Fixed point free action on groups of odd order

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Abstract

Let $A$ be a finite abelian group that acts fixed point freely on a finite (solvable) group $G$. Assume that $|G|$ is odd and $A$ is of squarefree exponent coprime to 6. We show that the Fitting length of $G$ is bounded by the length of the longest chain of subgroups of $A$.

Keywords: Solvable groups; Fixed point free action; Finite groups; Representations

Introduction

Let $G$ be a finite solvable group and $A$ be a finite group acting fixed point freely on $G$. A long-standing conjecture is that if $(|G|, |A|) = 1$, then the Fitting length $f(G)$ of $G$ is bounded by the length $\ell(A)$ of the longest chain of subgroups of $A$. By an elegant result due to Bell and Hartley [1], it is known that any finite nonnilpotent group $A$ can act fixed point freely on a solvable group $G$ of arbitrarily large Fitting length with $(|G|, |A|) \neq 1$. We expect that the conjecture is true when the coprimeness condition is replaced by the assumption that $A$ is nilpotent. This question is still unsettled except for cyclic groups $A$ of order $pq$ and $pqr$ for pairwise distinct primes $p, q$ and $r$ [3,4].

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In the present paper we establish the conjecture without the coprimeness condition when $A$ is a finite abelian group of squarefree odd exponent not divisible by 3 and $|G|$ is odd. This improves the bound given in Theorem 3.4 of [6]; as a by-product we also improve a bound given in Theorem 8.5 of [2].

Namely, we shall prove the following:

**Theorem A.** Let $A$ be a finite abelian group acting fixed point freely on a finite group $G$ of odd order. If $A$ has squarefree exponent coprime to 6, then $f(G) \leq \ell(A)$.

**Theorem B.** Let $G$ be a finite (solvable) group of order coprime to 6. If $C$ is a Carter subgroup of $G$, then $f(G) \leq 2(2\ell(C) - 1)$.

**Preliminary remarks.** All the groups considered in this paper are finite and solvable. Except for the following, the notation and terminology are as in [2].

Let $G$ be a group.

We denote by $\hat{G}$ the Frattini factor group of $G$.

If $S$ is a subgroup of $G$ and $a \in G$, then for any positive integer $n$, we denote by $[S, a]^n$ the commutator subgroup $[S, a, \ldots, a]$ with $a$ repeated $n$ times.

Let $K$ be a group acting on $G$, that is, there is a homomorphism from $K$ into Aut($G$). We write $(K \text{ on } G)$ to denote this action. If $x \in K$, then we write $g^x$ for the image of $g \in G$ under the automorphism of $G$ which is the image of $x$ in Aut($G$). Let another group $L$ act on $K$, and let $l \in L$. We write $(K \text{ on } G)^l$ to denote the action of $K$ on $G$ given by $x \to (K \text{ on } G)(x^{l^{-1}})$ for $x \in K$.

Let $K$ be a group acting on groups $H$ and $G$. We say $(K \text{ on } G)$ and $(K \text{ on } H)$ are weakly equivalent if each nontrivial irreducible section of $(K \text{ on } G)$ is $K$-isomorphic to an irreducible section of $(K \text{ on } H)$ and vice versa. We write $(K \text{ on } H) \equiv_w (K \text{ on } G)$ if $(K \text{ on } H)$ is weakly equivalent to $(K \text{ on } G)$.

Let $K$, $L$, $G$ and $H$ be groups.

(a) If $(K \text{ on } G) \equiv_w (K \text{ on } H)$, then $(L \text{ on } G) \equiv_w (L \text{ on } H)$ for each $L \leq K$.

(b) Let $L$ act on $K$ and $K$ act on $G$ and $H$. If $(K \text{ on } G) \equiv_w (K \text{ on } H)$, then $(K \text{ on } G)^l \equiv_w (K \text{ on } H)^l$ for each $l \in L$.

(c) Let $V$ be a completely reducible $kG$-module for a field $k$ and let $L$ act on $G$. Let $l \in L$ and $V_l$ denote the $kG$-module with respect to $(G \text{ on } V)^l$. Assume that $(G \text{ on } V) \equiv_w (G \text{ on } V)^l$. Let $M \leq G$ such that $M$ is $(l)$-invariant, and $W$ be the sum of all irreducible $kG$-submodules of $V$ on which $M$ acts nontrivially. Then $W = W^\# = W_l$ as subspaces where $W^\#$ stands for the sum of all irreducible $kG$-submodules of $V_l$ on which $M$ acts nontrivially.

Note that $W$ and $W_l$ need not be isomorphic as $kG$-modules.

**Lemma 1.** Let $S(\langle \alpha \rangle)$ be a group where $S < S(\langle \alpha \rangle)$, $S$ is an $s$-group for some prime $s$, $\Phi(S) \leq Z(S)$, $\langle \alpha \rangle$ is cyclic of order $p$ for an odd prime $p$. Suppose that $V$ is a $kS(\langle \alpha \rangle)$-module for a field $k$ of characteristic different from $s$. Then $C_V(\langle \alpha \rangle) \neq 0$ if one of the following is satisfied:

(i) $[Z(S), \alpha]$ is nontrivial on $V$.
(ii) $[S, \alpha]^{p^{-1}}$ is nontrivial on $V$ and $p = s$. 

Furthermore, if $S(\alpha)$ acts irreducibly on $V$ or the characteristic of $k$ is different from $p$, then we also have $(C \text{ on } C_V(\alpha)) \equiv_w (C \text{ on } V)$ where $C = C_D(\alpha)$ for

$$D = \begin{cases} S & \text{when (i) holds,} \\ [S, \alpha]^{p-1} & \text{when (ii) holds.} \end{cases}$$

**Proof.** See [2, Proposition 3.10]. □

**Lemma 2.** (See Lemma 5.30 in [2].) Let $S < S(\alpha)$ where $\langle \alpha \rangle$ is cyclic of prime order and let $V$ be an irreducible $kS(\alpha)$-module. If $E$ is an $\langle \alpha \rangle$-invariant subgroup of $Z(S)$ and $U$ is a nonzero $E\langle \alpha \rangle$-submodule of $V$, then $\ker(E \text{ on } V) = \ker(E \text{ on } U)$.

