study, is vital to improving teacher noticing. This is confirmed by the findings of other studies which revealed positive effects of coaching programs on teachers' classroom teaching practice (Aygün, 2019; Auletto & Stein, 2020; Polly, 2012; Russell et al., 2020). However, inconsistent with prior studies (Olson & Barrett, 2004; Saclarides & Lubienski, 2021), teachers could not improve their traditional way of instruction. The different findings might indicate that the role of coaches and the unique characteristics of the coaching program create a difference in the program's quality of teacher instruction. In other words, using authentic tasks and carrying out the cyclic process of coaching may not guarantee improvements in teacher learning in terms of responding to students' thinking since coaches with different focus and expertise might be a precursor for carrying out coaching activities successfully or not. In that respect, while discussing the effects of coaching on teachers' learning, both the characteristics of coaching activities and the coach's quality should be considered.

With regard to the influence of the cyclic model on the teacher's decision-making during teaching, it can be said that Aysu’s noticing skills in teaching might have affected the elements she noticed in planning and reflecting phases. The data revealed that she attended to issues that were pedagogically and mathematically less significant or attended to issues such as students' thinking and cognitive demand of the tasks with their superficial characteristics in the planning sessions; in line with these noticed issues, she demonstrated pedagogical decisions with lower noticing levels such as correcting students' thinking and not elaborating on students' ideas during teaching. In other words, her descriptive and evaluative stance while discussing tasks, students' thinking, and pedagogy in teaching are similar to her comments on how to notice elements while planning the lesson and after the lesson. Inversely, for instance, at the end of Cycle 3, her comments in the reflecting and planning phases, such as "How can we make students make sense of the geometric rate of change since asking the question of ‘what do you think about the ratio on graphs’ is not enough?" illustrate her focused noticing during the teaching of the idea of the geometric rate of change. Her enthusiasm to build this understanding guided her to make changes in her pedagogy and construct appropriate probing questions for further teaching. Likewise, Choy (2017) argued the reflexive effect of what and how teachers notice during planning on noticed issues while teaching. This bidirectional relationships among planning, teaching and reflecting regarding teacher noticing are also be discussed in the next section.

5.2.3. A middle school mathematics teacher's noticing in reflecting phases

One of the phases of coaching program was reflecting. The findings revealed that in the reflecting phase, Aysu's noticing varied between Level 1 and Level 2 early in the coaching program (Cycle 1 and Cycle 2). The teacher's focus was on issues related to teacher pedagogy, a specific moment of instruction including specific students' thinking, the coach's action, and whole class understanding. She missed the importance of students' responses and work and did not try to make sense of her pedagogy. Specifically, she described her behaviors in terms of suitability in implementing tasks as expected or students' responses in terms of variability and correctness. Her focus was on the whole class rather than particular students and her specific pedagogic responses. This finding is expected since many studies indicate that in-service and pre-service teachers have struggled to attend to critical incidents in instruction (Callejo & Zapatera, 2017; Derry, 2007; Lee & Lee, 2021; Spitzer et al., 2011); Schwarz et al., 2018; Teuscher et al., 2017). According to Callejo and Zapatera (2017), the reason for focusing on the general aspect of the instruction is that it is simple for teachers than identifying the difficulties or misconceptions of students in particular. Highlighting only what is correct or incorrect about students' answers or making evaluative comments about one's pedagogy and whole class learning require less mathematics and cognitive competency from teachers. This also might indicate that making sense of students' strategies and mathematical thinking are limited by teachers' own mathematical understanding and thinking (Dreher & Kuntze, 2015; Lee & Cross Francis, 2018; Schack et al., 2013; Schoenfeld, 2011).

The findings also showed that Aysu's noticing varied between Levels 2, 3, and 4 later in this study (in the third and fourth coaching program cycles). In particular, the percentages of levels 3 and 4 increased. Throughout the reflection meeting sessions, Aysu began to have a more nuanced understanding of student algebraic reasoning. She tried to understand whether the intended goal was reached regarding the quality of student discussions and her action in the face of unexpected student thinking or various student ideas. She also noticed that tasks and tools (e.g., Geogebra) are important aspects of practical algebraic thinking. In particular, she highlighted that some of the conceptualizations of slope required a higher level of thinking, and she claimed that "the reason for using graphics at last is because it is difficult. In my previous teaching, students were allowed to combine dots with a line while using graphics". Based on this comment, it can be deduced that she noticed the gap between students' conceptualizing different slope meanings by using graphic representations and teaching graphs in lessons. This noticed issue enabled Aysu to gain an insight into students' thinking on slope by relating the Slope Framework (Nagle et al., 2019) and task affordances. In that respect, she focused on eliminating students' misconceptions and improving their slope understanding by using appropriate tasks in planning based on reflective ideas from previous lessons. In that sense, reflection on enacted tasks might be seen as essential to increase teachers' noticing (Wickstrom, 2014). In addition, this shows that reflection and noticing may be conceptualized as a dichotomous pair of processes that could be mutually reinforcing (Criswell & Krall, 2017).

Reflection, one aspect of the cyclic model of the coaching program, is seen as an important aspect of teacher learning and coach hypotheses on teacher learning. Hence, one finding of the study indicated the power of reflection on teacher's noticing of students thinking. In the reflecting phase, the coach also used students' works and critical moments, which the teacher did not mention students thinking, and teacher pedagogy, as pedagogical tools to discuss relationships among tasks. Thus, the environment in which the teacher and coach discussed their views and suggestions in the reflection phase contributed to the in-service teacher's noticing. For example, the coach used one of the students' incomplete solutions by using a graph to discuss the differences between this solution and other students. This was an attempt to make her realize that beginning with a graph and an equation requires a different understanding of slope, and the sample of the student is an opportunity to advance students' understanding of the functional property and geometric ratio conceptualizations. It can be said that reflecting and collaborating positively impacted noticing. Likewise, other studies emphasized the importance of examining teachers' comments in reflection more closely (Barnhart & van Es, 2015; Criswell & Krall, 2017). At this point, it can be said that the collaborative, intensive, content-specific, and cyclic nature of coaching improves teachers to make sense of critical student thinking.

The previous section discusses the current study's finding of an increase in the noticing skills of Aysu by comparing earlier and later cycles. Regarding her attention on issues across cycles, it was seen that during cycles 1 and 2, she had difficulty sequencing students' ideas and extending their thinking in symbolic manipulation and covariational reasoning, while it was found in cycles 3 and 4 that she attempted to build conceptual understanding and reacted to the coach's action in a more robust way to improve students thinking. Overall, it might be said that the teacher's learning is gradual, although the transition from cycle 3 to 4 is not quite extensive as in from cycle 2 to cycle 3. This finding could be due to the differences between big ideas in the cycles. Tasks in Cycles 2 and 3 were related to connecting representations and building connections between algebraic ratio with geometric ratio, geometric ratio with parametric coefficient, and reasoning with functional property. These aspects are related to the action and process stages of students' thinking (Nagle et al., 2019); however, tasks in Cycle 4 required more sophisticated algebraic thinking (object stage) than the previous one. Therefore, the teacher might have followed a similar progression in slope as students do. As a result, her learning occurred gradually.

Lastly, one of the striking findings of the study is that, although an in-depth analysis about the frequency of the teacher’s action or her comments about *how teacher notice* dimension was not given, it was realized that she mostly attended to modifying tasks in the reflecting and planning phases based on students' thinking. However, at the moment of teaching, her action for unexpected student thinking was either revoicing students' thinking without elaborating on it or asking probing questions. She did not attempt to change the task context or sub-questions of the task in teaching as often as she did in the planning and reflecting phases. These findings are similar to the findings of Luna and Selmer (2021), who revealed that the teacher tended to focus on an individual students’ thinking and used questioning and revoicing while describing her past pedagogical response; on the other hand, she endeavored to the whole class, and her response involved modifying/adding a task while describing her future pedagogical response. This situation might have stemmed from her difficulty in changing the tasks at that moment of teaching. In fact, changing the task requires much more cognitive effort within a complex learning environment (Lee & Francis, 2018; Wilhelm, 2014). Another issue about this finding could be related to her belief that changing tasks is not a neccessary attempt to be done to respond to students' thinking appropriately. The tasks were also designed to increase student learning by the coach, so most of the sub-questions of the task were considered with respect to the possible variety of students' thinking. Hence, her reaction to unexpected situations mostly included revoicing the idea or asking probing questions rather than adding a new task.

5.3. The Important Features and Activities of Coaching that Influence Knowledge of Cognitive Demand and Noticing of Aysu

Another main finding of this study is that Aysu progressed in knowledge of cognitive demand of mathematical tasks and noticing, which suggests that coaching can support an in-service teachers’ conceptual structure for understanding differences across student thinking in slope and characteristics of tasks. In the previous section, the findings were discussed by relating them to the nature of the coaching program, whereas in this section I discussed the detailed characteristics of coaching specifically adopted in the current study that might support this shift. Coaching derives its strength from anticipating and recognizing students' thinking, understanding, and responses from research-based materials (Mudzimiri et al., 2014), which may be one of the reasons why Aysu's awareness of students' mathematical thinking has improved. The researchers confirmed the effectiveness of using the artifacts such as the TAG and instructional tasks on teachers' recognizing and implementation of high-level tasks. In line with this, Aysu started to attend the cognitive demands of tasks and relate task context with possible student thinking. Some studies additionally emphasize the importance of frameworks and protocols of observations or discussions that make professional development more effective and systematic (e.g., Amador & Carter, 2018; Scherrer & Stein, 2013; Tripp & Rich, 2012; Walkoe, 2015). For instance, Walkoe (2015) utilized a framework, the Algebraic Thinking Framework (ATF), adapted from the idea of Kaput (2000), and the author concluded that the ATF might encourage teachers to consider student algebraic thinking in greater depth. In a similar perspective, in the present study, the APOS-Slope Framework (Nagle et al., 2019) was given to the teacher as a guide to consider and discuss the nature of tasks and students' algebraic thinking. These tools and frameworks were intended to compensate for an in-service teacher's lack of knowledge in recognizing tasks and thinking algebraically. They were also intended to help her develop a practical understanding of students' mathematical reasoning. Congruent with previous studies, the study revealed that the framework is likely to increase teacher's attention to students' algebraic thinking and sense-making of their responses.

Previous studies indicated that coaches position themselves differently in relation to teachers, with two major distinctions: responsive stance versus directive stance (Ippolito, 2010). The former is about reflection on teacher practice, whereas the latter is concerned with a direct message about practice for teachers. Although these studies have not directly examined the effectiveness of these stances (Ippolito, 2010), Russell and colleagues (2020) found that coaches modified their usage of the inquiry stance in response to teachers' perceived responsiveness to coaching. Thus, they concluded that based on teachers' perceptions of coaching, the coaches' inquiry stance could be changed to either directive or responsive approach coaching. Similarly, the coach in the current study took an inquiry stance during conversations with the teacher; however, to create cognitive conflict in some aspects, the coach adapted the inquiry stance as a directive approach. It is believed that the opportunity of observing whether her assumption of student thinking is satisfactory or not might contribute to her learning rather than giving feedback such as "this is ok, but this part is not."

Apart from the roles of coaches in planning and reviewing, one possible most outstanding contribution to the coaching program's success is made through the coach's help in site-based observation. The coach in the current study enabled Aysu to acquire new viewpoints on students' slope thinking as she participated in the coaching process by interacting with the coach consistently. She had a chance to observe the coach's prompts in lessons or questions such as "What do you think about the sequence of students' thinking?" during teaching. Previous studies emphasized that coaches could teach the lesson together or model the lesson before the instruction. However, in the current study, the coach's role during teaching was to observe what the teacher attended and how she responded to and collected close evidence of student work. In addition, strategic and technical help was given to the teacher rather than modeling and co-teaching. The strategic and technical help referred to coach’s action to ask a question (e.g., Could you tell again?) to elicit different students' thinking, and to ask probing questions (e.g., Could you use Geogebra and show us your argument about increasing or decreasing function, or How can you conceptualize slope on graphs only?) to challenge the teacher to justify her answers. This help was strategic since it was provided only when a critical moment for students was observed, and the teacher did not acknowledge this opportunity. It is believed that this is very helpful in allowing Aysu to practice how to elicit thinking and to extend and make sense of thinking. If teachers were not allowed to challenge and redirect student thinking, they had difficulty to respond to unanticipated student responses (Hallman-Thrasher, 2017; Meschede, Fiebranz, Möller, & Steffensky, 2017)

The other common feature of coaching is collaborative work with teachers and coaches. The collaborative structure of the coaching program affected the in-service teacher's noticing positively. Aysu emphasized the importance of working together, and the coach's vision and prompts on students' algebraic thinking and task affordances contributed to her growth as a teacher. Several studies have indicated the crucial role of collaboration in coaching programs on teachers' learning, knowledge, and skills (Yopp et al., 2011). The teacher and the coach created a lesson with respect to the framework and envisioned a learning trajectory on students' cognitive development in slope through collaborative work. Consequently, the teacher was expected to strengthen her noticing abilities and acquire novel and diverse views based on the abundance of opportunities offered by this coaching program.

Finally, it is believed that focusing on the notion (slope) under the same specific mathematical domain (algebra) and the context of high-level tasks provides an opportunity for deep teacher learning with respect to the nature of high-level slope tasks and noticing of students' algebraic thinking. For example, it was observed that Aysu better noticed how students struggled to begin with specific representations (graphics or equations) to conceptualize particular slope meanings such as smooth reasoning or linearity. Likewise, it is stated in the literature that focusing on a specific mathematics topic improves teachers' noticing skills (Güner & Akyüz, 2020). It is believed that focusing on a particular topic provided an opportunity to improve Aysu’s specialized content knowledge regarding various slope conceptualizations and knowledge of content and students by analyzing tasks context and sequence based on students’ thinking and mathematical content.

5.4. Implications of the Study

In this section, the implications of this study are presented under two major sections: Implications for noticing framework and implications for coaching program.

5.4.1. Implications for noticing framework

The modified coaching framework of teacher noticing, adapted from van Es's (2011) work, proved beneficial for examining experienced teacher noticing in the context of hig-level mathematics tasks. The earlier research on teacher noticing informed the present study. The framework articulates what teachers focus on and how they analyze a noticing episode. However, I updated the framework to highlight high-level task context (for detailed description of the revised framework see section 3.1). Thus, the framework is beneficial in eliciting what and how the teacher notices by expanding the boundaries of noticing from reflecting to planning, teaching, and reviewing within the context of this study. Based on Jacobs and colleagues' three interrelated dimensions of noticing and mathematical tasks, Choy (2015) also devised a framework for productive noticing through planning, teaching, and reviewing. In particular, the teaching part seemed to begin with attending to students' confusion and understanding, and interpreting ideas to respond to students' thinking hierarchically. However, in a complex classroom environment, teachers' noticing could not be visible to an observer in this continuum. In that sense, documenting triggered reasons for teachers' actions in a real teaching environment might be beneficial to understanding how teachers react to which intentions (Luna & Sermer, 2021). Therefore, the dimension of "what teacher notices" was changed into "what is noticed" that triggered the pedagogical reaction comparable to the concept of Luna and Selmer (2021), given that van Es' framework is based on the general features of noticing and is more suitable for reflection on action than reflection in action. Besides, Choy mentioned that "teachers attend to refine mathematical task based on this new understanding how students may think about the concept "(p.453) as a how to respond dimension of reviewing the lesson. Although this is an important aspect of modifying tasks after the lesson, the issues related to the previous lesson and reflections on these issues should be considered in the planning phase since it is evident that teachers' predominant orientation toward student work is evaluative in that they use it to determine whether or not the educational activity is successful (Zhao & Cobb, 2007). The revised framework used in the current study took into consideration issues related to previous lessons and the next mathematical goal of the instructional sequence. Mathematics teacher educators and professional development facilitators or coaches might benefit from the framework and might assist teachers in considering elements of the previous lesson as a resource for the future planning of subsequent instruction. Similarly, further studies to investigate teacher noticing within three aspects of practices based on van Es's (2011) noticing framework should consider adjustments to have a thorough understanding of teachers' noticing in planning, teaching, and reflecting. Besides, the characterization of levels of noticing from baseline to extended in terms of the instructional decisions made by teachers provided a detailed portrait of a teacher's noticing. This can be used to identify opportunities for improvement of teachers in high responsive skills by taking into account the processes of noticing of the teacher before, during, and after the lesson.

Nevertheless, there are some challenges in identifying the teacher's noticing of algebraic thinking in the planning and reflecting phases. For instance, reflecting on noticing episodes related to students' learning, pedagogy, or the nature of tasks can also be an issue for the planning phase for the next lesson. Thus, comments of Aysu during the moment of reflection can be related to her plans for the next lesson based on prior experience. In other words, the comments in the reflecting and planning phases could not be separated since there is no clear distinction between planning and reflecting phases. Further studies might consider the nature of those phases and in what situations they might be separated or uniformed by looking at the methodology of the noticing studies. For instance, if researchers create a design with a non-consecutive lesson analysis, the level of teachers' noticing skills can be discussed whether these skills in planning would differ from those in reflecting with a clear perspective.

5.4.2. Implications for coaching program

The coaching program improved Aysu's noticing skills of students' algebraic thinking within the context of high cognitive demand. It also enabled her to produce a robust rationale for the task level through various slope conceptualizations, students' thinking, and task nature. In fact, her knowledge regarding mathematical task nature was enriched through the cycles since her prominent language for providing a rationale for task selection seemed to change from the criteria in the TAG to the context of the slope tasks with respect to students' thinking. Concerning noticing skills, it is also evident that she could attend to the contextual nature of the task and students' slope thinking and interpret these instructional elements by using the Framework on slope conceptualizations or following the coach's prompts. Mathematics teacher educators or policymakers may benefit from these findings. They might assist teachers in developing their ability to select, implement, and adapt tasks and notice students' thinking within a rich mathematical context with the help of a coach or coaches.

In the literature, several studies indicate that a coaching program is a beneficial professional development initiative for enriching teachers' practices, beliefs, and knowledge. However, their methodology lacks coaching activities in depth. Thus, researchers or mathematics teacher educators might struggle to identify coaching practices and how to locate teachers' needs along the learning trajectory of teachers. Although the aim of the study is not to portray principles or conjectures of coaching program within a rich mathematical tasks context, the coaching activities and the nature of the coaching program in the current study and the reason why those activities were selected were explained with a view to the general principles of teaching and learning mathematics. These coaching activities include detecting the teacher's need (demonstrating high-level students' thinking or asking for explanations by connecting with the TAG and slope conceptualization and representations), providing both high and low-level tasks at the beginning, deciding on the directive or responsive manner, collecting evidence from the classroom, and strategic and limited intervention during teaching. This evidence might indicate that the cyclic nature of coaching and specific activities embedded in this program have an important role in increasing the teacher's noticing skills and awareness of algebraic tasks nature. Research community and teacher educators should consider the specific aspects of the coaching program carried out in this study. Moreover, in the findings section, how the coach communicates with a teacher, which questions she asks related to slope notion, and which tasks she adapted in a sequence to enrich students' learning of slope might give a perspective for coach trainers and professional developers in schools and researchers to design and carry out coaching programs. To sum up, further studies might benefit from the study's design and findings focusing on the slope notion with rich mathematical tasks.