**Lemma 3.** Let $S(\alpha)$ be a group such that $S < S(\alpha)$ where $\langle \alpha \rangle$ is of prime order $p$. Suppose that $V$ is a $kS(\alpha)$-module for a field $k$ of characteristic different from $p$, and $\Omega$ is an $S(\alpha)$-stable subset of $V^*$. Set $V_0 = \bigcap \{ \ker f \mid f \in \Omega - C_{\Omega}(\alpha) \}$. If there exists a nonzero $f$ in $\Omega$ and $x \in S$ such that $f(V_0) \neq 0$ and $[x, a, \alpha] \notin C_S(f)$ for each $1 \neq a \in \langle \alpha \rangle$, then $C_V(\alpha) \not\subseteq V_0$.

**Proof.** Since $f(V_0) \neq 0$, it follows that $f \in C_{\Omega}(\alpha)$ and so $C_S(f)$ is normalized by $\langle \alpha \rangle$. The assumption $[x, a, \alpha] \notin C_S(f)$ for each $1 \neq a \in \langle \alpha \rangle$ yields that $[x, a] \notin C_S(f)$ for each $1 \neq a \in \langle \alpha \rangle$. Then $bx^{-1}f \notin C_{\Omega}(\alpha)$ for each $b \in \langle \alpha \rangle$. Set $g = \sum_{b \in \langle \alpha \rangle} bxf$. It is clear that $g \in C_{\Omega}(\alpha)$ and so $[V, \alpha] \subseteq \ker g$. Since $V = [V, \alpha] \oplus C_V(\alpha)$, either $g = 0$ or $C_V(\alpha) \not\subseteq \ker g$. If the latter holds, then $C_V(\alpha) \not\subseteq V_0$ as claimed, because $V_0 \subseteq \ker(bxf)$ for each $b \in \langle \alpha \rangle$. Hence we may assume that $g = 0$. Now $0 = x^{-1}g = f + \sum_{1 \neq b \in \langle \alpha \rangle} [x, b]f$ and then $f = -\sum_{1 \neq b \in \langle \alpha \rangle} [x, b]f$. Since $[x, b, \alpha] \notin C_S(f)$ by the hypothesis, we have $[x, b]f \notin C_{\Omega}(\alpha)$ for each $1 \neq b \in \langle \alpha \rangle$. Then $f(V_0) = 0$. This contradiction completes the proof. □

The following result is a generalization of Theorem 2.1.A in [5].

**Theorem 1.** Let $S(\alpha)$ be a group such that $S < S(\alpha)$, $S$ is an $s$-group, $\langle \alpha \rangle$ is cyclic of order $p$ for odd primes $s$ and $p$ with $p \geq 5$, $\Phi(\Phi(S)) = 1$, $\Phi(S) \subseteq Z(S)$.

Suppose that $k$ is a field of characteristic not dividing $sp$ and $V$ is a $kS(\alpha)$-module such that $[S, \alpha]^{p-1}$ acts nontrivially on each irreducible submodule of $V|S$.

Let $\Omega$ be an $S(\alpha)$-stable subset of $V^*$ which linearly spans $V^*$ and set $V_0 = \bigcap \{ \ker f \mid f \in \Omega - C_{\Omega}(\alpha) \}$. Then $C_V(\alpha) \not\subseteq V_0$ and

$$(C_D(\alpha) \text{ on } C_V(\alpha)/C_{V_0}(\alpha)) \equiv_w (C_D(\alpha) \text{ on } V) \quad \text{where}$$

$$D = \begin{cases} S & \text{when } s = p, \\ [S, \alpha]^{p-1} & \text{otherwise.} \end{cases}$$

**Proof.** Assume that the theorem is false and consider a counterexample with $\dim V + |S(\alpha)|$ minimal. Set $X = C_V(\alpha)/C_{V_0}(\alpha)$ and $C = C_D(\alpha)$.

**Claim 1.** We may assume that $S$ acts faithfully and $S(\alpha)$ acts irreducibly on $V$ and $k$ is a splitting field for all subgroups of $S(\alpha)$.
Put $\bar{S} = S/\text{Ker}(S \text{ on } V)$. By induction applied to the action of $\bar{S}(\alpha) \text{ on } V$, we get $C_{\bar{V}}(\alpha) \nsubseteq V_0$ and $(C_{\bar{V}}(\alpha) \text{ on } X) \equiv_w (C_{\bar{V}}(\alpha) \text{ on } V)$. As $\bar{C} = \bar{C}_D(\alpha) \leq C_{\bar{V}}(\alpha)$, we have obtained $(C \text{ on } X) \equiv_w (C \text{ on } V)$. Thus we may assume that $S$ is faithful on $V$.

Since $V$ is completely reducible as an $S(\alpha)$-module, we have a collection $\{V_1, \ldots, V_l\}$ of irreducible $S(\alpha)$-submodules of $V$ such that $V = \bigoplus_{i=1}^l V_i$. Now $[S, \alpha]^{p-1}$ acts nontrivially on each irreducible constituent of $V_i|S$ and hence $[S, \alpha]^{p-1}$ acts nontrivially on each $V_i$ for $i = 1, \ldots, l$. It is easy to observe that $\Omega_i^{|V_i}$ is an $S(\alpha)$-stable subset of $V_i^*$ and $\Omega_i^{|V_i} = V_i^*$ for each $i = 1, \ldots, l$. If $V$ is not irreducible as an $S(\alpha)$-module, we apply induction to the action of $S(\alpha)$ on $V_i$ for each $i$ and get $C_{V_i}(\alpha) \nsubseteq (V_i)_0$ and $(C \text{ on } C_{V_i}(\alpha) / (V_i)_0(\alpha)) \equiv_w (C \text{ on } V_i)$. Set $X_i = C_{V_i}(\alpha) / C_{V_i \cap (V_i)_0}(\alpha)$. Now $(C \text{ on } X_i) \equiv_w (C \text{ on } V_i)$ since $(V_i)_0 = \bigcap \{\text{Ker } g \mid g \in \Omega_i - C_{\Omega_i}(\alpha)\} \supseteq V_i \cap (V_i)_0$. As $V = \bigoplus_{i=1}^l V_i$ and $X \cong \bigoplus_{i=1}^l X_i$, it follows that $(C \text{ on } X) \equiv_w (C \text{ on } V)$. Therefore we can regard $V$ as an irreducible $S(\alpha)$-module.

Claim 2. $[Z(S), \alpha, \alpha] = 1$.