Relying on the findings of this study, I propose a few strategies for addressing the work of algebra teachers in the classroom. One is to broaden the relations with conceptualizations of slope notion teachers attend. For example, in the current study, Aysu struggled to conceptualize slope as a measure in graphics and functional property even in subsequent lessons. We might keep this in mind when designing a coaching professional development experience. We might ask teachers to discuss covariational reasoning and corresponded slope conceptualizations in more profound ways early in the coaching program. Moreover, we might ask teachers to discuss more students' thinking or videos in which teachers ask high-level questions and students try to make sense of the geometric rate of change. In conclusion, in accordance with design experiment methodology, I want to end my discussion with the following revised principle regarding coaching program within the context of cognitively high-level slope tasks: Coaching activities within this particular context should include more tasks possessing the idea of linear constant and geometric rate of change conceptualizations of slope to improve teachers' specialized content knowledge, pedagogical content knowledge and consequently noticing skills. Another issue related to the inefficient teaching in classrooms might be that tasks in textbooks give teachers an insight that graphs should be used at the final stage and high-level sub-questions related to slope conceptualizations on graphic display are missing. In fact, Aysu mentioned this as a limitation of curriculum materials. Therefore, curriculum developers should also provide teachers practical high-level tasks created based on students' learning progression on slope as a guide. In that respect, the tasks adapted in the current study might give a promising sample for stakeholders.

5.5. Limitations and Recommendations

In the current study, there are many limitations. First, it focused only on experienced mathematics teachers' noticing in the context of coaching due to the in-depth analysis of noticing skills and knowledge of mathematical tasks. Although investigating elements of one-to-one coaching is believed to give more insightful knowledge on coaching literature, large-scale studies can be conducted to show the practicality and effectiveness of coaching. To do so, tasks repository on specific content gathered from earlier implementation or literature could be an initial fruitful step to increase the practicality of coaching on teacher and student mathematics learning. In addition, Cobb and Jackson (2015) advocated for the designing coach teacher meetings through a regularly scheduled time periods across a large number of schools as a key support for teachers' improvement. With this limitation, coaches' and students' noticing should also be explored since coach or/and student noticing might be related to each other. For instance, in the current study, findings revealed that the teacher's noticing was shaped with respect to students' noticed elements regarding slope conceptualizations and the coach's prompts or actions on what they noticed through discourse in the class. Similarly, Lobato, Hohensen, and Rhodehamel (2013) highlight that students' noticing will help to identify the effectiveness of teachers' plans for student reasoning and the responsive skills of teachers. Therefore, future studies on these relationships can give valuable insight into the following questions, which are about the development of the noticing, through robust evidence: "How does student noticing impact teacher's noticing of student thinking? How does the coach's noticing impact teacher's noticing of students' thinking while implementing high-level tasks?"

Another limitation of conducting the current study with one teacher as a participant is related to her specific characteristics. These characteristics are being an experienced teacher and her interest in applying reform-based pedagogies in her classroom. Saclarides and Munson (2021) pointed out that noticing skills can be affected by contextual factors, and thus, future studies should be conducted with more in-service teachers with different characteristics. These characteristics may be associated with the level of enthusiasm to adapt new pedagogies or years of experience in the teaching profession. Besides, additional research is required to determine whether and under what conditions the work of coaches could be reformed to promote the learning of groups of teachers.

Gibbons, Kazemi, and Lewis (2017) anticipated that the techniques and expertise required for coaching groups of teachers differ from the practices required for assisting individual teachers. Nevertheless, there might be similarities between coaching activities for individuals and groups of teachers. Therefore, further research can help discover these differences and similarities.

Third, this study is limited to unique perspective of the researcher's (and coach's) regarding mathematics teaching and learning. Although different data sources such as video recordings of lessons and interviews, field notes observation were provided for validity of data and consistency across data and the teacher's own perspectives about her noticing were given, there may still be a researcher's bias. Another limitation of being a coach in the current study might be that aspects of specific qualifications or expertise of the coach in developing teachers' practices were not explained in depth. Hence, it is suggested to explore how the quality of the coaches affects teacher learning through further studies.

The fourth limitation of the current study relates to using a particular context to develop teacher's noticing. Future studies can identify how different settings or coaching activities influence noticing. Besides, similar settings in other cultures could be established to portray the possible effects of a coaching program on teacher noticing or learning. Therefore, replication of this study can be conducted with individual in-service teachers.

Another limitation is about studying teacher noticing within a specific mathematical domain of algebra and the notion of slope within the context of the coaching program. Although there are more studies on enriching teachers' noticing of various topics and ideas within the context of coaching, a few studies investigated the influence of coaching on teachers' learning of a specific mathematical topic or idea. In further research, mathematics coaching can be conducted in other learning areas (measurement, geometry) and ideas (spatial ability, proof) to detect patterns among coaching activities and various content domains. In addition, it would also be interesting to consider how teachers' thinking on the nature of mathematical tasks and noticing differ in various content domains such as geometry and measurement.

Finally, the current study explored the possible effects of the coaching program on an experienced mathematics teacher's noticing skills. Although her knowledge of slope conceptualizations and beliefs about the teaching and learning of algebra were assessed, motivation, orientation, knowledge, and attitudes might be other factors that could influence the teacher's noticing. In that respect, mixed method studies can be conducted with several teachers who could be regarded as multiple cases based on those factors. The findings of these studies might contribute to the field by exploring the relationships between noticing of teachers and teachers' knowledge or beliefs.

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APPENDICES

A. SAMPLE TASKS IN FOUR CYCLES USED SECOND YEAR EXPERIMENT

CYCLE 1

Task A-Level 3

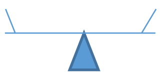
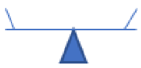
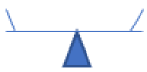
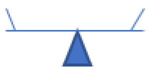
a. $7-\boxed{}=6-4$ eşitliğinin sağlanabilmesi için kutuya hangi sayı yazılmalıdır? Nasıl düşündüğünüzü açıklayın.

b. Aşağıda verilen eşitliklerden doğru olanları doğru; yanlış olanları ise yanlış olarak işaretleyin.

i. 674-389=664-380……………………………………. (D/Y)
ii. 5.84=10.168………………………………………...(D/Y)
iii. 37+54=38+53………………………………………(D/Y)
iv. 64 ÷ 4= 32 ÷ 8……………………………………..(D/Y)

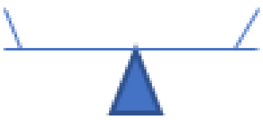
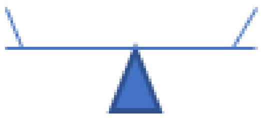
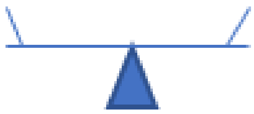
Task B-Level 1 & Level 2

a. Aşağıda verilen her bir durumun eşit kollu terazide kolları düşünerek dengede olup olmadığını işaretleyin.

Durumlar	
a. 3(50-42)=2(10+2) 	Dengede / Dengede Değil
b. $X^2-81=(x-9)(x+9)$ 	Dengede / Dengede Değil
c. 3(x+5)=3x+5 	Dengede / Dengede Değil
d. $X^2+5x+25=(x+5)^2$ 	Dengede / Dengede Değil

b. Eşitliğe ekleme, çıkarma,çarpma,bölme ve karesini aldığımızda hangi işlemden sonra yine eşitlik bozulmaz neden?

c. i.2x+5=15 için eşitliğin korunumu ilkesine göre terazi modelini kullanarak bilinmeyeni bulunuz. (Terazi kefelerinde yapılan herbir değişikliği aşağıda verilen terazi modellerini kullarak gösteriniz)

1. Adım	2. Adım	3. Adım
		

ii. 3x+4=2x+7eşitliğinde bilinmeyeni bulmak için terazi modelini kullanın. (Yukarıda verilen gibi bir terazi modeli çizebilirsiniz)

iii. 4x-5=6x-17 eşitliğindeki bilinmeyeni terazi modeli **kullanmayarak** bulun.

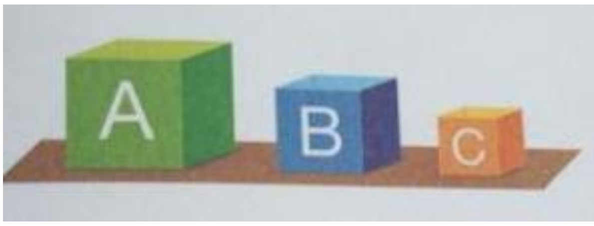
Task C-Level 4

$\frac{a+4}{4} = \frac{2a+2}{5}$ eşitliğini veren bir problem yazın.

Problem:

Çözüm:

Task D-Level 2



A, B, ve C kutularının içerisinde aynı maddeden farklı miktarlarda bulunmaktadır. B kutusunun içerisindeki maddenin miktarı, A kutusundaki madde miktarının yarısı kadardır. C kutusunun içerisindeki maddenin miktarı, B kutusundaki madde miktarının çeyreği kadardır.

Tüm kutulardaki maddenin toplam değeri 650 liradır. (1 kg 50 liradır). A kutusunun %25'ı, B kutusunun %40'ı ve C kutusunun %50'si doludur.

a. A kutusundaki madde kaç kilogramdır?

b. Kutulardaki toplam değeri veren denklemi kurun.
c. Yukarıdaki kullanmadığımız bilgiyi kullanmak istesek nasıl bir soru sormamız gerekirdi? (Adapted MoNE, 2018)

Task E-Level 4

Aşağıda verilen durumlar hakkında ne söyleyebilirsiniz? Yanlarına düşüncelerini yazınız.

a. p + 12 = s + 12
b. 2x+9=0
c. 2x+9=y
d. 2x+9
e. 3(x-4)
f. 2x+5=3x+1
g. 3 + 2y = 5y
h. q + 2 = q + 16

Task F-Level 4

Pirinç pilavı yapmak için 3 bardak su 2 bardak pirinç gerekli ise su ile pirinç arasındaki ilişkiyi nasıl ifade edersiniz?

Homework

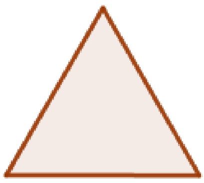
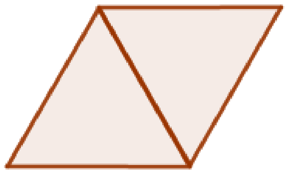
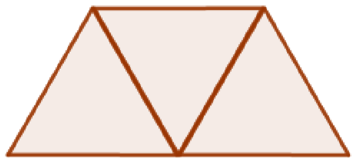
Bilim insanları, sera kalitesini ve ürünlerden elden edilen verimi artırmak için, yetiştirilen ürünler ile onların büyümesine etkileyen faktörleri araştırmışlardır. 1 hektar tarlada üretilen pirinç miktarının ortalama gün sıcaklığına bağlı değişimi aşağıda verilmiştir.

$P = -\frac{\sqrt{3}}{2}S + 33.2$ (S: celcius, P:kilogram)

a. 20 derece sıcaklıkta pirinç miktarı yaklaşık olarak kaçtır?
b. Bu sıcaklıkta üretilen pirinç miktarından 3/5’I kadar daha fazla üretilmek istense, hangi mevsimde üretilmesi doğru bir karar olur? Nedeninizi açıklayın.

CYCLE 2

Task A-Level 4

		
I. Adım	2. Adım	3. Adım

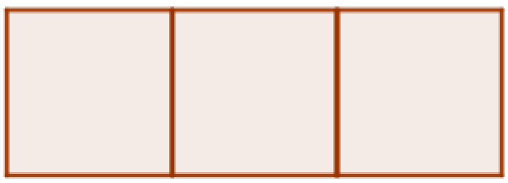
a. Çevre ile kullanılan materyal sayısı arasında nasıl bir ilişki vardır?

b. 20 tane üçgen kullandığımızda çevresi ne olur?

c. 100 tane üçgen kullanıldığında çevresi ne olur?

d. n tane üçgen kullandığımızda oluşan şeklin çevresini nasıl bulabilirsiniz? Bulduğunuz kuralı yazın.

e. Bulduğunuz yöntem dışında başka bir yöntem ile kuralı bulunuz.



f. Üçgen yerine yukarıdaki gibi kare kullansa idik, bu şekilde yanyana dizildiğinde, herhangi bir sayıdaki kare için çevresini veren bir kural bulun.

g. Altıgen kullandığımızda kural ne olur? Nasıl buldunuz?

h. Çevresi 120 birim ise kaç tane altıgen kullanılmıştır?

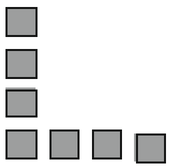
ı. Herhangi bir kenarlı çokgen için genel bir formül bulabilir misiniz?

i. Her bir üçgen, dörtgen ve altıgen için çevre ve kullanılan çokgen sayısına göre grafiklerini çizin. Aynı grafik üzerinde gösterin (Noktalı kağıt üzerinde)

j. Grafikte ne farkettiniz? (Adapted from Radford, 2008)

Homework:

1.



4 tabanlı oyuncak L harfinin yapımında 7 tane kare kullanılmaktadır.

a. 7 tabanlı oyuncak için kaç kere kullanılır?

b. n tabanlı oyuncak L yapmak için kullanılacak kule sayısı kaçtır? Kuralı yazın. Nasıl buldunuz açıklayın.

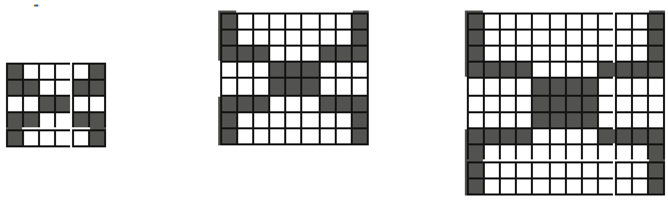
2.

a. Her bir adımda artan örüntülerle çalıştık. Her bir adımda azalan bir örüntü modeli oluşturun.

b. Bu duruma uygun kuralı bulun.

3.

Kurbağa çizelim.



1. adım   2. Adım   3. Adım

a. Gri kare sayısı ile adım sayısı arasında nasıl bir ilişki vardır? Adım sayısını t ile ifade edersek, t. ci adımda gri karenin sayısı ne olur?

Task B-Level 3

a. Aşağıda verilen her bir durumun sayı kullanmadan grafiklerini çiziniz.

i. Ayşe kumbarasına her gün belli miktarda para atmaktadır. Güne bağlı olarak kumbaradaki parayı veren grafik,

ii. Kiloya bağlı elmaya ödenen paranın miktarını veren grafik,

iii. Sabit hızla ilerleyen arabanın zamana bağlı hızını veren grafik,

iv. Sabit hızla ilerleyen arabanın zamana bağlı kat ettiği mesafeyi gösteren grafik,

v. Tamamı dolu olan varilden sabit hızla su akıtan bir musluk ile boşaltılmaya başlanınca varilde kalan su miktarı ile zaman arasındaki ilişki,

vi. Deniz seviyesindeki ölçülen sıcaklık 0 derece olarak kabul edilmektedir. Her bir metre deniz seviyesinden aşağı inildikçe sıcaklık sabit azaldığına göre, deniz seviyesinden aşağı inildikçe sıcaklık değişimi.

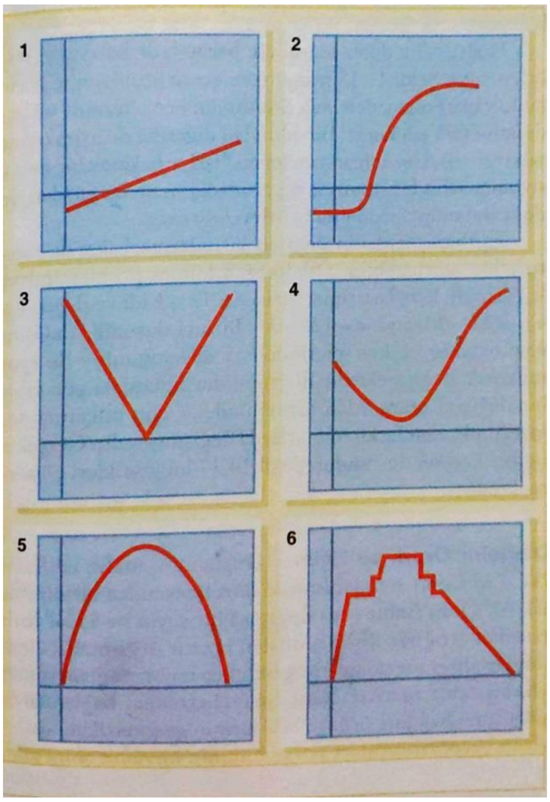
b. Aşağıda verilen dört durum ile grafiklerini eşleştirin.

i. Donmuş bir yemeğin buzluktan alınmasının 30 dk öncesinden başlayarak, mikrodalgaya belirli bir sure konulması ve çıkarılıp sofraya getirilmesi sürecindeki sıcaklık değişimi,

ii. Satılan maddelerin sayısı bakımından yapılan kar,

iii. Beyzbol topunun atıldıktan yere düşünceye kadarki zaman içerisinde yüksekliği,

iv. Beyzbol topunun c şıkkında verilen durumdaki hızı,

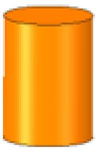
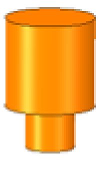
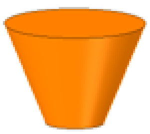
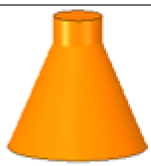


Neden böyle bir eşleştirme yaptın? Açıkla.

……………………………………………………………………………
……………………………………………………………………………
……………………………………………………………………………
……………………………………………………………………………
……………………………………………………………………………
……………………………………………………………………………
……………………………………………………………………………
………(Adapted from van de Walle, 2013)

Task C-Level 4

Aşağıda verilen variller eş musluklardan akan su ile doldurulmaktadır. Zamana bağlı olarak 1-2-3-4-5 numaralı varillerdeki su yüksekliğini veren grafiklerini çiziniz. Aşağıdaki soruları cevaplayınız.

1	2	3	4	5
				

i. Her bir varil için grafikleri nasıl çizdin? Açıkla.

Birinci varil için:

İkinci varil için:

Üçüncü varil için:

Dördüncü varil için:

Beşinci varil için:

ii. Grafik ile durum arasında nasıl bir ilişki kurdun? Açıkla.

iii. Çizdiğin grafikler ile durum arasında farklılıklar neler? Bu grafiğe nasıl yansıdı? Açıkla. (Adapted from Carlson, Michael Oehrtman, and Nicole Engelke, 2010)

CYCLE-3

Task A-Level 3

1. 20 Temmuz 1969'da Neil Armstrong ve Buzz Aldrin'i insanoğlunun en büyük uzay macerasına taşıyan Saturn V, devasa boyutlarının da hakkını veriyor. Dünya tarihinin en büyük, en uzun ve en ağır roketi Saturn V, işlev gördüğü zamanlarda tam 763 Asya fili, yani yaklaşık 2.000.000kg ağırlığında yakıt tüketiyordu. Yakıt tüketim hızı sabit olduğu varsayımından yola çıkarak, uçuşundan sonra 2. Saniyede 12.000 kg 6. Saniyede ise 36.000 kg yakıt tükettiğine göre bu saniyeler arası yakıt tüketim hızı ile 25.ve 50. Saniyeler arası yakıt tüketim hızını karşılaştırınız. Düşüncenizin doğru olduğunu matematiksel olarak ispatlayın.

Task B-Level 4

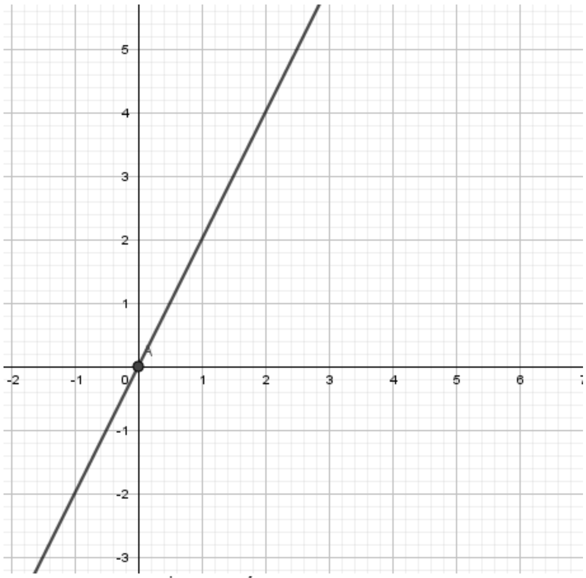
Bir araç 1 litre yakıtla 23 km yol almaktadır. Yakıt tankı 20 litre almaktadır. Bir yolculuğa çıkacağınızı ve başlangıçta tankı doldurduğunuzu hayal edin. Alınan yol verildiğinde kaç litre yakıt kaldığını gösteren bir matematiksel model oluşturunuz. (Taken from van de Walle, 2013)

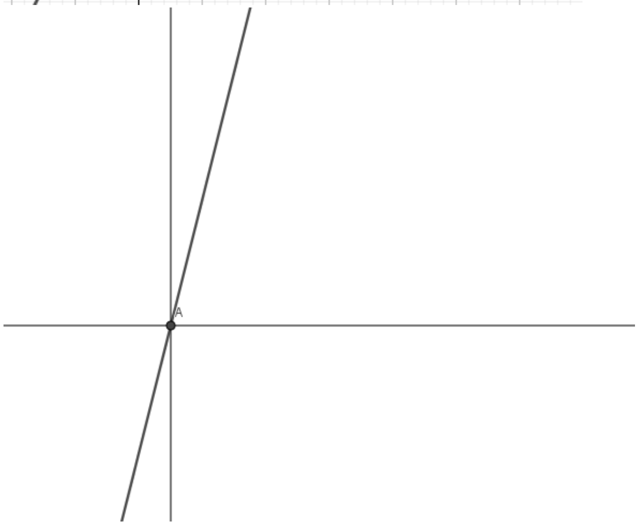
Task C-Level 3 (Adapted from Stump, 2001)

Task D-Level 4 (Adapted from Deniz &Tangül-Kabael, 2017)

Task E-Suggested

Herhangi bir doğrusal ilişki içeren iki değişkenin grafikleri aşağıdaki gibidir. Buna göre bu doğruların eğimleri hakkında ne söyleyebilirsiniz**? Açıklamalarınızı gerekçelendirin.**





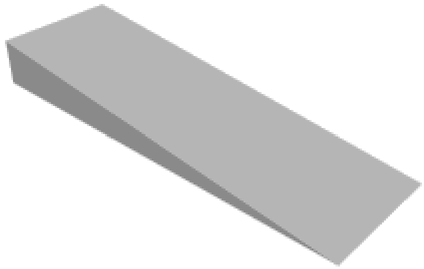
(Adapted from Nagle et al., 2019)

CYCLE-4

Task A- Level 3

Engelli Rampası Eğimi belirlenirken, tekerlekli sandalye kullanıcıları, yürüme zorluğu yaşayan yaşlılar, bebek arabası kullanan yayalar ve görme engellilerin de kullanacağı düşünülerek mümkün olan en az eğim dikkate alınmalıdır.

Yükseklik 51 cm – 100 cm arasında ise rampanın eğimin en fazla (%8) olması beklenmektedir.



Bu bilgilere göre bu eğime etki etkenler neler olabilir?

Siz mühendis olsanız ve rampa yapmak isteseniz bu rampanın özellikleri ne olabilir? Farklı bir rampa çizebilir misiniz?

Koordinat düzleminde gösterilmek istense bu durumu nasıl gösterirdiniz?

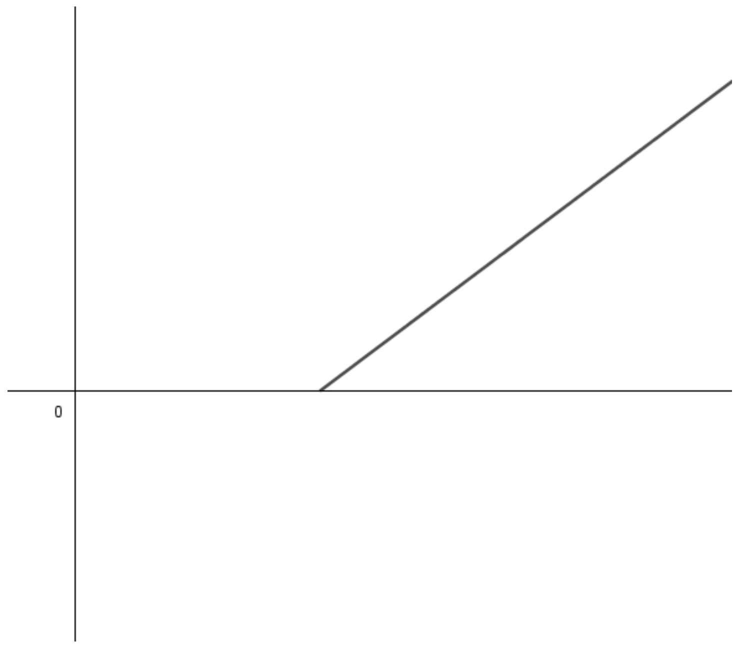
Task B-Level 4

Ayşe, iki telefon şirketinin aylık ücret reklamlarını görmüştür. A Şirket ayda 20, 00 TL sabit (8GB Internet) ve kullanılan her dakika konuşma için 0, 10 TL ücret karşılığında telefon hizmeti sunmaktadır. B Şirketinin ise aylık sabit ücreti yoktur, ancak konuşma dakikası 0, 35 TL'dir. B şirketinin de sunduğu internet paketi A şirketi ile aynıdır. Ayşe, bu iki şirketin ücretlerini, her ay kullanılan telefon sürelerine göre karşılaştırmak istiyor.

a. Sorudan ne anladığınızı kendi cümleleriniz ile ifade edin.

b. Sizce hangi şirket ile konuşma yapınca daha az ödenir? Neden? (PISA 2012, Released Item)

Task C-Level 3



Yukarıda verilen grafiği çizen bir öğrenci kağıdına kahve dökülmesi sonucunda grafiğin bir kısmı görülmemektedir. Bu grafiği verebilecek öğrencinin uğraştığı problem ne olabilir? Problemi yazınız.

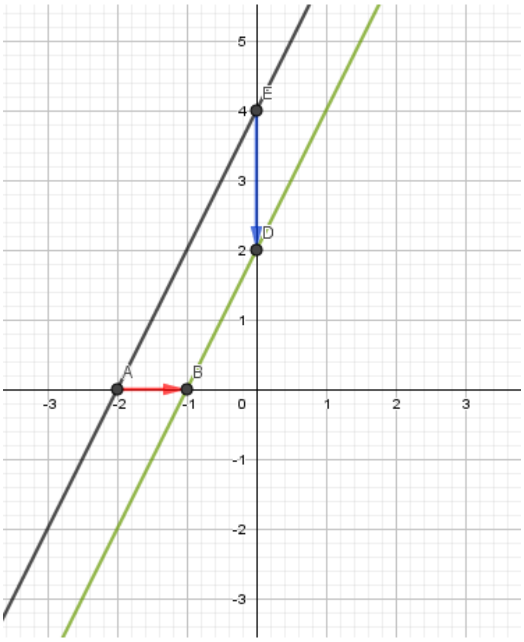
Task D: Level 3

If we scroll thorugh the line along the x axis, how much changes will occur in y axis? Explain in words at first.

…………………..

Then support your claim with graphs and algebraic notations?

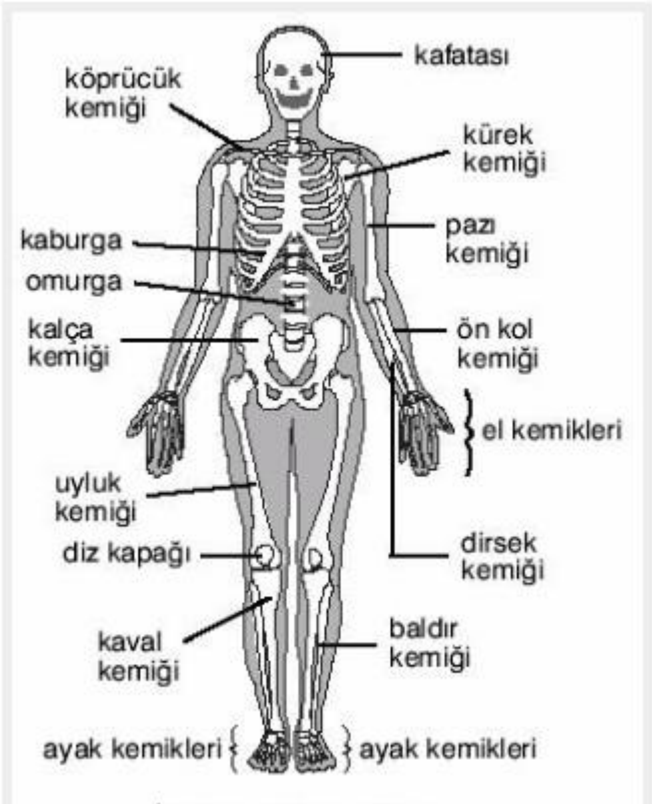
(For students who struggle to realize the relation between differences in x axis and y axis.)



(Adapted the idea from Byerley and Thompson, 2017)

Task E- Level 4

Bir yol inşaatı sırasında antik çağda yaşayan insanlara ait olduğu düşünülen kemikler bulundu. İskeletlerin tamamı bulunamamakla birlikte konumlarından, farklı kişilere ait oldukları anlaşıldı. Bulunan kemiklerden bir kişinin pazı kemiği uzunluğu 47.5 olarak ölçülmüştür; insan vücudundaki bu ve diğer kemiklerin yerleri Şekil 1'de gösterilmiştir.



İnsan iskeleti ve kısımları

Uzun yıllar önce yaşayan insanların fiziksel özelliklerinin belirlenmesi tarihçiler için önemlidir. Bu konuda tarihçiler sizden (sizin gibi matematikçilerden) yardım istiyorlar. Tabii ki yöntemleriniz bilimsel ve güvenilir olmalıdır. Bu konuda işinize yarayabilecek Türkiye Adli Tıp Veri Merkezi'nin (TATVEM) veri tabanından elde edilen istatistiksel bilgiler Şekil 1'de verilmiştir. Bu tabloda değişik yaş ve cinsiyetlerde kişilerden derlenmiş kemik ölçümleri verilmiştir.

Bulunan iskeletin pazı kemiğinin uz unluğu 47.5 olan kişinin boyunu hesaplayınız. Değişik kemiklerin insan boyu ile olan istatistiksel ilişkilerini kullanarak matematiksel modeller oluşturabilirsiniz. Bunun için TATVEM veri tabanında aşağıdaki verilen bilgilerden yararlanabilirsiniz.

A	B	C	D	E	F
1	150	30,5	39,6	39,6	20,5
1	154	30,2	38,4	38,4	21,7
1	150	31,5	38,6	38,6	20,4
1	155	31,9	38,9	38,9	19,6
1	150	32,1	38,6	38,6	20,3
1	154	30,6	39,8	39,8	20,2
1	151	33,9	39,4	39,4	22,1
1	152	32,4	38,4	38,4	21,8
1	147	33,1	39,8	39,8	21,8
1	159	34,7	41,0	41,0	22,8
1	153	35,3	42,1	42,1	22,0
1	172	37,2	44,8	44,8	24,1
1	153	32,5	41,2	41,2	21,5
1	165	34,4	42,4	42,4	22,9
1	154	33,5	41,2	41,2	22,4
1	157	32,8	39,8	39,8	21,5
1	144	31,4	38,1	38,1	20,5
1	164	34,3	42,6	42,6	22,8
1	143	31,3	36,2	36,2	21,6
1	160	36,6	41,9	41,9	24,0
1	152	33,2	40,7	40,7	22,4
1	150	31,7	38,5	38,5	21,6
1	154	33,7	41,6	41,6	22,6
1	162	34,0	41,8	41,8	25,4

Şekil 1. Türkiye Ali Tıp Veri Merkezi (TATVEM) Veri Tabanı [Anahtar, A sütunu 1: erkek, 2: kadın, B sütunu boy (cm), C sütunu kaval kemiği (cm), D sütunu uyluk kemiği (cm), E sütunu pazı kemiği (cm), F sütunu ön kol kemiği (cm)]

Bu iskeletin ait olduğu kişinin boyunu hesaplayınız. Değişik kemiklerin insan boyu ile olan istatistiksel ilişkilerini kullanarak matematiksel modeller oluşturabilirsiniz. [Geogebra kullanabilirsiniz]. (Adapted from Erbaş et al., 2016)

Task E-Suggested

Here is a table for the height versus the area of a rectangle that is growing in proportion

1. Explain the relationship between the height and area of the rectangle by using at least two different model or representations.

Height (cm)	Area(cm^2)
2	16
4	64
6	144
10	400
65	?
1/2	?

(Adapted from Ellis, 2011)

B. SAMPLE OF TASKS USED FOR TASK SORTING ACTIVITY BEFORE COACHING

Task A

Manipulatives/Tools: Counters

For homework Mark's teacher asked him to look at the pattern below and draw the figure that should come next.

Mark does not know how to find the next figure.

A. Draw the next figure for Mark.

B. Write a description for Mark telling him how you knew which figure comes next.

(QUASAR Project—QUASAR Cognitive Assessment Instrument—Release Task)

Task B

Manipulatives/Tools: None

Part A: After the first two games of the season, the best player on the girls' basketball team had made 12 out of 20 free throws. The best player on the boys' basketball team had made 14 out of 25 free throws. Which player had made the greater percent of free throws?

Part B: The "better" player had to sit out the third game because of an injury. How many baskets, out of an additional 10 free-throw "tries," would the other player need to make to take the lead in terms of greatest percentage of free throws?

(Adapted from *Investigating Mathematics* [New York: Glencoe Macmillan/McGraw-Hill, 1994])

Task C

Manipulatives/Tools: Calculator

Your school's science club has decided to do a special project on nature photography. They decided to take a few more than 300 outdoor photos in a variety of natural settings and in all different types of weather. They want to choose some of the best photographs and enter the state nature photography contest. The club was thinking of buying a 35 mm camera, but one member suggested that it might be better to buy disposable cameras instead. The regular camera with autofocus and automatic light meter would cost about $40.00, and film would cost $3.98 for 24 exposures and $5.95 for 36 exposures. The disposable cameras could be purchased in packs of three for $20.00, with two of the three taking 24 pictures and the third one taking 27 pictures. Single disposables could be purchased for $8.95. The club officers have to decide which would be the better option and justify their decisions to the club advisor. Do you think that they should purchase the regular camera or the disposable cameras? Write a justification that clearly explains your reasoning.

Task D

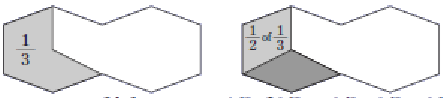
Manipulatives/Tools: None

The cost of a sweater at a department store was $45. At the store's "day and night" sale it was marked 30 percent off the original price. What was the price of the sweater during the sale? Explain the process you used to find the sale price.

Task E

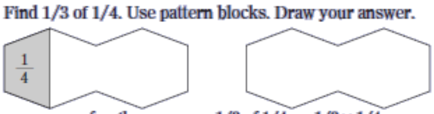
Manipulatives/Tools: Pattern blocks

1/2 of 1/3 means one of two equal parts of one-third



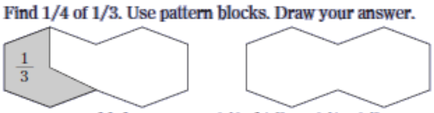
one-third   1/2 of 1/3, or 1/2 × 1/3, = 1/6

Find 1/3 of 1/4. Use pattern blocks. Draw your answer.



one-fourth   1/3 of 1/4, or 1/3 × 1/4, = ______.

Find 1/4 of 1/3. Use pattern blocks. Draw your answer.

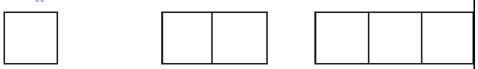


one-third   1/4 of 1/3, or 1/4 × 1/3, = ______.

Task F

Manipulatives/Tools: Square pattern tiles

Using the side of a square pattern tile as a measure, find th perimeter of, or distance around, each train in the pattern-bloc figure shown.

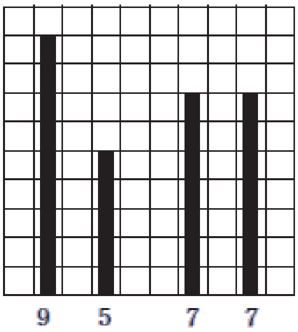


Train 1   Train 2   Train 3

Task G

Manipulatives/Tools: Grid paper

The pairs of numbers in (a)–(d) represent the heights of stacks of cubes to be leveled off. On grid paper, sketch the front views of the columns of cubes with these heights before and after they are leveled off. Write a statement under the sketches that explains how your method of leveling off is related to finding the average of the two numbers.



(a) 14 and 8 (b) 16 and 7 (c) 7 and 12 (d) 13 and 15

By taking two blocks off the first stack and giving them to the second stack, I've made the two stacks the same. So the total number of cubes is now distributed into two columns of equal height. And that is what average means.

(Taken from Bennett and Foreman [1989/1991])

Task H

Manipulatives/Tools: None

Give the fraction and percent for each decimal.

0.20 = ______ = ______.
0.25 = ______ = ______.
0.33 = ______ = ______.
0.50 = ______ = ______.
0.66 = ______ = ______.
0.75 = ______ = ______.

Task I

Shade 6 small squares in a 4 x 10 rectangle. Using the rectangle, explain how to determine each of the following: a) the percent of area that is shaded, b) the decimal part of area that is shaded, and c) the fractional part of area that is shaded.

Task J

Task K

Postal rates have been figured by the ounce since July 1, 1885. From that date until January 1, 1995, the rates were:

Nov. 3, 1917	3 cents	Dec. 31, 1975	13 cents
July 1, 1919	2 cents	May 29, 1978	15 cents
July 6, 1932	3 cents	March 22, 1981	18 cents
Aug. 1, 1953	4 cents	Nov. 1, 1981	20 cents
Jan. 7, 1963	5 cents	Feb. 17, 1985	22 cents
Jan. 7, 1968	6 cents	April 3, 1988	25 cents
May 16, 1971	8 cents	Feb. 3, 1991	29 cents
March 2, 1974	10 cents	Jan. 1, 1995	32 cents

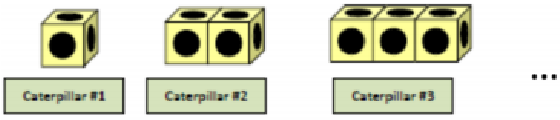
Based on the data, predict the cost of mailing a one ounce first class letter in 2010. Explain your reasoning.

Task L

C. CLINICAL INTERVIEW QUESTIONS FOR THE CONCEPT IMAGE AND MEANING FOR SLOPE

1. Eğim nedir?
2. Eğim nasıl temsil edilmektedir?
3. Eğim formülündeki bölme işaretinin anlamı nedir?

One day my little niece saw a clump of wriggling spotted caterpillars on the branch of a tree. Later she made her own collection of caterpillars with linking blocks and stickers. The first caterpillar was made with 1 block and 6 stickers. The second caterpillar was made with 2 blocks and 10 stickers. She continued to add to her collection:



i) How many stickers are needed for the next caterpillar's spots (Caterpillar #4)?

ii) How many stickers are needed for Caterpillar #7?

iii) How many stickers are needed for Caterpillar #17?

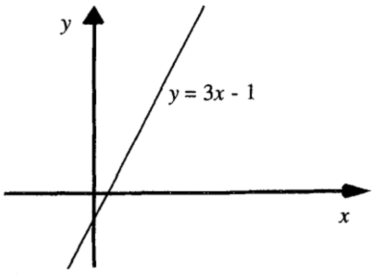
iv) For any caterpillar number you are given, how do you find the total number of stickers needed for its spots?

(The task taken from Wilkie, 2019, p. 24)

Önce bu görevi çözün ve ardından aşağıdaki alt soruları yanıtlayın.