Assume the contrary. Set $S_1 = Z(S)|C$. Then $S_1$ is an $\langle \alpha \rangle$-invariant subgroup of $S$, and $V|_{S_1(\alpha)}$ is completely reducible. Note that $C \nsubseteq S_1(\alpha)$. Let $V_i$ be an irreducible $S_1(\alpha)$-submodule of $V$ and $W$ be a homogeneous component of $V_i|C$.

Now $Z(S)(\alpha) \leq C_{S_1(\alpha)}(\alpha) \leq N_{S_1(\alpha)}(W)$. This yields that $V_i|C$ is homogeneous. We also observe that Ker$(Z(S) \text{ on } V_1) = \text{Ker}(Z(S) \text{ on } V) = 1$ by applying Lemma 2 to the action of $S(\alpha)$ on $V$.

Since $[Z(S), \alpha] \neq 1$, $[Z(S_1), \alpha] \neq 0$ is nontrivial on $V_i$. Applying Lemma 1 to the action of $S_1(\alpha)$ on $V_i$, we obtain $C_{V_i}(\alpha) \neq 0$. If $C_{V_i}(\alpha) \nsubseteq V_0$, it follows that $(C \text{ on } C_{V_i}(\alpha) / C_{V_i \cap V_0}(\alpha) ) \equiv_w (C \text{ on } V_1)$ as $V_i|C$ is homogeneous. This forces that there is an irreducible $S_1(\alpha)$-submodule $V_i$ of the completely reducible module $V|_{S_1(\alpha)}$ such that $C_{V_i}(\alpha) \subseteq V_0$. Since $0 \neq C_{V_i}(\alpha)$, we have $V_i \cap V_0 \neq 0$. Set $\Omega_i = \Omega_i^{|V_i}$. Now $\Omega_i$ is an $S_1(\alpha)$-stable subset of $V_i^*$, and $(V_i)_0 = \bigcap \{\text{Ker } h \mid h \in \Omega_i - C_{\Omega_i}(\alpha)\} \neq 0$ as $V_i \cap V_0 \subseteq (V_i)_0$. Let $f \in \Omega$ be such that $f((V_i)_0) \neq 0$. Then $f_i = f|_{V_i} \in C_{\Omega_i}(\alpha)$. Consider $\langle f_i \rangle = \{cf_i \mid c \in \mathbb{k}\}$, a $C_{Z(S)}(f_i)\langle \alpha \rangle$-submodule of $V_i^*$. Appealing to Lemma 2 together with $\langle f_i \rangle$ and $C_{Z(S)}(f_i)$, we get $C_{Z(S)}(f_i) = \text{Ker}(C_{Z(S)}(f_i)) \text{ on } V_i^* = 1$. On the other hand, there exists $x \in Z(S)$ such that $[x, \alpha, \alpha] \neq 1$, as $[Z(S), \alpha, \alpha] \neq 1$. It follows that $[x, \alpha, \alpha] \neq 1$ for any $1 \neq a \in \langle \alpha \rangle$, that is $[x, \alpha, \alpha] \notin C_{S_1}(f_i)$, for any $1 \neq a \in \langle \alpha \rangle$. Now Lemma 3 applied to the action of $S_1(\alpha)$ on $V_i$, together with $f_i$ and $\Omega_i$, gives that $C_{V_i}(\alpha) \nsubseteq (V_i)_0$. This is a contradiction as $V_i \cap V_0 \subseteq (V_i)_0$ and $C_{V_i}(\alpha) \subseteq V_0$. Thus we have the claim.

Claim 3. $s \neq p$.

Assume that $s = p$. Since $[S, \alpha]^{p-1} \neq 1$, $[S, \alpha]^{p-3} \neq 1$. Set $S_1 = [S, \alpha]^{p-3}$. We can prove that $[S_1, [S, \alpha]^{p-1}] \leq [\Phi(S), \alpha]^{p-3} = 1$ (see [2, 5.37]). Hence $[S, \alpha]^{p-1} \leq Z(S_1)$.

We have a collection $\{V_1, \ldots, V_l\}$ of irreducible $S_1(\alpha)$-modules such that $V = \bigoplus_{i=1}^l V_i$. Fix $i \in \{1, \ldots, l\}$. We notice that $C = C_{[S, \alpha]^{p-1}}(\alpha) \leq S_1(\alpha)$ implying $V|C$ is completely reducible. In particular, $C \leq Z(S_1(\alpha))$ and so $V_i|C$ is homogeneous.

Set $X_i = C_{V_i}(\alpha) / C_{V_i \cap (V_i)_0}(\alpha)$ and assume that $(C \text{ on } X_i) \not\equiv_w (C \text{ on } C_{V_i}(\alpha))$. If $[S, \alpha]^{p-1}$ is trivial on $V_i$, then $C$ acts trivially on $V_i$, and this contradicts the assumption. Hence $[S, \alpha]^{p-1}$ is not trivial on $V_i$. If $V_i \cap V_0 = 0$, then $(C \text{ on } X_i) \equiv_w (C \text{ on } C_{V_i}(\alpha))$, and again we have a contradiction. Hence, $V_i \cap V_0 \neq 0$, and there exists some $f \in \Omega$ such that $f(V_i \cap V_0) \neq 0$. Now $f \in C_{\Omega}(\alpha)$. Set $f|_{V_i} = f_i$. Now $\langle f_i \rangle = \{cf_i \mid c \in \mathbb{k}\}$ is a $C_{[S, \alpha]^{p-1}}(f_i)\langle \alpha \rangle$-submodule of $V_i^*$. Appealing to Lemma 2, we get $C_{Z(S_1)}(f_i) = \text{Ker}(C_{Z(S_1)}(f_i)) \text{ on } V_i^*$. We also have
Therefore we conclude that $(C \text{ on } C_V(\alpha)/C_{V_0}(\alpha)) \equiv_w (C \text{ on } C_V(\alpha))$. Appealing to Lemma 1 together with $V$ and $S(\alpha)$, we also see that $C_V(\alpha) \not\equiv 0$ and $(C \text{ on } C_V(\alpha)) \equiv_w (C \text{ on } V)$ hold. Thus $(C \text{ on } V) \equiv_w (C \text{ on } C_V(\alpha)/C_{V_0}(\alpha))$. Since $[S, \alpha]^{p-1} \neq 1$ and $s = p, C \neq 1$. Hence $C$ is nontrivial on $V$ and so is on $C_V(\alpha)/C_{V_0}(\alpha)$. This supplies $C_V(\alpha) \not\subseteq V_0$, a contradiction.