1. Öğrencilerin bu soruya verdikleri tipik cevaplar neler olabilir?
2. Öğrencilerden gelen diğer olası yanıtlar neler olabilir?
3. Eğer öğrencilerden bazıları şöyle yanıtlasaydı: "Tırtıl sayısı 4 ile çarpılır ve ardından her birine 2 eklenir" Bundan sonra ne sorarsınız?
4. Bu durumun cebirsel ifadesi düşünüldüğünde 4 ne anlama gelmektedir?

4.
5.



a. x 1 arttığında y'deki değişim ne olur?

b. x a'dan a+h'a değiştiğinde y deki değişiklik için ne söylenebilir?

D. INTERVIEW QUESTIONS FOR PRE-COACHING WORKSHOP ON IMPLEMENTATION OF HIGH LEVEL TASKS

VIDEO-Bir öğretmenin Uygulaması (Videoyu izledikten sonra her bir soruya tek tek cevap veriniz, videoyu dikkatli bir şekilde inceleyiniz)

Not: Aşağıdaki eklerde Smith ve Stein'ın dört kategorisi ve kriterlerini bulabilirsin hocam. Ayrıca Mathematical Task Framework' e (Eklerde) bakıp task'ın geçirdiği süreçler hakkında genel bilgiye ulaşabilirsin.

A

1. a. Sence bu videoda kullanılan task (derse başlamadan önce-task as plannedortask as selected) hangi seviyede? (Ek 1'de sunulan 4 seviyenin özelliklerini kullanarak cevaplayınız)

b. Dersin işleyiş sürecinde (task as enacted) task'ın seviyesi nedir?

2.Yukarıda değinilen her iki durum için taskın seviyesini nasıl belirledin? Bu düşünceni destekleyen kanıtlarını taskın özelinde açıklar mısın? Neden böyle düşündün?

3. Videoda en çok dikkatini çeken neydi?

4. Bunun dışında başka videoda fark ettiğin veya değinmek istediğin neler var?

B

Bu kısma geçtiğinde lütfen önceki yazdıklarını değiştirme!

1. Videoda öğrenciler neler söylüyor? Nasıl düşünüyorlar? Örnek vererek açıkla lütfen

2. Sence öğrenci düşünceleri kullandığımız seviyelere göre kaçıncı seviyede? Neden böyle düşündün?

Öğrenci düşünce örneği 1:

Öğrenci düşünce örneği 2:

…… Eğer öğrenci önceki bilgilerine dayanarak çözüyor diyorsan

Bu soruda bu bilgiyi nasıl kullanıyorlar?

3. Neden böyle düşündün? Bunun sebebi sence ne olabilir? Örnek verir misin? Öğrenci sence neden böyle demiştir/düşünmüştür/şaşırmıştır/yanılmıştır?

4. Öğretmen hakkında ne düşünüyorsun?

5. Öğrenci düşünmelerini etkileyecek neler yapıyor? Bu öğretmen sence öğrencilerin öğrenmelerine nasıl katkı sağlıyor veya sağlamıyor?

6. Hangi faktörler öğrenci öğrenmelerine pozitif veya negatif etki etmiştir? Ne gibi durumlar örneğin?

7. Sen bu dersi uygulayacak olsan neleri değiştirmek isterdin? Neden? Neler aynı kalırdı neden?

Yazılı sınıf içi diyalog örneği

1.a. Sence yazılı olarak sunulan task(lar) (as planned) hangi seviyede? (Eklerdeki 4 seviyenin özelliklerini kullanarak cevaplayınız)

b. Dersin işleniş sürecinde (task as enacted) taskın seviyesi nedir?

2. Neden bu seviyede olduğunu düşünüyorsun? Bu düşünceni destekleyen kanıtlarını taskın özelinde açıklar mısın?

3.Öğretmenin uygulamasında dikkatini ne çekti? Neden bu durum/olay senin dikkatini çekti?

4. Öğrenci düşünüşleri hakkında ne söyleyebilirsin? Öğrenci düşünüşleri ile ilgili vardığın kanıyı nasıl desteklersin? Örnek verir misin?

5. Öğrenciler neler söylüyor? Nasıl düşünüyorlar? Örnek vererek açıkla lütfen

6. Sence öğrenci düşünceleri kullandığımız seviyelere göre kaçıncı seviyede? Neden böyle düşündün?

Öğrenci düşünce örneği 1:

Öğrenci düşünce örneği 2:

……

7.Neden böyle düşündün? Bunun sebebi sence ne olabilir? Örnek verir misin? Öğrenci sence neden böyle demiştir/düşünmüştür/şaşırmıştır/yanılmıştır?

8. Öğretmen hakkında ne düşünüyorsun?

9. Öğrenci düşünmelerini etkileyecek neler yapıyor? Bu öğretmen sence öğrencilerin öğrenmelerine nasıl katkı sağlıyor veya sağlamıyor?

10. Hangi faktörler öğrenci öğrenmelerine pozitif veya negatif yönde etki etmiştir?

11. Sen bu dersi uygulayacak olsan neleri değiştirmek isterdin? Neden? Neler aynı kalırdı neden?

E. SAMPLE INTERVIEW QUESTIONS FOR PRE-OBSERVATIONS

1. Bu taska beklenen öğrenci cevapları ne olabilir?
2. Önceki öğrenmeleri düşündüğümüzde bir sonraki ders için nasıl bir görev hazırlanabilir?
3. Teknolojiyi kullanmak istersen nasıl dâhil edebilirsin?
4. Hangi sırada öğrenci cevapları verilmeli sence neden?
5. Bu göreve ilişkin örnek bir öğrenci cevabına bakalım. Bu cevapta dikkatinizi ne çekti?

F. SAMPLE INTERVIEW QUESTIONS FOR POST-OBSERVATIONS

1. Sizce ders nasıl geçti? Dersi kısaca özetleyebilir misiniz?
2. Dersin başarılı geçen kısımları nelerdir? Neden? Görev veya uygulayış daha nasıl geliştirilebilir?
3. Taskın ilk hali ile derste uygulanış biçimine göre bilişsel istem düzeyi hakkında neler söyleyebilirsin? Bilişsel istem düzeyini artıran etmenler sence neler? Örnek vererek açıklar mısın?
3. Ders esnasında dikkatini neler çekti? Bu durumun dikkat çekme sebepleri neler?
4. Dersin başarısız olan kısımları var mıydı? Neden? Daha başarılı olması için ne gibi değişiklikler yapılmalı?
5. *[Ders anlatımından bir video bölümü gösterilerek]* sence bu bölümde öğrenci anlamaları nasıl geliştirilebilirdi? Öğrencinin keşfetmesini sağlayabildin mi? Başka ne yapılabilir? … şeklinde teknoloji destekli etkinlik sence bu kavramı keşfetmesi için daha etkili olabilir mi? Neden?
6. Matematiksel düşüncenin gelişiminde görevi nasıl kullandın? Açıklayabilir misin? Nelerin değişmesini nelerin aynı kalmasını istersin?
7. *[Ders anlatımından bir video bölümü gösterilerek]* burada sence öğrencinin kavram yanılgısını giderebildin mi? Teknolojiyi başka nasıl kullanabilirdin? *[Ders anlatımından başka bir video bölümü gösterilerek]* peki burada öğrencide bir kavram yanılgısı oluşturmuş olabilir misin? Neden? Ne yapılması gerekiyordu sence? Nasık sorular sorabilirdi? Taskı değiştirmeyi düşünür müsün?*(öneriler)*
8.[*Ders anlatımından bir video bölümü gösterilerek]* bu derste öğrenciler matematiği öğretirken hangi soruları sordun? Hangi sorular işe yaradı hangileri işe yaramadı? Başka nasıl sorular sorabilirdin? Neden?
9. Sence hazırladığın bu görev dersi nasıl etkiledi? Planlanılan öğrenci cevapları alınabildi mi?
10. Bu dersi tekrardan işleyecek olsan neler aynı kalırdı, neleri değiştirirdin? Neden? (Derse eklemek ya da dersten çıkarmak istediğin bir şey var mı?) Açıklar mısın?
11. Uygulama esnasında zorlandığınız kısımlar oldu mu? Nereler?
12. Ders planına uymadığın oldu mu? Neden?
13. Derste beklenmedik bir olay ile karşılaştın mı? Olduysa bu durumu nasıl karşıladın?
14. Öğrencilerin derse tepkisi nasıldı? Beklediğin şekilde miydi?

    Bir soruda öğrencilerin çözümlerini tabloda göstermelerini istedin; sırasıyla almadın cevapları neden?
15. Problem sorusunda orandan hız yol veya musluklardan akan su hızı sence öğrencilere yeterli yönlendirme yapabildin mi? Genel toparlamayı sen

yaptığında öğrenciler bu üç tip örnek arasında farklılıkları sence anladı mı? Hangi soruların etkili oldu veya olmadı?

16. Öğrencilerin anlamalarını artırmak için sorularını başka nasıl düzenleyebilirdin? Örneğin her zaman doğru x bilinmeyen ve her zaman yanlış; değişken gibi dilin kullanılması daha iyi olabilir miydi? Neden? *(öneriler* Neden? *Öğrencileri böyle br tartışmaya yönlendirsen nasıl öğrenirlerdi? Sence etkili olur muydu?)*

G. DESCRIPTORS OF THE LEVELS OF COGNITIVE DEMAND TASKS FROM THE TASK ANALYSIS GUIDE (TAG; STEIN ET AL. 2000)

Doing Mathematics Tasks	-Require complex and non-algorithmic thinking (i.e., there is not a predictable, well-rehearsed approach or pathway explicitly suggested by the task, task instructions, or a worked-out example) -Require students to explore and to understand the nature of mathematical concepts, processes, or relationships -Demand self-monitoring or self-regulation of one's own cognitive processes -Require students to access relevant knowledge and experiences and make appropriate use of them in working through the task -Require students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions -Require considerable cognitive effort and may involve some level of anxiety for the student due to the unpredictable nature of the solution process required
Procedures with Connection	-Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas -Suggest pathways to follow (explicitly or implicitly) that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts -Usually are represented in multiple ways (e.g., visual diagrams, manipulative, symbols, problem situations). Making connections among multiple representations helps to develop meaning -Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with the conceptual ideas that underlie the procedures in order to successfully complete the task and develop understanding
Procedures without Connections	-Are algorithmic. Use of the procedure is either specifically called for or its use is evident based on prior instruction, experience, or placement of the task -Require limited cognitive demand for successful completion. There is little ambiguity about what needs to be done and how to do it -Have no connection to the concepts or meaning that underlie the procedure being used -Are focused on producing correct answers rather than developing mathematical understanding -Require no explanations, or explanations that focus solely on describing the procedure that was used
Memorization	-Involve either producing previously learned facts, rules, formulae, or definitions OR committing facts, rules, formulae, or definitions to memory -Cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure -Are not ambiguous—such tasks involve exact reproduction of previously seen material and what is to be reproduced is clearly and directly stated -Have no connection to the concepts or meaning that underlay the facts, rules, formulae, or definitions being learned or reproduced

H. APPROVAL OF THE ETHICS COMMITTEE OF METU RESEARCH CENTER FOR APPLIED ETHICS

UYGULAMALI ETİK ARAŞTIRMA MERKEZİ
APPLIED ETHICS RESEARCH CENTER

ORTA DOĞU TEKNİK ÜNİVERSİTESİ
MIDDLE EAST TECHNICAL UNIVERSITY

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02 Ocak 2020

Konu: Değerlendirme Sonucu

Gönderen: ODTÜ İnsan Araştırmaları Etik Kurulu (İAEK)

İlgi: İnsan Araştırmaları Etik Kurulu Başvurusu

Sayın Prof.Dr.Mine IŞIKSAL BOSTAN

Danışmanlığını yaptığınız **Emine Aytekin KAZANÇ'ın "Ortaokul Matematik Öğretmenlerinin Yüksek Bilişsel İstem Düzeyindeki Eğim, Denklemler ve Eşitsizlik Etkinliklerinin Hazırlama, Uygulama Süreçlerindeki Gelişimlerinin İncelenmesi"** başlıklı araştırması İnsan Araştırmaları Etik Kurulu tarafından uygun görülmüş ve **510 ODTU 2019** protokol numarası ile onaylanmıştır.

Saygılarımızla bilgilerinize sunarız.

Doç.Dr. Mine MISIRLISOY
Başkan

Prof. Dr. Tolga CAN
Üye

Doç.Dr. Pınar KAYGAN
Üye

Dr. Öğr. Üyesi Ali Emre TURGUT
Üye

Dr. Öğr. Üyesi Şerife SEVİNÇ
Üye

Dr. Öğr. Üyesi Müge GÜNDÜZ
Üye

Dr. Öğr. Üyesi Süreyya Özcan KABASAKAL
Üye

I. OFFICIAL PERMISSIONS OBTAINED FROM THE MINISTRY OF NATIONAL EDUCATION

T.C.
ANKARA VALİLİĞİ
Milli Eğitim Müdürlüğü

Sayı : 14588481-605.99-E.6353318 27.03.2019
Konu : Araştırma izni

ORTA DOĞU TEKNİK ÜNİVERSİTESİNE
(Öğrenci İşleri Daire Başkanlığı)

İlgi: a) MEB Yenilik ve Eğitim Teknolojileri Genel Müdürlüğünün 2017/25 nolu Genelgesi.
b) 28.02.2019 tarihli ve 100 sayılı yazınız.

Üniversiteniz Sosyal Bilimler Enstitüsü Doktora Öğrencisi Emine AYTEKİN' in **"Ortaokul Matematik Öğretmenlerinin Yüksek Bilişsel İstem Düzeyindeki Eğim, Denklemler ve Eşitsizlik Etkinliklerinin Hazırlama, Uygulama Süreçlerindeki Gelişimlerinin İncelenmesi"** konulu tezi kapsamında uygulama yapma talebi Müdürlüğümüzce uygun görülmüş ve uygulamanın yapılacağı İlçe Milli Eğitim Müdürlüğüne bilgi verilmiştir.

Uygulama formunun (10 sayfa) araştırmacı tarafından uygulama yapılacak sayıda çoğaltılması ve çalışmanın bitiminde bir örneğinin (cd ortamında) Müdürlüğümüz Strateji Geliştirme Şubesine gönderilmesini rica ederim.

Turan AKPINAR
Vali a.
Milli Eğitim Müdürü

Güvenli Elektronik İmzalı
Aslı İle Aynıdır.
27/03/2019

Adres: Emniyet Mah. Alparslan Türkeş Cad. 4/A Yenimahalle
Bilgi için: Emine KONUK

J. PARENT APPROVAL FORM

Sevgili Anne/Baba

Bu çalışma, Orta Doğu Teknik Üniversitesi Matematik ve Fen Bilimleri öğretim üyelerinden Prof. Dr. Mine Işıksal-Bostan'ın danışmanlığında; doktora tez öğrencisi Emine Aytekin Kazanç tarafından yürütülmektedir.

Çocuğunuzun katılımcı olarak ne yapmasını istiyoruz?: Bu amaç doğrultusunda, çocuğunuzdan Denklemler, Eğim ve Eşitsizlikler konuları ile ilgili açık uçlu soruları cevaplamasını isteyeceğiz ve cevaplarını/davranışlarını not ederek ve görüntü kaydı alarak toplayacağız. Sizden çocuğunuzun katılımcı olmasıyla ilgili izin istediğimiz gibi, çalışmaya başlamadan çocuğunuzdan da sözlü olarak katılımıyla ilgili rızası mutlaka alınacak.

Çocuğunuzdan alınan bilgiler ne amaçla ve nasıl kullanılacak?: Çocuğunuzdan alacağımız cevaplar tamamen gizli tutulacak ve sadece araştırmacılar tarafından değerlendirilecektir. Elde edilecek bilgiler sadece bilimsel amaçla (yayın, konferans sunumu, vb.) kullanılacak, çocuğunuzun ya da sizin ismi ve kimlik bilgileriniz, hiçbir şekilde kimseyle paylaşılmayacaktır.

Çocuğunuz ya da siz çalışmayı yarıda kesmek isterseniz ne yapmalısınız?: Katılım sırasında sorulan sorulardan ya da herhangi bir uygulama ile ilgili başka bir nedenden ötürü çocuğunuz kendisini rahatsız hissettiğini belirtirse, ya da kendi belirtmese de araştırmacı çocuğun rahatsız olduğunu öngörürse, çalışmaya sorular tamamlanmadan ve derhal son verilecektir.

Bu çalışmayla ilgili daha fazla bilgi almak isterseniz: Çalışmaya katılımınızın sonrasında, bu çalışmayla ilgili sorularınız yazılı biçimde cevaplandırılacaktır. Çalışma hakkında daha fazla bilgi almak için Matematik ve Fen Bilimleri Eğitimi Bölümü araştırma görevlisi Emine Aytekin ile (e-posta: ayemine@metu.edu.tr) ile iletişim kurabilirsiniz. Bu çalışmaya katılımınız için şimdiden teşekkür ederiz.

Yukarıdaki bilgileri okudum ve çocuğumun bu çalışmada yer almasını onaylıyorum (Lütfen alttaki iki seçenekten birini işaretleyiniz.

***Evet onaylıyorum*___**　　　　***Hayır, onaylamıyorum*___**

Annenin adı-soyadı: ______________　　　　Bugünün Tarihi:________________

Çocuğun adı soyadı ve doğum tarihi:________________

(Formu doldurup imzaladıktan sonra araştırmacıya ulaştırınız).

K. CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name: Aytekin Kazanç, Emine
Nationality: Turkish (TC)
Date and Place of Birth: 9 March 1989, Bor
Marital Status: Married
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email: aytekinem@gmail.com

EDUCATION

Degree	**Institution**	**Year of Graduation**
PhD	METU Elementary Education	2022
MS	Hacettepe UniversityAssessment and Evaluation in Education	2019
MS	METU Elementary Science and Mathematics Education	2015
BS	METU Elementary Math Education	2012
High School	Niğde Anatolian Teacher High School, Niğde	2007

WORK EXPERIENCE

Year	Place	Enrollment
2012-2022	Middle East Technical University, Department of Elementary Education	Research Assistant
2012 December-2012 March	Düzce University, Department of Elementary Mathematics Education	Research Assistant

FOREIGN LANGUAGES

Upper Intermediate English, Intermediate German

PUBLICATIONS

Journal Papers in International and National Indexed Journals

Aytekin, E. & Isiksal-Bostan, M. (2019). Middle school students' attitudes towards the use of technology in mathematics lessons: does gender make a difference? *International Journal of Mathematical Education in Science and Technology, 50*(5), 707-727, DOI: 10.1080/0020739X.2018.1535097.

Aytekin-Kazanç, E., Acar-Çakırca, & Işıksal-Bostan, M. (2021). 8.Sınıf öğrencilerinin eğim kavramına yönelik kavrayışları. *Cumhuriyet International Journal of Education, 10*(4), 1535-1561. http://dx.doi.org/10.30703/cije.874553

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Aytekin-Kazanç, **E.** & Işıksal-Bostan, M. (2021, July). Profesyonel Gelişim Modeli Matematik Koçluğuna İlişkin Algılar. *Proceedings of Internetional Eurasian Educational Research Congress Online (EJER, 2021)* (pp. 789-791). Anı Academy.

Aytekin Kazanç, E., Çakıroğlu, E., Sevinç, Ş., Işıksal Bostan, M., Kandil S. (2019, October). Strategies Used by First Grade Students in the Process of Solving Joing Problems. *Proceedings of 2nd International Elementary Education Congress (UTEK 2019)* (p.261), Muğla, Turkey.

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Aytekin, E., Ayan, R. Işıksal-Bostan, M. (2016, Eylül). Ortaokul Öğrencilerinin Ondalık Gösterimleri Verilen Sayılarla Çarpma Ve Bölme İşlemlerine İlişkin Kavram Yanılgıları Ve Çözüm Stratejileri. *12. Ulusal Fen Bilimleri ve Matematik Eğitimi Kongresi (12. UFBMEK) Bildiri Özetleri Kitapçığı*. (s. 268), Trabzon, Türkiye: Pegem Akademi

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Projects

Project Title: Investigation of Pre-service Teachers' Beliefs on Alternative Teaching Approaches in Science and Mathematics Lessons
Project length: 12 Month (2017-2018)
Funding institution: Middle East Technical University, Turkey (BAP-05-07-2017-002)
Role in the project: Researcher
Project coordinator: Prof. Dr. Elvan Şahin

Project Title: Developing a Learning Trajectory Based on Realistic Mathematics Education for 1st Grade Mathematics, Numbers and Operations Unit: A Design Research (First Step Project)Project length: 24 Month (2017-2019)
Funding institution: The Scientific and Technological Research Council of Turkey (1001-116K078)
Role in the project: Scholarship Holder
Project coordinator: Prof. Dr. Erdinç Çakıroğlu

Project Title: Investigation of Middle School Students' Attitudes towards the Use of Technology for Mathematics Lesson
Project length: 12 Month (2014-2015)
Funding institution: Middle East Technical University, Turkey (BAP-07-03-2014-003)
Role in the project: Researcher
Project coordinator: Prof. Dr. Mine Işıksal-Bostan

SCHOLARSHIPS, GRANTS AND AWARDS

- Scholarship of the Scientific and Technological Research Council of Turkey (TUBITAK 2210) for Master Degree, Turkey (2012-2014)

- Scholarship of the Scientific and Technological Research Council of Turkey (TUBITAK 2210) for PhD Degree, Turkey (2015-2019)

- Financial Support for CERME-11 from ERME Graham Litter Fund (2019, February)

- High Honor Roll (B.S.) – Middle East Technical University, Turkey, 2012.

L. TURKISH SUMMARY / TÜRKÇE ÖZET

BİR ÖĞRETMENİN KOÇLUK PROGRAMI VASITASIYLA ÖĞRENCİ DÜŞÜNÜŞLERİNİ İYİLEŞTİRME ADINA YÜKSEK BİLİŞSEL İSTEM DÜZEYİNDEKİ GÖREVLERİ UYGULAMAYA YÖNELİK ÖĞRENİMİ

GİRİŞ

Matematiksel görev, öğrencilerin önceden belirlenmiş bir matematiksel fikre ulaşmasını sağlayan herhangi bir matematiksel aktivite olarak tanımlanmaktadır (Stein, Grover ve Henningsen, 1996). Matematiksel görevlerin farklı doğası ve değişik düzeydeki bilişsel istemine (Stein, Smith, Henningsen ve Silver, 2000) dayalı olarak öğrencilerin bu görevlerle çalışırken, matematiksel düşünüşlerinde de farklılıklar görülmektedir. Görevlerin bilişsel istemi, öğrencilerin görevler üzerinde çalışırken katılmaları gereken "bilişsel süreçler" olarak tanımlanmaktadır (Doyle, 1988, s.170). Bilişsel süreçleri Görev Analiz Rehberi [Task Analysis Guide] (Stein ve Lane, 1996; Stein ve Smith, 1998), üç ana kategoride sınıflandırmaktadır: Düşük bilişsel istem seviyeli, Yüksek bilişsel istem seviyeli ve Sistematik olmayan keşifler. Ayrıca her bir ana ve alt kategorilerin özellikleri ayrıntılı olarak sunulmaktadır. Düşük bilişsel istem seviyesi, *ezber* görevleri ve *ilişkilendirmeye dayanmayan yöntem* görevleri olmak üzere iki alt seviyeye ayrılmaktadır. Yüksek bilişsel istem seviyesi ise *ilişkilendirmeye dayalı yöntem* görevleri ve *matematik yapma* görevlerinden oluşmaktadır. Matematiksel kuralları, gerçekleri ve tanımları hatırlama ve matematik prosedürleri uygulanması ile ilişkili olan düşük bilişsel istem seviyesindeki görevlerle karşılaştırıldığında, yüksek bilişsel istem seviyesindeki görevler; öğrencilerin çeşitli çözümler ve hipotezler üretmelerine, çözümlerini test etmelerine ve doğrulamalarına ve daha önce öğrenilen matematiksel fikir, konu ve kavramları birbirine bağlamalarına olanak tanımaktadır (ör, Boaler ve Staples, 2008). Üçüncü kategori, sistematik olmayan keşif (Stein ve Lane, 1996), daha üst düzey düşünme potansiyeline sahip olabilecek bir göreve atıfta bulunmaktadır. Ancak öğrenciler, kavramın anlaşılmasının engellenmesine yol açan sistematik olmayan bir yaklaşım geliştirerek görevle çalışırlar.

Matematik eğitimcileri ve öğretmenlerinin bir organizasyonu olan Amerikan Ulusal Matematik Öğretmenleri Konseyi [National Council of Teachers of Mathematics (NCTM)] öğretmenlere "değerli matematiksel görevleri" (1991, s. 25) seçme ve uygulama konusundaki tavsiyeleri, yüksek bilişsel istem seviyesindeki görevleri seçmenin ve uygulamanın önemini vurgulamaktadır. Çünkü üst bilişsel istemdeki görevler, öğrencilerin dikkatini görevlere çekmek ve onları matematiksel fikirler üretmeye zorlayan matematik yapmak için birer geçit olarak düşünülebilir. Bu doğrultuda Türkiye'de Milli Eğitim Bakanlığı (MEB, 2018) matematik öğretmenlerinin öğretme ve öğrenmede "görevlerin aracılık rolü" ne dayalı olarak öğrencilerin matematiği yapan kişiler olmaları için yüksek bilişsel gereksinime sahip matematiksel görevleri kullanmalarını önermiştir (Johnson, Coles ve Clarke, 2017, s.815). Daha önceki çalışmaların da ortaya koyduğu gibi, değerli görevlerin kullanılması öğrencilerin kavramsal anlamalarını ve başarılarını önemli ölçüde etkilemektedir (Tarr, Reys, Reys, Chavez, Shih ve Osterlind, 2008).

Ancak çalışmalar (Arbaugh ve Brown, 2002; González ve Eli, 2015; Graven ve Coles, 2017; Jackson, Garrison, Wilson, Gibbons ve Shahan, 2013; Lozano, 2017; Silver, Mesa, Morris, Star, ve Benken, 2009; Sullivan, Clarke, Clarke ve O'Shea, 2010; Ubuz ve Sarpkaya, 2014) öğretmenlerin görev doğasını tanımada ve yüksek bilişsel istem seviyesindeki görevleri uygulamada karşılaştıkları zorluklara işaret etmektedir. Daha özelde, öğretmenlerin tipik olarak görevlerin gerçek yaşam bağlamı, teknoloji, şekil ve temsiller içerip içermediği gibi görevlerin yüzeysel özelliklerine göre görevleri kategorize ettikleri görülmüştür. Ayrıca öğretmenlerin matematiksel içeriğe veya konusuna, görevin uzunluğuna ve öğrencilere göre görevin zorluğu veya kolaylığına göre görevleri sınıflandırmaya eğilimli oldukları belirlenmiştir. Böylece, bulgular öğretmenlerin görevleri öğrencilerin matematiksel düşünmeleriyle ilişkilendirmediğini göstermektedir (Arbaugh ve

Brown, 2005; Osana, Lacroix, Tucker ve Desrosiers, 2006; Tekkumru-Kısa, Stein ve Doyle 2020). Görev Analiz Rehberi (TAG) gibi bir kılavuz sağlamak ve görevleri sınıflandırırken öğretmenlerin bu kılavuza atıfta bulunmalarını istemek, öğretmenlerin görevlerle ilgili kapasitesini ve bilgilerini artırmak için etkili bir stratejidir (Arbaugh ve Brown, 2005; Boston, 2013; Boston ve Smith, 2009; Boston ve Smith, 2011; Estrella, Zakaryan, Olfos ve Espinoza, 2020). Diğer bir strateji ise öğretmenlere yüksek bilişsel istem düzeyinde görevler sunmaktır (ör., Guberman ve Leikin, 2013).

Ayrıca, öğretmenlerin yüksek bilişsel istem düzeyindeki görevlerin özellikleri belirlemeleri ve bu görevlerin nasıl uygulanabileceğini detaylandırmaları, örnek öğrenci çalışmalarını analiz etmeleri ve sınıf ortamında uygulamaları gibi etkinlikler, diğer önemli gelişim faaliyetlerini oluşturmaktadır. Bu mesleki gelişim faaliyetleri sayesinde, öğretmenler yüksek istem düzey görevlerin üst düzey öğrenci anlayışına yol açtığının farkına varmakta ve bu özellikleri olan görevleri daha iyi planlayıp uygulayabilmektedirler (Boston ve Smith, 2009; Parrish, Snider ve Creager, 2022). Bu çalışmalarda bahsi geçen faaliyetler öğretmenlerin üst düzey görevlerin planlanması ve uygulanmasına dair gelişimleri hakkında önemli çıktılar sağlamış olmasına rağmen, öğretmenlerin görevleri gerçek sınıfta sürekli uygulamalarına ilişkin tutarlı bir analizden yoksundur (Boston, 2013). Dolayısıyla, bu çalışmalar, her bir uygulamadan sonra sonraki dersin görevinin bilişsel istem düzeyine ilişkin öğretmenlerin bilgileri hakkında güçlü kanıtlar sağlayamamaktadır. Bu nedenle, bu çalışma, gerçek bir sınıf ortamında görevleri planlama ve uygulamayı içeren bir koçluk programı ile bir öğretmenin matematiksel görevlerin bilişsel istem düzeyine ilişkin bilgisinin nasıl değiştiğini araştırmayı amaçlamıştır.

Ek olarak, belirli bir matematiksel fikir veya konuya odaklanmadan herhangi bir konu ve kavramı içeren matematiksel görevler ile ilgili öğretmenlerin yeterliği veya gelişimi incelenmiştir (ör., Chrambalous, 2010; Choppin, 2011; Wilhelm, 2014). Ancak, Chrambalous (2010), belirli bir içerikteki matematik görevler hakkındaki öğretmen bilgisinin incelenmesini de önermiştir. Bu öneri temelinde, öğretmenlerin matematik görevlerinin bilişsel istem düzeylerine yönelik bilgi ve yeterliklerinin araştırılabilmesi için bu çalışmada eğim kavramı seçilmiştir. Nitekim eğim, diğer kavram ve disiplinlerle karmaşık bir şekilde bağlantılıdır (Peck, 2020). Bu nedenle öğretmenler ve öğrenciler eğimin çeşitli kavramlaştırmalarını tanımakta zorlanmaktadırlar (ör., Byerley ve Thompson, 2017; Lobato, Ellis ve Muñoz, 2003; Reiken, 2009; Stump, 2001; Thompson, 1994; Thompson vd., 2017; Wilkie, 2016; Zazlavsky vd., 2002) ve öğretmenler yüksek bilişsel istem düzeyindeki cebir görevlerini seçme ve uygulamada zorlukla karşılaşmaktadırlar (ör., Magiera, van den Kieboom ve Moyer, 2013; Nagle, Moore-Russo ve Styers, 2017; Rule ve Hallagan, 2007; Steele, Hillen ve Smith, 2013; Vlassis ve Fagnant, 2018; Warren, 2006; Wilkie, 2016). Bu noktada, eğim, matematiksel görev bağlamında aydınlatılması gereken hayati bir kavram olarak kabul edilebilir. Ayrıca, öğrencilerin belirli bir matematik kavramını (eğim) anlamlandırırken yaşadıkları zorluklarını fark etmek, yüksek bilişsel istem düzeyindeki matematik görevleri planlamak/seçmek ve uygulamak için bir anahtardır (Choppin, 2011).

Bu çalışma, Doyle'un (1988) matematiksel görevlerin bilişsel istem kavramına ve deneyimli bir öğretmenin matematik görevi planlarken, uygularken ve yansıtırken fark etmelerinin nasıl değiştiğini açıklayan van Es'in (2011) "fark etmeyi öğrenme" kavramı üzerine inşa edilmiştir. Van Es (2011) fark etmeyi, temelde iki boyutta değerlendirmektedir: *Öğretmenler ne fark eder* ve *Öğretmenler nasıl fark eder*. Her iki kategori (*Ne Fark Etti ve Nasıl Fark Etti*) dört alt düzeye ayrılarak öğretmenin zaman içerisindeki gelişimini ortaya koymaktadır: Düzey 1-Temel, Düzey 2-Karma, Düzey 3-Odaklanmış ve Düzey 4-Genişletilmiş. İlk boyutun, öğretmenlerin neyi fark ettiği (örneğin bir bütün olarak sınıf, bir grup olarak öğrenciler, belirli öğrenciler, öğretmen davranışları veya kendileri) ve odak konusu (örneğin pedagojik stratejiler, davranış veya düşünme) ile ilgili olduğu söylenebilir. İkinci boyut ise hem *analitik yaklaşım* [analytical stance] (değerlendirme ve yorumlama) hem de *analizin derinliği* [depth of analysis] (kanıt sağlama veya detaylandırma) dâhil olmak üzere, öğretmenlerin fark ettiklerini nasıl yorumladıkları ile ilişkilidir. Analitik yaklaşım tanımlama, değerlendirme ve yorumlama olarak üçe ayrılmaktadır. Tanımlama, gerçekleşen olayların tasvir edilmesini kapsamaktadır. Değerlendirme, öğretmenin bu olaylara dair sunduğu yargıları içermektedir. Yorumlama ise öğretmenin gözlemlerinden elde ettiği çıkarımlarını gerekçelendirme amacıyla derin açıklamalardan oluşmaktadır. Analizin derinliği ise öğretmenin düşünüşlerini gerekçelendirme için kanıtlar sunması veya sunmayarak genel gözlemlerinden bahsetmesine odaklanmaktadır.

Öte yandan Jacobs ve meslektaşları (2010), öğretmenlerin neyi farketttiklerine kıyasla, belirli öğrenci matematiksel düşünüşlerini nasıl ve ne derecede fark ettiklerine daha fazla dikkat çekerek bir çerçeve geliştirmişlerdir. Başka bir deyişle, Jacobs ve meslektaşları tarafından geliştirilen çerçevenin odak noktası özellikle öğrenci matematiksel düşünüşleri üzerine iken van Es'in teorik çerçevesi, öğretmenlerin fark ettiklerinin çeşitliliğine ve fark ettiklerini nasıl anlamlandırdıklarına eylemden sonra yansıtma (reflection on action) perspektifi ile odaklanmaktadır. Yüksek bilişsel istem seviyesindeki eğim görevleri bağlamında, mevcut çalışmanın amaçlarından diğeri ise, öğretmenin nelere katıldığını ve katıldığı konuları nasıl anlamlandıracağını sadece eyleme yansıtma değil, aynı zamanda eylem ve planlamaya yansıtma yoluyla incelemektir. Bu anlamda, fark etmenin odağı, öğrencilerin düşünüşü ile birlikte önemli sınıf durumlarına katılma, derslerin devamı hakkında karar verme ve alternatif öğrenci-öğretmen etkileşimleri de dâhil olmak üzere dersi oluşturan bileşenleri içermektedir. Bu nedenle bu çalışmada hem öğretmenin öğretim sırasında neleri ve nasıl fark ettiği ve öğretmenin öğretimden sonra belirli öğrenci düşünüşlerindeki fark etme becerisi işaret edilmiştir. Bu anlamda, Jacobs ve diğerleri'nin (2010) öğrenci düşünüşlerine özel bir vurguya dayanan fark etme tanımı ve van Es'in çerçevesindeki iki temel boyut (*öğretmenin neyi fark ettiği ve öğretmenin nasıl fark ettiği*) ve bu boyutlara ilişkin dört düzey bu çalışmada öğretmenin fark etme becerisindeki değişimi belirlemek için kullanılmıştır. Bununla birlikte, çalışmanın amacı doğrultusunda dersi planlama, öğretim ve ders sonrası yansıtma sırasında öğretmenin hangi unsurları fark ettiğini ve öğretmenin bilişsel olarak yüksek matematiksel görevler bağlamında bunları nasıl anlamlandırdığını anlamak için çerçevede bazı uyarlamalar kullanılmıştır.

Özellikle öğretim esnasında karar verme becerisi, karmaşık sınıf ortamında öğretmenlerin anlık ve etkili karar verebilmeleri deneyimli öğretmenler için bile (Lee ve Choy, 2017) en zorlayıcı beceridir (Choy, 2016; Lee ve Francis, 2018; Stahnke, Schueler ve Roesken-Winter, 2016). Öğrencilerin düşünce kalıplarının farkında olmak ve bu öğrenci düşüncelerinden öğretim için uygun ve önemli olanları seçmek ve yüksek bilişsel istem düzeyindeki görevleri yerine getirirken uygun pedagojik davranışlarda bulunmak önemlidir (Van Zoest vd., 2017). Bu noktada doğası gereği koçluk programı, öğretmenlerin kaliteli öğretim uygulamalarında bulunmalarını sağlayabilecek mesleki gelişim modellerinden biridir (Kraft, Blazar ve Hogan, 2018; Sailors ve Price, 2015).

Son yıllarda koçluk programı, öğretmenlerin profesyonel gelişimi için umut verici bir alan olarak ortaya çıkmaktadır (Ellington vd., 2017; Knapp, Moore ve Barrett, 2014; Kraft, Blazar ve Hogan, 2018; Sailors ve Price, 2015; Yopp, Burroughs , Sutton ve Greenwood, 2017). Genel olarak, koçluk saha temelli, sürekli, bireyselleştirilmiş, yoğun, bağlama özgü ve odaklıdır (Kraft, Blazar ve Hogan, 2018) ve üç ana aşamadan oluşan döngüsel bir süreci içerir: gözlem öncesi (planlama), gözlem (öğretim) ve gözlem (yansıtma) (McGatha vd., 2018; Russell, Correnti, Stein, Thomas, Bill ve Speranzo, 2019). Araştırmalar, koçların öğretmen gelişimini desteklerken uyguladığı çeşitli etkinlikleri/veya stratejilerin etkililiğini incelemişlerdir (Aygün, 2016; Gibbons & Cobb, 2016, 2017; Ellighton vd., 2017; Gibbons, Kazemi & Lewis, 2017; Hopkins, Ozimek & Sweet , 2017; Mudzimiri vd., 2014; Munson 2017; Neuberger, 2012, Polly 2012). Bu çalışmalar, öğretmenlerin öğretime ilişkin bilgi, inanışlarındaki değişime nispeten öğretim uygulamalarındaki değişime daha az odaklanıldığı görülmektedir. Benzer şekilde araştırmacılar "koçların öğretmenlerle birebir kendi sınıf ortamlarında nasıl çalışabileceği" ve etkili koçluk özellikleri ve uygulamaları üzerine araştırmaların sınırlı olduğu belirtilmektedir (Cobb ve Jackson, 2011, s. 19; Gibbons ve Cobb, 2017). Bu nedenle, yüksek bilişsel istem düzeyindeki matematik görevler bağlamında fark etmenin üçlü doğası (Choy vd., 2017; Amador vd., 2017; Bakker vd., 2022) ve koçluk döngüsünün tekrarlayan üç bileşenine (Russell vd., 2020) dayalı olarak, bir öğretmenin öğrenmesini zenginleştirmek için başarılı koçluk stratejilerinin ve etkinliklerinin nasıl yürütüldüğünün ve öğretmen öğrenmesine etkisinin araştırılması önerilmektedir. Bu anlamda, bu çalışmanın amaçları çok yönlüdür: (1) deneyimli bir ortaokul matematik öğretmenin bir koçluk programına katılımı yoluyla matematiksel görevlerin bilişsel istem düzeylerine ilişkin bilgisindeki değişiklikleri incelemek, ve (2) bir öğretmenin sekizinci sınıf öğrencilerinin cebirsel düşünme şekillerini, özellikle farklı eğim kavramlarını fark etme becerisinin, yüksek bilişsel istem düzeyindeki matematik görev kapsamında tekrarlanan koçluk döngüleri aracılığıyla nasıl geliştiğini belirlemek. Bu doğrultuda, çalışma aşağıdaki araştırma sorularını ele almayı amaçlamıştır:

1. Matematiksel görevleri seçme/uyarlama konusunda bir koçluk programına katılmasının ardından, bir ortaokul matematik öğretmenin matematiksel görevlerin bilişsel taleplerine ilişkin bilgisi ne şekilde değişir?
2. Bir ortaokul öğretmenin 8. sınıf öğrencilerinin cebirsel düşünme şekillerini, özellikle eğim kavramını fark etme becerisi, yüksek bilişsel istem düzeyindeki matematik görev kapsamında koçluk döngüleri aracılığıyla nasıl gelişir?