**Claim 4. The theorem follows.**

Now $s \neq p$ and $[\Phi(S), \alpha] = 1$. Then $\Phi(S) \leq Z(S(\alpha))$ and so $S$ is a central product of $[S, \alpha]$ and $C_S(\alpha)$. As $C = C_S(\alpha) \ast S(\alpha)$, $V|C$ is completely reducible. In fact, $V|C$ is homogeneous, because any homogeneous component is stabilized by $S(\alpha)$ as $C$ is centralized by $[S, \alpha](\alpha)$. It follows that $(C \text{ on } C_V(\alpha)/C_{V_0}(\alpha)) \equiv_w (C \text{ on } V)$ if $C_V(\alpha) \not\subseteq V_0$ holds. Hence $C_V(\alpha) \subseteq V_0$. Note that $C_V(\alpha) \neq 0$, because otherwise we would have obtained $s = 2$ as $[S, \alpha]$ is nontrivial on $V$. Then there exists $0 \neq f \in C_{Z}(\alpha)$ with $f(V_0) \neq 0$. Now $C_{Z}(S)(f) = \text{Ker}(C_{Z}(S)(f) \text{ on } V^*) = 1$ by Lemma 2. If follows that $C_{Z}[S, \alpha](f) = 1$, as $[\Phi(S), [S, \alpha]] = 1$. Then $C_{[S, \alpha]}(f)$ is properly contained in $[S, \alpha]$. Let $M$ be a maximal $\alpha$-invariant subgroup of $[S, \alpha]$ containing $C_{[S, \alpha]}(f)$.

The abelian group $[S, \alpha]/M = [\bar{S}, \bar{\alpha}]$ forms an irreducible $\langle \alpha \rangle$-module on which $\langle \alpha \rangle$ acts fixed point freely. Thus we have $[\bar{x}, \bar{a}, \alpha] \neq 0$ for any $0 \neq \bar{x} \in [\bar{S}, \bar{\alpha}]$. It follows that $[\bar{x}, \bar{a}, \alpha] \neq 0$ for each $1 \neq a \in \langle \alpha \rangle$. Put $\bar{x} = \alpha M$ for $x \in [S, \alpha]$. Then $[x, a, \alpha] \neq M$. In particular, $[x, a, \alpha] \neq C_{[S, \alpha]}(f)$ for each $1 \neq a \in \langle \alpha \rangle$. Recall that $V|C$ is homogeneous. Then Lemma 3 applied to the action of $S(\alpha)$ on $V$ gives that $C_V(\alpha) \not\subseteq V_0$. This contradiction completes the proof of Theorem 1. □

Let $V$ be an irreducible $G(\alpha)$-module where $G \triangleleft G(\alpha)$ and $\langle \alpha \rangle$ is cyclic of prime order $p$. We say $V$ is an amenable $G(\alpha)$-module if $[G, \alpha]^{p-1}$ acts nontrivially on $V$. Notice that when $|G|$ is odd, this coincides with the definition of an amenable module given in [2].

**Theorem 2.** Let $S(\alpha)$ be a group such that $S \leq S(\alpha)$, $S$ is an $s$-group, $\langle \alpha \rangle$ is cyclic of order $p$ for distinct primes $s$ and $p$, $\Phi(\Phi(S)) = 1, \Phi(S) \leq Z(S)$. Suppose that $V$ is an irreducible $kS(\alpha)$-module on which $[S, \alpha]$ acts nontrivially where $k$ is a field of characteristic different from $s$. Then

$$[V, \alpha]^{p-1} \neq 0 \quad \text{and} \quad (C_S(\alpha) \text{ on } V) \equiv_w (C_S(\alpha) \text{ on } [V, \alpha]^{p-1}).$$

unless $p$ is a Fermat prime, $s = 2$ and $[\bar{S}, \alpha]$ is an irreducible $\langle \alpha \rangle$-module.

**Proof.** See [2, Proposition 3.10]. □

Now we are ready to prove our key result, which improves Theorem 3.1 in [5] obtained by pursuing the idea in Dade’s work [2].

**Theorem 3.** Let $G \triangleleft GA$ and $\langle z \rangle \triangleleft A$ of prime order $p$ with $p \geq 5$. Suppose that $P_1, \ldots, P_t$ is an $A$-Fitting chain of $G$ such that $[P_1, z] \neq 1, P_i$ is a $p_i$-group where $p_i$ is an odd prime for each $i \leq t$. Then $C_{[S, \alpha]}(f)$ has a nontrivial irreducible $\langle \alpha \rangle$-submodule on which $\langle \alpha \rangle$ acts fixed point freely.
\[ i = 1, \ldots, t, \text{ and } t \geq 3. \] Then there are sections \( D_{i_0}, \ldots, D_t \) of \( P_{i_0}, \ldots, P_t \), respectively, forming an \( A \)-Fitting chain of \( G \) such that \( z \) centralizes each \( D_j \) for \( j = i_0, \ldots, t \) where

\[ i_0 = \begin{cases} 
2 & \text{if } p_1 \neq p, \\
3 & \text{if } p_1 = p.
\end{cases} \]

**Proof.** Let \( q \) be a prime number different from \( p_1 \), let \( p_{t+1} = q \) and let \( P_{t+1} \) stand for the regular \( \mathbb{Z}_q[P_1P_{t+1}] \)-module. We shall add \( P_{t+1} \) to the given chain and define subspaces \( E_i \) of \( P_t \) for each \( i = 1, \ldots, t+1 \) as follows: \( E_1 = P_1 \), \( E_i = \langle X_i, E_i-1 \rangle \) for \( i = 2, \ldots, t+1 \), where \( X_i/\Phi(P_i) \) is the sum of all ample irreducible \( E_i-1(z) \)-submodules of \( P_i \). It is easy to observe that for each \( i = 2, \ldots, t+1 \), \( E_i \) are all \( E_i-1A \)-invariant subgroups of \( P_t \) and \( E_i \) is a direct sum of ample irreducible \( E_i-1(z) \)-submodules.

We now define subgroups \( F_i \) of \( E_i \) for \( i = 1, \ldots, t+1 \) as follows:

\[
F_1 = \{1\}, \\
F_i = C_{E_i}(z) \quad \text{if } p_i \neq p \text{ and } i \geq 2, \\
F_2 = C_{[E_2,z]^{p-1}}(z) \quad \text{if } p_2 = p, \\
F_i = \left([E_i,z]^{p-1}, F_{i-1}\right) \quad \text{if } p_i = p \text{ and } i \geq 3.
\]

It can also be easily seen that for each \( i = 2, \ldots, t+1 \), \( F_i \) is \( F_{i-1}A \)-invariant and is centralized by \( z \).