Araştırmanın Önemi

Araştırmalar, mesleki gelişimin öğretmenlerin yüksek bilişsel istem seviyesindeki görevleri seçmesi ve uygulaması üzerindeki önemli etkisini göstermektedir (Boston & Smith, 2009). Ancak bulgular bazı öğretmenlerin, mesleki gelişim sonuna gelindiğinde dahi *İlişkilendirmeye Dayalı Yöntem* görevlerini tanımakta zorluk çekmeye devam ettilerini göstermiştir; bu nedenle araştırmacılar, öğretmenlerin üst düzey görevlerin seçimini ve uygulanmasını zenginleştirmek için öğretmenlerin kendi uygulamalarına ilişkin daha fazla kanıt toplama sürecine dahil oldukları yeni tasarımlar önermektedir. Bu nedenle, öğretmenlerin yüksek bilişsel istem seviyesindeki görevleri seçme ve uygulama sırasında öğrenci düşünüşleri ve görevler arasındaki etkileşimleri anlamlandırmayı öğrenmelerini desteklemek için, öğretmenlerin gerçek sınıf ortamında gözlemlenmelerine ve desteklenmelerine ihtiyaç vardır. Böyle bir girişim, etkinliklerin öğretmenlerin yüksek bilişsel istem seviyesindeki görevleri fark etmelerini ve bu görevlerin seviyesini koruyarak sınıf ortamında uygulamalarını nasıl sağladığı yönünde alan yazın katkı sağlayabilir. Bu nedenle, öğretmenlerin matematiksel görevlerin bilişsel istem düzeylerine ilişkin bilgisi, bu araştırmanın kritik bir bölümünü oluşturmaktadır. Ayrıca “görevlere odaklanmak, öğrencinin düşünüşlerini fark etmeyi nasıl destekleneceğini anlamamıza yardımcı olur” (Tekkumru-Kısa, Stein ve Doyle, 2020, s.3). Bu bağlamda, öğretmenin fark etmesi, öğrencilerin düşünüşleri (van Es, 2009) ve görevler de dahil olmak üzere öğretimin temel yönlerine katılmaya ve bunları detaylandırmaya dayanan kavramsal kavramlardan biridir. Fark etme ve mesleki gelişim ile ilgili alan yazın, öğretimin önemli yönlerini vurgularken, görevlere dikkat edilmemektedir (Santagata ve diğerleri, 2021; Tekkumru-Kısa, Stein ve Doyle, 2020). Görevlere dikkat etmemek, araştırmacıların detaylı öğrencilerin düşünüşlerini göz ardı etmesine neden olabilmekte ve öğrenci düşünüşlerini ilerletme fırsatları kaybolabilmektedir. Dolayısıyla bu çalışma, matematik görevler bağlamında bir öğretmenin öğrencilerinin düşünüşlerini fark etmesini inceleyerek alan yazınına katkı sağlamayı amaçlamaktadır. Fark etme becerisinin önemine rağmen (Jacobs ve Spangler, 2017), deneyimli öğretmenler, yorumlarını öğretimin genel yönlerinden öğrencilerin düşünme ve pedagojisinin çok daha özel yönlerine (Bonaiuti, Santagata ve Vivanet, 2020) değiştirmede ve yanıt vermede sorunlar yaşamaktadırlar (Lee ve Choy, 2017). Ayrıca, deneyimli öğretmenlerin öğrencilerin cebirsel muhakemelerine ilişkin farkındalıklarının sınırlı olduğunu da ortaya koymaktadır (Coe, 2007; Styers, Nagle ve Moore-Russo, 2020). Bu nedenle, alan, öğretmenlerin öğrencilerin eğim düşünüşlerini (La Rochelle vd., 2019) bir mesleki gelişim bağlamında güçlü bir şekilde fark etme derecesinin belgelenmesinden yoksundur. Bu nedenle, bu çalışmada, deneyimli bir öğretmenin matematik cebirsel görevler bağlamında fark etme becerisinin bir mesleki gelişim programı aracılığıyla desteklemeyi amaçlamaktadır.

Mevcut çalışma, fark etmenin sınırlarını genişleterek ve *Fark ederek Öğrenme Çerçevesini* (van Es, 2011) matematiksel görevler kapsamında bir koçluk programı bağlamında uyarlayarak fark etmeyi ortaya çıkarmak ve analiz etmek için farklı bir metodolojik yaklaşım içermektedir. Fark etme çalışmalarının birçoğunda, fark etme becerilerini incelemek adına öğretmenlerden öğrencilerin yazılı çalışmaları, öğretim videoları veya video klipsleri üzerinde düşüncelerini yansıtmalarını istedikleri görülmektedir. Diğer bir deyişle, bu çalışmalarda geriye dönük veya olay sonrasını yansıtan durumlar ele alınmıştır. Etkili öğretim için üretken fark etme becerilerinin önemli rolü (Spitzer vd., 2011) göz önüne alındığında, bu çalışmada, fark etme becerisinin sınırları derse ilişkin yansıtmaya ek olarak ders planlama (Amador vd., 2017) ve ders uygulamaları (Choy, 2017) olarak genişletilmiştir. Bu doğrultuda van Es'in teorik çerçevesinin iki temel boyutu olan *öğretmen ne fark etti ve nasıl fark ett*i boyutları uyarlanmıştır. Benzer şekilde bu çerçevede, iki boyuta ilişkin düzeylerin özellikleri genel ve bütüncül olarak betimlendiği ve eylem (ders) esnasında yansıtmadan ziyade eylem sonrası yansımaya odaklanıldığı görülmektedir. Ayrıca, çalışmanın içeriğini matematik görevleri ve koçluk mesleki gelişim programı oluşturmasından dolayı bu bağlamlar uyarlanmış teorik çerçeveye yerleştirilmiştir. Bu değişikliklerle birlikte çerçeve, bir öğretmenin fark etmesini hem araştırmacının hem de katılımcıların bakış açılarından değerlendirmeyi sağlamakta ve matematik öğretmeninin dersi planlarken, öğretim yaparken ve ders sonrası düşüncelerini belirtirken neye dikkat ettiklerini ve bunları nasıl yorumladıklarını içermektedir. Böylece bu çalışmanın, tekrarlı bir şekilde bu aşamaları barındıran koçluk programı sürecindeki öğretmenin fark etme becerilerini detaylı bir şekilde ortaya koyarak alana katkı sağlayacağı düşünülmektedir. Ayrıca yeni uyarlanan çerçeve, profesyonel mesleki gelişimle ilgilinen araştırmacılara, öğretmenlerin koçluk ve yüksek bilişsel istem düzeyindeki görevler bağlamında fark etme becerilerini değerlendirme konusunda rehberlik edebileceğine inanılmaktadır.

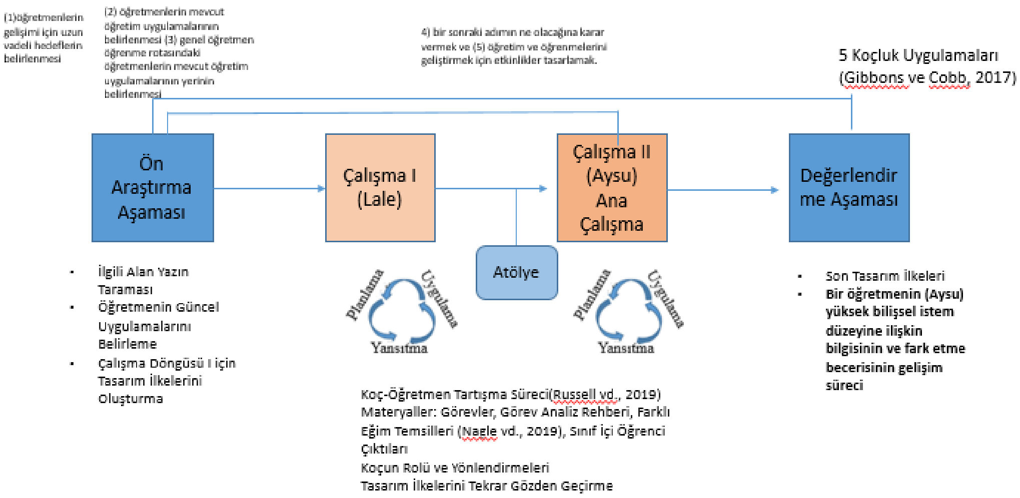
Her ne kadar çalışmalar, koçluk programının öğretmenin uygulamalarında iyileşme, inanışlarının değişmesi yönünde katkılar sağlandığını gösterse de, öğretmenlerin gerçek sınıf ortamlarında uygulamalarını önemli ölçüde gözlemleyerek koçluk programının etkisi üzerine derinlemesine inceleme ihtiyacının altını çizmektedir (Auletto ve Stein, 2020; Gibbons ve Cobb, 2017). Bu çalışmanın bağlamı olan matematiksel görevleri odağına alan Russell ve meslektaşlarının (2019) koçluğa dair modelleri temel alınmıştır. Böylece koçluk etkinlikleri geliştirilirken belli bir teorik çerçevenin kullanılması koçluk aktivitelerin nasıl geliştirildiğine dair teorik ve pratik altyapı sunacağı öngörülmüştür.

YÖNTEM

Araştırmanın amacı, araştırmacıların ve bir öğretmenin teori ve pratiği birleştirerek yüksek bilişsel istem düzeyindeki eğim görevlerini seçme ve uygulama üzerinde çalıştıkları işbirlikçi bir ortam yaratmayı gerektiren koçluk programının ilkeleriyle uyumludur. Bu nedenle, çalışma deseni olarak öğretim deneyi, bir öğretmenin görevlerin bilişsel istem düzeylerine ilişkin bilgisinin ve bu görevleri planlarken, uygularken ve üzerinde düşünürken öğrenci düşünüşlerine ilişkin fark etme becerisinin gelişim sürecini incelemek adına benimsenmiştir. Çünkü öğretim deneyleri öğrencilerin [öğrenenlerin] kavrayışlarının başlangıç ve sondaki durumlarının karşılaştırılması ile birlikte süreç içindeki öğrenmeyi nasıl yapılandırdıkları ve geliştirdikleriyle de ilgilenir (Steffe ve Thompson, 2000; Steffe ve Ulrich, 2014).

Her öğretim deneyininin ileriye dönük (prospective component) ve yansıtıcı bileşenleri (reflective component) bulunmaktadır (Steffe ve Thompson, 2000). Öngörülen öğrenmeyi (ileriye dönük bileşen) uygularken, araştırmacılar varsayımlarını gerçek öğrenme (yansıtıcı bileşen) ile test ederler. Bu yansıtıcı analiz, araştırmacılara yeni hipotezler oluşturma, bunları çürütme veya değiştirme konusunda rehberlik etmektedir (Bakker, 2018). Öğretim deneyi bir dersten uzun olsa bile her dersten sonra yansıtma yapılabilir. Bu tür yansıtıcı analiz, bir sonraki ders için orijinal planda değişikliklere yol açabilir. Koçluk, her bir ders veya birkaç dersi planlama, uygulama ve ders/dersler sonrası yansıtma aşamalarını içeren döngüsel bir doğaya sahiptir. Bu anlamda, mevcut çalışmada, haftada üç kez tekrarlanan iki saatlik derslerin planlanması, derslerin uygulanması ve ders sonrası yansıtılması aşamasındaki yapılan analizler, öğretmen öğrenmesine ilişkin çalışma ekibinin varsayımlarını çürütmek ve yenilerini oluşturmak için yol gösterebilir. Böyle bir analizin bulguları çoğunlukla yeni bir döngüyü (Bakker,

2018) veya çoklu mikro döngülerin ve makro döngünün toplu analizini (Çalışma 1) ayrıca öğretim deneyi ve sonraki makro döngüyü (Çalışma 2) bilgilendirir. Sonuç olarak, yüksek bilişsel istem düzeyinde matematiksel görevler bağlamında öğretmenin fark etme becerisini geliştirmek için planlama, öğretme ve yansıtma aşamalarından oluşan bir koçluk modeli tasarlanmış ve bir öğretim deneyi olarak kullanılmıştır. Detaylı olarak bakıldığında mevcut çalışma öğretim deneyinin üç aşamasını (ön aşama, uygulama ve değerlendirme aşaması) da içeren iki ana makro döngüden oluşmaktadır. Çalışmanın tasarımı, öğretim deneyi aşamaları ve koçluk uygulamaları bileşenlerine göre Şekil 1'de sunulmuştur.



Figür 1. Çalışmanın Genel Tasarımı

Ön Araştırma Aşaması

Öğretim deneyinin hazırlanma aşaması, görevlerin istem düzeyine yönelik öğretmen bilgisi gelişimine yönelik, eğimin öğretimi ve öğrenimine ilişkin alan yazın taramasını ve cebir ve eğim özelinde görevleri içeren task bankasının oluşturulmasını içermektedir. Ayrıca öğretmenin eğimin temsilleri ve anlamına ilişkin sahip olduğu kavram imajlarını ve anlamlandırmasını ve görevlerin sınıflandırılmasına ilişkin muhakemelerini saptamak amacıyla klinik görüşmeler yapılmıştır (Koichu ve Harel, 2007). Ek olarak, görüşmede mevcut cebir ve eğim öğretimine ilişkin sorular yöneltilmiştir. Klinik görüşmelerdeki sorular alanyazındaki eğimin farklı bir bağlamda yorumlanmasını gerektiren görev ve eğimin diğer anlamları ve temsilleri arasındaki bağlantıyı sorgulayan görevlerle ölçülmüştür (bknz. Ek C). Bu bağlamda görüşme soruları hem öğretmenin alan bilgisini, öğrenci düşünüşleriyle ilgili bilgisini öğrenmeye hem de öğretimsel yaklaşımlarını sorgulamaya yönelik tasarlanmıştır.

İlgili alan yazın önderliğinde ve öğretmenin mevcut eğim ve cebir öğretiminine ilişkin eksiklikleri göz önüne alındığında, son olarak koçluk uygulamasına ilişkin tasarım ilkeleri oluşturulmuştur. Bu bağlamda, matematik görevlerini seçme, uygulama veya değiştirme veya adapte etme aşamalarında öğretmenin yüksek bilişsel istem düzeyini düşürmemesi için koçluk uygulamalarına dair ilkeler geliştirilmiştir. Bu ilkeleri geliştirirken koçluk aktivitelerinden biri olan tekrarlanan planla-uygula-yansıtma döngüsü temel alınmıştır. Bu ilkelerin değişimi ve son hali çalışmanın diğer aşamalarında detaylı olarak ele alınmıştır.

Katılımcılar

Bu çalışma büyük bir çalışmanın parçası olup, toplamda iki deneyimli kadın 8. Sınıf matematik öğretmeni ile çalışılmıştır. Fakat bu çalışmanın amacına parallel olarak bir öğretmenin (Aysu) öğrenme çıktısı raporlaştırılmıştır. Çalışmanın katılımcılarını belirlemek için amaçlı örnekleme yöntemlerinden ölçüt örnekleme kullanılmıştır. Bu yöntem, "araştırmacının keşfetmek, anlamak ve içgörü kazanmak istediği ve bu nedenle en çok öğrenilebilecek bir örneklem seçmesi gerektiği" varsayımına dayanmaktadır (Merriam, 1998, s. 61). Bu doğrultuda, araştırma sorularına daha iyi yanıtlar bulmak için katılımcıların seçilmesine yönelik bazı kriterler belirlenmiştir. Bu kriterler şunlardır: (1) en az iki aylık bir süreyi içeren koçluk programında koçla işbirliği yapmaya istekli olmak, (2) doğrusal denklemler ünitesinin mevcut öğretiminde geleneksel ve algoritmik yöntemler kullanmak, (3) öğretime ilişkin yenilikleri öğrenme ve öğretmede hevesli olmak (4) deneyimli bir öğretmen olmak (orta okullarda 10 yıldan fazla öğretmenlik yapmak) ve (5) farklı okullarda çalışan öğretmenler ise çalıştıkları okulun ve öğrencilerinin sosyo-kültürel özelliklerinin benzer olmasına dikkat etmek.

Çalışma Döngüsü I

Çalışma döngüsü I'i öğretmen Lale ile birlikte öğretim deneyi uygulama ve bu öğretim deneyi esnasında ve geriye dönük analizler oluşturmaktadır. Bu analizler doğrultusunda, görev bankasındaki bazı görevlerin alt soruları, bağlamı veya şartları veya bazı görevlerin sırası değiştirilmiştir. Ayrıca bazı görevler de eklenerek görev bankası öğrenci ve öğretmen öğrenmelerine dair etkili bir materyal olarak ikinci uygulama için hazır hale getirilmiştir. Bu görevler Cebir Düşünme Çerçeve'sine (Walkoe, 2015) ve Eğim Kavramsallaştırma Kavramsal Çerçevesi'ne (Nagle vd., 2019) göre dört mezo döngüsüne ayrılmıştır. Her bir mezo döngüsü Cebir düşünme çerçevesinin dört alt boyutuna ve özelde son üç döngüde kullanılan görevler eğimin öğrencilerin farklı eğim kavramsallaştırmaları ile ilişkili olarak bilişsel gelişimlerinin göre sıralanmıştır. Her bir mezo döngü birden fazla tekrarlanan planlama-uygulama-yansıtma mini-döngülerini içermektedir. Buna ek olarak, öğretmen öğrenmesine ilişkin oluşturulan koçluğun doğası ve etkinliklerine ilişkin ilkelerde değişikliğe gidilmiştir. Örneğin, öğretmenin koçla birlikte planlamış oldukları yüksek bilişsel istem düzeyindeki görevleri içeren dersleri istem düzeyini düşürmeden uygulayabilecekleri çalışmalarla saptanmıştır (Boston ve Smith, 2011; Smith, 2001). Bu doğrultuda, koçun görevine ilişkin ilkelerden biri uygulama esnasında sadece gözlemleme olarak belirlenmiştir. Fakat öğretmenin ders esnasında öğrenci beklenmedik cevaplarında veya kavram yanılgılarında görevin bilişsel istem düzeyini korumak veya bu durumlardan öğretimsel olarak faydalanmakta zorluk çektiği gözlemlenmiştir. Bu nedenle ders uygulama esnasında koçun görevine gözlemlemeye ek olarak sınırlı-stratejik yardım sağlama da eklenmiştir.