We next define the sections \( D_i \) by \( D_i = F_i/\text{Ker}(F_i \text{ on } \tilde{E}_{i+1}) \) for \( i = 2, \ldots, t \) and claim that they form an \( A \)-chain each of its sections is centralized by \( z \), as desired.

We proceed from this point by assuming that we can prove the following two claims whose proofs will follow later.

**Claim 1.** Assume that \( i \geq 2 \) and \( p_i \neq p \). If \( E_i \neq 1 \), then \( D_i \) is a nontrivial \( F_{i-1} \)-invariant section such that \( (F_i \text{ on } \tilde{E}_i) \equiv_w (F_{i-1} \text{ on } \tilde{D}_i) \).

**Claim 2.** Assume that \( i \geq 2 \) and \( p_i = p \). If either \( i = 2 \) or \( D_{i-1} \neq 1 \), then \( \text{Ker}(F_i \text{ on } \tilde{E}_{i+1}) = 1 \), \( D_i = F_i \neq 1 \) and \( (F_{i-1} \text{ on } \tilde{E}_i) \equiv_w (F_{i-1} \text{ on } \tilde{F}_i) \).

We first prove the theorem in the case \( p_1 \neq p \).

Now \( E_1 = P_1 \) and \([E_1,z]^{p-1} = [E_1,z] \neq 1\). Then the faithful action of \( P_1 \) on \( \tilde{P}_2 = [\tilde{P}_2, [E_1,z]] \oplus C_{\tilde{P}_2}([E_1,z]) \) forces that \( \tilde{E}_2 \neq 0 \), that is, \( \tilde{P}_2 \) contains an irreducible ample \( E_1(z) \)-submodule. If \( p_2 \neq p \), we apply Claim 1 to the action of \( E_1(z) \) on \( \tilde{E}_2 \) and obtain that \( D_2 \) is a nontrivial section of \( E_2 \). If \( p_2 = p \), we also have \( D_2 = F_2 \neq 1 \) by Claim 2. Thus we have seen that \( D_2 \neq 1 \) in any case.

Suppose that \( D_{i-1} \neq 1 \) for some \( i \geq 3 \). Then \( E_i \neq 1 \). Appealing again to Claims 1 and 2, respectively, when \( p_i \neq p \) and \( p_i = p \), we see that \( D_i \) is a nontrivial \( F_{i-1} \)-invariant section and \( (F_i \text{ on } \tilde{E}_i) \equiv_w (F_{i-1} \text{ on } \tilde{D}_i) \) for each \( i \geq 2 \). It follows that \( D_{i-1} = F_{i-1}/\text{Ker}(F_{i-1} \text{ on } \tilde{D}_i) \) normalizes \( D_i = F_i/\text{Ker}(F_i \text{ on } \tilde{E}_i+1) \) and \( \text{Ker}(D_i-1 \text{ on } \tilde{D_i}) = 1 \) for each \( i = 3, \ldots, t \).

We also have \( \Phi(D_i) \leq Z(D_i), \Phi(D_i) = 1 \) and \([\Phi(D_i), D_{i-1}] = 1\) for \( i = 2, \ldots, t \).

It remains to prove that \( (D_{i-1} \text{ on } \tilde{D}_i) \) is weakly \( D_{i-1} \)-invariant for \( i = 4, \ldots, t \). Since \( (P_{i-1} \text{ on } \tilde{P}_i) \) is weakly \( P_{i-1} \)-invariant, \( (E_{i-1} \text{ on } \tilde{E}_i) \) is weakly \( F_{i-1} \)-invariant by Remark (a), that
is, \((E_{i-1} \text{ on } \tilde{P}_i) \equiv_w (E_{i-1} \text{ on } \tilde{P}_i)^x\) for each \(x \in F_{i-2}\). Then \(X_i/\Phi(P_i) = (X_i/\Phi(P_i))^x\) by Remark (c) and so \((E_{i-1} \text{ on } \tilde{E}_i) \equiv_w (E_{i-1} \text{ on } \tilde{E}_i)^x\). Hence \((E_{i-1} \text{ on } \tilde{E}_i)\) is weakly \(F_{i-2}\)-invariant. This gives that \((F_{i-1} \text{ on } \tilde{E}_i)\) is weakly \(F_{i-2}\)-invariant, too. As \((F_{i-1} \text{ on } \tilde{E}_i) \equiv_w (F_{i-1} \text{ on } \tilde{D}_i)\) holds, it also follows that \((F_{i-1} \text{ on } \tilde{D}_i)\) is weakly \(F_{i-2}\)-invariant by Remark (b). Consequently we have obtained that \((D_{i-1} \text{ on } \tilde{D}_i)\) is weakly \(D_{i-2}\)-invariant, proving the theorem when \(p_1 \neq p\).

Finally we assume that \(p_1 = p\), and consider the chain \(P_2, \ldots, P_l\). Note that \([P_2, z] \neq 1\), because otherwise \([P_1, z] = 1\) by the three subgroup lemma. Since \(p_2 \neq p\), the above argument gives an \(A\)-Fitting chain \(D_3, \ldots, D_l\) whose terms are all centralized by \(z\). This completes the proof of Theorem 3. \(\square\)

We shall need the following fact in proving Claim 1.

**Lemma 4.** Assume \(p_i \neq p\) and let \(W\) be an irreducible submodule of \(\tilde{P}_{i+1}|_{E_i}\). If \(\Phi(E_i)\) acts nontrivially on \(W\), then so does \([E_i, z]\).

**Proof.** Suppose that \(W\) is an irreducible submodule of \(\tilde{P}_{i+1}|_{E_i}\) on which \(\Phi(E_i)\) acts nontrivially and \([E_i, z]\) acts trivially. Then there exists an \(E_iA\)-submodule \(X\) of \(\tilde{P}_{i+1}\) such that \(W\) is isomorphic to an irreducible \(E_i\)-submodule of \(X\). Since \(X|_{E_i}\) is completely reducible, there is a collection \(\{U_1, \ldots, U_m\}\) of homogeneous \(E_i\)-modules such that \(X = \bigoplus_{i=1}^{m} U_i\). Assume that \(U_1\) is a sum of isomorphic copies of \(W\). Then \(\text{Ker}(E_i|_X) = \bigcap_{a \in A} \text{Ker}(E_i|_U) = \bigcap_{a \in A} \text{Ker}(E_i|_W)^a\).