Çalışma Döngüsü II

Çalışma döngüsü 2, koçluğun beş uygulamasının yanısıra koçluk teorik çerçevesi (Russell vd., 2019) kullanılarak uygulanmıştır. Koçluk Çerçevesi üç temel koçluk uygulamasını (1) öğretim üçgeninin (öğrenci düşüncesi, matematik ve pedagoji) derin ve özel tartışmaları, (2) matematiksel ve pedagojik hedeflerin oluşturulması ve (3) kanıt- tabanlı geri bildirim, sorgulayıcı duruş ve koç-öğretmen tartışma sürecini içermektedir. Russell ve meslektaşlarına göre (2019), Koç-Öğretmen tartışma süreci koç ve öğretmenin yüksek bilişsel isrtem düzeyindeki görevi çözmesi ve çözerken öğrencilerin olası düşüncelerini belirlemesiyle başlamaktadır. Sürecin sonraki aşamaları, West ve Staub'un (2003) döngüsel sürecine benzer şekilde ön gözlem (planlama), ders gözlemi (öğretim) ve konferans sonrası (ders sonrası yansıtma) aşamalardan oluşmaktadır. Mevcut çalışmada süreç, modelde vurgulanan sonraki adımlarla başlamıştır, bu nedenle matematiksel fikri belirleme ve görevi seçme/uyarlama ve öğrencilerin olası düşüncelerini belirleme aşamaları konferans öncesi aşamada gömülüdür. Temel koçluk uygulamalarından ikisi (1 ve 2) gözlem öncesi, gözlem ve gözlem sonrası aşamalarda kullanılırken, konferans sonrası aşamada ders analizi sırasında toplanan kanıta dayalı geri bildirim (3) koç tarafından kullanılmıştır. Önceki bölümde belirtilen çalışmanın ilke ve varsayımlarından hareketle koçluk uygulamalarının üç ana özelliği uygulanmıştır. İlk özellik, "örnek otantik görevler ve uygulamaları" nı temsil etmek için seçilen koçluk oturumlarında kullanılan görevlerdir (Smith, 2001, s.43). İlk öğretim deneyinde cebir ve eğim özelinde hazırlanan görev bankası öğretmene sunularak bu görevlerin çözülmesini, öğrencilerin örnek çalışmalarını analiz edilmesini, yazılı veya video formlardaki öğretim durumlarını incelenmesini ve öğretmenin kendi sınıfındaki uygulamayı yorumlanmasını içermiştir. İkinci özellik, öğretmenden görevlerin, Görev Analiz Rehberi (TAG) kullanılarak istem düzeyinin beklenen ve uygulanan arasındaki ilişkiyi kurmasının istenmesidir. Teori ve uygulamayı birleştirme adına (Tekkumru-Kısa vd., 2020) öğretmene görev bankası dışında görevler ekleyebileceği, görevlerde değişiklik yapabileceği veya yeniden oluşturabileceği koç tarafından teşvik edilmiştir. Görevlerin sunulması tek başına öğretmene etkili bir öğretim için yeterli gelmediğinden, daha özelde eğimle ilgili araştırma temelli çerçevelerden yararlanılmıştır. Nagle, Martínez-Planell ve Moore-Russo (2019) tarafından çeşitli eğim kavramsallaştırmalarında öğrencilerin eylem aşamasından nesne aşamasına kadar bilişsel gelişimlerini sunan çerçeve ve eğimin kovaryasyonel anlamlarına ilişkin örnek düşünüşleri içeren Thompson ve Byerley (2017)'in çalışması pedagojik araç olarak öğretmene verilmiştir.

Tekrarlanan koçluk döngülerinden planlama aşamasında, görüşmeler en az bir saat sürmekte olup bahsedildiği gibi öğretmen her dersin ana fikrine göre görevler seçmeye, değiştirmeye veya oluşturmaya teşvik edilmiştir. Koç bu esnada görevin bilişsel talebi, görevin bağlamı ve öğrencilerin beklenen matematiksel fikirleri arasındaki ilişkileri yorumlayabilmesi için Aysu'ya rehberlik etmiştir (Stein, Engle, Smith ve Hughes, 2008; Stein et al., 2009). Ayrıca bu planlama aşamasında öğretmenin zorlandığı kısımlarla ilişkili olarak örneğin, sınıf-içi etkili tartışma ortamı yaratabilmeyi sağlamaya yönelik olan Stein ve arkadaşlarının (2008) makalesindeki örnekler ve fikirler tartışılmıştır. Bu konferans öncesi aşama, öğretmenin neyi fark ettiği ve bunları nasıl fark ettiği ile ilgili ihtiyaçlarını ışık tutmuş ve bu işbirlikli oturumlar ile görevlerin seçimi, beklendik öğrenci cevaplarının tahlili ve bu görevlerin uygulanışı hakkında öğretmenin yorumlayıcı bir duruş sergilemesi desteklenmiştir.

Her döngüdeki bir öğretim aşamasında koç, öğretmenin derslerini iki saat boyunca gözlemlemiştir. Öğretmen, bir sekizinci sınıfta, planlama aşamasında tartışılan ve üzerinde değişiklik yapılan bir görevi uygulamıştır. Koç beklenmedik ve dikkat çeken öğrenci düşünüşleri ve öğretmenin pedagojisi hakkında notlar almış ve mümkünse öğretimin akışını kesmeden öğretmenle ne fark ettiği üzerine mini bir söyleşi yapmıştır. Ayrıca koç, her bir öğrencinin yazılı çalışmalarını toplamış ve görevlerin bilişsel istem düzeyinin korunmasında veya azalmasında olası etkili olan faktörleri ve öğretmenin öğrencilerin fikirleri arasında bağlantı kurmadaki kalitesi ve sorularının kalitesi hakkında notlar almıştır. Bu gözlem notları ve öğrenci cevapları, ders sonrası yansıtma aşamasında öğretmenle tartışabilme için materyal oluşturuken aynı zamanda öğretmen ihtiyaç ve eksikliklerini belirlemek adına araştırma ekibine sunulmuştur. Ayrıca koç, öğretim sırasında öğretmene sınırlı ve stratejik yardım sağlamıştır. Ayrıca koçluk döngüsünün ilk zamanlarında uygulama esnasında öğretmenin öğrenci düşünüşlerini sıraya koyma, bu düşünüşleri sınıf içi tartışmaya getirme gibi eylemlerde bulunmadığı görülmüş ve çalışma ekibi öğretim esnasında da öğretmenin farklı öğrenci düşünüşlerine dikkat edip; çok kısa notlar alması gerektiği önerilmiştir.

Yansıtma oturumlarında ise, öğretmenin dersle ilgili olarak ne farkettiği ve nasıl farkettiğine ilişkin fark etme becerisini ortaya çıkaracak sorular yöneltilmiştir. Ayrıca koç ile birlikte öğretmen görevin bilişsel istem düzeyi, pedagoji, belirli öğrenci düşünüş ve cevapları üzerine derinlemesine düşünmüşlerdir. Örneğin, öğrencilerin geometrik oran temsilini kullanmayı tercih etmediklerini fark ettiğini söyleyen Aysu'ya koç, bu durum için neler yapılabileceğini, bu zorluğun eğimin kavramsallaştırma sürecinde ne anlam ifade ettiği yönünde sorular yöneltmiştir. Ayrıca koç, yansıtma oturumlarından önce dersle ilgili önemli öğretimsel materyalleri (farklı öğrenci cevapları, dersin belirli bir bölümünü içeren video klips, öğretmenin spesifik pedagojisi veya sorusu) tartışmaya açması, sunulan durum için gerekçeler ve pedagojik cevaplar üretebilmesi için teşvik edici ve yönlendirici ek sorular sorması öğretmenin farketmediği veya üzerinde düşünmediği önemli konuları yeniden düşünmeye teşvik etmiştir. Özetle, bu oturumların amacı öğretmenin önceki dersi derinlemesine değerlendirmesine ve sonraki dersi bu önemli öğretimsel öğeler ışığında planlamasına destek sağlamaktır.

Değerlendirme Aşaması (Veri Analizi)

Öğretmenlerin bilgilerindeki değişiklikleri analiz etmek için (araştırma sorusu 1) öncelikle Görev Analiz Rehberi kullanılarak cevapları doğru veya yanlış olarak kodlanmıştır. Ayrıca, öğretmenin görevlerin düzeyi hakkında sunduğu gerekçe ve açıklamaları da nitel analiz yöntemleriyle analiz edilmiştir. Bu bağlamda Boston'un (2013) kodları, mevcut çalışmanın veri toplama araçları ve prosedürüne göre uyarlanmıştır. Bu kodlama üç ana bileşeni içermektedir (1) Rehberde sunulan dört kategorilerin belirli kullanımı (örn. ezber, ilişkilendirmeye dayanmayan yöntem, ilişkilendirmeye dayalı yöntem ve matematik yapma); (2) düşük ve yüksek bilişsel istem seviyelerin özelliklerini karşılaştırarak görevlerin sınıflandırılması için gerekçe sağlayan ifadelerin kullanımı (örn. "düşük seviyeli görevler diyagramlar içerir"); ve (3) öğretmen tarafından dört döngü boyunca kullanılan belirgin dil kullanımı. Birinci ve ikinci bileşenler görev düzeyine ilişkin genellemeleri veya inanışlarını TAG'da belirtilen kriterler ile ilişkilendirerek, görevlerin doğasına verdiği yanıtlara karşılık gelirken, üçüncü bileşen görevlerin planlama ve yansıtma aşamasında ortaya çıkan dili yansıtan ifadelerle ilgilidir.

Bu çalışmanın ikinci amacı doğrultusunda, her bir makro döngüdeki tekrarlayan planlama, öğretim ve yansıtma aşamalarında neyi ve nasıl fark ettiğine ilişkin fark etme düzeyleri ve unsurlarına ilişkin bir ön anlayış geliştirmek için bu çalışma için öğretim videosu ve görüşme verileri incelenmiştir. Van Es'in (2011) teorik çerçevesi kullanılarak, analiz iki bileşene ayrılmıştır (a) fark etme düzeylerini belirlemek, (b) bu örnekleri endüktif ve tümdengelimli analiz süreciyle iki geniş fark etme kategorisi (neyi ve nasıl fark edileceğini) içinde karakterize etmek. Fakat bu çalışma sadece ders sonrası yansıtma aşamasındaki fark etme becerisine değil, aynı zamanda dersi planlama ve uygulama aşaması sırasında öğretmenin fark etme becerilerinin bu çerçeveye göre nasıl geliştiğine odaklanmıştır. Bu nedenle, bu üç aşamada matematiksel görev ve koçluk bağlamında ne ve nasıl fark etti boyut ve seviyeleri revize edilmiştir. Öğretmenin-öğrenci diyalogları ve öğretmen söylem alıntıları anlamlı birimlere bölünerek analiz edilmiştir. Fakat Stockero ve van Zoest'in (2013) belirttiği gibi ders dışı yansıtma ile ders esnasındaki fark etme becerilerinin birbirinden farklı olduğuna dayanarak uygulama aşamasındaki ne fark etti boyutu öğrenci ihtiyacına göre pedagojik tepkiyi tetikleyen fark edilen (Luna ve Selmer 2021) boyutuna evrilmiştir. Bu anlamda öğretim için Stockero ve Van Zoest (2013) tarafından genişletme (extending), çelişki (contradiction), karışıklık (confusion), anlamlandırma (make sense) ve yanlış matematik (incorrect math) olmak üzere beş tür kritik öğretim anını içeren kodlama şeması verilere dayalı olarak değiştirilmiştir. Örneğin, ortaya çıkarma (eliciting), Van Es'in (2021) shaping kavramına benzer olarak öğretmenlerin bir öğrencinin düşüncesini geliştirmek yerine tüm sınıf için görünür kılma girişimlerine atıfta bulunmaktadır. Öte yandan, çelişki, genişletme, anlamlandırma ve kavramsal anlama, daha büyük bir çaba gerektirdiği için ortaya çıkarmaktan farklı olduğu görülmektedir.

Tablo 1. Fark etme becerilerinin analizinde kullanılan boyutlar

	Ne Fark Etti	Nasıl Fark Etti
Planlama	Olası Öğrenci Cevapları ve Görev [TI&CI] Belirli Öğrenci cevapları veya öğretim anı [CI &TI] Eğim Kavramsallaştırılması ve Görev [CI]	• Analitik Duruş (Tanımlayıcı, Değerlendirici, Yorumlayıcı) • Belirginlik (Genel, Özel)
Öğretim	Kısa doğru cevaplar Çelişki Ortaya Çıkarma Anlamlandırma Genişletme Belli öğrenci Kavram yanılgıları/zorluklar Koç Eylemi Kavramsal Anlama	• Davranış Biçimi (Talk moves Chapin, O'Conner ve Anderson, 2009)
Yansıtma	Tüm Sınıf Öğrenmesi[TI] Belirli Öğrenci cevapları veya öğretim anı [CI &TI] Eğim Kavramsallaştırılması ve Görev [CI]	• Analitik Duruş (Tanımlayıcı, Değerlendirici, Yorumlayıcı) • Belirginlik (Genel, Özel)

Koçluk programının planlama, öğretim ve yansıtma aşamalarından elde edilen veriler, yukarıda sunulan (özne, konu, öğrencinin ihtiyacına göre harekete geçiren an, tutum, davranış ve belirginlik) boyutlar ve kategoriler (Düzey 1, Düzey 2, Düzey3 ve Düzey 4) göz önünde bulundurularak analiz edilmiştir. Bu fark edilen durumlar koç veya öğretmen tarafından oluşturulabilir, bu nedenle [CI] kısaltması koç tarafından başlatılan fikir, [TI] ise öğretmen tarafından başlatılan bir olay veya fikre karşılık gelmektedir. Öğretmenin becerilerinin gelişimsel süreci her bir koçluk aşamasındaki planlama-öğretim ve ders sonrası yansıtma kısımları için fark etmeleri düzeyleri belirlenerek ortaya konmuştur. Araştırma verileri iki araştırmacı tarafından birbirinden bağımsız olarak incelenmiş ve Van Es (2011) tarafından oluşturulan kategoriler ve veriden gelen ek durumlar dikkate alınarak yorumlanmıştır.

BULGULAR VE TARTIŞMA

Koçluk programı sürecinde tekrarlayan birden fazla planlama-öğretim-yansıtma döngüsünü içeren dört koçluk döngüsü yürütülmüştür. Koç-öğretmen tartışma sürecinde her bir 2-saatlik dersi planlama, öğretim ve yansıtmayı içermektedir. Aysu'nun dersi planlarken görevlerin bilişsel istem düzeyine ilişkin bilgisindeki ve tekrarlanan mini-döngülerdeki fark etme becerisindeki gelişimine ilişkin bulgular aşağıda ele alınmıştır.

Veriler, Aysu'nun çalışmanın her iki aşamasında, koçluk öncesi ve sırasında düşük seviyeli görevleri belirlemede yetkin olduğunu ortaya koymuştur. Daha fazla *İlişkilendirmeye dayanmayan yöntem* görevleri koçluk porgramı öncesi çalıştayda Aysu tarafından tanınmıştır. Öte yandan koçluk programının ilk döngüsünde düşük seviyeli cebir görevlerinin seviyesini belirlemede Aysu'nun bazı zorluklar yaşadığı saptanmıştır. Öğretmenin bu sınıflandırmayı doğru yapamamasının en büyük sebeplerinden biri de görevleri yüzeysel özelliklerine göre örneğin gerçek yaşam durumu içermesi, teknoloji kullanılması sınıflandırma eğiliminden de kaynaklanmış olabilir (Arbaugh ve Brown, 2005; Parrish, 2022). Yüksek bilişsel istem düzeyindeki görevlerle (*İlişkilendirmeye dayanan yöntem ve Matematik yapma*) ilgili olarak, Aysu çalıştay sonrasında ve koçluk döngülerinde görevlerin çoğunu doğru sınıflayabildiği görülmüştür. Bu sınıflama detaylı olarak incelendiğinde ise çalıştayda yanlış olarak kodlanan sınıflandırmada, yüksek istem düzeyindeki görevlerin düşük istem düzeyi olarak düşünüldüğü fakat koçluk programı boyunca kullanılan görevlerin yanlış olarak kodlanmasında yüksek istem düzeyindeki alt kategoriler arasında kurulan yanlış eşleştirmenin olduğu saptanmıştır. Dolayısıyla koçluk döneminde üst düzey görevleri tanımada koçluk öncesine göre Aysu'nun daha yetkin olduğu söylenebilir. Bu bulgu, öğretmenlerin görev sınıflandırması ve sıralama performansları konusunda çalıştay öncesi ve sonrası gerekçeleri arasında önemli farklılıklar olduğunu gösteren önceki araştırmalarla tutarlıdır (Arbaugh ve Brown, 2005; Boston, 2013; Watson ve Mason, 2007). Özetle, Aysu'nun düşük ve yüksek seviyeler için TAG'deki kriterleri kullanabildiği görülmüştür.

Yukarıda bahsedilen bulgulardan bir tanesinin Aysu'nun yüksek bilişsel istem düzeyindeki görevleri tanıyabildiği görülmüştür. Aysu'yu eğim görevlerini yüksek düzeyde sınıflandırmaya teşvik eden unsurun eğimin doğası olabilir. Çünkü eğimin çeşitli kavramsallaştırmaları, temsilleri olan ve diğer matematiksel fikirlerle bağlantılı bir kavramdır. Fakat birinci koçluk döngüsünde eğilimi görevlerin uygulama öncesi potansiyel bilişsel istem düzeyi ile uygularken öğrenci düşünüşlerine göre görevin düzeyi arasındaki farklılık Aysu'nun önceki sınıflandırmasını gözden geçirmesini tetiklemiştir. Bu açıdan öğretmenlere yüksek ve düşük bilişsel istem düzeyinde görevlerin sunulması onların meraklarını veya şüphelerini tetikleyerek (Olson ve Barrett, 2004; Swan, 2007; Watson ve Mason, 2007) sürekli olarak görevin bilişsel istem düzeyi ile muhtemel öğrenci düşünüşü arasında ilişkiyi yorumlamasına neden olabilir. Bu anlamda, yüksek bilişsel istem düzeyindeki görevleri seçme ve uygulama bağlamında olan bu koçluk programında, ilk döngü için hem yüksek hem de düşük istem düzeyindeki görevleri öğretmene sunmanın sonraki döngülerde öğretmene sadece yüksek istem düzeyinde görevler sunulsa bile görevin doğasını detaylı bir şekilde analiz ettiği görülmüştür. Diğer bir bulgunun ise Aysu'nun *İlişkilendirmeye dayanan yöntem* ve *Matematik yapma* görevlerini ayırt edemediğini göstermiştir. Başka çalışmalarla da desteklenen bu bulgunun (örn. Pettersen ve Nortvedt, 2017) gerekçesi Aysu'nun öğrencilerin ön bilgileri ile görevin bağlamı arasındaki ilişkiyi kuramaması olabilir. Çünkü öğretmenlerin öğrencilerin ön bilgilerini nasıl kavradıkları, öğretimlerinin odağını (Schwartz vd., 2007) ve görevin potansiyeline ilişkin gerekçelerini etkiler. Zorluğun ikinci nedeni, görevlerin bilişsel istemlerinin Görev Analiz Rehberi'nde yetersiz ve genel operasyonelleştirilmesi olabilir. Örneğin, yüksek seviyeli görevlerde, "bir dereceye kadar bilişsel çaba" ve "önemli ölçüde bilişsel çaba" gibi göreceli terimler, bu iki yüksek bilişsel istem düzeyindeki görevlerin arasındaki farklılığı ortaya koyan kriterlerde değinilmiştir. Çaba miktarının anlamı farklı insanlar için farklı olabilir. Bu noktada Aysu'nun da bahsettiği üzere, *matematik yapma* ve *ilişkilendirmeye dayanmayan yöntem* görevleri arasında ayrım yapmak için daha spesifik kriterlerin tanımlanması gerekli olabilir (Osana vd., 2006). TAG kriterlerindeki belirsizliğe ek olarak, bu zorluk, görevlerdeki açıklık derecesini ifade eden yapı ikilemi (structure dilemma) (Barbosa ve de Oliveira, 2013) ile de ilgili olabilir. Aslında Klein ve Leikin (2020), öğretmenlerin birden çok sonucu olan göreve kıyasla birden çok strateji ile çözülebilen görevlere daha aşina olduklarını belirttiklerini bulmuşlardır. Klein ve Leikin, öğretmenlerin bu tür görevleri kullanma eğilimlerini, bu görevleri tasarlarken az çaba göstermeleri ile ilişkilendirerek bu sonucu tartışmışlardır. Mevcut çalışmada, benzer bir şekilde Aysu, çoklu stratejiler içeren görevleri ilişkilendirmeye dayanan yöntem olarak sınıflama eğilimindeyken (örn. Görev C, Döngü 4), daha açık olan görevleri (birden fazla sonucu olabilen) matematik yapma düzeyinde sınıflandırma eğiliminde olduğu görülmüştür. Bu nedenle, yüksek ve düşük istem seviyesindeki görevleri kullanmanın yanısıra farklı açıklıkta ve yapıda görevleri (çoklu stratejiler, çoklu sonuç veya araştırma görevleri) istem düzeylerinde zıtlıklar oluşacak şekilde öğretmene sınıflandırması için sunmak etkili bir yöntem olabilir.