Put \(K = \text{Ker}(\Phi(E_i)|_X)\). \(K\) is an \(A\)-invariant normal subgroup of \(E_i\). Furthermore, \(K\) is \(E_{i-1}\)-invariant because \(\Phi(E_i), E_{i-1} = 1\). Set \(\tilde{E}_i = E_i/K\) and \(\tilde{E}_i = E_i/\text{Ker}(E_i)\). Note that \(E_i' = \Phi(E_i)\) since \(C_{E_i/E_i'}(E_{i-1}) = 0\). Now \(\tilde{E}_i\) is nonabelian, because otherwise \(E_i' = \Phi(E_i) = K\), which is not the case. It follows that \(V = \tilde{E}_i/Z(\tilde{E}_i) \neq 0\). Obviously we have \(Z(\tilde{E}_i) \subseteq Z(\tilde{E}_i)\). On the other hand, if \(Z(\tilde{E}_i) = \tilde{C} = \tilde{C}/\text{Ker}(\tilde{C})\) on \(X\), then \([\tilde{C}, \tilde{E}_i]\) is \(\text{Ker}(\tilde{E}_i)\). Because \(\Phi(\tilde{E}_i) = \Phi(E_i/K)\) is faithful on \(X\). Therefore \(\tilde{C} \leq \text{Z}(\tilde{E}_i)\), that is, \(\text{Z}(\tilde{E}_i) = (\tilde{E}_i)\).

Also note that \(\text{Ker}(\tilde{E}_i|_X) \subset \text{Z}(\tilde{E}_i):\) Because otherwise there is \(\bar{x} \in \text{Ker}(\tilde{E}_i|_X) \setminus \text{Z}(\tilde{E}_i)\) and so there is \(\bar{y} \in \tilde{E}_i\) such that \(1 \neq [\bar{x}, \bar{y}]\). Now \([\bar{x}, \bar{y}]\) is a nontrivial element of \(\Phi(\tilde{E}_i)\) acting trivially on \(X\). This contradicts the fact that \(\Phi(\tilde{E}_i)\) is faithful on \(X\).

Thus \(\text{Z}(\tilde{E}_i) = \text{Z}(\tilde{E}_i)/\text{Ker}(\tilde{E}_i)\). We conclude that \(\tilde{E}_i/\text{Z}(\tilde{E}_i)\) and \(\tilde{E}_i/\text{Z}(\tilde{E}_i)\) are \(\langle z \rangle\)-isomorphic modules. Since \(\langle z \rangle\) is trivial on \(\tilde{E}_i\), it is trivial on \(V\) also. An application of the three subgroup lemma supplies that \([E_{i-1}, z]\) is also trivial on \(V\). It follows that \([E_{i-1}, z]\) is trivial on each of the \(E_{i-1}\langle z \rangle\)-composition factors of \(V\). Note that \(V\) is a nonzero quotient module of \(\tilde{E}_i\). Since \(\tilde{E}_i\) is a direct sum of ample irreducible \(E_{i-1}\langle z \rangle\)-submodules, so is \(V\), that is, \([E_{i-1}, z]\) is nontrivial on \(V\), a contradiction completing the proof of Lemma 4. \(\square\)

**Proof of Claim 1.** We have \(E_{i-1} \neq 1\) as \([E_i, E_{i-1}] = E_i\). Also \(\text{Ker}(E_i|_{E_{i+1}}/\Phi(P_{i+1})) = \text{Ker}(E_i|_{E_{i+1}}) = \text{Ker}(E_i|_{\tilde{E}_{i+1}})\). Appealing to Remark (c) together with \(V = \tilde{P}_{i+1}, G = P_i, L = F_{i-1}\) and \(M = [E_i, z]\), we see that \(\text{Ker}(E_i|_{\tilde{E}_{i+1}})\) is \(F_{i-1}\)-invariant. This yields that \(D_{i'} = F_i/\text{Ker}(F_i|_{\tilde{E}_{i+1}})\) is \(F_{i-1}\)-invariant, as \(F_{i-1}\) normalizes \(F_i\).

We know that \(\tilde{E}_i = \bigoplus_{j=1}^{l} W_{ij}\) where \(W_{i1}, \ldots, W_{il}\) are irreducible ample \(E_{i-1}\langle z \rangle\)-submodules. Set \(W_{ij} = U_j/\Phi(E_i)\) for each \(j = 1, \ldots, l\). Since \(\tilde{P}_{i+1}|_{E_i}\) is completely reducible and \(E_i\) is
faithful on \( \tilde{P}_{i+1} \), there exists at least one irreducible component of \( \tilde{P}_{i+1}|_{E_i} \) on which \( U_j \) acts nontrivially. Let \( \mathfrak{M}_j \) denote the set of all such components of \( \tilde{P}_{i+1}|_{E_i} \).

There are two cases: Either

(I) there is at least one \( N \) in \( \mathfrak{M}_j \) on which \( \Phi(E_i) \) acts trivially,

or

(II) there is no \( N \) in \( \mathfrak{M}_j \) on which \( \Phi(E_i) \) acts trivially.

In the latter case, as an immediate consequence of Lemma 4, we have the following:

Let \( N \) be an irreducible component of \( \tilde{E}_{i+1}|_{E_j} \). Then \( N \in \mathfrak{M}_j \) iff \( \Phi(E_i) \) acts nontrivially on \( N \).

Thus \( U_j \) is trivial on each irreducible component \( N \) of \( \tilde{P}_{i+1}|_{E_i} \), lying outside \( \tilde{E}_{i+1} \), because otherwise \( N \in \mathfrak{M}_j \) implying that \( \Phi(E_i) \) and hence \( [E_i, z] \) is nontrivial on \( N \), a contradiction. It follows that

\[
1 = \text{Ker}(U_j \text{ on } \tilde{P}_{i+1}) = \text{Ker}(U_j \text{ on } \tilde{E}_{i+1}) \quad \text{when (II) holds.}
\]

Now suppose that \( \text{Ker}(U_j \text{ on } \tilde{E}_{i+1}) = 1 \) for each \( j = 1, \ldots, l \) and \( \text{Ker}(U_j \text{ on } \tilde{E}_{i+1}) \neq 1 \) for each \( j = s + 1, \ldots, l \).

For each \( j = s + 1, \ldots, l \), set \( \Omega_j = \{ f \in W_{ij}^* | \text{there exists } N \in \mathfrak{M}_j \text{ on which } \Phi(E_i) \text{ acts trivially and } \text{Ker}(U_j \text{ on } N)/\Phi(E_i) \subseteq \text{Ker } f \} \). Now for each \( N \in \mathfrak{M}_j \) on which \( \Phi(E_i) \) acts trivially, \( \text{Ker}(U_j \text{ on } N)/\Phi(E_i) \) is proper in \( W_{ij} \) and hence is contained in a maximal subspace \( M \). Therefore \( \Omega_j \neq \emptyset \). Also \( \Omega_j \) is \( E_{i-1}(z) \)-invariant. This yields that \( \langle \Omega_j \rangle = W_{ij}^* \), by the irreducibility of \( W_{ij}^* \) as an \( E_{i-1}(z) \)-module.

Now for each \( j = 1, \ldots, l \), we set \( K_j = \text{Ker}(U_j \text{ on } \tilde{E}_{i+1}) \). Then \( K_j \Phi(E_i)/\Phi(E_i) \subseteq (W_{ij})_0 \). If not, then \( j \in \{s + 1, \ldots, l \} \) and there exist \( x \in K_j, f \in \Omega_j - C_{\Omega_j}(z) \text{ such that } f(x \Phi(E_i)) \neq 0 \).

By the definition of \( \Omega_j \), we can find an irreducible submodule \( N \) of \( \tilde{P}_{i+1}|_{E_j} \) on which \( U_j \) is nontrivial, \( \Phi(E_i) \) is trivial and \( \text{Ker}(U_j \text{ on } N)/\Phi(E_i) \subseteq \text{Ker } f \). Then \( x \notin \text{Ker}(U_j \text{ on } N) \) as \( x \in \text{Ker}(U_j \text{ on } \tilde{E}_{i+1}) \). As \( x \in \text{Ker}(U_j \text{ on } \tilde{E}_{i+1}) \), \( N \) lies outside \( \tilde{E}_{i+1}|_{E_i} \), that is, \( [E_i, z]^{p-1} = [E_i, z] \) acts trivially on \( N \). Thus \( [U_j, z] \) is trivial on \( N \) and so \( f \in C_{\Omega_j}(z) \), a contradiction.

Since \( W_{ij} \) is an irreducible \( E_{i-1}(z) \)-module, \( W_{ij}|_{E_{i-1}} \) decomposes into a direct sum of homogeneous \( E_{i-1} \)-modules which are permuted transitively by \( (z) \). Since \( [E_{i-1}, z]^{p-1} \) is nontrivial on at least one of these components, it is nontrivial on all of them. It follows that \( [E_{i-1}, z]^{p-1} \) acts nontrivially on each irreducible component of \( W_{ij}|_{E_{i-1}} \) for each \( j = 1, \ldots, l \).

Let \( \Omega_j \) denote the whole of \( W_{ij}^* \) when \( j \in \{1, \ldots, s \} \). Appealing to Theorem 1 for each \( j = 1, \ldots, l \) together with the action of \( E_{i-1}(z) \) on \( W_{ij} \) and the corresponding \( \Omega_j \), we see that \( C_{W_{ij}}(z) \nsubseteq (W_{ij})_0 \) and \( (F_{i-1} \text{ on } C_{W_{ij}}(z)/C(W_{ij})_0(z)) \equiv_w (F_{i-1} \text{ on } W_{ij}) \).

We shall now observe that for each \( j = 1, \ldots, l \), \( (F_{i-1} \text{ on } W_{ij}) \equiv_w (F_{i-1} \text{ on } C_{W_{ij}}(z)) \): If \( p_i - 1 = p \) or \( [Z(E_{i-1}), z] \) is nontrivial on \( W_{ij} \), this holds by Lemma 1. Assume that \( p_i - 1 \neq p \) and \( [Z(E_{i-1}), z] \leq K = \text{Ker}(E_{i-1} \text{ on } W_{ij}) \). Since \( [E_{i-1}, z] \) is nontrivial on \( W_{ij} \) and \( p_i - 1 \) is odd, it can be easily seen that \( C_{W_{ij}}(z) \neq 0 \). Put \( \tilde{E}_{i-1} = E_{i-1}/K \). As \( \Phi(\tilde{E}_{i-1}) = \Phi(E_{i-1}) \leq Z(\tilde{E}_{i-1}(z)) \), \( \tilde{E}_{i-1} \) is a central product of \( \tilde{E}_{i-1}(z)(z) \) and \( C_{\tilde{E}_{i-1}}(z) \). Then \( C_{\tilde{E}_{i-1}}(z) \) is the central product of \( \tilde{E}_{i-1}(z)(z) \) and \( C_{\tilde{E}_{i-1}}(z) \).
and $W_{ij}/C_{w_{ij}}(z)$ is homogeneous. We have $F_{i-1} - w C_{E_{i-1}}(z)$ yielding that $(F_{i-1} - w C_{w_{ij}}(z))$. Thus $(F_{i-1} - w C_{w_{ij}}(z))$ holds, for each $j = 1, \ldots, l$. Set $L_j = \text{Ker}(C_{U_j}(z)$ on $\tilde{E}_{i+1})$. Notice that any nontrivial irreducible $F_{i-1}$-submodule of $C_{U_j}(z)/C_{w_{ij}}(z)$ is $F_{i-1}$-isomorphic to an irreducible $F_{i-1}$-submodule of $C_{U_j}(z)/L_j$. Therefore any nontrivial irreducible $F_{i-1}$-submodule of $W_{ij}$ is $F_{i-1}$-isomorphic to an irreducible $F_{i-1}$-submodule of $C_{U_j}(z)/L_j$. On the other hand, any nontrivial irreducible $F_{i-1}$-submodule of $C_{U_j}(z)/L_j$ is $F_{i-1}$-isomorphic to an irreducible $F_{i-1}$-submodule of $W_{ij}$ and hence to an irreducible $F_{i-1}$-submodule of $W_j$. This shows that $(F_{i-1} - w C_{w_{ij}}(z))$ holds.