Sonuç olarak, veriler, öğretmenin görevlerin bilişsel istem düzeyine ilişkin fikirlerinin, görevlerin yüzeysel niteliklerini veya işlemsel bileşenlerini vurgulamaktan, spesifik öğrenci düşünüşleri ile görevlerin yapısal özellikleriyle ilişkilendirmeye doğru evrildiğini ortaya koymuştur. Aysu'nun görevlerin doğasına yönelik bilgisinin gelişiminde, pedagojik materyallerin kullanımının (Görev Analiz Rehberi, eğimin farklı temsilleri) uygulama esnasında yüksek bilişsel istem düzeyini koruma eğilimine odaklanılmasının (Johnson vd., 2017), koçun farklı seviyelerdeki örnek öğrenci düşünüşlerini tartışmaya açmasının ve sınıfta uygulanan görevlere ilişkin durumları derinlemesine analiz edilmesinin etkili olduğu söylenebilir.

İkinci araştırma sorusuna ilişkin bulgular aşağıdaki tabloda sunulmuştur.

Tablo 2. Koçluk döngülerinde tekrarlanan planlama, öğretim ve yansıtma aşamalarında Aysu'nun fark etme süreci

	Döngü 1/ Fark Etme Düzeyi Sıklığı	Döngü 2 Fark Etme Düzeyi Sıklığı	Döngü 3 Fark Etme Düzeyi Sıklığı	Döngü 4 Fark Etme Düzeyi Sıklığı
Planlama	Seviye 1:12 Seviye 2:13 Seviye 3:4 Seviye 4:0	Seviye 1:3 Seviye 2:12 Seviye 3:5 Seviye 4:0	Seviye 1:0 Seviye 2:4 Seviye 3:13 Seviye 4:0	Seviye 1:0 Seviye 2:3 Seviye 3:12 Seviye 4:0
Uygulama/Öğretim	Seviye 1:10 Seviye 2:22 Seviye 3:5 Seviye 4:0	Seviye 1:5 Seviye 2:12 Seviye 3:12 Seviye 4:2	Seviye 1:2 Seviye 2:13 Seviye 3:12 Seviye 4:5	Seviye 1:2 Seviye 2:9 Seviye 3:16 Seviye 4:6
Yansıtma	Seviye 1:8 Seviye 2:13 Seviye 3:5 Seviye 4:0	Seviye 1:1 Seviye 2:11 Seviye 3:8 Seviye 4:2	Seviye 1:1 Seviye 2:6 Seviye 3:9 Seviye 4:4	Seviye 1:1 Seviye 2:9 Seviye 3:16 Seviye 4:5

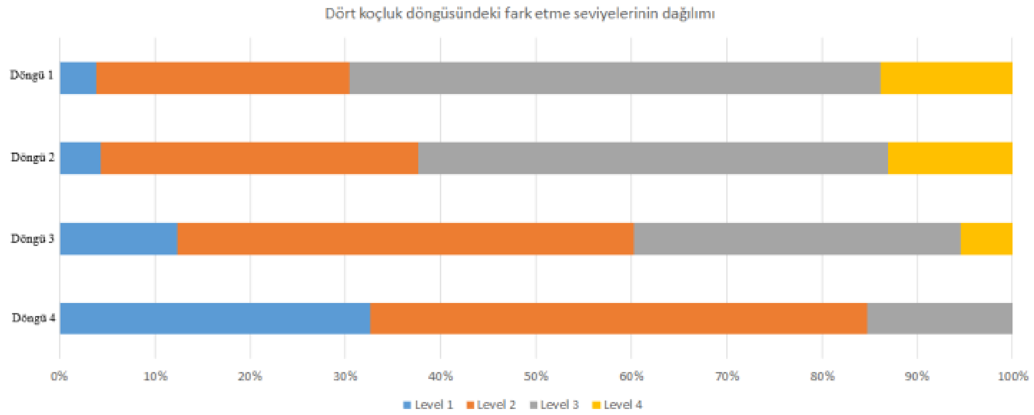
Tablo 2'ye göre koçluk programı süresi boyunca Aysu'nun fark etme becerileri Düzey 1, Düzey 2 ve Düzey 3 olarak çeşitlilik göstermiştir. Planlama, Öğretim ve Yansıtma aşamasında fark etme becerilerinin üst düzeylere doğru son iki döngüde arttığı görülmektedir. Diğer bir deyişle, Aysu görevin yüzeysel ve genel özellikleri ve genel sınıf öğrenmeleri gibi çeşitli konulardan daha özelde görevin doğası ile ilişkili olarak spesifik öğrenci düşünüşü, eğimin çeşitli kavramlarına ve öğretim pedagojisi gibi konulara dikkat etmeye başladığı saptanmıştır. Daha detaylı öğrenci düşünüşlerine ve görevin doğasına odaklanmak Aysu'nun yorumlarını gerekçelendirmek için öğrenci düşünüşü ve ilgili teorik çerçeveleri kullanmaya yöneldiği de görülmüştür. Böylece tanımlayıcı ve değerlendirici yaklaşımdan öte yorumlayıcı ve sorgulayıcı yaklaşımı kullandığı saptanmıştır. Öğretim aşamasında ise ilk döngülerde öğrenci cevaplarını rastgele dinleyip onay verme, yanlış cevaplar üzerinde fazla zaman harcamadan düzeltme, farklı öğrenci cevaplarını fark edememe veya sınıf-içi tartışmaya açmama, görevin bilişsel istem düzeyini düşürme eğiliminde olduğu görülmüştür. Diğer yandan öğrencilerin düşüncelerini veya cevaplarını açıklamaları için destekleyici sorular sormasına karşın etkili bir tartışma ortamı yaratabilecek ek destekleyici soruları çoğunlukla yöneltemediği saptanmıştır. Fakat son döngülerde bu ek destekleyici soruları sorabildiği ve görevin yüksek bilişsel istem düzeyini koruduğu görülmüştür.

Planlama aşamasında, Aysu ve koç, beklenen öğrenci tepkilerini, görevlerin bilişsel taleplerini, görevlerin sırasını, beklenen öğrenci düşüncesini ve görevlerin sunduğu matematiksel fikir gibi konuları birlikte tartışmışlardır. Bu bağlamda, Ne fark etti boyutunda bulgular onun materyal, örneğin teknoloji, manipülatif, zaman, pedagoji, görevin doğası ve öğrencilerin matematiksel düşünme ve anlamaları gibi çok çeşitli konulara odaklandığını ortaya koymuştur. Odaklandığı konulardaki çeşitlilik her döngüde görülmesine rağmen erken döngülerde, nasıl fark ettiğine ilişkin çoğu yorumunun betimleyici ve değerlendirmeci yaklaşıma sahip olduğu görülmüştür. Görevin bilişsel istem seviyesine ve görevin bağlamsal özelliklerine odaklanmasına rağmen, öğrencilerin farklı ve üst düzey düşünmelerini ortaya çıkaracak olası sorulara değinememiştir. Bu, görevin bağlamsal özellikleri ile öğrencilerin düşünmesini sağlam bir şekilde ilişkilendiremediğini göstermiştir. Daha önceki çalışmalarda da belirtildiği gibi (Star, Lynch ve Perova, 2011; Vondrova ve Zalska, 2013), öğretmenin görevlerin matematiksel yönlerini tespit etmesi zor olabilir veya öğretmen olası öğrencilerin düşünce ve stratejilerini fark edebilir; ancak bu stratejileri görevin temel özellikleriyle ilişkilendirmede zorluk yaşarlar (Fernandez, Llinares ve Valls, 2012). Fakat, koçluk döngüleri devam ettikçe, öğretmen görevin bağlamsal özellikleri, sıralaması ve bilişsel istem düzeyi ile öğrenci düşünüşleriyle bağdaştırabilmiştir. Aysu'nun yorumlarının çoğu, sonraki döngülerde Seviye 1 ve 2'den Seviye 3'e ve 4' e doğru ilerleme göstermiştir. Ayrıca görevin uygulanmasına ilişkin alternatif pedagojik kararlar önermeye çalışmıştır. Benzer şekilde Choy (2017), öğretmen adaylarının öğrencilerin kesirlerdeki güçlüklerine göre görevleri değiştirmeye yöneldiklerini göstermiştir. Bu gelişim yüksek bilişsel istem düzeyindeki görevlerin öğrenci düşünüşleri ile ilgili ilgisi bağlamındaki koçluk programının içeriği ve koçun destekleyici soruları ile ilişkili olabilir. Çünkü koç görevin bilişsel seviyesinin tanımasının yanısıra olası veya karşılaşılan öğrenci zorluklarında veya farklı düşünüşlerinde göreve ve uygulanmasına ilişkin değişikliklerle (Sullivan, Clarke ve Clarke, 2013) ilgili öğretmenin düşünmesini istemiştir.

Öğretim aşamasında ise, mevcut çalışmanın verileri, genel olarak Aysu'nun farklı öğrenci düşünüşlerini dinlediğini, fakat tartışmayı yönetmek için harekete geçmediğini göstermiştir. Bu nedenle, fark etme becerilerinin düzeyi, çoğunlukla koçluk programının ilk dönemlerinde Düzey 1 ve Düzey 2 olarak kodlanmıştır. Bu bulgu, hem öğretmen adaylarının hem de öğretmenlerin matematik öğretimi sırasında fark etme yeteneklerinin öğrencilerin düşüncelerine yanıt vermedeki yetersizliklerini ortaya çıkaran diğer çalışmaların bulgularıyla paralellik göstermektedir (ör., Goldsmith ve Seago, 2011; Kazemi ve Franke, 2004; La Rochelle vd., 2019; Luna ve Selmer, 2021; van Es, 2011). Öğretmenler görevleri sınıflandırmada iyi olsalar bile, uygulama sırasında görevin yüksek bilişsel istemini sürdürmekte zorlandıkları için bu şaşırtıcı bir bulgu değildir. Benzer şekilde Aysu, görevlerin neden yüksek bilişsel istem düzeyinde olduğunu açıklasa da öğrencilerin görevlerle ilgili beklenmedik düşünüşlerine etkili öğrenme ortamı yaratmakta zorlanmıştır. Bu bulgu, öğretimin karmaşık olduğunu ve deneyimli öğretmenlerin bile öğrencilerin düşüncelerini zenginleştirmekte zorlanabileceğini göstermektedir. Bu nedenle, öğretmenlerin diğer özelliklerinin deneyimli olmaktan ziyade öğretim sırasında etkili karar vermeyi engellemesi muhtemeldir. Bu özellikler, öğretmenlerin öğretim sırasında karar vermelerini ve davranışlarını etkileyen sahip oldukları bilgileri, kaynakları ve yönelimleri olabilir (Dreher ve Kuntze, 2015; Schoenfeld, 2010). Ayrıca, Lee ve Francis (2017), öğrencilerin düşünmesini sağlama ve öğrencileri alternatif stratejileri keşfetmeye dahil etme gibi uzmanlaşmış içerik bilgisi ve yanıt verme becerilerinin de fark etme ve etkili öğretim ile ilişkisi olduğunu iddia etmiştir. Sonuç olarak, Aysu'nun sınırlı tepki verme becerileri ve uzmanlaşmış içerik bilgisi (Wilhelm, 2014), öğrencilerin düşünüşlerine etkili cevap verme ve fark etmedeki zorluklarının diğer sebepleri olabilir. Bu iddianın en önemli göstergelerinden biri, koçluk programının başlangıcında cebirsel düşünmeyi değişkenlerin birlikte değişmesinden soyutlanmış olarak algılaması ve çeşitli eğim kavramsallaştırmalarına yönelik sınırlı anlamlara sahip olmasıdır. Ayrıca Aysu'nun ders esnasında öğrencilerin sınavlarda başarılı olmalarına yardımcı olabilmek adına görevlerin doğru çözüm yolunu göstermeye ve yanlış öğrenci çözümlerini anında düzeltmeye özen göstermiş olabilir. Bu öğretmenin ulusal değerlendirmelerde başarılı öğrenciler yetiştirme sorumluluğuna olan inancı olabilir. Nitekim Brown ve meslektaşları (2011) öğretmenlerin sorumluluklarını çoğunlukla okul veya değerlendirmeye ilişkin ulusal kararla ilişkilendirdiğini iddia etmektedir. Bu nedenle Aysu öğretmenin sınavlarda başarılı öğrenciler yetiştirmeye yönelik sorumluluk bilinci farklı öğretim tekniklerini uygulamasının önüne geçiyor olabilir.

Fakat sonraki döngülerde "ne farketti" boyutuyla ilgili olarak, farklı öğrenci cevaplarını veya düşünüşlerine daha çok odaklandığı görülmüştür. Daha detaylı olarak, öğrenci cevaplarından veya görevin temel matematiksel fikrinden yola çıkarak öğrencilerin kavramsal anlama, çelişkili durumlar yaratma, genişletme gibi ihtiyaçlarını fark ettiği görülmüştür. Erken dönemlere kıyasla "nasıl farketti" ile ilgili olarak, bilişsel istem düzeyini artırmak için görevleri değiştirme, daha etkili yönlendirici sorular sorma, öğrencilerin çeşitli çalışmalarını ortaya çıkarma ve sıralama gibi davranışlarda bulunmuştur. Bu çalışmadaki koçluk programının yeterince uzun bir süre içinde öğretmenin fark etme yeteneğinin gelişimine yönelik sürekli ve yoğun müdahalesinin etkin bir rol oynadığı söylenebilir. Koçluk programlarının öğretmenlerin sınıf içi uygulamaları üzerindeki olumlu etkilerini ortaya koyan diğer araştırmaların bulguları da bunu doğrulamaktadır (Aygün, 2019; Auletto ve Stein; 2020; Polly, 2012; Russell ve diğerleri, 2020). Nitekim bu çalışmada da öğretmen Aysu’nun koçun hareketini fark edip, öğrenci ihtiyaçlarına göre etkili ve kaliteli cevaplar geliştirdiği görülmüştür. Ancak, önceki çalışmalarla tutarsız olması (Olson ve Barrett, 2004; Saclarides ve Lubienski, 2021), koç rolünün ve koçluk programı etkinliklerinin farklılığı öğretmen eğitimi kalitesinde bir fark yarattığını gösterebilir. Bu açıdan koçluğun öğretmenlerin öğrenmeleri üzerindeki etkileri tartışılırken hem koçluk faaliyetlerinin özellikleri hem de koçun yeterliği göz önünde bulundurulmalıdır.

Yansıtma aşamasında ise, bulgular, Aysu'nun fark etmesinin koçluk programının başlarında Seviye 1 ve Seviye 2 arasında değiştiğini ortaya koymuştur. Öğretmenin odak noktasını, öğretmen pedagojisi, belirli öğrenci düşünüşleri ve genel sınıf kavrayışı, koçun eylemi, belli bir öğretim anı gibi konular oluşturmaktadır. Her ne kadar bu odaklar fark etme becerisinin yüksek olduğunu belirtse de diğer önemli öğrenci cevaplarını kaçırdığı ve bahsedilen öğrenci düşünüşlerini ve pedagojiyi çok fazla anlamlandırmaya ve yorumlamaya çalışmadıklarını göstermektedir. Birçok araştırma hizmet içi ve öğretmen adaylarının öğretimde kritik olaylara katılmakta zorlandıklarını gösterdiği için bu bulgu beklenen bir bulgudur (Callejo ve Zapatera, 2017; Derry, 2007; Lee ve Lee, 2021; Spitzer vd., 2011; Schwarz vd., 2018; Teuscher vd., 2017). Callejo ve Zapatera'ya (2017) göre öğretmenlerin öğretimin genel yönüne odaklanmaları, özellikle öğrencilerin zorluklarını veya kavram yanılgılarını fark etme ve buna uygun olarak ileriki cevaplar sunmalarına göre daha kolaydır. Bu, aynı zamanda öğrencilerin stratejilerini ve matematiksel düşünmelerini anlamlandırmanın öğretmenlerin kendi matematiksel anlayışları ve düşünmeleriyle sınırlı olduğunu gösterebilir (Dreher ve Kuntze, 2015; Lee ve Cross Francis, 2018; Schack vd., 2013; Schoenfeld, 2011). Fakat koçluk döngüleri ilerledikçe Aysu'nun fark etme becerilerinin daha yüksek düzeyde olduğu görülmüştür. Daha özelde, Aysu'nun belirli öğrenci cevaplarına ve görevin başlangıçtaki amacı ile uygulama esnasındaki göreve ilişkin öğrenci düşünüşlerini kıyaslamaya daha fazla odaklandığı belirlenmiştir. Yansıma toplantısı oturumları boyunca, Aysu, öğrencilerin cebirsel muhakemesi ve görevin doğası arasındaki ilişkiyi Eğim Kavramsallaştırma Çerçevesini (Nagle vd., 2019) kullanarak açıklamıştır. Bu anlamda, uygulanmış görevler üzerinde düşünmelerini sağlamanın, öğretmenlerin fark etmeleri için önemi görülmüştür (Wickstrom, 2014). Yansıtma aşamasında, koç aynı zamanda, öğretmenin bahsetmediği önemli öğrenci çalışmaları ve kritik öğretim anların görevlerle ilişkilerini tartışmak için pedagojik araçlar olarak kullanmıştır.



Figür 2. Dört koçluk döngüsündeki fark etme seviyelerinin dağılımı

Figür 2'de görüldüğü üzere öğretmenin öğrencilerin cebirsel düşünmelerini fark etme becerilerinde matematiksel görev bağlamındaki bu koçluk programı süresince ilerleme kaydetmiştir. Böylece mesleki gelişim modellerinden koçluk uygulamalarının etkili olduğu söylenebilir (fark etme becerilerin mesleki gelişim modelleri yardımıyla geliştirilebileceği söylenebilir (Goldsmith ve Seago, 2011; Jacobs, Lamb, ve Philipp, 2010; Jacopovic, 2021; Munson, 2020; Reinke, et al., 2021; Sherin & van Es, 2011).

Bu çalışmanın bağlamı doğrultusunda Van Es'in (2011) fark etme teorik çerçevesi genişletilmiştir. Bu çerçeve, öğretmenlerin yüksek bilişsel istem düzeyindeki görevlerin planlanması, uygulanması ve ders sonrası üzerinde tekrar düşünme (yansıtma) aşamalarındaki fark etme düzeylerini karakterize etmek için bir araç olarak kullanılabilir. Ayrıca bulgular bölümünde koçun öğretmenle nasıl iletişim kurduğu, eğim kavramıyla ilgili hangi soruları sorduğu ve öğrencilerin eğimi öğrenmesini zenginleştirmek için hangi görevleri sırayla uyarladığı ile ilgili bölümlerin sunulması okul müdürlere ve araştırmacılara koçluk programı etkinlikleri ve uygulanışı hakkında detaylı ipucu sunabilir.

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