As $\tilde{E}_i = \bigoplus_{j=1}^l W_{ij}$ and $C_{\tilde{E}_i}(z) = \bigoplus_{j=1}^l C_{w_{ij}}(z) = \bigoplus_{j=1}^l C_{U_j}(z)\Phi(E_i)/\Phi(E_i)$, we have $(F_{i-1} - w C_{w_{ij}}(z)) \equiv w (F_{i-1} - w C_{U_j}(z)/\text{Ker}(C_{E_i}(z)\Phi(E_i)/\Phi(E_i)))$. Notice that $D_i = C_{E_i}(z)/\text{Ker}(C_{E_i}(z)$ on $\tilde{E}_{i+1})$. Hence $(F_{i-1} - w C_{w_{ij}}(z)) \equiv w (F_{i-1} - w D_i)$, because $[\Phi(D_i), F_{i-1}] = 1$. Since $C_{U_j}(z) \not\subseteq (W_{ij})_0$ we have $F_i = C_{E_i}(z) \not\subseteq \text{Ker}(E_i)$ on $\tilde{E}_{i+1})$ and so $D_i \not= 1$, completing the proof of Claim 1.

**Proof of Claim 2.** Suppose that $p_i = p$ for some $i \geq 2$. If $i \not= 2$, assume that $D_{i-1} \not= 1$. Now Ker($[E_i, z]^{p-1}$ on $\tilde{E}_{i+1}$) = Ker([E_i, z]^{p-1} on $\tilde{E}_{i+1}$). Since $F_i \not= [E_i, z]^{p-1}$, we have Ker($F_i$ on $\tilde{E}_{i+1}$) = 1, that is $F_i$.

We first consider the case $i = 2$. Then $p_2 = p$ and so $p_1 \not= p$. Since $E_1 = P_1$ and [E_1, z] \not= 1, we see that $E_2 \not= 0$. Applying Theorem 2 to the action of $E_1(z)$ on each irreducible $E_1(z)$-component of $E_2$, we get $[E_2, z]^{p-1} \not= 0$. This yields that $[E_2, z]^{p-1} \not= 1$ and so $F_2 = C_{E_2, z}^{p-1}(z) \not= 1$. As $F_1 = 1$, this completes the proof of Claim 2 when $i = 2$.

We next assume that $i > 2$. Now $p_{i-1} \not= p$ and $F_{i-1} = C_{E_{i-1}, z}$. Since $D_{i-1} \not= 1, F_{i-1} \not= 1$ and $\tilde{E}_i \not= 0$. We apply Theorem 2 to the action of $E_{i-1}(z)$ on each irreducible $E_{i-1}(z)$-component of $\tilde{E}_i$ to get $[\tilde{E}_i, z]^{p-1} \not= 0$ and $(F_{i-1} - w \tilde{E}_i) \equiv w (F_{i-1} - \tilde{E}_i, z]^{p-1})$. This gives that $(F_{i-1} - w \tilde{E}_i) \equiv w (F_{i-1} - [\tilde{E}_i, z]^{p-1}, F_{i-1})$ as $[\tilde{E}_i, z]^{p-1} = [\tilde{E}_i, z]^{p-1}, F_{i-1}] \not= C_{E_{i-1}, z}^{p-1}(F_{i-1})$. Now $(F_{i-1} - w \tilde{E}_i) \equiv w (F_{i-1} - \tilde{E}_i)$ holds, because $[\Phi(E_i), F_{i-1}] = 1$. This finishes the proof of Claim 2.

**Proofs of theorems.**

**Theorem A.** Let $A$ be an abelian group acting fixed point freely on a group $G$ of odd order. If $A$ has squarefree exponent coprime to 6, then $f(G) \leq \ell(A)$.

**Proof.** Set $f = f(G)$. By Lemmas 8.1 and 8.2 in [2], there is an $A$-Fitting chain of length $f$ in $G$. Since $A$ is nilpotent, it is a Carter subgroup of any semidirect product of it with a section of $G$. Thus $A$ acts fixed point freely on any section of this chain. Hence once the following assertion referring only to $A$-Fitting chains is proved, the theorem will follow immediately.

Let $A$ be an abelian group of squarefree exponent coprime to 6, and let $P_1, \ldots, P_t$ be an $A$-Fitting chain of a finite solvable group $G$ such that $P_i$ has odd order and $A$ acts fixed point freely on $P_i$ for each $i = 1, \ldots, t$. Then $t \leq \ell(A)$.

We shall use induction on $t$. We may assume that $P_1$ is an irreducible $A$-module. As $A$ acts fixed point freely on $P_1$, there exists $z \in A$ of prime order $p$ such that $[P_1, z] \not= 1$. Then $[P_1, z] =$
Let \( \ell(\langle z \rangle) \in [2, \ell(C)] \). Then \( f(G) \leq 2^{\ell(C)} - 1 \).

**Proof.** Set \( f = f(G) \). By Lemmas 8.1 and 8.2 in [2], there is a \( C \)-Fitting chain \( P_1, \ldots, P_f \). Since \( C \) is a Carter subgroup of \( H \) with \( G \cap C = 1 \), it centralizes no nontrivial section of \( G \). By Lemma 5, we obtain that \( f \leq 2^{\ell(C)} - 1 \). \( \square \)

**Theorem B.** Let \( C \) be a Carter subgroup of a group \( G \). If \( G \) has order coprime to 6, then \( f(G) \leq 2^{\ell(C)} - 1 \).

**Proof.** Set \( f = f(G) \). We use induction on \( \ell(C) \). If \( \ell(C) = 0 \), then \( C = 1 \) and so the theorem follows. Assume that \( \ell(C) > 0 \). Fix a Carter subgroup \( C \) of \( G \). There is an integer \( k \geq 0 \) such that \( F_k(G) \cap C = 1 \) and \( F_{k+1}(G) \cap C \neq 1 \). Put \( \overline{G} = G/F_{k+1}(G) \). Since \( \overline{C} \) is a Carter subgroup of \( \overline{G} \) and \( F_{k+1}(G) \cap C \neq 1 \), \( \ell(\overline{C}) < \ell(C) \). So by induction

\[
\ell(\overline{C}) = f - k - 1 \leq 2^{\ell(C)} - 1.
\]

Now \( C \) is a Carter subgroup of \( K = CF_k(G) \) and \( F_k(G) \) is a normal complement to each Carter subgroup of \( K \). Thus \( k = f(F_k(G)) \leq 2^{\ell(C)} - 1 \) by Theorem 4.

It follows that

\[
f = 1 + k + (f - k - 1) \leq 1 + 2^{\ell(C)} - 1 + 2(2^{\ell(C)-1} - 1) = 2^{\ell(C)} - 1.
\]  \( \square \)
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